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Stability of a boundary permeation model for Navier-Stokes fluids

by

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DEDICATION

This thesis is dedicated to the memory of the following people:

My son, NKOSINATHI (Nathi!), who passed away at the tender age of sixteen months, in 1984.

My mother, MRS EVELYN ZOLOZI HLOMUKA (Intombi ka Hobosha!).

My School Science Master, MR PETER CROFT (Think people, think!).

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ABSTRACT

Title:	Stability of a boundary permeation model for Navier-Stokes fluids
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The stability of a boundary permeation model for incompressible second grade fluids was formulated and solved for bounded regions, by Maritz and Sauer. Under an additional boundary condition they proved exponential decay to the rest state provided the initial energy of the system is not too large. In this exposition, we apply the same model to the boundary permeation problem, but for incompressible first grade fluids, also known as Navier-Stokes fluids. In our situation, the additional boundary condition is not necessary, but in contrast to the second grade fluid, the decay of perturbed solutions is first order polynomial.

CHAPTER 1

THE SETTING

1.1. Introduction.

A Navier-Stokes fluid is an incompressible Newtonian fluid. When the fluid is not a mixture, incompressibility means a constant fluid density. We shall consider only such fluids. Newtonian fluids are characterized by the linear occurrence of the rate of deformation in the stress tensor — a term which contains the viscosity of the fluid as a physical constant. Viscous fluids are often modelled as having the property that they stick to a solid boundary and therefore moves with it.

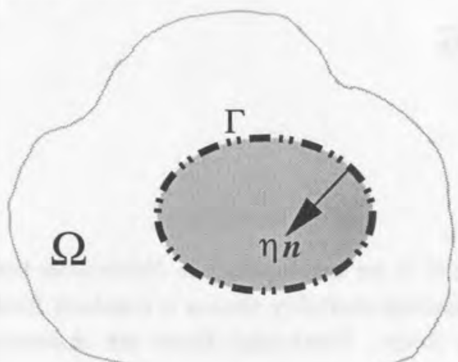
A phenomenon observed in non-Newtonian fluids which is absent in Navier-Stokes fluids, is the presence of a normal stress component, additional to the pressure, determined by the velocity field. For non-Newtonian incompressibility fluids, which adhere to a boundary, this component has been calculated explicitly by Berker [2], and the expression shows that for incompressible Navier-Stokes flows, the component is zero.

A question to be asked then is: if the boundary is permeable, i.e. allows the motion of a fluid through it, is it possible for a fluid to pass through the boundary as a result of fluid motion in the region bounded by the particular wall? A model for studying this situation has, for incompressible non-Newtonian (second grade) fluids been proposed and studied in [9]. It was found that for second grade fluids, the problem is well-posed if certain additional boundary conditions were imposed.

The model mentioned above is formulated for arbitrary fluids and the question arises as to its applicability for Navier-Stokes fluids, which are less complex than the non-linear fluids. Once this is established, the behaviour at the permeable boundary can be compared to what, seemingly, has been observed experimentally.

1.2. Navier-Stokes Flows.

The setting for the problem mentioned in the previous section, is a Navier-Stokes fluid in a container, which in turn is immersed in a larger body of the same fluid. The boundary of the container is permeable and allows fluid flow into and out of the same container.



The outside of the smaller container is assumed to be the bounded domain $\Omega \subset \mathrm{R}^n$ and the flows are normal to the permeable walls(boundaries). The boundary of the container is denoted by Γ and assumed smooth.

Velocity at the boundary, denoted by $\gamma_0 \boldsymbol{v}(x,t)$, for $x \in \Gamma$, is prescribed in terms of an unknown function η_v which can only be found from its evolution equation derived later. Our permeability model, adopted from [9], assumes that the velocity at the interface Γ is non-tangential: I.e.

$$\gamma_0 \boldsymbol{v}(x,t) = -\eta_v(x,t)\boldsymbol{n}(x)$$

where $\boldsymbol{n}(x)$ is the normal unit vector on Γ. The function η_v, defined only on Γ is unknown. In order to determine it, a dynamic boundary condition leads to an additional evolution equation which is intimately coupled to the Navier-Stokes equations in the domain Ω. The basic idea behind the boundary equation is that the boundary flow is caused by normal stresses at the boundary.

Later it is found that, due to the incompressibility of our fluid,

$$\int_\Gamma \eta_v \, dx = 0.$$

It is assumed that

$$v(x,t) = 0,$$

on the outer boundary.

1.3. Goals of the thesis.

In the case of the fluid flows under consideration, we wish to establish:

a. at least the aspect of well-posedness which deals with the stability of the rest state;

b. the conditions under which the stability of the rest state of the problem occurs.

In Chapter 2 we deal with the mathematical modelling of the permeation problem. This deals with the continuum-mechanical assumptions underlying the equations, the concept of permeability modelled by means of an *effective surface measure*, and the final expression of the dynamic boundary equation in terms of geometric entities such as curvature.

In Chapter 3 the stability of the rest state, which is our main aim, is studied. This entails intricate estimates based on a-priori estimates for the classical Stokes operator, use of a very general, recently developed form of the Helmholtz-Weyl projection and an unusual energy method.

CHAPTER 2

MODELLING OF NAVIER-STOKES FLOWS

In the previous chapter, we presented the setting for the problem. In this chapter, we wish to follow up to the setting by mathematically modelling the situation. Before we proceed with the modelling, we wish to explain the symbols for the quantities to be used in the model itself. The units for some quantities are in brackets.

2.1. Explanation of the Symbols.

$x = (x_1, x_2, x_3)$	:	position in 3-dimensional space
$\boldsymbol{v}(x,t)$	:	the velocity field in the fluid(ms^{-1})
ρ	:	fluid volume density(kgm^{-3})
μ	:	coefficient of viscosity(Nm^{-2}s)
Γ	:	boundary of the bounded domain $\Omega \subset \mathrm{R}^3$
$\boldsymbol{n}(x)$	:	the unit exterior normal to Γ
ds	:	Lebesgue measure of surface area on $\Gamma(m^2)$
$da(x)$	:	the effective area measure on Γ
$\zeta(x)$	:	density function in terms of the area measure da
	:	i.e. $da = \zeta\, ds$; $0 < \zeta(x) < 1$
$\delta(x)$	:	surface thickness at any point $x \in \Gamma$(m)
$\sigma(x)$	:	surface density of the fluid at any $x \in \Gamma(kg^{-2})$
	:	i.e. $\sigma(x) = \delta(x)\zeta(x)\rho$
$\gamma_0 \boldsymbol{v}(x,t)$	:	velocity at $x \in \Gamma$ (ms^{-1})
p	:	pressure(Nm^{-2})
$\eta_v(x,t)$	:	the normal velocity component at $x \in \Gamma$.
	:	We shall, in fact, assume that $\gamma_0 \boldsymbol{v} = -\eta_v(x,t)\boldsymbol{n}(x)$
$[\nabla \boldsymbol{v}]_{i,j} = \partial_j v_i$	:	the velocity gradient (s^{-1})
$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{v}$	:	the vorticity (s^{-1}) (wedge denotes vector product)
$\boldsymbol{D}(\boldsymbol{v})$	:	the rate of deformation tensor

2.2. Modelling in Ω.

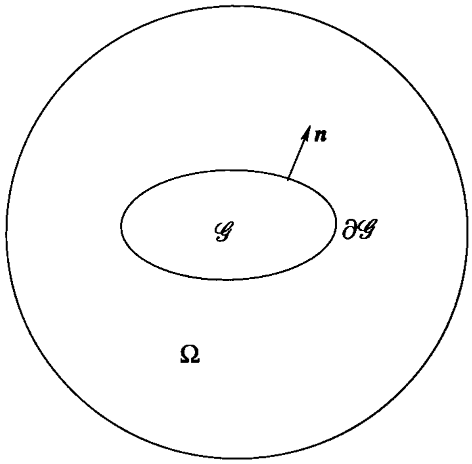


Figure 1

We consider an arbitrary fixed region in the fluid in Ω denoted by $\mathscr{G}$ and with a boundary $\partial\mathscr{G}$ (Figure 1). Fluid particles enter and leave this volume, so that at different times there may be different fluid particles in $\mathscr{G}$.

Let $\boldsymbol{n}$ be the unit normal to $\partial\mathscr{G}$, as shown. If $\rho(x,t)$ is the density of the fluid in $\mathscr{G}$, and $\boldsymbol{v}(x,t)$ the velocity of the particle which, at time t, is in position x, then the conservation of fluid mass in $\mathscr{G}$, may be expressed as:

$$\frac{d}{dt}\int_{\mathscr{G}} \rho\, dx = -\int_{\partial\mathscr{G}} \rho\boldsymbol{v}\cdot\boldsymbol{n}\, ds.$$

Since the volume is fixed, formal differentiation under the integral sign, and use of the divergence theorem yields

$$\int_{\mathscr{G}} \rho_t\, dx = -\int_{\mathscr{G}} \nabla\cdot(\rho\boldsymbol{v})\, dx.$$

Thus,

$$\int_{\mathscr{G}} [\rho_t + \nabla\cdot(\rho\boldsymbol{v})]\, dx = 0,$$

which gives, because $\mathscr{G}$ is arbitrary,

$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{1}$$

If ρ is constant, it follows that $\nabla \cdot \boldsymbol{v} = 0$. We shall be studying this case which is referred to as an incompressible (isochoric) motion.

Next, we consider the conservation of linear momentum in $\mathscr{G}$. This is expressed in the following way:

Rate of change of momentum in $\mathscr{G}$ = *Rate of change of momentum over* $\partial\mathscr{G}$+*Total force on* $\partial\mathscr{G}$+*Total force on* $\mathscr{G}$.

Since we deal with a fixed domain $\mathscr{G}$, we introduce the momentum flux $\boldsymbol{M}$ so that the rate of change of linear momentum over $\partial\mathscr{G}$ in the direction of $\boldsymbol{n}$ is given by

$$\int_{\partial\mathscr{G}} \boldsymbol{M}\boldsymbol{n}\, ds = \int_{\mathscr{G}} \nabla \cdot \boldsymbol{M}\, dx.$$

Let $\boldsymbol{T}$ denote the stress tensor on $\partial\mathscr{G}$. The (internal) force per unit area at $y \in \partial\mathscr{G}$, may then be expressed in the form $\boldsymbol{T}(y)\boldsymbol{n}(y)$. Thus, the total force on $\partial\mathscr{G}$ is

$$\int_{\partial\mathscr{G}} \boldsymbol{T}\boldsymbol{n}\, ds = \int_{\mathscr{G}} \nabla \cdot \boldsymbol{T}\, dx,$$

by the divergence theorem.

We then write the conservation law in mathematical terms:

$$\frac{d}{dt}\int_{\mathscr{G}} \rho\boldsymbol{v}\, dx = -\int_{\mathscr{G}} \nabla \cdot \boldsymbol{M} dx + \int_{\mathscr{G}} \nabla \cdot \boldsymbol{T} dx + \int_{\mathscr{G}} \rho \boldsymbol{f}\, dx.$$

Since $\mathscr{G}$ is arbitrary,

$$\partial_t(\rho\boldsymbol{v}) + \nabla \cdot \boldsymbol{M} = \nabla \cdot \boldsymbol{T} + \rho\boldsymbol{f}. \tag{2}$$

We choose $\boldsymbol{M}$ as follows:

$$\boldsymbol{M} := \rho\boldsymbol{v} \otimes \boldsymbol{v},$$

where the symbol $\otimes$ denotes the tensor product. Having made this choice, we have:

$$\nabla \cdot \boldsymbol{M} = [\nabla \cdot (\rho\boldsymbol{v} \otimes \boldsymbol{v})],$$

$$\begin{aligned}
[\nabla \cdot (\rho\boldsymbol{v} \otimes \boldsymbol{v})] &= \sum_{i=1}^{3} \partial_i(\rho\boldsymbol{v}_i \cdot \boldsymbol{v}_j) \\
&= \sum_{i=1}^{3}[\partial_i(\rho\boldsymbol{v}_i)]\boldsymbol{v}_j + \sum_{j=1}^{3} \rho\boldsymbol{v}_i\partial_i\boldsymbol{v}_j \\
&= [\nabla \cdot (\rho\boldsymbol{v})]\boldsymbol{v} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}.
\end{aligned}$$

Thus,

$$\nabla \cdot \boldsymbol{M} = [\nabla \cdot (\rho \boldsymbol{v})]\boldsymbol{v} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \tag{3}$$

However,

$$\begin{aligned} \partial_t(\rho \boldsymbol{v}) &= \rho_t \boldsymbol{v} + \rho \boldsymbol{v}_t \\ &= (-\nabla \cdot \rho \boldsymbol{v})\boldsymbol{v} + \rho \boldsymbol{v}_t, \end{aligned} \tag{4}$$

by (1). We rewrite (2), using (3) and (4) to obtain the expression:

$$-[\nabla \cdot (\rho \boldsymbol{v})]\boldsymbol{v} + \rho \boldsymbol{v}_t + (\nabla \cdot \rho \boldsymbol{v})\boldsymbol{v} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla \cdot \boldsymbol{T} + \rho f,$$

from which we obtain:

$$\rho \boldsymbol{v}_t + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \nabla \cdot \boldsymbol{T} + \rho f,$$

which is the mathematical statement for the conservation of momentum combined with the conservation of mass.

We shall only consider the case where body forces are absent, i.e $f = 0$ in Ω. Thus,

$$\rho[\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] = \nabla \cdot \boldsymbol{T}. \tag{5}$$

2.3. Modelling on Γ.

We return to the setting described in Section 1.1. Like before, the 3-dimensional space between the two containers is denoted by Ω, and the interior of the immersed container is denoted by Ω_0. The permeable interface boundary is denoted by Γ and the outer boundary of Ω is denoted by Γ_0. This boundary is supposed to be impermeable and sticky. A representation of the "geometry" of the situation is given in Figure 2.

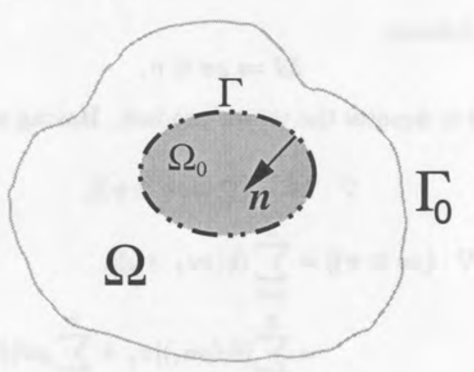


Figure 2

The unit exterior normal to Γ is denoted by $\boldsymbol{n}$ and the trace operator γ_0 will be used to denote restriction to Γ. Our model assumes that

fluid particles are accelerated from rest in Ω_0 across Γ into Ω, or they are decelerated from Ω across Γ and come to rest in Ω_0.

Let $\boldsymbol{v}(x,t)$ and $p(x,t)$ denote the velocity and pressure fields respectively, at $x \in \Omega$ and time $t > 0$. We have already derived the equations of motion as (6) in the previous section. We have for incompressible motions:

$$\begin{aligned} \rho D_t \boldsymbol{v}(x,t) &= \nabla \cdot \boldsymbol{T}(\boldsymbol{p},\boldsymbol{v}); x, \boldsymbol{v} \in \Omega \\ \nabla \cdot \boldsymbol{v}(x,t) &= 0 \end{aligned} \qquad (1).$$

In equation (1):

$$D_t := \partial_t + \boldsymbol{v} \cdot \nabla;$$

and for a Navier-Stokes fluid, the stress tensor is chosen as

$$\boldsymbol{T}(\boldsymbol{p},\boldsymbol{v}) := -p\boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{v});$$

where,

$$\boldsymbol{D}(\boldsymbol{v}) := \tfrac{1}{2}[\nabla \boldsymbol{v} + \nabla \boldsymbol{v}].$$

$\boldsymbol{D}(\boldsymbol{v})$ is called *the rate of deformation tensor.*

It is assumed that the velocity field $\boldsymbol{v}(x,t)$ always satisfies the homogeneous Dirichlet boundary condition: $\boldsymbol{v}(.,t) = 0$, on Γ_0 for $t > 0$. At the permeable boundary Γ, we shall assume that

$$\gamma_0 \boldsymbol{v}(x,t) = -\eta_v(x,t)\boldsymbol{n}(x). \qquad (2)$$

The scalar-valued function η_v defined on Γ is unknown, and is determined by a dynamic boundary condition, which is an evolution equation. The said evolution equation will be derived later in this chapter. Also, the condition $\nabla \cdot \boldsymbol{v}(x,t) = 0$ (for incompressible fluids), leads to

$$\int_\Gamma \eta_v \, ds = 0.$$

The permeability of the boundary Γ is brought into play by means of the concept of effective area denoted by *da*. The measure *da* expresses the surface area through which fluid particles can move, while the measure *ds* relates to the total surface area. Since the effective area cannot exceed the total area, it is assumed that they are related by

$$da = \zeta(y)\,ds$$

and that ζ is restricted by $0 < \zeta(y) < 1$ (for $y \in \Gamma$). It may be thought of as a probability-density function which expresses the probability of finding a "hole" in Γ.

Next, we assume that, at any time t, the surface Γ contains fluid particles. Since there are fluid particles present in the surface Γ, it is necessary to introduce a surface fluid density $\sigma(y)$ from which the total mass of fluid in any boundary patch $\Gamma_1 \subset \Gamma$ can be calculated as:

$$m(\Gamma_1) = \int_{\Gamma_1} \sigma(y)\,ds(y) \tag{3}$$

One may introduce a "surface thickness" $\delta(y)$ by means of a dimensional argument and obtain the formal relationship

$$\sigma(y) = \delta(y)\zeta(y)\rho \text{ for every } y \in \Gamma.$$

We shall assume that σ is a measurable function which is bounded away from zero by a positive number.

The law of conservation of linear momentum in the surface Γ is, in the absence of body forces, expressed in terms of the momentum transfer tensor $\boldsymbol{M}$ and the stress tensor $\boldsymbol{T}$.

We state the law of conservation of linear momentum in Γ:

For every measurable boundary patch $\Gamma_1 \subset \Gamma$, the rate of change of linear momentum in Γ_1 is explained by the net influx of momentum into Γ_1 and the resultant force due to stress on the patch.

Thus, if $\boldsymbol{M}_0$ and $\boldsymbol{T}_0$ represent, respectively, the momentum and stress tensor on the Ω_0 - side, then

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_1} \sigma(y)\gamma_o \boldsymbol{v}(y,t)\,ds(y) &= \int_{\Gamma_1} [\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n}\,da + \int_{\Gamma_1} [\boldsymbol{T}_0 - \boldsymbol{T}]\boldsymbol{n}\,ds \\ &= \int_{\Gamma_1} \zeta[\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n}\,ds + \int_{\Gamma_1} [\boldsymbol{T} - \boldsymbol{T}_0]\boldsymbol{n}\,ds\end{aligned}$$

by equation (2). Since Γ_1 is arbitrary, it follows that

$$\sigma[\gamma_o \boldsymbol{v}]_t = \zeta[\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n} + [\boldsymbol{T} - \boldsymbol{T}_0]\boldsymbol{n} \tag{4}$$

We use the measure da in the momentum term because momentum flux is restricted by the permeability.

2.4. Problem formulation.

Having modelled the motion of the fluid in Ω and derived the boundary conditions in Γ, we are now in a position to formulate our problem.

Equation (2.3–1) is the equation of motion. Equations (2.3–2) and (2.3–4) form our boundary conditions. We formulate the problem as follows:

To find $\boldsymbol{v} \in \Omega \subset R^n$ which satisfies the following equations:

$$\rho D_t \boldsymbol{v}(x,t) = \nabla \cdot \boldsymbol{T}(p, \boldsymbol{v}) \tag{1}$$

where,

$$\begin{aligned} \boldsymbol{T}(p,\boldsymbol{v}) &:= -p\boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{v}), \\ \boldsymbol{D}(\boldsymbol{v}) &:= \tfrac{1}{2}[\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}], \end{aligned}$$

subject to the following conditions:

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{v}(x,t) &= 0; \\ \gamma_0 \boldsymbol{v}(x,t) &= -\eta_v(x,t)\boldsymbol{n}(x) \text{ on } \Gamma; \\ -[\sigma(x)\partial_t \eta_v(x,t)]\boldsymbol{n} &= \zeta[\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n} + [\boldsymbol{T} - \boldsymbol{T}_0]\boldsymbol{n}; \\ \boldsymbol{v} &= 0 \text{ on } \Gamma_0. \end{aligned}\right\} \tag{2}$$

2.5. The boundary equation.

The third equation in (2.4–2) is an evolution equation for η_v, which, together with (2.4–1), should be used to solve for $\boldsymbol{v}$. In order to put this equation in a more convenient form, we shall derive an expression for $\gamma_0[\boldsymbol{D}(\boldsymbol{v})]$:

We consider a local orthogonal system, in Γ, consisting of normal curves $x_1(s_1)$ and $x_2(s_2)$ with s_1 and s_2 denoting the arc length. Let $\boldsymbol{\tau}_k = x'_k (k = 1, 2)$ denote the unit tangent vectors to the curve x_k.

We choose $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ such that $\boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_2 = \boldsymbol{n}$. In such a coordinate system, the surface gradient of a scalar field f on Γ is defined as

$$\nabla_s f := \frac{\partial f}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial f}{\partial s_2}\boldsymbol{\tau}_2. \tag{1}$$

For a function defined on Ω, the relationship between gradient and surface gradient is given by

$$\gamma_0 \nabla f := \nabla_s \gamma_0 f + [\gamma_1 f]\boldsymbol{n} \tag{2}$$

where $\gamma_1 f := \frac{\partial f}{\partial n}$ denotes the normal derivative.

The surface gradient of a vector field $\boldsymbol{f}$ on Γ is defined as

$$\nabla_s \boldsymbol{f} := \frac{\partial \boldsymbol{f}}{\partial s_1} \otimes \boldsymbol{\tau}_1 + \frac{\partial \boldsymbol{f}}{\partial s_2} \otimes \boldsymbol{\tau}_2. \tag{3}$$

The relationship between gradient and surface gradient for the vector field $\boldsymbol{f}$ is given by

$$\gamma_0 \nabla \boldsymbol{f} = \nabla_s \gamma_0 \boldsymbol{f} + [\gamma_1 \boldsymbol{f}] \otimes \boldsymbol{n}. \tag{4}$$

Similarly, for the vector field $\boldsymbol{f}$, we have the following definitions and relationships:

$$\nabla_s \wedge \boldsymbol{f} := \boldsymbol{\tau}_1 \wedge \frac{\partial \boldsymbol{f}}{\partial s_1} + \boldsymbol{\tau}_2 \wedge \frac{\partial \boldsymbol{f}}{\partial s_2}; \tag{5}$$

$$\gamma_0 \nabla \wedge \boldsymbol{f} = \nabla_s \wedge \gamma_0 \boldsymbol{f} + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{f}; \tag{6}$$

$$\nabla_s \cdot \boldsymbol{f} := \boldsymbol{\tau}_1 \cdot \frac{\partial \boldsymbol{f}}{\partial s_1} + \boldsymbol{\tau}_2 \cdot \frac{\partial \boldsymbol{f}}{\partial s_2}; \tag{7}$$

$$\gamma_0 \nabla \cdot \boldsymbol{f} = \nabla_s \cdot \gamma_0 \boldsymbol{f} + \boldsymbol{n} \cdot \gamma_1 \boldsymbol{f}. \tag{8}$$

By (1), (3), (5) and (7), taking into account the Serret-Frenet formulas for curves without torsion as well as the chosen orientation of tangent vectors, we derive the following expressions:

$$\begin{aligned}
\nabla_s[\eta_v \boldsymbol{n}] &= \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_1} \otimes \boldsymbol{\tau}_1 + \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_2} \otimes \boldsymbol{\tau}_2 \\
&= (\frac{\partial \eta_v}{\partial s_1} \boldsymbol{n} + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_2}) \otimes \boldsymbol{\tau}_1 + (\frac{\partial \eta_v}{\partial s_1} \boldsymbol{n} + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_2}) \otimes \boldsymbol{\tau}_2 \\
&= \frac{\partial \eta_v}{\partial s_1} \boldsymbol{n} \otimes \boldsymbol{\tau}_1 + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_1} \otimes \boldsymbol{\tau}_1 + \frac{\partial \eta_v}{\partial s_2} \boldsymbol{n} \otimes \boldsymbol{\tau}_2 + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_2} \otimes \boldsymbol{\tau}_2 \\
&= \frac{\partial \eta_v}{\partial s_1} \boldsymbol{n} \otimes \boldsymbol{\tau}_1 + \frac{\partial \eta_v}{\partial s_2} \boldsymbol{n} \otimes \boldsymbol{\tau}_2 + \eta_v(-\kappa_1 \boldsymbol{\tau}_1) \otimes \boldsymbol{\tau}_1 + \eta_v(-\kappa_2 \boldsymbol{\tau}_2) \otimes \boldsymbol{\tau}_2 \\
&= \boldsymbol{n} \otimes \frac{\partial \eta_v}{\partial s_1} \boldsymbol{\tau}_1 + \boldsymbol{n} \otimes \frac{\partial \eta_v}{\partial s_2} \boldsymbol{\tau}_2 - \eta_v[[\kappa_1 \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2 \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] \\
&= \boldsymbol{n} \otimes [\frac{\partial \eta_v}{\partial s_1} \boldsymbol{\tau}_1 + \frac{\partial \eta_v}{\partial s_2} \boldsymbol{\tau}_2] - \eta_v[\kappa_1 \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2 \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2] \\
&= \boldsymbol{n} \otimes \nabla_s \eta_v - \eta_v[\kappa_1 \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1 + \kappa_2 \boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2].
\end{aligned} \tag{9}$$

We also have:

$$\begin{aligned}
\nabla_s \wedge [\eta_v \boldsymbol{n}] &= \boldsymbol{\tau}_1 \wedge \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_1} + \boldsymbol{\tau}_2 \wedge \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_2} \\
&= \boldsymbol{\tau}_1 \wedge [\frac{\partial \eta_v}{\partial s_1} + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_1}] + \boldsymbol{\tau}_2 \wedge [\frac{\partial \eta_v}{\partial s_2}\boldsymbol{n} + \eta_v \frac{\partial \boldsymbol{n}}{\partial s_2}] \\
&= \boldsymbol{\tau}_1 \wedge [\frac{\partial \eta_v}{\partial s_1}\boldsymbol{n} - \eta_v \kappa_1 \boldsymbol{\tau}_1] + \boldsymbol{\tau}_2 \wedge [\frac{\partial \eta_v}{\partial s_2}\boldsymbol{n} - \eta_v \kappa_2 \boldsymbol{\tau}_2] \\
&= \frac{\partial \eta_v}{\partial s_1}(\boldsymbol{\tau}_1 \wedge \boldsymbol{n}) - \eta_v \kappa_1 (\boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_1) + \frac{\partial \eta_v}{\partial s_2}(\boldsymbol{\tau}_2 \wedge \boldsymbol{n}) - \eta_v \kappa_2 (\boldsymbol{\tau}_2 \wedge \boldsymbol{\tau}_2) \\
&= (\frac{\partial \eta_v}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial \eta_v}{\partial s_2}\boldsymbol{\tau}_2) \wedge \boldsymbol{n} \\
&= \nabla_s \eta_v \wedge \boldsymbol{n}.
\end{aligned}$$

$$\begin{aligned}
\nabla_s \cdot [\eta_v \boldsymbol{n}] &= \boldsymbol{\tau}_1 \cdot \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_1} + \boldsymbol{\tau}_2 \cdot \frac{\partial(\eta_v \boldsymbol{n})}{\partial s_2} \\
&= \boldsymbol{\tau}_1 \cdot [\frac{\partial \eta_v}{\partial s_1}\boldsymbol{n} - \eta_v \kappa_1 \boldsymbol{\tau}_1] + \boldsymbol{\tau}_2 \cdot [\frac{\partial \eta_v}{\partial s_2}\boldsymbol{n} - \eta_v \kappa_2 \boldsymbol{\tau}_2] \\
&= -\eta_v \kappa_1 + (-\eta_v \kappa_2) \\
&= -\eta_v \kappa.
\end{aligned}$$

From $\nabla \cdot \boldsymbol{v} = 0$, we have $\gamma_0 \nabla \cdot \boldsymbol{v} = 0$. Therefore,

$$\gamma_0 \nabla \cdot \boldsymbol{v} = \nabla_s \cdot \gamma_0 \boldsymbol{v} + \boldsymbol{n} \cdot \gamma_1 \boldsymbol{v} = 0.$$

Hence,

$$\begin{aligned}
\boldsymbol{n} \cdot \gamma_1 \boldsymbol{v} &= -\nabla_s \cdot \gamma_0 \boldsymbol{v} \\
&= -\nabla_s \cdot (-\eta_v \boldsymbol{n}) \\
&= \nabla_s \cdot (\eta_v \boldsymbol{n}) \\
&= -\eta_v \kappa.
\end{aligned}$$

Next, we have for the vorticity $\boldsymbol{\omega}$,

$$\begin{aligned}
\gamma_0 \boldsymbol{\omega} &= \gamma_0 \nabla \wedge \boldsymbol{v} \\
&= \nabla_s \wedge \gamma_0 \boldsymbol{v} + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{v} \\
&= \nabla_s \wedge (-\eta_v \boldsymbol{n}) + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{v} \\
&= -\nabla_s \wedge (\eta_v \boldsymbol{n}) + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{v} \qquad (10) \\
&= -\nabla_s \eta_v \wedge \boldsymbol{n} + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{v} \\
&= \boldsymbol{n} \wedge \nabla_s \eta_v + \boldsymbol{n} \wedge \gamma_1 \boldsymbol{v} \\
&= \boldsymbol{n} \wedge [\nabla_s \eta_v + \gamma_1 \boldsymbol{v}].
\end{aligned}$$

Now,

$$\begin{aligned}\gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n} &= \boldsymbol{n}\wedge[\gamma_1\boldsymbol{v}+\nabla_s\eta_v]\wedge\boldsymbol{n}\\ &= (\boldsymbol{n}\cdot\boldsymbol{n})(\gamma_1\boldsymbol{v}+\nabla_s\eta_v)-(\boldsymbol{n}\cdot[\gamma_1\boldsymbol{v}+\nabla_s\eta_v])\boldsymbol{n}\\ &= \gamma_1\boldsymbol{v}+\nabla_s\eta_v-[\boldsymbol{n}\cdot\gamma_1\boldsymbol{v}+\boldsymbol{n}\cdot\nabla_s\eta_v]\boldsymbol{n}\\ &= \gamma_1\boldsymbol{v}+\nabla_s\eta_v-(\boldsymbol{n}\cdot\gamma_1\boldsymbol{v})\boldsymbol{n}\end{aligned}$$

since $\boldsymbol{n}\cdot\nabla_s\eta_v = 0$. From this, we have that

$$\begin{aligned}\gamma_1\boldsymbol{v} &= \gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n}-\nabla_s\eta_v+(\boldsymbol{n}\cdot\gamma_1\boldsymbol{v})\boldsymbol{n}\\ &= \gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n}-\nabla_s\eta_v-\eta_v\kappa\boldsymbol{n}.\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_s\gamma_0\boldsymbol{v} &= \nabla_s(-\eta_v\boldsymbol{n})\\ &= \eta_v[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2]-\boldsymbol{n}\otimes\nabla_s\eta_v.\end{aligned}\tag{11}$$

But,

$$\gamma_0\nabla\boldsymbol{v} = \nabla_s\gamma_0\boldsymbol{v}+[\gamma_1\boldsymbol{v}]\otimes\boldsymbol{n}.$$

Thus,

$$\gamma_0\nabla\boldsymbol{v} = \eta_v[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2]-\boldsymbol{n}\otimes\nabla_s\eta_v+(\gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n})\otimes\boldsymbol{n}-[\nabla_s\eta_v+\eta_v\kappa\boldsymbol{n}]\otimes\boldsymbol{n}.\tag{12}$$

From (12), we derive an explicit expression for $\gamma_0\boldsymbol{D}(\boldsymbol{v})$ in the following way: First we observe that

$$\gamma_0\nabla^T\boldsymbol{v} = \eta_v[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2]-\nabla_s\eta_v\otimes\boldsymbol{n}+\boldsymbol{n}\otimes(\gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n})-\boldsymbol{n}\otimes[\nabla_s\eta_v+\eta_v\kappa\boldsymbol{n}]$$

Therefore,

$$\begin{aligned}\gamma_0\boldsymbol{D}(\boldsymbol{v}) &= \tfrac{1}{2}[\gamma_0\nabla\boldsymbol{v}+\gamma_0\nabla^T\boldsymbol{v}]\\ &= \eta_v[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2-\eta_v\kappa\boldsymbol{n}\otimes\boldsymbol{n}]\\ &\quad+\tfrac{1}{2}[\gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n}-2\nabla_s\eta_v]\otimes\boldsymbol{n}\\ &\quad+\tfrac{1}{2}\boldsymbol{n}\otimes[\gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n}-2\nabla_s\eta_v].\end{aligned}$$

Hence,

$$\gamma_0\boldsymbol{D}(\boldsymbol{v}) = \eta_v[\kappa_1\boldsymbol{\tau}_1\otimes\boldsymbol{\tau}_1+\kappa_2\boldsymbol{\tau}_2\otimes\boldsymbol{\tau}_2-\kappa\boldsymbol{n}\otimes\boldsymbol{n}]+\tfrac{1}{2}\boldsymbol{\Psi}\otimes\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{n}\otimes\boldsymbol{\Psi},\tag{13}$$

where $\boldsymbol{\Psi} = \gamma_0\boldsymbol{\omega}\wedge\boldsymbol{n}-2\nabla_s\eta_v$. It is easily seen that $\boldsymbol{\Psi}$ is the sum of tangential vectors, and therefore tangential. In addition, it is seen that $\gamma_0\boldsymbol{D}$ is symmetrical.

It is important to calculate the normal component of deformation at the boundary Γ. Towards this (and other objectives) we deduce from (13) the following relations:

$$\gamma_0 \boldsymbol{D}(\boldsymbol{v})\boldsymbol{\tau}_1 = \eta_v \kappa_1 \boldsymbol{\tau}_1 + \tfrac{1}{2}(\boldsymbol{\Psi}\cdot\boldsymbol{\tau}_1)\boldsymbol{n}; \tag{14}$$

$$\gamma_0 \boldsymbol{D}(\boldsymbol{v})\boldsymbol{\tau}_2 = \eta_v \kappa_2 \boldsymbol{\tau}_2 + \tfrac{1}{2}(\boldsymbol{\Psi}\cdot\boldsymbol{\tau}_2)\boldsymbol{n}; \tag{15}$$

$$\gamma_0 \boldsymbol{D}(\boldsymbol{v})\boldsymbol{n} = -\eta_v \kappa \boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\Psi}. \tag{16}$$

From (16) and the tangentiality of $\boldsymbol{\Psi}$ it follows that

$$\begin{aligned} \boldsymbol{n}\cdot\gamma_0[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n} &= -\eta_v\kappa + \tfrac{1}{2}\boldsymbol{n}\cdot\boldsymbol{\Psi} \\ &= -\eta_v\kappa. \end{aligned} \tag{17}$$

It is instructive to obtain the matrix representation of the tensor $\gamma_0\boldsymbol{D}$ relative to the local basis on Γ defined by $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n})$. For this we use we use the expressions (14) , (15) and (16) as well as the symmetry of $\gamma_0\boldsymbol{D}$:

$$\gamma_0[\boldsymbol{D}(\boldsymbol{v})] = \tfrac{1}{2}\begin{pmatrix} \boldsymbol{\tau}_1\cdot\boldsymbol{D}\boldsymbol{\tau}_1 & \boldsymbol{\tau}_2\cdot\boldsymbol{D}\boldsymbol{\tau}_1 & \boldsymbol{n}\cdot\boldsymbol{D}\boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_1\cdot\boldsymbol{D}\boldsymbol{\tau}_2 & \boldsymbol{\tau}_2\cdot\boldsymbol{D}\boldsymbol{\tau}_2 & \boldsymbol{n}\cdot\boldsymbol{D}\boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_1\cdot\boldsymbol{D}\boldsymbol{n} & \boldsymbol{\tau}_2\cdot\boldsymbol{D}\boldsymbol{n} & \boldsymbol{n}\cdot\boldsymbol{D}\boldsymbol{n} \end{pmatrix}$$

$$= \tfrac{1}{2}\begin{pmatrix} 2\eta_v\kappa_1 & 0 & (\boldsymbol{n}\wedge\boldsymbol{\omega} - 2\nabla\eta_v)\cdot\boldsymbol{\tau}_1 \\ 0 & 2\eta_v\kappa_2 & (\boldsymbol{n}\wedge\boldsymbol{\omega} - 2\nabla\eta_v)\cdot\boldsymbol{\tau}_2 \\ (\boldsymbol{n}\wedge\boldsymbol{\omega} - 2\nabla\eta_v)\cdot\boldsymbol{\tau}_1 & (\boldsymbol{n}\wedge\boldsymbol{\omega} - 2\nabla\eta_v)\cdot\boldsymbol{\tau}_2 & \nabla\cdot\boldsymbol{v} - 2\eta_v\kappa \end{pmatrix}$$

$$= -\eta_v\begin{pmatrix} -\kappa_x & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & \kappa \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \Psi\cdot\boldsymbol{\tau}_1 \\ 0 & 0 & \Psi\cdot\boldsymbol{\tau}_2 \\ \boldsymbol{\Psi}\cdot\boldsymbol{\tau}_1 & \Psi\cdot\boldsymbol{\tau}_2 & 0 \end{pmatrix}. \tag{18}$$

We may now write the differential equation for η_v in a simpler form:
From (4) we have that

$$\sigma[\gamma_0\boldsymbol{v}]_t = \zeta\gamma_0[\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n} + \gamma_0[\boldsymbol{T}_0 - \boldsymbol{T}]\boldsymbol{n}.$$

Therefore,

$$-\sigma\partial_t\eta_v\boldsymbol{n} - \zeta\gamma_0[\boldsymbol{M} - \boldsymbol{M}_0]\boldsymbol{n} + \gamma_0\boldsymbol{T}\boldsymbol{n} = \gamma_0\boldsymbol{T}_0\boldsymbol{n} \tag{19}$$

From our previous choice of $\boldsymbol{M}$, we have

$$\gamma_0\boldsymbol{M}\boldsymbol{n} = \rho\eta_v^2\boldsymbol{n};$$

so that,

$$\boldsymbol{n} \cdot \gamma_0 \boldsymbol{M n} = \rho \eta_v^2.$$

Our stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -\gamma_0 \boldsymbol{p} + 2\mu \boldsymbol{D}(\boldsymbol{v}).$$

Hence,

$$\gamma_0 \boldsymbol{T n} = -\gamma_0 \boldsymbol{p n} + 2\mu\gamma_0[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n}.$$

Eventually,

$$\boldsymbol{n} \cdot \boldsymbol{T n} = -\gamma_0 p + 2\mu\{\boldsymbol{n} \cdot \gamma_0[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n}\} = -\gamma_0 p - 2\mu\eta_v\kappa$$

by (17).

By our assumption, fluid particles are at rest in Ω_0 I.e. $\boldsymbol{v} = 0$, in Ω_0.
Thus,

$$\boldsymbol{D}(\boldsymbol{v}) = \nabla \boldsymbol{v} + \nabla^T \boldsymbol{v} = 0 \text{ in } \Omega_0.$$

Also, the conservation of momentum equation (2.2–5) implies that $\nabla p = 0$ from whence it follows that $p = p_0(t)$ in Ω_0.
Therefore,

$$\boldsymbol{T}_0 = -p_0 \boldsymbol{I}.$$

Taking the scalar product of the equation (19) with $\boldsymbol{n}$ we obtain:

$$-\sigma\partial_t\eta_v - \zeta(\boldsymbol{n} \cdot \boldsymbol{M n}) + \boldsymbol{n} \cdot \boldsymbol{T n} = \boldsymbol{n} \cdot \boldsymbol{T}_0\boldsymbol{n}.$$

With the appropriate substitution, we obtain

$$-\sigma\partial_t\eta_v - \zeta(\rho\eta_v^2) + (-\gamma_0 p - 2\mu\eta_v\kappa) = p_0.$$

That is,

$$\sigma\partial_t\eta_v + \rho\zeta\eta_v^2 + \gamma_0 p + 2\mu\eta_v\kappa = -p_0(t). \tag{20}$$

Equation (20), whilst being an evolution equation for η_v, also depicts the conservation of linear momentum for the region Ω_0

2.6. Reformulation of the problem.
In view of (2.4–1),

$$\begin{aligned}\rho[\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] &= \nabla\cdot\boldsymbol{T}; \boldsymbol{v}\in\Omega\\ &= -\nabla\boldsymbol{p} + \mu\nabla\cdot\boldsymbol{A}(\boldsymbol{v})\\ &= -\nabla\boldsymbol{p} + \mu\Delta\boldsymbol{v}.\end{aligned}$$

Then,

$$\rho\boldsymbol{v}_t + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \mu\Delta\boldsymbol{v} + \nabla\boldsymbol{p} = 0,$$

is the new form of the equation of motion in Ω.

The boundary conditions are incorporated in the evolution equation (2.5–19) for η_v. We can now reformulate the problem as follows: We look for $\boldsymbol{v}$ such that

$$\left.\begin{aligned}\rho\boldsymbol{v}_t + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \mu\Delta\boldsymbol{v} + \nabla p = 0;\\ \sigma\eta_t + \rho\zeta\eta_v^2 + 2\eta_v\mu\kappa + \gamma_0 p = -p_0(t)\end{aligned}\right\} \tag{1}$$

In the next chapter, we will show that the null solution $\boldsymbol{v}\equiv 0$ for (1), which corresponds to the rest state, is stable under the criterion established in [7] by Le Roux and Sauer. According to this criterion, we will use the coupling in (1) to construct some canonical operators. To do this, we have to transform the two equations by multiplying the first by $\rho^{-\frac{1}{2}}$ and the second by $\sigma^{-\frac{1}{2}}$, to obtain the following forms:

$$\left.\begin{aligned}&\rho^{\frac{1}{2}}\boldsymbol{v}_t + \rho^{\frac{1}{2}}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \mu\rho^{-\frac{1}{2}}\Delta\boldsymbol{v} + \rho^{-\frac{1}{2}}\nabla\boldsymbol{p} = 0\\ &\sigma^{\frac{1}{2}}\eta_t + \sigma^{-\frac{1}{2}}\rho\zeta\eta_v^2 + 2\sigma^{-\frac{1}{2}}\eta_v\mu\kappa + \sigma^{-\frac{1}{2}}\gamma_0\boldsymbol{p} = -\sigma^{-\frac{1}{2}}p_0(t).\end{aligned}\right\} \tag{2}$$

In the next chapter, we will rewrite (2) as an implicit equation in terms of certain canonical operators.

CHAPTER 3

THE STABILITY OF NAVIER-STOKES FLOWS THROUGH PERMEABLE BOUNDARIES

3.1. The system as an implicit equation. The system defined by (1) in Section 2.6 may be viewed as a coupled system, with the coupling situated in the pressure term which appears in both equations. There is also some additional coupling within the system when the constraint

$$\nabla \cdot \boldsymbol{v} = 0$$

is taken into account. The system is more closely coupled once we observe that

$$\eta = \eta_v := -[\gamma_0 \boldsymbol{v}] \cdot \boldsymbol{n}\,.$$

Thus, the evolution equation at the boundary cannot be solved separately from the Navier-Stokes equations (which are strongly coupled) in the domain Ω.

The modelling described in Chapter 2, therefore, leads to the following constraints on the velocity field $\boldsymbol{v}$:

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0; \\ \boldsymbol{v} &= 0, \text{ on } \Gamma_0; \\ \gamma_0 \boldsymbol{v} &= -\eta_v \boldsymbol{n} \text{ on } \Gamma. \end{aligned}\right\} \tag{1}$$

The system (1) in Section 2.6 may now be expressed as an implicit evolution equation, by defining the mappings L, B, N and $\mathscr{L}$ as follows:

$$\left.\begin{aligned} L\boldsymbol{v} &:= -\mu\langle 2\rho^{-1/2}\nabla \cdot \boldsymbol{D}(\boldsymbol{v}), -\sigma^{-1/2}\kappa\eta_v\rangle; \\ B\boldsymbol{v} &:= \langle \rho^{1/2}\boldsymbol{v}, \sigma^{1/2}\eta_v\rangle; \\ N(\boldsymbol{v}) &:= \langle \rho^{1/2}\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \zeta\rho\sigma^{-1/2}\eta_v^2\rangle; \\ \mathscr{L}p &:= \langle \rho^{-1/2}\nabla p, \sigma^{-1/2}\gamma_0 p\rangle. \end{aligned}\right\} \tag{2}$$

The notation $\langle a, b\rangle$ indicates an element of some product space.

The mappings L, B and $\mathscr{L}$ are evidently linear if the class of functions on which they act is a linear space. The system (2) in Section 2.6 can now be rewritten in implicit form as follows:

$$\partial_t[Bv] + Lv + N(v) + \mathscr{L}p = \langle 0, -p_0(t)\rangle = -\mathscr{L}p_0(t). \tag{3}$$

In this equation, the only unknowns are the velocity field v and the pressure p. In order to be more explicit about the mappings defined in (2) and the equation (3), we define the following spaces:

1. $\boldsymbol{X} = \boldsymbol{L}^2(\Omega)$; the space of measurable square integrable vector fields on Ω. We will not be concerned about the topology induced by the "natural" norm $\|\ \|_X$ which makes $\boldsymbol{X}$ a Banach space.
2. We define $\mathscr{D} \subset \boldsymbol{X}$ as follows:
$$\mathscr{D} := \{v \in H^2(\Omega) : \nabla \cdot v = 0;\ v = 0 \text{ on } \Gamma_0; \gamma_0 v = -\eta_v \boldsymbol{n} \text{ on } \Gamma\}.$$
3. $\boldsymbol{Y} := \boldsymbol{L}^2(\Omega) \times H^{\frac{3}{2}}(\Gamma)$, with the inner product
$$(\langle a, b\rangle, \langle c, d\rangle)_Y := (a, c)_X + \theta(b, d)_{H^{3/2}(\Gamma)}.$$
The corresponding norm would be defined (by taking the square root) of:
$$\|\langle a, b\rangle\|_Y^2 = \|a\|_X^2 + \theta\|b\|_{H^{3/2}(\Gamma)}^2.$$
Here the 'thickness-parameter' $\theta > 0$ is introduced to ensure that the metric in the product space is dimensionally correct provided that a and b have the same physical units. With these properties, $\boldsymbol{Y}$ becomes a Hilbert space.
The following should also be noted:

A. By the assumption of boundedness (Chapter 2; Section 2.3) on the surface density σ, the metric in $L^2(\Omega)$ can be defined with σ as the weight. Occasionally, we will use the equivalent norm:
$$(f, g)_\sigma = \int_\Gamma \sigma(y) f(y) g(y)\, ds(y),$$
for $\boldsymbol{L}^2(\Gamma)$.

B. The space $\mathscr{D}$ shall be taken as the domain for the operators defined in equation (2)

C. By the trace theorem (Lions and Magenes [8], p. 39) L and B are well-defined linear operators on $\mathscr{D}$ with range $\boldsymbol{Y}$ (See the appendix)

D. By the Sobolev embedding theorems ([1], Theorem 5.23, p. 115; Theorem 7.57, p. 217), the non-linear operator N is also defined on $\mathscr{D}$; with range in $\boldsymbol{Y}$. Indeed,

$$N(\boldsymbol{v}) = \langle \rho^{1/2}\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \zeta\rho\sigma^{-1/2}\eta_v^2 \rangle;$$

where $\rho^{1/2}\boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L^2(\Omega)$; $\zeta\rho\sigma^{-1/2}\eta_v \in L^2(\Gamma)$.

E. It is seen that (2.5–17) holds. I.e.,

$$\boldsymbol{n} \cdot \gamma_0 \boldsymbol{D}(\boldsymbol{v})\boldsymbol{n} = -\kappa\eta_v$$

for every $\boldsymbol{v} \in \mathscr{D}$.

We shall use the notation $\boldsymbol{v}(t)$ for the function $\boldsymbol{v}(t) : x \in \Omega \mapsto \boldsymbol{v}(x,t)$, for $t > 0$. A similar notation will be used for the pressure p.

For $t > 0$, let $\boldsymbol{v} : t \mapsto \boldsymbol{v}(t) \in \mathscr{D}$ be a given function. We shall say: *$B\boldsymbol{v}$ is differentiable* in $\boldsymbol{Y}$, if for every $t > 0$,

$$\lim_{h \to 0} h^{-1}[B\boldsymbol{v}(t+h) - B\boldsymbol{v}(t)]$$

exists in the norm of $\boldsymbol{Y}$. This limit will be denoted by $[B\boldsymbol{v}(t)]'$.

The pair of functions $\boldsymbol{v}(t) \in \mathscr{D}$ and $p(t) \in H^1(\Omega)$ is said to be a solution of the *Implicit Cauchy Problem* (ICP) if $B\boldsymbol{v}(t)$ is differentiable in $\boldsymbol{Y}$ and for a given $p_0(t)$ and $y \in \boldsymbol{Y}$:

$$\left.\begin{aligned} [B\boldsymbol{v}(t)]' + L\boldsymbol{v}(t) + N\boldsymbol{v}(t) + \mathscr{L}p(t) &= -\mathscr{L}p_0(t) \\ \lim_{t \to 0+} \|B\boldsymbol{v}(t) - y\|_Y &= 0 \end{aligned}\right\} \tag{4}$$

3.2. The Helmholtz-Weyl Projection.

Terms of the form $\mathscr{L}p$ in the implicit system (3.1–4) cannot be handled with the aid of the traditional Helmholtz-Weyl projection because of the presence of a boundary component.

There exists, however, a projection $P : \boldsymbol{Y} \to \boldsymbol{Y}$; which annihilates terms of the form:

$$\mathscr{L}p = \langle \rho^{-1/2}\nabla p, \sigma^{-1/2}\gamma_0 p\rangle;$$

and leaves intact, terms of the form:

$$B\boldsymbol{v} = \langle \rho^{1/2}\boldsymbol{v}, \sigma^{1/2}\eta_v\rangle;$$

when $\nabla \cdot \boldsymbol{v} = 0$. (See Sauer, [10]). We now apply the projection P on (4) to obtain :

$$[PB\boldsymbol{v}(t)]' + PL\boldsymbol{v}(t) + PN(\boldsymbol{v}(t)) + P\mathscr{L}p(t) = -P\mathscr{L}p_0(t)$$
$$\lim_{t\to 0^+} \|PB\boldsymbol{v}(t) - Py\|_Y = 0.$$

This leads to the *Projected Implicit Cauchy Problem* PICP:

$$\left.\begin{aligned} [B\boldsymbol{v}(t)]' + PL\boldsymbol{v}(t) + PN(\boldsymbol{v}(t)) = 0 \\ \lim_{t\to 0^+} \|B\boldsymbol{v}(t) - Py\|_Y = 0, \end{aligned}\right\} \tag{1}$$

from which the pressure has been eliminated.

3.3. Some Identities and Inequalities.

Before proceeding with the study of stability, some identities and inequalities, derived from integration by parts, need to be discussed. We shall use the 'colon product' between second order tensors defined by

$$\boldsymbol{A} : \boldsymbol{B} := \sum_{i,j} A_{ij}B_{ij}$$

in what follows.

PROPOSITION 3.3.1. *For $\varphi, \boldsymbol{\theta} \in \mathscr{D}$,*

$$(PL\varphi, B\boldsymbol{\theta})_Y = 2\mu((\boldsymbol{D}(\varphi)), \boldsymbol{D}(\boldsymbol{\theta}))_{L^2(\Omega)}, \tag{1}$$

$$(PN(\varphi), B(\varphi))_Y = \rho\int_\Gamma [\zeta - \tfrac{1}{2}]\eta_\varphi^3\, ds, \tag{2}$$

where,

$$(\boldsymbol{D}(\varphi), \boldsymbol{D}(\boldsymbol{\theta}))_{L^2(\Omega)} = \int_\Omega \boldsymbol{D}(\varphi) : \boldsymbol{D}(\boldsymbol{\theta})\, dx.$$

Proof.
We have

$$(PL\varphi, B\theta)_Y = (L\varphi, PB\theta)_Y = (L\varphi, B\theta)_Y,$$

since P, as a projection operator, is self-adjoint and $PB\theta = B\theta$. Also,

$$\begin{aligned}(L\varphi, B\theta)_Y &= (\langle -\rho^{-\frac{1}{2}}\mu\nabla\cdot 2D(\varphi), 2\mu\kappa\sigma^{-\frac{1}{2}}\eta_\varphi\rangle, \langle \rho^{\frac{1}{2}}\theta, \sigma^{\frac{1}{2}}\eta_\theta\rangle)_Y \\ &= -\mu(\nabla\cdot 2D(\varphi), \theta)_{L^2(\Omega)} + 2\mu(\kappa\eta_\varphi, \eta_\theta)_{L^2(\Gamma)}\end{aligned}$$

Using integration by parts,

$$(\nabla\cdot 2D(\varphi), \theta)_{L^2(\Omega)} = \int_\Gamma \gamma_0\theta\cdot 2D(\varphi)n\,ds - \int_\Omega 2D(\varphi)\nabla\theta dx.$$

But,

$$2D(\varphi) : \nabla\theta = \tfrac{1}{2}[2D(\varphi) : 2D(\theta)];$$

using the properties of the 'colon product'. Therefore,

$$\begin{aligned}-\mu(\nabla\cdot 2D(\varphi), \theta)_{L^2(\Omega)} &= \tfrac{1}{2}\mu(2D(\varphi), 2D(\theta))_{L^2(\Omega)} - \mu\int_\Gamma \gamma_0\theta\cdot\gamma_0 2D(\varphi)n\,ds \\ &= 2\mu(D(\varphi), D(\theta))_{L^2(\Omega)} - 2\mu\int_\Gamma(-\eta_\theta n)\cdot(-\kappa\eta_\varphi)n\,ds \\ &= 2\mu(D(\varphi), D(\theta))_{L^2(\Omega)} - 2\mu\int_\Gamma \kappa\eta_\varphi\eta_\theta\,ds;\end{aligned}$$

using (1). Thus,

$$\begin{aligned}(PL(\varphi), B\theta)_Y &= 2\mu(D(\varphi), D(\theta))_{L^2(\Omega)} - 2\mu\int_\Gamma \kappa\eta_\varphi\eta_\theta\,ds + 2\mu\int_\Gamma \kappa\eta_\varphi\eta_\theta\,ds \\ &= 2\mu(D(\varphi), D(\theta))_{L^2(\Omega)}\end{aligned}$$

This proves (1).

Next we prove (2):

$$(PN(\varphi), B\varphi)_Y = (N(\varphi), B\varphi)_Y;$$

since P is self-adjoint. Therefore,

$$\begin{aligned}(PN(\varphi), B(\varphi))_Y &= (N(\varphi), B\varphi)_Y, \\ &= (\langle\rho^{1/2}\varphi\cdot\nabla\varphi, \zeta\rho\sigma^{-1/2}\eta_\varphi^2\rangle, \langle\rho^{1/2}\varphi, \sigma^{1/2}\eta_\varphi\rangle)_Y \\ &= \rho(\varphi\cdot\nabla\varphi, \varphi)_{L^2(\Omega)} + \rho(\zeta\eta_\varphi^2, \eta_\varphi)_{L^2(\Gamma)}.\end{aligned}$$

Next, we derive the expression for $(\varphi \cdot \nabla\varphi, \varphi)_{L^2(\Omega)}$: Since $\sum_{k=1}^{3} \partial_k \varphi_k = \nabla \cdot \varphi = 0$,

$$\begin{aligned}
(\varphi \cdot \nabla\varphi, \varphi)_{L^2(\Omega)} &= \sum_{k,j=1}^{3} \int_\Omega (\varphi_k \partial_k \varphi_j)\varphi_j \, dx \\
&= \sum_{k,j=1}^{3} \int_\Omega (\partial_k[\varphi_k \varphi_j] \cdot \varphi_j) \, dx \\
&= \sum_{k,j=1}^{3} \int_\Gamma \eta_k \varphi_k \varphi_j \varphi_j \, ds - \sum_{k,j=1} \int_\Omega \varphi_k \varphi_j \partial_k \varphi_j \, dx \\
&= \int_\Gamma (\varphi \cdot \boldsymbol{n}) \gamma_0 \varphi \cdot \gamma_0 \varphi \, ds - (\varphi, (\varphi \cdot \nabla)\varphi)_\Omega.
\end{aligned}$$

Therefore,

$$\rho(\varphi \cdot \nabla\varphi, \varphi)_{L^2(\Omega)} = \rho \int_\Gamma (\varphi \cdot \boldsymbol{n}) \gamma_0 \varphi \cdot \gamma_0 \varphi \, ds - \rho(\varphi, (\varphi\nabla)\varphi)_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned}
2\rho(\varphi \cdot \nabla\varphi, \varphi)_{L^2(\Omega)} &= \rho \int_\Gamma (\varphi \cdot \boldsymbol{n}) \gamma_0 \varphi \cdot \gamma_0 \varphi \, ds \\
&= \rho \int_\Gamma (-\eta_\varphi) \eta_\varphi^2 \, ds.
\end{aligned}$$

We then obtain

$$\rho(\varphi \cdot \nabla\varphi, \varphi)_{L^2(\Omega)} = -\tfrac{1}{2}\rho \int_\Gamma \eta_\varphi^3 \, ds.$$

Finally, we have:

$$\begin{aligned}
(PN(\varphi), B\varphi)_Y &= -\tfrac{1}{2}\rho \int_\Gamma \eta_\varphi^3 \, ds + \rho \int_\Gamma \zeta \eta_\varphi^3 \, ds \\
&= \rho \int_\Gamma [\zeta - \tfrac{1}{2}] \eta_\varphi^3 \, ds.
\end{aligned}$$

□

The following identity will have important consequences when the mean curvature of the surface Γ is nonnegative:

PROPOSITION 3.3.2. *For* $\varphi \in \mathscr{D}$,

$$\|\boldsymbol{D}(\varphi)\|^2_{L^2(\Omega)} = \tfrac{1}{2}[\|\nabla\varphi\|^2_{L^2(\Omega)} + \int_\Gamma \kappa(y)\eta^2_\varphi(y)\,ds(y)].$$

Proof.

Since $\boldsymbol{D}(\varphi) = \frac{1}{2}[\nabla\varphi + \nabla^T\varphi]$,we have,

$$\begin{aligned}(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)} &= \tfrac{1}{4}(\nabla\varphi + \nabla^T\varphi, \nabla\varphi + \nabla\varphi)_{L^2(\Omega)} \\ &= \tfrac{1}{4}[\|\nabla\varphi\|^2_{L^2(\Omega)} + 2(\nabla\varphi, \nabla^T\varphi) + \|\nabla^T(\varphi)\|^2_{L^2(\Omega)}] \\ &= \tfrac{1}{4}[2\|\nabla\varphi\|^2_{L^2(\Omega)} + 2(\nabla\varphi, \nabla^T\varphi)_{L^2(\Omega)}].\end{aligned}$$

Next, we derive an expression for $(\nabla\varphi, \nabla^T\varphi)_{L^2(\Omega)}$; using the Gauss divergence theorem and the Stokes theorem:

$$\begin{aligned}(\nabla\varphi, \nabla^T\varphi)_{L^2(\Omega)} &= \sum_{i.j}^{3} \int_\Omega \partial_i\varphi_j\partial_j\varphi_i\,dx \\ &= \sum_{i,j}^{3} \partial_i[\varphi_j\partial_j\varphi_i]\,dx;\end{aligned}$$

since $\nabla \cdot \varphi = 0$. However,

$$\int_\Omega \partial_i[\varphi_j\varphi_i]\,dx = \int_\Gamma n_i\varphi_j\partial_j\varphi_i\,ds;$$

by the divergence theorem.

Therefore,

$$\begin{aligned}(\nabla\varphi, \nabla^T\varphi)_{L^2(\Omega)} &= \sum_{i,j=1}^{3} \int_\Gamma n_i\varphi_j\partial_j\varphi_i\,ds \\ &= -\sum_{i,j=1}^{3} \int_\Gamma \eta_\varphi n_i n_j(\partial_j\varphi_i)\,ds \\ &= \int_\Gamma \eta_\varphi \boldsymbol{n} \cdot (\nabla\varphi)\boldsymbol{n}\,ds \\ &= \int_\Gamma \eta_\varphi \boldsymbol{n} \cdot [\tfrac{1}{2}(\nabla\varphi + \nabla^T\varphi)]\boldsymbol{n}\,ds \\ &= \int_\Gamma \eta_\varphi \boldsymbol{n} \cdot \boldsymbol{D}(\varphi)\boldsymbol{n}\,ds \\ &= \int_\Gamma \eta_\varphi(-\kappa\eta_\varphi)\,ds.\end{aligned}$$

Finally,we have:

$$(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)} = \tfrac{1}{4}[2\|\nabla\varphi\|^2_{L^2(\Omega)} - 2\int_\Gamma \eta_\varphi(-\eta_\varphi\kappa)\,ds]$$
$$= \tfrac{1}{2}[\|\nabla\varphi\|^2_{L^2(\Omega)} + \int_\Omega \kappa\eta_\varphi^2\,ds].$$

$$(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)} = \|\boldsymbol{D}(\varphi)\|^2_{L^2(\Omega)}$$
$$= \tfrac{1}{2}[\|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} + \int_\Gamma \kappa\eta_\varphi^2\,ds];$$

provided $\kappa \geq 0$; and the result follows. □

Before we continue, it is appropriate that we derive the following identities. These will be used in proofs of some of the inequalities.

First, we define the bilinear form $R(\varphi,\psi)$; for $\varphi,\psi \in \mathscr{D}$ by

$$R(\varphi,\psi) := (PL\varphi, B\psi)_Y.$$

By (3.3.1),

$$(PL\varphi, B\varphi)_Y = 2\mu(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned} R(\varphi,\varphi) &= 2\mu(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)} \\ &= 2\mu\|\boldsymbol{D}(\varphi)\|^2_{L^2(\Omega)} \end{aligned} \quad (3)$$

We also note that

$$(L\varphi, L\varphi)_Y = \tfrac{\mu}{\rho}\|\Delta\varphi\|^2_{L^2(\Omega)} + 4\mu^2\|\kappa^{1/2}\sigma^{-1/2}\eta\varphi\|^2_{H^{3/2}(\Gamma)}. \quad (4)$$

We now continue with the inequalities that will help us later in our study of stability.

LEMMA 3.3.3. *For $\varphi \in \mathscr{D}$, $|\varphi\cdot\nabla\varphi| \leq |\varphi|\cdot|\nabla\varphi|$.*

Proof.

$$|\varphi\cdot\nabla\varphi|^2 = \sum_j[\sum_i \varphi_i\partial_i\varphi_j]^2$$
$$\leq \sum_j[(\sum_i(\varphi_i^2))^{1/2}\cdot(\sum_i(\partial_i\varphi_j)^2)^{\frac{1}{2}}]^2,$$

by the Schwartz inequality. Eventually,

$$\begin{aligned} |\varphi \cdot \nabla\varphi|^2 &\leq \sum_j (\sum_i \varphi_i^2)(\sum_i (\partial_i \varphi_j)^2) \\ &= (\sum_i \varphi_i^2)(\sum_j \sum_i (\partial_i \varphi)^2) \\ &= |\varphi|^2 \cdot |\nabla\varphi|^2, \end{aligned}$$

and the result follows. □

Next, we also note that: Since in our case, $m = 2$, $p = 2$ and $n = 3$, making $mp > n$; in accordance with the Sobolev embedding theorem (Adams, [1], Theorem 5.23, p. 115) for $\varphi \in \mathscr{D}$, there exists $c > 0$ such that

$$\begin{aligned} |\varphi(x)| &\leq \sup_{x\in\Omega} |\varphi(x)| \\ &\leq c\|\varphi\|_{H^2(\Omega)}. \end{aligned} \tag{5}$$

Similarly, the following lemma will be crucial in our study of stability:

LEMMA 3.3.4. *For $\varphi \in \mathscr{D}$, $(\varphi \cdot \nabla)\varphi \in L^2(\Omega)$ and*

$$\|(\varphi \cdot \nabla)\varphi\|_{L^2(\Omega)}^2 \leq C_2^2 \|\varphi\|_{H^2(\Omega)}^2 \|\nabla\varphi\|_{L^2(\Omega)}^2.$$

Proof. We have that:

$$\begin{aligned} \|(\varphi \cdot \nabla)\varphi\|_{L^2(\Omega)}^2 &= \int_\Omega |(\varphi \cdot \nabla)\varphi|^2 \, dx \\ &\leq \int_\Omega |\varphi|^2 |\nabla\varphi|^2 \, dx \end{aligned}$$

by Lemma 3.3.3. But,

$$\begin{aligned} \int_\Omega |\varphi|^2 |\nabla\varphi|^2 \, dx &\leq \sup |\varphi|^2 \int_\Omega |\nabla\varphi|^2 \, dx \\ &\leq c\|\varphi\|_{H^2(\Omega)} \int_\Omega |\nabla\varphi|^2 \, dx \\ &= c\|\varphi\|_{H^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)}. \end{aligned}$$

Thus,

$$\|(\varphi \cdot \nabla)\varphi\|_{L^2(\Omega)}^2 \leq c\|\varphi\|_{H^2(\Omega)}^2 \cdot \|\nabla\varphi\|_{L^2(\Omega)}^2.$$

□

We next observe that, since the elements of $\mathscr{D}$ vanish on the outer surface Γ_0, and since Ω is bounded, the Poincaré inequality holds:

For any $\varphi \in \mathscr{D}$,there exists a constant $c_p > 0$ such that

$$||\nabla\varphi||^2_{L^2(\Omega)} \geq c_p||\varphi||^2_{L^2(\Omega)} \tag{6}$$

(See e.g. Dunn & Fosdick, [3], pp. 248–249)

We define the Sobolev $H^1(\Omega)$ norm in accordance with the Poincaré inequality so that the units are balanced, by

$$||\varphi||_{H^1(\Omega)} := ||\nabla\varphi||^2_{L^2(\Omega)} + c_p||\varphi||^2_{L^2(\Omega)};$$

for $\varphi \in \mathscr{D}$.

We now demonstrate the equivalence of some norms:

LEMMA 3.3.5. *For $\varphi \in \mathscr{D}$, if the mean curvature κ of the surface Γ is nonnegative and bounded, then $||\boldsymbol{D}(\varphi)||_{L^2(\Omega)}$ is a norm on $\mathscr{D}$, equivalent to $||\ \ ||_{H^1(\Omega)}$.*

Proof.

From Proposition 3.3.2, we have that:

$$||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} = \tfrac{1}{2}[||\nabla\varphi||^2_{L^2(\Omega)} + \int_\Gamma \kappa\eta^2_\varphi\, ds].$$

Then, if $\kappa \geq 0$,

$$||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} \geq \tfrac{1}{2}||\nabla\varphi||^2_{L^2(\Omega)}. \tag{7}$$

That is,

$$\begin{aligned} ||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} &\geq \tfrac{1}{4}[||\nabla\varphi||^2_{L^2(\Omega)} + \tfrac{1}{4}||\nabla\varphi||^2_{L^2(\Omega)}] \\ &\geq \tfrac{1}{4}||\nabla\varphi||^2_{L^2(\Omega)} + \tfrac{1}{4}c_p||\varphi||_{L^2(\Omega)} \\ &= \tfrac{1}{4}||\varphi||^2_{H^1(\Omega)}; \end{aligned} \tag{8}$$

from the Poincaré inequality and the definition of the Sobolev $H^1(\Omega)$ norm.

Also,

$$\int_\Gamma \kappa\eta^2_\varphi\, ds \leq \left|\int_\Gamma \kappa\eta^2_\varphi\, ds\right| \leq K||\eta_\varphi||^2_{L^2(\Gamma)}.$$

where $K = \sup_{y\in\Gamma}\kappa(y) > 0$. By the trace theorem (Lions-Magenes [8], Theorem 3.2, p.21; Adams [1], Theorem 7.55, p. 216), there exists $m > 0$ such that

$$||\eta_\varphi||^2_{L^2(\Gamma)} \leq m||\varphi||^2_{H^1(\Omega)}.$$

Thus, there exists $M = Km > 0$ such that

$$\int_\Gamma \kappa\eta_\varphi^2\, ds \leq M||\varphi||^2_{H^1(\Omega)}.$$

By Proposition 3.3.2,

$$||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} = \tfrac{1}{2}[||\nabla\varphi||^2_{L^2(\Omega)} + \int_\Gamma \kappa\eta_\varphi^2\, ds].$$

Therefore,

$$\begin{aligned}||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} &\leq \tfrac{1}{2}[||\nabla\varphi||^2_{L^2(\Omega)} + M||\varphi||^2_{H^1(\Omega)}]\\ &\leq \tfrac{1}{2}[||\nabla\varphi||^2_{L^2(\Omega)} + M\{||\nabla\varphi||^2_{L^2(\Omega)} + C_p||\varphi||^2_{L^2(\Omega)}\}];\end{aligned}$$

by the definition of the Sobolev $H^1(\Omega)$ norm. Eventually, we have

$$\begin{aligned}||\boldsymbol{D}(\varphi)||_{L^2(\Omega)} &\leq \tfrac{1}{2}[(1+M)||\nabla\varphi||^2_{L^2(\Omega)} + Mc_p||\varphi||^2_{L^2(\Omega)}]\\ &\leq \tfrac{1}{2}(1+2M)[||\nabla\varphi||_{L^2(\Omega)} + c_p||\varphi||^2_{L^2(\Omega)}]\\ &= \tfrac{1}{2}(1+2M)||\varphi||^2_{H^1(\Omega)};\end{aligned}$$

by the definition of the Sobolev norm. Combining the two preceding inequalities involving $\boldsymbol{D}$, we finally obtain

$$\tfrac{1}{4}||\varphi||^2_{H^1(\Omega)} \leq ||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} \leq (M + \tfrac{1}{2})||\varphi||^2_{H^1(\Omega)};$$

and the result follows. □

The following assumption will be made from now on:

the mean curvature κ of Γ and the surface density σ are positive constants.

We now define the norms, $||\varphi||_1^2$ and $||\varphi||_2^2$ in accordance with Le Roux and Sauer [7]:

$$\begin{aligned}||\varphi||_1^2 &:= \hat{R}(\varphi,\varphi)\\ &= 2\mu||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)}.\end{aligned} \tag{9}$$

By Lemma 3.3.5 it is a norm. The other norm is defined by

$$\begin{aligned}||\varphi||_2^2 &:= ||\varphi||_1^2 + \tfrac{1}{2}(L\varphi, L\varphi)_{L^2(\Omega)}\\ &= ||\varphi||_1^2 + \tfrac{\mu^2}{2\rho}||\Delta\varphi||^2_{L^2(\Omega)} + \tfrac{\mu^2\kappa^2}{2\sigma}||\eta_\varphi||^2_{H^{3/2}(\Gamma)}.\end{aligned} \tag{10}$$

3.4. Estimates.

We will use the following well-known result (see e.g. Galdi [4], p. 227; Temam [12], p. 33):

If φ and p solves the Stokes problem,

$$-\Delta\varphi + \nabla p = \boldsymbol{f} \text{ on } \Omega;$$
$$\nabla \cdot \varphi = 0, \text{ on } \Omega;$$
$$\gamma_0 \varphi = \boldsymbol{\Psi}; \text{ on } \Gamma$$

and if $\boldsymbol{f} \in L^2(\Omega)$; $\boldsymbol{\Psi} \in H^{3/2}(\Gamma)$, then, $\varphi \in H^2(\Omega)$, $p \in H^1(\Omega)$, and there exists C_Δ, $C_\Gamma > 0$ such that

$$||\varphi||^2_{H^2(\Omega)} \le ||\nabla p||^2_{L^2(\Omega)} + ||\varphi||^2_{H^2(\Omega)} \le C_\Omega ||\boldsymbol{f}||^2_{L^2(\Omega)} + C_\Gamma ||\gamma_0 \boldsymbol{\Psi}||^2_{H^{3/2}(\Gamma)}. \tag{1}$$

We proceed by applying the traditional Helmholtz decomposition to the vector field $\boldsymbol{f}$. The result is:

$$-\Delta\varphi + \nabla p = P_0 \boldsymbol{f} + \nabla q; q \in H^1(\Omega), \tag{2}$$

with P_0 the traditional Helmholtz-Weyl projection. Another application of P_0 gives

$$-P_0\Delta\varphi + P_0\nabla(p-q) = P_0^2 f = P_0 f.$$

Thus,

$$-P_0\Delta\varphi = P_0 f. \tag{3}$$

We then use (2), (3) and (1), to obtain

$$||\varphi||^2_{H^2(\Omega)} \le C_\Delta ||P_0\Delta\varphi||^2_{L^2(\Omega)} + C_\Gamma ||\boldsymbol{\Psi}||_{H^{3/2}(\Gamma)}. \tag{4}$$

In our situation,

$$\boldsymbol{\Psi} = -\eta_\psi \boldsymbol{n}. \tag{5}$$

Then, we rewrite (4) in terms of (5) to obtain

$$||\varphi||^2_{H^2(\Omega)} \le C_\Delta ||P_0\Delta\varphi||^2_{L^2(\Omega)} + C_\Gamma ||\eta_\varphi \boldsymbol{n}||^2_{H^{3/2}(\Gamma)}. \tag{6}$$

Since $||P_0|| \le 1$,

$$C_\Delta ||P_0\Delta\varphi||_{L^2(\Omega)} \le C_\Delta ||\Delta\varphi||_{L^2(\Omega)}.$$

We therefore obtain from (6)

$$||\varphi||^2_{H^2(\Omega)} \le C_\Delta ||\Delta\varphi||^2_{L^2(\Omega)} + C_\gamma ||\eta_\varphi||^2_{H^{3/2}(\Gamma)}. \tag{7}$$

The inequality (7) is the a-priori estimate we shall need.

The first result on equivalence of norms is:

LEMMA 3.4.1. *For $\varphi \in \mathscr{D}$, the norms of the Sobolev space $H^2(\Omega)$ and the norm defined by $||L\varphi||_Y$ are equivalent.*

Proof.

From (3.3–10),

$$||L\varphi||_Y^2 = \mu^2[\tfrac{1}{\rho}||\Delta\varphi||^2_{L^2(\Omega)} + \tfrac{\kappa^2}{\sigma}||\eta_\varphi||^2_{H^{3/2}(\Gamma)}].$$

We first estimate $||L\varphi||_Y^2$ from above by using the trace theorem. In fact, from the identity above,

$$\begin{aligned}||L\varphi||_Y^2 &\leq \mu^2[\tfrac{1}{\rho}||\varphi||^2_{H^2(\Omega)} + \tfrac{\kappa^2}{\sigma}m||\varphi||^2_{H^2(\Gamma)}]\\ &\leq \mu^2[\tfrac{3}{\rho} + \tfrac{\kappa^2 m}{\sigma}||\varphi||^2_{H^{3/2}(\Gamma)}].\end{aligned}$$

The lower bound is obtained by making use of the a-piori estimate (6):

$$\begin{aligned}||L\varphi||_Y^2 &= \mu^2[\tfrac{1}{\rho C_\Delta}C_\Delta||\Delta\varphi||^2_{L^2(\Omega)} + \tfrac{\kappa^2}{\sigma C_\delta}C_\delta||\eta_\varphi||^2_{H^{3/2}(\Gamma)}]\\ &\geq \mu^2\min\{\tfrac{1}{\rho C_\Delta}, \tfrac{\kappa^2}{\sigma C_\delta}\}[C_\Delta||\Delta\varphi||^2_{L^2(\Omega)} + C_\delta||\eta_\varphi||^2_{H^{3/2}(\Gamma)}]\\ &\geq \mu^2\min\{\tfrac{1}{\rho C_\Delta}, \tfrac{\kappa^2}{\sigma C_\delta}\}||\varphi||^2_{H^2(\Omega)}.\end{aligned}$$

We have used the estimates

$$\begin{aligned}||\eta_\varphi||_{H^{3/2}(\Gamma)} &\leq m||\varphi||_{H^1(\Omega)}\\ ||\Delta\varphi||^2_{L^2(\Omega)} &\leq 3||\varphi||^2_{H^2(\Omega)}\end{aligned}$$

derived from the trace theorem and elementary considerations. The proof is complete. □

The following results involve $||\varphi||_1$ and $||\varphi||_2$ defined at the end of the previous section, and will be equally crucial in our study of the stability of Navier-Stokes flows:

PROPOSITION 3.4.2. *For $\varphi \in \mathscr{D}$,*

$$K||\varphi||^2_{H^2(\Omega)} \leq ||\varphi||_2^2,$$

where

$$K := \tfrac{\mu^2}{2}\min\{\tfrac{1}{\rho C_\Delta}, \tfrac{\kappa^2}{\sigma C_\delta}\}.$$

Proof.

The proof is a direct consequence of Lemma 3.4.1.

We end this section with an estimate of the nonlinear operator $N : \mathscr{D} \to Y$. For this purpose, we shall need the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. For our situation, shall also need the embedding

$$H^1(\Omega) \hookrightarrow L^4(\Gamma).$$

(Adams [1], Theorem 5.22, page 114) for the combination $m = 1, p = 2$ and $n = 3$, so that $mp < n$, $2 \le q \le 4$.) Thus, there exist $C_E > 0$ and $C_{E,4} > 0$, such that

$$\begin{aligned} |\varphi(y)| &\le C_E \|\varphi\|_{H^2(\Omega)}, \\ \|\eta_\varphi\|_{L^4(\Omega)} &\le C_{E,4}\|\varphi\|_{H^1(\Omega)}, \text{ for } \varphi \in \mathscr{D}. \end{aligned} \tag{8}$$

We now proceed to estimate N:

LEMMA 3.4.3. *There exists a constant $C_1 > 0$ such that, for all $\varphi \in \mathscr{D}$,*

$$\|N(\varphi)\|_Y^2 \le [C_1 \|\boldsymbol{D}(\varphi)\|_{L^2(\Omega)} \|L\varphi\|_Y]^2,$$

with

$$C_1 := \tfrac{2\rho}{K}[C_E^2 + \tfrac{2\rho C_{E,4}^4}{\sigma}]$$

Proof.

From the definition of N,

$$\begin{aligned} \|N(\varphi)\|_Y^2 &= \rho \int_\Omega |\varphi \cdot \nabla\varphi|^2 \, dx + \rho^2 \int_\Gamma \tfrac{\zeta^2}{\sigma} \eta_\varphi^4 \, ds \\ &\le \rho \int_\Omega |\varphi|^2 |\nabla\varphi|^2 \, dx + \tfrac{\rho^2}{\sigma} \int_\Gamma \eta_\varphi^4 \, dx; \end{aligned} \tag{9}$$

by Lemma 3.3.3 and since $0 < \zeta < 1$. From (8) we now obtain

$$\|N(\varphi)\|_Y^2 \le \rho C_E \|\varphi\|^2_{H^2(\Omega)} \int_\Omega |\nabla\varphi|^2 \, dx + \tfrac{\rho^2}{\sigma} \int_\Gamma \eta_\varphi^4 \, dx. \tag{10}$$

By (8),

$$\|\eta_\varphi\|^4_{L^4(\Omega)} \le C^4_{E,4}\|\varphi\|^4_{H^1(\Omega)}.$$

Therefore,

$$\begin{aligned} \|N(\varphi)\|_Y^2 &\le \rho C_E{}^2 \|\varphi\|^2_{H^1(\Omega)} \|\nabla\varphi\|^2_{L^2(\Omega)} + \tfrac{\rho^2}{\sigma} C^4_{E,4} \|\varphi\|^4_{H^1(\Omega)} \\ &= \rho C_E^2 \|\varphi\|^2_{H^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} + \tfrac{\rho^2}{\sigma} C^4_{E,4} \|\varphi\|^2_{H^1(\Omega)} \|\varphi\|^2_{H^1(\Omega)} \\ &\le \rho C_E^2 \|\varphi\|^2_{H^2(\Omega)} \|\nabla\varphi\|^2_{L^2(\Omega)} + \tfrac{\rho^2}{\sigma} C^4_{E,4} \|\varphi\|^2_{H^2(\Omega)} \|\varphi\|^2_{H^1(\Omega)} \\ &= \|\varphi\|^2_{H^2(\Omega)} [\rho C_E^2 \|\nabla\varphi\|^2_{L^2(\Omega)} + \tfrac{\rho^2}{\sigma} C^4_{E,4} \|\varphi\|^2_{H^1(\Omega)}] \\ &= \rho \|\varphi\|^2_{H^2(\Omega)} [C_E^2 \|\nabla\varphi\|^2_{L^2(\Omega)} + \tfrac{\rho}{\sigma} C^4_{E,4} \|\varphi\|^2_{H^1(\Omega)}]. \end{aligned}$$

By (3.3–7) and (3.3–8), we have, taking into acount Proposition 3.4.2

$$||N(\varphi)||_Y^2 \leq \frac{\rho}{K}||L\varphi||_Y^2[2C_E^2||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} + \frac{4\rho C_{E,4}^2}{\sigma}||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)}],$$

so that the result is true. □

3.5. Stability.

In our study of stability, we shall follow the method of Le Roux and Sauer [7], which is an adaptation of the method of Galdi and Padula [5], to implicit evolution equations. The method applies the theory of bilinear forms as opposed to spectral theory in Galdi and Padula [5].

As required by the theory in Le Roux and Sauer [7], we have already constructed the bilinear form R defined by R

$$R(\varphi, \boldsymbol{\theta}) = (PL\varphi, B\boldsymbol{\theta})_Y;$$

for $\varphi, \boldsymbol{\theta} \in \mathscr{D}$.

We now define another bilinear form S:

$$\begin{aligned} S(\varphi, \boldsymbol{\theta}) &:= (PB\varphi, B\boldsymbol{\theta})_Y \\ &= \rho(\varphi, \boldsymbol{\theta})_{L^2(\Omega)} + \sigma(\eta_\varphi, \eta_\theta)_{L^2(\Gamma)} \end{aligned}$$

The following proposition establishes an important property of the bilinear form, R.

Proposition 3.5.1. *For $\varphi, \boldsymbol{\theta} \in \mathscr{D}$,the quadratic form $\hat{R}$ associated with the bilinear form R is positive definite.*

Proof.

We have

$$\begin{aligned} \hat{R}(\varphi) &:= R(\varphi, \varphi) \\ &= 2\mu(\boldsymbol{D}(\varphi), \boldsymbol{D}(\varphi))_{L^2(\Omega)} \\ &= 2\mu||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} \end{aligned}$$

Since $\kappa > 0$, it follows from Lemma 3.3.5 that $\hat{R}$ is positive definite in $H^1(\Omega)$. □

The next Proposition may be viewed as a corollary of Lemma 6. However, together with Proposition 4, it will be crucial in terms of the stability criterion in Le Roux and Sauer [7]:

PROPOSITION 3.5.2. *There is a constant $C_2 = C_1\mu^{-1/2}$ such that*

$$||N(\varphi)||_Y^2 \leq [C_2||\varphi||_1||\varphi||_2]^2$$

for every $\varphi \in \mathscr{D}$.

Proof.

From Lemma 3.4.3, we have

$$||N(\varphi)||_Y^2 \leq [C_1||L\varphi||_Y||\boldsymbol{D}(\varphi)||_{L^2(\Omega)}]^2.$$

From (9) and (10) we have

$$||\boldsymbol{D}(\varphi)||^2_{L^2(\Omega)} = \frac{1}{2\mu}||\varphi||_1^2$$
$$||L\varphi||_Y^2 \leq 2||\varphi||_2^2.$$

Hence,

$$||N(\varphi)||_Y^2 \leq \frac{C_1^2}{\mu}||\varphi||_1^2||\varphi||_2^2.$$

□

In terms of the stability criterion in Le Roux and Sauer,[7], the estimate for $||N(\varphi)||_Y$ is of the form

$$||N(\varphi)||_Y^2 \leq [C_2||\varphi||_1^\alpha||\varphi||_2]^2.$$

In our situation, as shown by Proposition 3.5.2, $\alpha = 1$. The 'energy' associated with $\varphi \in \mathscr{D}$, is defined as follows:

$$\begin{aligned} E_\varphi &:= \tfrac{1}{2}[\hat{R} + \hat{S}] \\ &= \tfrac{1}{2}\rho||\varphi||^2_{L^2(\Omega)} + \tfrac{1}{2}||\sigma\eta_\varphi||^2_{L^2(\Gamma)} + \tfrac{1}{2}||\varphi||_1^2 \end{aligned} \tag{1}$$

From (1) we have that

$$2E_\varphi = \rho||\varphi||^2_{L^2(\varphi)} + ||\sigma\eta_\varphi||^2_{L^2(\Gamma)} + ||\varphi||_1^2,$$

from which we deduce the very important inequalities:

$$||\varphi||_1^2 \leq 2E_\varphi; \tag{2}$$

$$\hat{S}(\varphi) \leq 2E_\varphi \tag{3}$$

If $v(t)$ is the solution of the PICP in Section 3.2, we shall use the following notation:

$$E(t) := E_{v(t)};\, t > 0$$
$$E_0 := E(0)$$

Before we conclude the stability theorem, we consider the rate of change of energy associated with the PICP problem. We follow an energy method which leads us to the setting of Le Roux and Sauer [7]. Towards this, we have the following abstract result:

LEMMA 3.5.3. *Let H be a real Hilbert space and $A : H \to H$ a linear (not necessarily bounded or closed) operator. For $v \in \mathfrak{D}(A)$, let the bilinear form r be defined by $r(u, v) := (u, Av)$. Let us suppose that r is symmetrical for $u, v \in \mathfrak{D}(A)$. If $t \mapsto v(t) \in \mathfrak{D}(A)$ is differentiable in the norm topology and $t \mapsto Av(t)$ is weakly continuous, then the quadratic form $\hat{r}(v(t))$ is differentiable and*

$$\tfrac{d}{dt}\hat{r}(v(t)) = 2r(v'(t), v(t)).$$

Proof.

$$\begin{aligned}
\tfrac{d}{dt}\hat{r}(v(t)) &= \lim_{h\to 0} h^{-1}[(v(t+h), Av(t+h)) - (v(t), Av(t))] \\
&= \lim_{h\to 0} h^{-1}[(v(t+h), Av(t+h)) - (v(t), Av(t+h)) \\
&\qquad + (v(t), Av(t+h)) - (v(t), Av(t))] \\
&= (\lim_{h\to 0} h^{-1}[v(t+h) - v(t)], \lim_{h\to 0} Av(t+h)) \\
&\qquad + \lim_{h\to 0}(v(t), h^{-1}[Av(t+h) - Av(t)]) \\
&= (\tfrac{d}{dt}v, Av(t)) + (\lim_{h\to 0} h^{-1}[v(t+h) - v(t)], Av(t)) \\
&= 2(\tfrac{d}{dt}v, Av(t));
\end{aligned}$$

since $\hat{r}(u, v)$ is symmetrical. Thus,

$$(\tfrac{d}{dt}v, Av(t)) = \tfrac{1}{2}\tfrac{d}{dt}\hat{r}(v(t)).$$

□

From the definition of $R(\varphi, \boldsymbol{\theta})$ and $S(\varphi, \boldsymbol{\theta})$, we have

$$R(\varphi, \boldsymbol{\theta}) = (PL\varphi, B\boldsymbol{\theta}) = R(\boldsymbol{\theta}, \varphi).$$

$$S(\varphi, \boldsymbol{\theta}) = (B\varphi, B\boldsymbol{\theta}) = S(\boldsymbol{\theta}, \varphi).$$

This shows that $R(\varphi, \boldsymbol{\theta})$ and $S(\varphi, \boldsymbol{\theta})$ are symmetrical.

REMARKS:

1. From the definition of $R(\varphi, \boldsymbol{\theta})$ and $S(\varphi, \boldsymbol{\theta})$, we conclude that

$$\hat{S} \leq 2E_\varphi;$$

$$||\varphi||_1 \leq 2E_\varphi.$$

2. From the definition of $||\varphi||_1$ and $||\varphi||_2$, we conclude that

$$||\varphi||_1 \leq ||\varphi||_2.$$

3. In the derivation of the energy inequality, we shall use the Schwartz inequality and the fact that

$$ab \leq \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2.$$

Next, we rewrite the $PICP$ as follows:

$$\tfrac{d}{dt}Bv(t) = -PLv(t) - PN(v(t)).$$

Firstly, we take the scalar product of the PICP with $Bv(t)$ to obtain

$$\begin{aligned}(\tfrac{d}{dt}Bv(t), Bv(t))_Y &= -(PLv(t), Bv(t))_Y - (PN(v(t)), Bv(t))_Y \\ &= -\hat{R}(v(t)) - (N(v(t)), Bv(t))_Y,\end{aligned}$$

since P is self-adjoint and leaves B intact. Thus we have, according to Lemma 3.5.3,

$$\tfrac{1}{2}\tfrac{d}{dt}\hat{S}(v(t)) = -\hat{R}(v(t)) - (N(v(t)), Bv(t))_Y. \tag{4}$$

Secondly, we take the scalar product of the $PICP$ with $PLv(t)$ and obtain

$$(\tfrac{d}{dt}Bv(t), PLv(t))_Y = -||PLv(t)||_Y^2 - (PN(v(t)), PLv(t))_Y.$$

We observe, again by Lemma 3.5.3, that if $PLv(t)$ is weakly continuous in Y, $(\frac{d}{dt}Bv(t), PLv(t))_Y = \frac{1}{2}\frac{d}{dt}\hat{R}(v(t))$ since $Bv(t)$ is differentiable. Therefore,

$$\tfrac{1}{2}\tfrac{d}{dt}(Bv(t), Lv(t))_Y = -||PLv(t)||_Y^2 - (PN(v(t)), PLv(t))_Y.$$

From (1) and (2), we have,

$$\tfrac{1}{2}\tfrac{d}{dt}\hat{S}(v(t) = -\hat{R}(v(t)) - (N(v(t)), Bv(t))_Y;$$

$$\tfrac{1}{2}\tfrac{d}{dt}\hat{R}(v(t)) = -||PLv(t)||_Y^2 - (PN(v(t)), PLv(t))_Y. \tag{5}$$

We know that $E_v' = \frac{1}{2}\frac{d}{dt}\hat{S}(v) + \frac{1}{2}\frac{d}{dt}\hat{R}(v)$. Therefore,

$$\begin{aligned} E'(t) &= -\hat{R}(v(t)) + (PN(v), Bv)_Y - \|PLv\|_Y^2 + (PLv, N(v))_Y \\ &\le -\|v\|_1^2 + \|Bv\|_Y\|N(v)\|_Y - \|Lv\|_Y^2 + \|Lv\|_Y\|N(v)\|_Y \\ &\le -\|v\|_1^2 + [\hat{S}(v)]^{\frac{1}{2}}\|N(v)\|_Y - \|Lv\|_Y^2 + \tfrac{1}{2}\|Lv\|_Y^2 + \tfrac{1}{2}\|N(v)\|_Y^2 \\ &= -\|v\|_1^2 - \tfrac{1}{2}\|Lv\|_Y^2 + [\hat{S}(v)]^{\frac{1}{2}}\|N(v)\|_Y + \tfrac{1}{2}\|N(v)\|_Y^2. \end{aligned}$$

Also, by Proposition 3.5.2, (2) and (3)

$$\begin{aligned} E'(t) &\le -\|v\|_2^2 + [2E(t)]^{\frac{1}{2}} C_2\|v\|_1\|v\|_2 + \tfrac{1}{2}C_2^2\|v\|_1^2\|v\|_2^2 \\ &\le -\|v\|_2^2 + [2E(t)]^{\frac{1}{2}} C_2\|v\|_2^2 + \tfrac{1}{2}C_2^2\|v\|_1^2\|v\|_2^2. \end{aligned}$$

Thus,

$$\begin{aligned} E'(t) &\le -\|v\|_2^2 + [2E]^{\frac{1}{2}} C_2\|v\|_2^2 + C_2^2\|v\|_1^2\|v\|_2^2 \\ &= -\|v\|_2^2[1 - C_2[2E]^{\frac{1}{2}} - \tfrac{1}{2}C_2^2\|v\|_1^2] \\ &\le -\|v\|_2^2[1 - C_2[2E]^{\frac{1}{2}} - \tfrac{1}{2}C_2^2 \cdot 2E(t)]; \end{aligned}$$

by (2). That is,

$$E'(t) \le -\|v\|_2^2[1 - C_2[2E(t)]^{\frac{1}{2}} - C_2^2E(t)]. \tag{6}$$

We now put $\beta(t) = 1 - C_2[2E_(t)]^{\frac{1}{2}} - C_2^2E(t)$ and $\beta_0 = \beta(0)$. If we can show that $\beta(t) > 0$, then $E\prime(t)$ will be bounded above by a negative quantity. Indeed, we have more:

Proposition 3.5.4. *If* $\beta_0 = 1 - C_2[2E_0]^{\frac{1}{2}} - C_2^2E_0 > 0$, *then* $\beta(t) > 0$ *for all* $t > 0$.

Proof.

Suppose there exists a smallest $t_0 > 0$ for which $\beta(t_0) = 0$. From (6) we have

$$E(t_0) - E_0 \le -\int_0^{t_0} \|v(t)\|_2^2\beta(t)\,dt \le 0$$

since β is nonnegative in $(0, t_0)$. Therefore $E(t_0) \le E_0$. Hence,

$$\begin{aligned} 0 &= 1 - C_2[2E(t_0)]^{\frac{1}{2}} - C_2^2E(t_0) \\ &\ge 1 - C_2[2E_0]^{\frac{1}{2}} - C_2^2E_0 = \beta_0 > 0. \end{aligned}$$

The result is proved by *reductio ad absurdum.* □

Our stability theorem can now be obtained in a way similar to the stability theorem in Le Roux and Sauer [7]:

THEOREM 3.5.5. (Polynomial stability) *Suppose that the mean curvature κ of the surface Γ and the surface density of the fluid in Γ are positive constants. Let $v(t)$ be a solution of the Projected Implicit Cauchy Problem which has the property that the mapping $t \to PLv(t)$ is weakly continuous in Y. If*

$$E_0 < \tfrac{1}{2}[\tfrac{\sqrt{3}-1}{C_2}]^2,$$

then there exists a constant $C > 0$, such that for $t > 0$,

$$\|v(t)\|_1^2 \le \tfrac{C}{1+t}.$$

Proof.

Firstly, we observe that the condition $E_0 < \frac{1}{2}[\frac{\sqrt{3}-1}{C_2}]$ is equivalent to $\beta_0 > 0$. Secondly, we observe from Proposition 3.5.4, that $E'(t) \le 0$ and hence $E(t) \le E_0$ so that $\beta(t) \ge \beta_0$. Thus, when we integrate (6) from 0 to t, we obtain

$$E(t) - E_0 = \int_0^t E'(t)\,dt \le -\int_0^t \|v(s)\|_2^2 \beta(s)\,ds \le -\int_0^t \|v(s)\|_2^2 \beta_0\,ds.$$

Hence,

$$\int_0^t \|v(s)\|_2^2\,ds \le \tfrac{E_0 - E(t)}{\beta_0} \le \tfrac{E_0}{\beta_0}. \qquad (7)$$

Therefore,

$$\begin{aligned} \tfrac{d}{dt}\|v\|_1^2 = \tfrac{d}{dt}\hat{R}(v) &= -2\|PLv\|_Y^2 - 2(PLv, PN(v))_Y \\ &\le -2\|PLv\|_Y^2 + 2\|PLv\|_Y\|PN(v)\|_Y \\ &\le -2\|PLv\|_Y^2 + \|PLv\|_Y^2 + \|PN(v)\|_Y^2 \\ &\le \|PN(v)\|_Y^2 \\ &\le C_2^2\|v\|_1^2\|v\|_2^2. \end{aligned} \qquad (8)$$

Next, with (8) in mind, let

$$\begin{aligned} h_1(t) &:= \|v\|_1^2 \\ h_2(t) &:= \|v\|_2^2 \end{aligned} \qquad (9)$$

and

$$\Phi(t) := (1+t)h_1(t). \qquad (10)$$

We differentiate (10) with respect to t and obtain

$$\begin{aligned}\tfrac{d}{dt}\Phi(t) &= h_1(t) + (1+t)\tfrac{d}{dt}h_1(t)\\ &= h_1(t) + (1+t)\tfrac{d}{dt}\|v(t)\|_1^2\\ &\leq h_1(t) + (1+t)C_2^2\|v\|_1^2\|v\|_2^2 \qquad (11)\\ &= h_1(t) + C_2^2(1+t)h_1(t)h_2(t)\\ &= h_1(t) + C_2^2 h_2(t)\Phi(t);\end{aligned}$$

by (8) and (9). We note that (11) is a first order ordinary differential inequality, which we solve by using an integrating factor; to obtain:

$$\frac{d}{dt}[\Phi e^{-C_2^2\int_0^t h_2(\tau)\,d\tau}] \leq h_1(t)e^{-C_2^2\int_0^s h_2(\tau)\,d\tau}ds. \qquad (12)$$

Integration of (12) from 0 to t, we find the following expression:

$$\Phi(t)e^{-C_2^2\int_0^t h_2(\tau)\,d\tau} - \Phi(0) \leq \int_0^t h_1(s)e^{-C_2^2\int_0^s h_2(\tau)\,d\tau}ds.$$

Then, eventually,

$$\Phi(t) \leq \left[\Phi(0) + \int_0^t h_1(s)e^{-C_2^2\int_0^s h_2(\tau)\,d\tau}ds\right]e^{C_2^2\int_0^t h_2(\tau)\,d\tau}. \qquad (13)$$

We observe that for all $t \geq 0$,

$$e^{-C_2^2\int_0^t h_2(\tau)\,d\tau} \leq 1.$$

In view of the above, we rewrite (13) as follows:

$$\Phi(t) \leq \left[\Phi(0) + \int_0^t h_1(s)ds\right]e^{C_2^2\int_0^t h_2(\tau)\,d\tau} \qquad (14)$$

Using (7), (9) and (10), we rewrite (14):

$$\begin{aligned}(1+t)h_1(t) &\leq \left[h_1(0) + \int_0^t \|v(s)\|_1^2\,ds\right]e^{C_2^2\int_0^t \|v(\tau)\|_2^2\,d\tau}\\ &\leq \left[\|v(0)\|_1^2 + \int_0^t \|v(s)\|_2^2 ds\right]e^{C_2^2\int_0^t \|v(\tau)\|_2^2\,d\tau}\\ &\leq [\|v(0)\|_1^2 + \tfrac{E_0}{\beta_0}]e^{\frac{C_2^2E_0}{\beta_0}}.\end{aligned}$$

Finally, we have:

$$\begin{aligned}\|v(t)\|_1^2 &\leq \frac{[\|v(0)\|_1^2 + \frac{E_0}{\beta}]e^{\frac{C_2^2E_0}{\beta}}}{1+t}\\ &\leq \tfrac{C}{1+t}.\end{aligned}$$

□

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