



Process capability indices for Marshall–Olkin inverse log-logistic distribution

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Abstract

Process capability analysis is a vital tool in quality management that enables organizations to evaluate and enhance their processes. Real-world data are mostly non-normal, they often deviate from the assumption of normality. The estimators of process capability indices (PCIs) for normal processes are not sufficient to characterize non-normal processes and can give misleading results. The Marshall–Olkin inverse log-logistic (MO-ILL) distribution is a flexible distribution that can effectively model data exhibiting positive skewness, asymmetry and heavy tails. In this paper, we derived the process capability indices (PCIs) based on the MO-ILL distribution when the process is assumed to be in a state of statistical control. Two PCIs based on MO-ILL mean and variance, and MO-ILL quantiles are proposed. The proposed PCIs were compared with the traditional PCIs and percentile-based PCIs using two real life data and data generated from MO-ILL distribution. Moreover, the effect of the sample size and parameters of the MO-ILL distribution on the PCI measures is also investigated. The results showed that PCIs values based on the proposed MO-ILL mean and variance, and MO-ILL quantiles are respectively lower and better than the traditional PCIs and percentile-based PCIs. This is an indication that MO-ILL distribution-based methods developed have narrow margin of error and are more appropriate in assessing the performance of a skewed process.

Keywords Capability · MO-ILLD · Non-normal · Process performance · Process skewness

Abbreviations

BS Birnbaum–Saunders

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CDF	Cumulative distribution function
CVM	Cramer–von Mises
EDF	Empirical distribution function
K-S	Kolmogorov–Smirnov
Ks	Kurtosis
LCL	Lower control limit
LSD	Location-scale distribution
LSL	Lower specification limit
Max	Maximum
Min	Minimum
MLE	Maximum likelihood estimation
MO-ILL	Marshall–Olkin inverse log-logistic
PCA	Process capability analysis
PCI	Process capability index
Q1	First quartile
Q3	Third quartile
SD	Standard deviation
SK	Skewness
UCL	Upper control limit
USL	Upper specification limit

1 Introduction

Process capability analysis (PCA) is a valuable statistical tool used by organizations to assess and improve the performance of their processes. The main objective of PCA is to determine whether or not a process is capable enough to meet customers' requirements in terms of quality and cost. It is very important to know that process instability deteriorates the quality of the outputs and services. Thus, PCA presumes that if a process has high variation, then it will be hard to maintain consistent quality. Traditionally, PCA assumes that the underlying data follows a normal distribution. However, in real-world scenarios, processes often exhibit non-normality, posing challenges to effective analysis and decision-making. Non-normal data can arise due to various reasons, such as process instability, measurement errors, or the presence of outliers. In recent years, there has been growing interest in analyzing process capability indices (PCIs) for non-normal distributions. Various traditional PCIs such as C_p , C_{pk} , C_{pm} and C_{pmk} have been developed specifically for normal data; see, for example, Juran [10–12]. When the underlying process distribution departs from normality, researchers recommend the use of two different approaches [20, 21]. The first approach suggests the use of normal based PCIs after transforming non-normal data into normal, and the second one suggests the use of PCIs defined for non-normal data.

Some authors have used transformation methods, non-parametric techniques, and robust estimation to leverage non-normal capability indices. Adekeye [2] used the median absolute deviation (MAD) to develop PCIs for non-normal distribution processes. Safdar and Ahmed [18] used the shape parameter of Weibull distribution to calculate PCIs and assess the efficacy of the technique by bootstrapping the results of estimate and standard error of the shape parameter. Leiva et al. [13] proposed a method for PCIs based on Birnbaum-

Saunders (BS) processes using parametric inference, bootstrap, and optimization tools. Piña-Monarez et al. [17] proposed non-normal PCIs as a function of the mean and standard deviation of Weibull and lognormal distributions which have the same practical meaning as those of the normal distribution. Safdar et al. [19] proposed a method to estimate four basic indices for non-normal processes using the Johnson system, which comprises three types of equations that translate a continuous non-normal distribution to normal. Taib and Alani [22] discussed four methods for evaluating non-normal PCIs, which include Box-Cox power transformation, weighted variance method, Clements' method, and Darling-Anderson goodness of fit test. The results showed that the process is stable and under statistical control but not capable based on the value of the PCIs, which did not exceed 65%.

Wang et al. [25] proposed modified Clements' PCIs based on a model selection approach, named robust PCIs method, for the location-scale distribution (LSD) family to evaluate the process capability of a production process. Tsai et al. [23] utilized maximum likelihood estimation (MLE) method and the Bayesian approaches using informative and non-informative prior distributions to infer the parameters of the Weibull distribution, and they proposed life performance index with a Type-I hybrid censoring scheme. Li et al. [14] proposed an estimation method of PCIs for the two-parameter exponential distribution with measurement errors using full error model, MLE and bootstrap methods to construct confidence intervals of PCIs. Alevizakos [3] computed the classical indices for discrete data following Poisson, binomial or negative binomial distribution using various transformation techniques. A simulation study under different situations of a process and comparisons with other existing PCIs for discrete data were also presented.

Several researchers advocated that skewness and kurtosis in process data can negatively affect traditional PCI measures' performance (or accuracy). Few decades ago, Wright [27] reported that traditional PCI measures do not adequately reflect process capability in the presence of skewness. Later on, Pearn and Chang [16] investigated the performance of CPIs on skewed distributions (see also Chang and Bai [4, 5]). Their study considered distributions like lognormal, chi-square, and Weibull and reported that the estimator's bias increases as the skewness coefficient increases. Wu and Swain [28] analyzed the performance of different PCIs under non-normal distributions using a simulation study. Their findings revealed that traditional PCIs are sensitive to the violation of the normality assumption and recommended the use of the weighted variance approach to estimate the process capability in the presence of skewness and kurtosis. Wooluru et al. [26] recently investigated the use of traditional C_p , C_{pk} , C_{pm} , and C_{pmk} PCIs to model non-normal process data. They reported that traditional CPIs neglect changes in the distribution's shape. Thus, it is important to consider PCIs that incorporate skewness to assess the process capability more accurately.

This article focuses on the MO-ILL distribution proposed by Aako et al. [1]. The MO-ILL distribution is a flexible distribution that can effectively model data exhibiting positive skewness, asymmetry and heavy tails, to explore the intricacies of analyzing process capability and highlight key considerations and techniques for accurate assessments. The MLE method would be employed to estimate the parameters from the available data to developed specialized PCIs, such as C_{Mp} , C_{Mpk} , C_{Mpm} and C_{Mpmk} , which focused on the distributional characteristics of the data.

The remainder of this article is organized as follows. Section 2 gives a mathematical background of MO-ILL distribution. In addition, Sect. 2 introduces the new CPIs based on the mean, variance and quantiles of MO-ILL distribution. The results of this study are pro-

vided and discussed in Sect. 3. In addition, Sect. 3 investigates the effect of the sample size and parameters of the MO-ILL distribution on the PCIs considered in this article. Section 4 provides numerical examples using real-world data. Section 5 gives the concluding remarks based of the results of the study and findings from real-world examples.

2 Process capability indices based on MO-ILL distribution

MO-ILL distribution proposed by Aako et al. [1] is a positively skewed unimodal distribution with cumulative distribution function and probability density function defined by

$$G(x) = \frac{1}{1 + \alpha x^{-\gamma}}; x > 0, \gamma > 0, \alpha > 0 \tag{1}$$

and

$$g(x) = \frac{\alpha \gamma}{x^{\gamma+1} (1 + \alpha x^{-\gamma})^2}; x > 0, \gamma > 0, \alpha > 0, \tag{2}$$

respectively, where α is a tilt parameter and γ is a shape parameter.

The mean, variance, and quantile of the MO-ILL distribution are given by

$$\bar{X} = E(X) = \alpha^{\frac{1}{\gamma}} \frac{\pi}{\gamma} \operatorname{csc} \left(\frac{\pi (\gamma - 1)}{\gamma} \right), \tag{3}$$

$$\operatorname{Var}(X) = \alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \operatorname{csc} \left(\frac{\pi (\gamma - 2)}{\gamma} \right) \left(2 - \frac{\pi}{\gamma} \operatorname{csc} \left(\frac{\pi (\gamma - 2)}{\gamma} \right) \right) \tag{4}$$

and

$$x_q = \left[\frac{1}{\alpha} \left(\frac{1}{q} - 1 \right) \right]^{-\frac{1}{\gamma}}, \tag{5}$$

respectively; where q is in the domain (0,1) and x_q represents the $(100q)^{th}$ percentile.

2.1 PCI based on mean and variance of MO-ILL distribution

The proposed method is based on Vännman [24]’s approach which unifies the four basic PCIs, namely C_p , C_{pk} , C_{pm} and C_{pmk} as follows:

$$C_p(u, v) = \frac{d - u |\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \tag{6}$$

where u and v are non-negative constants which takes the value 0 or 1, $d = \frac{USL - LSL}{2}$, $m = \frac{USL + LSL}{2}$ and T is the target value. T is usually unknown; it is taken as equal to m . μ and σ are respectively the process mean and process standard deviation.

From (6), the four main PCIs can be deduced as follows:

$$C_p(u, v) = \begin{cases} C_p, & u = 0, v = 0 \\ C_{pk}, & u = 1, v = 0 \\ C_{pm}, & u = 0, v = 1 \\ C_{pmk}, & u = 1, v = 1 \end{cases} \quad (7)$$

Thus, PCIs based on the mean and variance of MO-ILL distribution using (3) and (4) will be determined by using the generic expression defined by

$$C_{Mp}(u, v) = \frac{d - u \left| \left(\alpha^{\frac{1}{\gamma}} \frac{\pi}{\gamma} \csc \left(\frac{\pi(\gamma-1)}{\gamma} \right) \right) - m \right|}{3 \sqrt{\alpha^{\frac{2}{\gamma}} \frac{\pi}{\gamma} \csc \left(\frac{\pi(\gamma-2)}{\gamma} \right) \left(2 - \frac{\pi}{\gamma} \csc \left(\frac{\pi(\gamma-2)}{\gamma} \right) \right) + v \left(\left(\alpha^{\frac{1}{\gamma}} \frac{\pi}{\gamma} \csc \left(\frac{\pi(\gamma-1)}{\gamma} \right) \right) - T \right)^2}}, \quad (8)$$

where d , m and T are as defined in (6).

Using (8), the four PCIs based on the mean and variance of MO-ILL are therefore,

$$C_{Mp}(u, v) = \begin{cases} C_{Mp}, & u = 0, v = 0 \\ C_{Mpk}, & u = 1, v = 0 \\ C_{Mpm}, & u = 0, v = 1 \\ C_{Mpmk}, & u = 1, v = 1 \end{cases} \quad (9)$$

2.2 PCI based on quantiles of MO-ILL distribution

PCIs based on MO-ILL quantiles will be determined by using Chen and Pearn [6]’s approach which modified (6) without implicitly assuming normality of the underlying distribution. Thus, the PCIs based on quantiles for a two-sided specification are defined by

$$C_{p(q)}(u, v) = \frac{d - u |q_{50} - m|}{3 \sqrt{\left(\frac{q_{99.865} - q_{0.135}}{6} \right)^2 + v(q_{50} - T)^2}}, \quad (10)$$

where $q_{99.865}$, q_{50} and $q_{0.135}$ are the 99.865th quantile, 50th quantile and 0.135th quantiles, respectively.

From (10), the four PCIs are deduced as follows:

$$C_{p(q)}(u, v) = \begin{cases} C_{p(q)}, & u = 0, v = 0 \\ C_{p(q)k}, & u = 1, v = 0 \\ C_{p(q)m}, & u = 0, v = 1 \\ C_{p(q)mk}, & u = 1, v = 1 \end{cases} \quad (11)$$

$$\text{From (5), } x_{99.865} = \left[\frac{0.001352}{\alpha} \right]^{-\frac{1}{\gamma}} \text{ and } x_{0.135} = \left[\frac{739.74}{\alpha} \right]^{-\frac{1}{\gamma}}.$$

Therefore,

$$x_{99.865} - x_{0.135} = \alpha^{\frac{1}{\gamma}} \left(0.00135^{-\frac{1}{\gamma}} - 739.74^{-\frac{1}{\gamma}} \right), \tag{12}$$

and

$$x_{0.5} = \left(\frac{1}{\alpha} \right)^{-\frac{1}{\gamma}}. \tag{13}$$

Thus, the PCIs based on MO-ILL quantiles using (12) and (13) is defined as

$$C_{Mp(q)}(u, v) = \frac{d - u \left| \left(\frac{1}{\alpha} \right)^{-\frac{1}{\gamma}} - m \right|}{3 \sqrt{\left(\frac{\alpha^{\frac{1}{\gamma}} \left(0.00135^{-\frac{1}{\gamma}} - 739.74^{-\frac{1}{\gamma}} \right)}{6} \right)^2 + v \left(\left(\frac{1}{\alpha} \right)^{-\frac{1}{\gamma}} - T \right)^2}}, \tag{14}$$

where d , m and T are as defined in (6).

The four PCIs are then defined as

$$C_{M(q)p}(u, v) = \begin{cases} C_{M(q)p}, & u = 0, v = 0 \\ C_{M(q)pk}, & u = 1, v = 0 \\ C_{M(q)pm}, & u = 0, v = 1 \\ C_{M(q)pmk}, & u = 1, v = 1 \end{cases}. \tag{15}$$

The MLE method would be used to estimate the parameters of MO-ILL distribution.

2.3 Determination of specification limits

The guard bands proposed by Lombardo, da Silva, Lourenço [15], and da Silva, Lourenço [7, 8] were used to determine the upper specification limit (USL) and the lower specification limit (LSL). That is,

$$USL = UCL + SD \times CF. \tag{16}$$

and

$$LSL = LCL - SD \times CF,$$

where UCL represents the upper control limit, LCL represents the lower control limit, SD is the standard deviation of the data set and CF is the coverage factor which should be chosen to ensure appropriate specification limits and a reduces consumer’s risk less than 0.08.

Table 1 PCIs based on normal, quantile and MO-ILL for MO-ILL (4,3.5) distribution for different sample sizes

<i>N</i>	Type	Normal PCI	MO-ILL PCI	Quantile PCI	MO-ILLq PCI
100	C _p	1.116971	1.0261583	1.0469404	0.7185426
	C _{pk}	1.116971	1.0261583	0.9866384	0.6751382
	C _{pm}	1.116971	1.0261583	1.2187552	0.7667610
	C _{pmk}	1.116971	1.0261583	1.1485568	0.7204439
200	C _p	1.056704	0.9955655	0.9419524	0.6886421
	C _{pk}	1.056704	0.9955655	0.8816280	0.6443377
	C _{pm}	1.056704	0.9955655	1.0713001	0.7325730
	C _{pmk}	1.056704	0.9955655	1.0026920	0.6854423
300	C _p	1.090801	1.0037479	0.8103980	0.6945960
	C _{pk}	1.090801	1.0037479	0.7447225	0.6503208
	C _{pm}	1.090801	1.0037479	0.9160706	0.7396219
	C _{pmk}	1.090801	1.0037479	0.8418312	0.6924767

Table 2 PCIs based on normal, quantile and MO-ILL for MO-ILL (4, 5.5) distribution for different sample sizes

<i>N</i>	Type	Normal PCI	MO-ILL PCI	Quantile PCI	MO-ILLq PCI
100	C _p	1.0454100	0.9643006	0.9798662	0.6752284
	C _{pk}	1.0454100	0.9643006	0.9195641	0.6318241
	C _{pm}	1.0454100	0.9643006	1.1196278	0.7152592
	C _{pmk}	1.0454100	0.9643006	1.0507247	0.6692816
200	C _p	0.9915826	0.9369756	0.8839026	0.6481147
	C _{pk}	0.9915826	0.9369756	0.8235782	0.6038103
	C _{pm}	0.9915826	0.9369756	0.9898781	0.6846615
	C _{pmk}	0.9915826	0.9369756	0.9223211	0.6378588
300	C _p	1.0220360	0.9442843	0.7593100	0.6534470
	C _{pk}	1.0220360	0.9442843	0.6936345	0.6091719
	C _{pm}	1.0220360	0.9442843	0.8458597	0.6908828
	C _{pmk}	1.0220360	0.9442843	0.7726981	0.6440712

3 Results and discussion

3.1 The proposed MO-ILL PCIs versus the existing normal and quantile-based PCIs

Three data sets consisting of 100, 200 and 300 observations grouped into sub-samples of size 5 each, were generated from MO-ILL (4, 3.5), MO-ILL (4, 5.5) and MO-ILL (5, 7.5) distributions, respectively. The generated data are used to confirm the behaviours of the proposed PCIs based on MO-ILL distribution in comparison with the normal and quantile-based capability indices. The data was tested for stability using control charts which confirmed the stability of the process; i.e., the in-control state of the process. The information collected from the process control analysis reveal that the data sets are in statistical control as no point falls outside the limits. The computed capability indices are presented in Tables 1, 2 and 3 for MO-ILL (4, 3.5), MO-ILL (4, 5.5), and MO-ILL (5, 7.5), respectively.

Table 1 presents PCIs across four estimation methods: normal, MO-ILL, quantile, and MO-ILLq when the sample size $N = 100, 200$ and 300 , using indices C_p, C_{pk}, C_{pm} , and C_{pmk} . The skewness and kurtosis of the data are 1.07 and 5.06, respectively, showing

Table 3 PCIs based on normal, quantile and MO-ILL for MO-ILL (5, 7.5) distribution for different sample sizes

<i>N</i>	Type	Normal PCI	MO-ILL PCI	Quantile PCI	MO-ILLq PCI
100	C _p	1.240264	1.168738	1.2104456	0.8581205
	C _{pk}	1.240264	1.168738	1.1604724	0.8193990
	C _{pm}	1.240264	1.168738	1.4145642	0.9264210
	C _{pmk}	1.240264	1.168738	1.3561640	0.8846175
200	C _p	1.168011	1.134669	1.1134626	0.8245479
	C _{pk}	1.168011	1.134669	1.0609337	0.7846232
	C _{pm}	1.168011	1.134669	1.2856171	0.8881423
	C _{pmk}	1.168011	1.134669	1.2249666	0.8451383
300	C _p	1.214851	1.142970	0.9679879	0.8308845
	C _{pk}	1.214851	1.142970	0.9080817	0.7910005
	C _{pm}	1.214851	1.142970	1.1204832	0.8957869
	C _{pmk}	1.214851	1.142970	1.0511394	0.8527875

that the generated data are skewed and not normal. Normal PCI values are reliably higher, indicating a more optimistic view of process capability, while MO-ILL provides slightly lower but more stable estimates. Quantile PCI values vary more with the sample size, showing sensitivity to data distribution, while MO-ILLq gives the most conventional estimates, especially for small samples. Despite changes in sample size, the PCI trends differ across methods, highlighting the significance of method choice for accurate process capability assessment. This is an indication that the target and the spread of normal and MO-ILL distribution-based methods are within the specification limits while that of quantile and MO-ILL quantile methods process mean are not perfectly centered.

Table 2 compares PCIs using normal, MO-ILL, quantile, and MO-ILLq methods under the MO-ILL (4, 5.5) distribution when $N = 100, 200, \text{ and } 300$. The skewness and kurtosis of the data are 1.95 and 10.71, respectively, showing that the generated data are skewed and not normal. The normal method consistently yields the highest PCI values, suggesting a more optimistic view of process capability, while MO-ILL values are slightly lower and more stable, offering a well-adjusted perspective. Quantile PCI values decrease with as the sample size increases, indicating greater sensitivity to data distribution, while MO-ILLq values are the lowest, reflecting a highly conservative approach. Across indices like C_p, C_{pk}, C_{pm} and C_{pmk} , the choice of method significantly impacts the perceived process capability, with the normal method being less sensitive to deviations from normality and MO-ILLq emphasizing stringent criteria. The variation across methods highlights the importance of selecting a suitable estimation method based on the data’s distribution characteristics. It can be concluded that MO-ILL distribution-based methods have lower margin of error than the normal and quantile-based methods and that normal method will lead to a wrong conclusion.

Table 3 presents the comparison of PCIs using normal, MO-ILL, quantile, and MO-ILLq methods for MO-ILL (5, 7.5) distribution across different sample sizes (100, 200, and 300). The skewness and kurtosis of the data are 1.4 and 3.8, respectively, showing that the generated data are skewed and not normal. The normal method consistently provides the highest PCI values across all indices (C_p, C_{pk}, C_{pm} , and C_{pmk}), suggesting a more optimistic assessment of process capability. The MO-ILL method yields slightly lower but relatively stable estimates, indicating a more moderate approach. Quantile PCI values tend to decrease as the sample size increases, reflecting sensitivity to the data distribution. The MO-ILLq

method produces the lowest estimates, indicating a conservative approach that may account for more distributional discrepancies. As the sample size increases, the differences between the methods become more apparent, especially for the Quantile method, where the PCI values tend to decrease, while the Normal and MO-ILL methods remain closer in value. Overall, the choice of method significantly influences the perceived process capability, with the Normal method showing the highest values and the MO-ILLq method being the most conservative. It can be resolved that MO-ILL distribution-based methods have lower margin of error and more appropriate than the normal and quantile-based methods.

3.2 Effect of the sample size and parameters of the MO-ILL distribution on different PCIs

In this section, the effect of α , γ and N on the C_p , C_{pk} , C_{pm} and C_{pmk} PCIs is investigated when $\alpha \in \{4,5\}$, $\gamma \in \{3.5, 4.5, 7.5\}$ and $N \in \{100, 200, 300\}$. Figure 1- Panel 1 shows that when α is kept constant, the larger the value of γ , the lower the PCIs values. For instance, when $\alpha=4$, if the value of γ increases from 3.5 to 5.5, the normal PCI decreases from 1.117 to 1.0454 when $N=100$. In other words, the normal PCI decreases by 6.41%. Similarly, the MO-ILL PCI decreases by 6.03%, i.e., decreases from 1.0262 to 0.9643. A similar pattern is observed for other sample sizes (i.e., when $N=200$ and 300). When both α and γ increase, the PCIs values increase as well. This is indicated by the larger values of the PCIs for the MO-ILL (5,7.5) distribution. In terms of the sample size, the PCI values are higher for small sample sizes and they decrease as N increases between 100 and 300. However, the values of the normal and MO-ILL PCIs increase as the sample size increases to 300. Figure 1-Panel 2 shows a similar pattern of the effects of α and γ on the PCIs when the quantile PCI method is used. However, as the N increases, the PCI values decrease as well (see, Fig. 1-Panel 2(a)-(d)). Figure 1-Panel 3 shows that the PCIs values increase when either α or γ increases, or when both α and γ increase. In terms of N , the PCI values increase when $100 \leq N < 300$, and start increasing again when $N = 300$.

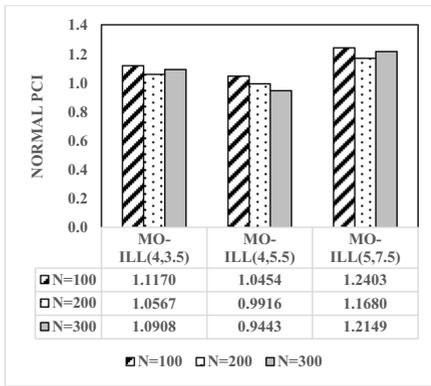
4 Application to real data

Two data sets are used to illustrate the proposed methods. The first data set consist of 100 measurements from a process earlier presented by Safdar et al. [19], while the second data set is the ball size (in mm) taken from Hsu et al. [9]. The two data sets are displayed in Table 4.

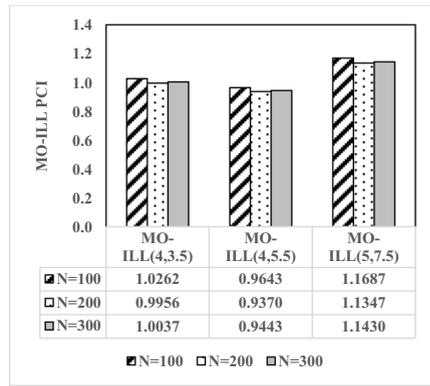
The descriptive statistics of data set I and II are presented in Table 5.

From Table 5, the mean is greater than the median for the two data sets, which is an indication that the two data sets are positively skewed which was confirmed by the histogram and theoretical density for the two data sets in Fig. 2. The coefficient of skewness (Sk) and kurtosis (Ks) values in Table 5 further strengthen the conclusion that the two data sets are from non-normal process.

The Kolmogorov–Smirnov (K–S) values which is the measures of the maximum vertical distance between the empirical distribution function (EDF) of the sample and the cumulative distribution function (CDF) of the theoretical distribution and Cramer-von Mises (CVM) values which is the squared differences between the observed and expected CDFs in

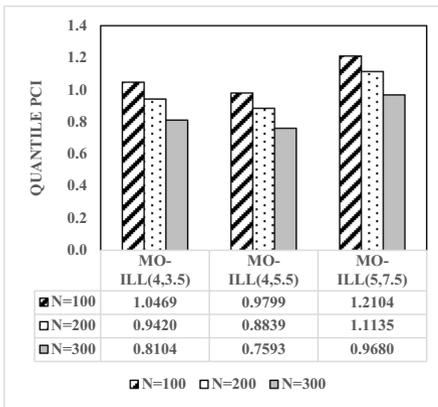


(a) Normal PCI

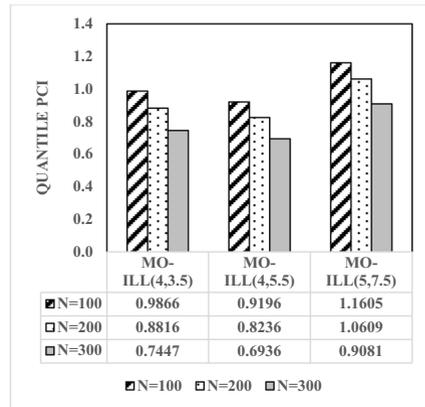


(b) MO-ILL PCI

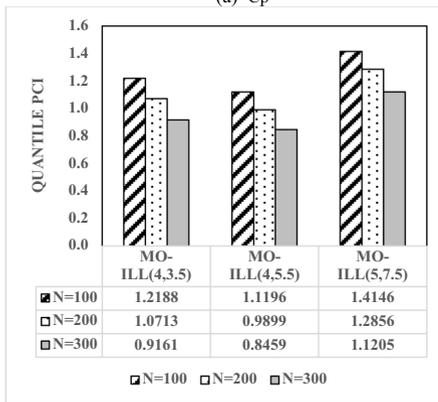
Panel 1: PCIs using the normal and MO-ILL methods ($C_p = C_{pk} = C_{pm} = C_{pmk}$)



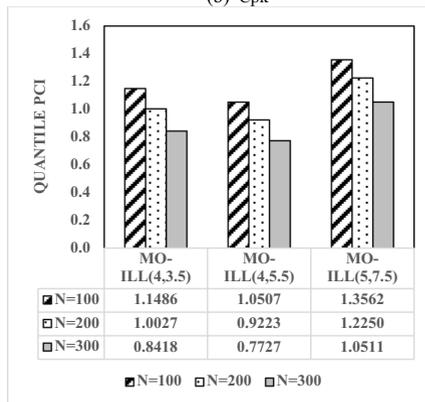
(a) C_p



(b) C_{pk}



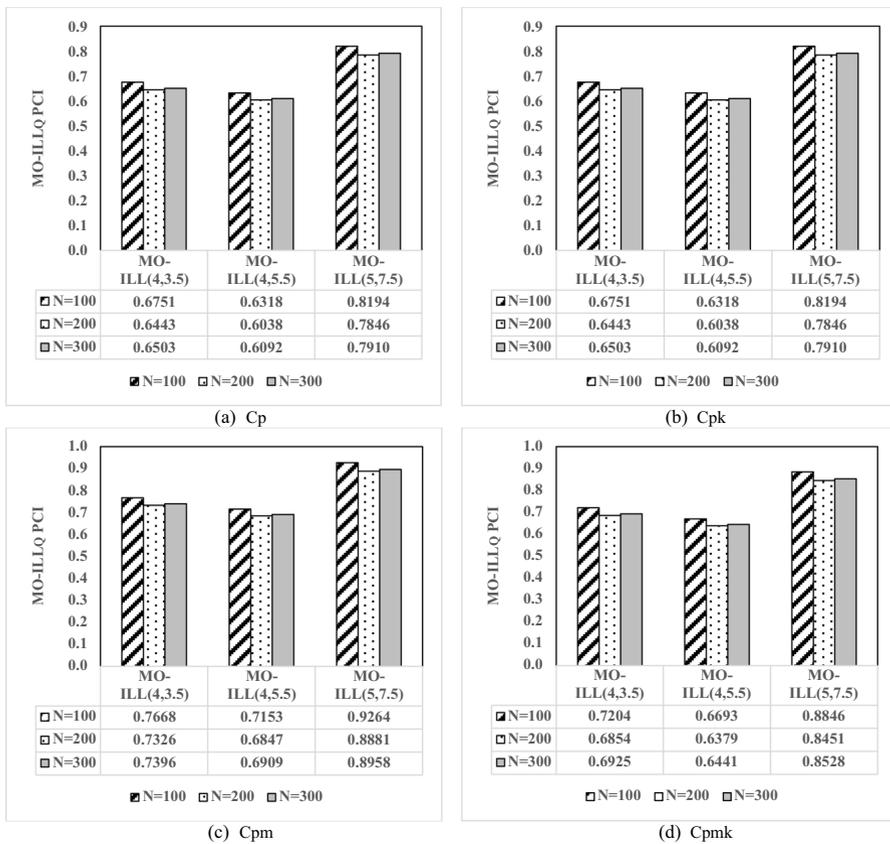
(c) C_{pm}



(d) C_{pmk}

Panel 2: C_p , C_{pk} , C_{pm} and C_{pmk} PCIs using the quantile method

Fig. 1 Effects of parameters (α , γ) and sample size (N) on different PCIs



Panel 3: Cp, Cpk, Cpm and Cpmk PCIs using the MO-ILLq method

Figure 1 (continued)

Table 5 reflected that the two data sets can be perfectly modelled with the MO-ILL distribution. The EDF and CDF graphs of the two data sets are presented in Fig. 3. Thus, it is very clear that the MO-ILL distribution better fit the two data sets.

Table 6 presents the PCIs based on normal, quantile and MO-ILL distribution for data sets I and II.

From the results in Table 6, it is clear that the four PCIs are not capable to measure the performance of the data sets as their indices are less than 1. The indices of MO-ILL and MO-ILL quantile-based methods are considerably lower than the normal based indices. The C_p , C_{pk} , C_{pm} and C_{pmk} based on MO-ILL quantile are less than those based on quantile of the data. These suggest MO-ILL distribution-based methods have little room for error compare to normal and quantile-based methods and would give accurate performance indices of the data sets.

Table 4 Real-world data

Data set I									
6.3	6.8	9.3	10.4	11.1	11.6	12.2	12.5	12.5	12.6
12.9	13.2	13.2	13.3	13.3	13.5	13.5	13.9	14	14.4
14.8	14.8	15.2	15.4	15.7	15.8	15.9	16.2	16.3	16.5
16.5	16.7	16.9	17	17.1	17.7	17.8	17.9	18	18.1
18.1	18.1	18.1	18.1	18.1	18.4	18.4	18.7	18.7	18.8
19.1	19.3	19.3	19.5	19.6	19.7	19.7	19.9	20.2	20.3
20.6	20.6	20.7	20.8	21.4	21.5	21.9	22	22	22.1
22.3	22.6	22.7	22.9	23	23.3	23.3	23.5	24	24.2
24.7	25	25.1	25.5	25.5	25.7	25.9	26	26.1	29.3
29.4	29.6	29.6	29.8	29.9	29.9	31.4	34	34.9	40.6
Data set II									
2.891	4.035	4.495	2.890	2.312	3.158	5.228	3.334	5.896	5.639
3.842	1.590	1.954	1.842	0.680	2.752	1.301	2.260	0.889	2.381
0.619	2.788	1.050	3.750	3.508	6.123	6.549	5.954	2.207	4.417
4.805	1.516	2.227	2.797	1.636	1.066	0.940	4.101	4.542	1.295
1.770	3.492	5.706	3.722	6.644	2.472	1.383	4.494	1.694	2.892
2.111	3.591	2.093	3.222	2.891	2.582	0.665	3.234	1.102	1.083
1.508	1.811	2.803	6.659	0.923	6.229	3.177	2.333	1.311	4.419
2.495	0.921	4.061	9.725	1.600	4.281	3.360	1.131	1.618	4.489
3.696	1.982	2.413	5.480	1.992	2.573	1.845	4.620	6.221	1.694
4.882	1.380	3.982	2.260	2.366	2.899	3.782	2.336	1.175	3.055

Table 5 Descriptive statistics of data set I and II

Data	Min	Q1	Median	Mean	Q3	Max	SD	Sk	Ks	K-S	CVM
I	6.30	15.78	18.95	19.74	23.08	40.60	6.10	0.59	0.63	0.0502	0.0261
II	0.62	1.69	2.77	3.04	4.04	9.73	1.72	0.98	1.05	0.05378	0.0543

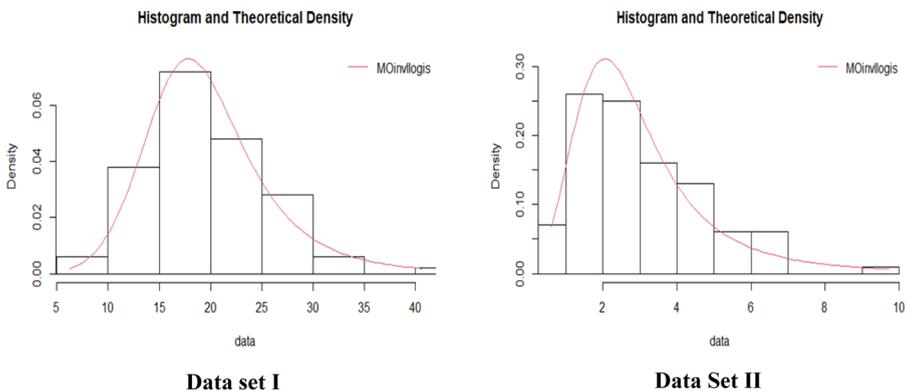


Fig. 2 Histogram and theoretical densities

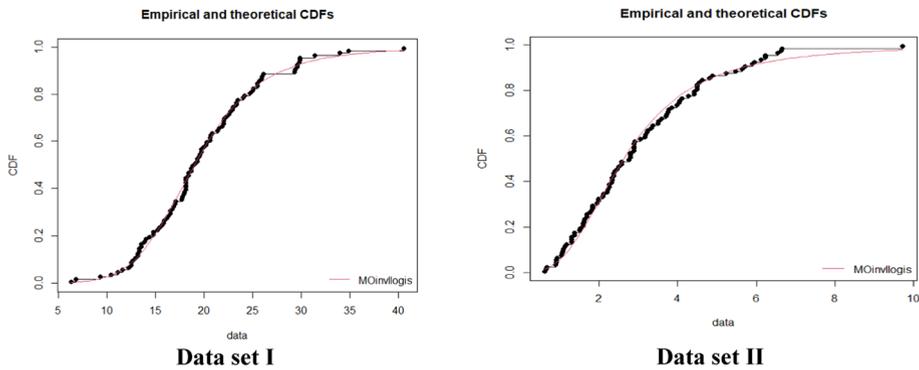


Fig. 3 Histogram and theoretical densities, and Empirical and theoretical CDFs of data II

Table 6 PCIs based on normal, quantile and MO-ILL for data sets I & II

Data Set	PCI	Normal	Quantile	MO-ILL	MO-ILLq
I	Cp	0.4562	0.5149	0.0018	0.0012
	Cpk	0.4562	0.4701	0.0018	- 0.0446
	Cpm	0.4562	0.5185	0.0018	0.0012
	Cpmk	0.4562	0.4733	0.0018	- 0.0446
II	Cp	0.7287	0.8631	0.4482	0.2940
	Cpk	0.4928	0.5224	0.3265	0.1673
	Cpm	0.9507	1.5904	0.4807	0.3081
	Cpmk	0.6429	0.9627	0.3502	0.1753

5 Conclusion

Process capability indices (CPIs) is a critical tool for organizations striving to enhance their processes and ensure consistent quality. This paper proposes PCIs based on Marshall–Olkin inverse log-logistic (MO-ILL) distribution for a process that is controlled and has predetermined requirements by the product designer. When dealing with non-normal data, it is essential to acknowledge the limitations of traditional approaches and adopt specialized techniques that account for the distributional characteristics. The mean, standard deviation and quantiles of MO-ILL distribution were used to develop two PCIs. Three data sets of size 100, 200 and 300 generated from MO-ILL (4, 3.5), MO-ILL (4, 5.5) and MO-ILL (5, 7.5) distributions, respectively, and two real-world data were used to test the superiority of the proposed methods in comparison with the normal and quantile-based PCIs. The results showed that PCI methods based on MO-ILL distribution have low C_p , C_{pk} , C_{pm} and C_{pmk} . This is an indication that the methods have little room for error and would be able to accurately measure the performance of skewed processes. Therefore, users of PCI are encouraged to use the proposed methods when the process data is skewed. The proposed PCIs are designed to be used for skewed processes. When there is not enough evidence of the skewness of the process, it is recommended to use PCIs based on more robust nonparametric methods. This topic is under investigation, and the results will be reported in a separate article. Researchers are recommended to develop and investigate the performance of PCIs under the violation of the assumption of normality with a combined effect of measurement

error and autocorrelation. Interest can also be drawn toward developing PCIs for nonlinear profiles where the quality characteristic is represented by a nonlinear relationship between a response variable and one or several predictors. Researchers are also invited to introduce PCIs for insurance companies using skewed compound distributions.

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