



# Epidemics: towards understanding undulation and decay

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## Abstract

Undulation (usually called *waves*) of infection levels in epidemics, is not well understood. In this paper we propose a mathematical model that exhibits undulation (oscillation) and decay towards a stable state. The model is a re-interpretation of the original SIR-model obtained by postulating different constitutive relations whereby classical logistic growth with recovery is obtained. The recovery relation is based on the premise that it is only achieved after some time. This leads to a differential–difference (delay) equation which intrinsically exhibits periodicity in its solutions but not necessarily decay to asymptotic equilibrium. Limit cycles can indeed occur. An appropriate linearization of the governing equation provides a firm basis for heuristic reasoning as well as confidence in numerical calculations.

**Keywords** Epidemic · Waves · Oscillation · Decay

**Mathematics Subject Classification** 39A21 · 92D30

## 1 Introduction

In a pioneering paper Hutchinson [12] states: “...that circular paths often exist which tend to be self-correcting within certain limits, but which break down, producing violent oscillations ...” An equation to model this situation, is given in a footnote to the paper as

$$\frac{dy}{dt}(t) = y(t)[1 - y(t - \tau)] \quad (1.1)$$

with  $\tau$  a ‘time lag’. This, with some constants added, came to be known as *Hutchinson’s equation*. The oscillatory nature of its solutions was the subject of a number of mathematical studies, the earliest of which are Cunningham [5], Wright [20] and Jones [13], sometimes in an equivalent form. A generalization dealing with several time lags is treated in Gopalsamy [9]. This was anteceded by Van der Plank [17, 18] who considered plant diseases in which dormant as well as an infectious periods are taken into account. The equation that carries his

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For my soul-mate Adri Prinsloo (1974–2021) whose penetrating questions—why, not how—contributed immensely to the development and understanding of this work. She left this life too soon.

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name, suitably transformed, is

$$\frac{dy}{dt}(t) = Ry(t)[y(t - \tau_d) - y(t - \tau_i)], \quad (1.2)$$

which is of interest to us when the period  $\tau_d$  of dormancy is taken to be zero. By and large, studies of equations such as these have concentrated on long term behaviour of solutions, particularly decay to a point of equilibrium; “flattening of the curve”. A phenomenological model for the prediction of “waves” is presented by Cacciapaglia et al. [4].

In present-day clinical contexts, the phenomenon of “waves of infection” seems to be very much at the forefront but does not appear to be well-understood or even defined. It is the purpose of this paper to align this with Hutchinson’s “violent oscillations” which have been mathematically shown to be exhibited by equations such as (1.1) and its generalizations. The term *wave* seems to be inappropriate since wave phenomena inseparably involve both time and space. For this reason we have chosen the word *undulation*. This phenomenon is known to occur in plant as well as animal populations.

Instead of (1.1) we shall use the “logistic delay equation” as stated in Ruan [16]. It is of the form  $x'(t) = rx(t)[1 - a_1x(t) - a_2x(t - \tau)]$ , with prime denoting the time derivative  $d/dt$ . This equation turns up occasionally without indication of the assumptions made to derive it. In Sect. 2 we give a systematic derivation based on a general view of the SIR model introduced by Kermack and McKendrick [14]. This view also leads to the “theta-model” which can be used to obtain better correspondence to observed data. Moreover, this approach establishes parameters with specific significance, easily lost when the treatment is entirely mathematical. Section 3 deals with normalization and scaling of the equations to arrive at more familiar forms. It is also shown there that the theta-model equation can be transformed to the ‘standard’ form with constants and variables having different meanings.

Some general results are obtained in Sect. 4 such as positivity of solutions when the ‘initial history’ is so, and an upper bound which implies that solutions cannot grow in an unbridled manner. In Sect. 5 we obtain a ‘natural’ linearization and in Sect. 6 treat the linear homogeneous problem by means of the Laplace transform. This elaborate treatment serves to augment rather sketchy treatments found in the literature by constantly keeping track of the model parameters. Inversion of the Laplace transform is discussed in Sect. 7. This leads to a series representation of the solution of the linear homogeneous problem. Section 8 gives sharp estimates of the position of poles and terms in the series solution. It is shown that convergence to equilibrium is guaranteed if a constant, given in terms of the parameters, is sufficiently small.

Significance of linear analysis for the non-linear problem is discussed in Sect. 9. In Sect. 10 we present a numerical example to illustrate that undulations, as exhibited by the model and supported by the results of Sects. 6–9, can qualitatively be in accordance with observed phenomena. The example shows that decay to an asymptotic equilibrium point is possible, but also that stable limit cycles can possibly occur. The numerical procedure, based on a construction in Sect. 4, is sketched in Sect. 11.

The concluding remarks of Sect. 12, although unscientific, have some seriousness about them.

## 2 SIR, Verhulst and more

Fundamental to many mathematical descriptions of epidemics is the SIR model and its variants. Here three quantities, the number of susceptibles  $S(t)$ , the number of infectious

individuals  $I(t)$ , and the number recovered (restored?)  $R(t)$  at time  $t$ , are related by the ‘conservation principle’

$$S(t) + I(t) + R(t) = N \tag{2.1}$$

with the constant  $N$  denoting the total ‘population’ considered. The dynamics of the epidemic is in the system of ordinary differential equations

$$I'(t) = -[S'(t) + R'(t)]; \tag{2.2}$$

$$S'(t) = \mathfrak{F}_S(S(t), I(t), R(t)); \tag{2.3}$$

$$R'(t) = \mathfrak{F}_R(S(t), I(t), R(t)). \tag{2.4}$$

Equation (2.2) is simply (2.1) differentiated. For specific purposes the ‘driving forces’  $\mathfrak{F}_S$  and  $\mathfrak{F}_R$  are chosen by postulates known as *constitutive relations*. In the original SIR-model, the postulates are:  $\mathfrak{F}_S = -\beta I(t)[S(t)/N]$  and  $\mathfrak{F}_R = \gamma I(t)$  with  $\beta$  and  $\gamma$  positive ‘rate’ constants (dimension [time]<sup>-1</sup>).

The Verhulst logistic growth model [19], originally aimed at population growth, is sometimes used for epidemics when recoveries do not occur. It may be considered as a fundamental principle of population dynamics. In this model the population size  $N$  is called the *carrying capacity*, the number of individuals that can be infected, and recovery is ignored. Let  $P(t)$  be the probability of finding an infectious individual at time  $t$ . Then the relevant constitutive relation, to which is added an assignment of probability, is expressed as follows:

$$\mathfrak{F}_S = -\beta I(t)[1 - P(t)]; \tag{2.5}$$

$$P(t) = I(t)/N, \tag{2.6}$$

where  $1 - P(t)$  signifies the probability of finding a ‘healthy’ individual. This, combined with (2.2) and (2.3), yields the classical *logistic equation*

$$I'(t) = \beta I(t)[1 - I(t)/N]$$

which is at the core of many informed speculations.

We now turn to the situation where *recovery* is also taken into account and postulate, in (2.4), the relation

$$\mathfrak{F}_R = \gamma I(t)P(t - \tau), \tag{2.7}$$

with  $\tau > 0$  the *period of infectiousness*. This means that the rate of recovery is proportional to the probability of infectiousness occurring at the earlier instant  $t - \tau$ . If  $P(t)$  is still specified according to (2.6), the relations (2.5), (2.7) gives the *logistic-recovery equation*

$$I'(t) = I(t)\{\beta[1 - (I(t)/N)] - \gamma(I(t - \tau)/N)\}. \tag{2.8}$$

This is the equation we shall study, although expressed differently.

In a model for competing species Gilpin and Ayala [8] essentially chose  $P(t) = [I(t)/N]^\theta$  with  $\theta$  a positive constant. In the context of our discussion here, this gives rise to the *theta-logistic-recovery equation*

$$I'(t) = I(t)\{\beta[1 - (I(t)/N)^\theta] - \gamma(I(t - \tau)/N)^\theta\}. \tag{2.9}$$

One may view this as a mathematical generalization of the logistic model ( $\theta = 1$ ) to manipulate the sigmoidal curve, but it can be grounded in probability theory (Feller [7, II.5–8, Randomization]).

Constitutive relations in SIR models can be found in Della Morte and Sannino [6] and Buonomo and Cerasuolo [3]. In the latter a time lag is introduced in  $\mathfrak{F}_S$ . A SIR-model with delay in  $\mathfrak{F}_R$ , not the same as (2.7), has been suggested by Reiser [15].

### 3 Normalization, scaling and a reduction

We first normalize the equation (2.8) by the setting  $F(t) = I(t)/N$  so that  $I = N$  corresponds to  $F = 1$ , and  $P(t) = F(t)$ . One may think of  $F(t)$  as the *level of infectiousness* at time  $t$ . The result is

$$F'(t) = F(t)\{\beta[1 - F(t)] - \gamma F(t - \tau)\}. \tag{3.1}$$

This is a *differential–difference equation* or *delay equation*. To solve the equation for times  $t > 0$  we need to know the state (history) of  $F$  for times  $t \in [-\tau, 0]$  (see e.g., Bellman and Cooke [2] or Hale [10]). Thus we have the *initial condition*

$$F(t) = \Phi(t) \text{ for } -\tau \leq t \leq 0, \tag{3.2}$$

with  $\Phi$  a given function defined on  $[-\tau, 0]$ .

If it is assumed that an asymptotically, non-zero stable state for solutions of (3.1) exists, that is, if  $F'(t) \rightarrow 0$  as  $t \rightarrow \infty$  and the limit  $F_\infty = \lim_{t \rightarrow \infty} F(t)$  exists, it follows that  $F_\infty = \beta/(\beta + \gamma)$ . We use this parameter to scale the equation (3.1). In addition the parameter  $\tau$  is used as unit of time to obtain a completely dimensionless formulation. Thus we define the new variables  $t_\tau = t/\tau$  and  $f(t_\tau) = F(t)/F_\infty = F(\tau t_\tau)/F_\infty$  to obtain, in place of (3.1),

$$\begin{aligned} f'(t_\tau) &= \beta\tau f(t_\tau) \left\{ \left[ 1 - \left( \frac{\beta}{\beta + \gamma} \right) f(t_\tau) \right] - \left( \frac{\gamma}{\beta + \gamma} \right) f(t_\tau - 1) \right\} \\ &= \tilde{\beta} f(t_\tau) \{ [1 - \beta^* f(t_\tau)] - \gamma^* f(t_\tau - 1) \} \end{aligned} \tag{3.3}$$

with  $\tilde{\beta} := \beta\tau$ ,  $\beta^* := \beta/(\beta + \gamma)$ ,  $\gamma^* := \gamma/(\beta + \gamma)$  and the prime denoting  $d/dt_\tau$ . We note also the identity  $\beta^* + \gamma^* = 1$  which will be important at a later stage and gives rise to an alternative form discussed in Sect. 12. The initial condition (3.2) now has the form

$$f(t_\tau) = \phi(t_\tau) := \frac{\Phi(\tau t_\tau)}{F_\infty} \text{ for } -1 \leq t_\tau \leq 0. \tag{3.4}$$

It is clear that if a non-zero stable asymptote exists, it has the value  $f_\infty = 1$ . Also note that (3.3) has the same form as the “logistic delay equation” mentioned in Sect. 1 but the constants can be interpreted in terms of the constitutive relations (2.5), (2.7).

Under the transformations just described, Eq. (2.9) has the form

$$f'(t_\tau) = \tilde{\beta} f(t_\tau) \{ 1 - \beta^* f^\theta(t_\tau) - \gamma^* f^\theta(t_\tau - 1) \}.$$

This equation can assume a less intimidating look if we replace  $f^\theta(t_\tau)$  with  $f(t_\tau)$  to obtain an equation of precisely the same form as (3.3) except that now  $\tilde{\beta} = \theta\beta\tau$ . Thus we go on to study (3.3).

### 4 Some general considerations

In this section we consider the scaled equation (3.3) together with the initial condition (3.4). We shall abuse notation by writing  $t$  instead of the dimensionless time  $t_\tau$ . The initial level  $f_0 := f(0)$  will be of significance.

First we obtain a formal representation of the solution by the substitution  $g(t) = 1/f(t)$ , familiar for equations of the Bernoulli-kind. This leads to

$$g'(t) + \tilde{\beta}[1 - \gamma^* f(t - 1)]g(t) = \beta^* \tilde{\beta}. \tag{4.1}$$

If we interpret the term  $f(t - 1)$  as  $1/g(t - 1)$  this is a differential–difference equation for  $g$  with initial condition

$$g(t) = 1/\phi(t) \text{ for } t \in [-1, 0]. \tag{4.2}$$

Associated with this equation we have the integrating factor

$$i(t) := \tilde{\beta}[t - \gamma^* \psi(t)]; \text{ with } \psi(t) := \int_0^t f(s - 1) ds = \int_{-1}^{t-1} f(s) ds, \tag{4.3}$$

so that

$$\frac{d}{dt}[\exp\{i(t)\}g(t)] = \beta^* \tilde{\beta} \exp\{i(t)\}. \tag{4.4}$$

This can be integrated directly. However, further integration by parts of the term on the right yields:

$$\begin{aligned} \exp\{i(t)\}g(t) &= g_0 - \beta^* + \beta^* \exp\{i(t)\} \\ &\quad + \tilde{\beta}\beta^* \gamma^* \int_0^t \exp\{i(s)\}f(s - 1) ds, \end{aligned} \tag{4.5}$$

where  $g_0 = 1/f_0$ .

The formal calculations above can be placed on a firmer footing under the following hypotheses about the initial state which will be taken for granted from now on:

H1. *The initial state  $\phi$  is continuous on  $[-1, 0]$ .*

H2.  *$\phi(t) > 0$  for  $t \in [-1, 0]$ .*

We immediately note that the integrating factor  $i(t)$  exists and is differentiable for  $t > 0$ .

The initial value problem we study is well-posed in the sense of the following result.

**Theorem 4.1** *Under Assumptions H1 and H2:*

- (a) *The function  $g(t)$  as represented in (4.5) is a positive solution of (4.1), (4.2) with  $g(0) = g_0$ .*
- (b) *The function  $f(t) = 1/g(t)$  is a positive solution of (3.3), (3.4).*

**Proof** The proof of the two assertions will simultaneously unfold by progression over the time intervals  $[0, 1)$ ,  $[1, 2)$ , ... as in many instances to be found in [2].

We begin with  $0 \leq t < 1$ . Here the function  $\psi$  is totally determined by the initial state  $\phi$ ; it is in fact differentiable and positive. Thus the integrating factor is of suitable nature and the function  $g(t)$ , as determined by (4.5), is positive and solves the initial value problem (4.1), (4.2). It follows that  $f(t) > 0$  solves (3.3), (3.4). Thus (a) and (b) are established in  $[0, 1)$ . In addition the limit as  $t \rightarrow 1$  defines the functions  $g(1)$  and  $f(1)$

The same argument can be followed in the interval  $[1, 2)$ , as the crucial properties have been established in  $[0, 1]$ . It is clear that a formal induction argument will lead to the required outcome. □

**Theorem 4.2** *Under Assumptions H1 and H2 the solution of the initial value problem (3.3), (3.4) is restricted in the following ways:*

- (a) *It is bounded. Specifically,*

$$0 < f(t) < \frac{f_0}{\beta^* f_0 + (1 - \beta^* f_0) \exp\{-\tilde{\beta}t\}} \text{ for } t > 0.$$

- (b) *If the limit  $f_\infty = \lim_{t \rightarrow \infty} f(t)$  exists, it is equal to 1*

**Proof** Since  $f(t) > 0$  for  $t \geq -1$ , we see from (4.3) that  $i(t) < \tilde{\beta}t$ . From the representation (4.5) follows that

$$g(t) > \beta^* + (g_0 - \beta^*) \exp\{-i(t)\} > \beta^* + (g_0 - \beta^*) \exp\{-\tilde{\beta}t\}.$$

The stated upper bound is obtained by reciprocation and further manipulation.

To prove (b) let us assume that the limit is zero. Then for given  $\varepsilon \in (0, 1)$  there exists  $t_0$  such that for  $t > t_0$  both  $f(t)$  and  $f(t - 1)$  are less than  $\varepsilon$ . This implies that  $f'(t) > \tilde{\beta}f(t)[1 - \varepsilon]$  since  $\beta^* + \gamma^* = 1$ . It now follows that  $f(t) > f(t_0) \exp\{\tilde{\beta}[1 - \varepsilon](t - t_0)\}$  for all  $t > t_0$  which, according to (a), is absurd.

It is seen from (3.3) that the limit  $\lim_{t \rightarrow \infty} f'(t)$  exists. The mean value theorem shows that this limit is zero, and hence  $f_\infty [1 - f_\infty] = 0$ . □

We note that  $f(t) < M := \max\{f_0, 1/\beta^*\}$ .

### 5 Linearization

For the problem at hand we arrive at a suitable linearization by shifting the (expected) equilibrium level from  $f = 1$  to  $f = 0$ . To avoid an undue proliferation of symbols, we once again abuse notation by replacing  $f(t) - 1$  by  $f(t)$ . The governing equation (3.3) then has the form

$$f'(t) + \tilde{\beta}[\beta^* f(t) + \gamma^* f(t - 1)] = -\tilde{\beta}f(t)[\beta^* f(t) + \gamma^* f(t - 1)]. \tag{5.1}$$

Of course, the initial condition (3.4) is adapted accordingly.

One immediately notes that the left of (5.1), as opposed to the right, is linear. Intuitively, if  $f(t)$  is close to zero for large  $t$ , the non-linear term which is quadratic in  $f$  will be insignificant in the long run. Formally, the right of (5.1) linearizes to zero. This leads us to the linear homogeneous equation

$$f'(t) + \tilde{\beta}[\beta^* f(t) + \gamma^* f(t - 1)] = 0, \tag{5.2}$$

which will be studied in detail.

Some simplifying notation is introduced:  $b := \tilde{\beta}\beta^*$ ,  $c := \tilde{\beta}\gamma^*$  and  $B := c \exp\{b\}$ . We obtain an estimate for solutions of the non-homogeneous problem

$$\left. \begin{aligned} f'(t) + bf(t) + cf(t - 1) &= v(t) \text{ for } t > 0; \\ f(t) &= \phi(t) \text{ for } -1 \leq t \leq 0, \end{aligned} \right\} \tag{5.3}$$

with  $v$  a given function, continuous on  $[0, \infty)$ .

**Theorem 5.1** *Every solution of (5.3) satisfies*

$$|f(t)| \leq [F_0 + V(t)] \exp\{(B - b)t\}; \quad t \geq 0,$$

with  $F_0 := |f_0| + c \int_{-1}^0 \exp\{bs\} |\phi(s)| ds$  and  $V(t) = \int_0^t \exp\{bs\} |v(s)| ds$ .

**Proof** Let us write the first equation in (5.3) in the form

$$\frac{d}{dt} [\exp\{bt\} f(t)] = \exp\{bt\} [v(t) - cf(t - 1)],$$

integration of which yields

$$\begin{aligned} [\exp\{bt\}f(t)] &= f_0 - c \int_0^t \exp\{bs\}f(s-1)ds + \int_0^t \exp\{bs\}v(s)ds \\ &= f_0 - c \int_{-1}^0 \exp\{bs\}\phi(s)ds - c \exp\{b\} \int_0^{t-1} \exp\{bs\}f(s)ds \\ &\quad + \int_0^t \exp\{bs\}v(s)ds. \end{aligned}$$

With  $F(t) := \exp\{bt\}|f(t)|$ , we obtain the integral inequality

$$F(t) \leq F_0 + V(t) + B \int_0^t F(s)ds.$$

Since  $V$  is monotonically increasing, the Grönwall-Bellman inequality [1], [2, Lemma 3.1] applies, i.e.,  $F(t) \leq [F_0 + V(t)] \exp\{Bt\}$ , and the proof is complete.  $\square$

This result shows that the solution of (5.3) is unique and depends continuously on the initial data.

### 6 The linear homogeneous equation

Our attention now turns to the linear equation (5.2) under the initial condition  $f(t) = \phi(t)$  for  $-1 \leq t \leq 0$ ;  $f(0) = f_0$ . The aim of this and the following two sections is to obtain precise conditions on the parameters  $\beta$ ,  $\gamma$  and  $\tau$  for asymptotic stability, instability and oscillation of solutions.

General stability results for first order differential–difference equations are well-known (see e.g. [2, Chap. 6] or [10, Chap. 5]). For equations with constant coefficients—such as (5.2)—a necessary and sufficient condition for asymptotic stability is that the roots of the *characteristic equation* all have positive real part e.g. [2, Cor. 4.2]. Our aim, however, is to relate this condition to the parameters of the problem in such a way that transition from stable to unstable is revealed in a precise way. At the same time the analysis will reveal that oscillations are intrinsic to solutions of the equation. In the literature oscillations are treated as a separate topic e.g. [11, Chap.5].

Our approach is by means of the Laplace transform defined as  $\hat{f}(s) := \int_0^\infty \exp\{-st\}f(t)dt$  for complex  $s$  with positive real part. This approach is sketched in [2, §3.7].

Theorem 5.1 with  $v \equiv 0$  shows that for  $\text{Re } s > B - b$  the Laplace transform of (5.2) may be taken. The result is

$$\left. \begin{aligned} h(s)\hat{f}(s) &= H(s); \\ h(s) &:= s + b + c \exp\{-s\}; \\ H(s) &:= f_0 - c \exp\{-s\} \int_{-1}^0 \exp\{-st\}\phi(t)dt. \end{aligned} \right\} \tag{6.1}$$

For inversion of  $\hat{f}$  we need to study the zeros of the complex-valued function  $h(s)$  for  $s = x + iy$  ( $i = \sqrt{-1}$ ) in the complex plane, which is related to the roots of the characteristic equation. For this the real and imaginary parts of  $h$  must vanish and we have

$$(x + b) + c \exp\{-x\} \cos y = 0; \tag{6.2}$$

$$y - c \exp\{-x\} \sin y = 0, \tag{6.3}$$

both of which need to be satisfied. The real part  $x$  determines asymptotic stability while oscillation is reflected by the imaginary part  $y$ . The criterion for stability is, in fact, that all such  $x$  are negative. As a first step we eliminate the trigonometric terms in (6.2), (6.3) to obtain

$$y^2 = c^2 \exp\{-2x\} - (x + b)^2. \tag{6.4}$$

This defines a curve on which the solution points  $s = x + iy$  must lie, but not every point on the curve is necessarily a solution of (6.2) and (6.3). Moreover, the curve so obtained is only defined for those  $x$  for which the right of (6.4) is non-negative. We investigate this question first.

**Lemma 6.1** *With  $B$  a positive constant:*

- (a) *The equation  $B \exp\{-u\} = u$  has a unique positive solution  $u_M$  and  $B/(1 + B) \leq u_M < B$ .*
- (b) *If  $B > e^{-1}$  the equation  $B \exp\{u\} = u$  has no positive solution.*

**Proof** Assertion (a): The existence, uniqueness, positivity and the upper bound of a solution is straightforward. To obtain the lower bound we note that  $\exp\{-u\} \geq 1 - u$  for  $u > 0$ . Hence  $u_M = B \exp\{-u_M\} \geq B(1 - u_M)$  and the result follows.

Assertion (b) follows from the inequality  $u \exp\{-u\} \leq e^{-1}$ . □

In accordance with (6.4), let  $k^{(2)}(x) := c^2 \exp\{-2x\} - (x + b)^2 = [c \exp\{b\}]^2 \exp\{-2(x + b)\} - (x + b)^2$ .

**Proposition 6.2** *Let  $B = c \exp\{b\}$  and  $u_M$  as in Lemma 6.1. Then*

- (a) *For  $-b \leq x \leq x_M := u_M - b$ ,  $k^{(2)}(x) \geq 0$  with equality only if  $x = x_M$ .*
- (b) *For  $x + b < 0$  the function  $k^{(2)}(x) > 0$  if  $B > e^{-1}$ .*

**Proof** For statement (a), let  $u = x + b \geq 0$ . Then  $k^{(2)}(x) = B^2 \exp\{-2u\} - u^2 = [B \exp\{-u\} + u][B \exp\{-u\} - u]$  which, by Lemma 6.1(a) is non-negative for  $u \leq u_M$ . To prove (b) let  $u = -(x + b) > 0$ . Then  $k^{(2)}(x) = B^2 \exp\{2u\} - u^2$ . The result follows from Lemma 6.1(b). □

It is of importance to introduce the parameter  $\rho := \gamma/\beta = \gamma^*/\beta^*$ . It corresponds to the reciprocal of the *basic reproduction number*  $R_0 = \beta/\gamma$  in the classical SIR model. This leads to the relation  $c = \rho b$ . From now on we make an assumption stronger than suggested by Lemma 6.1(b) namely

$$B = c \exp\{b\} = \rho b \exp\{b\} > 1. \tag{6.5}$$

Under this assumption the function  $k^{(2)}(x) \geq 0$  for  $x \leq x_M$  so that  $k^{(2)}(x)$  can actually be considered a square. We define the function  $k$  by

$$\begin{aligned} k^2(x) &:= k^{(2)}(x) = c^2 \exp\{-2x\} - (x + b)^2 \\ &= B^2 \exp\{-2(x + b)\} - (x + b)^2 \\ &= \rho^2 b^2 \exp\{-2x\} - (x + b)^2; \quad x \leq x_M. \end{aligned} \tag{6.6}$$

The curve  $K$  defined in the complex plane by  $s = x \pm ik(x)$  will be our next concern. Because of symmetry we shall deal mainly with the positive branch which will also be referred to as  $K$ .

**Proposition 6.3** *The curve  $K$  has the following properties:*

- (a)  $k(x) \rightarrow \infty$  as  $x \rightarrow -\infty$ .
- (b) At  $x < x_M$  the tangent is negative.
- (c)  $x_M$  is positive if and only if  $\gamma > \beta$ , i.e.,  $\rho > 1$ .

**Proof** The assertion (a) is, by (6.6), straightforward.

From (6.6) we see that  $k(x)k'(x) = -[B^2 \exp\{-2(x + b)\} + (x + b)]$ . If  $x + b > 0$ , the term in brackets is positive. Since  $k(x) > 0$  it follows that  $k'(x) < 0$  for such  $x$ . If  $x + b < 0$  let  $u = -(x + b)$  and it follows that  $B^2 \exp\{-2(x + b)\} + (x + b) = B^2 \exp\{2u\} - u > B^2(1 + 2u) - u > 1 + u > 0$ , by Assumption (6.5). Thus (b) is established.

To prove (c) we notice that  $k^{(2)}(0) = c^2 - b^2$ . Hence if  $c > b$ , the point  $(0, [c^2 - b^2]^{1/2})$  is on the curve  $K$ . From (b) we see that  $x_M$ , where  $k(x) = 0$ , must be positive. But  $c > b$  means the same as  $\gamma > \beta$ . This argument can also be reversed. □

From what we have established so far, the following is significant:

- A. Under Assumption (6.5) the zeros of the function  $h$  are all to the left of the vertical line  $x = B$ .
- B. If  $\beta \geq \gamma$  the zeros in question are to the left of the line  $x = \delta$  for arbitrary  $\delta > 0$ . In fact, if  $\beta = \gamma$ ,  $x_M = 0$ .

Thus inversion of the Laplace transform becomes a distinct possibility if the zeros on the curve  $K$  can be located. For that we need to obtain information about the points  $(x, k(x))$  for which the Eqs. (6.2) and (6.3) are both satisfied. Towards this we eliminate the exponential terms from these equations to obtain, additional to (6.4), the relation  $y + (x + b) \tan y = 0$ . This defines another (multi-branched) curve  $L$  which has to meet the curve  $K$  in certain points. The function to be considered is

$$\ell(x) := -(x + b) \tan k(x). \tag{6.7}$$

The zeros we look for will occur at points where  $\ell(x) = k(x)$ . It is seen from Proposition 6.3(a) that this will happen at points where  $k(x)$  is near  $(n + 1/2)\pi$ ;  $n = 0, 1, 2, \dots$  and this results in a discrete sequence of zeros of  $h$ , each of the form  $s = x + ik(x)$ .

We next examine the possibility that zeros with non-negative real part may occur. According to Proposition 6.3(c) this can only happen if  $\rho > 1$ . It will be necessary to indicate the dependence of functions and derived parameters on  $b = \tilde{\beta}\beta^* = \beta^*\beta\tau$  and  $\rho = \gamma/\beta$ . Thus we write  $B = B(b, \rho)$ ,  $k(x) = k(x; b, \rho)$  in accordance with (6.5), (6.6). Also note that  $x_M$  depends on  $b$  and  $\rho$ .

It is convenient to consider equation (6.2) on the curve  $K$  instead of (6.7). This yields, after some manipulation,

$$r(x; b, \rho) := (x + b) \exp\{x + b\} + B(b, \rho) \cos k(x; b, \rho) = 0. \tag{6.8}$$

To investigate this equation we consider  $r(-b; b, \rho) = B(b, \rho) \cos k(-b; b, \rho) = B(b, \rho) \cos B(b, \rho)$  as can be seen from (6.6). Thus, if  $3\pi/2 > B(b, \rho) > \pi/2 > 1$ ,  $r(-b; b, \rho) < 0$ . Also at  $x = x_M$ , we find that  $r(x_M; b, \rho) = (x_M + b) \exp\{x_M + b\} + B(b, \rho) > 0$  since  $k(x_M; b, \rho) = 0$ . We conclude that  $r(x; b, \rho)$  has zeros in the interval  $(-b, x_M)$ . These zeros may still be negative.

We can, however, find a very interesting value of  $b$  by noticing that  $k(0; b, \rho) = (\rho^2 - 1)^{-1/2}b$  and

$$r(0; b, \rho) = b \exp\{b\} [1 + \rho \cos\{[\rho^2 - 1]^{1/2}b\}]. \tag{6.9}$$

Thus, if we take  $b$  as

$$b_0 = b_0(\rho) := (\rho^2 - 1)^{-1/2} \arccos(-\rho^{-1}) = (\rho^2 - 1)^{-1/2}[\pi - \arccos(\rho^{-1})], \quad (6.10)$$

it is seen that  $r(0; b_0, \rho) = 0$ . Moreover, since  $\rho > 1$ ,  $0 < \arccos(\rho^{-1}) < \pi/2$  so that  $\pi/2 < (\rho^2 - 1)^{1/2}b_0 < \pi$ . Thus,  $\rho b_0 > \pi/2$  and hence  $B(b_0, \rho) > (\pi/2) \exp\{b_0\} > \pi/2$ . For this particular choice of  $b$ ,  $x = 0$  therefore is a zero of  $r$ . Corresponding to  $b_0$  is a critical value of  $\tau$ :

$$\tau_0 = b_0/\beta\beta^* = (1 + \rho)b_0(\rho)/\beta. \quad (6.11)$$

A positive zero may be contrived by incrementing  $b_0$  without changing  $\beta$  and  $\gamma$  (i.e.,  $\rho$  fixed). This amounts to letting  $\tau = \tau_0 + \sigma$ ;  $\sigma > 0$ . Then  $b = b_0 + \beta^* \beta \sigma$ . From (6.9) we see that if

$$\beta\sigma \leq \left[ \frac{\rho + 1}{\rho - 1} \right]^{1/2} [\pi - (\rho^2 - 1)^{1/2}b_0],$$

$r(0; b) < 0$  which means that a positive zero of (6.8) exists.

The zeros of  $h$  are simple. Indeed, if  $s^*$  is a zero, then  $s^* + b = -c \exp\{-s^*\}$  and  $h'(s^*) = 1 - c \exp\{-s^*\} = 1 + b + s^*$  which cannot be zero. Also, when the negative branch of  $K$  namely,  $s = x - ik(x)$  is considered, we see from (6.7) that if  $s = x + ik(x)$  is a zero on the positive branch, its complex conjugate  $\bar{s} = x - ik(x)$  is a zero on the negative branch.

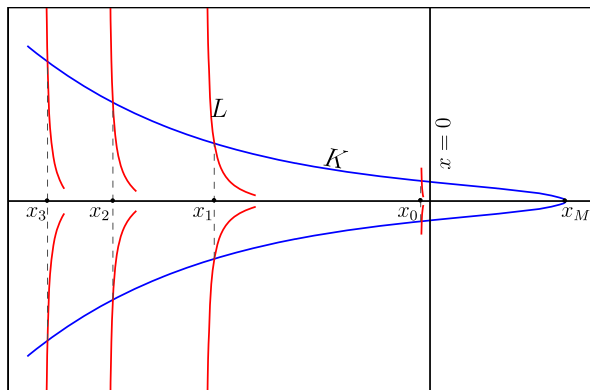
Positioning of the zeros of  $h$  is illustrated in Fig. 1. Increasing  $\tau$  could shift  $x_0$  to ‘the other side’. We order the roots of  $h$  according to their real parts:  $x_n > x_{n+1}$ ;  $n = 0, 1, \dots$  and note that  $x_n \rightarrow -\infty$  as  $n \rightarrow \infty$ .

We summarize the findings above:

**Theorem 6.4** *In terms of the parameters  $b = \tilde{\beta}\beta^*$ ,  $\rho = \gamma/\beta$  and  $B(b) = \rho b \exp\{b\}$  the following is known about the zeros of the function  $h(s) = s + b[1 + \rho \exp\{-s\}]$ :*

- (a) All zeros are simple.
- (b) There is a constant  $p > 0$  such that all zeros occur to the left of the contour  $C = \{s = p + iy : y \in \mathbb{R}\}$ .
- (c) If  $B(b) > 1$  there is a decreasing unbounded sequence  $\{x_n < x_M : n = 0, 1, 2, \dots\}$  such that  $s_n = x_n + ik(x_n; b)$  are zeros. The complex conjugates are also roots.
- (d) If  $\rho \leq 1$  all zeros have negative real part.  
If  $\rho > 1$  and  $3\pi/2 > B(b) > \pi/2$ , a finite number zeros with non-negative real part can occur.

**Fig. 1** Zeros of  $h(s)$



(e) If  $\rho > 1$  there is a critical value  $b_0$ , given by (6.10), such that for  $b < b_0$  all zeros have negative real part and for  $b > b_0$  a zero with positive real part exists. This corresponds to a critical value  $\tau_0$ , given by (6.11), of the delay parameter  $\tau$  with the same effect.

### 7 Laplace inversion

The significance of Theorem 6.4 is seen by considering the Mellin inversion of  $\hat{f}$ . From (6.1),  $\hat{f} = H(s)/h(s)$  and the inversion yields the representation

$$f(t) = \frac{1}{2\pi i} \int_C \frac{H(s)}{h(s)} \exp\{st\} ds.$$

From the residue theorem the integral on the right equals  $2\pi i$  times the sum of the residues of the integrand at its poles, and the poles are precisely the zeros of  $h$ . Now, let  $s_n = x_n + iy_n = x_n + ik(x_n)$  be one of the zeros of  $h$  obtained from the positive branch of the curve  $K$  as discussed so far. To calculate the residue, we consider the Taylor expansion of  $h$  about  $s_n$ . By taking into account that  $s_n + b = -c \exp\{-s_n\}$ ,  $h'(s) = 1 + b - c \exp\{-s\}$  and  $h^{(j)}(s) = (-1)^j c \exp\{-s\}$  for  $j \geq 2$ , we obtain

$$h(s) = (s - s_n) \left[ 1 + b + s_n + (b + s_n) \sum_{j=2}^{\infty} (-1)^{j-1} \frac{(s - s_n)^{j-1}}{j!} \right].$$

The pole at  $s = s_n$  therefore has the residue

$$R(s_n) = \frac{H(s_n)}{1+b+s_n} \exp\{s_n t\}.$$

However, the negative branch of the curve  $K$  also contributes. In fact, if  $s_n = x_n + iy_n$  with  $y_n = k(x_n)$ , its complex conjugate  $\bar{s}_n$  is also a zero of  $h$ . The residue at this pole turns out to be the complex conjugate of  $R(s_n)$ . If we write  $H(s_n)/(1 + b + s_n) = |H(s_n)|/|1 + b + s_n| \exp\{i\sigma_n\}$ , the two residues together contribute to the solution by the term

$$f_n(t) = 2 \frac{|H(s_n)|}{|1 + b + s_n|} \exp\{x_n t\} \cos(y_n t + \sigma_n). \tag{7.1}$$

We immediately note that, since  $y_n > 0$ , there is undulation in every such term. Also, if  $x_n < 0$  the term decays to zero exponentially. This is not the case when  $x_n \geq 0$ . Thus the dominant term in the solution will correspond to  $s_0 = x_0 + ik(x_0)$ .

We conclude this section by estimating the period  $T$  of the principal mode of undulation, namely that associated with  $x_0$ . The ‘angular velocity’ is  $y_0 = k(x_0)$  and the scaled period  $T_\tau = 2\pi/y_0$  so that in unscaled time,

$$T = T_\tau \tau = \left( \frac{2\pi}{y_0} \right) \tau. \tag{7.2}$$

The value of  $x_0$  can be obtained numerically by (carefully) solving (6.8) for  $x$ , making sure that the obtained value is the largest.

### 8 Estimates

The aim of this section is to obtain information about long-term behaviour of the solution of the homogeneous equation  $f'(t) + bf(t) + cf(t - 1) = 0$  under the initial condition  $f(t) = \phi(t)$  for  $t \in [-1, 0]$ ;  $f(0) = f_0$ . To begin with we notice that, at least formally,  $f(t) = \sum_{n=0}^{\infty} f_n(t)$  with the terms  $f_n$  given by (7.1). Since  $x_n \rightarrow -\infty$ , there is a smallest  $m \geq 0$  such that  $x_m < -b$ . With is in mind, we define the (possibly) *principal part* of the solution as  $f_{pr}(t) := \sum_{n=0}^m f_n(t)$  and the *remainder* as  $f_{rm}(t) := \sum_{n=m+1}^{\infty} f_n(t)$  so that  $f(t) = f_{pr}(t) + f_{rm}(t)$ . We obtain estimates for the two components under the assumption that  $x_0 < 0$  which means that all  $x_n$  are negative.

First to be considered is the coefficient  $A_n := 2|H(s_n)|/|1 + b + s_n|$  in (7.1). We introduce the symbolism  $\|\phi\| := \sup_{t \in [-1, 0]} |\phi(t)|$  to obtain from (6.1)

$$\begin{aligned}
 |H(s_n)| &\leq |f_0| + c \exp\{-x_n\} \|\phi\| \int_{-1}^0 \exp\{-x_n t\} dt \\
 &< |f_0| + c|x_n|^{-1} \|\phi\| \exp\{|x_n|\}.
 \end{aligned}
 \tag{8.1}$$

Also,  $|1 + b + s_n|^2 = y^2 + (x_n + b)^2 + 2(x_n + b) + 1 > B^2 \exp\{-2(x_n + b)\} + 2(x_n + b)$  as can be seen from (6.6). We therefore have

$$|1 + b + s_n| > c \exp\{|x_n|\} \text{ if } x_n + b \geq 0. \tag{8.2}$$

Cases where  $x_n + b < 0$  are treated differently. We (temporarily) set  $u_n = -(x_n + b) > 0$  to obtain  $|1 + b + s_n|^2 > B^2 \exp\{u_n\}[1 - 2B^{-2}u_n \exp\{-2u_n\}]$ . From the inequality  $2u \exp\{-2u\} \leq e^{-1}$  we obtain

$$|1 + b + s_n| > cE \exp\{|x_n|\} \text{ if } x_n + b < 0. \tag{8.3}$$

with  $E^2 = 1 - B^{-2}e^{-1}$ . Combination of (8.1) with (8.2) and (8.3) yields

$$\frac{1}{2}A_n < (c|x_0|)^{-1} \exp\{-|x_n|\} |f_0| + \|\phi\| \text{ if } x_n + b \geq 0; \tag{8.4}$$

$$\frac{1}{2}A_n < E^{-1}[c^{-1}|f_0| \exp\{-|x_n|\} + b^{-1}\|\phi\|] \text{ if } x_n + b < 0. \tag{8.5}$$

Here use have been made of the inequalities  $|x_n| \geq |x_0|$  and  $|x_n| > b$  in the two different cases.

From (8.4) and (7.1) it is seen that  $|f_{pr}(t)| < C_{pr} \exp\{-|x_0|t\}$  with  $C_{pr}$  a positive constant. The infinite series  $f_{rm}(t)$  needs more attention. For this it is necessary to obtain information about the behaviour of  $x_n$  for  $n \geq m + 1$ . Our arguments will hinge on the equations (6.2) and (6.4) expressed in the form  $\cos y_n = -B^{-1}(x_n + b) \exp\{x_n + b\}$  and  $y_n^2 = B^2 \exp\{-2(x_n + b)\} - (x_n + b)^2$ . Since  $u_n := -(x_n + b) > 0$  the equations are

$$\cos y_n = B^{-1}u_n \exp\{-u_n\}; \tag{8.6}$$

$$y_n^2 = B^2 \exp\{2u_n\} - u_n^2. \tag{8.7}$$

The constant  $\epsilon := [\arccos(B^{-1}e^{-1})]/\pi < 1/2$  will provide some clarity. In Sect. 6 it is suggested that  $y_n$  should be near  $y_n^* := (n + 1/2)\pi$ , and this we shall make more precise.

From (8.6) we see that  $0 < \cos y_n < B^{-1}e^{-1} = \cos(\epsilon\pi)$ . Standard trigonometry (even a good sketch) shows that these inequalities can only be satisfied by  $y_n$  in the intervals  $(y_n^*, y_n^* + \epsilon\pi]$  for  $n$  odd and  $[y_n^* - \epsilon\pi, y_n^*)$  for  $n$  even. The endpoints  $y_n^*$  are excluded since  $\cos y_n^* = 0$ .

**Theorem 8.1** *If  $x_n + b < 0$  then*

$$\ln[\exp\{-(b + 1)/(n + 1)\pi\}] < x_n < \ln[c/n\pi].$$

**Proof** From (8.7) we see that  $y_n^2 \exp\{-2u_n\} = B^2[1 - (B^{-1}u_n \exp\{-u_n\})^2] = B^2[1 - \cos^2 y_n] = B^2 \sin^2 y_n$  (having used (8.6) again). This, in turn gives (after some manipulation)  $x_n = \ln[c|\sin y_n|/y_n]$ . Careful consideration of the cases  $n$  even/odd leads to  $B^{-1}e^{-1} = \cos \epsilon\pi \leq |\sin y_n| < 1$  and it follows that  $-\ln[\exp\{b + 1\}y_n] \leq x_n \leq \ln[c/y_n]$ .

Since  $0 < \epsilon < 1/2$  we have  $y_n^* - \epsilon\pi > n\pi$  and  $y_n^* + \epsilon\pi < (n + 1)\pi$ . It follows that  $n\pi < y_n < (n + 1)\pi$  regardless of the parity of  $n$ . □

The inequality (8.4) may now be employed to estimate  $f_n(t)$  in (7.1) when  $n \leq m$ . This results in

$$|f_n(t)| < 2[(c|x_0|)^{-1}|f_0| \exp\{-|x_n|t\} + \|\phi\|] \exp\{-|x_n|t\} \text{ if } n \leq m.$$

For the case  $n > m$  we apply (8.5) together with Theorem 8.1 to obtain

$$|f_n(t)| < 2E^{-1} \left[ c^{-1}|f_0| \left(\frac{c}{\pi}\right)^{t+1} \left(\frac{1}{n}\right)^{t+1} + b^{-1}\|\phi\| \left(\frac{c}{\pi}\right)^t \left(\frac{1}{n}\right)^t \right] \text{ if } n > m.$$

The solution  $f$  of the linearized homogeneous problem can now be estimated in terms of the parameters of the problem:

**Theorem 8.2** *If  $x_0 < 0$  then:*

(a) *There exists a constant  $C_{pr} > 0$  such that*

$$|f_{pr}(t)| < C_{pr} \exp\{-|x_0|t\}.$$

(b) *There exist positive constants  $C_{rm,1}, C_{rm,2}$  such that for  $t > 1$*

$$|f_{rm}(t)| < C_{rm,1} \zeta(t + 1) \left(\frac{c}{\pi}\right)^{t+1} + C_{rm,2} \zeta(t) \left(\frac{c}{\pi}\right)^t$$

*with  $\zeta$  the Euler–Riemann zeta function.*

**Corollary 8.3** *If  $x_0 < 0$  and  $c < \pi$ ,  $f(t) \rightarrow 0$  uniformly and exponentially as  $t \rightarrow \infty$ .*

We note that the convergence is determined by  $c/\pi$  which factors out of the partial sums in the series expansion.

### 9 Non-linear behaviour

We now turn to the non-linear equation (5.1), with the usual initial condition, expressed in the following way:

$$\left. \begin{aligned} f'(t) + [bf(t) + cf(t - 1)] &= F(f(t), f(t - 1)), \quad t > 0; \\ F(u, v) &:= -u[bu + cv]. \end{aligned} \right\}$$

Our first aim is to obtain a non-linear integral representation. Towards this we consider the non-homogeneous problem (5.3) once again by writing  $f = f_{[0]} + f_{[v]}$  with  $f_{[0]}$  the solution of the homogeneous problem studied in Sect. 6 and  $f_{[v]}$  the solution of the non-homogeneous

problem under the homogeneous initial condition  $f_{[v]}(t) = 0$  for  $-1 \leq t < 0$ . For this purpose we introduce the kernel function  $\mathfrak{K}$  as the solution of the problem

$$\left. \begin{aligned} \mathfrak{K}'(t) + b\mathfrak{K}(t) + c\mathfrak{K}(t - 1) &= 0, \text{ for } t > 0; \\ \mathfrak{K}(t) &= 0, \text{ for } t < 0; \\ \mathfrak{K}(0) &= 1. \end{aligned} \right\}$$

This differs from the homogeneous problems studied in Sects. 6–8 in the jump discontinuity at  $t = 0$  which is not serious. In fact,  $\mathfrak{K}(t) = \exp\{-bt\}$  for  $0 \leq t \leq 1$ . Theorems 8.1 and 8.2 apply in this case as well with  $f_0 = 1$  and  $\|\phi\| = 0$ . If  $\lambda := \ln\{\pi/c\}$  this leads to:

$$|\mathfrak{K}(t)| < 2(c|x_0|)^{-1} \exp\{-|x_0|(t + 1)\} + 2(Ec)^{-1} \zeta(t + 1) \exp\{-\lambda(t + 1)\} \text{ for } t > 1.$$

Therefore there are constants  $C > 0, \mu > 0$  such that

$$|\mathfrak{K}(t)| \leq C \exp\{-\mu t\} \text{ for } t \geq 0. \tag{9.1}$$

Repetition of the Laplace transform procedure of Sect. 6 for  $f_{[v]}$  yields, in the notation of (6.1),  $\widehat{f_{[v]}}(s) = \widehat{v}(s)/h(s)$ .

From the convolution theorem we now have  $f_{[v]}(t) = \int_0^t \mathfrak{K}(t-t')v(t')dt' = \int_0^t \mathfrak{K}(t')v(t-t')dt'$ , and hence,

$$f(t) = f_{[0]}(t) + \int_0^t \mathfrak{K}(t')v(t-t')dt'. \tag{9.2}$$

The formal calculations above can be justified by Theorem 5.1 and the (tacit) assumption that  $v$  has a Laplace transform. We continue with the formalism by letting  $v(t) = F(f(t), f(t - 1))$ .

To see that the integral representation (9.2) is more than formal, we note that by virtue of Theorem 4.2(a) (with  $f, f_0$  replaced by  $f - 1, f_0 - 1$ ), that the function  $t \rightarrow v(t)$  is bounded. In fact, if  $|f(t)| \leq M$  then

$$|v(t)| \leq \tilde{\beta}M|f(t)| \leq \tilde{\beta}M^2, \tag{9.3}$$

so that the Laplace transform exists.

It is now possible to consider the asymptotic stability of the non-linear equation (5.1) under the initial condition  $f(t) = \phi(t)$  for  $-1 \leq t \leq 0$ .

**Theorem 9.1** *If  $\lim_{t \rightarrow \infty} v(t)$  exists,  $x_0 < 0$  and  $c < \pi$ , then  $f(t) - f_{[0]} \rightarrow 0$  as  $t \rightarrow \infty$ .*

**Proof** From (9.2) we see that

$$\begin{aligned} f_{[v]}(r + t) - f_{[v]}(r) &= \int_r^{r+t} \mathfrak{K}(t')v(r + t - t')dt' \\ &\quad + \int_0^r \mathfrak{K}(t')[v(r + t - t') - v(r - t')]dt'. \end{aligned} \tag{9.4}$$

To estimate the terms on the right we rely on the inequalities (9.1) and (9.3).

For the first term on the right of (9.4) one has

$$\left| \int_r^{r+t} \mathfrak{K}(t')v(r + t - t')dt' \right| \leq \tilde{\beta}M^2C\mu^{-1} \exp\{-\mu r\}[1 - \exp\{-\mu t\}]$$

and this tends to zero as  $r \rightarrow \infty$ .

The second term is treated differently by splitting the integral in two, one over  $[0, r/2]$  and the other over  $[r/2, r]$ . Since the limit of  $v(t)$  exists, there is for given  $\varepsilon > 0$ ,  $t_\varepsilon$  such that for  $r, t > t_\varepsilon$ ,  $|v(r+t) - v(r)| < \varepsilon$ . Now

$$\left| \int_0^{r/2} \mathfrak{K}(t') [v(r+t-t') - v(r-t')] dt' \right| \leq C\mu^{-1} [1 - \exp\{-\mu r/2\}] \varepsilon \text{ for } r, t > 2t_\varepsilon.$$

The integral over  $[r/2, r]$  is treated the same as the first term on the right of (9.4) and we conclude that  $\lim_{t \rightarrow \infty} [f(t) - f_{[0]}(t)]$  exists. The conclusion follows from Corollary 8.3 and Theorem 4.2. □

The heuristic argument in Sect. 5 for the linear homogeneous equation with solution  $f_{[0]}$  is justified under the hypotheses of Theorem 9.1. In fact, since  $f_{[0]} \rightarrow 0$  as  $t \rightarrow \infty$ , the theorem shows that under a fairly weak hypothesis the non-linear part in the representation (9.2) decays to zero for large  $t$ . More explicit conditions for asymptotic stability are known. In [2, Chap.11] Lyapunov stability of general delay equations is demonstrated under the assumptions that  $x_0 < 0$  and the initial state  $\phi$  is near the equilibrium level. In [16, Thm. 3] it is shown, by construction of a Lyapunov function, that asymptotic equilibrium occurs if  $c < b$ .

### 10 An example

After the somewhat daunting mathematical sections above, it is appropriate to give an example in which the results are applied. The one presented here is based on data of the COVID-19 epidemic collected in South Africa. Specifically, the *positivity rate* (new infections per day) has been taken as a measure of the rate  $f'(t) = \beta f(t_\tau) \{ [1 - \beta^* f(t_\tau)] - \gamma^* f(t_\tau - 1) \}$  according to (3.3). Available data suggested that the theta-model (Sects. 2, 3) is necessary. Least squares estimates based on data in initial stages indicated the parameter values  $\theta = 2.8$ ,  $\beta = 0.017/\text{day}$  and  $f_0 = 0.046$ . The parameters  $\tau$  and  $\gamma$  have been manipulated experimentally to obtain results that correspond reasonably to observed data. The choices are  $\tau = 32$  days,  $\gamma = 0.9/\text{day}$ . This gives  $B = c \exp\{b\} = 1.53778 < \pi/2$  and  $c/\pi = 0.475$ . All zeros have negative real part even though  $\rho = \gamma/\beta = 52.941$ , much larger than 1. Numerical computation based on (6.8) yields  $x_0 = -0.043 < -b = -0.028 \dots$  and  $y_0 = 1.561$

Numerical calculation of the solution of the initial value problem (3.3), (3.4), with the initial state taken as constant, namely  $\phi \equiv f_0$ , (remembering that  $f$  really means  $f^\theta$ ) resulted in Fig. 2. The period of undulation, estimated from the linearization, according to (7.2), is  $T \approx 2\pi \tau / y_0 = 128.9$  days = 4.37 lunar months. The equilibrium level corresponds to  $F_\infty = [\beta^*]^{1/\theta} = 0.238$ .

The computation reported above shows undulation and suggests decay of infectivity levels as time goes on. Our analysis of the linear problem in Sect. 6 indicates that decay could be to the asymptotic equilibrium state  $f_\infty = 1$ . We have also shown that by increasing only  $\tau$  a zero with non-negative real part can occur and then, according to the linearized version, there will be no decay. According to (6.11) the critical value in the present example is  $\tau = \tau_0 = 34.033$  days. For this value of  $\tau$ ,  $x_0 = 0$ . The question is: would this be so for the non-linear equation?

First, Theorem 4.2(a) states that infectivity levels cannot run away, but it can happen in the linearization (according to (7.1)). To come closer to answers it is instructive to calculate trajectories in the phase portrait ( $f'$  as a function of  $f$ ). This is shown in Fig. 3 for the case

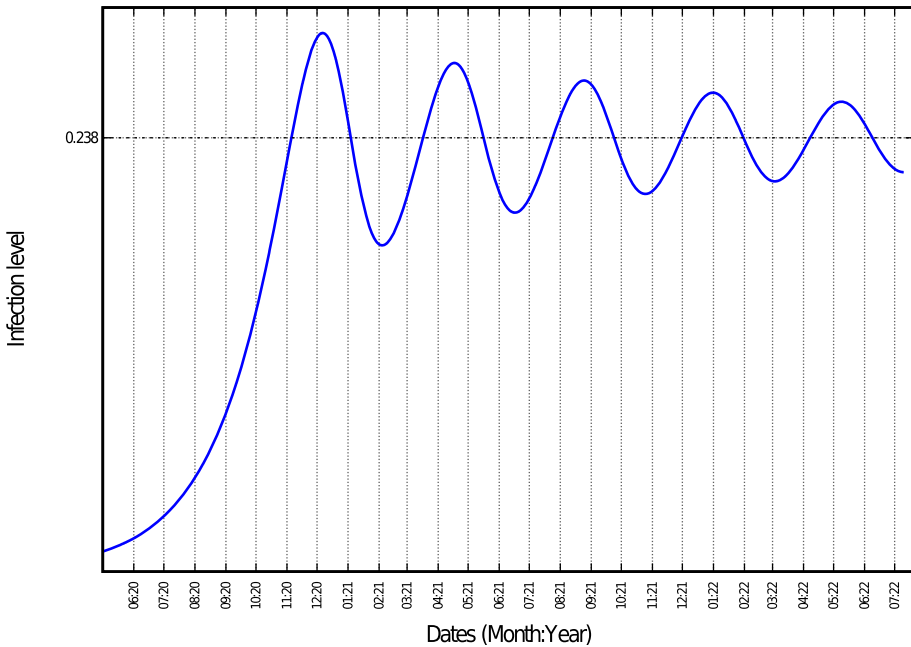


Fig. 2 Undulation and decay

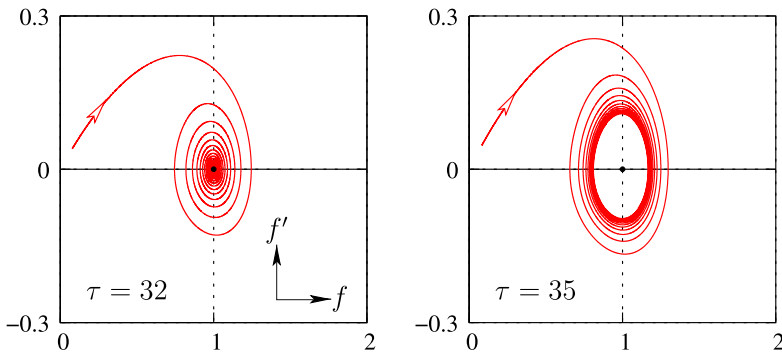


Fig. 3 Decay to equilibrium (left) and a limit cycle (right)

discussed above and with  $\tau$  alone increased to 35, slightly above  $\tau_0$ . The result seems to confirm that in one case ( $\tau = 32$ ) decay is to the asymptotic point  $(1, 0)$  and in the other case ( $\tau = 35$ ) to a limit cycle about this point. Linearization indeed leads to some clarification.

### 11 The numerical algorithm

A brief note on the method used for the numerical calculations is in order. We refer to the representation obtained in Sect. 4 by the substitution  $f = 1/g$  and specifically note the expressions (4.3) and (4.4) used to obtain qualitative results. For the numerical solution we

integrate the latter expression over the interval  $[t, t + \eta]$  to obtain:

$$g(t + \eta) = \exp\{-\delta i(t)\}g(t) + \beta^* [1 - \exp\{-\delta i(t)\}] + \tilde{\beta}\beta^* \gamma^* \exp\{-i(t + \eta)\} \int_t^{t+\eta} \exp\{i(t')\}f(t' - 1)dt',$$

with

$$\delta i(t) := i(t + \eta) - i(t) = \tilde{\beta}[\eta - \gamma^* \int_t^{t+\eta} f(t' - 1)dt'].$$

Approximation of the integrals above by the trapezium rule gives

$$\begin{aligned} & \int_t^{t+\eta} \exp\{i(t')\}f(t' - 1)dt' \\ & \approx \frac{1}{2}\eta[\exp\{i(t + \eta)\}f(t + \eta - 1) + \exp\{i(t)\}f(t - 1)]; \\ & \delta i(t) \approx \tilde{\beta}\eta[1 - \frac{1}{2}\gamma^*[f(t + \eta - 1) + f(t - 1)]]. \end{aligned}$$

Combination of everything results in the computational algorithm

$$g(t + \eta) \approx g(t) \exp\{-\delta i(t)\} + \beta^* [1 - \exp\{-\delta i(t)\}] + \frac{1}{2}\eta\tilde{\beta}\beta^* \gamma^* [f(t + \eta - 1) + \exp\{-\delta i(t)\}f(t - 1)].$$

Since  $f(t)$  is given for  $-1 \leq t \leq 0$ , this can be computed (coding is straightforward). The algorithm is grounded in the problem.

## 12 Concluding unscientific remarks

The caption above is borrowed from a similar-sounding title by Johannes Climacus (Søren Kierkegaard), published in 1846 which, in a way, echoes the Socratic aphorism that *the only wisdom we have is knowing that we do not know*—a paradox that can be (partly) resolved if ‘knowing’ is replaced by ‘understanding’. Understanding, it has been said, expands when horizons of knowing meet. It is never complete.

A fundamental tenet for the mathematical description of growth is the Verhulst logistic model which states that growth is determined by what is left to grow upon. It has the property that growth will increase to devour available resources. If the initial level is below equilibrium levels will increase towards equilibrium. On the other hand, if the level is initially above equilibrium population will decrease towards equilibrium. Since  $\beta^* = 1 - \gamma^*$ , the logistic-recovery equation (3.3) may be re-phrased as

$$f'(t) = \tilde{\beta}f(t)[1 - f(t)] + cf(t)[f(t) - f(t - 1)],$$

which is the logistic equation perturbed by a recovery term that occurs in *Van der Plank’s equation* (1.2) without dormancy. Recovery has the effect of overshooting the equilibrium state. Logistic growth, acting like a counterweight, then forces growth to decrease. Once the level is below equilibrium, growth will turn upwards again. This cyclic process can decay towards stable equilibrium, but may also become repetitive like the motion of a pendulum or a planet orbiting the sun.

It is interesting to note that unfettered undulation can only occur when  $\rho = 1/R_0 > 1$  (Theorem 6.4). The idea that  $R_0 < 1$  is prudent, may be questioned. Within the present discussion this can lead to lowering of the equilibrium level, but at the

price of an undulation in which equilibrium could become a spectre. But then,  $R_0$  in the classical SIR model cannot have the same meaning as  $1/\rho$  in the model discussed here.

The notion of recovery as used here should not be confused with the clinical use of the word. One might ask: what is recovering, the patient or the pathogen? The long recovery period (32 days) used in the example makes the question more incisive; so does the loss of asymptotic equilibrium when recovery takes longer. One could argue that a sufficiently short recovery period provides less opportunity for transmission (or evolution) of the pathogen so that decay to equilibrium would be the result. We should heed the view of James van der Plank in [18]: Observation of infected subjects merely reflect the state of the pathogen.

Mathematical models serve as a basis for motivated speculation and not much more. They count among the many metaphors we invent to explain and understand what is called *reality*. Computational experiments with such models, not supported by mathematical insight, are similar to searching for "...two grains of wheat hid in two bushels of chaff ...". Search for grains of truth may be long and arduous.

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**Data availability** Data used in Sect. 10 was released on a daily basis by the National Institute For Communicable Diseases, South Africa, during the period 2020–2022. That was the data processed and used by the author.

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