ON MID (p, r)-COMPACT OPERATORS

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Abstract

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$ where p^* is the conjugate index of p. We introduce and study mid (p, r)-compact sets and operators. We begin by introducing and defining the mid (p, r)-compact subsets of a Banach space X and the mid (p, r)-compact operators between Banach spaces X and Y. The set of mid (p, r)-compact operators between Banach spaces X and Y is denoted by $\mathcal{K}^{\text{mid}}_{(p,r)}(X, Y)$. We prove that the ideal $(\mathcal{K}^{\text{mid}}_{(p,r)}(X, Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is a quasi-Banach operator ideal. We also introduce and study (p, r)-limited subsets in Banach spaces. We prove that every mid (p, r)-compact subset of X is (p, r)-limited and that the set $\mathcal{K}^{\text{mid}}_{(p,r)}(X, Y)$ consists of (p, r)-limited sets.

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1 Introduction and Notation

We denote Banach spaces by X and Y, and by $\mathbb{K} = \mathbb{R}/\mathbb{C}$ the field of real or complex numbers, respectively. For a Banach space X, denote by B_X the closed unit ball of X and by S_X the unit sphere of X. The space of continuous bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. We refer to the element of $\mathcal{L}(X, Y)$ as *operators*. Denoted by $X^* := \mathcal{L}(X, \mathbb{K})$ the dual of X; its typical element will be denoted by x^*. For $x \in X$, we write $\langle x^*, x\rangle = x^*(x)$ (or $\langle x, x^*\rangle$) for the action of x^* on x. We also denote by κ_X the canonical isometric embedding from X into X^{**} where X^{**} is the dual of X^*. The subspaces of all finite rank and all compact operators from X to Y are denoted by $\mathcal{F}(X, Y)$ and $\mathcal{K}(X, Y)$, respectively. For any operator $T \in \mathcal{L}(X, Y)$, $N(T)$ and $R(T)$ denote the null space and range (space) of T, respectively. For any $T \in \mathcal{L}(X, Y)$, T^* denotes the adjoint operator of T from Y^* into X^*.
For the characterization of relatively compact sets in Banach spaces, we recall the

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following: Let $p \geq 1$. We denote by $\ell_p^w(X)$ the set of all weakly p-summable sequences in X. Then $\ell_p^w(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup\{\|(x^*(x_n))\|_p : x^* \in B_{X^*}\}.$$

$\ell_p^w(X)$ is isometrically isomorphic to $\mathcal{L}(\ell_{p^*}, X)$ if $p > 1$ and $p^* = p(p-1)^{-1}$. For $p = 1$, $\ell_1^w(X)$ is isometrically isomorphic to $\mathcal{L}(c_0, X)$. Denote these isometries by $\Phi : \ell_p^w(X) \longrightarrow \mathcal{L}(\ell_{p^*}, X)$ (respectively, $\Phi : \ell_1^w(X) \longrightarrow \mathcal{L}(c_0, X)$), where $\Phi_{(x_n)}((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n x_n$ and $(\alpha_n) \in \ell_{p^*}$ (respectively, $(\alpha_n) \in c_0$). We write p-conv$(x_n) := \Phi_{(x_n)}(B_{\ell_{p^*}})$ for $1 < p < \infty$, 1-conv$(x_n) := \Phi_{(x_n)}(B_{c_0})$, and ∞-conv$(x_n) :=$ abs-conv$(x_n) = \Phi_{(x_n)}(B_{\ell_1})$, the absolute convex hull of (x_n).

Let $\ell_p(X)$ be the subspace of $\ell_p^w(X)$ of all p-summable sequences in X ($1 \leq p < \infty$), and $c_0(X)$ the space of all norm null sequences in X. Then $\ell_p(X)$ is a Banach space with the norm

$$\|(x_n)\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Also, $\ell_\infty(X)$ is a Banach space in the norm

$$\|(x_n)\|_\infty = \|(x_n)\|_\infty^w = \sup_{n\in\mathbb{N}} \|x_n\|.$$

Moreover, $c_0(X)$ is a closed subspace of $\ell_\infty(X)$ in the above norm. We write $\Phi(\ell_\infty^p(X)) = \Phi_p(X)$ ($1 \leq p \leq \infty$) and $\Phi(c_0(X)) = \Phi_0(X)$.

Let $1 \leq p < \infty$. Given a Banach space X, consider the vector space

$$\ell_p^{\text{mid}}(X) := \{(x_j)_{j=1}^\infty \in \ell_p^w(X) \ : \ ((x_n^*(x_j))_{j=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p)$$
$$\text{whenever } (x_n^*)_{n=1}^\infty \in \ell_p^w(X^*)\}$$

under pointwise operations and equipped with the norm

$$\|(x_j)_{j=1}^\infty\|_{mid,p} := \sup_{(x_n^*)_{n=1}^\infty \in B_{\ell_p^w(X^*)}} \Big\{\Big(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |x_n^*(x_j)|^p\Big)^{1/p}\Big\},$$

under which it is a Banach space [4].

If $p = \infty$, then $\ell_p(X)$, $\ell_p^w(X)$ and $\ell_p^{mid}(X)$ are replaced by $c_0(X)$, $c_0^w(X)$ and $c_0^{mid}(X)$, and $\ell_p(\ell_p)$ takes the form $c_0(c_0)$. The norm on $c_0^{mid}(X)$ takes the form

$$\|(x_j)_{j=1}^\infty\|_{mid,\infty} := \sup_{(x_n^*)_{n=1}^\infty \in B_{c_0^w(X^*)}} \Big\{\sup_n \sup_j |x_n^*(x_j)|\Big\}. \tag{1.1}$$

The norm that the mid p-summable sequence space $\ell_p^{mid}(X)$ inherits from $\ell_p^w(X)$ does not, in general, complete the mid p-summable sequence space, and the above norm is the appropriate one on $\ell_p^{mid}(X)$.

The symbol $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_Y \leq \|x\|_X$ for every $x \in X$. It is shown in [4] that

$$\ell_p(X) \overset{1}{\hookrightarrow} \ell_p^{mid}(X) \overset{1}{\hookrightarrow} \ell_p^w(X). \tag{1.2}$$

The definition of the (p,r)-convex hull of $(x_n) \in \ell_p^w(X)$ is extended in the following way (see [1, Definition 7.3] and [3]).

Definition 1.1. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let $(x_n) \in \ell_p^w(X)$. If $r \neq \infty$, the (p,r)-convex hull of (x_n) is given by*

$$(p,r)\text{-conv}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \Big\}.$$

If $r = \infty$ and $(x_n) \in \ell_1^u(X)$, the sequence space of unconditionally 1-summable sequences, we define

$$1\text{-conv}(x_n) := (1,\infty)\text{-conv}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_\infty} \Big\}.$$

If $r = \infty$ and $(x_n) \in \ell_1^w(X)$, then define

$$1\text{-co}(x_n) := (1,\infty)\text{-co}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \Big\}.$$

Given $(x_n) \in \ell_p^w(X)$, the operator $\Phi_{(x_n)} : \ell_r \longrightarrow X$ ($\Phi_{(x_n)} : c_0 \longrightarrow X$ when $r = \infty$) is also well defined and $(p,r)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_r})$ if $r \neq \infty$, while

$$1\text{-co}(x_n) = (1,\infty)\text{-co}(x_n) = \Phi_{(x_n)}(B_{c_0}).$$

However, $\Phi_{(x_n)}$ need not be compact, unless $(x_n) \in \ell_p^u(X)$, the Banach space of unconditionally p-summable sequences [5, §§8.2 and 8.3] and [7, Theorem 1.4] (also see [1, §7.3]). As described in [1, p.60], the space $\ell_p^u(X)$ was defined as the (closed) subspace of $\ell_p^w(X)$, consisting of $(x_n) \in \ell_p^w(X)$ such that $(x_n) = \lim_{N\to\infty}(x_1, \ldots, x_N, 0, 0, \ldots,) \in \ell_p^w(X)$. It is known that $\ell_\infty^u(X) = c_0(X)$ as Banach spaces (details may be found in [7, p.351] and [5, 8.2]). We recall the following definition from [1, Definition 7.4].

Definition 1.2. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A subset K of a Banach space X is said to be relatively unconditionally (p,r)-compact if $K \subset (p,r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p^u(X)$. It is said that K is relatively weakly (p,r)-compact if $K \subset (p,r)\text{-conv}(x_n)$ ($K \subset 1\text{-co}(x_n)$ when $r = \infty$) for some $(x_n) \in \ell_p^w(X)$ ($(x_n) \in c_0^w(X)$ when $p = \infty$).*

Concerning the operators, we have the following [1, Definition 7.5].

Definition 1.3. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Given Banach spaces X and Y, an operator $T \in \mathcal{L}(X,Y)$ is said to be unconditionally (respectively, weakly) (p,r)-compact if $T(B_X)$ is a relatively unconditionally (respectively, weakly) (p,r)-compact subset of Y. Denote by $\mathcal{U}_{(p,r)}$ and $\mathcal{W}_{(p,r)}$ the classes of all unconditionally and weakly (p,r)-compact operators acting between arbitrary Banach spaces.*

By [4, Example 1.7], the Banach spaces $\ell_p^u(X)$ and $\ell_p^{\text{mid}}(X)$ are generally not comparable. In particular, $\ell_p^{\text{mid}}(X)$ is not contained in $\ell_p^u(X)$ despite the fact that both sequence spaces satisfy

$$\ell_p(X) \subset \ell_p^u(X), \ell_p^{\text{mid}}(X) \subset \ell_p^w(X).$$

We recall the following concept in [6]:

Definition 1.4. *Let X and Y be Banach spaces. An operator $T : X \to Y$ is* p-summing *precisely when there exists a constant $C \geq 0$ such that for $x_1, \cdots, x_n \in X$ and regardless of the natural number n, we have for all $1 \leq p < \infty$ that*

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq \ C \cdot \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}}.$$

The least C for which the above inequality holds will be denoted by $\pi_p(T)$, and we also denote by $\Pi_p(X,Y)$ the set of all p-summing operators from X into Y.

2 Main Results

Our main concern is to expound a theory of *mid (p,r)-compact sets and operators*. This we do by defining and introducing the *relatively mid (p,r)-compact sets* in X by defining the (p,r)-convex hull of $(x_n) \in \ell_p^{mid}(X)$ as in Definition 1.1 above. Alternatively, by replacing $\ell_p(X)$ (as in [1, Definition 3.2]) with $\ell_p^{\text{mid}}(X)$.

Definition 2.1. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_n) \in \ell_p^{\text{mid}}(X)$. Then call*

$$(p,r)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \right\}$$

the (p,r)-convex hull of (x_n).

Definition 2.2. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A subset K of X is said to be relatively mid (p,r)-compact if $K \subset (p,r)$-conv(x_n) for some $(x_n) \in \ell_p^{\text{mid}}(X)$.*

The operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j$$

is bounded. Indeed, let $(x_j) \in \ell_p^{\text{mid}}(X)$. Fix $\varphi \in B_{X^*}$ arbitrarily and note that $(x_j^*) := (\varphi, 0, 0, \ldots) \in B_{\ell_p^w(X^*)}$. So if $(\alpha_j) \in B_{\ell_r}$, then

$$
\begin{aligned}
|\varphi(\Phi_{(x_j)}(\alpha_j))|^p &= |\sum_{j=1}^{\infty}\alpha_j\varphi(x_j)|^p \le (\sum_{j=1}^{\infty}|\alpha_j||\varphi(x_j)|)^p \\
&\le \Big((\sum_{j=1}^{\infty}|\alpha_j|^r)^{1/r}(\sum_{j=1}^{\infty}|\varphi(x_j)|^{r^*})^{1/r^*}\Big)^p \\
&\le \Big((\sum_{j=1}^{\infty}|\alpha_j|^r)^{1/r}(\sum_{j=1}^{\infty}|\varphi(x_j)|^{p})^{1/p}\Big)^p \\
&\le \sum_{j=1}^{\infty}|\varphi(x_j)|^p = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}|x_n^*(x_j)|^p \\
&\le (\|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p})^p.
\end{aligned}
$$

Hence

$$\|\Phi_{(x_j)}(\alpha_j)\| \le \|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p},$$

and so,

$$\|\Phi_{(x_j)}\| \le \|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p}. \tag{2.1}$$

Coincidence of the mid (p,r)-compactness with $\ell_p^w(X)$ and $\ell_p(X)$ occurs in the following extreme cases which are stated in [4] and originally come from [13, Proposition 3.1 and Theorem 4.5].

Theorem 2.3. *[[4], Theorem 1.2] Let X be a Banach space and $1 \le p < \infty$. Then*

(i) $\ell_p^{\mathrm{mid}}(X) = \ell_p^w(X)$ *if and only if* $\Pi_p(X,\ell_p) = \mathcal{L}(X,\ell_p)$.

(ii) $\ell_p^{\mathrm{mid}}(X) = \ell_p(X)$ *if and only if X is a subspace of $L_p(\mu)$ for some measure μ.*

The Banach spaces X for which Theorem 2.3 holds are what the authors of [4] have termed *weak mid p-spaces* and *strong mid p-spaces*: A Banach space X is said to be a *weak mid p-space* if $\ell_p^{\mathrm{mid}}(X) = \ell_p^w(X)$, and it is said to be a *strong mid p-space* if $\ell_p^{\mathrm{mid}}(X) = \ell_p(X)$. Therefore, our relatively mid (p,r)-compact sets are relatively (p,r)-compact (respectively, relatively weakly (p,r)-compact) whenever X is a strong mid p-space (respectively, weak mid p-space) by the definition of strong mid p-space (respectively, weak mid p-space) and the weakly (p,r)-compact sets treated in [1, Chap 7].

Definition 2.4. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. Given Banach spaces X and Y, an operator $T : X \longrightarrow Y$ is said to be mid* (p,r)-compact *if $T(B_X)$ is a relatively mid* (p,r)-compact *subset of Y. Denote by $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$ the set of mid* (p,r)-compact *operators from X to Y.*

Let X and Y be Banach spaces, $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$ and $T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$. Then the quantity

$$\begin{aligned}\kappa^{\text{mid}}_{(p,r)}(T) &= \inf\{\|(y_n)\|_{\text{mid},p} : (y_n) \in \ell_p^{\text{mid}}(Y)\,((y_n) \in c_0^{\text{mid}}(Y) \,\text{when}\, p=\infty), \\ &\quad T(B_X) \subset \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_r}\}\} \end{aligned} \tag{2.2}$$

defines, as will be shown below, a complete quasi-norm on $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$. Recall from equation (1.2) that $(y_n) \in \ell_p(Y)$ implies that $(y_n) \in \ell_p^{mid}(Y)$ and $\|(y_n)\|_{\text{mid},p} \leq \|(y_n)\|_p$. Also recall from [9, Theorem 4.6] that, if $T \in \mathcal{K}_{(p,r)}(X,Y)$, then

$$\begin{aligned}\kappa_{(p,r)}(T) &= \inf\{\|(y_n)\|_p : (y_n) \in \ell_p(Y)\,((y_n) \in c_0(Y) \,\text{when}\ \ p=\infty), \\ &\quad T(B_X) \subset \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_r}\}\}. \end{aligned}$$

The class $\mathcal{K}^{\text{mid}}_{(p,r)}$ is an operator ideal and the proof follows a similar style to that for $\mathcal{K}_{(p,r)}$ (see [1, Proposition 3.8] and [2]) and uses the definition of $\ell_p^{\text{mid}}(Y)$ as we show next.

Proposition 2.5. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. The class $\mathcal{K}^{\text{mid}}_{(p,r)}$ of mid (p,r)-compact operators is an operator ideal.*

Proof. Let $S, T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$. Then there exist $(x_n), (y_n) \in \ell_p^{\text{mid}}(Y)$ ($(x_n), (y_n) \in c_0^{\text{mid}}(Y)$ when $p=\infty$) such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_r})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. Let $c \in \mathbb{K}$ and define

$$z_n = \begin{cases} 2^{1/r} c x_{(n+1)/2}, & n \text{ odd} \\ 2^{1/r} y_{n/2}, & n \text{ even.} \end{cases}$$

Then for every $(y_n^*) \in B_{\ell_p^w(Y^*)}$ we have

$$\begin{aligned}\sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(z_k)|^p &= \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(z_{2k-1})|^p + \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(z_{2k})|^p \\ &= \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(2^{1/r} c x_k)|^p + \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(2^{1/r} y_k)|^p \\ &= 2^{p/r}|c|^p \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(x_k)|^p + 2^{p/r} \sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(y_k)|^p \\ &\leq 2^{p/r}|c|^p (\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + 2^{p/r} (\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p \\ &= 2^{p/r}\Big(|c|^p (\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + (\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p\Big), \end{aligned}$$

so that

$$(\sum_{n=1}^\infty \sum_{k=1}^\infty |y_n^*(z_k)|^p)^{1/p} \leq 2^{1/r}\Big(|c|^p (\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + (\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p\Big)^{1/p} < \infty.$$

This proves that $(z_n) \in \ell_p^{\mathrm{mid}}(Y)$ ($(z_n) \in c_0(Y)$ after the appropriate adjustments when $p = \infty$).

Let $w \in (cS + T)(B_X)$. Then there exist $x \in B_X$ such that $w = cS(x) + T(x)$. Hence, for this x, there exist $(a_n), (b_n) \in B_{\ell_r}$ such that

$$S(x) = \Phi_{(x_n)}(a_n) = \sum_{n=1}^{\infty} a_n x_n \quad T(x) = \Phi_{(y_n)}(b_n) = \sum_{n=1}^{\infty} b_n y_n$$

respectively. Define

$$c_n = \begin{cases} 2^{-1/r} a_{(n+1)/2}, & n \text{ odd} \\ 2^{-1/r} b_{n/2}, & n \text{ even.} \end{cases}$$

Then $(c_n) \in B_{\ell_r}$. For,

$$\begin{aligned} \sum_{n=1}^{\infty} |c_n|^r &= \sum_{n=1}^{\infty} |c_{2n-1}|^r + \sum_{n=1}^{\infty} |c_{2n}|^r = \sum_{n=1}^{\infty} |2^{-1/r} a_n|^r + \sum_{n=1}^{\infty} |2^{-1/r} b_n|^r \\ &= 2^{-1}\Big(\sum_{n=1}^{\infty} |a_n|^r + \sum_{n=1}^{\infty} |b_n|^r \Big) \\ &\leq 2^{-1}\Big(\|(a_n)\|_r^r + \|(b_n)\|_r^r \Big) \\ &\leq 2^{-1}(2) = 1, \end{aligned}$$

so that $(\sum_{n=1}^{\infty} |c_n|^r)^{1/r} \leq 1$, as claimed.
Thus,

$$\begin{aligned} \Phi_{(z_n)}(c_n) &= \sum_{n=1}^{\infty} c_n z_n \\ &= \sum_{n=1}^{\infty} c_{2n-1} z_{2n-1} + \sum_{n=1}^{\infty} c_{2n} z_{2n} \\ &= \sum_{n=1}^{\infty} 2^{-1/r} a_n 2^{1/r} c x_n + \sum_{n=1}^{\infty} 2^{-1/r} b_n 2^{1/r} y_n \\ &= \sum_{n=1}^{\infty} 2^{-1/r} a_n 2^{1/r} c x_n + \sum_{n=1}^{\infty} 2^{-1/r} b_n 2^{1/r} y_n \\ &= c \sum_{n=1}^{\infty} a_n x_n + \sum_{n=1}^{\infty} b_n y_n \\ &= c\Phi_{(x_n)}(a_n) + \Phi_{(y_n)}(b_n) \\ &= cS(x) + T(x) = w. \end{aligned}$$

This proves that $w \in \Phi_{(z_n)}(B_{\ell_r})$. Therefore $(cS + T)(B_X) \subset \Phi_{(z_n)}(B_{\ell_r})$, so that $(cS+T) \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(X, Y)$, showing that $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$.

On defining $(z_n) \in \ell_p^{\mathrm{mid}}(Y)$ by $z_1 = \|x^*\|y, z_2 = z_3 = \ldots = 0$, it is clear that, if $(x_n^*) \in \ell_p^w(Y^*)$, then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|x_n^*(z_k)|^p = (\sum_{n=1}^{\infty}|x_n^*(z_1)|^p \leq \|z_1\|^p(\|(x_n^*)\|_p^w)^p,$$

whence

$$\|(z_n)\|_{\mathrm{mid},p} \leq \|z_1\| = \|\|x^*\|y\| = \|x^*\|\|y\|.$$

Furthermore, given $x \in B_X$, it holds that

$$x^* \otimes y(x) = x^*(x)y = \frac{x^*(x)}{\|x^*\|}\|x^*\|y = \sum_{n=1}^{\infty} a_n z_n \in \Phi_{(z_n)}(B_{\ell_r}),$$

with $a_1 = \frac{x^*(x)}{\|x^*\|}, a_2 = a_3 = \ldots = 0$, and so $(a_n) \in B_{\ell_r}$. Therefore

$$x^* \otimes y(B_X) \subset \Phi_{(z_n)}(B_{\ell_r}),$$

and so $x^* \otimes y \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$; that is, the space $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$ contains finite rank operators.

The ideal property also holds. That is, if $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(Y,W)$, then $BTA \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(Z,W)$. For, let $z \in B_Z$. Then $\frac{Az}{\|A\|} \in B_X$. So there exists $(t_n) \in B_{\ell_r}$ such that

$$T\Big(\frac{Az}{\|A\|}\Big) = \Phi_{(y_n)}(t_n).$$

We thus have

$$\begin{aligned} B\Big(T\Big(\frac{Az}{\|A\|}\Big)\Big) &= B(\Phi_{(y_n)}(t_n)) \\ &= B(\sum_{n=1}^{\infty} t_n y_n) \\ &= \sum_{n=1}^{\infty} t_n B y_n. \end{aligned}$$

Therefore,

$$(BTA)z = \sum_{n=1}^{\infty} t_n \|A\| B y_n = \Phi_{(w_n)}(t_n),$$

where $(w_n) := (\|A\|By_n) \in \ell_p^{\mathrm{mid}}(Y)$ since for every $(y_n^*) \in \ell_p^w(Y^*)$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(\|A\|By_k)|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p |y_n^*(By_k)|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p |B^* y_n^*(y_k)|^p \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p \|B^*\|^p \, |y_n^*(y_k)|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p \|B\|^p \, |y_n^*(y_k)|^p \\
&= \|A\|^p \|B\|^p \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(y_k)|^p < \infty.
\end{aligned}$$

□

Next, we will show that $(\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y), \kappa_{(p,r)}^{\mathrm{mid}}(\cdot))$ is a quasi-normed operator ideal (see [5, 9.3] for the definition of a quasi-normed operator ideal).

Proposition 2.6. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. The linear space $\mathcal{K}_{(p,r)}^{\mathrm{mid}}$ of mid (p,r)-compact operators is a quasi-normed operator ideal.*

Proof. The restriction of $\kappa_{(p,r)}^{\mathrm{mid}}(\cdot)$ to $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$ is a quasi-norm. For consider $T \in (\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y), \kappa_{(p,r)}^{\mathrm{mid}}(\cdot))$ such that $\kappa_{(p,r)}^{\mathrm{mid}}(T) = 0$, and fix $\epsilon > 0$. Then there exists $(y_n) \in \ell_p^{\mathrm{mid}}(Y)$ such that $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$ and

$$\|(y_n)\|_{\mathrm{mid},p} < \epsilon. \tag{2.3}$$

Fix $x \in B_X$ and choose $\alpha^x = (\alpha_n^x) \in B_{\ell_r}$ so that $Tx = \sum_{n=1}^{\infty} \alpha_n^x y_n$. Observe that, if $\varphi \in B_{Y^*}$, then $f = (f_1, f_2, \ldots, f_k, \ldots,) := (\varphi, 0, 0, \ldots) \in B_{\ell_p^w(Y^*)}$. Moreover,

$$\begin{aligned}
\langle Tx, \varphi \rangle &= \sum_{n=1}^{\infty} \alpha_n^x \langle y_n, \varphi \rangle \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_n^x \langle y_n, f_k \rangle,
\end{aligned}$$

whence

$$\begin{aligned}
|\langle Tx,\varphi\rangle| &= |\sum_{n=1}^{\infty}\alpha_n^x\langle y_n,\varphi\rangle| \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\alpha_n^x||\langle y_n,f_k\rangle| \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\alpha_n^x|\ |\langle y_n,f_k\rangle| \\
&\le \sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty}|\alpha_n^x|^r\Big)^{1/r}\Big(\sum_{n=1}^{\infty}|\langle y_n,f_k\rangle|^{r^*}\Big)^{1/r^*} \\
&\le \sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty}|\langle y_n,f_k\rangle|^{r^*}\Big)^{1/r^*} \quad (\text{since } \alpha^x\in B_{\ell_r}) \\
&= \Big(\sum_{n=1}^{\infty}|\langle y_n,\varphi\rangle|^{r^*}\Big)^{1/r^*} \\
&\le \Big(\sum_{n=1}^{\infty}|\langle y_n,\varphi\rangle|^{p}\Big)^{1/p} \quad (\text{since } r^*\ge p, \text{ and so } \ell_p\subset\ell_{r^*}) \\
&\le \|(y_n)\|^w \quad (\text{by (1.2)}) \\
&< \epsilon. \quad (\text{by (2.3)}) \qquad (2.4)
\end{aligned}$$

Hence, $\|Tx\|\le\epsilon$ by (2.4), and so

$$\|T\|\le\epsilon. \qquad (2.5)$$

Letting ϵ tend to zero, we conclude from (2.5) that $\|T\|=0$. Since $\|\cdot\|$ is a norm on $\mathcal{L}(X,Y)$, it follows that $T=0$, as was to be proved.
Now, fix $c\in\mathbb{K}$. Since $cT\in\mathcal{K}^{\rm mid}_{(p,r)}(X,Y)$, as seen above in Proposition 2.5, it follows that there exists $(y_n)\in\ell_p^{\rm mid}(Y)$ such that $T(B_X)\subset\Phi_{(y_n)}(B_{\ell_r})=\{\sum_{n=1}^{\infty}\alpha_n y_n : \alpha_n\in B_{\ell_r}\}$. Fix $c\neq 0$ in $\mathbb{K}$ and for every $x\in B_X$, choose $\alpha_n^x\in B_{\ell_r}$ such that $Tx=\sum_{n=1}^{\infty}\alpha_n^x y_n$. Then $cTx=\sum_{n=1}^{\infty}\alpha_n^x cy_n$. Since $(cy_n)\in\ell_p^{\rm mid}(Y)$, it follows that

$$\begin{aligned}
\kappa^{\rm mid}_{(p,r)}(cT) &\le \|(cy_n)\|_{{\rm mid},p} \\
&= |c|\|(y_n)\|_{{\rm mid},p} \quad (\text{since } \|\cdot\|_{{\rm mid},p} \text{ is a norm}). \qquad (2.6)
\end{aligned}$$

Hence

$$\kappa^{\rm mid}_{(p,r)}(cT) \le |c|\kappa^{\rm mid}_{(p,r)}(T). \qquad (2.7)$$

On the other hand, fix $\epsilon>0$. Since $cT\in\mathcal{K}^{\rm mid}_{(p,r)}(X,Y)$, choose $(z_n)\in\ell_p^{\rm mid}(Y)$ such that

$$\kappa^{\rm mid}_{(p,r)}(cT)>(1+\epsilon)\|(z_n)\|_{{\rm mid},p}.$$

Then

$$\begin{aligned}\kappa^{\mathrm{mid}}_{(p,r)}(cT) &> (1+\epsilon)\|(z_n)\|_{\mathrm{mid},p}\\ &= (1+\epsilon)\|c(\frac{1}{c}z_n)\|_{\mathrm{mid},p}\\ &= (1+\epsilon)|c|\|(\frac{1}{c}z_n)\|_{\mathrm{mid},p} \quad (\text{since } \|\cdot\|_{\mathrm{mid},p} \text{ is a norm}). \qquad (2.8)\end{aligned}$$

But $(cT)(B_X) \subset \{\sum_{n=1}^{\infty} \alpha_n z_n : (\alpha_n) \in B_{\ell_r}\}$ implies that $(T)(B_X) \subset \{\sum_{n=1}^{\infty} \frac{\alpha_n}{c} z_n : (\alpha_n) \in B_{\ell_r}\}$ with $(\frac{\alpha_n}{c}) \in \ell_r$ (which is a linear space). Therefore

$$\kappa^{\mathrm{mid}}_{(p,r)}(T) \leq \|(\frac{1}{c}z_n)\|_{\mathrm{mid},p}.$$

This together with (2.8) implies that

$$\kappa^{\mathrm{mid}}_{(p,r)}(cT) \geq (1+\epsilon)|c|\kappa^{\mathrm{mid}}_{(p,r)}(T).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\kappa^{\mathrm{mid}}_{(p,r)}(cT) \geq |c|\kappa^{\mathrm{mid}}_{(p,r)}(T). \qquad (2.9)$$

From (2.9) and (2.7) it holds that

$$\kappa^{\mathrm{mid}}_{(p,r)}(cT) = |c|\kappa^{\mathrm{mid}}_{(p,r)}(T), \qquad (2.10)$$

as was to be shown.

Let $S, T \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$ and choose $(x_n), (y_n) \in \ell_p^{\mathrm{mid}}(Y)$ such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_r})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. Fix $x \in B_X$. As above (in the proof of Proposition 2.5) with $c = 1$, we can define $(z_n) \in \ell_p^{\mathrm{mid}}(Y)$ and $(c_n) \in B_{\ell_r}$ (with appropriate adjustments when $p = \infty$) such that

$$(S+T)x = \Phi_{(z_n)}(c_n).$$

Then

$$
\begin{aligned}
\kappa^{\mathrm{mid}}_{(p,r)}(S+T) \;&\le\; \|(z_n)\|_{\mathrm{mid},p} \\
&= \sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_k)|^p\Big)^{1/p} \\
&= \sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_{2k-1})|^p\Big)^{1/p} + \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_{2k})|^p\Big)^{1/p}\Big) \\
&= \sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(2^{1/r}x_k)|^p\Big)^{1/p} \\
&\quad + \sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(2^{1/r}y_k)|^p\Big)^{1/p} \\
&= 2^{1/r}\sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(x_k)|^p\Big)^{1/p} \\
&\quad + 2^{1/r}\sup_{(w_n^*)\in B_{\ell_p^{\mathrm{mid}}(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(y_k)|^p\Big)^{1/p} \\
&= 2^{1/r}(\|(x_k)_{k=1}^{\infty}\|_{\mathrm{mid},p}) + 2^{1/r}(\|(y_k)_{k=1}^{\infty}\|_{\mathrm{mid},p}) \\
&= 2^{1/r}(\|(x_k)_{k=1}^{\infty}\|_{\mathrm{mid},p} + \|(y_k)_{k=1}^{\infty}\|_{\mathrm{mid},p}). \qquad (2.11)
\end{aligned}
$$

On taking the infimum on both sides, first over all the (x_n) defining S and then over all the (y_n) defining T we obtain

$$
\kappa^{\mathrm{mid}}_{(p,r)}(S+T) \;\le\; 2^{1/r}\Big(\kappa^{\mathrm{mid}}_{(p,r)}(S) + \kappa^{\mathrm{mid}}_{(p,r)}(T)\Big).
$$

This proves that $(\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is indeed a quasi-normed linear space.

Next, we show that $(\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is actually a quasi-normed operator ideal. To this end, recall from Proposition 2.5 that $\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$ admits finite rank operators: Given $x \in B_X$ and $y \in B_Y$, it holds that $x^* \otimes y \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$ with $\kappa^{\mathrm{mid}}_{(p,r)}(x^* \otimes y) \le 1$ (see the representation in the proof of Proposition 2.5). In particular, since $\mathbb{K}^* = \mathbb{K}$, and for all $\alpha \in \mathbb{K}$ it holds that

$$
\begin{aligned}
id_{\mathbb{K}}(\alpha) = \alpha \;&=\; 1 \cdot \alpha \cdot 1 \\
&=\; 1 \otimes 1(\alpha) \quad (1 \in \mathbb{K}^*, 1 \in \mathbb{K}).
\end{aligned}
$$

We then have

$$
id_{\mathbb{K}} = 1 \otimes 1 \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y),
$$

and whence

$$
\kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}) \;\le\; 1. \qquad (2.12)
$$

Alternatively, as quasi-normed spaces,

$$\mathcal{K}_{(p,r)}(X,Y) \hookrightarrow \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$$

and

$$\kappa^{\mathrm{mid}}_{(p,r)}(T) \leq \kappa_{(p,r)}(T).$$

Since $id_{\mathbb{K}} \in \mathcal{K}_{(p,r)}$ with $\kappa_{(p,r)}(id_{\mathbb{K}}) = 1$, it follows that

$$\kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}) \leq \kappa_{(p,r)}(id_{\mathbb{K}}) = 1.$$

On the other hand fix $\epsilon > 0$. Then there exists $(z_n) \in S_{\ell_p^{\mathrm{mid}}(Y)}$ such that

$$\|(z_n)\|_{\mathrm{mid},p} < \kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon. \tag{2.13}$$

Fix $\varphi \in B_{Y^*}$ arbitrarily and observe that $f_n := (\varphi, 0, \ldots) \in B_{\ell_p^w(Y^*)}$ where $f_1 = \varphi$ and $f_n = 0$ for all $n \geq 2$. Hence,

$$\begin{aligned}\Big(\sum_{k=1}^{\infty} |\langle \varphi, z_k\rangle|^p\Big)^{1/p} &= \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle f_n, z_k\rangle|^p\Big)^{1/p} \\ &\leq \|(z_n)\|_{\mathrm{mid},p} < \kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon \quad \text{(by (2.13))}.\end{aligned}$$

Taking the supremum over φ in the previous inequality gives

$$1 = \|(z_n)\|_{\ell_p^w} \leq \kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon.$$

Letting ϵ tend to 0 leads to

$$1 \leq \kappa^{\mathrm{mid}}_{(p,r)}(id_{\mathbb{K}}). \tag{2.14}$$

By (2.12) and (2.14) we have

$$\|id_{\mathbb{K}}\|_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} = 1,$$

as was to be shown.

Lastly, it follows (again) from Proposition 2.5 that, if $T \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$, $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(Y,W)$, then $BTA \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(Z,W)$, where $(\|A\|By_n) \in \ell_p^{\mathrm{mid}}(Y)$ defines BTA whenever $(y_n) \in \ell_p^{\mathrm{mid}}(Y)$ defines T. So

$$\begin{aligned}\kappa^{\mathrm{mid}}_{(p,r)}(BTA) &\leq \|(\|A\|By_n)\|_{\mathrm{mid},p} \\ &= \sup_{(y_n^*)\in B_{\ell_p^w(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(\|A\|By_k)|^p\Big)^{1/p} \quad \text{(by (2.1))} \\ &\leq \|A\|\|B\|\Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(y_k)|^p\Big)^{1/p}.\end{aligned}$$

On taking the infimum on both sides of the preceding inequality over the (y_n)'s in $\ell_p^{\mathrm{mid}}(Y)$ that define T we obtain

$$\kappa^{\mathrm{mid}}_{(p,r)}(BTA) \leq \|A\|\kappa^{\mathrm{mid}}_{(p,r)}(T)\|B\|,$$

as was to be proved, and this affirms the claim that $(\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is a quasi-normed operator ideal.

□

That $(\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is a quasi-Banach operator ideal will follow from proving that it is an s-Banach operator ideal where $s = \frac{pr}{p+r}$ and this definition of s leads to $0 < s \leq 1$.

Proposition 2.7. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then $(\mathcal{K}^{\mathrm{mid}}_{(p,r)}, \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is an s-Banach operator ideal.*

Proof. Assume that $(T_n) \subset \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$ is a sequence such that $\sum_{n=1}^{\infty} \kappa^{\mathrm{mid}}_{(p,r)}(T_n)^s < \infty$. Given the ideal quasi-norm $\kappa^{\mathrm{mid}}_{(p,r)}(\cdot)$ with the universal constant $c := 2^{\frac{1}{s}-1}$, a classical renorming process yields an equivalent ideal s-norm

$$|||T|||_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} := \inf\{\Big(\sum_{i=1}^{n} \kappa^{\mathrm{mid}}_{(p,r)}(T_i)^s\Big)^{1/s} \ : \ T := \sum_{i=1}^{n} T_i, T_1, \ldots, T_n \in \mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)\}$$

(see [11], 6.2.5)). Using this s-norm yields

$$\begin{aligned} |||\sum_{k=1}^{n} T_k|||^s_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} &\leq \sum_{k=1}^{n} \kappa^{\mathrm{mid}}_{(p,r)}(T_k)^s \\ &\leq \sum_{k=1}^{\infty} \kappa^{\mathrm{mid}}_{(p,r)}(T_k)^s \end{aligned}$$

for each $n \in \mathbb{N}$. Taking the $|||\cdot|||_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)}$-limit on both sides as n tends to ∞, yields $|||\sum_{k=1}^{\infty} T_k|||^s_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} < \infty$, whence $U = \sum_{n=1}^{\infty} T_n \in \mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$, by the equivalence of the s-norm $|||\cdot|||_{\mathcal{K}^{\mathrm{mid}}_{(p,r)}(X,Y)}$ with the quasinorm $\kappa^{\mathrm{mid}}_{(p,r)}(\cdot)$, as was to be proved.

Since for each $x \in B_X$ and $y \in B_Y$ it holds that $\kappa^{\mathrm{mid}}_{(p,r)}(x^* \otimes y) = \|x^*\| \, \|y\|$, for any quasi-Banach operator ideal [10, 6.1.5], and the ideal property is also satisfied as seen in the proof of Proposition 2.5, it follows that $(\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is indeed an s-Banach operator ideal.

□

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$, $(x_j) \in \ell_p^{\mathrm{mid}}(X)$ and X be a Banach space. Recall from the earlier discussion that $(e_n) \in S_{\ell_r^w(\ell_r^*)}$, where we considered the unit vectors $e_n \in \ell_{r^*} \subset (\ell_r)^*$ as coordinate functionals for ℓ_r. The operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j$$

admits a representation

$$\Phi_{(x_j)} \quad = \quad \sum_{j=1}^{\infty} e_j \otimes x_j \tag{2.15}$$

Moreover, $\Phi_{(x_j)}$ is approximable:

Proposition 2.8. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell_p^{\mathrm{mid}}(X)$. Let X be a Banach space. Then $\Phi_{(x_j)}$ with the representation in equation 2.15 is approximable. Hence, $\Phi_{(x_j)} \in \mathcal{K}(\ell_r, X)$.*

Proof. Let $(x_j)_{j\leq m} := (x_1, \ldots, x_m, 0, 0, \ldots)$ and define

$$\Phi_{(x_j)_{j\leq m}} = \sum_{j=1}^{m} e_j \otimes x_j.$$

Then

$$\begin{aligned}
&\|\Phi_{(x_j)} - \Phi_{(x_j)_{j\leq m}}\| \;=\; \sup_{(a_j)\in B_{\ell_r}} \| \sum_{j=m+1}^{\infty} a_j x_j\| \\
= \quad & \sup_{(a_j)\in B_{\ell_r}} \sup_{x^*\in B_{X^*}} |\sum_{j=m+1}^{\infty} a_j x^*(x_j)| \\
= \quad & \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} |\sum_{j=m+1}^{\infty} a_j x^*(x_j)| \leq \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \sum_{j=m+1}^{\infty} |a_j||x^*(x_j)| \\
\leq \quad & \begin{cases} \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sum_{j=m+1}^{\infty} |a_j|^{p^*}\Big)^{1/p^*} \Big(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\Big)^{1/p}, \\ 1 < p < \infty,\ 1 \leq r \leq p^* \\ \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sup_{j\geq m+1} |a_j|\Big)\Big(\sum_{j=m+1}^{\infty} |x^*(x_j)|\Big), \\ p = 1 \\ \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sum_{j=m+1}^{\infty} |a_j|\Big)\Big(\sup_{j\geq m+1} |x^*(x_j)|\Big), \\ p = \infty \end{cases} \\
\leq \quad & \begin{cases} \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sum_{j=m+1}^{\infty} |a_j|^{r}\Big)^{1/r} \Big(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\Big)^{1/p}, \\ 1 < p < \infty,\ 1 \leq r \leq p^* \\ \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sup_{j\geq m+1} |a_j|\Big)\Big(\sum_{j=m+1}^{\infty} |x^*(x_j)|\Big), \\ p = 1,\ r = p^* \\ \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \Big(\sum_{j=m+1}^{\infty} |a_j|\Big)\Big(\sup_{j\geq m+1} |x^*(x_j)|\Big), \\ p = \infty,\ r = p^* \end{cases} \\
\leq \quad & \begin{cases} \sup_{x^*\in B_{X^*}} \Big(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\Big)^{1/p}, & 1 \leq p < \infty,\ 1 \leq r \leq p^* \\ \sup_{x^*\in B_{X^*}} \Big(\sup_{j\geq m+1} |x^*(x_j)|\Big), & p = \infty \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \sup_{(z_n^*)\in B_{\ell_p^w(X^*)}} \left(\sum_{n=1}^{\infty}\sum_{j=m+1}^{\infty}|z_n^*(x_j)|^p\right)^{1/p}, \\ 1\le p<\infty,\ 1\le r\le p^* \quad (z_1^*:=x^*, z_n^*:=0, n\neq 1) \\ \sup_{x^*\in B_{X^*}}\left(\sup_{n\ge 1}\sup_{j\ge m+1}|z_n^*(x_j)|\right), \\ p=\infty \quad (z_1^*:=x^*, z_n^*:=0, n\neq 1) \end{cases} \\
&\le \begin{cases} \sup_{(x_n^*)\in B_{\ell_p^w(X^*)}} \left(\sum_{n=1}^{\infty}\sum_{j=m+1}^{\infty}|x_n^*(x_j)|^p\right)^{1/p}, \\ 1\le p<\infty,\ 1\le r\le p^* \\ \sup_{x^*\in B_{X^*}}\left(\sup_{n\ge 1}\sup_{j\ge m+1}|x_n^*(x_j)|\right), \\ p=\infty \end{cases} \\
&= \begin{cases} \|(x_j)_{j=1}^{\infty}-(x_j)_{j=1}^{m}\|_{\mathrm{mid}\,p}, \quad 1\le p<\infty \\ \|(x_j)_{j=1}^{\infty}-(x_j)_{j=1}^{m}\|_{\mathrm{mid}\,\infty}, \quad p=\infty \end{cases} \\
&\overset{m}{\to} \ 0.
\end{aligned}
$$

This proves that $\Phi_{(x_j)_{j\le m}}$ converges to $\Phi_{(x_j)}$ in the operator norm as $m\to\infty$, whence $\Phi_{(x_j)}\in\mathcal{K}(\ell_r,X)$ whenever $(x_j)\in\ell_p^{\mathrm{mid}}(X)$ ($(x_j)\in c_0^{\mathrm{mid}}(X)$ when $p=\infty$). □

If $(y_j)\in\ell_p^{\mathrm{mid}}(Y)$ for a given Banach space Y, $1\le p\le\infty$ and $1\le r\le p^*$, the operator $\Phi_{(y_j)}$ admits a factorization

$$\Phi_{(y_j)}=\overline{\Phi}_{(y_j)}\,q,$$

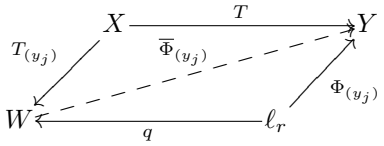
where $q:\ell_r\longrightarrow W:=\ell_r/N(\Phi_{(y_j)})$ is the linear quotient mapping and $\overline{\Phi}_{(y_j)}:W\longrightarrow Y$ is the linear injective associate of $\Phi_{(y_j)}$. This kind of factorization is a property of every operator in $\mathcal{L}(X,Y)$ for all Banach spaces X and Y (see [1], p.19). Furthermore, suppose that $T(B_X)\subset\Phi_{(x_j)}(B_{\ell_r})$ so that for each $x\in B_X$ there exists $\alpha\in B_{\ell_r}$ such that $Tx=\Phi_{(x_j)}\alpha$. It is known that each such x in X determines an $\alpha\in\ell_r$ uniquely modulo the null space $N(\Phi_{(x_j)})$ of $\Phi_{(x_j)}$ such that $\|\alpha\|_r\le\|x\|$. This was first observed and introduced by the authors of [12] for $r=p^*$ and then later improved on in [1] and [2] for any $1\le r\le p^*$. Since the involved factorizations are basically algebraic, the foregoing also hold in our setting of $(y_j)\in\ell_p^{\mathrm{mid}}(Y)$. This, *said and done*, we define

$$
\begin{aligned}
T_{(y_j)}:\ X &\longrightarrow W \\
x &\longmapsto q\alpha
\end{aligned}
$$

where $x\in X$, and $\alpha\in\ell_r$ satisfies $\|\alpha\|_r\le\|x\|$ and $Tx=\Phi_{(y_j)}\alpha$. If $\|x\|\le 1$, then $\|T_{(y_j)}x\|\le\|q\|=1$. Following the dictum from [12], we call the injective operator $T_{(y_n)}$ the *essential quotient* of T with respect to (y_j). Hence

$$T = \overline{\Phi}_{(y_j)}T_{(y_j)} \tag{2.16}$$

where $T_{(y_j)} \in \mathcal{L}(X,W)$, $\|T_{(y_j)}\| \leq 1$, and $N(T_{(y_j)}) = N(T)$ and (as in [12]) we shall say that T is quotiented in ℓ_r by $(y_j) \in \ell_p^{\text{mid}}(Y)$. Then each of the two triangles in the following diagram is commutative:



Given Banach spaces X and Y, $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, fix $T \in \mathcal{K}_{(p,r)}^{\text{mid}}(X,Y)$. Then the operator T has a natural factorization similar to the case of $\mathcal{K}_{(p,r)}(X,Y)$ [1, pp.33 – 34].

Recall the following definition in [4, Definition 2.1]:

Definition 2.9. *Given Banach spaces X and Y, an operator $T \in \mathcal{L}(X,Y)$ is said to be absolutely mid p-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^{\text{mid}}(X)$. It is said to be weakly mid p-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p^{mid}(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$.*

We now show that the operator

$$\begin{array}{rccc} \Phi_{(x_j)}^* : & X^* & \longrightarrow & (\ell_r)^* \\ & x^* & \longmapsto & (\langle x^*, x_j \rangle)_{j=1}^{\infty} \end{array}$$

is weakly mid p-summing whenever $(x_j) \in \ell_p^{\text{mid}}(X)$

Proposition 2.10. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell_p^{\text{mid}}(X)$. Consider the operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by*

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j.$$

Then the operator

$$\begin{array}{rccc} \Phi_{(x_j)}^* : & X^* & \longrightarrow & (\ell_r)^* \\ & x^* & \longmapsto & (\langle x^*, x_j \rangle)_{j=1}^{\infty} \end{array}$$

is weakly mid p-summing.

Proof. Let $(x_j) \in \ell_p^{\text{mid}}(X)$ and consider the linear operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j.$$

Let $(x_n^*) \in \ell_p^w(X^*)$. Fix $\varphi = (g_n)$ where $(g_n) \in B_{\ell_p^w((\ell_r)^{**})}$ is arbitrary. Then

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_n,\Phi^*_{(x_j)}(x_k^*)\rangle|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\langle \Phi^{**}_{(x_j)}(g_n),x_k^*\rangle|^p\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\|\Phi^{**}_{(x_j)}(g_n)\|^p\,|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|},x_k^*\rangle|^p\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sup_{\phi\in B_{(\ell_r)^*}}|\langle \phi,\Phi^{**}_{(x_j)}(g_n)\rangle|^p|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|},x_k^*\rangle|^p\\
&= \sum_{n=1}^{\infty}\sup_{\phi\in B_{(\ell_r)^*}}|\langle \phi,\Phi^{**}_{(x_j)}(g_n)\rangle|^p\sum_{k=1}^{\infty}|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|},x_k^*\rangle|^p\\
&\le (\|(x_k^*)\|_p^w)^p\sup_{\phi\in B_{(\ell_r)^*}}\sum_{n=1}^{\infty}|\langle \phi,\Phi^{**}_{(x_j)}(g_n)\rangle|^p\\
&= (\|(x_k^*)\|_p^w)^p(\|(\Phi^{**}_{(x_j)}(g_n))\|_p^w)^p\\
&= (\|(x_k^*)\|_p^w)^p(\|\widehat{\Phi^{**}_{(x_j)}}((g_n)_{n=1}^{\infty})\|_p^w)^p\\
&\le (\|(x_k^*)\|_p^w)^p\|\Phi^{**}_{(x_j)}\|^p \quad (\text{ since } (g_n)\in B_{\ell_p^w((\ell_r)^{**})}\ )\\
&= (\|(x_k^*)\|_p^w)^p\|\Phi_{(x_j)}\|^p\\
&\le (\|(x_k^*)\|_p^w)^p(\|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p})^p \quad (\text{using (2.1)}).
\end{aligned}$$

Therefore $\Phi^*_{(x_j)}$ is weakly mid p-summing by Definition 2.9. □

Concerning the operator $\Phi_{(x_n)_{n=1}^{\infty}}$, we take note of the following: By [4, Example 1.7] and since $(e_k)\in\ell_2^w(c_0)=\ell_2^{\mathrm{mid}}(c_0)$, we may consider

$$\begin{aligned}
\Phi_{(e_n)}:\ c_0 &\longrightarrow c_0\\
(a_n) &\longmapsto \sum_{n=1}^{\infty}a_ne_n=(a_n).
\end{aligned}$$

Moreover, $(\Phi_{(e_n)}(e_k))_{k=1}^{\infty}=(e_k)_{k=1}^{\infty}\notin\ell_2(c_0)$, so that $\Phi_{(e_n)}$ is not absolutely mid 2-summing by Definition 2.9.

Furthermore, since $(e_n)\in\ell_1^{\mathrm{mid}}(\ell_1)$ and $\|(e_j)\|_{\mathrm{mid},1}=1$, consider

$$\begin{aligned}
\Phi_{(e_n)}:\ c_0 &\longrightarrow \ell_1\\
(a_n) &\longmapsto \sum_{n=1}^{\infty}a_ne_n=(a_n).
\end{aligned}$$

and recall that $(e_k)\in\ell_1^w(c_0)$. But $((\Phi_{(e_n)}(e_k))_{k=1}^{\infty}=(e_k)_{k=1}^{\infty}\notin\ell_1^{\mathrm{mid}}(\ell_1)$ since $\ell_1^{\mathrm{mid}}(\ell_1)=\ell_1(\ell_1)$, by Theorem 2.3(ii) and $(e_k)_{k=1}^{\infty}$ is not strongly ℓ_1-summable. This proves that $\Phi_{(e_n)}$ is not weakly mid 1-summing.

Now, observe that

$$\begin{aligned} \Phi_{(e_n)}:\ c_0 &\longrightarrow c_0 \\ (a_n) &\longmapsto \sum_{n=1}^{\infty} a_n e_n = (a_n) \end{aligned}$$

gives rise to

$$\begin{aligned} \Phi^*_{(e_n)}: \ell_1 &\longrightarrow \ell_1 \\ \varphi &\longmapsto (\langle \varphi, e_n \rangle)_{n=1}^{\infty}. \end{aligned}$$

Note that $(e_k) \in \ell_1^{\mathrm{mid}}(\ell_1)$ and $\|(e_k)\|_{\mathrm{mid},1} = 1$, as before, and $(\Phi^*_{(e_n)}(e_k))_{k=1}^{\infty} = ((\langle e_k, e_n \rangle)_{n=1}^{\infty})_{k=1}^{\infty} = (e_k)_{k=1}^{\infty} \notin \ell_1(\ell_1)$. This shows that $(\Phi^*_{(e_n)})$ is not absolutely mid 1-summing.

3 (p, r)-limited Sets

Recall that a nonempty subset A of a Banach space X is called *limited* in X if for every weak*-null sequence (f_n) in X^* (that is, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$), it holds that $f_n \to 0$ uniformly on A. Alternatively [13], for every weak*-null sequence (f_n) in X^*, there exists $(\alpha_n) \in c_0$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in A$ and all $n \in \mathbb{N}$. It is known that every compact set is limited. The converse is however false and this was initially mistakenly thought to be true by Gelfand [8] but was later refuted by Phillips [10] who produced an example of a non-compact limited set (see [13, §2, second paragraph]).

Recently, the concept of a limited set has been extended to the p-level in [13]: A subset A of X is said to be *p-limited* in X $(1 \leq p < \infty)$ if for every weak*-p-summable sequence (f_n) in X^* (that is, $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all $x \in X$), there exists an $(\alpha_n) \in \ell_p$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$. Subsequently, the following result was proved in [13, Proposition 2.1].

Proposition 3.1. *Let $1 \leq p < \infty$ and X be a Banach space. Then every p-compact subset of X is p-limited.*

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. In analogy with the concept of (p, r)-compact operators, we'll extend the concept of a limited set to the (p, r)-level, where the (p, p^*)-level is the p-level mentioned above.

Definition 3.2. *Let $1 \leq p < \infty$, $1 \leq r \leq p^*$, and X and Y be Banach spaces. A subset A of X is said to be (p, r)-limited in X if for every weak*-p-summable sequence (f_n) in X^* (that is, $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all x in X), there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(x)| \leq \alpha_n$ for $x \in A$ and $n \in \mathbb{N}$. An operator $T \in \mathfrak{L}(X, Y)$ is said to be (p, r)-limited if $T(B_X)$ is (p, r)-limited in Y.*

It follows from the paragraph preceding [4, Proposition 2.12] that the weakly mid p-summing operators defined in Proposition 2.9 are precisely the sequentially p-limited operators defined in [13, Definition 4.1]. These are the (p, p^*)-limited

operators in terms of Definition 3.2.
It is to be noted, though, that by virtue of how they are defined the absolutely (respectively, weakly) mid (p,r)-summing operators (in [4]) are not related to the (p,r)-limited operators given in Definition 3.2.

Now, Proposition 3.1 can be extended as follows.

Theorem 3.3. *Let $1 \leq p < \infty$, $1 \leq r \leq p^*$ and X be a Banach space. Then every mid (p,r)-compact subset of X is (p,r)-limited.*

Proof. Let $K \subset X$ be a mid (p,r)-compact set and choose $(f_n) \in \ell_p^{w^*}(X^*)$ arbitrarily. Then there exists $(x_n) \in \ell_p^{\mathrm{mid}}(X)$ such that $K \subset \Phi_{(x_n)}(B_{\ell_r})$. Furthermore,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\|(f_n)\|_p^{w^*})^p\,|\frac{f_n}{\|(f_n)\|_p^{w^*}}(x_k)|^p \\ &\leq (\|(f_n)\|_p^{w^*}\ \|(x_k)\|_{\mathrm{mid},p})^p. \\ &\quad (\text{ since } \ell_p^{w^*}(X^*) = \ell_p^{w}(X^*) \text{ isometrically}).\end{aligned}$$

Since $\ell_p^{w^*}(X^*) \subset \ell_{r^*}^{w^*}(X^*)$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^{r^*} &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^p \\ &\leq (\|(f_n)\|_p^{w^*}\ \|(x_k)\|_{\mathrm{mid},p})^p\end{aligned}$$

by the previous inequality.

Let

$$(\sum_{k=1}^{\infty}|f_n(x_k)|^{r^*})^{1/r^*} = \alpha_n$$

for all n. Then $(\alpha_n) \in \ell_{r^*}$.

Fix $z \in K$ arbitrarily. Then $z = \sum_{k=1}^{\infty}\gamma_k x_k$ for some $(\gamma_k) \in B_{\ell_r}$. Now, for each $n \in \mathbb{N}$ it holds that

$$\begin{aligned}|f_n(z)| &= |\sum_{k=1}^{\infty}\gamma_k f_n(x_k)| \\ &\leq \sum_{k=1}^{\infty}|\gamma_k|\,|f_n(x_k)| \\ &\leq (\sum_{k=1}^{\infty}|\gamma_k|^r)^{1/r}(\sum_{k=1}^{\infty}|f_n(x_k)|^{r^*})^{1/r^*} \\ &\leq (\sum_{k=1}^{\infty}|f_n(x_k)|^{r^*})^{1/r^*} \\ &= \alpha_n.\end{aligned}$$

Therefore K is (p,r)-limited, as was to be proven. □

Corollary 3.4. *Let $1 \leq p < \infty, 1 \leq r \leq p^*$ and X be a Banach space. Suppose that $(x_n) \in \ell_p^{\mathrm{mid}}(X)$. Then $\Phi_{(x_n)}(B_{\ell_r})$ is a (p,r)-limited set. In particular, $\Phi_{(x_n)}$ is (p,r)-limited.*

Proof. If $(x_n) \in \ell_p^{mid}(X)$, then $\Phi_{(x_n)}(B_{\ell_r})$ is a mid (p,r)-compact set in X. □

Theorem 3.5. *Let X and Y be Banach spaces, $1 \leq p < \infty$, and $1 \leq r \leq p^*$. If $T \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$, then T is (p,r)-limited.*

Proof. This follows from Theorem 3.3 since $T(B_X)$ is a mid (p,r)-limited set in Y. □

Another property of (p,r)-limited sets that generalizes the p-level one in [13, Proposition 2.2] is as follows:

Proposition 3.6. *Let $1 \leq p < \infty, 1 \leq r \leq p^*$ and X be a Banach space. Given any two subsets A and B of X it holds that:*

(a) If B is (p,r)-limited and $A \subset B$, then A is also (p,r)-limited.

(b) If A is (p,r)-limited, then $\bar{A}$ is (p,r)-limited.

(c) If A and B are (p,r)-limited, so are $A \cup B$, $A + B$ and $A \cap B$.

(d) If A is (p,r)-limited and $T \in \mathcal{L}(X,Y)$, then $T(A)$ is (p,r)-limited.

Proof.

(a) Let $(f_n) \in \ell_p^{w^*}(X^*)$. Since B is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(x)| \leq \alpha_n$ for every $x \in B$. Fix $a \in A$ arbitrarily. Since $A \subset B$, it follows that $a \in B$ so that $|f_n(a)| \leq \alpha_n$ for every $n \in \mathbb{N}$. This proves that A is (p,r)-limited, as was to be proven.

(b) Let $(f_n) \in \ell_p^{w^*}(X^*)$ and fix $x \in \bar{A}$ arbitrarily. Then there exists (x_k) in A such that $x_k \to x$ as $k \to \infty$. Thus, for each $n \in \mathbb{N}$, it holds that $f_n(x_k) \to f_n(x)$. Since A is (p,r)-limited, it follows that there is $(\alpha_n) \in \ell_{r^*}$ such that for each fixed $n \in \mathbb{N}$, $|f_n(x_k)| \leq \alpha_n$ for every $k \in \mathbb{N}$. Hence, for each fixed $n \in \mathbb{N}$, $|f_n(x)| \leq \alpha_n$, so that $\bar{A}$ is (p,r)-limited, as was to be shown.

(c) Let $(f_n) \in \ell_p^{w^*}(X^*)$. Since A is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(a)| \leq \alpha_n$ for every $a \in A$ and $n \in \mathbb{N}$. Since B is (p,r)-limited, there exists $(\beta_n) \in \ell_{r^*}$ such that $|f_n(b)| \leq \beta_n$ for every $b \in B$ and $n \in \mathbb{N}$. Define (γ_n) by

$$\gamma_n = \begin{cases} \alpha_n & , n \text{ odd} \\ \\ \beta_n & , n \text{ even.} \end{cases}$$

Then $(\gamma_n) \in \ell_{r^*}$ since

$$\begin{aligned}\sum_{n=1}^{\infty}|\gamma_n|^{r^*} &= \sum_{n=1}^{\infty}|\gamma_{2n-1}|^{r^*}+\sum_{n=1}^{\infty}|\gamma_{2n}|^{r^*}=\sum_{n=1}^{\infty}|\alpha_{2n-1}|^{r^*}+\sum_{n=1}^{\infty}|\beta_{2n}|^{r^*}\\ &\le \sum_{n=1}^{\infty}|\alpha_{2n-1}|^{r^*}+\sum_{n=1}^{\infty}|\alpha_{2n}|^{r^*}+\sum_{n=1}^{\infty}|\beta_{2n-1}|^{r^*}+\sum_{n=1}^{\infty}|\beta_{2n}|^{r^*}\\ &= \|(\alpha_n)\|_{r^*}^{r^*}+\|(\beta_n)\|_{r^*}^{r^*}\\ &< \infty.\end{aligned}$$

Fix $z \in A \cup B$ arbitrarily. Then either $z \in A$ or $z \in B$. If $z \in A$ then by the preceding paragraphs it holds that for every $n \in \mathbb{N}$,

$$|f_{2n-1}(z)| \le \alpha_{2n-1} = \gamma_{2n-1}.$$

If $z \in B$ then by the preceding paragraphs it holds that for every $n \in \mathbb{N}$,

$$|f_{2n}(z)| \le \beta_{2n} = \gamma_{2n}.$$

Hence,

$$|f_n(z)| \le \gamma_n$$

for every $z \in A \cup B$ and $n \in \mathbb{N}$. This proves that $A \cup B$ is (p,r)-limited.

Next, let $(f_n) \in \ell_p^{w^*}(X^*)$ and fix $z \in (A+B)$ arbitrarily. Then $z = a + b$ for some $a \in A$ and $b \in B$. Since A is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(a)| \le \alpha_n$ for every n. Since B is (p,r)-limited, there exists $(\beta_n) \in \ell_{r^*}$ such that $|f_n(b)| \le \beta_n$ for every n. Let $\gamma_n := \alpha_n + \beta_n$ for each $n \in \mathbb{N}$. Then $(\gamma_n) \in \ell_{r^*}$ too. Moreover, it holds that for every $n \in \mathbb{N}$,

$$|f_n(z)| = |f_n(a) + f_n(b)| \le |f_n(a)| + |f_n(b)| \le \alpha_n + \beta_n = \gamma_n.$$

This proves that $A + B$ is (p,r)-limited.

That $A \cap B$ is (p,r)-limited follows from (a) by considering $A \cap B \subset A$ or $A \cap B \subset B$ whenever A and B are (p,r)-limited.

(d) Let $(f_n) \in \ell_p^{w^*}(Y^*)$. Fix $y \in T(A)$ arbitrarily where $T \in \mathcal{L}(X,Y)$. Then there exists $x \in A$ such that $y = T(x)$. Define $g_n \in X^*$ by $g_n = T^* f_n$ for each $n \in \mathbb{N}$. Then $(g_n) \in \ell_p^{w^*}(X^*)$ since for every $x \in X$ it follows that

$$\sum_{n=1}^{\infty}|g_n(x)|^p = \sum_{n=1}^{\infty}|T^* f_n(x)|^p = \sum_{n=1}^{\infty}|f_n(Tx)|^p = \sum_{n=1}^{\infty}|f_n(y)|^p < \infty.$$

Since A is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|g_n(z)| \le \alpha_n$ for every $z \in A$ and $n \in \mathbb{N}$. With this $(\alpha_n) \in \ell_{r^*}$,

$$|f_n(y)| = |f_n(T(x))| = |T^* f_n(x)| = |g_n(x)| \le \alpha_n$$

for every $n \in \mathbb{N}$. This proves that $T(A)$ is (p,r)-limited whenever A is (p,r)-limited.

□

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