Finite element analysis of vibrating elastic structures

by

Sonja du Toit

Submitted in partial fulfilment of the requirements for the degree

Magister Scientiae

in the Department of Mathematics and Applied Mathematics
in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

December 2014

DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Name: Sonja du Toit

Date: December 2014

Title	Finite element analysis of vibrating elastic structures
Name	Sonja du Toit
Supervisor	Prof N F Janse van Rensburg
Co-Supervisor	Dr M Basson
Department	Mathematics and Applied Mathematics
Degree	Magister Scientiae

Summary

0.1 Overview

In the research for the Masters degree, principally a literature study, the focus was on the Finite Element Method applied to vibrating systems: vibrating beams, plates, multi-dimensional elastic solids, hybrid models and linked vibrating solids. The emphasis of this dissertation was on the Finite Element Method applied to these model problems.

An important aspect was the comparison of different models for the same structure. The chosen method to compare models is numerical simulation. Therefore convergence of the finite element approximation is important. Specific attention was given to:

- determining whether error estimates are available in general and any conditions necessary for convergence;
- error analysis and convergence of solutions;
- existence of solutions.

A search through the literature revealed that a substantial number of publications deals with the finite element method applied to the multi-dimensional wave equation but less on vibrating beams, plates and multi-dimensional elastic solids. Hardly anything has been published on error estimates and finite element approximations for hybrid models and structures, linked systems or structures with boundary damping or damping in joints.

In Chapter 1 we present an introduction to the theory of vibrating beams, plates and multi-dimensional elastic solids. For beams we start with the most general linear model. In particular we looked at the differences between the Euler Bernoulli, Rayleigh and Timoshenko models for the transverse vibration of a beam. We show how the Reissner-Mindlin plate model is derived from the equations of motion and how the classical plate model follows by making additional assumptions.

Also in Chapter 1 we mention briefly comparison results in the literature concerning the Euler Bernoulli, Rayleigh and Timoshenko models for the transverse vibration of a beam. In [SP06] and [LVV09a] the Timoshenko theory is compared to a multi-dimensional model. It was found that the Timoshenko theory is an excellent approximation in the case of beam applications, i.e. for transverse loads. In the articles [VV06] and [LVV09a] the Timoshenko model is compared to Rayleigh or Euler-Bernoulli models. The Rayleigh and Euler-Bernoulli models can be useful when the parameter α is large. A value of α of anything between 2000 and 5000 may be sufficient, but it depends on the initial data or the manner of excitation. This means that it is not possible to determine beforehand whether the Euler-Bernoulli or Rayleigh model is suitable for a given application (unless α is extremely large). The Euler-Bernoulli and Rayleigh models are compared in [VV06]. The values for the first 5 eigenvalues indicate that the difference between the models is probably not significant.

In Chapter 2 we present general existence results for hyperbolic equations which may be applied to the model problems. These results are from the article [VV02]. In this article the existence theorems are given in a convenient form for application to the finite element method. Unfortunately only weak solutions are guaranteed. For higher regularity properties one may consult [Eva98] but the model problem does not include damping and only homogenous Dirichlet boundary conditions. Strict compatibility conditions must be imposed on the initial data and regularity conditions on the forcing function to yield these higher regularity properties.

In Chapter 3 we study the available convergence theory for hyperbolic equations to determine whether error estimates are available in general and any conditions necessary for convergence. A general theory is required that can be applied to any linear vibrating system. We considered only [OR76], [BV13], [BV14] and [Bas14] since other important articles (e.g. [Bak76] and [Dup73]) are already considered in these publications. These results were then applied to each of the different model problems in Chapters 4 to 7.

In the article [BV13] convergence is proved but with weaker assumptions than the other articles considered. This is achieved by splitting the error into semi-discrete and fully discrete errors. In the article semi-discrete and fully discrete error estimates for the Galerkin approximation of a general linear second order hyperbolic problem are derived. Viscous type damping is also incorporated and so the results in [BV13] could be applied to problems like the Reissner-Mindlin plate model. The results and proofs in the article [BV13] are mostly given in sufficient detail in the article, hence the focus in this dissertation was to state the results and apply them. The results can not be applied to problems with boundary damping.

Hybrid beam models with boundary damping are considered in Chapters 4 and 5 (where the theory in [BV13] is not applicable). The model problem in Chapter 4 was introduced in [ZVV04] where the finite element method is used to compute natural frequencies and modes of vibration. In [Bas14, Section 8.2] the model problem is considered briefly. Aspects covered are existence and convergence of the finite element approximation. The idea in this dissertation was to apply the general theory for convergence but this is only available for the semi-discrete approximation in [BV14]. We present a complete derivation for the error estimate for the fully discrete approximation using [OR76] and [Bas14].

In Chapter 5 we study [BDV14] which is based on the work in [AS02]. In the article [AS02] the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. The model is more realistic than other articles: the fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account. The Euler-Bernoulli theory for a beam with Kelvin-Voigt damping is used. The beam is clamped at one end and the tip body is attached to the other end. The existence of a unique solution for the model problem is established. In [BDV14] the authors present a complete analysis of the model. Firstly they derive a correct linear approximation for the model problem. Next they also prove the existence of a unique solution. Their approach differs from that in [AS02]: they write the model problem in variational form and use results from [VV02] where general linear vibration models in variational form are considered. We show that it makes virtually no difference if the Rayleigh model is used instead of the Euler-Bernoulli model.

We observe that in [ZVV04] the Timoshenko beam theory is used which is a better model but the interface condition for the tip body is more realistic in

[BDV14].

Vertical structures are the focus of Chapter 6. Model problems for high rise buildings, towers and other vertical "beams" e.g. industrial chimneys are analyzed. Beam models appear to be popular but in the articles considered no justification was offered. In this chapter we compare Timoshenko, Rayleigh and Euler-Bernoulli models for vertical structures. Beam models for chimneys seem reasonable but beam models for buildings are questionable. (Tall buildings are often modeled as vertical beams, see e.g. [HV07] and [WFH01] and the references in [WFH01].)

Recall that our objective is to compare models. To start, we consider the most general linear model for a vertical beam where gravity is taken into account.

Simulation is necessary to determine displacements, acceleration and stresses in structures subjected to earthquake loads or wind loads.

In Section 6.2 we compare beam models for buildings. Simulation of Timoshenko and Rayleigh models were done. We did not consider existence of solutions and convergence since this is done in Section 6.4 for a more complex model.

In Section 6.3 a vertical Euler-Bernoulli model but with concentrated masses to represent the floors in [WFH01] is mentioned. We consider multiple beam models with concentrated masses to represent the floors, which we believe is a better model.

A vertical slender structure with resilient seating is the topic in Section 6.4. We consider hybrid Timoshenko and Rayleigh beam models for the structure. A Rayleigh model is in [LVV05], where the origin of the problem is discussed and references are given. Models that correspond to Newland's lumped parameter system models [New89, p 129-132] and [New84] are presented. The structure is modelled as a beam mounted vertically and gravity is taken into account. Modeling the behavior of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations. The modeling is done in great detail in [Lab06] where the finite element method is used for a comprehensive modal analysis, and convergence of finite element approximations for eigenvalue problems is considered. Existence results can be found in [LVV09b].

In Chapter 7 plate models and model problems for elastic solids are considered.

From the introduction to the article [Wu05], we conclude that this is the first result on convergence of the finite element method for the vibration of a Reissner-Mindlin plate. With no damping or viscous damping the theory in [BV13] can be applied. In [Wu05] the fully discrete problem is formulated in Section 3 and the explicit scheme explained. The algorithm is virtually the same as central differences. Therefore the theory is a special case of the theory in Section 4.6 of this dissertation. In [LVV09c] a plate-beam system is considered. In out opinion the notation is preferable and we followed their presentation for the plate problem in [Wu05].

Linear elasto-dynamics [Wu03] is also briefly discussed in Section 7.3. The theory is almost the same as for the plate problem.

0.2 Results

As this is a literature study, there are few results that are really new. The main conclusion is that there is scope for research in modeling and finite element analysis and implementation.

Recall that one objective was to compare models. In Chapter 6 vertical beam models with the effect of gravity were compared. To start, we considered the most general linear model and then simplified to obtain workable Timoshenko and Rayleigh models. It turned out that there is a lot more to do than anticipated. It now seems that a complete dissertation can be written on beam models for vertical structures. The work done in Section 6.3 is promising and is worth following up. The comparison between the Timoshenko and Rayleigh beams is only a start. Also, multi-beam models should be compared to single beam models.

The error estimates obtained in [BV13] require weaker regularity assumptions than other articles but the assumptions are still too restrictive. Consequently convergence results for even classical solutions are not available. Note that the existence theorems in Chapter 2, which are convenient to apply, guarantee only weak solutions.

0.3 Further research

The comparison of the Timoshenko and Rayleigh models in Section 6.2 is an obvious topic for further research. Systematic experiments are required to reach a conclusion. Simulation of earthquakes is another possibility since the excitation is different from that caused by wind.

Another topic which should definitely be investigated is the comparison of multi-beam models with single beam models for buildings. A vertical Euler-Bernoulli model but with concentrated masses to represent the floors, is used in [WFH01]. Our proposed multiple beam model should be compared to the model in [WFH01] and single beam models.

Clearly it is essential to have reliable approximations for solutions, and the problems mentioned with error estimates should be addressed. For a start a continued search of the literature is indicated.

Contents

Chapter 1

Vibrating elastic bodies and structures

1.1 Introduction

The following paragraph is from [Wu03]:

"Structural dynamics is one of the most active and attractive research fields with a long history of development. Elastic multi-structures are usually composed of a number of elastic substructures with the same or different dimensions (three-dimensional bodies, plates, rods, etc.) coupled by some proper junctions, frequently used in automobile and aeroplane structures and motion- and force-transmitting machines and mechanisms."

A well-known challenge to engineers is to prevent or to minimize the effect of unwanted vibrations. To do this effectively, theory, mathematical analysis, simulations and numerical experiments are indispensable and motivates ongoing research.

The structures mentioned above are often modelled as ordinary or partial differential equations or even a (complex) system of partial differential equations. It has become common practice to use numerical methods (often pre-programmed computer software) to solve problems.

Research is necessary to improve mathematical models, compare existing or new mathematical models and to improve or create numerical algorithms. This is particularly relevant for elastic multi-structure problems. The finite element method is used for steady-state problems, modal value analysis of a structure and simulating the transient response. Clearly it is essential to analyze the convergence of the finite element approximation and derive error estimates.

Many articles have been published on error estimates and convergence for hyperbolic partial differential equations, i.e. the multi-dimensional wave equation. But less has been published on problems from continuum mechanics such as vibrating beams, plates and 3-dimensional elastic bodies.

As far as can be ascertained not much has been published on error estimates and finite element approximations for hybrid models and structures or linked systems. This is also the case for structures with boundary damping or damping in joints.

In the research for the Masters degree, principally a literature study, the focus is on the Finite Element Method applied to models of hybrid or linked vibrating systems. Specific attention was given to:

- comparing different models for the same structure;
- determining whether error estimates are available in general and any conditions necessary for convergence;
- error analysis and convergence of solutions;
- existence of solutions.

In Chapter 2 we present general existence results for hyperbolic equations which may be applied to the model problems. In Chapter 3 we study the existing convergence theory for hyperbolic equations.

Hybrid beam models are considered in Chapters 4 and 5.

Vertical structures are the focus of Chapter 6. Model problems for high rise buildings, towers and other vertical "beams" e.g. steel chimneys are analyzed. Here we compare the Timoshenko theory to the Rayleigh and Euler-Bernoulli theories.

In Chapter 7 **plate models** and model problems for elastic solids are considered.

Before we consider the research objectives in detail, it is necessary to consider the theory of vibrating beams, plates and multi-dimensional elastic solids.

1.2 Beam theory

1.2.1 Transverse vibration

Consider the transverse vibration of a prismatic slender solid with density ρ, length ℓ and cross-sectional area A (see Appendix F). Recall the assumption that every cross-section executes a rigid motion: We assume that the position of a point (x, y, z) at time t is given by

$$\bar{R}(x, y, z, t) = \bar{r}_0(x, t) + y\bar{e}_y(x, t) + z\bar{e}_z(x, t),$$

where $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_z$ are orthogonal unit vectors moving with the cross-section.

Suppose the points of reference for the space and the reference configuration coincide and let $\bar{e}_i$ be the fixed right handed system.

For pure transverse vibration the motion of a centroid is given by

$$\bar{r}_0(x, t) = x\bar{e}_1 + w(x, t)\bar{e}_2,$$

with $w(x, t)$ transversal displacement in the $X - y$ plane.

The assumption of transverse vibration as formulated above does not correspond to reality unless the displacements are very small. To allow for significant transverse displacements it is necessary to consider two-dimensional motion:

$$\bar{r}_0(x, t) = (x + u(x, t))\bar{e}_1 + w(x, t)\bar{e}_2.$$

Consider the rotation of cross-sections about $\bar{e}_3$, i.e. $\bar{e}_z(t) = \bar{e}_3$ for each t. (We say that the cross-sections rotate about the $z-$axis.) Note that the motion of each point is in a plane perpendicular to $\bar{e}_3$. To be specific, we assume that

$$\bar{r}(x, t) = y\bar{e}_y(x, t) + z\bar{e}_3,$$

where

$$\bar{e}_y(x, t) = -\sin\phi(x, t)\bar{e}_1 + \cos\phi(x, t)\bar{e}_2.$$

The unit vector $\bar{e}_y(x, t)$ is used to describe the rotation of a cross section and ϕ is the angle of rotation of the cross-section.

Angular momentum density

If $\mathcal{D}$ is symmetric with respect to the $y-$axis, then the angular momentum density (see Appendix F) is

$$\bar{H}(x,t) = \rho I \ \partial_t \phi(x,t)\bar{e}_3, \tag{1.2.1}$$

where I is the area moment of inertia $\left(I = \iint_D y^2 dA\right)$.

To derive the equations of motion, we use Propositions F.3 and F.4 in Appendix F. Note that

$$\partial_t^2 \bar{r}_0 = \partial_t^2 u \bar{e}_1 + \partial_t^2 w \bar{e}_2, \quad \partial_t \bar{H} = \rho I \ \partial_t^2 \phi \bar{e}_3$$

and

$$\partial_x \bar{r}_0 \times \bar{F} = (1 + \partial_x u) F_2 \bar{e}_3 - (1 + \partial_x u) F_3 \bar{e}_2 - \partial_x w F_1 \bar{e}_3 + \partial_x w F_3 \bar{e}_1.$$

Note that

$$\begin{aligned} \partial_t^2 \bar{r}_0 \cdot \bar{e}_3 &= 0, \\ \partial_t \bar{H} \cdot \bar{e}_1 &= 0, \\ \partial_t \bar{H}_0 \cdot \bar{e}_2 &= 0. \end{aligned}$$

The assumed motion is only possible if there are no forces in the z-direction: $\bar{P} \cdot \bar{e}_3 = 0$.

Equations of motion

$$\begin{aligned} \rho A \partial_t^2 u &= \partial_x F_1 + P_1, && (1.2.2) \\ \rho A \partial_t^2 w &= \partial_x F_2 + P_2, && (1.2.3) \\ \rho I \partial_t^2 \phi &= (1 + \partial_x u) F_2 - \partial_x w F_1 + \partial_x M_3. && (1.2.4) \end{aligned}$$

It also follows that

$$\begin{aligned} 0 &= \partial_x F_3, \\ 0 &= \partial_x w F_3 + \partial_x M_1, \\ 0 &= -(1 + \partial_x u) F_3 + \partial_x M_2. && (1.2.5) \end{aligned}$$

The system of equations above will be satisfied if $F_3 = M_1 = M_2 = 0$ at the boundary points. As a consequence $F_3 = M_1 = M_2 = 0$ throughout the beam.

Constitutive equations and linear approximation

To obtain a mathematical model the equations of motion must be supplemented by constitutive equations. A natural choice for shear and bending are the constitutive equations of the Timoshenko theory (see e.g. [Inm94], [Cow66]) and for the longitudinal strain Hooke's law in its simplest form. However, the tensile force is not equal to F_1 and the shear force is not equal to F_2 due to the rotation of the cross-section.

Linear approximation

If we assume that ϕ and $\partial_x w$ are both small, then the following constitutive equations may be used

$$\begin{aligned} M_3 &= EI\partial_x\phi, \\ F_2 &= \kappa^2 AG(\partial_x w - \phi), \\ F_1 &= AE\partial_x u. \end{aligned}$$

1.2.2 Timoshenko beam theory

The equations of motion are derived in the previous subsection. Consider the case where $\partial_t^2 u = 0$ and $\partial_t P_1 = 0$. The equations of motion are

$$0 = \partial_x S + P_1, \tag{1.2.6}$$
$$\rho A\partial_t^2 w = \partial_x V + P_2, \tag{1.2.7}$$
$$\rho I\partial_t^2 \phi = (1 + \partial_x u)V + \partial_x M + L, \tag{1.2.8}$$

where $M = M_3,\ V = F_2,\ S = F_1$ and $L = -\partial_x w S$.

If $P_1 = 0$ and $S(0)$ or $S(\ell) = 0$, then $S = 0$ and consequently $\partial_x u = 0$ and $L = 0$. The simplified equations of motion are

$$\rho A\partial_t^2 w = \partial_x V + P_2, \tag{1.2.9}$$
$$\rho I\partial_t^2 \phi = V + \partial_x M. \tag{1.2.10}$$

Remarks

1. Equations (1.2.9) and (1.2.10) are the equations of motion for the well known Timoshenko beam.

2. Note the unusual term L present in Equation (1.2.8). This term represents a moment density term that will be used in some of the mathematical models.

3. The term $\rho I \partial_t^2 \phi$ in Equation (1.2.8) is usually referred to as the rotary inertia term.

The following constitutive equations for the moment M and the shear force V are used in Timoshenko's theory.

$$\begin{aligned} M &= EI\partial_x\phi, && (1.2.11)\\ V &= AG\kappa^2\big(\partial_x w - \phi\big). && (1.2.12) \end{aligned}$$

In these equations, E and G are elastic constants and κ^2 the shear coefficient or shear correction factor. We refer the reader to [Tim37, p 337-338], [Fun65, p 323-324],[Cow66], [Inm94, p 337-338] and [New89, p 392-395].

The Euler-Bernoulli theory for a beam originated in the 18-th century. An improvement was introduced by Rayleigh in the 19-th century. In 1921, Timoshenko proposed his theory where shear is taken into account.

Substituting the constitutive equations (1.2.11) and (1.2.12) into the equations of motion (1.2.7) and (1.2.8), yield the well known Timoshenko model for the free vibration of a beam.

$$\begin{aligned} \rho A\partial_t^2 w &= \partial_x\big(AG\kappa^2(\partial_x w - \phi)\big) + P_2,\\ \rho I\partial_t^2 \phi &= AG\kappa^2\big(\partial_x w - \phi\big) + \partial_x\big(EI\partial_x\phi\big) + L. \end{aligned}$$

When confronted by complex interface conditions, it is advisable to use the equations of motion and constitutive equations (Equations (1.2.7) – (1.2.12)), rather than the partial differential equations above.

Applications are given in Chapters 4 and 6.

1.2.3 The Euler-Bernoulli and Rayleigh models

The **Rayleigh model** can be derived formally from the Timoshenko model. Combining Equations (1.2.7) and (1.2.8), we find that

$$\rho A\partial_t^2 w = \rho I\partial_t^2\partial_x\phi - \partial_x^2 M + P - \partial_x L.$$

For this model, it is assumed that a cross section remains perpendicular to the neutral plane. This implies that $\partial_x w = \phi$, and the equation reduces to

$$\rho A \partial_t^2 w = \rho I \partial_t^2 \partial_x^2 w - \partial_x^2 M + P - \partial_x L.$$

This is the equation of motion for the Rayleigh model. The constitutive equation for the shear force V is now redundant and the constitutive equation for the bending moment is

$$M = EI\partial_x^2 w.$$

The partial differential equation for the Rayleigh model is

$$\rho A \partial_t^2 w - \rho I \partial_t^2 \partial_x^2 w = -EI\partial_x^4 w + P - \partial_x L.$$

The **Euler-Bernoulli model** is a special case of the Rayleigh model where rotary inertia is ignored and the result is

$$\rho A \partial_t^2 w = -EI\partial_x^4 w + P - \partial_x L.$$

Applications are given in Chapters 5 and 6.

1.2.4 Dimensionless form

In this subsection we write the equations of motion and constitutive equations in dimensionless form. Set

$$\tau = \frac{t}{t_0}, \quad \xi = \frac{x}{\ell}, \quad w^*(\xi, \tau) = \frac{w(x,t)}{\ell} \quad \text{and} \quad \phi^*(\xi, \tau) = \phi(x,t).$$

We introduce the dimensionless constants

$$\alpha = \frac{A\ell^2}{I}, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}.$$

The constant γ depends on the elastic constants and the shear correction factor κ^2 that is determined by the shape of the cross section. The values of κ^2 range between $\frac{1}{2}$ and 1 (see [Cow66] or [APBS78, p 173]). On the other hand, for isotropic materials we assume that $\frac{G}{E} = \frac{1}{2(1+\nu)}$ (see [Myk66, p

174] or [Fun65, Sec 7.2]). Realistic values for γ range between $\frac{1}{6}$ and $\frac{1}{2}$. $\Big($Timoshenko ([Tim37, p 342]) used $\frac{2}{3}$ for κ^2 and $\frac{G}{E} = \frac{3}{8}$.$\Big)$

The constant α is subject to significant variation. With r the radius of gyration we have $\alpha = \frac{A\ell^2}{I} = \frac{\ell^2}{r^2}$.

The forces and moments in dimensionless form are

$$L^*(\xi,\tau) = \frac{L(x,t)}{G\kappa^2 A}, \quad P_i^*(\xi,\tau) = \frac{\ell P_i(x,t)}{G\kappa^2 A},$$

$$S^*(\xi,\tau) = \frac{S(x,t)}{G\kappa^2 A} \quad \text{so} \quad L^*(\xi,\tau) = -\partial_x w S^*,$$

$$V^*(\xi,\tau) = \frac{V(x,t)}{G\kappa^2 A} \quad \text{and} \quad M^*(\xi,\tau) = \frac{M(x,t)}{\ell G\kappa^2 A}.$$

A convenient choice for t_0 is

$$t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}\,.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

Timoshenko model

$$0 = \partial_x S + P_1 \tag{1.2.13}$$

$$\partial_t^2 w = \partial_x V + P_2, \tag{1.2.14}$$

$$\frac{1}{\alpha}\,\partial_t^2 \phi = (1 + \partial_x u)V + \partial_x M + L, \tag{1.2.15}$$

$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{1.2.16}$$

$$V = \partial_x w - \phi. \tag{1.2.17}$$

Rayleigh model

$$
\begin{aligned}
\partial_t^2 w &= \partial_x V + P_2, && (1.2.18)\\
\frac{1}{\alpha}\,\partial_t^2 \partial_x w &= (1+\partial_x u)V + \partial_x M + L, && (1.2.19)\\
M &= \frac{1}{\beta}\,\partial_x^2 w. && (1.2.20)
\end{aligned}
$$

Remark

In many applications $P_1, S, \partial_x u$ and L are zero. The equations of motion are then simplified.

Timoshenko model

$$
\begin{aligned}
\partial_t^2 w &= \partial_x V + P_2, && (1.2.21)\\
\frac{1}{\alpha}\,\partial_t^2 \phi &= V + \partial_x M. && (1.2.22)
\end{aligned}
$$

Rayleigh model

$$
\begin{aligned}
\partial_t^2 w &= \partial_x V + P_2, && (1.2.23)\\
\frac{1}{\alpha}\,\partial_t^2 \partial_x w &= V + \partial_x M. && (1.2.24)
\end{aligned}
$$

Euler-Bernoulli model

The Euler-Bernoulli model is obtained from the Rayleigh model by omitting the rotary inertia term $\frac{1}{\alpha}\,\partial_t^2 \partial_x w$.

Remark

Note that the rotary inertia term is simply omitted. It is not correct to reason that $\frac{1}{\alpha} \approx 0$, since that would imply that $\frac{1}{\beta} \approx 0$.

Since the Euler-Bernoulli model is a special case of the Rayleigh model, we will not refer to this model again in the theoretical discussions that follow. To obtain results for the Euler-Bernoulli model, one uses the relevant equations for the Rayleigh model with the modification mentioned above.

1.2.5 Timoshenko model vs Rayleigh or Euler-Bernoulli models

In [SP06] and [LVV09a] the Timoshenko theory is compared to a multi-dimensional model. It was found that the Timoshenko theory is an excellent approximation in the case of beam applications, i.e. for transverse loads. In the articles [VV06] and [LVV09a] the Timoshenko model is compared to Rayleigh or Euler-Bernoulli. The Rayleigh and Euler-Bernoulli models can be useful when the parameter α is large. Depending on the initial data or the manner of excitation, a value of α of anything between 2000 and 5000 may be sufficient.

When the weight of a beam is taken into account it is necessary to distinguish between vertical and horizontal beams. If the beam is horizontal then $P_2 = \rho A g + q(x,t)$, where q is the load. If the beam is vertical then $P_1 = -\rho g A$ and hence $S' = \rho g A$. See Chapters 4, 5 and 6 for more detail.

1.3 Plate theory

1.3.1 Equations of motion

Consider small transverse vibration of a thin plate with thickness h and density ρ. The reference configuration for the plate is a domain Ω in the plane.

The transverse displacement of $\boldsymbol{x}$ at time t is denoted by $w(\boldsymbol{x},t)$. The angle between a "material line" and a perpendicular to the plane is $\psi(\boldsymbol{x},t)$ and the angle between the projection of the material line in the plane and the unit vector $\boldsymbol{e}_1$ is $\phi(\boldsymbol{x},t)$ (see [Rei88, Sec 3.2, Sec 3.5]). For a linear model $\boldsymbol{\psi}$ is approximated by

$$\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T = [\psi \cos\phi \quad \psi \sin\phi]^T.$$

The equations of motion (see [Min51] and [Rei88, p 152]) are given by

$$\begin{aligned} \rho h \partial_t^2 w &= \operatorname{div} \boldsymbol{Q} + q, & (1.3.1)\\ \rho I \partial_t^2 \boldsymbol{\psi} &= \operatorname{div} M - \boldsymbol{Q}, & (1.3.2) \end{aligned}$$

where $I = \dfrac{h^3}{12}$ is the length moment of inertia. $\boldsymbol{Q}$ represents a force density, $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a moment density and q an external load on the plate.

1.3.2 The Reissner-Mindlin and Kirchhoff models

Constitutive equations

We restrict our attention to the linear theory. The following assumptions are made for small curvature and small partial derivatives (see [Rei88, p 61] and [Min51]).

$$\boldsymbol{Q} = \kappa^2 G h (\nabla w + \boldsymbol{\psi}), \tag{1.3.3}$$

where G is the shear modulus and κ^2 a correction factor.

$$M = \frac{1}{2} D \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix}. \tag{1.3.4}$$

D is a measure of stiffness for the plate and is given by

$$D = \frac{EI}{1-\nu^2},$$

where E is Young's modulus and ν Poisson's ratio.

The correction factor κ^2 is chosen in such a way that the solution of the plate model compares well with the solution of the three-dimensional model. (See [Min51]).

The equations of motion and the constitutive equations above are known as the **Reissner-Mindlin plate model**.

The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [Rei88, p 152] and [Min51]). In our approach these partial differential equations are not used.

For classical plate theory, $\boldsymbol{\psi}$ is replaced by $-\nabla w$ and the constitutive equation for $\boldsymbol{Q}$ is no longer necessary. This is sometimes referred to as the **Kirchhoff plate model**.

1.3.3 Dimensionless forms

We introduce the dimensionless variables

$$\tau = \frac{t}{t_0}, \quad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell}\,,$$

where ℓ is a suitable length and t_0 must still be specified.

The dimensionless variables, with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, are

$$w^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell}\right) w(\boldsymbol{x}, t), \quad \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) = \boldsymbol{\psi}(\boldsymbol{x}, t),$$

$$\boldsymbol{Q}^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell G\kappa^2}\right) \boldsymbol{Q}(\boldsymbol{x}, t), \quad M^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell^2 G\kappa^2}\right) M(\boldsymbol{x}, t)$$

$$\text{and} \quad q^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{G\kappa^2}\right) q(\boldsymbol{x}, t).$$

The dimensionless constants that are used are given by

$$h_p = \frac{h}{\ell}\,, \quad I_p = \frac{h_p^3}{12} \quad \text{and} \quad \beta_p = \frac{\ell^3 G\kappa^2}{EI}\,.$$

The constant h_p denotes the dimensionless thickness of the plate and I_p the dimensionless length moment of inertia.

We choose $t_0 = \ell\sqrt{\dfrac{\rho}{G\kappa^2}}$ (as in Subsection 1.2.4).

Using the original notation for the corresponding dimensionless quantities, the equations of motion and constitutive equations in dimensionless form are presented below.

Reissner-Mindlin plate model

$$h_p\,\partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{1.3.5}$$

$$I_p\,\partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \boldsymbol{Q}, \tag{1.3.6}$$

$$\boldsymbol{Q} = h_p\big(\nabla w + \boldsymbol{\psi}\big), \tag{1.3.7}$$

$$M = \frac{1}{2\beta_p(1-\nu^2)} \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix}. \tag{1.3.8}$$

Classical plate model

As before, replacing ψ by $-\nabla w$, yields the Kirchoff theory

$$h_p\,\partial_t^2 w \;=\; \operatorname{div}\boldsymbol{Q} + q, \tag{1.3.9}$$

$$I_p\,\partial_t^2\left(\boldsymbol{\nabla} w\right) \;=\; \boldsymbol{Q} - \operatorname{div} M, \tag{1.3.10}$$

$$M = -\frac{1}{\beta_p(1-\nu^2)}\begin{bmatrix} \left(\partial_1^2 w + \nu\partial_2^2\right)w & (1-\nu)\partial_1\partial_2 w \\ (1-\nu)\partial_1\partial_2 w & \left(\partial_2^2 w + \nu\partial_1^2\right)w \end{bmatrix}. \tag{1.3.11}$$

Generally the rotary inertia term $I_p\partial_t^2\big(\nabla w\big)$ in Equation (1.3.10) is ignored.

The two models for the vibration of a plate is considered in Chapter 7.

The partial differential equations for classical plate theory became known when Sophie Germain received a prize in 1816 for the derivation but only after Lagrange made some corrections. The boundary conditions posed a challenge and it took another 34 years before the theory was complete. Contributions were made by Kirchhoff and Lord Kelvin (William Thompson), see [Rei88].

From 1945 to 1950 improvements to classical plate theory were made by E. Reissner, H. Hencky, Y. S. Uflyand and R. D. Mindlin (see [Min51] for references). The improved theory became known as the Reissner-Mindlin theory.

1.4 Multi-dimensional elasto-dynamics

Consider the case of so called small vibrations of a three-dimensional elastic body. The assumption is that the local displacements are small. The equation of motion is derived from the conservation law for momentum

$$\rho\partial_t^2\mathbf{u} = \operatorname{div}\,\mathbf{T} + \rho\mathbf{b},$$

where $\mathbf{u}$ is the displacement, ρ the density, $\mathbf{T}$ the stress tensor and $\mathbf{b}$ the body force (density), see [Fun65]. It is possible to derive a system of partial differential equations using Hooke's law

$$\mathbf{T} \;=\; \frac{E}{1+\nu}\,\boldsymbol{\mathcal{E}} + \frac{\nu E}{(1+\nu)(1-2\nu)}\,\operatorname{tr}(\boldsymbol{\mathcal{E}})I,$$

where $\boldsymbol{\mathcal{E}}$ is the strain tensor, E Young's modulus and ν Poisson's ratio.

Linear vibration problem

Suppose $\Omega \subset E_3$ is the reference configuration for a solid executing small vibrations. The boundary of Ω consists of two parts Σ and Γ. The problem is to find $\mathbf{u}$ such that

> The equation of motion is satisfied in Ω;
>
> Hooke's law is satisfied in Ω;
>
> The specified displacement for $\mathbf{u}$ is satisfied on Σ;
>
> The specified traction $\mathbf{T}n$ is satisfied on Γ.

Variational forms and existence of solutions are considered in Section 7.2.

Chapter 2

Existence of solutions

2.1 Variational form of a model problem

To illustrate the variational form and weak variational form, we use the example of a cantilever beam.

Problem R

Consider the Rayleigh model for the vibration of a beam, Equations (1.2.20), (1.2.23) and (1.2.24).

To obtain the variational form, multiply the (dimensionless) equation of motion (1.2.23) by an arbitrary function u and integrate.

$$\int_0^1 \partial_t^2 w(\cdot,t)u = -\int_0^1 V(\cdot,t)u' + V(1,t)u'(1) - V(0,t)u(0) + \int_0^1 P_2(\cdot,t)u. \tag{2.1.1}$$

Next, multiply the (dimensionless) equation of motion (1.2.24) (where $L = 0$), by an arbitrary function v and integrate.

$$\frac{1}{\alpha}\int_0^1 \partial_t^2 \partial_x w(\cdot,t)v = -\int_0^1 M(\cdot,t)v' + M(1,t)v(1) - M(0,t)v(0) + \int_0^1 V(\cdot,t)v. \tag{2.1.2}$$

Next we use the fact that $M(1,t) = V(1,t) = 0$ for a Cantilever beam. Let $v = u'$, and add (2.1.1) and (2.1.2):

$$\int_0^1 \partial_t^2 w(\cdot,t)u + \frac{1}{\alpha}\int_0^1 \partial_t^2\partial_x w(\cdot,t)u'$$
$$= -\int_0^1 M(\cdot,t)u'' + \int_0^1 q(\cdot,t)u - M(0,t)u'(0) - V(0,t)u(0),$$

where $q = P_2$. For a Cantilever beam $u(0) = u'(0) = 0$. The space of these test functions is denoted by $T[0,1]$.

Test functions

$$T[0,1] = \{u \in C^1[0,1] : u(0) = u'(0) = 0\}.$$

Substituting the constitutive equation, we obtain

$$\int_0^1 \partial_t^2 wu + \frac{1}{\alpha}\int_0^1 \partial_t^2\partial_x wu' = -\frac{1}{\beta}\int_0^1 \partial_x^2 wu'' + \int_0^1 qu \quad \text{for each } u \in T[0,1].$$

Notation

The inner product of $\mathcal{L}^2(0,1)$ is denoted by (f,g), i.e.

$$(f,g) = \int_0^1 fg.$$

Bilinear forms:

$$c(u,v) = \int_0^1 uv + \int_0^1 \frac{1}{\alpha}u'v',$$
$$b(u,v) = \int_0^1 \frac{1}{\beta}u''v''.$$

Problem RV

Find a function w such that for each $t > 0$, $w(\cdot,t) \in T[0,1]$ and

$$c(\partial_t^2 w(\cdot,t),v) + b(w(\cdot,t),v) = (q(\cdot,t),v)$$

for each $v \in T[0,1]$, with initial conditions $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_d$.

2.2 Weak variational form

For the weak variational form it is necessary to consider Sobolev spaces, treated briefly in Appendix A.

We define a function $u(t) = w(\cdot, t)$ and spaces V, W and X as follows:

> V is the closure of the test functions in the Sobolev space $H^2(0,1)$,
>
> W is the closure of the test functions in $H^1(0,1)$,
>
> and X is $\mathcal{L}^2(0,1)$.

To analyse a differential equation of a function with values in a general Hilbert space, we need the definition of a derivative. Let J be a bounded or unbounded interval. J is either an open interval containing zero or it is of the form $[0,\tau)$ or $[0,\infty)$. Let Y be any Hilbert space and consider a function u on J with values in Y.

Definition Derivative

Let t be any interior point of J. Suppose there exists a $v \in Y$ such that

$$\lim_{h \to 0} \left\| h^{-1}\big(u(t+h) - u(t)\big) - v \right\|_Y = 0,$$

then v is the derivative of u at t. We write $u'(t)$ for the derivative and $u'(t) \in Y$ to show that the derivative exists with respect to the norm of Y. The derivative (function) u' is defined in the usual way as $u'(t)$ for every $t \in J$, with u'' defined by $(u')'$.

Notation $C^k\big(J, Y\big)$

$u \in C^k\big(J, Y\big)$ if $u^{(k)}(t) \in Y$ for each $t \in J$.

With the definitions of derivatives of functions with values in a Hilbert space available, it is possible to formulate the weak variational problem.

Problem RW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (q(\cdot, t), v) \quad \text{for each} \quad v \in V,$$

while $u(0) = w_0$ and $u'(0) = w_d$.

2.3 General existence results

For existence of a solution we use the theory in [VV02].

Consider Hilbert spaces V, W and X where $V \subset W \subset X$. The inner products and norms for V, W and X are $b(\cdot,\cdot)$, $c(\cdot,\ \cdot)$, and $(\cdot,\ \cdot)_X$ and $\|\cdot\|_V$, $\|\cdot\|_W$ and $\|\cdot\|_X$ respectively.

The following general problem is considered in [VV02].

Problem PG

Given a function $f : J \to X$, find a function $u \in C(J, V)$ such that u' is continuous at 0 and for each $t \in J$, $u'(t) \in V$, $u''(t) \in W$,

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = \big(f(t), v\big)_X \quad \text{for each } v \in V$$
$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Note

For Problem RW the spaces V, W and X are defined in Section 2.2 and the bilinear form $a = 0$.

Assumptions

The following assumptions are made for the theory in [VV02].

E1 V is dense in W and W is dense in X.

E2 There exist a positive constant κ_1 such that $\|v\|_W \leq \kappa_1\|v\|_V$ for each $v \in V$.

E3 There exist a positive constant κ_2 such that $\|w\|_X \leq \kappa_2\|w\|_W$ for each $w \in W$.

E4 The bilinear form a is non-negative and symmetric.

2.3.1 General damping term

Theorem 2.3.1. *Suppose Assumptions E1, E2, E3 and E4 hold. If for $u_0 \in V$ and $u_1 \in V$ there exists some $y \in W$ such that*

$$b\big(u_0, v\big) + a\big(u_1, v\big) = c\big(y, v\big) \quad \textit{for each } v \in V, \tag{2.3.1}$$

then for each $f \in C^1\big([0,\tau);X\big)$ *there exists a unique solution*

$$u \in C\big([0,\tau);V\big) \cap C^1\big([0,\tau);W\big) \cap C^1\big((0,\tau);V\big) \cap C^2\big((0,\tau);W\big),$$

for Problem G. If $f = 0$ *then* $u \in C^1\big([0,\infty);V\big) \cap C^2\big([0,\infty);W\big)$.

Proof. See [VV02]. □

The theorem above is convenient to use since the assumptions are in terms of the bilinear forms a, b and c and it is not necessary to construct linear operators with suitable properties as in [Eva98].

Problem G is equivalent to a first order system in the product space $H = V \times W$. For $x \in H$ we write $x = \langle x_1, x_2\rangle$ with $x_1 \in V$ and $x_2 \in W$. The system is

$$\begin{aligned} &x_2 = x_1', \\ &c(x_2', v) + a(x_2, v) + b(x_1, v) = (f(t), v)_X \quad \text{for each } v \in V \end{aligned}$$

with $\langle x_1(0), x_2(0)\rangle = \langle u_0, u_1\rangle$.

To make this precise a linear operator A is constructed in [VV02]. The operator is determined by the three bilinear forms c, b and a. The domain is $\mathcal{D}(A) \subset V \times W$ and $\langle u_0, u_1\rangle \in \mathcal{D}(A)$ if and only if u_0, $u_1 \in V$ and there exists a $y \in W$ such that condition (2.3.1) holds.

Now, if u is a solution of Problem G, $g(t) = \langle 0, f(t)\rangle$ and $w(t) = \langle u(t), u'(t)\rangle$, then

$$w' = Aw + g \text{ with } w(0) = \langle u_0, u_1\rangle. \tag{2.3.2}$$

Conversely, if w is a solution of the initial value problem (2.3.2), then w_1 is a solution of Problem G. To determine the solvability of the problem in (2.3.2) semigroup theory can be used. One possibility is the book [Paz83].

Due to Assumptions E1, E2 and E4, A is the infinitesimal generator of a C_0 semigroup of contractions. For the existence of a solution of (2.3.2) conditions are imposed on $w(0)$ and g: $w(0) \in \mathcal{D}(A)$ and $g \in C^1(J,H)$. Now $w(0) \in \mathcal{D}(A)$ is equivalent to the condition imposed in the theorem and $g \in C^1(J,H)$ if and only if $f \in C^1(J,W)$.

2.3.2 Weak damping and strong damping

In [VV02] three types of damping are identified. It is shown that the type of damping determines the assumptions used in proofs and the properties of the solution.

Stronger existence results can be obtained if the damping term a has special properties. For example, Condition (2.3.1) may be replaced by simpler conditions.

We first consider so called **weak damping**.

Assumption E4W

The bilinear form a is nonnegative, symmetric and bounded on W, i.e.

$$|a(u,v)| \leq K_a \|u\|_W \|v\|_W.$$

Definition

$$E_b = \{\, x \in V \;\big|\; \text{there exists a } y \in W \text{ such that } c(y,v) = b(x,v) \text{ for all } v \in V \,\}.$$

Due to Assumptions E1, E2 and E4W, $\mathcal{D}(A) = E_b \times V$ and A is the infinitesimal generator of a C_0 group of contractions (see [VV02]).

Theorem 2.3.2. *Suppose Assumptions E1, E2, E3 and E4 hold and a is bounded with respect to the norm in W. Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1\big(J, V\big) \cap C^2\big(J, W\big)$$

for Problem PG for each $u_0 \in E_b$, $u_1 \in V$ and each $f \in C^1\big(J, X\big)$. If $f = 0$ then $u \in C^1\big((-\infty,\infty), V\big) \cap C^2\big((-\infty,\infty), W\big)$.

Proof. See [VV02] □

Remark

The theorem above is valid for the case where there is no damping, i.e. $a = 0$. This is the case in Problem RW (in Section 2.2).

Next we consider **strong damping** where the bilinear form a satisfies an additional condition. In this case it is also possible to improve the conclusion of Theorem 2.3.1.

Assumption E5S

The bilinear form a is positive definite, i.e. there exists a positive constant C_a such that $a(u,u) \geq C_a \|u\|_V^2$ for each $u \in V$.

Note that the operator A is the infinitesimal generator of an analytic semigroup (see [VV02]).

Theorem 2.3.3. *Suppose Assumptions E1, E2, E3, E4 and E5S hold. If f is locally Lipschitz continuous on $(0,\tau)$ and $u_0 \in V$, $u_1 \in W$, then Problem G has a unique solution*

$$u \in C([0,\tau), V) \cap C^1([0,\tau), W) \cap C^2((0,\tau), W).$$

If $f = 0$ then $u \in C\big([0,\infty); V\big) \cap C^1\big([0,\infty); W\big) \cap C^\infty\big((0,\infty); V\big)$.

Proof. See [VV02]. $\square$

Remarks

1. Locally Lipschitz may be replaced by $f \in C^1(0,\tau)$ in Theorem 2.3.3. For the corresponding result in [Paz83] f is assumed to be locally Hölder continuous.

2. It is important to note that boundary damping is neither weak nor strong. (See Chapter 4).

Existence results may also be found in other publications e.g. [Eva98] but the results in [VV02] are convenient for this dissertation as we explain below. (The model problems in [Eva98] do not include damping.)

Application to Problem RW

In this problem $a = 0$ and Assumption E4W holds trivially. Consequently we may use Theorem 2.3.2.

It is necessary to show that Assumptions E1, E2 and E3 hold. We will not do it since Problem RW is a special case of the problem in Chapter 5.

Chapter 3

Galerkin finite element approximation

3.1 Galerkin approximation

In this section we consider the Galerkin approximation. Suppose S^h is a finite dimensional subspace of V. The notation is customary, the symbol h has no meaning at this stage.

Problem G^h

Given a function $f : J \to X$, find a function $u_h \in C^2(J)$ such that u_h' is continuous at 0 and for each $t \in J$, $u_h(t) \in S^h$ and

$$c\big(u_h''(t), v\big) + a\big(u_h'(t), v\big) + b\big(u_h(t), v\big) = \big(f(t), v\big)_X \text{ for each } v \in S^h, \tag{3.1.1}$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

The initial values u_0^h and u_1^h are elements of S^h as close as possible to u_0 and u_1. The possibilities for the choices are discussed in Sections 3.3 and 3.4.

Example

Consider Problem RW in Section 2.2. We construct a finite dimensional subspace S^h using piecewise hermite cubics. See Appendix D.

Problem RW^h

Find a function w_h such that for each $t > 0$, $w_h(x,t) = \sum_{j=1}^{n} w_j(t)\delta_i(x)$ and

$$c(w_h(\cdot,t),\delta_i) + b(w_h(\cdot,t),\delta_i) = (q(\cdot,t),\delta_i)$$

for $i = 1, 2, \cdots, n$ while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

3.1.1 A system of ordinary differential equations

Problem G^h is equivalent to a system of ordinary differential equations as we prove below. This equivalence enables us to prove that the problem has a unique solution and to derive differentiability properties of the solution.

Suppose that S^h is the span of the set of basis functions $\{\phi_1,\ \phi_2, \cdots,\ \phi_n\}$. It is convenient to introduce the following transformation. For any $\bar{x} \in R_n$, let

$$T_h\bar{x} = \sum_{i=1}^{n}\ x_i\phi_i \in S^h.$$

If the basis functions are linearly independent, the mapping T_h is a linear bijection.

For each u in S^h let $\bar{u} = T_h^{-1}u$. Then, for a function w with values in S^h we define a function $\bar{w}$ with values in R_n by

$$\bar{w}(t) = T_h^{-1}w(t).$$

Consequently $\bar{w}$ is differentiable if and only if w_j is differentiable for each j and

$$\frac{d}{dt}\left(\sum_{j=1}^{n}\ w_j(t)\phi_j\right) = \sum_{j=1}^{n}\ w_j'(t)\phi_j.$$

The same is true for the second order derivative.

Now, for u and v in S^h,

$$c(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_j v_i\, c(\phi_i,\phi_j).$$

If the matrix M is defined by $M_{ij} = c(\phi_i,\phi_j)$, then $c(u,v) = M\bar{u}\cdot\bar{v}$. Similarly, if the matrices K and C are defined by $K_{ij} = b(\phi_j,\phi_i)$ and $C_{ij} = a(\phi_j,\phi_i)$, then $a(u,v) = C\bar{u}\cdot\bar{v}$ and $b(u,v) = K\bar{u}\cdot\bar{v}$

Define $\bar{u}$ by $\bar{u}(t) = T_h^{-1} u_h(t)$, then

$$\begin{aligned} c(u_h''(t), \phi_i) &= \sum_{j=1}^{n} u_j''(t) c(\phi_j, \phi_i), \\ a(u_h'(t), \phi_i) &= \sum_{j=1}^{n} u_j'(t) a(\phi_j, \phi_i) \quad \text{and} \\ b(u_h(t), \phi_i) &= \sum_{j=1}^{n} u_j(t) b(\phi_j, \phi_i) \end{aligned}$$

for each i.

For the function f in (3.1.1), define a function F with values in $\mathbb{R}_n$ by $F_i(t) = (f(\cdot, t), \phi_i)_X$. We find that u_h satisfies (3.1.1) if and only if

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F.$$

Problem G-ODE

Find $\bar{u} \in C^2(J)$ such that

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F(t) \quad \text{with} \quad \bar{u}(0) = \bar{d} \quad \text{and} \quad \bar{u}'(0) = \bar{v},$$

where $\bar{d} = T_h^{-1} u_0^h$ and $\bar{v} = T_h^{-1} u_1^h$.

We have proved the following result.

Proposition 3.1.1. *Suppose M, K, C, F, $\bar{d}$ and $\bar{v}$ are defined as above. Then, the function u_h is a solution of Problem G^h if and only if the function $\bar{u}$ is a solution of Problem G-ODE.*

Proposition 3.1.2. *If $F \in C(J)$, then Problem G-ODE has a unique solution for each pair of vectors $\bar{d}$ and $\bar{v}$. If $F \in C^k(J)$, then $\bar{u} \in C^{2+k}(J)$.*

Proof. Since c is an inner product, the matrix M is symmetric and positive definite. Consequently M is invertible and the differential equation in Problem G-ODE may be written in the form $\bar{u}'' + M^{-1}C\bar{u}' + M^{-1}K\bar{u} = M^{-1}F(t)$. It follows from the theory of linear differential equations that the initial value problem has a unique solution.
If $F \in C^1(J)$, then $u'' \in C^1(J)$ and hence $u \in C^3(J)$. The result follows by induction. $\square$

Theorem 3.1.1. *Let J be an interval containing zero. If $f \in C(J, X)$, then there exists a unique solution $u_h \in C^2(J)$ for Problem G^h for each u_0^h and u_1^h in S^h. If $f \in C^k(J, X)$, then $u_h \in C^{2+k}(J)$ and if $f = 0$ then $u_h \in C^\infty\big((-\infty, \infty)\big)$.*

Proof. Since S^h is finite dimensional, $f \in C(J, X)$ implies that $F \in C(J)$. It follows that Problem G-ODE has a unique solution (by Proposition 3.1.2) and hence Problem G^h has a unique solution (by Proposition 3.1.1).

Note that $F \in C^k(J)$ if $f \in C^k(J, X)$. Then, obviously $\bar{u} \in C^{2+k}(J)$ which implies that $u_h \in C^{2+k}(J)$. □

In the next sections we consider the error for the semi-discrete approximation u_h. To begin we introduce the projection operator.

3.2 Projection

To find an estimate for the discretization error $\|u(t) - u_h(t)\|_V$, a projection is used. The projection operator P_h has domain V and is defined by

$$b(u - P_h u, v) = 0 \quad \text{for all } v \in S^h.$$

We will write P for P_h if no confusion is possible. For a function with values in V, we define a function Pu by $(Pu)(t) = Pu(t)$ for each $t \in J$.

Consider the error $e_h(t) = u(t) - u_h(t)$. Let $e(t) = Pu(t) - u_h(t)$ and $e_p(t) = u(t) - Pu(t)$, then

$$e_h(t) = e_p(t) + e(t). \tag{3.2.1}$$

Note that

$$\|u(t) - u_h(t)\|_V \leq \|e_p(t)\|_V + \|e(t)\|_V. \tag{3.2.2}$$

In applications estimates for the norm of $e_p(t)$ are obtained from approximation theory. The challenge is to find an estimate for the difference between the projection of u and the Galerkin approximation $(e(t))$. The estimates that can be obtained depend on the differentiability of Pu and that depends on the properties of the solution.

The usual way to start is to subtract Equation (3.1.1) from the variational equation in Problem PG and then use the fact that $e_p(t)$ is orthogonal to S^h.

Lemma 3.2.1. *If u is the solution of Problem PG and u_h is the solution of Problem G^h, then*

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h. \tag{3.2.3}$$

Proof. By subtracting Equation (3.1.1) from the variational equation in Problem PG we obtain the following identity. For any $v \in S^h$,

$$c(u''(t) - u_h''(t), v) + a(u'(t) - u_h'(t), v) + b(u(t) - u_h(t), v) = 0.$$

The result follows from (3.2.1) and the fact that

$$b(u(t) - Pu(t), v) = 0 \text{ for all } v \in S^h.$$

□

We now consider the differentiability of Pu.

Proposition 3.2.1. *If $u \in C^{(k)}(J, V)$, then $Pu \in C^{(k)}(J, V)$ and $(Pu)^{(k)}(t) = Pu^{(k)}(t)$.*

Proof. From the defintion of P_h we have

$$\|P_h u\|_V^2 = b(P_h u, P_h u) = b(u, P_h u) \leq \|u\|_V \|P_h u\|_V.$$

Consequently P is a bounded linear operator with norm less than one. It follows that

$$\|(\delta t)^{-1} (Pu(t + \delta t) - Pu(t)) - Pu'(t)\|_V \leq \|(\delta t)^{-1}(u(t+\delta t) - u(t)) - u'(t)\|_V.$$

Since the term on the right converges to 0 as $\delta t \to 0$, $Pu \in C^1(J, V)$ and $(Pu)'(t) = Pu'(t)$. Now use induction to complete the proof. □

To prove Lemma 3.2.2 below the existence of $(Pu)''$ is required. From the existence theorem, Theorem 2.3.1, we have that if u is a solution of Problem G, then $u \in C^1(J, V)$ and hence $Pu \in C^1(J)$. However, $(Pu)''$ may not exist.

Assumption A1

The solution u of Problem G has the property that $(Pu) \in C^2(J)$.

Assumption A2

The solution u of Problem G satisfies $u \in C^2(J, V)$.

Remark

If Assumption A2 is used it is possible to prove that $Pu \in C^2(J)$. The assumption is sufficient to prove A1 but not necessary.

Proposition 3.2.2. *If $u \in C^2(J, W)$ and u satisfies Assumption A1, then*

$$e_p \in C^2(J, W).$$

Proof. Recall that $e_p(t) = u(t) - (Pu)(t)$. Since norms in the finite dimensional space S^h are equivalent, $Pu \in C^2(J, W)$. □

Lemma 3.2.2. *If the solution u of Problem G satisfies Assumption A1, then*

$$c(e_p''(t), v) + a(e_p'(t), v) + c(e''(t), v) + a(e'(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h.$$

Proof. Since $e_h = e_p + e$, the result follows from Proposition 3.2.1. □

3.2.1 Projection error

To prove the final convergence results, it is necessary to estimate the projection errors e_p, e_p' and e_p''. We use interpolation errors.

It is necessary to generalize the approach in the literature to accommodate systems for example. There exists a subspace $H(V, k)$ of V, a positive constant $\widehat{C}$ and a positive integer α such that for any $u \in H(V, k)$

$$\inf_{v \in S^h} \|u - v\|_V \leq \widehat{C} h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V, k)$. This is a generalization of the assumptions in [Dup73] and [Bak76].

The following generalization (from [Bas14]) is more useful.

Definition Generalized interpolation operator

$\Pi u = \sum_{k=1}^{n} \phi_k(u) w_k$ where $\{w_k\}$ is a basis for S^h and ϕ_k are linear functionals.

To formulate an assumption regarding the error when an element of V is approximated by an element of S^h, we suppose h is a parameter related to the dimension n of S^h and $h \to 0$ as $n \to \infty$.

Assumption A3

There exists a subspace $H(V,k)$ of V, an interpolation operator Π and positive constants C_Π and α (depending on V and k) such that for $u \in H(V,k)$:

$$\|u - \Pi u\|_V \leq C_\Pi h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V,k)$.

The following consequence will be used to estimate the projection errors in the next sections.

Proposition 3.2.3. *There exists a subspace $H(V,k)$ of V and positive constants C_Π and α (depending on V and k) such that for $u \in H(V,k)$:*

$$\|u - Pu\|_V \leq C_\Pi h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V,k)$.

As mentioned before, e needs to be estimated in order to find an estimate for the error e_h. Estimates for e are in terms of e_p. The estimates are obtained in the following 2 sections, starting with either Lemma 3.2.1 or Lemma 3.2.2. This idea is from [Bas14].

Different methods are used for different types of damping. For weak damping the article [BV13] is used (see Section 3.3). The results in this article have the advantage that less regularity is required from the solution u. For general damping we investigate the technical report [BV14] (see Section 3.4). For these results the only assumption on the damping is Assumption E4, and this holds for boundary damping for example. In this approach an extra regularity assumption is needed from the solution u however. Finally, in Section 3.5 the fully discrete approximation is considered, using [BV13].

3.3 Semi-discrete error estimate for weak damping

In this section we consider the recent article [BV13]. The authors generalized the results of Baker [Bak76] and included weak damping. The proofs in [BV13] are done in detail and we cannot elaborate on it. The objective in this section is to give the results and convey the main ideas of the proofs.

3.3.1 Fundamental estimate

Recall that, since we consider weak damping in this section, Assumptions E1, E2, E3 and E4W are basic and assumed to hold.

Lemma 3.3.1. *If the damping term a satisfies Assumption E4W, then for $t \in [0,T]$,*

$$\begin{aligned}\|e(t)\|_W \;\le\; \sqrt{2}\Bigg(&\|e(0)\|_W + 3T\|e_h'(0)\|_W + 3\int_0^T \|e_p'(t)\|_W \\ &+3K_aT\|e_h(0)\|_W + 3K_a\int_0^T \|e_p(t)\|_W\Bigg).\end{aligned}$$

As mentioned, the proof is done in detail in [BV13] (see proof of Lemma 4.1). Note that for this Lemma no extra assumption is made on the solution u. In particular it is not necessary to make Assumption A1. This is due to the fact that Lemma 3.2.1 is used in the proof and not Lemma 3.2.2. In [Bak76] the result in Lemma 3.2.2 is used and he implicitly assumes that $Pu \in C^2[0,T]$ without commenting on it. This is an important improvement made in the article [BV13].

Suppose that $a = 0$, i.e. there is no damping. Using the same arguments as for the result above, the following corollary is obtained.

Corollary 3.3.1. *If $a = 0$ in Problem G, then*

$$\|e(t)\|_W \le \sqrt{2}\|e(0)\|_W + 2T\|e_h'(0)\|_W + 4\sqrt{T}\max_{t\in[0,T]}\|e_p'(t)\|_W,$$

for $t \in [0,T]$.

In [Bak76] error estimates are obtained "using $\mathcal{L}^2$-projections of the initial data as starting values". He considers the undamped case with the additional assumption that $c(u_1 - u_1^h, v(0)) = c(e_h'(0), v(0)) = 0$.

Using this assumption the following result, which is a generalization of the estimate for $e = Pu - u$ in [Bak76], is obtained.

Corollary 3.3.2. *Consider Problem G with $a = 0$. If $c(u_1 - u_1^h, v) = 0$ for all $v \in S^h$, then*

$$\|e(t)\|_W \leq \sqrt{2}\|e(0)\|_W + 2\sqrt{T} \max_{t\in[0,T]} \|e_p'(t)\|_W,$$

for $t \in [0, T]$.

In [Bak76] The fact that the starting values for the approximation are projections, is used to derive estimates. But it is not always possible to find the projections exactly. It is not clear in such an event how the error estimates can be interpreted. In the approach in [BV13] projections are not used as initial approximations in the fundamental estimate. The errors for the initial approximations are dealt with later.

3.3.2 Convergence and error estimates for the semi-discrete approximation

The error estimates follow from Lemma 3.3.1.

Lemma 3.3.2. *Assume that $u'(t) \in \mathcal{L}^2([0,T], H(V,k))$. If u is the solution of Problem G and the damping term satisfies Assumption E4W, then*

$$\begin{aligned}\|u(t) - u_h(t)\|_W \;\; \leq \;\; & \|u(t) - Pu(t)\|_W + \sqrt{2}\big(\|Pu_0 - u_0\|_W + 3T\|u_1 - u_1^h\|_W \\ & +3\int_0^T \|u'(t) - Pu'(t)\|_W + (1 + 3K_aT)\|u_0 - u_0^h\|_W \\ & +3K_a\int_0^T \|u(t) - Pu(t)\|_W\bigg), \qquad (3.3.1)\end{aligned}$$

for each $t \in [0, T]$.

Proof. Recall that $\|u(t) - u_h(t)\|_W \leq \|e_p(t)\|_W + \|e(t)\|_W$. The result follows from Lemma 3.3.1 since $\|Pu_0 - u_0^h\|_W \leq \|Pu_0 - u_0\|_W + \|u_0 - u_0^h\|_W$. □

It is necessary to consider the errors on the right hand side of the Estimate (3.3.1) in applications.

Initial conditions

The error depends on the unknown initial conditions u_0^h and u_1^h. In applications u_0^h and u_1^h must be chosen. If we choose $Pu_0 = u_0^h$, for example, then $\|Pu_0 - u_0^h\|_V = 0$. This is difficult if not impossible in practice. An obvious choice is to use the interpolants of u_0 and u_1.

Corollary 3.3.3. *Suppose u_0 and u_1 are in $H(V,k)$, Assumption A3 holds and $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem G satisfies $u'(t) \in \mathcal{L}^2([0,T], H(V,k))$, then*

$$\begin{aligned}
&\|u(t) - u_h(t)\|_W \\
&\leq C_b C_\Pi h^\alpha \|u(t)\|_{H(V,k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V,k)} \right. \\
&\left. +3K_a \int_0^T \|u(t)\|_{H(V,k)} + (2 + 3K_a T)\|u_0\|_{H(V,k)} + 3T\|u_1\|_{H(V,k)} \right),
\end{aligned}$$

for each $t \in [0,T]$.

Proof. The result follows from Lemma 3.3.2 and Assumptions E2 and A3. $\square$

Higher order estimates

As explained in [BV13], the Aubin-Nitche trick can be used to obtain higher order estimates for the projection error. However, the improved estimates serve no purpose if the interpolants are used to approximate the initial values. Additionally, the value of Baker's theory lies in the fact that convergence is proved for a weak solution and the Aubin-Nitsche trick requires higher order regularity which is not always available.

See Subsection 3.4.3 for a discussion on the possibilities to obtain higher order estimates for general damping.

3.4 Semi-discrete error estimate for general damping

In the technical report [BV14] the work of [Dup73] is generalized. The convergence of the Galerkin approximation for a general linear second order hyperbolic equation with general damping (i.e. only assumption on the damping term is Assumption E4) is investigated. This technical report is not published and therefore, even though the proofs are done in detail in the report, it is presented in this section for completeness.

3.4.1 General energy method

Recall that an estimate for $e(t) = Pu(t) - u_h(t)$ is required.

The following "energy" expression is convenient:

$$\begin{aligned} E(t) &= \frac{1}{2}c(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) \\ &= \frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2. \end{aligned} \tag{3.4.1}$$

Remark Note that Lemma 3.2.2 is used in this proof and therefore Assumption A1 is needed, in contrast with the previous section.

Lemma 3.4.1. *If the solution u of Problem G satisfies Assumption A1, then*

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \tag{3.4.2}$$

Proof. We have that $e \in C^2[0,T]$, since $u^h \in C^2[0,T]$ and $Pu \in C^2[0,T]$ (from Assumption A1). Also, since Pu and u_h are in S^h, $e(t) \in S^h$. It follows that $e'(t) \in S^h$.

Choose $v = e'(t)$ in Lemma 3.2.2, then

$$c(e_p''(t), e'(t)) + c(e''(t), e'(t)) + a(e_p'(t), e'(t)) + a(e'(t), e'(t)) + b(e(t), e'(t)) = 0. \tag{3.4.3}$$

Since b and c are bounded, it is easy to prove the "product rule" for differentiation. From (3.4.1):

$$E'(t) = c(e''(t), e'(t)) + b(e(t), e'(t)).$$

Combining this with (3.4.3) gives us

$$E'(t) = -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)) - a(e'(t), e'(t)).$$

The result follows since $a(e'(t), e'(t)) \geq 0$. $\square$

Now, the aim is to obtain an estimate for $E(t)$ in terms of projection errors. This will enable us to find an estimate for $e(t)$, since $\|e'(t)\|_W^2 + \|e(t)\|_V^2 = 2E(t)$, and therefore an estimate for the error $e_h(t)$. As mentioned, in the technical report [BV14] the work done in [Dup73] is generalized.

3.4.2 Fundamental estimate

Recall Assumptions E1, E2, E3 and E4. We also assume that the unique solution satisfies Assumption A1.

When trying to obtain estimates from Equation (3.4.2), there is a problem with the last term since

$$a(e_p'(t), e'(t)) \leq K\|e_p'(t)\|_V \|e'(t)\|_V,$$

and $\|e'(t)\|_V$ is not bounded by $E(t)$. We need the following auxiliary result to enable us to estimate this term.

Proposition 3.4.1. *If $e_p \in C^2([0,T], V)$, then for any $t \in [0,T]$,*

$$\int_0^t a\big(e_p'(\cdot), e'(\cdot)\big) = a\big(e_p'(t), e(t)\big) - a\big(e_p'(0), e(0)\big) - \int_0^t a\big(e_p''(\cdot), e(\cdot)\big).$$

Proof. If $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is a basis for S^h, then

$$e(t) = \sum_{i=1}^{n} e_i(t)\phi_i$$

and therefore

$$a(e_p', e) = \sum_{i=1}^{n} e_i a(e_p', \phi_i). \tag{3.4.4}$$

Since $e \in C^1[0,T]$, it follows that $e_i \in C^1[0,T]$ for $i = 1, 2, \ldots, n$ and

$$a(e_p', e') = \sum_{i=1}^{n} e_i' a(e_p', \phi_i). \tag{3.4.5}$$

Since $e_p \in C^2([0,T],V)$, for any $t \in [0,T]$,

$$\begin{aligned}\big|a\big(e_p'(t+h),\phi_i\big) - a\big(e_p'(t),\phi_i\big) - a\big(e_p''(t),\phi_i\big)\big| \\ \leq K\|e_p'(t+h) - e_p'(t) - e_p''(t)\|_V\|\phi_i\|_V\end{aligned}$$

and it follows that, for $i = 1, 2, \ldots, n$, $a\big(e_p'(\cdot),\phi_i\big)$ is differentiable and

$$\frac{d}{dt}a\big(e_p'(t),\phi_i\big) = a\big(e_p''(t),\phi_i\big). \tag{3.4.6}$$

Using (3.4.4), (3.4.5) and (3.4.6) we have

$$\begin{aligned}\frac{d}{dt}a\big(e_p'(t),e(t)\big) &= \sum_{i=1}^{n}\frac{d}{dt}\{e_i(t)a(e_p',\phi_i)\} \\ &= \sum_{i=1}^{n}e_i'(t)a(e_p',\phi_i) + \sum_{i=1}^{n}e_i(t)a(e_p'',\phi_i) \\ &= a\big(e_p''(t),e(t)\big) + a\big(e_p'(t),e'(t)\big).\end{aligned}$$

The result follows by integrating both sides and using the fundamental theorem of calculus. □

Remark Note that $e_p \in C^2([0,T],V)$ is required in the proposition above and therefore Assumption A2 is needed.

The following lemma from the technical report generalizes the approach in [Dup73].

Lemma 3.4.2. *If the solution u of Problem G satisfies $u \in C^2([0,T],V)$ (Assumption A2), then for any $t \in [0,T]$*

$$\|e(t)\|_V + \|e'(t)\|_W \leq \sqrt{24e^{3t}}K_T, \quad \textit{where}$$

$$\begin{aligned}K_T = \textstyle\int_0^T \|e_p''(\cdot)\|_W + 3K_a \max\|e_p'(t)\|_V + 3K_a \int_0^T \|e_p''(\cdot)\|_V + \|e'(0)\|_W \\ +\sqrt{1+K_a}\|e(0)\|_V + \sqrt{K_a}\|e_p'(0)\|_V.\end{aligned}$$

Proof. Consider the Inequality (3.4.2) in Lemma 3.4.1. We have to estimate the terms on the right. The estimate

$$|c(e_p''(t),e'(t))| \;\; \leq \;\; \|e_p''(t)\|_W\,\|e'(t)\|_W \leq \frac{1}{2}\|e_p''(t)\|_W^2 + \frac{1}{2}\|e'(t)\|_W^2$$

follows from the Cauchy-Schwartz inequality and Young's inequality. Since $\|e'(t)\|_W^2 \leq 2E(t)$,

$$E'(t) \leq \frac{1}{2}\|e_p''(t)\|_W^2 + E(t) - a(e_p'(t), e'(t)).$$

Integrating on both sides yields

$$E(t) \leq E(0) + \frac{1}{2}\int_0^t \|e_p''(\cdot)\|_W^2 + \int_0^t E(\cdot) - \int_0^t a(e_p'(\cdot), e'(\cdot)). \qquad (3.4.7)$$

From Proposition 3.4.1 we obtain

$$\begin{aligned} E(t) \leq\; & E(0) + \frac{1}{2}\int_0^t \|e_p''(\cdot)\|_W^2 + \int_0^t E(\cdot) \\ & - \left(a\big(e_p'(t), e(t)\big) - a\big(e_p'(0), e(0)\big) - \int_0^t a(e_p''(\cdot), e(\cdot)) \right). \end{aligned}$$

We need estimates for the terms in brackets. Using Young's inequality and the definition of $E(t)$ we have

$$\begin{aligned} \left|a\big(e_p'(t), e(t)\big)\right| & \leq K_a\|e_p'(t)\|_V\,\|e(t)\|_V \\ & \leq K_a\varepsilon^{-2}\|e_p'(t)\|_V^2 + K_a\varepsilon^2\|e(t)\|_V^2 \\ & \leq K_a\varepsilon^{-2}\|e_p'(t)\|_V^2 + 2K_a\varepsilon^2 E(t), \end{aligned}$$

with ε arbitrary. Also,

$$\begin{aligned} \left|\int_0^t a\big(e_p''(\cdot), e(\cdot)\big)\right| & \leq \int_0^t \left|a\big(e_p''(\cdot), e(\cdot)\big)\right| \\ & \leq K_a\varepsilon^{-2}\int_0^t \|e_p''(\cdot)\|_V^2 + K_a\varepsilon^2\int_0^t \|e(\cdot)\|_V^2 \\ & \leq K_a\varepsilon^{-2}\int_0^t \|e_p''(\cdot)\|_V^2 + 2K_a\varepsilon^2\int_0^t E(\cdot). \end{aligned}$$

Therefore we have that

$$\begin{aligned} E(t) \leq\; & E(0) + \frac{1}{2}\int_0^t \|e_p''(\cdot)\|_W^2 + \int_0^t E(\cdot) + 2K_a\varepsilon^2\int_0^t E(\cdot) + 2K_a\varepsilon^2 E(t) \\ & + K_a\varepsilon^{-2}\|e_p'(t)\|_V^2 + a\big(e_p'(0), e(0)\big) + K_a\varepsilon^{-2}\int_0^t \|e_p''(\cdot)\|_V^2. \end{aligned}$$

By choosing ε such that $2K_a\varepsilon^2 = \frac{1}{2}$ we obtain

$$\begin{aligned} E(t) &\leq E(0) + \frac{1}{2}\int_0^t \|e_p''(\cdot)\|_W^2 + \frac{3}{2}\int_0^t E(\cdot) + \frac{1}{2}E(t) + 4K_a^2\|e_p'(t)\|_V^2 \\ &\quad + a\big(e_p'(0), e(0)\big) + 4K_a^2\int_0^t \|e_p''(\cdot)\|_V^2. \end{aligned} \tag{3.4.8}$$

Note that $\left|a\big(e_p'(0), e(0)\big)\right| \leq \frac{1}{2}K_a\|e_p'(0)\|_V^2 + \frac{1}{2}K_a\|e(0)\|_V^2$. Rewriting (3.4.8) we obtain

$$\begin{aligned} \frac{1}{2}E(t) &\leq \frac{3}{2}\int_0^t E(\cdot) + \frac{1}{2}\int_0^t \|e_p''(\cdot)\|_W^2 + 4K_a^2\|e_p'(t)\|_V^2 \\ &\quad + 4K_a^2\int_0^t \|e_p''(\cdot)\|_V^2 + E(0) + \frac{1}{2}K_a\|e_p'(0)\|_V^2 + \frac{1}{2}K_a\|e(0)\|_V^2. \end{aligned}$$

This can be written in the form needed for the Gronwall inequality (see Appendix B):

$$\begin{aligned} &E(t) \leq 3\int_0^t E(\cdot) + K_T^*, \text{ where the constant} \\ &K_T^* = \int_0^T \|e_p''(\cdot)\|_W^2 + 8K_a^2 \max\|e_p'(t)\|_V^2 + 8K_a^2\int_0^T \|e_p''(\cdot)\|_V^2 \\ &\qquad + \|e'(0)\|_W^2 + (1+K_a)\|e(0)\|_V^2 + K_a\|e_p'(0)\|_V^2. \end{aligned}$$

By applying the Gronwall inequality, we have $E(t) \leq e^{3t}K_T^*$, and therefore

$$\frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2 \leq e^{3t}K_T^*.$$

Finally, using the fact that $\|\cdot\|_2^2 \leq n\|\cdot\|_\infty^2 \leq n\|\cdot\|_1^2$, we obtain

$$\begin{aligned} K_T^* &\leq 6\Bigg[\int_0^T \|e_p''(\cdot)\|_W + \sqrt{8}K_a \max\|e_p'(t)\|_V + \sqrt{8}K_a\int_0^T \|e_p''(\cdot)\|_V \\ &\quad + \|e'(0)\|_W + \sqrt{1+K_a}\|e(0)\|_V + \sqrt{K_a}\|e_p'(0)\|_V\Bigg]^2, \end{aligned}$$

and the result follows from the above inequality and the fact that $(a+b)^2 \leq 2a^2 + 2b^2$. $\square$

3.4.3 Convergence and error estimates

Lemma 3.4.3. *Assume that the solution u of Problem G satisfies $u \in C^2([0,T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0,T], H(V,k))$. Then, for any*

$t \in [0, T]$,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \|u(t) - Pu(t)\|_V + \|u'(t) - Pu'(t)\|_W + \sqrt{24e^{3t}} K_T, \quad \textit{where}
\end{aligned}$$

$$\begin{aligned}
K_T = \textstyle\int_0^T \|u'' - Pu''\|_W + 3K_a \max \|u'(t) - Pu'(t)\|_V + 3K_a \int_0^T \|u'' - Pu''\|_V \\
+\|Pu_1 - u_1^h\|_W + \sqrt{1 + K_a}\|Pu_0 - u_0^h\|_V + \sqrt{K_a}\|u_1 - Pu_1\|_V.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \|e(t)\|_V + \|e_p(t)\|_V + \|e'(t)\|_W + \|e_p'(t)\|_W.
\end{aligned}$$

The result follows from Proposition 3.2.1 and Lemma 3.4.2. □

To obtain the final theorem we substitute the estimates for the projection errors.

Theorem 3.4.1. *Suppose Assumption A3 holds for the space V. Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,*

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + C_b\|u'(t)\|_{H(V,k)}\right) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b\|u''(\cdot)\|_{H(V,k)}\right. \\
&\quad \left. +3K_a \max \|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)}\right] \\
&\quad + \sqrt{24e^{3t}} \left[\|Pu_1 - u_1^h\|_W + \sqrt{1 + K_a}\|Pu_0 - u_0^h\|_V + \sqrt{K_a}\|u_1 - Pu_1\|_V\right].
\end{aligned}$$

Proof. The result follows from Lemma 3.4.3, Assumption E2 ($\|u\|_W \leq C_b\|u\|_V$) and Proposition 3.2.3. □

Initial conditions

For a discussion on initial conditions, see Subsection 3.3.

Corollary 3.4.1. *Suppose Assumption A3 holds for the space V and u_0 and u_1 are in $H(V, k)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume also that the* *solution u of Problem G satisfies $u \in C^2([0,T],V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0,T],H(V,k))$. Then,*

$$
\begin{aligned}
&\|u(t)-u_h(t)\|_V + \|u'(t)-u_h'(t)\|_W \\
&\quad \leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + C_b\|u'(t)\|_{H(V,k)}\right) + \sqrt{24e^{3t}}C_\Pi h^\alpha \left[\int_0^T C_b\|u''(\cdot)\|_{H(V,k)}\right. \\
&\qquad +3K_a \max\|u'(t)\|_{H(V,k)} + 3K_a\int_0^T \|u''(\cdot)\|_{H(V,k)} \\
&\qquad \left. + \, 2C_b\|u_1\|_{H(V,k)} + 2\sqrt{1+K_a}\|u_0\|_{H(V,k)} + \sqrt{K_a}\|u_1\|_{H(V,k)}\right].
\end{aligned}
$$

for $t \in [0,T]$.

Proof. We have that

$$
\begin{aligned}
\|Pu_0 - \Pi u_0\|_V &\leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\
&\leq 2C_\Pi h^\alpha\|u_0\|_{H(V,k)} \quad \text{and} \\
\|Pu_1 - \Pi u_1\|_W &\leq C_b\|Pu_1 - \Pi u_1\|_V \\
&\leq C_b\|Pu_1 - u_1\|_V + C_b\|u_1 - \Pi u_1\|_V \\
&\leq 2C_bC_\Pi h^\alpha\|u_1\|_{H(V,k)},
\end{aligned}
$$

using the triangle inequality and Proposition 3.2.3. The result follows from Theorem 3.4.1. □

Higher order inertia norm estimate

The Aubin-Nitsche trick can be applied to find higher order estimates in the inertia norm for the discretization error for general damping. The results are not new and proofs can be found in [SF73], [Zie00] and [Bas11].

The regularity of the weak solution u is the main factor that determines the rate of convergence of the solution u_h of Problem G^h to the solution u of Problem G as h tends to zero. However, from Theorem 2.3.1 only $u \in C^1(J,V)$ can be guaranteed.

Recall the definition of the set E_b in Section 2.3.2.

Assumption A4

$E_b \subset H(V,k)$ and for any $u \in E_b$ there exists a constant $\widehat{c}_b$ such that $\|u\|_{H(V,k)} \leq \widehat{c}_b\|y\|_W$, where

$$b(u,v) = (y,v) \ \forall \ v \in V.$$

Lemma 3.4.4. *Suppose that Assumptions A3 and A4 hold for the space V. Then, for all $u \in E_b$,*

$$\|u - Pu\|_W \leq C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u\|_{H(V,k)},$$

where $\widehat{c}_b$ is the constant from Assumption A4.

To obtain higher order estimates, the initial values u_0^h and u_1^h must be chosen appropriately. Either $u_0^h = Pu_0$ and $u_1^h = Pu_1$ or a sufficiently accurate approximation must be used.

Following [Dup73], if we assume $\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$ the result below is obtained.

Corollary 3.4.2. *Assume that*

a) Assumptions A3 and A4 hold for the space V,

b) the solution u of Problem G satisfies $u \in C^2([0,T], V)$ (Assumption A2),

c) $u'' \in \mathcal{L}^2([0,T], H(V,k))$ and

d) $\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$.

Then, for any $t > 0$,

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V,k)} + \sqrt{24e^{3t}} C_b C_\Pi^2 h^{2\alpha} \widehat{c}_b \left(\int_0^T C_b \|u''(\cdot)\|_{H(V,k)} \right. \\ & \left. + 3K_a \max \|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)} + \sqrt{K_a} \|u_1\|_{H(V,k)} \right) \\ & + \sqrt{24e^{3t}} C_b K h^{2\alpha},\end{aligned}$$

where K depends on C_0 and K_a.

The special case where $u_0^h = Pu_0$ and $u_1^h = Pu_1$ are considered by some authors.

Corollary 3.4.3. *Assume $u_0^h = Pu_0$ and $u_1^h = Pu_1$ and that Assumptions A3 and A4 hold for the space V. Assume also that the solution u of Problem G* *satisfies* $u \in C^2([0,T],V)$ *(Assumption A2) and* $u'' \in \mathcal{L}^2([0,T],H(V,k))$. *Then, for any* $t > 0$,

$$\begin{aligned}&\|u(t)-u_h(t)\|_W \\ &\leq C_\Pi^2 h^{2\alpha}\widehat{c}_b\|u(t)\|_{H(V,k)} + \sqrt{24e^{3t}}C_b C_\Pi^2 h^{2\alpha}\widehat{c}_b\left(\int_0^T C_b\|u''(\cdot)\|_{H(V,k)}\right. \\ &\left. +3K_a \max\|u'(t)\|_{H(V,k)} + 3K_a\int_0^T \|u''(\cdot)\|_{H(V,k)} + \sqrt{K_a}\|u_1\|_{H(V,k)}\right).\end{aligned}$$

3.5 An implicit fully discrete scheme for weak damping

In this section we introduce Baker's algorithm. In [BV13] it was shown that it can also be used for weak damping and the authors generalized the proofs in [Bak76] and included the damping term.

The results of this section are from [BV13] and are completely general. The spaces and norms are as in Section 2.3 and the additional assumption of weak damping, i.e. Assumption E4W, is made. The proofs are done in detail in [BV13] and therefore some of it will be omitted.

3.5.1 Baker's fully discrete Galerkin scheme

Notation

For any sequence $\{x_k\} \subset R_n$:

$$\delta_t x_k = \tau^{-1}[x_{k+1} - x_k] \quad \text{and}$$

$$x_{k+\frac{1}{2}} = \frac{1}{2}[x_{k+1} + x_k].$$

Problem G^h-D-Baker

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \ldots, N-1$,

$$\delta_t u_k^h = v_{k+\frac{1}{2}}, \tag{3.5.1}$$

$$c(\delta_t v_k, \psi) + a(v_{k+\frac{1}{2}}, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) = \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \tag{3.5.2}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Adapting the proof in [Bak76], it is proved in [BV13] that the solution of Problem G^h-D is well defined. In the process an algorithm is derived. If confusion is not possible we write u_k for u_k^h.

Proposition 3.5.1. *Problem G^h-D-Baker has a unique solution for any pair of vectors d^h and v^h in S^h.*

Proof. For any $\psi \in S^h$,

$$\begin{aligned}
\frac{\tau}{2}b(u_{k+1} + u_k, \psi) &= \frac{\tau^2}{2}b(\tau^{-1}[u_{k+1} - u_k], \psi) + \frac{\tau}{2}b(u_k, \psi) + \frac{\tau}{2}b(u_k, \psi) \\
&= \frac{\tau^2}{2}b(\frac{1}{2}[v_{k+1} + v_k], \psi) + \tau b(u_k, \psi) \\
&= \frac{\tau^2}{4}b(v_{k+1}, \psi) + \frac{\tau^2}{4}b(v_k, \psi) + \tau b(u_k, \psi).
\end{aligned}$$

Substitution into (3.5.2) yields

$$\begin{aligned}
c(v_{k+1}, \psi) - c(v_k, \psi) + \frac{\tau}{2}\, a(v_{k+1} + v_k, \psi) + \frac{\tau^2}{4}\, b(v_{k+1}, \psi) + \frac{\tau^2}{4}\, b(v_k, \psi) \\
+\tau b(u_k, \psi) = \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X.
\end{aligned}$$

Consider

$$\begin{aligned}
&c(v_{k+1}, \psi) + \frac{\tau}{2}a(v_{k+1}, \psi) + \frac{\tau^2}{4}b(v_{k+1}, \psi) \\
&= c(v_k, \psi) - \frac{\tau}{2}a(v_k, \psi) - \frac{\tau^2}{4}b(v_k, \psi) - \tau b(u_k, \psi) + \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X
\end{aligned} \tag{3.5.3}$$

for each $\psi \in S^h$. The bilinear forms b and c are positive definite and as a consequence $c + \frac{\tau}{2}\, a + \frac{\tau^2}{4}\, b$ is positive definite. Therefore v_{k+1} is uniquely determined by (3.5.3) (using a special case of Riesz's theorem). Lastly, $u_{k+1} = u_k + \tau v_{k+\frac{1}{2}}$ from (3.5.1). □

The following algorithm may be used for simulation.

Problem FD

Find a sequence $\{\bar{u}_k\} \subset R_n$ such that for each k,

$$\bar{u}_{k+1} = \bar{u}_k + \tau \bar{v}_{k+\frac{1}{2}},$$

$$\left(M + \frac{\tau}{2}\, C + \frac{\tau^2}{4} K\right) \bar{v}_{k+1} = \left(M - \frac{\tau}{2} C - \frac{\tau^2}{4} K\right) \bar{v}_k - \tau K \bar{u}_k + \frac{\tau}{2}\left[F(t_k) + F(t_{k+1})\right]$$

with $\bar{u}_0 = \bar{d}$ and $\bar{v}_0 = \bar{v}$.

Truncation

Consider Problem G^h-D-Baker. Substituting $t = t_k$ and $t = t_{k+1}$ respectively into Equation (3.5.2), we have that

$$\begin{aligned} &c(\tau^{-1}[v_h(t_{k+1}) - v_h(t_k)], \psi) + \frac{1}{2}\, a([v_h(t_{k+1}) + v_h(t_k)], \psi) \\ &\quad + \frac{1}{2}\, b([u_h(t_{k+1}) + u_h(t_k)], \psi) = \frac{1}{2}\left([f(t_{k+1}) + f(t_k)], \psi\right)_X + c(\rho_k, \psi) \quad (3.5.4) \end{aligned}$$

where $v_h(t) = u_h'(t)$ and

$$\rho_k = \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - \frac{1}{2}\,[v_h'(t_{k+1}) + v_h'(t_k)].$$

The errors are denoted by e_k and q_k:

$$e_k = u_h(t_k) - u_k \quad \text{and} \quad q_k = u_h'(t_k) - v_k.$$

Note that in the approach in [BV13], $e_0 = 0$ and $q_0 = 0$.

Subtracting (3.5.2) from (3.5.4) results in

$$c(\tau^{-1}[q_{k+1} - q_k], \psi) + a(\frac{1}{2}[q_{k+1} + q_k], \psi) + b(\frac{1}{2}[e_{k+1} + e_k], \psi) = c(\rho_k, \psi),$$

which can be rewritten as follows

$$c(\delta_t q_k, \psi) + a(q_{k+\frac{1}{2}}, \psi) + b(e_{k+\frac{1}{2}}, \psi) \quad = \quad c(\rho_k, \psi). \qquad (3.5.5)$$

Substitution of u_h and v_h into (3.5.1) yields

$$\tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] = \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)] + \sigma_k, \qquad (3.5.6)$$

where (obviously)

$$\sigma_k = \tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] - \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)].$$

It follows from (3.5.1) and (3.5.6) that

$$\delta_t e_k \quad = \quad q_{k+\frac{1}{2}} + \sigma_k. \tag{3.5.7}$$

3.5.2 Stability

Using the truncation above and adapting the proof in [Bak76] (which is for the wave equation), [BV13] obtained the general result.

Lemma 3.5.1. *Stability*

$$\max \|e_n\|_W^2 \leq 8T\tau \sum_{n=0}^{N-1} \|\epsilon_n\|_W^2 + 2\tau^4\|\rho_0\|_W^2 + (8\tau^2 + 2\tau^4 K_a)\|\sigma_0\|_W^2,$$

where $\epsilon_n = \frac{\tau}{2}\rho_n + \tau \sum_{k=0}^{n-1} \rho_k + \sigma_n$ *for* $n = 1, 2, \ldots N-1$.

The proof in [BV13, Section 6.3] is long and in sufficient detail.

3.5.3 Error estimates

In this subsection we present the estimate for the error $u_h(t_k) - u_k^h$ in [BV13]. Recall that it must be with respect to $\|\cdot\|_W$.

It remains to find estimates for ρ_k and σ_k in the stability result. To find an estimate for ρ_k, we consider the following two parts separately

$$\begin{aligned} \rho_A &= \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - v_h'(t_k + \frac{\tau}{2}) \quad \text{and} \\ \rho_B &= -\frac{1}{2}[v_h'(t_{k+1}) + v_h'(t_k)] + v_h'(t_k + \frac{\tau}{2}). \end{aligned}$$

Recall that $v_h(t_k) = \sum_{i=1}^n v_i(t_k)\phi_i$. Using Taylor expansions we obtain

$$\begin{aligned} &\tau^{-1}[v_i(t_{k+1}) - v_i(t_k)] - v_i'(t_k + \frac{\tau}{2}) \\ &= \frac{1}{4\tau}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^2 v_i'''(\theta)d\theta + \frac{1}{4\tau}\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k - \theta)^2 v_i'''(\theta)d\theta. \end{aligned}$$

From the Cauchy-Schwartz inequality

$$\begin{aligned}
\left(\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2 v_i'''(\theta)\, d\theta\right)^2 &\leq \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^4\, d\theta \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \\
&= \frac{\tau^5}{160} \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \qquad \text{and} \\
\left(\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i'''(\theta)\, d\theta\right)^2 &\leq \frac{\tau^5}{160} \int_{t_k}^{t_k+\frac{\tau}{2}} (v_i'''(\theta))^2\, d\theta.
\end{aligned}$$

Using the estimate above and assuming that $\{\phi_j\}$ is orthonormal, we have

$$\begin{aligned}
\|\rho_A\|_W^2 &= \left\|\tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - v_h'(t_k + \frac{\tau}{2})\right\|_W^2 \\
&\leq \sum_{i=1}^{n} 2\left(\frac{1}{4\tau}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2 v_i'''(\theta) d\theta\right)^2 \\
&\quad + \sum_{i=1}^{n} 2\left(\frac{1}{4\tau}\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i'''(\theta) d\theta\right)^2 \\
&\leq \frac{\tau^3}{1280}\left[\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} \|v_h'''(\theta)\|_W^2\, d\theta + \int_{t_k}^{t_k+\frac{\tau}{2}} \|v_h'''(\theta))\|_W^2\, d\theta\right] \\
&\leq \frac{\tau^4}{1280} \max \|v_h'''\|_W^2,
\end{aligned}$$

where $\max\limits_{\theta\in[0,T]}$ is denoted by max.

To find an estimate for ρ_B, the same steps are followed. From Taylor expansions we have

$$\begin{aligned}
&\frac{1}{2}[v_i'(t_{k+1}) + v_i'(t_k)] - v_i'(t_k + \frac{\tau}{2}) \\
&\quad = \frac{1}{2}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta) v_i'''(\theta) d\theta - \frac{1}{2}\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta) v_i'''(\theta) d\theta.
\end{aligned}$$

Using the Cauchy-Schwartz inequality again, we obtain

$$
\begin{aligned}
\left(\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)v_i'''(\theta)d\theta\right)^2 &\leq \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2\,d\theta \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\,d\theta \\
&= \frac{\tau^3}{24}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\,d\theta \qquad \text{and} \\
\left(\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i^{(4)}(\theta)d\theta\right)^2 &\leq \frac{\tau^3}{24}\int_{t_k}^{t_k+\frac{\tau}{2}} (v_i'''(\theta))^2\,d\theta.
\end{aligned}
$$

Therefore

$$
\|\rho_B\|_W^2 \quad \leq \quad \frac{\tau^4}{48}\max\|v_h'''\|_W^2.
$$

Combining the estimates for ρ_A and ρ_B yields

$$
\|\rho_k\|_W^2 \quad \leq \quad \tau^4\max\|v_h'''\|_W^2. \tag{3.5.8}
$$

Finally consider $\sigma_k = \tau^{-1}[u_h(t_{k+1})-u_h(t_k)] - \frac{1}{2}[u_h'(t_{k+1})+u_h'(t_k)]$. Following the same method as for ρ_k, we obtain

$$
\|\sigma_k\|_W^2 \quad \leq \quad \tau^4\max\|u_h'''\|_W^2. \tag{3.5.9}
$$

The estimate for ϵ_n in the stability result is then obtained from (3.5.8) and (3.5.9):

$$
\|\epsilon_n\|_W^2 \quad \leq \quad 5T^2\tau^4\max\|v_h'''\|_W^2 + 4\tau^4\max\|u_h'''\|_W^2.
$$

Using the result in Lemma 3.5.1 yields

$$
\begin{aligned}
\max\|e_n\|_W^2 \quad \leq \quad & 40T^4\tau^4\max\|v_h'''\|_W^2 + 32T^2\tau^4\max\|u_h'''\|_W^2 \\
& +2\tau^8\max\|v_h'''\|_W^2 + 8\tau^6\max\|u_h'''\|_W^2 + 2\tau^8K_a\max\|u_h'''\|_W^2.
\end{aligned}
$$

Note that if $f \in C^2([0,T],X)$, then $u_h \in C^4([0,T],W)$ and consequently the following error estimate is obtained.

Theorem 3.5.1. *Error estimate.*

If $f \in C^2([0,T],X)$*, then*

$$\begin{aligned}\|u_h(t_k) - u_k^h\|_W \quad \leq \quad & 7T^2\tau^2 \max \|u_h^{(4)}\|_W + 7T\tau^2 \max \|u_h'''\|_W \\ & + \sqrt{2K_a}\, \tau^4 \max \|u_h'''\|_W \end{aligned} \tag{3.5.10}$$

for each $t_k \in (0,T)$*.*

3.5.4 Convergence of the fully discrete approximation for weak damping

Finally, by combining the semi-discrete estimates from Section 3.3 with Theorem 3.5.1, error estimates for the fully discrete approximation of the solution of Problem G are obtained.

Recall that from Corollary 3.3.3 we have: If $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, then

$$\begin{aligned} & \|u(t) - u_h(t)\|_W \\ & \leq C_b C_\Pi h^\alpha \|u(t)\|_{H(V,k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V,k)} \right. \\ & \left. +3K_a \int_0^T \|u(t)\|_{H(V,k)} + (2 + 3K_a T)\|u_0\|_{H(V,k)} + 3T\|u_1\|_{H(V,k)} \right), \end{aligned}$$

for each $t \in [0,T]$.

Using the triangle inequality we combine this with the result of Theorem 3.5.1, and obtain the final estimate.

Theorem 3.5.2. *Suppose*

1. *the damping term satisfies Assumption E4,*
2. *Assumption A3 holds for the space* V*,*
3. $u_0^h = \Pi u_0$ *and* $u_1^h = \Pi u_1$*,*
4. $u \in C^2([0,T],V)$ *(Assumption A2),*
5. $u''(t) \in \mathcal{L}^2([0,T],H(V,k))$*,*
6. $f \in C^2([0,T],X)$ *and*
7. *the sequence* $\{u_k^h\}$ *is a solution of Problem* G^h*-D-Central.*

Then

$$
\begin{aligned}
&\|u(t_k) - u_k^h\|_V \\
&\quad \leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + C_b\|u'(t)\|_{H(V,k)}\right) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b\|u''(\cdot)\|_{H(V,k)}\right. \\
&\qquad +3K_a \max\|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)} \\
&\qquad \left. +2C_b\|u_1\|_{H(V,k)} + 2\sqrt{1+K_a}\|u_0\|_{H(V,k)} + \sqrt{K_a}\|u_1\|_{H(V,k)}\right] \\
&\qquad + C_T \left(\tau^2 \max\left\|u_h^{(4)}(s)\right\|_W + \tau^2 \max\|u_h'''(s)\|_W\right),
\end{aligned}
$$

where C_T is a constant depending on T.

Remark

Other semi-discrete estimates can also be combined with Theorem 3.5.1. Note that if the energy estimates from Section 3.4 are used, Assumption A2 ($u \in C^2([0,T],V)$) must be made.

Chapter 4

Timoshenko cantilever beam with tip body and boundary damping

4.1 The model problem

The model problem was introduced in [ZVV04] where the finite element method is used to compute natural frequencies and modes of vibration. In [Bas14, Section 8.2] the model problem is considered briefly. Aspects covered are existence and convergence of the finite element approximation.

The equations of motion and constitutive equations (in dimensionless form) are in Subsection 1.2.4, equations (1.2.16), (1.2.17), (1.2.21) and (1.2.22). For convenience they are repeated here:

$$\partial_t^2 w = \partial_x V + q, \tag{4.1.1}$$

$$\frac{1}{\alpha}\partial_t^2 \phi = V + \partial_x M, \tag{4.1.2}$$

$$M = \frac{1}{\beta}\partial_x \phi, \tag{4.1.3}$$

$$V = \partial_x w - \phi. \tag{4.1.4}$$

The boundary damping (or control) is considered in [ZVV04]. The authors proposed the following interface conditions:

$$m\partial_t^2 w(\ell,t) = -V(\ell,t) - k_1\partial_t w(\ell,t),$$
$$I_m\partial_t^2\phi(\ell,t) = -M(\ell,t) - k_2\partial_t\phi(\ell,t).$$

Dimensionless form

Let $m^* = \dfrac{m}{\rho A\ell}$ and $I_m^* = \dfrac{I_m}{\ell\rho AI}$, then the interface conditions in dimensionless form are

$$m\partial_t^2 w(1,t) = -V(1,t) - \gamma_1\partial_t w(1,t) \quad \text{and} \tag{4.1.5}$$
$$I_m\partial_t^2\phi(1,t) = -M(1,t) - \gamma_2\partial_t\phi(1,t), \tag{4.1.6}$$

where $\gamma_1 = \dfrac{k_1\ell}{TAG\kappa^2}$ and $\gamma_2 = \dfrac{k_2}{AG\kappa^2\ell T}$.

Returning to the original notation we present the model problem.

Problem TB

The mathematical model consists of

the equations of motion (4.1.1) and (4.1.2),

the constitutive equations (4.1.3) and (4.1.4),

the boundary conditions $w(0,t) = \phi(0,t) = 0$ and

the interface conditions

$$m\partial_t^2 w(1,t) = -V(1,t) - \gamma_1\partial_t w(1,t) \text{ and} \tag{4.1.7}$$
$$I_m\partial_t^2\phi(1,t) = -M(1,t) - \gamma_2\partial_t\phi(1,t). \tag{4.1.8}$$

The initial conditions are $w(\cdot,0) = w_0$, $\phi(\cdot,0) = \phi_0$, $\partial_t w(\cdot,0) = w_d$ and $\partial_t\phi(\cdot,0) = \phi_d$.

4.2 Variational form

Consider equation (4.1.1), multiply by an arbitrary test function v.

$$\begin{aligned}&\int_0^1 \partial_t^2 w(\cdot,t)v \\ = \;& -\int_0^1 V(\cdot,t)v' + \int_0^1 q(\cdot,t)v + V(1,t)v(1) - V(0,t)v(0).\end{aligned} \tag{4.2.1}$$

(4.1.2) multiply by ψ.

$$\begin{aligned}
&\int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi \\
= \;& -\int_0^1 M(\cdot,t)\psi' + \int_0^1 V(\cdot,t)\psi + M(1,t)\psi(1) - M(0,t)\psi(0). \quad (4.2.2)
\end{aligned}$$

Add

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi \\
= \;& -\int_0^1 M(\cdot,t)\psi' - \int_0^1 V(\cdot,t)(v'-\psi) + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \;-\; V(0,t)v(0) \;+\; M(1,t)\psi(1) \;-\; M(0,t)\psi(0). \quad (4.2.3)
\end{aligned}$$

Test functions

$T[0,1] = \{v \in C^1 \mid v(0) = 0\}$.

Using (4.1.7) and (4.1.8) we find that

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi + m\partial_t^2 w(1,t)v(1) + I_m\partial_t^2\phi(1,t)\psi(1) \\
= \;& -\int_0^1 \frac{1}{\beta}\,\partial_x\phi(\cdot,t)\psi' - \int_0^1 \big(\partial_x w(\cdot,t) - \phi(\cdot,t)\big)(v'-\psi) + \int_0^1 q(\cdot,t)v \\
& - V(0,t)v(0) \;-\; M(0,t)\psi(0) - \gamma_1\partial_t w(1,t)v(1) - \gamma_2\partial_t\phi(1,t)\psi(1) \\
= \;& -\int_0^1 \frac{1}{\beta}\,\partial_x\phi(\cdot,t)\psi' - \int_0^1 \big(\partial_x w(\cdot,t) - \phi(\cdot,t)\big)(v'-\psi) + \int_0^1 q(\cdot,t)v \\
& -\gamma_1\partial_t w(1,t)v(1) - \gamma_2\partial_t\phi(1,t)\psi(1),
\end{aligned}$$

if $v \in T[0,1]$ and $\psi \in T[0,1]$.

Problem TBV

Find a pair of functions $\langle w, \phi\rangle$ such that $w(\cdot,t) \in T[0,1]$, $\phi(\cdot,t) \in T[0,1]$ and

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi + m\partial_t^2 w(1,t)v(1) + I_m\partial_t^2\phi(1,t)\psi(1) \\
&= -\int_0^1 \frac{1}{\beta}\,\partial_x\phi(\cdot,t)\psi' - \int_0^1 \big(\partial_x w(\cdot,t) - \phi(\cdot,t)\big)(v'-\psi) \\
&\quad -\gamma_1\partial_t w(1,t)v(1) - \gamma_2\partial_t\phi(1,t)\psi(1) + \int_0^1 q(\cdot,t)v.
\end{aligned}$$

for each pair $\langle v, \psi \rangle \in T[0,1] \times T[0,1]$.

4.3 Product spaces

Note that existence of solutions was not considered in [ZVV04] and we are not aware of any existence result for this problem except in [Bas14, Section 8.2]. However the presentation is rather brief and could be considered incomplete in places. To apply the existence theory in Section 2.3, we need the following product spaces:

$$X = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1) \times \mathbb{R}^2,$$

$$H^m = H^m(0,1) \times H^m(0,1) \times \mathbb{R}^2.$$

Recall that $(\cdot, \cdot)$ is used for the inner product in $\mathcal{L}^2(0,1)$.

Notation

An element $y \in X$ is written as $y = \langle y_1, y_2, y_3, y_4 \rangle$.

A natural inner product for X is

$$(x, y)_X = (x_1, y_1) + (x_2, y_2) + x_3 y_3 + x_4 y_4,$$

and the corresponding norm is denoted by $\| \cdot \|_X$.

The natural inner product for the product space H^m is

$$(x, y)_{H^m} = (x_1, y_1)_m + (x_2, y_2)_m + x_3 y_3 + x_4 y_4$$

and the corresponding norm is denoted by $\| \cdot \|_{H^m}$.

For the weak variational form, we define the following bilinear forms:

For u and v in X

$$c(u, v) = \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2 + m u_3 v_3 + I_m u_4 v_4.$$

For u and v in H^1

$$b(u,v) = \int_0^1 \frac{1}{\beta}\, u_2' v_2' + \int_0^1 (u_1' - u_2)(v_1' - v_2),$$

where the derivatives are weak derivatives.

For u and v in V

$$a(u,v) = \gamma_1 u_1(1,t) v_1(1,t) + \gamma_2 u_2(1,t) v_2(1,t).$$

Remark

Since $u_i(\cdot,t)$ is not necessarily continuous, $u_i(1,t)$ should be understood in the sense of trace, i.e. $u_i(1,t) = \Gamma u_i(\cdot,t)$. Consequently $\partial_t u_i(1,t) = \frac{d}{dt}\Gamma u_i(\cdot,t)$. (See Appendix C on trace.)

Note that the bilinear forms a, b and c are symmetric.

Proposition 4.3.1. *The bilinear form c is an inner product for the space X.*

Proof. Since c is a symmetric bilinear form, it is an inner product if $c(u,u) = 0$ implies $u = 0$. Now,

$$\begin{aligned} c(u,u) &= \int_0^1 u_1^2 + \frac{1}{\alpha}\int_0^1 u_2^2 + m u_3^2 + I_m u_4^2 \\ &\geq c^* \left(\|u_1\|^2 + \|u_2\|^2 + u_3^2 + u_4^2\right) \end{aligned} \tag{4.3.1}$$

where $c^* = \min\{1,\ \frac{1}{\alpha},\ m,\ I_m\}$. Consequently $u = 0$ if $c(u,u) = 0$. □

Definition Inertia space W

We refer to the vector space X equipped with the inner product c as the space W. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.

Proposition 4.3.2. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof. From (4.3.1) $\|u\|_W \geq c^* \|u\|_X$. Now,

$$c(u,u) = \int_0^1 u_1^2 + \frac{1}{\alpha}\int_0^1 u_2^2 + m u_3^2 + I_m u_4^2 \leq c_1 \left(\|u_1\|^2 + \|u_2\|^2 + u_3^2 + u_4^2\right)$$

where $c_1 = \max\{1,\ \frac{1}{\alpha},\ m,\ I_m\}$. The result follows. □

Let $V(0,1)$ be the closure of $T[0,1]$ in $H^1(0,1)$.

The energy space for this problem is

$$V = \{y \in V[0,1] \times V[0,1] \times \mathbb{R}^2 \mid y_3 = \Gamma y_1, y_4 = \Gamma y_2\},$$

In the next Proposition we prove that W is dense in V. The proof in [Bas14] is incomplete. In the proof below we use an idea from [BDV14], see Chapter 5 of this dissertation.

Proposition 4.3.3. *V is a dense subset of W.*

Proof. Consider any $y \in W$. Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ there exist sequences $\{\phi_n\} \subset C_0^\infty(0,1)$ and $\{\psi_n\} \subset C_0^\infty(0,1)$ such that $\|\phi_n - y_1\| \to 0$ and $\|\psi_n - y_2\| \to 0$.

We construct a sequence $\{\eta_n\}$ in $H^1(0,1)$ with the following properties

$$\Gamma\eta_n \quad = \quad 1 \;\text{ and }\; \|\eta_n\| \to 0.$$

For any natural number $n > 1$, let $\eta_n(x) = 0$ for $x \leq 1 - \frac{1}{n}$ and $\eta_n(x) = n(x + \frac{1}{n} - 1)$, then η_n has the desired properties.

Now, let $v_n = \langle f_n, g_n, \Gamma f_n, \Gamma g_n\rangle$, where $f_n = \phi_n + y_3\eta_n$ and $\Gamma f_n = y_3$, and $g_n = \psi_n + y_4\eta_n$ and $\Gamma g_n = y_4$. It follows that $v_n \in V$.

$$\begin{aligned} \|v_n - y\|_X^2 \quad &= \quad \|f_n - y_1\|^2 + \|g_n - y_2\|^2 \\ &\quad\; +(y_3 - f_n(1))^2 + (y_4 - g_n(1))^2 \\ &\quad\; \longrightarrow 0. \end{aligned}$$

□

The next result differs from [Bas14] where Corollary 4.3.2 was stated first and the proof of it is incomplete.

Proposition 4.3.4. *The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V.*

Proof. From the definition of $b(u,u)$ we have for each $u \in H^1$

$$b(u,u) \leq \frac{1}{\beta}K_2\|u_2'\|^2 + 2\|u_1'\|^2 + 2\|u_2\|^2 \leq K_b\|u\|_{H^1}^2, \tag{4.3.2}$$

when $K_b = \max\{\frac{1}{\beta}, 2\}$.

It remains to show that $\|\cdot\|_{H^1}$ is bounded below by $\|\cdot\|_V$. To start, use

$$|u_3| = |\Gamma u_1| \leq \|u_1\|_1 \text{ and } |u_4| = |\Gamma u_2| \leq \|u_2\|_1,$$

(Appendix C). It follows that

$$\|u\|_{H^1}^2 \leq 2\|u_1\|_1^2 + 2\|u_2\|_1^2.$$

Next, if $v \in V(0,1)$, then $\|v\| \leq \|v'\|$ (Poincaré inequality, see Appendix B). Therefore

$$\|u_i\|_1^2 = \|u_i\|^2 + \|u_i'\|^2 \leq 2\|u_i'\|^2.$$

and consequently

$$\|u\|_{H^m}^2 \leq 4\|u_1'\|^2 + 4\|u_2'\|^2.$$

Now,

$$\begin{aligned} \|u_1'\|^2 + \|u_2'\|^2 &\leq \|u_1' - u_2\|^2 + \|u_2\|^2 + \|u_2'\|^2 \\ &\leq \|u_1' - u_2\|^2 + 2\|u_2'\|^2, \end{aligned} \tag{4.3.3}$$

using the Poincaré inequality again. From the definition of b we have

$$b(u,u) \geq k_b \left(\|u_1' - u_2\|^2 + \|u_2'\|^2\right), \tag{4.3.4}$$

where $k_b = \max\{1, \frac{1}{\beta}\}$.

Combining (4.3.3) and (4.3.4) we have

$$\frac{2}{k}\ b(u,u) \ \geq\ 2\left(\|u_1' - u_2\|^2 + \|u_2'\|^2\right) \geq \|u_1'\|^2 + \|u_2'\|^2$$

and hence

$$\frac{8}{k} b(u,u) \geq \|u\|_{H^1}^2.$$

$\square$

Remark

Regarding the constants K_b and k_b, note that for a real beam $\beta > 1$, hence $K_b = 1$ and $k_b = \frac{1}{\beta}$.

Corollary 4.3.1. *The bilinear form b is an inner product for V.*

Proof. Since b is a symmetric bilinear form, this follows from Proposition 4.3.4. $\square$

Corollary 4.3.2. *There exists a constant C such that*

$$b(u,u) \quad \geq \quad C\|u\|_X^2 \quad \textit{for each } u \in V.$$

Proof. This is trivial since $\|u\|_X \leq \|u\|_{H^1}$. $\square$

Corollary 4.3.3. *There exists a constant C such that*

$$b(u,u) \quad \geq \quad C\|u\|_W^2 \quad \textit{for each } u \in V.$$

Proof. The norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent. $\square$

Definition Energy space V

We refer to the space V equipped with the inner product b as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

4.4 Weak variational form and existence

In this section we show how the existence and uniqueness of a weak solution for the model problem follow from Section 2.3. The approach followed is the same as that in [BDV14] and [Bas14]. We are not aware of any other existence results for this problem.

Notation

Let $\tilde{q}$ be the mapping $t \to q(\cdot,t)$ and $q_X = \langle \tilde{q}, 0, 0, 0 \rangle$.

Problem TBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t),v) + a_B(u'(t),v) + b(u(t),v) = (q_X(t),v)_X \quad \text{for each} \quad v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0, \Gamma w_0, \Gamma\phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d, \Gamma w_d, \Gamma\phi_d \rangle$.

Remark

The initial states u_0 and u_d is formulated incorrectly in [Bas14].

Proposition 4.4.1. *There exists a constant K such that, for all u and v in V,*

$$|\,a(u,v)| \quad \leq \quad K\|u\|_V\|v\|_V.$$

Proof.

$$\begin{aligned}
|a(u,v)| &\leq a(u,u)a(v,v) \\
&= \big(\gamma_1(u_1(1))^2+\gamma_2(u_2(1))^2\big)\,\big(\gamma_1(v_1(1))^2+\gamma_2(v_2(1))^2\big) \\
&\leq K^2(\|u_1\|^2+\|u_2\|^2)(\|v_1\|^2+\|v_2\|^2) \leq K^2\|u\|_{H_1}^2\|v\|_{H_1}^2,
\end{aligned}$$

where $K = \max\{\gamma_1, \gamma_2\}$. Since the norms $\|\cdot\|_V$ and $\|\cdot\|_{H_1}$ are equivalent, the result follows. $\square$

Theorem 4.4.1. *Suppose $\tilde{q} \in C^1([0,T), \mathcal{L}^2(0,1))$. If $u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that*

$$b(u_0, v) + a(u_d, v) = c(y, v) \textit{ for each } v \in V,$$

then Problem TBW has a unique solution,

$$u \in C\big([0,T);V\big) \cap C^1\big([0,T);W\big) \cap C^1\big((0,T);V\big) \cap C^2\big((0,T);W\big).$$

Proof. Clearly $q_X \in C^1([0,\tau), X)$. The result follows from Theorem 2.3.2. $\square$

4.5 Convergence of the semi-discrete approximation

Galerkin approximation

Suppose the space $S^h(0,1)$ consists of piecewise Hermite cubic basis functions (see Appendix D) such that the functions are zero at $x = 0$. Let S^h be the subspace of $S^h(0,1) \times S^h(0,1) \times \mathbb{R}^2$ such that $u_3 = u_1(1)$ and $u_4 = u_2(1)$ for each $u \in S^h$.

Problem TB$\boldsymbol{W}^h$

Find a function $u_h \in C^2(0,T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (q_X, v)_X \quad \text{for each} \quad v \in S^h, \tag{4.5.1}$$

while $u_h(0) = u_0^h = \langle w_0^h, \phi_0^h, \Gamma w_0^h, \Gamma \phi_0^h \rangle$ and $u_h'(0) = u_d^h = \langle w_d^h, \phi_d^h, \Gamma w_d^h, \Gamma \phi_d^h \rangle$.

The existence of a unique solution follows from Theorem 3.1.1 provided that q_X is continuous w.r.t. the norm of X, which requires $\tilde{q}$ to be continuous with respect to the norm of $\mathcal{L}^2(0,1)$.

In [ZVV04] it is shown how to construct the relevant matrices for the semi-discrete problem and derive the system of ordinary differential equations. However, the paper is concerned with modal analysis and the system ordinary differential equations is not considered.

Interpolation

Instead of Assumption A3 regarding the existence of an interpolation operator, we now construct an interpolation operator.

The interpolation operator Π is defined on the product spaces H^k by

$$\Pi u = \langle \Pi_c\, u_1,\ \Pi_c\, u_2,\ u_3,\ u_4 \rangle \quad \text{for } u \in H^k,$$

where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions. A **seminorm** for H^m is defined by

$$|u|_{k,H^m} \;=\; \sqrt{|u_1|_k^2 + |u_2|_k^2}\,,$$

with $|\cdot|_k$ the seminorm in $H^m(\Omega)$.

Proposition 4.5.1. *If $u \in H^k$, with $k \geq 2$, then*

$$\|u - \Pi u\|_V \leq \widehat{C} h^{\alpha-1} |u|_{\alpha,H^m}, \tag{4.5.2}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Proof. Since $u_i \in H^k(0,1)$ for $i = 1, 2$,

$$\|u_i - \Pi_c u_i\|_1 \leq \widehat{C}_c h^{\alpha-1} |u_i|_\alpha, \tag{4.5.3}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$ (see Appendix E).
From the definition of Π we have that $u - \Pi u = \langle u_1 - \Pi_c u_1, u_2 - \Pi_c u_2, 0, 0\rangle$, therefore

$$\|u - \Pi u\|_V^2 \quad \leq \quad K\left(\|u_1 - \Pi_c\, u_1\|_1^2 + \|u_2 - \Pi_c\, u_2\|_1^2\right), \qquad (4.5.4)$$

by (4.3.2). We combine (4.5.4) and (4.5.3) to obtain the result, using the definition of the semi-norm above. □

In the applications below we apply the results for general damping (Section 3.4). We assume that the solution u satisfies Assumption A2 and use the estimate (4.5.2). Note that the symbol α does not have the same meaning as in Assumption A3. Also, note that for Problem TBW, $H(V,k) = H^k \cap V$.

Application of Corollary 3.4.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2\big([0,T], H^k \cap V\big)$ with $k \geq 2$. Then, for any $t \in [0,T]$,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \widehat{C}h^{\alpha-1}\big(|u(t)|_{\alpha,H^k} + C|u'(t)|_{\alpha,H^k}\big) \\
&\qquad + \sqrt{12e^{3t}}\widehat{C}h^{\alpha-1}\left[\int_0^T C|u''(\cdot)|_{\alpha,H^k} + 3K_a \max|u'(t)|_{\alpha,H^k} + 3K_a\int_0^T |u''(\cdot)|_{\alpha,H^k}\right. \\
&\qquad \left. + 2C|u_d|_{\alpha,H^k} + 2\sqrt{1+K_a}|u_0|_{\alpha,H^k} + \sqrt{K_a}|u_d|_{\alpha,H^k}\right], \qquad (4.5.5)
\end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

As discussed before, Assumption A4 is satisfied in some cases and a **higher order estimate** can be obtained. Recall the definition of the set E_b in Section 2.3.2.

Assumption A4

$E_b \subset H(V,k)$ and for any $u \in E_b$ there exists a constant $\widehat{c}_b$ such that $\|u\|_{H(V,k)} \leq \widehat{c}_b\|y\|_W$, where

$$b(u,v) = (y,v) \;\forall\; v \in V.$$

If $E_b \subset H^k \cap V$, then Assumption A4 is satisfied.

Application of Corollary 3.4.2

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^{2\alpha}$ and $u'' \in \mathcal{L}^2\big([0,T], H^k \cap V\big)$ with $k \geq 2$. Then, for any $t > 0$,

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & \widehat{C}^2 h^{2\alpha} \widehat{c} |u_1(t)|_{\alpha+2} + \sqrt{12e^{3t}}\, C\widehat{C}^2 h^{2\alpha} \widehat{c} \bigg(\int_0^T C|u_1''(\cdot)|_{\alpha+2} \\ & +3K_a \max |u_1'(t)|_{\alpha+2} + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} + \sqrt{K_a} |u_d|_{\alpha+2} \bigg) \\ & + \sqrt{12e^{3t}} CKh^{2\alpha},\end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Remarks

1. Results on the regularity of weak solutions for the Timoshenko beam equations are not common knowledge and further study of the literature is required.
2. In theory, piecewise linear basis functions are suitable for the Timoshenko beam. However, due to the phenomenon of "locking", the rate of convergence in practice is not as predicted by the theory. For the Timoshenko beam locking can be avoided by using cubics according to [Bra01, p.302]. This fact is confirmed in [VVR10].

4.6 Fully discrete approximation

Suppose an approximation for the solution is required on the time interval $[0, T]$. The interval is divided into N steps resulting in a step length $\tau = T/N$. The approximation of $u_h(t_k)$ is denoted by u_k^h.

Consider central differences for the fully discrete approximation.

Problem $\boldsymbol{TBW^h}$-FD

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \ldots, N-1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) + b(u_k^h, v) \\ = (f(t_k), v)_X, \quad (4.6.1) \end{aligned}$$

for each $v \in S^h$, while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

We aim to prove convergence of the solution of Problem TBW^h-FD to the solution of Problem TBW and derive error estimates. Consider the error

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h]. \tag{4.6.2}$$

In Section 4.5 an error estimate was derived for $[u(t_k) - u_h(t_k)]$. We still need to derive an estimate for $[u_h(t_k) - u_k^h]$. The final estimate follows trivially by the triangle inequality.

We consider **central differences** for this problem. It is conditionally stable but has the advantage that it is "almost" explicit. In [Bas14] a generalized version of the theory in [OR76] can be found. It has not been published. The theory was also modified to include damping. In parts [Bas14] was difficult to read and we changed the presentation in those parts.

If confusion is not possible we write u_k for u_k^h. Denote the error by e_k:

$$e_k = u_h(t_k) - u_k.$$

Truncation

Substituting $t = t_k$ into Problem TBW^h, we obtain

$$c(u_h''(t_k), v) + a(u_h'(t_k), v) + b(u_h(t_k), v) = (f(t_k), v)_X. \tag{4.6.3}$$

Subtract (4.6.1) from (4.6.3) to obtain:

$$\begin{aligned} c(\tau^{-2}[e_{k+1} - 2e_k + e_{k-1}], v) + \frac{1}{2\tau} a(e_{k+1} - e_{k-1}, v) + b(e_k, v) \\ = c(\varepsilon_k, v) + a(\rho_k, v), \end{aligned} \tag{4.6.4}$$

where

$$\begin{aligned} \varepsilon_k &= \tau^{-2}[u_h(t_{k+1}) - 2u_h(t_k) + u_h(t_{k-1})] - u_h''(t_k) \quad \text{and} \\ \rho_k &= (2\tau)^{-1}[u_h(t_{k+1}) - u_h(t_{k-1})] - u_h'(t_k). \end{aligned}$$

Note that $e_0 = 0$ and $u_1 - u_{-1} = 2\tau u'(t_0)$. Therefore $e_1 - e_{-1} = 2\tau\rho_0$ and hence (4.6.4) reduce to

$$c(e_1 + e_{-1}, v) = \tau^2 c(\varepsilon_0, v) \quad \text{for each } v \in S^h.$$

Since $c(e_1 - e_{-1}, v) = 2\tau c(\rho_0, v)$, it follows that

$$c(e_1, v) = \frac{1}{2}\tau^2 c(\varepsilon_0, v) + \tau c(\rho_0, v) \quad \text{for each } v \in S^h.$$

Consequently

$$\|e_1\|_W \leq \frac{1}{2}\tau^2\|\varepsilon_0\| + \tau\|\rho_0\|.$$

4.6.1 Stability

The following inverse property follows from the one in [OR76]

$$\|u\|_V \leq C_I h^{-1} \|u\|_W. \tag{4.6.5}$$

This is so since $\|\cdot\|_V$ is equivalent to $\|\cdot\|_{H^1}$ and $\|\cdot\|_W$ is equivalent to $\|\cdot\|_X$.

Next it is shown that a sufficient condition for stability is

$$\tau^2 h^{-1} < C_I^{-1}$$

or

$$\left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right) > 0. \tag{4.6.6}$$

Most of the proof is clear in [OR76] and therefore little has been changed. However, after Equation (4.6.14) the proof is rather confusing and incomplete.

Lemma 4.6.1. *If $\tau^2 h^{-1} < C_I^{-1}$, then*

$$\begin{aligned}
&R\|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2}\|e_N\|_V^2 \\
&\quad \leq \left(1 + \frac{Te^{CT}}{R}\right)\left(2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau\sum_{k=2}^{N-2}\|\varepsilon_k\|_W^2\right) \\
&\qquad + \frac{\tau^2 e^{CT}}{R}\sum_{k=1}^{N-2}\|\varepsilon_k\|_W^2 + \tau\|\varepsilon_{N-1}\|_W^2,
\end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right)$.

Proof. Let $v = \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]$ in (4.6.4). Since $a \geq 0$ it follows that

$$\begin{aligned}
&c\left(\tau^{-2}[e_{k+1} - 2e_k + e_{k-1}], \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right) \\
&\quad + b\left(e_k, \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right) \\
&\qquad = \left(\varepsilon_k, \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right).
\end{aligned} \tag{4.6.7}$$

But

$$
\begin{aligned}
b\left(e_k, \tau^{-1}[e_{k+1}-e_k]\right) &= -\frac{\tau}{2} b\left(\tau^{-1}[e_{k+1}-e_k], \tau^{-1}[e_{k+1}-e_k]\right) \\
&\quad + \frac{1}{2\tau}[b(e_{k+1}, e_{k+1}) - b(e_k, e_k)] \\
&= -\frac{\tau}{2}\|\tau^{-1}(e_{k+1}-e_k)\|_V^2 + \frac{1}{2\tau}[\|e_{k+1}\|_V^2 - \|e_k\|_V^2].
\end{aligned}
\tag{4.6.8}
$$

Similarly,

$$
\begin{aligned}
b\left(e_k, \tau^{-1}[e_k-e_{k-1}]\right) &= \frac{\tau}{2} b\left(\tau^{-1}[e_k-e_{k-1}], \tau^{-1}[e_k-e_{k-1}]\right) \\
&\quad + \frac{1}{2\tau}[b(e_k, e_k) - b(e_{k-1}, e_{k-1})] \\
&= \frac{\tau}{2}\|\tau^{-1}(e_k-e_{k-1})\|_V^2 + \frac{1}{2\tau}[\|e_k\|_V^2 - \|e_{k-1}\|_V^2]
\end{aligned}
\tag{4.6.9}
$$

and

$$
\begin{aligned}
&c\left(\tau^{-2}[e_{k+1}-2e_k+e_{k-1}], \tau^{-1}[e_{k+1}-e_k] + \tau^{-1}[e_k-e_{k-1}]\right) \\
&\quad = \frac{1}{\tau}\left[\|\tau^{-1}[e_{k+1}-e_k]\|_W^2 - \|\tau^{-1}[e_k-e_{k-1}]\|_W^2\right].
\end{aligned}
\tag{4.6.10}
$$

Substituting the results from (4.6.8), (4.6.9) and (4.6.10) into (4.6.7), we obtain the following

$$
\begin{aligned}
&\frac{1}{\tau}\left[\|\tau^{-1}[e_{k+1}-e_k]\|_W^2 - \|\tau^{-1}[e_k-e_{k-1}]\|_W^2\right] - \frac{\tau}{2}\|\tau^{-1}(e_{k+1}-e_k)\|_V^2 \\
&\quad + \frac{\tau}{2}\|\tau^{-1}(e_k-e_{k-1})\|_V^2 + \frac{1}{2\tau}\left[\|e_{k+1}\|_V^2 - \|e_k\|_V^2\right] + \frac{1}{2\tau}\left[\|e_k\|_V^2 - \|e_{k-1}\|_V^2\right] \\
&\quad = c\left(\varepsilon_k, \tau^{-1}[e_{k+1}-e_k] + \tau^{-1}[e_k-e_{k-1}]\right).
\end{aligned}
$$

Summing from 1 to $n-1 \leq N-1$, we find

$$
\begin{aligned}
&\|\tau^{-1}[e_n-e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1-e_0]\|_W^2 - \frac{\tau^2}{2}\|\tau^{-1}(e_n-e_{n-1})\|_V^2 \\
&\quad + \frac{\tau^2}{2}\|\tau^{-1}(e_1-e_0)\|_V^2 + \frac{1}{2}\left[\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2\right] \\
&\quad \leq \tau c\left(\varepsilon_1, \tau^{-1}[e_1-e_0]\right) + \tau c\left(\varepsilon_{n-1}, \tau^{-1}[e_n-e_{n-1}]\right) \\
&\quad + \tau \sum_{k=1}^{n-2} c\left(\varepsilon_k, \tau^{-1}[e_{k+1}-e_k]\right) + \tau \sum_{k=1}^{n-2} c\left(\varepsilon_{k+1}, \tau^{-1}[e_{k+1}-e_k]\right).
\end{aligned}
\tag{4.6.11}
$$

Using the Schwartz inequality and Young's inequality (see Appendix B), it follows that

$$\begin{aligned} c(u, v) &\leq \|u\|_W \|v\|_W \\ &\leq \frac{\epsilon}{2}\|u\|_W^2 + \frac{2}{\epsilon}\|v\|_W^2. \end{aligned}$$

We apply this to (4.6.11) for various choices for ϵ:

$$\begin{aligned} &\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 - \frac{\tau^2}{2}\|\tau^{-1}(e_n - e_{n-1})\|_V^2 \\ &+ \frac{\tau^2}{2}\|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2}\left[\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2\right] \\ &\leq \frac{\tau}{2}\left(\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2\right) + \frac{\tau}{2\kappa}\|\varepsilon_{n-1}\|_W^2 + \frac{\tau\kappa}{2}\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 \\ &+ 2\sum_{k=1}^{n-2}\frac{\tau}{2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 + 2\sum_{k=2}^{n-2}\frac{\tau}{2}\|\varepsilon_k\|_W^2 + \frac{\tau}{2}\|\varepsilon_1\|_W^2 + \frac{\tau}{2}\|\varepsilon_{n-1}\|_W^2. \end{aligned}$$

If we choose κ such that $\frac{\tau\kappa}{2} = \frac{1}{2}$, then we have

$$\begin{aligned} &\frac{1}{2}\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 - \frac{\tau^2}{2}\|\tau^{-1}(e_n - e_{n-1})\|_V^2 \\ &+ \frac{\tau^2}{2}\|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2}\left[\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2\right] \\ &\leq \frac{\tau}{2}\left(\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2\right) + \frac{\tau^2}{2}\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=1}^{n-2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\ &+ \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \frac{\tau}{2}\|\varepsilon_1\|_W^2 + \frac{\tau}{2}\|\varepsilon_{n-1}\|_W^2. \end{aligned} \tag{4.6.12}$$

Since

$$\|\tau^{-1}(e_n - e_{n-1})\|_V^2 \leq C_I h^{-1}\|\tau^{-1}(e_n - e_{n-1})\|_W^2 \tag{4.6.13}$$

from the inverse property (4.6.5), (4.6.12) can now be written as

$$\begin{aligned} &\left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right)\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 \\ &+ \frac{\tau^2}{2}\|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2}\left[\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2\right] \\ &\leq \frac{\tau}{2}\left(\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2\right) + \frac{\tau^2}{2}\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=1}^{n-2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\ &+ \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \frac{\tau}{2}\|\varepsilon_1\|_W^2 + \frac{\tau}{2}\|\varepsilon_{n-1}\|_W^2. \end{aligned} \tag{4.6.14}$$

Note that $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right) > 0$ in (4.6.14), due to the assumption (4.6.6). From (4.6.14),

$$\begin{aligned} R\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 &\leq \|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \frac{\tau}{2}\left(\|\varepsilon_1\|_W^2 + \|\tau^{-1}e_1\|_W^2\right) \\ &+ \frac{\tau^2}{2}\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \frac{\tau}{2}\|\varepsilon_1\|_W^2 + \frac{\tau}{2}\|\varepsilon_{n-1}\|_W^2 \\ &+ \tau\sum_{k=1}^{n-2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2. \end{aligned}$$

Using the discrete Gronwall inequality (see Appendix B) and $\tau < 1$, we obtain

$$\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 \leq R^{-1}K_n\, e^{CT},$$

where

$$\begin{aligned} K_n &= (1 + \frac{\tau}{2})\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \frac{\tau^2 + \tau}{2}\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 \\ &\leq 2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2. \quad (4.6.15) \end{aligned}$$

Now, going back to (4.6.14) and rewriting it again we obtain

$$\begin{aligned} &R\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 + \frac{1}{2}\|e_n\|_V^2 \\ &\leq \|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \frac{\tau}{2}\left(\|\varepsilon_1\|_W^2 + \|\tau^{-1}e_1\|_W^2\right) + \frac{\tau^2}{2}\|\varepsilon_{n-1}\|_W^2 \\ &\quad + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \frac{\tau}{2}\|\varepsilon_1\|_W^2 + \frac{\tau}{2}\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=1}^{n-2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\ &\leq 2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau\|\varepsilon_{n-1}\|_W^2 \\ &\quad + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \tau\sum_{k=1}^{n-2}\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \end{aligned}$$

Therefore

$$R\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 + \frac{1}{2}\|e_n\|_V^2 \leq 2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau\|\varepsilon_{n-1}\|_W^2 + \tau\sum_{k=2}^{n-2}\|\varepsilon_k\|_W^2 + \frac{\tau e^{CT}}{R}\sum_{k=1}^{n-2}K_{k+1}. \tag{4.6.16}$$

Using the fact that $\tau N = T$ and substituting (4.6.15) into (4.6.16), we find that

$$\begin{aligned} &R\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 + \frac{1}{2}\|e_n\|_V^2 \\ &\quad\leq \left(1 + \frac{Te^{CT}}{R}\right)\left(2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau\sum_{k=2}^{N-2}\|\varepsilon_k\|_W^2\right) \\ &\quad\quad + \frac{\tau^2 e^{CT}}{R}\sum_{k=1}^{N-2}\|\varepsilon_k\|_W^2 + \tau\|\varepsilon_{n-1}\|_W^2. \end{aligned}$$

□

Corollary 4.6.1. *If $\tau^2 h^{-1} < C_I^{-1}$, then*

$$\begin{aligned} &R\|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2}\|e_N\|_V^2 \\ &\quad\leq \left(1 + \frac{Te^{CT}}{R}\right)\left(2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2\tau^4}\|e_1\|_W^2 + \tau\|\varepsilon_1\|_W^2 + \tau\sum_{k=2}^{N-2}\|\varepsilon_k\|_W^2\right) \\ &\quad\quad + \frac{\tau^2 e^{CT}}{R}\sum_{k=1}^{N-2}\|\varepsilon_k\|_W^2 + \tau\|\varepsilon_{N-1}\|_W^2. \end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right)$.

Proof. From (4.6.5) we have that $\|e_1\|_V^2 \leq C_I^2 h^{-2}\|e_1\|_W^2$, and therefore

$$\|e_1\|_V^2 \leq \tau^{-4}\|e_1\|_W^2,$$

since $\tau^2 h^{-1} < C_I^{-1}$. □

Corollary 4.6.2. *If $\tau^2 h^{-1} < C_I^{-1}$, then*

$$
\begin{aligned}
&R\|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2}\|e_N\|_W^2 \\
&\quad \leq \left(1 + \frac{Te^{CT}}{R}\right)\left(2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_W^2 + \tau\|\varepsilon_1\|_W^2 + \tau\sum_{k=2}^{N-2}\|\varepsilon_k\|_W^2\right) \\
&\qquad + \frac{\tau^2 e^{CT}}{R}\sum_{k=1}^{N-2}\|\varepsilon_k\|_W^2 + \tau\|\varepsilon_{N-1}\|_W^2.
\end{aligned}
$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h}\right)$.

4.6.2 Error estimate

In this subsection t_k is often denoted by t to simplify the notation.

To start it is necessary to estimate ε_k and ρ_k. Consider $u_h(t) = \sum_{j=1}^n u_j(t)\phi_j$, where $\{\phi_j\}$ is an orthonormal basis. Using Taylor expansions on the real valued functions u_j we obtain the following.

$$
u_j(t+\tau) = u_j(t) + \tau u_j'(t) + \frac{\tau^2}{2!}u_j''(t) + \frac{\tau^3}{3!}u_j'''(t) + \frac{1}{3!}\int_t^{t+\tau}(t+\tau-s)^3 u_j^{(4)}(s)\,ds \tag{4.6.17}
$$

and

$$
u_j(t-\tau) = u_j(t) - \tau u_j'(t) + \frac{\tau^2}{2!}u_j''(t) - \frac{\tau^3}{3!}u_j'''(t) + \frac{1}{3!}\int_t^{t-\tau}(t-\tau-s)^3 u_j^{(4)}(s)\,ds. \tag{4.6.18}
$$

Adding (4.6.17) and (4.6.18) yields

$$
\begin{aligned}
&\tau^{-2}\left(u_j(t+\tau) - 2u_j(t) + u_j(t-\tau)\right) - u_j''(t) \\
&\quad = \frac{1}{6\tau^2}\left(\int_t^{t+\tau}(t+\tau-s)^3 u_j^{(4)}(s)\,ds - \int_{t-\tau}^{t}(t-\tau-s)^3 u_j^{(4)}(s)\,ds\right) \\
&\quad = \varepsilon_{k,j}.
\end{aligned}
$$

Note that $\boldsymbol{\varepsilon}_k = \sum_{j=1}^n \varepsilon_{k,j}\phi_j$, therefore

$$
\begin{aligned}
\varepsilon_k &= \frac{1}{6\tau^2}\sum_{j=1}^n \left(\int_t^{t+\tau}(t+\tau-s)^3 u_j^{(4)}(s)\,ds - \int_{t-\tau}^t (t-\tau-s)^3 u_j^{(4)}(s)\,ds\right)\phi_j \\
&= \frac{1}{6\tau^2}\int_t^{t+\tau}(t+\tau-s)^3\sum_{j=1}^n u_j^{(4)}(s)\phi_j\,ds - \frac{1}{6\tau^2}\int_{t-\tau}^t (t-\tau-s)^3\sum_{j=1}^n u_j^{(4)}(s)\phi_j\,ds \\
&= \frac{1}{6\tau^2}\left(\int_t^{t+\tau}(t+\tau-s)^3 u_h^{(4)}(s)\,ds - \int_{t-\tau}^t (t-\tau-s)^3 u_h^{(4)}(s)\,ds\right).
\end{aligned}
$$

Using the above, we obtain

$$
\begin{aligned}
\|\varepsilon_k\|^2 &\le \left[\frac{1}{6\tau^2}\left\|\int_t^{t+\tau}(t+\tau-s)^3 u_h^{(4)}(s)\,ds + \int_{t-\tau}^t (t-\tau-s)^3 u_h^{(4)}(s)\,ds\right\|\right]^2 \\
&\le \frac{1}{36\tau^4}\left[\left\|\int_t^{t+\tau}(t+\tau-s)^3 u_h^{(4)}(s)ds\right\| + \left\|\int_{t-\tau}^t (t-\tau-s)^3 u_h^{(4)}(s)ds\right\|\right]^2 \\
&\le \frac{1}{36\tau^4}\left[\int_t^{t+\tau}\left\|(t+\tau-s)^3 u_h^{(4)}(s)\right\|ds + \int_{t-\tau}^t\left\|(t-\tau-s)^3 u_h^{(4)}(s)\right\|ds\right]^2 \\
&\le \frac{1}{36\tau^4}\Bigg[2\left(\int_t^{t+\tau}\left\|(t+\tau-s)^3 u_h^{(4)}(s)\right\|ds\right)^2 \\
&\qquad\qquad +2\left(\int_{t-\tau}^t\left\|(t-\tau-s)^3 u_h^{(4)}(s)\right\|ds\right)^2\Bigg] \\
&\le \frac{1}{18\tau^4}\Bigg[\int_t^{t+\tau}(t+\tau-s)^6\,ds\int_t^{t+\tau}\left\|u_h^{(4)}(s)\right\|^2 ds \\
&\qquad\qquad + \int_{t-\tau}^t (t-\tau-s)^6\,ds\int_{t-\tau}^t\left\|u_h^{(4)}(s)\right\|^2 ds\Bigg] \\
&= \frac{1}{18\tau^4}\left(\frac{\tau^7}{7}\right)\int_t^{t+\tau}\left\|u_h^{(4)}(s)\right\|^2 ds + \frac{1}{18\tau^4}\left(\frac{\tau^7}{7}\right)\int_{t-\tau}^t\left\|u_h^{(4)}(s)\right\|^2 ds \\
&= \frac{\tau^3}{126}\int_{t-\tau}^{t+\tau}\left\|u_h^{(4)}(s)\right\|^2 ds. \qquad (4.6.19)
\end{aligned}
$$

Using similar arguments we have

$$
\begin{aligned}
&2\tau^{-1}\big(u_j(t+\tau)-u_j(t-\tau)\big) - u_j'(t) \\
&\quad = \frac{1}{4\tau}\left(\int_t^{t+\tau}(t+\tau-s)^2 u_j'''(s)\,ds + \int_{t-\tau}^t (t-\tau-s)^2 u_j'''(s)\,ds\right) \\
&\quad = \rho_{k,j}.
\end{aligned}
$$

Now, $\rho_k = \sum_{j=1}^{n} \rho_{k,j}\phi_j$ and therefore

$$
\begin{aligned}
\|\rho_k\|^2 &\le \left[\frac{1}{4\tau}\left\|\int_t^{t+\tau}(t+\tau-s)^2 u_h'''(s)\,ds - \int_{t-\tau}^{t}(t-\tau-s)^2 u_h'''(s)\,ds\right\|\right]^2 \\
&\le \frac{1}{16\tau^2}\left[\left\|\int_t^{t+\tau}(t+\tau-s)^2 u_h'''(s)ds\right\| + \left\|\int_{t-\tau}^{t}(t-\tau-s)^2 u_h'''(s)ds\right\|\right]^2 \\
&\le \frac{1}{16\tau^2}\left[\int_t^{t+\tau}\left\|(t+\tau-s)^2 u_h'''(s)\right\|ds + \int_{t-\tau}^{t}\left\|(t-\tau-s)^2 u_h'''(s)\right\|ds\right]^2 \\
&\le \frac{1}{16\tau^2}\left[2\left(\int_t^{t+\tau}\left\|(t+\tau-s)^2 u_h'''(s)\right\|ds\right)^2\right. \\
&\qquad\qquad \left. +2\left(\int_{t-\tau}^{t}\left\|(t-\tau-s)^2 u_h'''(s)\right\|ds\right)^2\right] \\
&\le \frac{1}{8\tau^2}\left[\int_t^{t+\tau}(t+\tau-s)^4\,ds\int_t^{t+\tau}\|u_h'''(s)\|^2\,ds\right. \\
&\qquad\qquad \left. +\int_{t-\tau}^{t}(t-\tau-s)^4\,ds\int_{t-\tau}^{t}\|u_h'''(s)\|^2\,ds\right] \\
&= \frac{1}{8\tau^2}\left(\frac{\tau^5}{5}\right)\int_t^{t+\tau}\|u_h'''(s)\|^2\,ds + \frac{1}{18\tau^2}\left(\frac{\tau^5}{5}\right)\int_{t-\tau}^{t}\|u_h'''(s)\|^2\,ds \\
&= \frac{\tau^3}{40}\int_{t-\tau}^{t+\tau}\|u_h'''(s)\|^2\,ds. \qquad (4.6.20)
\end{aligned}
$$

Theorem 4.6.1. *Error estimate.*

If $f \in C^2([0,T],X)$, then, for each $t_k \in (0,T)$,

$$\|u_h(t_k) - u_k^h\|_V \le C_T R^{-\frac{1}{2}}\left(\tau^2 \max\left\|u_h^{(4)}(s)\right\|_W + \tau\max\|u_h'''(s)\|_W\right),$$

where C_T depends on T.

Proof. Recall that $\|e_1\|_W \le \frac{1}{2}\tau^2\|\varepsilon_0\| + \tau\|\rho_0\|$. From (4.6.19) and (4.6.20) we have that

$$\|\varepsilon_0\|^2 \le \frac{\tau^3}{126}\int_{-\tau}^{+\tau}\left\|u_h^{(4)}(s)\right\|^2 ds \quad \text{and} \quad \|\rho_0\|^2 \le \frac{\tau^3}{40}\int_{-\tau}^{+\tau}\|u_h'''(s)\|^2\,ds.$$

Therefore

$$\|e_1\|_W^2 \le \frac{\tau^7}{252}\int_{-\tau}^{\tau}\left\|u_h^{(4)}(s)\right\|^2 ds + \frac{\tau^5}{20}\int_{-\tau}^{\tau}\|u_h'''(s)\|^2\,ds. \qquad (4.6.21)$$

Substitution of (4.6.19), (4.6.20) and (4.6.21) into Corollary 4.6.1 yields

$$
\begin{aligned}
\|e_N\|_V^2 \leq & \left(1+\frac{Te^{CT}}{R}\right)\left[\frac{4\tau^2+1}{\tau^4}\left(\frac{\tau^7}{252}\int_{-\tau}^{\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds+\frac{\tau^5}{20}\int_{-\tau}^{\tau}\|u_h'''(s)\|_W^2 ds\right)\right.\\
& \left.+2\tau\left(\frac{\tau^3}{126}\int_{t_1-\tau}^{t_1+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds\right)+2\tau\sum_{k=2}^{N-2}\frac{\tau^3}{126}\int_{t-\tau}^{t+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds\right]\\
& +\frac{2\tau^2 e^{CT}}{R}\sum_{k=1}^{N-2}\frac{\tau^3}{126}\int_{t-\tau}^{t+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds+2\tau\left(\frac{\tau^3}{126}\int_{t_{N-1}-\tau}^{t_{N-1}+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds\right)\\
\leq & \left(1+\frac{Te^{CT}}{R}\right)\left[\frac{4\tau^5+\tau^3}{252}\int_{-\tau}^{\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds+\frac{4\tau^3+\tau}{20}\int_{-\tau}^{\tau}\|u_h'''(s)\|_W^2 ds\right.\\
& \left.+\frac{\tau^4}{63}\int_{t_1-\tau}^{t_1+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds+\frac{\tau^4}{63}\sum_{k=2}^{N-2}\int_{t-\tau}^{t+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds\right]\\
& +\frac{\tau^5 e^{CT}}{63R}\sum_{k=1}^{N-2}\int_{t-\tau}^{t+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds+\frac{\tau^4}{63}\int_{t_{N-1}-\tau}^{t_{N-1}+\tau}\left\|u_h^{(4)}(s)\right\|_W^2 ds\\
\leq & \left(1+\frac{Te^{CT}}{R}\right)\left[\frac{4\tau^6+\tau^4}{126}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{4\tau^4+\tau^2}{10}\max\|u_h'''(s)\|_W^2\right.\\
& \left.+\frac{2\tau^5}{63}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{2\tau^5}{63}\sum_{k=2}^{N-2}\max\left\|u_h^{(4)}(s)\right\|_W^2\right]\\
& +\frac{2\tau^6 e^{CT}}{63R}\sum_{k=1}^{N-2}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{2\tau^5}{63}\max\left\|u_h^{(4)}(s)\right\|_W^2\\
\leq & \left(1+\frac{Te^{CT}}{R}\right)\left[\frac{4\tau^6+\tau^4}{126}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{4\tau^4+\tau^2}{10}\max\|u_h'''(s)\|_W^2\right.\\
& \left.+\frac{2\tau^5}{63}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{2T\tau^4}{63}\max\left\|u_h^{(4)}(s)\right\|_W^2\right]\\
& +\frac{2T\tau^5 e^{CT}}{63R}\max\left\|u_h^{(4)}(s)\right\|_W^2+\frac{2\tau^5}{63}\max\left\|u_h^{(4)}(s)\right\|_W^2 .
\end{aligned}
$$

Since $\tau < 1$,

$$
\begin{aligned}
\|e_N\|_V^2 \leq & \left[\left(1+\frac{Te^{CT}}{R}\right)\frac{11\tau^4+4T\tau^4}{126}+\frac{2T\tau^5 e^{CT}+2\tau^5 R}{63R}\right]\max\left\|u_h^{(4)}(s)\right\|_W^2\\
& +\left(1+\frac{Te^{CT}}{R}\right)\frac{5\tau^2}{10}\max\|u_h'''(s)\|_W^2 ,
\end{aligned}
$$

where max denotes the maximum over the entire interval $[0, T]$. $\square$

4.7 Convergence of the fully discrete approximation for general damping

Finally, error estimates for the fully discrete approximation of the solution of Problem TBW is obtained by combining results from Section 4.5 with Theorem 4.6.1.

Estimate (4.5.5) is presented for convenience.

$$
\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \widehat{C}h^{\alpha-1}\big(|u(t)|_{\alpha,H^k} + C|u'(t)|_{\alpha,H^k}\big) \\
&\qquad + \sqrt{12e^{3t}}\widehat{C}h^{\alpha-1}\left[\int_0^T C|u''(\cdot)|_{\alpha,H^k} + 3K_a \max|u'(t)|_{\alpha,H^k} + 3K_a\int_0^T |u''(\cdot)|_{\alpha,H^k}\right. \\
&\qquad \left. + 2C|u_d|_{\alpha,H^k} + 2\sqrt{1+K_a}|u_0|_{\alpha,H^k} + \sqrt{K_a}|u_d|_{\alpha,H^k}\right], \qquad (4.7.1)
\end{aligned}
$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

From Theorem 4.6.1, if $f \in C^2([0,T], X)$, then

$$
\|u_h(t_k) - u_k^h\|_V \leq C_T R^{-\frac{1}{2}} \left(\tau^2 \max\left\|u_h^{(4)}(s)\right\|_W + \tau \max\|u_h'''(s)\|_W\right), \quad (4.7.2)
$$

for each $t_k \in (0,T)$.

Theorem 4.7.1. *Suppose*

(a) $u_0^h = \Pi u_0$ *and* $u_1^h = \Pi u_1$,

(b) $u \in C^2([0,T], V)$ *(Assumption A2),*

(c) $u''(t) \in \mathcal{L}^2([0,T], H^k \cap V)$,

(d) $f \in C^2([0,T], X)$ *and*

(e) the sequence $\{u_k^h\}$ *is a solution of Problem* TBW^h*-FD.*

Then

$$
\begin{aligned}
&\|u(t_k) - u_k^h\|_V \\
&\qquad \le \widehat{C} h^{\alpha-1}\big(|u(t)|_{\alpha,H^k} + C|u'(t)|_{\alpha,H^k}\big) \\
&\qquad + \sqrt{12e^{3t}}\widehat{C} h^{\alpha-1}\left[\int_0^T C|u''(\cdot)|_{\alpha,H^k} + 3K_a \max|u'(t)|_{\alpha,H^k}\right. \\
&\qquad \left. +3K_a \int_0^T |u''(\cdot)|_{\alpha,H^k} + 2C|u_d|_{\alpha,H^k} + 2\sqrt{1+K_a}|u_0|_{\alpha,H^k} + \sqrt{K_a}|u_d|_{\alpha,H^k}\right] \\
&\qquad + C_T R^{-\frac{1}{2}}\left(\tau^2 \max\left\|u_h^{(4)}(s)\right\|_W + \tau \max\|u_h'''(s)\|_W\right),
\end{aligned}
$$

where $\alpha = k$ *for* $2 \le k \le 4$ *and* $\alpha = 4$ *for* $k > 4$.

Proof. Use (4.7.1) and (4.7.2) and the triangle inequality. $\square$

Chapter 5

Beam with a damping tip body

5.1 Model problem

In the article [AS02] the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. The Euler-Bernoulli theory for a beam with Kelvin-Voigt damping is used. The beam is clamped at one end and the tip body is attached to the other end. The existence of a unique solution for the model problem is established.

The model is more realistic than other models discussed in articles: the fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account.

In [BDV14] the authors present a complete analysis of the model. Firstly they derive a correct linear approximation for the model problem. Next they also prove the existence of a unique solution. Their approach differs from that in [AS02]: they write the model problem in variational form and use results from [VV02] where general linear vibration models in variational form are considered.

In this chapter we study [BDV14], but we show that it makes virtually no difference if the Rayleigh model is used instead of the Euler-Bernoulli model. The Rayleigh model for the transverse vibration of a beam is derived in Subsection 1.2.4, equations (1.2.20), (1.2.23) and (1.2.24). For convenience the dimensionless equations of motion and constitutive equation is repeated here.

$$\begin{aligned} \partial_t^2 w &= \partial_x V + P_2, & (5.1.1) \\ \frac{1}{\alpha}\,\partial_t^2 \partial_x w &= V + \partial_x M, & (5.1.2) \\ M &= \frac{1}{\beta}\,\partial_x^2 w. & (5.1.3) \end{aligned}$$

Remark

It is convenient to use (5.1.1) and (5.1.2) to derive the variational form and in the process eliminate the shear force V.

The usual constitutive equation is (5.1.3) above. In [AS02] the authors include Kelvin-Voigt damping in the model. Equation (5.1.3) changes to

$$M = \frac{1}{\beta}\partial_x^2 w + \lambda \partial_t \partial_x^2 w, \tag{5.1.4}$$

where λ denotes a dimensionless damping parameter.

In [AS02] and [BDV14] the authors used the Euler-Bernoulli model instead of the Rayleigh model. So (5.1.2) reduce to

$$\partial_x M + V = 0. \tag{5.1.5}$$

In Section 5.3 we discuss how this affects the theory.

The constitutive equation for the moment M and the relation between the moment and shear force V are also used to model the interface conditions.

The left endpoint of the beam is clamped where the boundary conditions are the usual

$$w(0,t) = \partial_x w(0,t) = 0. \tag{5.1.6}$$

The interface conditions at the other endpoint are determined by the interaction between the beam and the rigid body. This is explained in [AS02] in some detail. It is necessary to consider the equations of motion for the rigid body carefully when deriving these conditions.

The position of the center of mass of the tip body relative to the endpoint of the beam is

$$d\cos\theta\ \mathbf{i} + d\sin\theta\ \mathbf{j},$$

where θ is the angle of the neutral plane with the horizontal. Therefore the velocity $\mathbf{v}_C$ and acceleration $\mathbf{a}_C$ of the center of mass are given by

$$\begin{aligned}\mathbf{v}_C &= \partial_t w(\ell,t)\,\mathbf{j} - d\dot{\theta}\sin\theta\,\mathbf{i} + d\dot{\theta}\cos\theta\,\mathbf{j},\\ \mathbf{a}_C &= \partial_t^2 w(\ell,t)\,\mathbf{j} - d\ddot{\theta}\sin\theta\,\mathbf{i} + d\ddot{\theta}\cos\theta\,\mathbf{j} - d\dot{\theta}^2\cos\theta\,\mathbf{i} - d\dot{\theta}^2\sin\theta\,\mathbf{j},\end{aligned}$$

where ℓ denotes the length of the beam. For the linear approximation it is assumed that the term $\dot{\theta}^2\sin\theta\,\mathbf{j}$ may be neglected, $\dot{\theta} \approx \partial_t\partial_x w(\ell,t)$, $\ddot{\theta} \approx \partial_t^2\partial_x w(\ell,t)$ and $\cos\theta \approx 1$. Using these approximations, we have the following expressions for the vertical components of the velocity and acceleration:

$$\partial_t w(\ell,t) + d\partial_t\partial_x w(\ell,t) \quad \text{and} \quad \partial_t^2 w(\ell,t) + d\partial_t^2\partial_x w(\ell,t).$$

The explanation above follows [BDV14].

In [AS02] the term $d\partial_t\partial_x w(\ell,t)$ in the expressions for the vertical component of the velocity is neglected. This should not be done. This point of view is motivated in the next section where we discuss the decay of energy for the system.

An external force $f_B(t)$ may act on the rigid body, e.g. gravity. Using Newton's second law for the motion of the center of mass we have

$$\begin{aligned}&m\partial_t^2 w(\ell,t) + md\,\partial_t^2\partial_x w(\ell,t)\\ &\quad = -V(\ell,t) + f_B(t) - \gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t).\end{aligned} \tag{5.1.7}$$

Taking moments about the center of mass we have

$$J\partial_t^2\partial_x w(\ell,t) = -M(\ell,t) + dV(\ell,t) - d\gamma^*\partial_t\partial_x w(\ell,t). \tag{5.1.8}$$

Combining Equations (5.1.7) and (5.1.8) results in

$$\begin{aligned}&md\,\partial_t^2 w(\ell,t) + (J+md^2)\partial_t^2\partial_x w(\ell,t)\\ &\quad = -M(\ell,t) - \gamma d\partial_t w(\ell,t) - (\gamma d + \gamma^*)d\partial_t\partial_x w(\ell,t) + df_B(t)\end{aligned} \tag{5.1.9}$$

It is convenient to rewrite Equations (5.1.7) and (5.1.9):

$$\begin{aligned}V(\ell,t) &= -m\partial_t^2 w(\ell,t) - md\,\partial_t^2\partial_x w(\ell,t)\\ &\qquad -\gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t) + f_B(t), \qquad (5.1.10)\\ M(\ell,t) &= -md\,\partial_t^2 w(\ell,t) - (J+md^2)\partial_t^2\partial_x w(\ell,t)\\ &\qquad -\gamma d\partial_t w(\ell,t) - (\gamma d + \gamma^*)d\partial_t\partial_x w(\ell,t) + df_B(t). \qquad (5.1.11)\end{aligned}$$

Problem Beam-DB

The mathematical model consists of

the Equations of motion (5.1.1) and (5.1.2),

the Constitutive equation (5.1.4) for the beam and

the Boundary conditions (5.1.6) and the Interface conditions (5.1.10) and (5.1.11).

Initial conditions $w(\cdot, 0) = w_0$ and $\partial_t w(\cdot, 0) = w_1$ need to be specified.

It is interesting that the Timoshenko theory is not considered in [AS02] and [BDV14] as in [ZVV04]. However, here the interface conditions are more realistic.

The variational form and weak variational form are derived in Sections 5.2 and 5.3. Auxillary results are proved in 5.4 and the existence theorems are in Section 5.5. Finally we present convergence analysis of the Galerkin approximation in Section 5.6.

5.2 Variational form

Exactly as in Section 2.1 we find that

$$\begin{aligned}
& \int_0^1 \partial_t^2 w(\cdot, t)v + \frac{1}{\alpha}\int_0^1 \partial_t^2 \partial_x w(\cdot, t)v' \\
= & -\int_0^1 M(\cdot, t)v'' + [V(x, t)v(x)]_0^1 + [M(x, t)v'(x)]_0^1 + \int_0^1 f(\cdot, t)v
\end{aligned}$$

Test functions

A function v is a test function if $v \in C^2[0, 1]$ and
$v(0) = v'(0) = 0$. The space of test functions is denoted by $T[0, 1]$.

It follows that

$$\begin{aligned}
& \int_0^1 \partial_t^2 w(\cdot, t)v + \frac{1}{\alpha}\int_0^1 \partial_t^2 \partial_x w(\cdot, t)v' \\
& \quad = -\int_0^1 M(\cdot, t)v'' + V(1, t)v(1) + M(1, t)v'(1) + \int_0^1 f(\cdot, t)v
\end{aligned}$$

for each $v \in T[0, 1]$.

We now use the constitutive equation $M = \frac{1}{\beta}\partial_x^2 w + \lambda\partial_t\partial_x^2 w$ and the interface conditions (5.1.10) and (5.1.11) to derive the variational form of the model problem.

It is convenient to introduce the following bilinear forms:

$$\begin{aligned}
\bar{b}(u,v) &= \int_0^1 \frac{1}{\beta} u''v'', \\
\bar{c}_{EB}(u,v) &= \int_0^1 uv + mu(1)v(1) + mdu'(1)v(1) + mdu(1)v'(1) \\
&\quad +(J+md^2)u'(1)v'(1), \\
\bar{c}_R(u,v) &= \int_0^1 uv + \frac{1}{\alpha}\int_0^1 u_1'v_1' + mu(1)v(1) + mdu'(1)v(1) \\
&\quad +mdu(1)v'(1) + (J+md^2)u'(1)v'(1), \\
\bar{a}(u,v) &= \int_0^1 \lambda u''v'' + \gamma u(1)v(1) + \gamma du'(1)v(1) + \gamma du(1)v'(1) \\
&\quad +(\gamma d + \gamma^*)du'(1)v'(1).
\end{aligned}$$

Notation

The bilinear forms $\bar{c}_{EB}(u,v)$ and $\bar{c}_R(u,v)$ correspond to the Euler-Bernoulli and Rayleigh models respectively. The notation $\bar{c}(u,v)$ will be used with the understanding that it can be either $\bar{c}_{EB}(u,v)$ or $\bar{c}_R(u,v)$.

We now have the variational form of the model problem.

Problem Beam-DBV

Find w such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$ and

$$\begin{aligned}
&\bar{c}(\partial_t^2 w(\cdot,t),v) + \bar{a}(\partial_t w(\cdot,t),v) + \bar{b}(w(\cdot,t),v) \\
&\quad = (f(\cdot,t),v) + f_B(t)v(1) + df_B(t)v'(1)
\end{aligned}$$

for each $v \in T(0,1)$ with $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$.

Remark

Note that the bilinear forms $\bar{a}$, $\bar{b}$ and $\bar{c}$ are symmetric. The additional term $\gamma du'(1)v(1)$ in the definition of $\bar{a}$ is necessary for symmetry. Problem BeamDBV may be used to compute finite element approximations.

Mechanical energy

The mechanical energy (kinetic energy plus elastic potential energy) of the system is

$$E(t) = \frac{1}{2}\,\bar{c}(\partial_t w(\cdot,t), \partial_t w(\cdot,t)) + \frac{1}{2}\,\bar{b}(w(\cdot,t), w(\cdot,t)).$$

Using the symmetry of $\bar{b}$ and $\bar{c}$ and assuming that w is sufficiently smooth, we have

$$E'(t) = \bar{c}(\partial_t^2 w, \partial_t w) + \bar{b}(w, \partial_t w) = -\bar{a}(\partial_t w, \partial_t w)$$

for the homogeneous case. It is obvious that $\bar{b}(u,u)$ is nonnegative and not difficult to show that $\bar{a}(u,u)$ and $\bar{c}(u,u)$ are nonnegative.

$$\begin{aligned}
\bar{a}(u,u) &= \int_0^1 \lambda(u'')^2 + \gamma[u(1)]^2 + 2\gamma d u(1)u'(1) + (\gamma d + \gamma^*)d[u'(1)]^2 \\
&= \int_0^1 \lambda(u'')^2 + \gamma[u(1) + du'(1)]^2 + \gamma^* d[u'(1)]^2 \geq 0, \\
\bar{c}_{EB}(u,u) &= \int_0^1 u^2 + m[u(1)]^2 + 2mdu'(1)u(1) + (J + md^2)[u'(1)]^2 \\
&= \int_0^1 u^2 + m[u(1) + du'(1)]^2 + J[u'(1)]^2 \geq 0.
\end{aligned}$$

Since $\bar{c}_R(u,u) \geq \bar{c}_{EB}(u,u)$, it follows that $\bar{c}_R(u,u)$ is also nonnegative. As a result $E'(t) \leq 0$. This result is to be expected from Physics. The fact that $\bar{a}$ is symmetric and $\bar{a}(u,u)$ nonnegative, is due to the additional term.

5.3 Weak variational form

Consider Problem Beam-DBV. We start of as usual by considering the closure of the space of test functions. Let $V(0,1)$ be the closure of $T[0,1]$ in $H^2(0,1)$, then $V(0,1)$ is a Hilbert space (being a closed subspace of a Hilbert space).

We require the **trace operator** which we denote by Γ, see Appendix C. Clearly $\Gamma u'$ is defined for $u \in H^2(0,1)$.

The following product spaces are necessary for the abstract problem.

Product spaces

$$X = \mathcal{L}^2(0,1) \times \mathbb{R} \times \mathbb{R}$$

$$H^m = H^m(0,1) \times \mathbb{R} \times \mathbb{R}$$

$$V_P = V(0,1) \times \mathbb{R} \times \mathbb{R}$$

$$V = \{ v \in V_P \;\big|\; v_2 = \Gamma v_1,\ \ v_3 = \Gamma v_1' \ \}$$

An element $y \in X$ is written as $y = \langle y_1, y_2, y_3 \rangle$. An obvious inner product for X is

$$(u,v)_X = \int_0^1 \ u_1 v_1 + u_2 v_2 + u_3 v_3,$$

and we denote the corresponding norm by $\| \cdot \|_X$.

Definition Bilinear forms

For u and v in X

$$\begin{aligned} c_{EB}(u,v) &= \int_0^1 \ u_1 v_1 + m u_2 v_2 + md(u_3 v_2 + u_2 v_3) + (J + md^2) u_3 v_3, \\ c_R(u,v) &= \int_0^1 \ u_1 v_1 + \frac{1}{\alpha} \int_0^1 u_1' v_1' + m u_2 v_2 \\ &\qquad + \ md(u_3 v_2 + u_2 v_3) + (J + md^2) u_3 v_3, \end{aligned}$$

and for u and v in H^2

$$\begin{aligned} b(u,v) &= \int_0^1 \ \frac{1}{\beta} u_1'' v_1'' \quad \text{and} \\ a(u,v) &= \int_0^1 \ \lambda u_1'' v_1'' + \gamma u_2 v_2 + \gamma d(u_3 v_2 + u_2 v_3) + (\gamma d + \gamma^*) d u_3 v_3. \end{aligned}$$

Remarks

1. $c_R(u,u) \geq c_{EB}(u,u)$.
2. The bilinear forms a, b and c are symmetric.
3. For u and v in V, $a(u,v) = \bar{a}(u_1,v_1)$, $b(u,v) = \bar{b}(u_1,v_1)$ and $c(u,v) = \bar{c}(u_1,v_1)$.

To apply the existence theory and prove convergence of the finite element approximation, we need to prove that $c(u,u) \geq K\|u\|_X^2$. In the rest of the chapter we will show that the theory is the same for both models. From a theoretical perspective this is an improvement on both articles ([AS02] and [BDV14]). In practical applications it does not matter much which model is used.

For the weak variational form of the model problem we need to show that the bilinear forms c and b are inner products for X and V respectively. We use the well known Poincare type inequalities, see Appendix B, the boundedness of Γ is also required.

Proposition 5.3.1. *For each $u \in V(0,1)$,*

$$\begin{aligned} \|u\| &\leq \|u'\| \leq \|u''\| \quad \textit{and} \\ |\Gamma u| &\leq \|u'\|. \end{aligned}$$

Proof. Using the fundamental theorem of calculus, the inequalities are easy to prove for the space of test functions $T[0,1]$. Since $V(0,1)$ is the closure of $T[0,1]$ with respect to the norm of $H^2(0,1)$ the result follows (see Appendix B). □

Proposition 5.3.2. *There exists a constant K such that*

$$c_{EB}(u,u) \quad \geq \quad K(u,u)_X \quad \textit{for each } u \in X.$$

Proof. From Young's inequality

$$m(u_2 + du_3)^2 \geq m(1-\epsilon)u_2^2 + m(1 - \frac{1}{\epsilon})d^2u_3^2.$$

By the definition of the moment of inertia of a rigid body, $J \geq \beta md^2$ for some $\beta > 0$. Therefore

$$m(u_2 + du_3)^2 + Ju_3^2 \geq m(1-\epsilon)u_2^2 + m(1 - \frac{1}{\epsilon})d^2u_3^2 + \beta md^2u_3^2.$$

Now choose $0 < \epsilon < 1$ and $\epsilon < (1+\beta)^{-1}$, it follows that there exists a constant K_1 such that $m(u_2 + du_3)^2 + Ju_3^2 \geq K_1(u_2^2 + u_3^2)$ for each $(u_2, u_3) \in \mathbb{R}_2$ and the result follows. □

Remark

The result above also holds for c_R since $c_R(u,u) \geq c_{EB}(u,u)$.

Corollary 5.3.1. *The bilinear form c is an inner product for the space X.*

Definition Inertia space

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$. We refer to the vector space X equipped with this norm as the inertia space and denote it by W.

Proposition 5.3.3. *There exists a constant K such that*

$$b(u,u) \;\geq\; K\|u_1\|_2^2 \;\; \textit{for each } u \in V.$$

Proof. We use Proposition 5.3.1.

$$\begin{aligned}\|u_1\|_2^2 &= \|u_1\|^2 + \|u_1'\|^2 + \|u_1''\|^2 \\ &\leq 3\|u_1''\|^2 \\ &= 3\beta\, b(u,u).\end{aligned}$$

□

Corollary 5.3.2. *The bilinear form b is an inner product for V.*

Proof. Clearly $b(u,u) = 0$ implies that $u_1 = 0$ and therefore $u_2 = \Gamma u_1 = 0$ and $u_3 = \Gamma u_1' = 0$. □

Definition Energy space

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

We proceed to determine a weak variational form of the model problem. Denote the function $t \to f(\cdot,t)$ by $\tilde{f}$ and let $f_X(t) = \langle \tilde{f}(t), f_B(t), df_B(t)\rangle$, $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3}\rangle$ and $u_d = \langle w_1, \tilde{u}_{d,2}, \tilde{u}_{d,3}\rangle$, with $\tilde{u}_{0,2},\ \tilde{u}_{0,3},\ \tilde{u}_{d,2}$ and $\tilde{u}_{d,3}$ arbitrary.

Problem Beam-DBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (f_X(t),v)_X \quad \text{for each } v \in V$$

with $u(0) = u_0$ and $u'(0) = u_d$.

Remark

The restrictions to $\tilde{u}_{0,2},\ \tilde{u}_{0,3},\ \tilde{u}_{d,2}$ and $\tilde{u}_{d,3}$ are discussed in Section 5.5.

5.4 Auxiliary results

We need the results of this section to apply the existence theorem in Section 2.2.

Proposition 5.4.1. *The space V is a dense subset of X.*

Proof. Consider any $y \in W$. Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ there exists a sequence $\{\phi_n\} \subset C_0^\infty(0,1)$ such that $\|\phi_n - y_1\| \to 0$.

It is not difficult to construct sequences $\{\eta_n\}$ and $\{\zeta_n\}$ in $H^2(0,1)$ with the following properties

$$\begin{aligned} \Gamma\eta_n &= 1, \ \ \Gamma\eta_n' = 0 \ \text{ and } \ \|\eta_n\| \to 0, \\ \Gamma\zeta_n &= 0, \ \ \Gamma\zeta_n' = 1 \ \text{ and } \ \|\zeta_n\| \to 0. \end{aligned}$$

For example, let $\eta_n = 0$ on $[0, 1 - \frac{1}{n}]$ and equal to a cubic polynomial on $(1 - \frac{1}{n}, 1]$ where $\eta_n(1 - \frac{1}{n}) = \eta_n'(1 - \frac{1}{n}) = 0$, $\eta_n(1) = 1$ and $\eta_n'(1) = 0$.

Now, let $v_n = \phi_n + y_2\eta_n + y_3\zeta_n$, then $v_n \in V(0,1)$, $\Gamma v_n = y_2$ and $\Gamma v_n' = y_3$.

Consequently $u_n = \langle v_n, v_n(1), v_n'(1)\rangle \in V$ and $\|u_n - y\|_X \to 0$. □

Proposition 5.4.2. *There exists a constant K such that*

$$b(u,u) \ \geq \ Kc_R(u,u) \ \text{ for each } u \in V.$$

Proof. We use Proposition 5.3.1.

$$\begin{aligned} c_R(u,u) &= \int_0^1 u_1^2 + \frac{1}{\alpha}\int_0^1 (u_1')^2 + m(u_2 + du_3)^2 + Ju_3^2 \\ &= \int_0^1 u_1^2 + \frac{1}{\alpha}\int_0^1 (u_1')^2 + m(\Gamma u_1 + d\Gamma u_1')^2 + J(\Gamma u_1')^2 \\ &= \|u_1\|^2 + \frac{1}{\alpha}\|u_1'\|^2 + m(\Gamma u_1 + d\Gamma u_1')^2 + J(\Gamma u_1')^2 \\ &\leq \|u_1''\|^2 + \frac{1}{\alpha}\|u_1''\|^2 + 2m(\Gamma u_1)^2 + 2md^2(\Gamma u_1')^2 + J(\Gamma u_1')^2 \\ &\leq \|u_1''\|^2 \ (1 + \frac{1}{\alpha} + 2m + 2md^2 + J) \\ &\leq \|u_1\|_2^2 \ (1 + \frac{1}{\alpha} + 2m + 2md^2 + J) \end{aligned}$$

Now apply Proposition 5.3.3. □

Remark

Since $c_R(u,u) \geq c_{EB}(u,u)$, the estimate above also holds for c_{EB}.

Proposition 5.4.3. *There exists a constant K such that*

$$|a(u,v)| \leq K\|u\|_V \, \|v\|_V.$$

for each u and v in V.

Proof. We use Proposition 5.3.1.

$$\begin{aligned}
|a(u,v)| &\leq \lambda\|u_1''\| \, \|v_1''\| + \gamma|u_2v_2| + \gamma d|u_3v_2| + \gamma d|u_2v_3| \\
&\quad +(\gamma d + \gamma^*)d|u_3v_3| \\
&= \lambda\|u_1''\| \, \|v_1''\| + \gamma|\Gamma u_1\Gamma v_1| + \gamma d|\Gamma u_1'\Gamma v_1| + \gamma d|\Gamma u_1\Gamma v_1'| \\
&\quad +(\gamma d + \gamma^*)d|\Gamma u_1'\Gamma v_1'| \\
&\leq \|u_1\|_2\|v_1\|_2(\lambda + \gamma + 2\gamma d + (\gamma d + \gamma^*)d \ ).
\end{aligned}$$

Now use Proposition 5.3.3. □

The result above is true for $\lambda \geq 0$. If $\lambda > 0$, the bilinear form a is positive definite on V and this has implications for existence results.

Proposition 5.4.4.

$$a(u,u) \geq \lambda\beta\|u\|_V^2$$

Proof. From the definitions of a and $\|\cdot\|_V$,

$$a(u,u) \geq \lambda\|u_1''\|^2 \geq \lambda\beta\|u\|_V^2.$$

□

5.5 Existence

Theorem 2.3.1 is applied to the case where $\lambda = 0$ and Theorem 2.3.3 to the case where $\lambda > 0$. Note that the Assumptions E1, E2, E3 and E4 are satisfied due to Propositions 5.4.1, 5.4.2, 5.3.2 and 5.4.3 respectively.

Theorem 5.5.1. *Suppose* $\lambda = 0$ *and*

(a) $\tilde{f} \in C^1([0,\tau), \mathcal{L}^2(0,1))$ *and* $f_B \in C^1([0,\tau), \mathbb{R})$,

(b) $u_0 \in V$, $u_d \in V$ *and there exists a* $y \in W$ *such that*

$$b(u_0, v) + a(u_d, v) = c(y, v) \text{ for each } v \in V.$$

Then Problem PW has a unique solution
$u \in C([0,\tau), V) \cap C^1([0,\tau), W) \cap C^1((0,\tau), V) \cap C^2((0,\tau), W)$.

Proof. Clearly $f_X \in C^1([0,\tau), X)$. The result follows from Theorem 2.3.1. □

Now consider the case $\lambda > 0$. The bilinear form a satisfies Assumption E5S (Proposition 5.4.4).

Theorem 5.5.2. *Suppose* $\lambda > 0$ *and*

(a) $\tilde{f}$ *is locally Lipschitz on* $[0,\tau)$ *with respect to the norm of* $\mathcal{L}^2(0,1)$ *and* f_B *is locally Lipschitz on* $[0,\tau)$,

(b) $u_0 \in V$ *and* $u_d \in W$.

Then Problem PW has a unique solution
$u \in C([0,\tau), V) \cap C^1([0,\tau), W) \cap C^2((0,\tau), W)$.
If $\tilde{f} = f_B = 0$, *then* $u \in C\big([0,\infty); V\big) \cap C^1\big([0,\infty); W\big) \cap C^\infty\big((0,\infty); V\big)$.

Proof. Clearly f_X is locally Lipschitz on $[0,\tau)$ with respect to the norm $\|\cdot\|_X$ and the bilinear form a is positive definite by Proposition 5.4.4. The result follows from Theorem 2.3.3. □

Sufficient conditions for existence

The case $\lambda > 0$ is trivial. It is sufficient that $\tilde{f}$ and f_B have continuous derivatives.

If $\lambda = 0$ sufficient conditions on $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3}\rangle$ and $u_d = \langle w_1, \tilde{u}_{d,2}, \tilde{u}_{d,3}\rangle$, are required to satisfy Condition (b) in Theorem 5.5.1. It is obviously necessary to assume that u_0 and u_d are in V which implies that $u_0 = \langle w_0, \Gamma w_0, \Gamma w_0'\rangle$ and $u_d = \langle w_1, \Gamma w_1, \Gamma w_1'\rangle$. First, suppose $w_0 \in C^4[0,1]$ and $w_1 \in C^2[0,1]$.

From the definition of the bilinear form b, using integration by parts we obtain

$$b(u,v) = \int_0^1 EIw_0^{(4)}v_1 - EIw_0'''(1)v_1(1) + EIw_0''(1)v_1'(1).$$

From the definition of a, we have

$$\begin{aligned} & b(u,v) + a(u,v) \\ &= \int_0^1 EIw_0^{(4)}v_1 - EIw_0'''(1)v_1(1) + EIw_0''(1)v_1'(1) \\ &\quad +\gamma w_1(1)v_1(1) + \gamma dw_1'(1)v_1(1) + \gamma dw_1(1)v_1'(1) + (\gamma d + \gamma^*)dw_1'(1)v_1'(1). \end{aligned}$$

Therefore

$$b(u,v) + a(u,v) = \int_0^1 EIw_0^{(4)}v_1 \quad \text{for each} \quad v \in V$$

if and only if

$$\begin{aligned} 0 \;=\; & -EIw_0'''(1)v_1(1) + EIw_0''(1)v_1'(1) \\ & +\gamma w_1(1)v_1(1) + \gamma dw_1'(1)v_1(1) + \gamma dw_1(1)v_1'(1) + (\gamma d + \gamma^*)dw_1'(1)v_1'(1) \end{aligned}$$

for each $v \in V$. Since $v_1(1)$ and $v_1'(1)$ are arbitrary, it follows that

$$\begin{aligned} -EIw_0'''(1) + \gamma w_1(1) + \gamma dw_1'(1) &= 0; \\ EIw_0''(1) + \gamma dw_1(1) + (\gamma d + \gamma^*)dw_1'(1) &= 0. \end{aligned}$$

Therefore a sufficient condition for existence is $w_0 \in C^4[0,1]$, $w_1 \in C^2[0,1]$ and

$$\begin{aligned} w_0(0) &= w_0'(0) = w_1(0) = w_1'(0) = 0; \\ V(1,0) &= -EIw_0'''(1) = -\gamma w_1(1) - \gamma dw_1'(1); \\ M(1,0) &= EIw_0''(1) = -\gamma dw_1(1) - (\gamma d + \gamma^*)dw_1'(1). \end{aligned}$$

Up to this point we followed [BDV14]. But it is still sufficient that $u_0 \in H^2 \cap V$ and $u_d \in V$. Taking limits we see that the equations above are satisfied in the sense of trace.

The conditions for the shear force $V(1,0)$ and bending moment $M(1,0)$ is important. Comparing them to Equations (5.1.7) and (5.1.9) we see that the force and moment at the endpoint must match the force and moment due to damping otherwise there will be discontinuities in the force and moment.

5.6 Galerkin approximation

Consider the dimensionless form of the model problem, see Section 5.3. To start, construct a finite dimensional subspace $S^h(0,1)$ of $V(0,1)$, using piecewise Hermite cubic basis functions. Let S^h be the subspace of $S^h(0,1)\times\mathbb{R}\times\mathbb{R}$ where $u_2 = u(1)$ and $u_3 = u'(1)$ for each $u \in S^h$. It follows that S^h is a finite dimensional subspace of V. Consider the Galerkin approximation.

Problem Beam-DBW^h

Find a function $u_h \in C^2(0,T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f_X(t), v)_X \quad \text{for each } v \in S^h$$

with $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$. The functions u_0^h and u_d^h must be suitable approximations for u_0 and u_d in S^h.

The existence of a unique solution follows from Theorem 3.1.1 provided that $\tilde{f}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$ and f_B is continuous.

5.6.1 Interpolation estimates for the one-dimensional hybrid models

An interpolation operator on the product spaces H^k can now be defined.

Definition

$$\Pi u = \langle \Pi_c\, u_1,\, u_2,\, u_3\rangle \quad \text{for } u \in H^k.$$

Note that $u - \Pi u = \langle u_1 - \Pi_c u_1, 0, 0\rangle$. From Propositions 5.3.3 and 5.4.2, we have that $\|u\|_V$ and $\|u_1\|_2$ are equivalent. Therefore

$$\|u - \Pi u\|_V^2 \le C\|u_1 - \Pi_c\, u_1\|_2^2.$$

For Hermite cubic basis functions:
If $u \in H^k(0,1)$ with $k \ge 3$, then there exists a constant $\widehat{C}_c$ such that,

$$\|u - \Pi_c u\|_2 \le \widehat{C}_c h^\alpha |u|_{\alpha+2}\,,$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \ge 4$. (Appendix E.)

Using the result above, we obtain an estimate for the interpolation operator Π:
If $u \in H^k \cap V$ with $k \geq 3$, then

$$\|u - \Pi u\|_V \leq \widehat{C} h^\alpha |u_1|_{\alpha+2}, \tag{5.6.1}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

The above inequality replaces Assumption A3. Assumption 2 is still needed, i.e. assume that $u_1 \in C^2\left([0,\tau),\ V(0,1)\right)$.

5.6.2 Convergence for the semi-discrete approximation

In a similar way to what was done in Section 4.5, we now apply the results from Section 3.4 (general damping). In the applications below, we assume that $u \in C^2([0,T],V)$ and use estimate (5.6.1).

Instead of Assumption A3 regarding the existence of an interpolation operator, we now construct an interpolation operator.

The interpolation operator Π is defined on the product spaces H^k by

$$\Pi u = \langle \Pi_c\, u_1,\, \Pi_c\, u_2,\, u_3,\, u_4 \rangle \quad \text{for } u \in H^k,$$

where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions. A **seminorm** for H^m is defined by

$$|u|_{k,H^m} \;=\; \sqrt{|u_1|_k^2 + |u_2|_k^2}\,,$$

with $|\cdot|_k$ the seminorm in $H^m(\Omega)$.

Recall that the symbol α does not have the same meaning as in Assumption A3 and note that for Problem Beam-DBW^h $H(V,k) = H^k \cap V$.

Application of Corollary 3.4.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0,T], H^k(\Omega) \cap V)$ with $k \geq 3$. Then, for $t \in [0,T]$,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \widehat{C} h^\alpha \big(|u_1(t)|_{\alpha+2} + C_b |u_1'(t)|_{\alpha+2}\big) \\
&\qquad + \sqrt{12 e^{3t}}\,\widehat{C} h^\alpha \left[\int_0^T C_b |u_1''(\cdot)|_{\alpha+2} + 3K_a \max |u_1'(t)|_{\alpha+2} + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} \right. \\
&\qquad \left. + 2C_b |u_d|_{\alpha+2} + 2\sqrt{1+K_a}\,|u_0|_{\alpha+2} + \sqrt{K_a}\,|u_d|_{\alpha+2}\right],
\end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

As mentioned before, in some cases we have that Assumption A4 is satisfied and therefore it is possible to use the Aubin-Nitsche trick to obtain **higher order estimates** in the inertia norm for the projection error.

Assumption A4

$E_b \subset H(V,k)$ and for any $u \in E_b$ there exists a constant $\widehat{c}_b$ such that $\|u\|_{H(V,k)} \leq \widehat{c}_b \|y\|_W$, where

$$b(u,v) = (y,v) \ \forall \ v \in V.$$

If $E_b \subset H^k \cap V$, then Assumption A4 is satisfied. (Recall the definition of E_b is Subsection 2.3.2)

Application of Corollary 3.4.2

Assume that $K\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0,T], H^k(\Omega) \cap V)$ with $k \geq 3$. Then, for any $t > 0$,

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & \widehat{C}^2 h^{2\alpha} \widehat{c}_b |u_1(t)|_{\alpha+2} + \sqrt{12e^{3t}}\, C_b \widehat{C}^2 h^{2\alpha} \widehat{c}_b \left(\int_0^T C_b |u_1''(\cdot)|_{\alpha+2} \right. \\ & \left. + 3K_a \max |u_1'(t)|_{\alpha+2} + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} + \sqrt{K_a} |u_d|_{\alpha+2} \right) \\ & + \sqrt{12e^{3t}} C_b K h^{2\alpha},\end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

Error estimates and convergence for the fully discrete approximation is the same as in Section 4.6.

Chapter 6

Vertical structures

6.1 Beam models for vertical structures

6.1.1 Introduction

For industrial chimneys, towers and high-rise buildings the height is large compared to the diameter. We refer to them as vertical structures.

Clearly, there exist a number of possibilities to model a vertical structure. Beam models appear to be popular but in the articles considered no justification was offered. In this chapter we compare Timoshenko, Rayleigh and Euler-Bernoulli models for vertical structures.

We consider buildings and slender structures separately. When is it slender? Beam models for chimneys seem reasonable but beam models for buildings are questionable. (Tall buildings are often modeled as vertical beams, see e.g. [HV07] and [WFH01] and the references in [WFH01].) We return to this matter in Subsection 6.1.2.

The oscillation of structures due to steady winds has been studied for a long time now: " ... *free-standing welded steel structures are prone to oscillate in the wind.*" [New89].

Modeling the effect of wind is for example discussed in [FL10] and [HV07]. The load caused by vortex shedding leads to oscilation of the structure. In recent years, more and more supertall constructions have been built and these are also affected by the wind.

Modeling of earthquakes are considered in [WFH01].

In [WFH01] the authors refer to earlier studies where it is recommended that " ... for increasing the levels of structural safety, integrity and occupant comfort, it is necessary to reduce the levels of earthquake- or wind-induced displacements and accelerations in tall buildings."

Simulation is necessary to determine displacements, acceleration and stresses in structures subjected to earthquake loads or wind loads.

6.1.2 Beam models for structures

Recall that our objective is to compare models. To start, consider the most general linear model. We use the equations of motion (1.2.6), (1.2.7) and (1.2.8) with constitutive equations (1.2.11) and (1.2.12). From (1.2.6),

$$0 = \partial_x S + P_1,$$

we see that the axial force due to gravity is given by

$$S(x) = -\rho A g \ell (1-x).$$

As a consequence we have a moment density (measured in Newton) $L = -S\partial_x w$ and equation (1.2.15) changes to (6.1.4) below. With $\mu = \dfrac{\rho g \ell}{G\kappa^2}$ and using the original notation, the dimensionless moment density is given by

$$L(x,t) = \mu(1-x)\partial_x w(x,t). \tag{6.1.1}$$

Note that $\partial_x u = \frac{S}{AE}$ is dimensionless. Now,

$$\frac{S}{AE} = -\frac{\rho A g \ell (1-x)}{A\kappa^2 G}\frac{\kappa^2 G}{E} = -\mu\gamma(1-x).$$

Therefore

$$\partial_x u = -\mu\gamma(1-x). \tag{6.1.2}$$

One may now consider Timoshenko, Rayleigh and Euler-Bernoulli models. In each case $\partial_x u$ and L are given by (6.1.2) and (6.1.1) respectively. For convenience the equations of motion and constitutive equation in dimensionless form are presented on the following page.

Timoshenko model

$$\begin{aligned}
\partial_t^2 w &= \partial_x V + q, && (6.1.3)\\
\frac{1}{\alpha}\,\partial_t^2 \phi &= (1+\partial_x u)V + \partial_x M + L, && (6.1.4)\\
M &= \frac{1}{\beta}\,\partial_x \phi, && (6.1.5)\\
V &= \partial_x w - \phi. && (6.1.6)
\end{aligned}$$

Rayleigh model

$$\begin{aligned}
\partial_t^2 w &= \partial_x V + q, && (6.1.7)\\
\frac{1}{\alpha}\,\partial_t^2 \partial_x w &= (1+\partial_x u)V + \partial_x M + L, && (6.1.8)\\
M &= \frac{1}{\beta}\,\partial_x^2 w. && (6.1.9)
\end{aligned}$$

Remark

It is necessary to eliminate the shear force V in the Rayleigh model since shear strain is not in the mathematical model, see Subsection 6.1.4.

We return to the question: When is a vertical structure slender? As mentioned in Subsection 1.2.5, the Timoshenko model compares well to two-dimensional and three-dimensional models but the Rayleigh model (which is practically the same as the Euler Bernoulli model) can only be used for slender structures. Using the articles [VV06] and [LVV09a] it is clear that $\alpha = 1000$ is an absolute minimum, preferably $\alpha > 5000$. There is no clear threshold for α, it depends on the number of relevant modes and this in turn depends also on the dynamic load.

6.1.3 Applications

In Section 6.2 we compare beam models for buildings. Simulation of Timoshenko and Rayleigh models are carried out. We mention briefly a vertical Euler-Bernoulli model with damping mechanism on top in [HV07].

In Section 6.3 a vertical Euler-Bernoulli model but with concentrated masses to represent the floors in [WFH01] is mentioned. We consider multiple beam models with concentrated masses to represent the floors, which we believe is a better model.

6.1.4 Variational form

In this subsection we derive two general variational forms that are used in the applications. Consider Equation (4.2.3), the variational form for the Timoshenko model in Section 4.2. We need to change it to accomodate the differences between equations (4.1.2) and (6.1.4). So, for the most general linear model we have the following variational form

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi \\
= \; & -\int_0^1 M(\cdot,t)\psi' - \int_0^1 V(\cdot,t)(v' - (1+\partial_x u)\psi) + \int_0^1 L(\cdot,t)\psi + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \; - \; V(0,t)v(0) \; + \; M(1,t)\psi(1) \\
& - M(0,t)\psi(0). \qquad (6.1.10)
\end{aligned}$$

To obtain the variational form for the Rayleigh theory, let $\phi = \partial_x w$ and $\psi = v'$. The result is

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\partial_x w(\cdot,t)v' \\
= \; & -\int_0^1 M(\cdot,t)v'' + \int_0^1 \partial_x u V(\cdot,t)v' + \int_0^1 L(\cdot,t)v' + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \; - \; V(0,t)v(0) \; + \; M(1,t)v'(1) - \; M(0,t)v'(0). \qquad (6.1.11)
\end{aligned}$$

The term $\int_0^1 \partial_x u V(\cdot,t)v$ is not admissable in the Rayleigh theory since only one constitutive equation (6.1.9) can be used. However, the parameter μ is extremely small hence $1 + \partial_x u \approx 1$.

Using this approximation in (6.1.10), we find that

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi \\
= \; & -\int_0^1 M(\cdot,t)\psi' - \int_0^1 V(\cdot,t)(v' - \psi) + \int_0^1 L(\cdot,t)\psi + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \; - \; V(0,t)v(0) \; + \; M(1,t)\psi(1) - \; M(0,t)\psi(0). \qquad (6.1.12)
\end{aligned}$$

This equation is used for the theory.

To obtain the variational form for the Rayleigh theory, let $\phi = \partial_x w$ and $\psi = v'$.

To do the finite element computations, we split (6.1.12):

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v &= -\int_0^1 V(\cdot,t)v' + \int_0^1 q(\cdot,t)v \\ &\quad + V(1,t)v(1) - V(0,t)v(0) \end{aligned} \tag{6.1.13}$$

and

$$\begin{aligned}\int_0^1 \frac{1}{\alpha}\,\partial_t^2 \phi(\cdot,t)\psi &= -\int_0^1 M(\cdot,t)\psi' + \int_0^1 V(\cdot,t)\psi + \int_0^1 L(\cdot,t)\psi \\ &\quad + M(1,t)\psi(1) - M(0,t)\psi(0). \end{aligned} \tag{6.1.14}$$

6.2 Beam models for high rise buildings

In this section we simulate the vibration of high-rise buildings subjected to certain dynamic loadings, such as earthquake loads or wind loads.

Our first objective is to compare Timoshenko and Rayleigh models for a high-rise building. The method to simulate motion with the finite element method is explained in Subsections 6.2.1 to 6.2.3.

Damping of vibrations is briefly discussed in Subsection 6.2.4. So called tuned mass dampers is one method ([WFH01] and [HV07]).

6.2.1 Timoshenko model

The general variational form was derived in the previous section, Equations (6.1.13) and (6.1.14).

Boundary conditions

At ground level $x = 0$ we have

$$w(0,t) = \phi(0,t) = 0$$

and at the top

$$M(1,t) = 0 \text{ and } V(1,t) = 0.$$

Test functions

$$T[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}.$$

Problem HBT

Find functions w and ϕ such that for $t > 0$, $w(\cdot, t) \in T[0,1]$, and $\phi(\cdot, t) \in T[0,1]$,

$$\int_0^1 \partial_t^2 w(\cdot, t)v = \int_0^1 V(\cdot, t)v' + \int_0^1 q(\cdot, t)v, \tag{6.2.1}$$

for each $v \in T[0,1]$.

$$\int_0^1 \frac{1}{\alpha}\, \partial_t^2 \phi(\cdot, t)\psi = -\int_0^1 M(\cdot, t)\psi' + \int_0^1 V(\cdot, t)\psi + \int_0^1 L(\cdot, t)\psi \tag{6.2.2}$$

for each $\psi \in T[0,1]$.

Galerkin approximation

As in Chapter 4 we use Hermite cubics for the Galerkin approximation (Appendix D).

Ordering of basis functions

The interval $[0,1]$ is divided in n subintervals of the same length. The basis functions δ_i are ordered in such a way that $\delta_1, \cdots, \delta_{n+1}$ are Type 1 basis functions and $\delta_{n+2}, \cdots, \delta_{2n+2}$ are Type 2 basis functions. The detail is in Appendix D.

The approximate solutions are denoted by w^h and ϕ^h.

We first consider Equation (6.2.1). Recall that a test function must satisfy the condition $v(0) = 0$. From this it follows that the first basis function is not admissable, i.e. the first Type 1 basis function δ_1 is not admissable.

Set $v = \delta_i$ in (6.2.1), then

$$\int_0^1 \partial_t^2 w^h(\cdot, t)\delta_i = \int_0^1 V(\cdot, t)\delta_i' + \int_0^1 q(\cdot, t)\delta_i, \tag{6.2.3}$$

for $i = 2, \ \cdots, \ 2n+2$.

If we write w^h in terms of the basis functions we have

$$w^h(x,t) = \sum_{j=1}^{2n+2} \delta_j(x)\, w_j(t),$$

where n is the number of subintervals on $[0,1]$. Substitute w^h into (6.2.3) to obtain

$$\sum_{j=2}^{2n+2}(\delta_j,\delta_i)\ddot{w}_j + \sum_{j=2}^{2n+2}(\delta_j',\delta_i')w_j + \sum_{j=2}^{2n+2}(\delta_j,\delta_i')\phi_j = (q(\cdot,t),\delta_i) \tag{6.2.4}$$

for $i = 2,\cdots,2n+2$. Next we define the relevant matrices. For $i = 2,\cdots,2n+2$ and $j = 2,\cdots,2n+2$:

$$M_{ij} = (\delta_j,\delta_i),\ K_{ij} = (\delta_j',\delta_i') \text{ and } L_{ij} = (\delta_j,\delta_i'), \tag{6.2.5}$$

see also Appendix D. This means that the first row and column are deleted from each matrix in Appendix D.

If we approximate $q(x,t)$ by $\sum_j q(x_j,t)\delta_j(x)$ then $(q(\cdot,t),\delta_i)$ is approximated by $\sum_j M_{ij}q(x_i,t)$. From (6.2.13) it now follows that

$$M\ddot{\bar{w}} + K\bar{w} - L\bar{\phi} = M\bar{q}(t), \tag{6.2.6}$$

where $q_i(t) = q(x_i,t)$.

Next we consider Equation (6.2.2) in Problem HBT. Since $\psi \in T[0,1]$ it follows that the first Type 1 basis function in (6.2.2) is not admissable.

Set $\psi = \delta_i$ in (6.2.2), then

$$\begin{aligned}\int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi^h(\cdot,t)\delta_i \;&=\; -\int_0^1 M(\cdot,t)\delta_i' + \int_0^1 V(\cdot,t)\delta_i \\ &\quad + \int_0^1 L(\cdot,t)\delta_i,\end{aligned} \tag{6.2.7}$$

for $i = 2,\cdots,\ 2n+2$.

We write ϕ^h in terms of the basis functions

$$\phi^h(x,t) \;=\; \sum_{j=2}^{2n+2}\delta_j(x)\phi_j(t).$$

where n is the number of subintervals on $[0,1]$. Substitute ϕ^h in terms of basis functions into (6.2.7) to obtain

$$\begin{aligned}&\frac{1}{\alpha}\sum_{j=2}^{2n+2}(\delta_j,\delta_i)\ddot{\phi}_j \;+\; \frac{1}{\beta}\sum_{j=2}^{2n+2}(\delta_j',\delta_i')\phi_j \;+\; \sum_{j=2}^{2n+2}(\delta_j',\delta_i)w_j \;+\; \sum_{j=2}^{2n+2}(\delta_j,\delta_i)\phi_j \\ &-\mu\sum_{j=2}^{2n+2}\Big(\int_0^1(1-x)\delta_j'(x)\delta_i(x)\,dx\Big)w_j \;=\; 0,\end{aligned} \tag{6.2.8}$$

for $i = 2, \cdots, 2n+2$.

The $M,\ K$ and L matrices are defined in (6.2.5). Now we define

$$N_{ij} \;=\; \mu \int_0^1 (1-x_i)\delta_j'(x)\delta_i(x).$$

As before the corresponding row and column (first row and column) are deleted from the $M,\ K,\ N$ and L matrices. The system of ordinary differential equations is

$$\frac{1}{\alpha} M\ddot{\bar{\phi}} \;+\; \frac{1}{\beta} K\bar{\phi} \;-\; L^T\bar{w} \;+\; M\bar{\phi} \;-\; \mu N\bar{w} \;=\; 0. \tag{6.2.9}$$

It remains to approximate the combined system (6.2.6)-(6.2.9) with finite differences.

6.2.2 Rayleigh model

Consider the variational form of the Timoshenko model given in (6.1.12). For the Rayleigh model let $\phi = \partial_x w$ and $\psi = v'$. The result is

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\partial_x w(\cdot,t)v' \\
= \;& -\int_0^1 M(\cdot,t)v'' + \int_0^1 L(\cdot,t)v' + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \;-\; V(0,t)v(0) \;+\; M(1,t)v'(1) - \; M(0,t)v'(0).
\end{aligned} \tag{6.2.10}$$

From the boundary conditions at $x = 0$ we obtain the space of test functions.

Test functions

$$T_R[0,1] = \{v \in C^1[0,1] \mid v(0) = v'(0) = 0\}.$$

At the top of the beam we have $M(1,t) = V(1,t) = 0$.

Problem HBR

$w(\cdot,t) \in T_R[0,1]$ and

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\partial_x w(\cdot,t)v' \\
= \;& -\int_0^1 \frac{1}{\beta}\partial_x^2 w(\cdot,t)v'' + \int_0^1 \mu(1-x)\partial_x w(\cdot,t)v' + \int_0^1 q(\cdot,t)v,
\end{aligned} \tag{6.2.11}$$

for each $v \in T_R[0,1]$.

Galerkin approximation

We again use Hermite cubics. The ordering of the basis functions are the same as in the previous subsection. From the definition of the space of test functions $T_R[0,1]$ it follows that the first Type 1 and the first Type 2 basis functions are not admissable. These two basis functions are represented by δ_1 and δ_{n+2}. Therefore the corresponding rows and columns are deleted from the M, K and L matrices i.e. the first and $(n+2)$ row and column are deleted.

Set $v = \delta_i$ in (6.2.11), then

$$\begin{aligned} &\int_0^1 \partial_t^2 w^h(\cdot,t)\delta_i + \int_0^1 \frac{1}{\alpha}\, \partial_t^2 \partial_x w^h(\cdot,t)\delta_i' \\ = \; & -\int_0^1 \frac{1}{\beta}\partial_x^2 w^h(\cdot,t)\delta_i'' + \int_0^1 \mu(1-x)\partial_x w^h(\cdot,t)\delta_i' + \int_0^1 q(\cdot,t)\delta_i, \end{aligned} \tag{6.2.12}$$

for each $i = 2, \cdots, n+1, n+3, \cdots 2n+2$.

If we write w^h in terms of the basis functions we have

$$w^h(x,t) = \sum_{j=1}^{2n+2} \delta_j(x)\, w_j(t),$$

where n is the number of subintervals on $[0,1]$.

Substitute w^h into (6.2.12) to obtain

$$\begin{aligned} &\sum_{j=2, j\neq n+2}^{2n+2} (\delta_j, \delta_i)\ddot{w}_j \;+\; \frac{1}{\alpha} \sum_{j=2, j\neq n+2}^{2n+2} (\delta_j', \delta_i')\ddot{w}_j \;+\; \frac{1}{\beta} \sum_{j=2, j\neq n+2}^{2n+2} (\delta_j'', \delta_i'')w_j \\ &\qquad = \; \mu \sum_{j=2, j\neq n+2}^{2n+2} \Big(\int_0^1 (1-x)\delta_j'(x)\delta_i'(x)\, dx \Big) w_j \;+\; (q(\cdot,t), \delta_i) \end{aligned} \tag{6.2.13}$$

for $i = 2, \cdots, n+1, n+3, \cdots 2n+2$.

Recall the definition of the matrices in (6.2.5). We now define the K_2 matrix as follows

$$[K_2]_{ij} = (\delta_j'', \delta_i'').$$

Note that the K matrix as defined in (6.2.5) will be referred to as the K_1 matrix.

By following the same procedure as in the previous subsection we obtain

$$M\ddot{\bar{w}} + \frac{1}{\alpha}K_1\ddot{\bar{w}} = -\frac{1}{\beta}K_2\bar{w} + \mu K_1\bar{w} + M\bar{q}(t). \qquad (6.2.14)$$

Euler-Bernoulli model

If the term $\frac{1}{\alpha}K_1\ddot{\bar{w}}$ is omitted from (6.2.14), we have the Euler Bernoulli model.

6.2.3 Comparison of models

In this subsection we consider wind induced oscillations in high-rise buildings, using the dynamic wind load model in [FL10]. Considering a number of prominent high-rise buildings a rough calculation yielded a value of less than 800 for the parameter α. In our numerical experiments we compare the Timoshenko and Rayleigh models. The idea is to use the Timoshenko model to determine the usefullness of the Rayleigh model. Therefore the relative errors for the Rayleigh model is calculated using the Timoshenko approximations as correct values.

A forcing function simulating the wind is applied between $x = 0.4$ and $x = 0.5$. To be precise, the wind is modeled by $q(x,t) = g(x)f(t)$ where

$$g(x) = \begin{cases} 1 & \text{if} \quad 0.4 \leq x \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

and $f(t) = A\sin\omega t$. This idea is from [FL10].

Figures 6.1 and 6.2 below illustrates the effect of the wind on a building. The Timoshenko model with $\omega = 2$ is used for both figures. In Figure 6.1, $\alpha = 4800$ which corresponds to a mast or tower, and $\alpha = 800$ is used for Figure 6.2 which is realistic for a high-rise building.

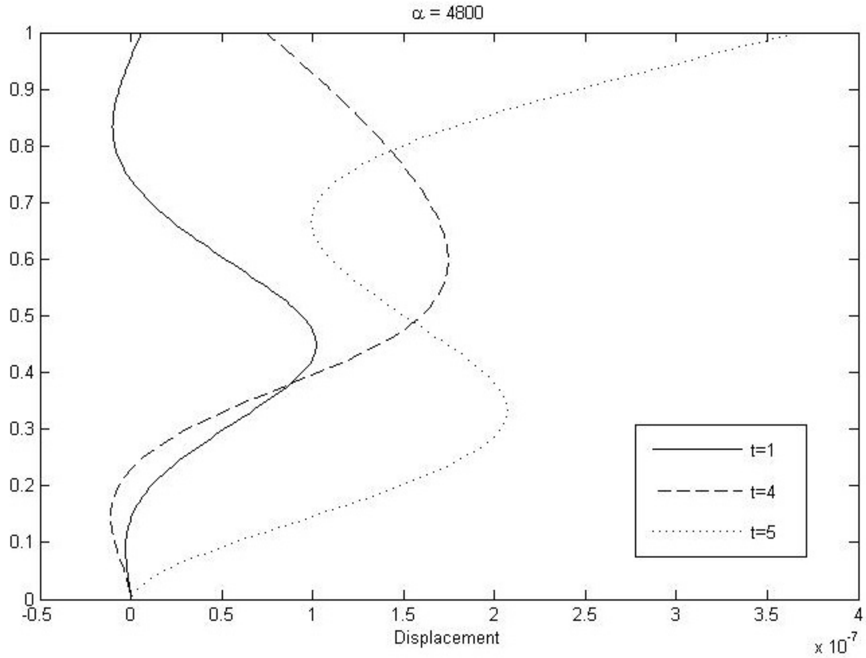


Figure 6.1: Timoshenko with $\alpha = 4800$ for different t values

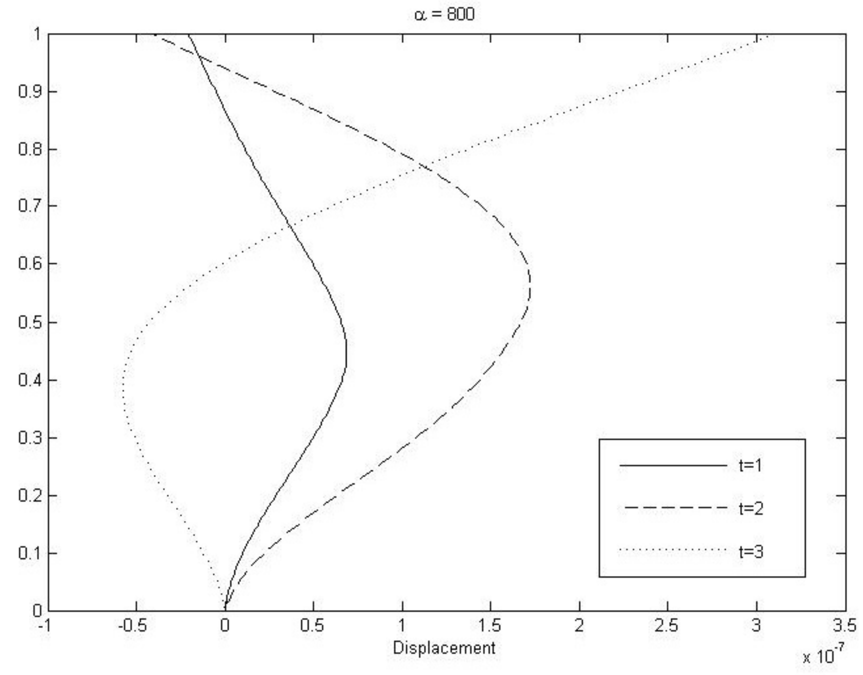


Figure 6.2: Timoshenko with $\alpha = 800$ for different t values

Note that initially the displacement is mainly localized to the position where the wind is applied. The wave then moves towards the top of the building. For the tower ($\alpha = 4800$) the top of the building moves significantly between times 4 and 5, and for the high-rise building ($\alpha = 800$) something similar happens between times 2 and 3. The acceleration there is of interest and should be considered in future experiments. Note that the scales on the axis are not the same, the displacements are very small.

We now consider the displacement of the top of the structure to compare the two models. At $t = 5$ the following results are obtained for different values of α and angular frequency (ω).

At t=5		
α	ω	Relative error at the top
4800	0.5	0.0369
	2	0.0832
1200	0.5	0.0660
	2	0.1868
800	0.5	0.0106
	2	2.8777

It appears that for large values of α the two models are very similar. However, as mentioned before, for a building a realistic value of α is less than 1000. For these values of α the Rayleigh model does not give accurate approximations. This should be considered as a preliminary investigation, further experiments are required for a scientific conclusion.

6.2.4 Tuned mass damper

By ignoring the rotary inertia term in (6.2.9), the Euler Bernoulli model in variational form is obtained:

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v = &- \int_0^1 M(\cdot,t)v'' + \int_0^1 L(\cdot,t)v' + \int_0^1 q(\cdot,t)v \\ &+ V(1,t)v(1) - V(0,t)v(0) + M(1,t)v'(1) \\ &- M(0,t)v'(0). \end{aligned} \tag{6.2.15}$$

Using our notation we derive the model from the equations of motion together with the constitutive equation. In (6.1.8), since $\partial_x u$ is small, $1+\partial_x u \approx 1$. Use

(6.1.7), (6.1.8) together with the constitutive equation in (6.1.9) to obtain

$$\partial_t^2 w = -\frac{1}{\beta}\partial_x^4 w + \partial_x L + q. \tag{6.2.16}$$

However in the article [HV07] by Hijmissen and Van Horssen they make use of the partial differential equation for the Euler Bernoulli model without given any justification for their choice. The mass damper is modeled as a simple linear spring, mass, dashpot system. In their model the load $q = F_D + F_L$ where F_D and F_L refer to nonlinear drag and lift forces acting on the structure due to a uniform wind-flow.

Their aim is to find asymptotic solutions for the problem.

Boundary conditions

At ground level $x = 0$ we have

$$w(0,t) = \partial_x w(0,t) = 0$$

and at the top

$$M(1,t) = 0.$$

Test functions

$$T[0,1] = \{v \in C^1[0,1] \mid v(0) = v'(0) = 0\}.$$

If the displacement of the damper is $u(t)$, then

$$m\ddot{u}(t) \;=\; -V(1,t) - c\dot{u}(t) - k\big(u(t) - w(1,t)\big).$$

Dimensionless form

Due to the mass of the damper the moment density L in (6.2.15) and (6.2.16) changes to

$$L \;=\; \mu(1-x)\partial_x w - w_D \partial_x w,$$

where

$$w_D \;=\; \frac{mg}{G\kappa^2}.$$

Variational form of the model problem

For each $t > 0$, $w(\cdot, t) \in T[0,1]$ and

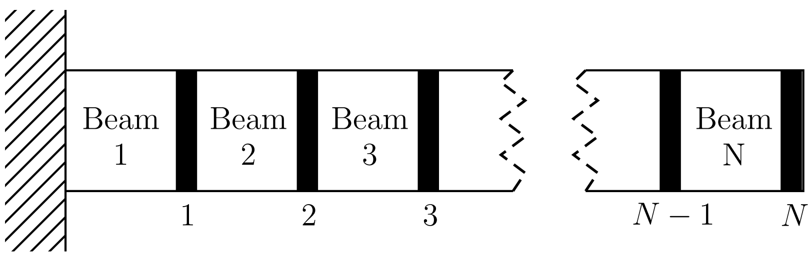
$$\int_0^1 \partial_t^2 w(\cdot,t)v = -\int_0^1 M(\cdot,t)v'' + \int_0^1 L(\cdot,t)v' + \int_0^1 q(\cdot,t)v \\ + \big(- m\ddot{u}(t) - c\dot{u}(t) - ku(t) + w(1,t)\big)v(1),$$

for each $v \in T[0,1]$.

6.3 Multiple beam models for high rise buildings

To study the dynamic behavior of a building it is suggested in [WFH01] that a more realistic model should be used. In the article a building is modeled as a beam with several lumped masses. The author used Hamilton's principle to derive the partial differential equation for the Euler Bernoulli theory and uses the finite element method to analyze the dynamic behavior.

We consider an alternative model which is more realistic in several ways. First of all an Euler Bernoulli or Rayleigh model should not be used for a building but a Timoshenko model. Secondly, rather than one beam we use a number of beams in series linked by rigid bodies. The diagram below illustrates the setup.



Using the result of Section 6.1, we start directly with the variational formulation. For each beam we use (6.1.12)

$$
\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi \\
= \; & -\int_0^1 M(\cdot,t)\psi' - \int_0^1 V(\cdot,t)(v'-\psi) + \int_0^1 L(\cdot,t)\psi + \int_0^1 q(\cdot,t)v \\
&+ V(1,t)v(1) \;-\; V(0,t)v(0) \;+\; M(1,t)\psi(1) \;-\; M(0,t)\psi(0).
\end{aligned}
$$

Follow the procedure in Section 4.2, let

$$
\begin{aligned}
L^2 &= \mathcal{L}^2(0,1)\times\mathcal{L}^2(0,1),\\
H^m &= H^m(0,1)\times H^m(0,1).
\end{aligned}
$$

Recall that $(\cdot,\cdot)$ is used for the inner product in $\mathcal{L}^2(0,1)$.

Notation

An element $y \in L^2$ is written as $y = \langle y_1, y_2\rangle$.

A natural inner product for L^2 is

$$(x,y)_{L^2} = (x_1,y_1) + (x_2,y_2),$$

and the corresponding norm is denoted by $\|\cdot\|_{L^2}$.

The natural inner product for the product space H^m is

$$(x,y)_{H^m} = (x_1,y_1)_m + (x_2,y_2)_m$$

and the corresponding norm is denoted by $\|\cdot\|_{H^m}$.

For the weak variational form, we define the following bilinear forms:

For u and v in L^2

$$c(u,v) = \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha}\,u_2 v_2.$$

For u and v in H^1

$$b(u,v) = \int_0^1 \frac{1}{\beta}\,u_2' v_2' + \int_0^1 (u_1'-u_2)(v_1'-v_2) + \int_0^1 \mu(1-x)u_1' u_2,$$

where the derivatives are weak derivatives.

For beam number k we then have

$$\begin{aligned} c(u_k'', v) + b(u_k, v) &= (\tilde{q}(t), v)_{L^2} + V_k(1,t)v(1) - V_{k+1}(0,t)v(0) \\ &+ M_k(1,t)\psi(1) - M_{k+1}(0,t)\psi(0), \end{aligned}$$

where $\tilde{q}(t) = \langle q(\cdot, t), 0\rangle$.

For the body (the floor) between beam k and $k+1$, we have

$$\begin{aligned} m_k u_{k,1}''(1,t) = V_{k+1}(0,t) - V_k(1,t) \\ I_k u_{k,2}''(1,t) = M_{k+1}(0,t) - M_k(1,t). \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{k=1}^{N} c(u_k''(t), v) + \sum_{k=1}^{N} b(u_k(t), v) &= \sum_{k=1}^{N} (\tilde{q}_k(t), v)_{L^2} - \sum_{k=1}^{N} m_k u_{k,1}''(1,t) v_{k,1}(1) \\ &- \sum_{k=1}^{N} I_k u_{k,2}''(1,t) v_{k,2}(1) + V_N(1,t) v_{N,1} \\ &+ M_N(1,t) v_{N,2} - V_1(0,t) v_1(0) \\ &- M_1(0,t) v_2(0). \end{aligned}$$

Now consider the structure. For the first beam the boundary conditions are $w(0,t) = \phi(0,t) = 0$. For the last beam $M(1,t) = V(1,t) = 0$. Boundary conditions at the top are

$$V_N(1,t) = M_N(1,t) = 0.$$

Choose $v_1(0) = v_2(0) = 0$, then the variational equation is

$$\begin{aligned} \sum_{k=1}^{N} c(u_k''(t), v) + \sum_{k=1}^{N} b(u_k(t), v) &= \sum_{k=1}^{N} (\tilde{q}_k(t), v)_{L^2} - \sum_{k=1}^{N} m_k u_{k,1}''(1,t) v_{k,1}(1) \\ &- \sum_{k=1}^{N} I_k u_{k,2}''(1,t) v_{k,2}(1). \end{aligned}$$

To formulate the problem for the structure some preparation is necessary. For the first beam the test functions are zero at $x = 0$: $V_1(0,1)$ is the closure of the space of test functions in $H^1(0,1)$. Let $V_1 = V_1(0,1) \times V_1(0,1)$. For the other beams $V_k = H^1$ for $k = 2, \cdots N$.

We introduce the product spaces X and V.

$$X = \Pi_{k=1}^{N} L^2 \times I\!R^N,$$
$$V_P = \Pi_{k=1}^{N} V_k \times I\!R^N,$$
$$V = \{u \in V_P | u_{N+k} = \Gamma u_k\}.$$

We use the trace operator Γ on $H^1(0,1)$ where $\Gamma y = y(1)$ if y is continuous. For $u \in H^1$ let $\Gamma_2 u = \langle \Gamma u_1, \Gamma u_2 \rangle$.

Let u be a function with values in X. Denote the derivative by $\dot{u}(t)$. For the first N components $\dot{u}_k(t) \in L^2$ but $\dot{u}_k(t) = d_t(\Gamma u_{k-N})(t)$ for $k > N$.

Notation

$$M_k u_k = \langle m_k \Gamma u_{k,1}, I_k \Gamma u_{k,2} \rangle,$$
$$c_{HB}(u, v) = \sum_{k=1}^{N} c(u_k, v_k) + \sum_{k=1}^{N} M_k u_k \cdot v_k,$$
$$b_{HB}(u, v) = \sum_{k=1}^{N} b(u_k, v_k).$$

The **weak variational form** is to find a function $u \in (C[0, \tau), L^2)$ such that

$$c_{HB}(u''(t), v) + b_{HB}(u(t), v) = \sum_{k=1}^{N} (\tilde{q}_k(t), v)_{L^2}$$

for each $v \in V$.

For finite element computations we need a system of differential equations. For beam number k we find, as in Section 6.2

$$M\ddot{\bar{w}}_k + K\bar{w}_k - L\bar{\phi}_k = M\bar{q}_k(t), \tag{6.3.1}$$

where $q_i(t) = q(x_i, t)$ and

$$\frac{1}{\alpha} M\ddot{\bar{\phi}}_k + \frac{1}{\beta} K\bar{\phi}_k - L^T\bar{w}_k + M\bar{\phi}_k - \mu N\bar{w}_k = 0. \tag{6.3.2}$$

To simulate the vibration of the structure the Systems (6.3.1) and (6.3.2) are used where $k = 1, 2, \cdots, N$.

6.4 Vertical slender structure with resilient seating

In this section we consider hybrid Timoshenko and Rayleigh beam models for a vertical slender structure. A Rayleigh model is in [LVV05], where the origin of the problem is discussed and references are given.

Due to the oscillations a steel structure may be damaged.

> *“ One method of artificially increasing the damping is to mount the chimney on a resilient foundation incorporating bearing pads made of a high-damping material.”* [New89, p 129-132]

In [LVV05] models that correspond to Newland’s lumped parameter system models [New89, p 129-132] and [New84] are presented. The slender structure (e.g. a steel chimney) is modelled as a Euler-Bernoulli or a Rayleigh beam mounted vertically and gravity is taken into account. Modelling the behavior of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations.

The modeling is done in greater detail in [Lab06] where the finite element method is used for a comprehensive modal analysis, and convergence of finite element approximations for eigenvalue problems is considered. Existence results can be found in [LVV09b].

6.4.1 The dynamics of the foundation block and resilient seating

From Sections 6.1 and 6.2 we have the relevant equations of motion for the Rayleigh and Timoshenko theories in dimensionless form.

To formulate the interface conditions at the base, it is necessary to consider the equations of motion for the resilient seating and foundation block. Both are modelled as rigid bodies connected to linear elastic springs and linear damping mechanisms. In this subsection we present the relevant equations following [Lab06].

Figure 1: Simplified sketch of the system

The springs and damping mechanisms in the sketch are schematic.

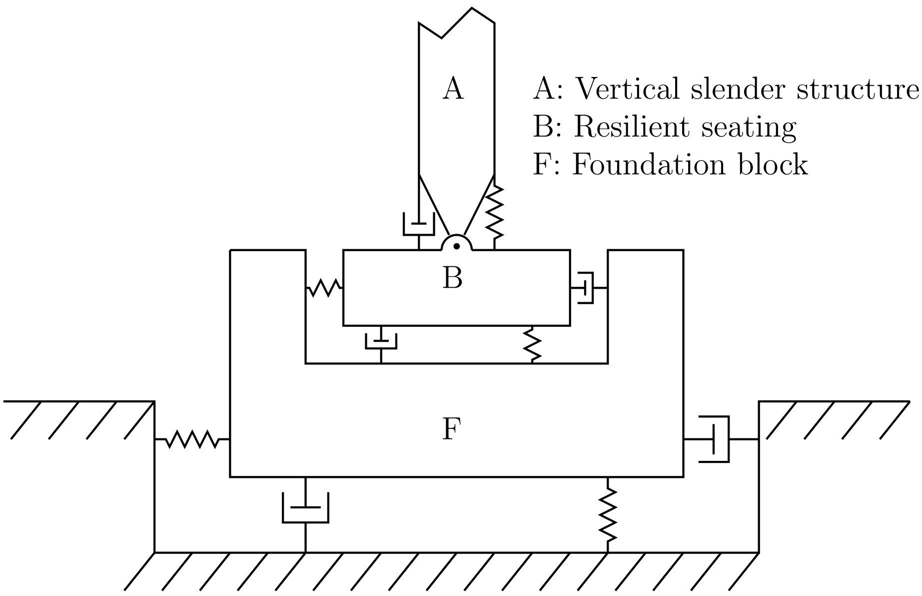
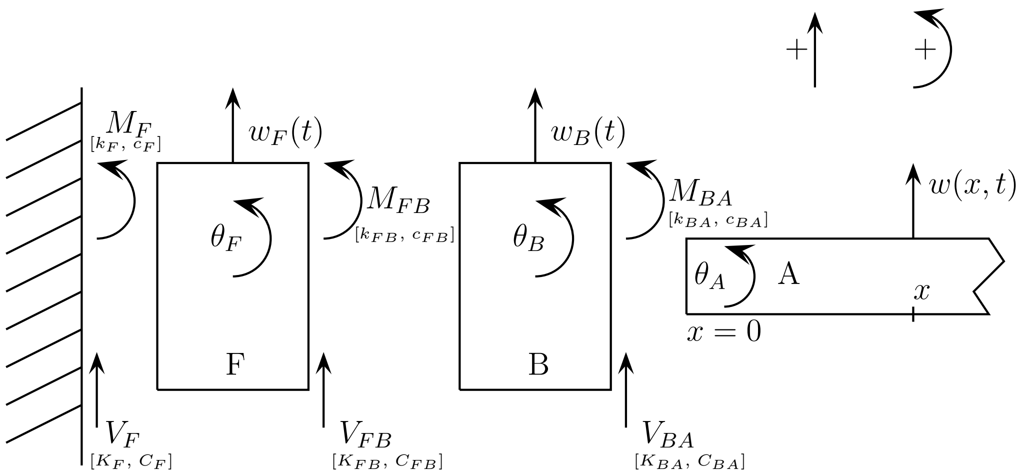


Figure 2: Displacements, angles of rotation, moments and forces



Convention: Moments and forces are denoted by the action of right on left. For instance, M_{FB} denotes the moment exerted by B on F.

Equations of motion

$$\begin{aligned}
m_F \ddot{w}_F &= V_{FB} - V_F, \\
m_B \ddot{w}_B &= V_{BA} - V_{FB}, \\
I_F \ddot{\theta}_F &= M_{FB} - M_F, \\
I_B \ddot{\theta}_B &= M_{BA} - M_{FB}.
\end{aligned}$$

Constitutive equations

$$\begin{aligned}
V_F &= K_F w_F + C_F \dot{w}_F, \\
V_{FB} &= K_{FB}(w_B - w_F) + C_{FB}(\dot{w}_B - \dot{w}_F), \\
M_F &= k_F \theta_F + c_F \dot{\theta}_F, \\
M_{FB} &= k_{FB}(\theta_B - \theta_F) + c_{FB}(\dot{\theta}_B - \dot{\theta}_F).
\end{aligned}$$

Interface conditions

Let $\theta_A(t)$ denote the rotation of the end point of the vertical structure.

$$\begin{aligned}
M_{BA}(t) &= k_{BA}\Big(\theta_A(t) - \theta_B(t)\Big) + c_{BA}\Big(\dot{\theta}_A(t) - \dot{\theta}_B(t)\Big), \\
M_{BA}(t) &= M(0,t), \\
V_{BA}(t) &= V(0,t), \\
w_B(t) &= w(0,t), \\
\theta_B(t) &\neq \theta_A(t) \quad \text{(in general)}.
\end{aligned}$$

The following assumption is made for $\theta_A(t)$:

$$\begin{aligned}
&\theta_A(t) = \partial_x w(0,t) \ \text{ for the Rayleigh model,} \\
&\theta_A(t) = \phi(0,t) \ \text{ for the Timoshenko model.}
\end{aligned}$$

Dimensionless constants

The dimensionless constants for the foundation block and the resilient seating are

$$m^* = \frac{m}{\ell \rho A}, \quad \text{and} \quad I^* = \frac{I}{\ell^3 \rho A}.$$

The different elastic and damping constants are

$$K^* = \frac{K\ell^3}{EI}, \quad k^* = \frac{k\ell}{EI}, \quad C^* = \frac{C\ell^3}{EIt_0} \quad \text{and} \quad c^* = \frac{c\ell}{EIt_0}.$$

Returning to the original notation, all the constants in the equations of motion for the foundation block and resilient seating and the equations for the interface conditions, are replaced by the corresponding dimensionless constants.

Problem VST

Equations of motion: (6.1.3) and (6.1.4) with (6.1.1) and (6.1.2).

Constitutive equations: (6.1.5) and (6.1.6).

Boundary conditions at $x = 1$: $M(1, t) = V(1, t) = 0$.

The interface conditions at $x = 0$:

$$\begin{aligned} m_B \partial_t^2 w(0,t) &= V(0,t) - K_{FB}\Big(w(0,t) - w_F(t)\Big) \\ &\quad - C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big), \end{aligned} \tag{6.4.1}$$

$$\begin{aligned} I_B \ddot{\theta}_B(t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t \phi(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad - k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{6.4.2}$$

$$\begin{aligned} M(0,t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) \\ &\quad + c_{BA}\Big(\partial_t \phi(0,t) - \dot{\theta}_B(t)\Big), \end{aligned} \tag{6.4.3}$$

$$\begin{aligned} m_F \ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad - K_F w_F(t) - C_F \dot{w}_F(t), \end{aligned} \tag{6.4.4}$$

$$\begin{aligned} I_F \ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad - k_F \theta_F(t) - c_F \dot{\theta}_F(t). \end{aligned} \tag{6.4.5}$$

The diagrams and equations in this subsection are from [LVV05] and [Lab06].

6.4.2 Variational form

Recall the variational equation (6.1.12):

$$
\begin{aligned}
& \int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\, \partial_t^2 \phi(\cdot,t)\psi \\
= & -\int_0^1 M(\cdot,t)\psi' - \int_0^1 V(\cdot,t)(v'-\psi) + \int_0^1 L(\cdot,t)\psi + \int_0^1 q(\cdot,t)v \\
& + V(1,t)v(1) \;-\; V(0,t)v(0) \;+\; M(1,t)\psi(1) -\; M(0,t)\psi(0). \qquad (6.4.6)
\end{aligned}
$$

Remark

The term in (6.1.3) containing $L(\cdot,t)$ is not always taken into account. In the specific application to a steel chimney from [New84] the effect of gravity was insignificant. This is due to the fact that M_F is extremely large and need not always be the case.

Using the boundary conditions at $x = 1$ and the notation for the $\mathcal{L}^2$ inner product in (6.4.8) yields

$$
\begin{aligned}
& (\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}(\partial_t^2 \phi(\cdot,t),\psi) + (M(\cdot,t),\psi') + (V(\cdot,t), v'-\psi) \\
& \quad = (L(\cdot,t),\psi) + (q(\cdot,t),v) \;-\; V(0,t)v(0) \;-\; M(0,t)\psi(0). \qquad (6.4.7)
\end{aligned}
$$

Substite the constitutive equations (6.1.1), (6.1.5) and (6.1.6) into (6.4.7):

$$
\begin{aligned}
& (\partial_t^2 w(\cdot,t), v) + \frac{1}{\alpha}(\partial_t^2 \phi(\cdot,t),\psi) + \frac{1}{\beta}(\partial_x \phi(\cdot,t),\psi') + (\partial_x w(\cdot,t) - \phi(\cdot,t), v'-\psi) \\
& = \mu \int_0^1 (1-x)\partial_x w(x,t)\psi(x)dx + (q(\cdot,t),v) \;-\; V(0,t)v(0) \;-\; M(0,t)\psi(0). \\
& \qquad (6.4.8)
\end{aligned}
$$

There are no forced boundary conditions on the test functions. Therefore, for the variational form of Problem VSTV, both v and ψ are in $T(0,1) = C^1[0,1]$.

Problem VSTV

Find w, ϕ, θ_B, w_F and θ_F such that for each $t > 0$, $w(\cdot,t) \in T(0,1)$, $\phi(\cdot,t) \in T(0,1)$ and Equation (6.4.8) holds for each $v \in T(0,1)$ and $\psi \in T(0,1)$ and Equations (6.4.1) to (6.4.5) hold.

6.4.3 Weak variational form

Following the procedure in Section 4.2, let

$$L^2 = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1),$$
$$H^m = H^m(0,1) \times H^m(0,1).$$

Recall that $(\cdot,\cdot)$ is used for the inner product in $\mathcal{L}^2(0,1)$.

Notation

An element $y \in L^2$ is written as $y = \langle y_1, y_2 \rangle$.

A natural inner product for L^2 is

$$(x,y)_{L^2} = (x_1,y_1) + (x_2,y_2),$$

and the corresponding norm is denoted by $\|\cdot\|_{L^2}$.

The natural inner product for the product space H^m is

$$(x,y)_{H^m} = (x_1,y_1)_m + (x_2,y_2)_m$$

and the corresponding norm is denoted by $\|\cdot\|_{H^m}$.

Define a function u with values in L^2 as follows:

$$u_1(t) = w(\cdot,t) \text{ and } u_2(t) = \phi(\cdot,t).$$

We rewrite Equation (6.4.8) in terms of this notation.

For the weak variational form, we define the following bilinear forms:

For u and v in L^2

$$c_A(u,v) = \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha}\, u_2 v_2.$$

For u and v in H^1

$$b_A(u,v) = \int_0^1 \frac{1}{\beta}\, u_2' v_2' + \int_0^1 (u_1' - u_2)(v_1' - v_2) + \int_0^1 \mu(1-x) u_1' u_2,$$

where the derivatives are weak derivatives.

For Problem VST there are no restrictions on the space of test functions and the solution w must satisfy Equation (6.4.8) for arbitrary functions $v \in C^1[0,1]$ and $\psi \in C^1[0,1]$.

For the analysis of the vibration problem we consider the weak variational form. For the weak variational form we use c_A and b_A as defined above.

Let $y(t) = w(\cdot,t)$ and let $\dot{y}(t)$ denote the derivative w.r.t. the norm of $\mathcal{L}^2(0,1)$ or $H^m(0,1)$ (see Section 2.1). We use the dot for the time derivative here to avoid confusion.

With the new notation, Equation (6.4.8) becomes

$$\begin{aligned} &c_A(\ddot{y}(t),v) + b_A(y(t),v) + m_B\ddot{w}_B(t)v(0) = -V(0,t)v(0) \\ &- M(0,t)\psi(0) + \big(q(\cdot,t),v\big). \end{aligned} \tag{6.4.9}$$

Equations (6.4.1) and (6.4.3) are used to substitute for $V(0,t)$ and $M(0,t)$. The result is that

$$\begin{aligned} &c_A(\ddot{y}(t),v) + b_A(y(t),v) + m_B\ddot{w}_B(t)v(0) = -K_{FB}\Big(\Gamma\big(y_1(t)\big) - w_F(t)\Big)\Gamma v_1 \\ &- C_{FB}\Big(\big(d_t\Gamma y_1(t)\big) - \dot{w}_F(t)\Big)\Gamma v_1 - k_{BA}\Big(\Gamma\big(y_2(t)\big) - \theta_B(t)\Big)\Gamma v_2 \\ &- c_{BA}\Big(\big(d_t\Gamma y_2(t)\big) - \dot{\theta}_B(t)\Big)\Gamma v_2 + \big(q(\cdot,t),v\big). \end{aligned} \tag{6.4.10}$$

Remark

Note that $\partial_t w(0,t)$ is replaced by $d_t(\Gamma y)(t)$ and not $\Gamma\big(\dot{y}(t)\big)$. This is necessary for the weak variational form.

The **weak variational form** of Problem VST is to find functions y, θ_B, w_F and θ_F such that (6.4.9) holds for any $v \in H^2(0,1)$ and equations (6.4.2), (6.4.4) and (6.4.5) are satisfied.

Product spaces

Define the product spaces

$$X = L^2 \times I\!R^3 \quad \text{and} \quad V = H^2 \times I\!R^3.$$

Bilinear forms

For u and v in X,

$$\begin{aligned}
c(u,v) &= c_A(\langle u_1,u_2\rangle,\langle v_1,v_2\rangle)+m_B\Gamma u_1\Gamma v_1+I_Bu_2v_2+m_Fu_3v_3+I_Fu_4v_4\,,\\
a(u,v) &= C_{FB}\,(\Gamma u_1-u_3)\,(\Gamma v_1-v_3)+c_{BA}(\Gamma u_1'-u_2)(\Gamma v_1'-v_2)\\
&\quad +c_{FB}(u_2-u_4)(v_2-v_4)+C_Fu_3v_3+c_Fu_4v_4\,.
\end{aligned}$$

For u and v in V,

$$\begin{aligned}
b(u,v) &= b_A(\langle u_1,u_2\rangle,\langle v_1,v_2\rangle)+K_{FB}\,(\Gamma u_1-u_3)\,(\Gamma v_1-v_3)\\
&\quad +k_{BA}(\Gamma u_1'-u_2)(\Gamma v_1'-v_2)+k_{FB}(u_2-u_4)(v_2-v_4)\\
&\quad +K_Fu_3v_3+k_Fu_4v_4\,.
\end{aligned}$$

Note that a, b and c are all symmetric.

We are now ready to reformulate the weak variational form of Problem VR. It is no longer necessary to denote time derivatives by dots.

Problem VSTW

Find u such that for each $t>0$, $u'(t)\in V$, $u''(t)\in X$, and

$$c(u''(t),v)+a(u'(t),v)+b(u(t),v)=(q(\cdot,t),v_1)\quad \text{for each } v\in V \tag{6.4.11}$$

while $u(0)=u_0$ and $u'(0)=u_d$.

Remark

The vector $\langle u_1(t),u_2(t),u_3(t),u_4(t),u_5(t)\rangle$ represent $\langle y_1(t),y_2(t),\theta_B(t),w_F(t),\theta_F(t)\rangle$ and Equations (6.4.10) and (6.4.11) are the same.

In the next Subsection we consider the Rayleigh theory. For the Timoshenko theory there is a problem to apply the existence theorem since the bilinear form b is not symmetric.

6.4.4 Variational form for the Rayleigh model

To start let $\phi(\cdot,t)=\partial_x w(\cdot,t)$ and $\psi=v'$ in (6.4.8).

$$(\partial_t^2 w(\cdot,t),v) + \frac{1}{\alpha}(\partial_t^2\partial_x w(\cdot,t),v') + \frac{1}{\beta}(\partial_x^2 w(\cdot,t),v'')$$
$$= \mu\int_0^1 (1-x)\partial_x w(x,t)v'(x)dx + (q(\cdot,t),v) \;-\; V(0,t)v(0) \;-\; M(0,t)v'(0). \tag{6.4.12}$$

Following [LVV09b] and [Lab06], we define the following **bilinear forms**:

$$\begin{aligned} c_A(u,v) &= (u,v) + \frac{1}{\alpha}\,(u',v'), \\ b_A(u,v) &= \frac{1}{\beta}\,(u'',v'') - \mu\int_0^1 (1-x)u'(x)v'(x)dx. \end{aligned}$$

Equations (6.4.12), (6.4.1) and (6.4.3) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} &c_A(\partial_t^2 w(\cdot,t),v) + b_A(w(\cdot,t),v) + m_B\ddot{w}_B(t)v(0) \\ &= K_{FB}\big(w_F(t) - w(0,t)\big)v(0) - C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big)v(0) \\ &\quad - c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big)v'(0) + k_{BA}\big(\theta_B(t) - \partial_x w(0,t)\big)v'(0) \\ &\quad + \big(q(\cdot,t),v\big), \end{aligned} \tag{6.4.13}$$

for each $v \in C^2[0,1]$. It is still necessary to consider Equations (6.4.2), (6.4.4) and (6.4.5):

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{6.4.14}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{6.4.15}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{6.4.16}$$

We derived the variational form of Problem VR.

Problem VRV

Find w, θ_B, w_F and θ_F such that for each $t > 0$, $w(\cdot,t) \in C^2[0,1]$, Equation (6.4.13) holds for each $v \in C^2[0,1]$ and

Equations (6.4.14), (6.4.15) and (6.4.16) hold.

Let $y(t) = w(\cdot, t)$ and let $\ddot{y}(t)$ denote the second order derivative with respect to the norm of $\mathcal{L}^2(0,1)$. Recall the definition of c_A and b_A but where the derivatives are weak derivatives.

For the finite element analysis, we reformulate the problem.

$$
\begin{aligned}
c_A(\ddot{y}(t), v_1) &+ m_B \ddot{w}_B(t) v(0) + I_B \ddot{\theta}_B(t) v_2 + m_F \ddot{w}_F(t) v_3 + I_F \ddot{\theta}_F(t) v_4 \\
&= - b_A(y(t), v_1) - K_{FB}\Big(\Gamma y(t) - w_F(t)\Big)\Big(\Gamma v_1 - v_3\Big) \\
&\quad - C_{FB}\Big(d_t\big(\Gamma y(t)\big) - \dot{w}_F(t)\Big)\Big(\Gamma v_1 - v_3\Big) \\
&\quad - k_{BA}\Big(\Gamma[\big(y(t)\big)'] - \theta_B(t)\Big)\Big(\Gamma v_1' - v_2\Big) \\
&\quad - c_{BA}\Big(d_t\big(\Gamma[\big(y(t)\big)']\big) - \dot{\theta}_B(t)\Big)\Big(\Gamma v_1' - v_2\Big) \\
&\quad - k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big)\Big(v_2 - v_4\Big) \\
&\quad - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big)\Big(v_2 - v_4\Big) \\
&\quad - K_F w_F(t) v_3 - C_F \dot{w}_F(t) v_3 - k_F \theta_F(t) v_4 - c_F \dot{\theta}_F(t) v_4 \\
&\quad + \big(q(\cdot, t), v_1\big),
\end{aligned}
\tag{6.4.17}
$$

for each $v_1 \in H^2(0,1)$ and any real numbers v_2, v_3 and v_4.

Next we construct the product spaces for the weak form of the variational problem.

Product spaces

Define the product spaces

$$X = \mathcal{L}^2(0,1) \times I\!R^3\,, \quad W = H^1(0,1) \times I\!R^3 \quad \text{and} \quad V = H^2(0,1) \times I\!R^3.$$

Bilinear forms

For u and v in W,

$$
\begin{aligned}
c(u,v) &= c_A(u_1, v_1) + m_B \Gamma u_1 \Gamma v_1 + I_B u_2 v_2 + m_F u_3 v_3 + I_F u_4 v_4\,, \\
a(u,v) &= C_{FB}\,(\Gamma u_1 - u_3)\,(\Gamma v_1 - v_3) + c_{BA}(\Gamma u_1' - u_2)(\Gamma v_1' - v_2) \\
&\quad + c_{FB}(u_2 - u_4)(v_2 - v_4) + C_F u_3 v_3 + c_F u_4 v_4\,.
\end{aligned}
$$

For u and v in V,

$$\begin{aligned} b(u,v) &= b_A(u_1,v_1) + K_{FB}\,(\Gamma u_1 - u_3)\,(\Gamma v_1 - v_3) \\ &\quad + k_{BA}(\Gamma u_1' - u_2)(\Gamma v_1' - v_2) + k_{FB}(u_2 - u_4)(v_2 - v_4) \\ &\quad + K_F u_3 v_3 + k_F u_4 v_4\,. \end{aligned}$$

Note that a, b and c are all symmetric.

We are now ready to reformulate the weak variational form of Problem VR. It is no longer necessary to denote time derivatives by dots.

To formulate the problem, let $\tilde{q}(t) = q(\cdot,t)$ and $q_X(t) = \langle \tilde{q}(t), 0, 0, 0\rangle$.

Problem VRW

Find u such that for each $t > 0$, $u'(t) \in V$, $u''(t) \in W$, and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (q_X, v)_X \quad \text{for each } v \in V \tag{6.4.18}$$

while $u(0) = u_0$ and $u'(0) = u_d$.

Remark

The vector $\langle u_1(t), u_2(t), u_3(t), u_4(t)\rangle$ represent $\langle y(t), \theta_B(t), w_F(t), \theta_F(t)\rangle$ and Equations (6.4.17) and (6.4.11) are the same.

6.4.5 Existence of a unique solution for Problem VRW

To apply the existence theorem (Theorem 2.3.1), we need Assumptions E1, E2, E3 and E4. In this subsection we present results from [Lab06] and [LVV09b] without proof. The proofs are similar to the proofs in Chapters 4 and 5.

Stability condition

In the following results, it is assumed throughout that the inequalities below hold for the physical constants.

$$1 > 2\mu, \quad k_{BA} > 4\mu, \quad k_{FB} > 8\mu \;\text{ and }\; k_F > 8\mu.$$

These inequalities are necessary to ensure the stability of the equilibrium and are physically realistic, as can be seen in the applications in [LVV05] and [Lab06].

Proposition 6.4.1. *The bilinear form c is an inner product for the space W.*

Definition Inertia space W

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.

Proposition 6.4.2. *There exists a constant C_c such that*

$$\|\cdot\|_X \leq C_c\|\cdot\|_W.$$

Proposition 6.4.3. *V is a dense subset of W and W is dense in X.*

Proposition 6.4.4. *There exists a constant C_b such that*

$$b(u,u) \quad \geq \quad C_b\|u\|_W^2 \quad \textit{for each } u \in V.$$

Corollary 6.4.1. *The bilinear form b is an inner product for V.*

Definition Energy space V

The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

Proposition 6.4.5. *There exists a constant K_{bc} such that*

$$\|u\|_X^2 \leq K_{bc}\, b(u,u)$$

for each $u \in V$.

Proposition 6.4.6. *There exists a constant K_{ba} such that for any $u \in V$ and $v \in V$,*

$$|a(u,v)| \leq K_{ba}\|u\|_V\|v\|_V.$$

Theorem 2.3.1 may now be applied. Note that the Assumptions E1, E2, E3 and E4 are satisfied due to Propositions 6.4.1 to 6.4.6.

Theorem 6.4.1. *Suppose*

(a) *the stability condition is satisfied,*

(b) $\tilde{q} \in C^1([0,\tau), \mathcal{L}^2(0,1))$,

(c) $u_0 \in V$, $u_d \in V$ *and there exists a* $y \in W$ *such that*

$$b(u_0,v) + a(u_d,v) = c(y,v) \textit{ for each } v \in V.$$

Then Problem VRW has a unique solution
$u \in C([0,\tau),V) \cap C^1([0,\tau),W) \cap C^1((0,\tau),V) \cap C^2((0,\tau),W)$.

Proof. Clearly $q_X = \langle \tilde{q}, 0, 0, 0\rangle \in C^1([0,\tau),X)$. The result follows from Theorem 2.3.1. $\square$

Remark

As in [Lab06] we note that $y = u_1 \in C^1\big((0,T); H^2(0,1)\big)$ and we find for example that $(\dot{y}(t))' = d_t\,(y'(t))$ and consequently $d_t\,[\Gamma y(t)] = \Gamma \dot{y}(t)$. However, second order time derivatives are not mentioned and from Theorem 2.3.1 we have $u_1 \in C^2((0,T);\mathcal{L}^2(0,1))$ which means that $\Gamma \ddot{u}_1$ need not exist. This matter should be investigated further.

6.4.6 Finite element method

Let $S^h(0,1)$ denote the subspace of $H^2(0,1)$ consisting of piecewise Hermite cubic basis functions. We define the finite dimensional subspace S^h of V by $S^h = S^h(0,1) \times \mathbb{R}^3$.

Problem VRW^h

Find a function $u_h \in C^2(0,T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c({u_h}''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (q_X, v)_X \quad \text{for each } v \in S^h$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

From Theorem 3.1.1 we have the existence of a unique solution, provided that $\tilde{q}$ is continuous w.r.t. the norm of $\mathcal{L}^2(0,1)$.

Interpolation

Notation: $H^k = H^k(0,1) \times \mathbb{R}^3$. As before, we need to consider Assumption A3. We have that $H(V,k) = H^k \cap V$ for this problem. An interpolation operator on the product space H^k can now be defined.

Definition

$$\Pi u = \langle \Pi_c\, u_1,\, u_2,\, u_3,\, u_4\rangle \quad \text{for } u \in H^k(\Omega),$$

where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions (see Appendix E).

From the definition of Πu it follows that

$$u - \Pi u = \langle u_1 - \Pi_c u_1, 0, 0, 0\rangle. \tag{6.4.19}$$

Proposition 6.4.7. *There exists a constant* $\widetilde{C}_b$ *such that*

$$\|u - \Pi u\|_V^2 \leq \widetilde{C}_b \|u_1 - \Pi_c u_1\|_2^2.$$

Proof. Consider $v = \langle v_1, 0, 0, 0\rangle$ where $v_1 \in H^k(0,1)$. Using the definition of b we have

$$b(v,v) = b_A(v_1, v_1) + K_{FB}(\Gamma v_1)^2 + k_{BA}(\Gamma v_1')^2.$$

From the definition of b_A and the estimate for the trace operator (see Appendix C), we obtain

$$\begin{aligned} b(v,v) &\leq \frac{1}{\beta}(v_1'')^2 - \mu \int_0^1 (1-x)(v_1')^2 dx + 2K_{FB}\|v_1\|_1^2 + 2k_{BA}\|v_1'\|_1^2 \\ &\leq \widetilde{C}_b \|v_1\|_2^2, \end{aligned}$$

where $\widetilde{C}_b$ depends on the constants β, μ, K_{FB} and k_{BA}. The result now follows from (6.4.19). $\square$

Now the results for the piecewise Hermite cubic functions in Appendix E can be used:
If $u \in H^k(0,1)$ for $k \geq 3$, then

$$\|u - \Pi_c u\|_2 \leq \widehat{C}_c h^\alpha |u|_{\alpha+2}\,,$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Combining Proposition 6.4.7 and the result above, the following estimate for the interpolation operator Π is obtained:
If $u \in H^k(\Omega)$ for $k \geq 3$

$$\|u - \Pi u\|_V \leq \widehat{C} h^\alpha |u_1|_{\alpha+2}\,, \tag{6.4.20}$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$ and the constant $\widehat{C}$ depends on $\widetilde{C}_b$ and $\widehat{C}_c$.

6.4.7 Convergence of the semi-discrete approximation

We now apply the results for general damping from Section 3.4. We use estimate (6.4.20) and assume $u \in C^2([0,T],V)$ (Assumption A2) in the applications below.

Application of Corollary 3.4.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0,T],H^k(\Omega)\cap V)$ with $k \geq 3$. Then, for $t \in [0,T]$,

$$\begin{aligned}
&\|u(t)-u_h(t)\|_V + \|u'(t)-u_h'(t)\|_W \\
&\quad \leq \widehat{C}h^\alpha\big(|u_1(t)|_{\alpha+2} + C_b|u_1'(t)|_{\alpha+2}\big) \\
&\qquad + \sqrt{12e^{3t}}\widehat{C}h^\alpha\left[\int_0^T C_b|u_1''(\cdot)|_{\alpha+2} + 3K_a\max|u_1'(t)|_{\alpha+2} + 3K_a\int_0^T|u_1''(\cdot)|_{\alpha+2}\right. \\
&\qquad \left. + 2C_b|u_d|_{\alpha+2} + 2\sqrt{1+K_a}|u_0|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2}\right],
\end{aligned}$$

where $\alpha = 1$ if $k=3$ and $\alpha = 2$ if $k \geq 4$.

As mentioned before, it is possible to use the Aubin-Nitsche trick to obtain **higher order estimates** in the inertia norm for the projection error in the cases where Assumption A4 is satisfied. In those instances, we have the following estimate.

Application of Corollary 3.4.2

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0,T],H^k(\Omega)\cap V)$ with $k \geq 3$. Then, for any $t > 0$,

$$\begin{aligned}
\|u(t)-u_h(t)\|_W \leq & \widehat{C}^2h^{2\alpha}\widehat{c}_b|u_1(t)|_{\alpha+2} + \sqrt{12e^{3t}}\,C_b\widehat{C}^2h^{2\alpha}\widehat{c}_b\left(\int_0^T C_b|u_1''(\cdot)|_{\alpha+2}\right. \\
&\left. +3K_a\max|u_1'(t)|_{\alpha+2} + 3K_a\int_0^T|u_1''(\cdot)|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2}\right) \\
& + \sqrt{12e^{3t}}C_bKh^{2\alpha},
\end{aligned}$$

where $\alpha = 1$ if $k=3$ and $\alpha = 2$ if $k \geq 4$.

6.4.8 Fully discrete

Consider the algorithm for central differences.

Problem VRW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \ldots, N-1$,

$$\begin{aligned}c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v)\\ +b(u_k^h, v) = (q(t_k), v_1),\end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

The work done in Chapter 4 for the central difference scheme can be adapted for this problem. We have the following result.

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume $u \in C^2([0,T], V)$. If $u'' \in \mathcal{L}^2([0,T], H^k(\Omega) \cap V)$ with $k \geq 3$, $q \in C^2([0,T], \mathcal{L}^2(0,1))$ and the sequence $\{u_k^h\}$ is a solution of Problem VRW^h-D, then

$$\begin{aligned}&\|u(t_k) - u_k^h\|_V\\ &\quad \leq \widehat{C} h^\alpha \big(|u_1(t)|_{\alpha+2} + C_b |u_1'(t)|_{\alpha+2}\big)\\ &\qquad + \sqrt{12e^{3t}}\widehat{C} h^\alpha \left[\int_0^T C_b |u_1''(\cdot)|_{\alpha+2} + 3K_a \max |u_1'(t)|_{\alpha+2}\right.\\ &\qquad \left. + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} + 2C_b |u_d|_{\alpha+2} + 2\sqrt{1+K_a}|u_0|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2}\right],\\ &\qquad + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \left\|u_h^{(4)}(s)\right\|_W + \tau \max \|u_h'''(s)\|_W\right).\end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

Chapter 7

Plates and multi-dimensional solids

7.1 Reissner-Mindlin plate

7.1.1 Model problem in variational form

Problem Plate

In Section 1.3 we have the Equations of motion (1.3.5) and (1.3.6) together with the constitutive equations (1.3.7) and (1.3.8), all in dimensionless form. Where the plate is clamped the boundary conditions are $w = \psi_1 = \psi_2 = 0$. The initial conditions are

$$w(\cdot, 0) = w_0, \quad \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_0,$$

$$\partial_t w(\cdot, 0) = w_1 \quad \text{and} \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1.$$

This model problem is considered in [Wu05] and [Wu06]. It is a special case of a plate-beam system considered in [LVV09c] and [VZV09]. The notation and layout in Sections 7.1.1 and 7.1.2 follows [LVV09c] and [VZV09] closely.

We make use of the following Green formulas which follows from the divergence theorem: For any scalar valued function v

$$\iint_\Omega (\operatorname{div} \mathbf{Q}) v \, dA = -\iint_\Omega \mathbf{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v \, ds. \qquad (7.1.1)$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T$,

$$\iint_\Omega \operatorname{div} M \cdot \boldsymbol{\phi}\, dA = -\iint_\Omega \operatorname{tr}(MD\bar{\phi})\, dA + \int_{\partial\Omega} M\mathbf{n} \cdot \boldsymbol{\phi}\, ds. \qquad (7.1.2)$$

where $D\bar{\phi} = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$ and "tr" denotes the trace of the matrix. Note that we used the symmetry of M.

Multiply equation (1.3.5) by an arbitrary scalar valued function v and integrate. By using the first Green formula in (7.1.1), we find that

$$h \iint_\Omega \partial_t^2 wv\, dA + \iint_\Omega \mathbf{Q} \cdot \nabla\, v\, dA - \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v\, ds = \iint_\Omega qv\, dA. \qquad (7.1.3)$$

From equation (1.3.6) we have

$$I\partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} = \operatorname{div} \mathrm{M} \cdot \boldsymbol{\phi} - \mathrm{Q} \cdot \boldsymbol{\phi},$$

where $\boldsymbol{\phi}$ is an arbitrary vector valued function. Using (7.1.2) we find that

$$I \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA + \iint_\Omega \operatorname{tr}(MD\bar{\phi})\, dA - \int_{\partial\Omega} M\mathbf{n} \cdot \boldsymbol{\phi}\, ds + \iint_\Omega \boldsymbol{Q} \cdot \boldsymbol{\phi}\, dA = 0. \qquad (7.1.4)$$

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$:

$$T_1(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega\},$$

$$T_2(\Omega) = \{\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T \mid \phi_1, \phi_2 \in C^1(\bar{\Omega}),\ \boldsymbol{\phi} = 0 \text{ on } \partial\Omega\}.$$

From (7.1.3)

$$h \iint_\Omega \partial_t^2 wv\, dA + \iint_\Omega \mathbf{Q} \cdot \nabla\, v\, dA = \iint_\Omega qv\, dA, \qquad (7.1.5)$$

for each $v \in T_1(\Omega)$.

From (7.1.4)

$$I \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA + \iint_\Omega \operatorname{tr}(MD\bar{\phi})\, dA + \iint_\Omega \boldsymbol{Q} \cdot \boldsymbol{\phi}\, dA = 0, \qquad (7.1.6)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

We define a bilinear form b_B by

$$\begin{aligned}
b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_\Omega \mathrm{tr}(MD\bar{\phi})\, dA \\
&= \frac{1}{\beta(1-\nu^2)} \iint_\Omega \big((\partial_1\psi_1 + \nu\partial_2\psi_2)\partial_1\phi_1 + (\partial_2\psi_2 + \nu\partial_1\psi_1)\partial_2\phi_2\big)\, dA \\
&\quad + \frac{1}{2\beta(1+\nu)} \iint_\Omega (\partial_1\psi_2 + \partial_2\psi_1)(\partial_1\phi_2 + \partial_2\phi_1)\, dA,
\end{aligned}$$

for each $\boldsymbol{\psi}, \boldsymbol{\phi}$ in $H^1(\Omega)^2$.

Finally, the second variational equation is given by

$$I \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA + b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi}\, dA = 0, \qquad (7.1.7)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Problem Plate-V

Find w and $\boldsymbol{\psi}$ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\boldsymbol{\psi}(\cdot, t) \in T_2(\Omega)$ and Equations (7.1.5) and (7.1.7) hold for each $v \in T_1(\Omega)$ and each $\boldsymbol{\phi} \in T_2(\Omega)$. The initial conditions are

$$w(\cdot, 0) = w_0, \quad \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_0,$$

$$\partial_t w(\cdot, 0) = w_1 \quad \text{and} \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1.$$

The variational form above is used to apply the finite element method. We consider the weak form of the variational problem in the next section.

7.1.2 Weak variational form

To see that this problem is of the same form as Problem G, add Equations (7.1.5) and (7.1.7).

$$\begin{aligned}
& h \iint_\Omega \partial_t^2 w v\, dA + I \iint_\Omega \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi}\, dA \\
&\quad = -b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) - h \iint_\Omega (\nabla w + \boldsymbol{\psi})(\nabla v + \boldsymbol{\phi})\, dA + \iint_\Omega qv\, dA
\end{aligned}$$

This is of the form $c(\partial_t^2 u, z) + b(u, z) = (q, v)_X$.

Product spaces

$$\begin{aligned} X &= \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)^2 \\ H^k &= H^k(\Omega) \times H^k(\Omega)^2 \\ T &= T_1(\Omega) \times T_2(\Omega) \end{aligned}$$

Let V be the closure of T in H^1.

Bilinear forms For u and v in V, define

$$\begin{aligned} c(u, v) &= h(u_1, v_1)_\Omega + I(u_2, v_2)_{0,2}^\Omega \text{ and} \\ b(u, v) &= b_B(u_2, v_2) + h\big(\nabla u_1 + u_2, \nabla v_1 + v_2\big)_{0,2}^\Omega . \end{aligned}$$

The norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$. A natural inner product for X is

$$(x, y)_X = (x_1, y_1)_0 + (x_2, y_2)_{0,2}.$$

(See Appendix A on Sobolev spaces of vector valued functions.) We denote the corresponding norm by $\|\cdot\|_X$.

Proposition 7.1.1. *There exist positive constants K_i such that*

$$K_1\|u\|_X \leq c(u, u) \leq K_2\|u\|_X^2.$$

Proof. It follows from the fact that $c(u, u) = h\|u_1\|_\Omega^2 + I(\|u_2\|_{0,2}^\Omega)^2$. □

Proposition 7.1.2. *The bilinear form c is an inner product for the space X.*

Proof. The bilinear form c is clearly symmetric and $c(u, u) \geq K_1\|u\|_X^2$ by Proposition 7.1.1. □

Definition Inertia space

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W.

Let $f(t) = \langle q(\cdot, t), \mathbf{0}\rangle$ and J any open interval containing zero.

Problem Plate-W

Find u such that for each $t \in J$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each} \quad v \in V, \tag{7.1.8}$$

while $u(0) = u_0 = \langle w_0, \boldsymbol{\psi}_0 \rangle$ and $u'(0) = u_1 = \langle w_1, \boldsymbol{\psi}_1 \rangle$.

Before we discuss existence and convergence we need the following results.

Proposition 7.1.3. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof. It follows directly from Proposition 7.1.1. □

Proposition 7.1.4. *V is a dense subset of W.*

Proof. $T_1(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ and $T_2(\Omega)$ is dense in $\mathcal{L}^2(\Omega)^2$. Consequently T is dense in X. □

Theorem 7.1.1. *There exist a constant K_b such that*

$$|b(u, v)| \leq K_b \|u\|_{H^1} \, \|v\|_{H^1},$$

for each $u \in V$.

Proof. The terms in $b_B(u_2, v_2)$ are all of the form $\iint_\Omega \partial_i u_{2,i} \, \partial_j v_{2,j} \, dA$. Therefore

$$|b_B(u_2, v_2)| \leq K \|u_2\|_{1,2} \, \|v_2\|_{1,2}.$$

To the other term in b we apply the Cauchy-Schwartz inequality.

$$|(\nabla u_1 + u_2, \nabla v_1 + v_2)_{0,2}| \leq \|\nabla u_1 + u_2\|_{0,2} \, \|\nabla v_1 + v_2\|_{0,2}.$$

Now

$$\begin{aligned}
\|\nabla u_1 + u_2\|_{0,2}^2 &\leq \|\nabla u_1\|_{0,2}^2 + \|\nabla u_1\|_{0,2} \|u_2\|_{0,2} + \|u_2\|_{0,2}^2 \\
&\leq 2\|\nabla u_1\|_{0,2}^2 + 2\|u_2\|_{0,2}^2 \\
&\leq 2|u_1|_1^2 + 2\|u_2\|_{0,2}^2 \\
&\leq 2\|u_1\|_1^2 + 2\|u_2\|_{0,2}^2 \leq 2\|u\|_{H^1}^2.
\end{aligned}$$

□

Next we use Korn's lemma.

Theorem 7.1.2. *There exists a constant C_b such that*

$$b_B(u, u) \quad \geq \quad C_b \|u\|_{1,2}^2 \quad \textit{for each } u \in V.$$

Proof. See [Bra01, p.289]. $\square$

Corollary 7.1.1. *The bilinear form b is an inner product for V.*

Definition Energy space

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

Corollary 7.1.2. *The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V.*

We have shown that Assumptions E1, E2 and E4 hold. (Since the vibration is undamped, E3 is not relevant.) Recall that J is an interval containing zero.

Theorem 7.1.3. *Suppose $f \in C^1(J, X)$, then there exists a unique solution*

$$u \in C^1\big(J; V\big) \cap C^2\big(J; W\big),$$

for Problem Plate-W for each $u_0 \in E_b$ and $u_1 \in V$.

Proof. It follows from Theorem 2.2.2. $\square$

7.1.3 Finite element method

Let S^h be a finite dimensional subspace of V. To construct S^h the four-node Bathe-Dvorkin elements are used in [Wu05]. (They are piecewise bilinear.) Another possibility is piecewise bicubic basis functions.

Problem Plate-$\boldsymbol{W}^h$
Find $u_h \in C(J, S^h)$ such that u_h' is continuous at 0 and for each $t \in J$

$$c(u_h''(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each} \quad v \in S^h,$$

while $u(0) = u_0^h = \langle w_0^h, \boldsymbol{\psi}_0^h \rangle$ and $u'(0) = u_1^h = \langle w_1^h, \boldsymbol{\psi}_1^h \rangle$.

Note that u_0^h and u_1^h are approximations for u_0 and u_1 in the subspace S^h.

Since no damping is present all the results from Sections 4.3 and 4.5 may be used. If we follow [Wu05] we may only use an explicit scheme but the results in Section 4.5 are for an implicit scheme. The results for the semi-discrete approximation in Section 4.3 may still be used but it remains to derive the estimate for the fully discrete approximation.

Interpolation

In [Wu05] the domain Ω is a rectangle. We must define an interpolation operator on $H^k = H^k(\Omega) \times H^k(\Omega)^2$. To illustrate how this is done, we use piecewise bilinear basis functions on rectangular elements. Πu for $u \in H^k$ is defined by: $\Pi u = \langle \Pi_b u_1, \Pi_B u_2 \rangle$ where Π_b and Π_B are defined in Appendix E.

Proposition 7.1.5. *There exists a $\widehat{C} > 0$ such that*

$$\|\Pi u - u\|_{H^1} \leq \widehat{C} h \|u\|_{H^2} \text{ for each } u \in H^2.$$

Proof. From Appendix E we have that

$$\|\Pi_b f - f\|_{1,\Omega} \leq \widehat{C} h |f|_{2,\Omega} \text{ for } f \in H^2(\Omega) \text{ and}$$
$$\|\Pi_B f - f\|_{1,2} \leq \widehat{C} h |f|_{2,2} \text{ for } f \in H^2(\Omega)^2.$$

Therefore, from the definition of Π, we have

$$\begin{aligned} \|\Pi u - u\|_{H^1}^2 &= \|\Pi_b u_1 - u_1\|_1^2 + \|\Pi_B u_2 - u_2\|_{1,2}^2 \\ &\leq \widehat{C}^2 h^2 |u_1|_{2,\Omega}^2 + \widehat{C}^2 h^2 |u_2|_{2,2}^2 \\ &\leq \widehat{C}^2 h^2 \|u\|_{H^2}^2. \end{aligned}$$

□

Lemma 7.1.1. *For $u \in H^2$, $\|Pu - u\|_E \leq \widehat{C} h \|u\|_2$.*

Proof. We use Theorem 7.1.1 and Proposition 7.1.5.

$$\|Pu - u\|_E \leq \sqrt{K_b} \|\Pi u - u\|_{H^1} \leq \sqrt{K_b} \widehat{C} h \|u\|_{H^2}.$$

□

Remark

Theoretically convergence follows from Lemma 3.3.2 but in practice locking may be a problem as observed in the numerical experiments in [Wu05]. However, with piecewise bicubic basis functions satisfactory results were obtained in [LVV09c].

7.1.4 Error estimates for the semi-discrete approximation

For Problem Plate-W^h we have that $H(V,k) = H^k \cap V$. Since no damping is present, the results for weak damping in Sections 3.3 and 3.5 are valid.

Instead of Assumption A3 regarding the existence of an interpolation operator, we constructed an interpolation operator and derived error estimates (Proposition 7.1.5 and Lemma 7.1.1).

For inertia norm error estimates (Section 3.3), we apply Corollary 3.3.3.

Initial conditions

The error depends on the choice of initial conditions u_0^h and u_1^h. An obvious choice is to use the interpolants of u_0 and u_1.

Theorem 7.1.4. *Suppose u_0 and u_1 are in $H^k \cap V$ and $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem G satisfies $u'(t) \in \mathcal{L}^2([0,T], H^k \cap V)$, then*

$$\begin{aligned}
&\|u(t) - u_h(t)\|_W \\
&\leq C_b\widehat{C}h^\alpha \|u(t)\|_{H^2} + \sqrt{2}C_b\widehat{C}h^\alpha \left(3\int_0^T \|u'(t)\|_{H^2}\right. \\
&\left. +3K_a\int_0^T \|u(t)\|_{H^2} + (2+3K_aT)\|u_0\|_{H^2} + 3T\|u_1\|_{H^2}\right),
\end{aligned}$$

for each $t \in [0,T]$.

Proof. The result follows from Lemma 7.1.1 and Corollary 3.3.3. □

7.1.5 Fully discrete approximation

In this section the aim is to obtain an estimate for $e_k = u_h(t_k) - u_k^h$. First consider the algorithm in Problem G^h-D-Baker.

Problem Plate-$\boldsymbol{W}^h$-FD
Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \ldots, N-1$,

$$\delta_t u_k^h = v_{k+\frac{1}{2}}, \tag{7.1.9}$$

$$c(\delta_t v_k, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) = \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \tag{7.1.10}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0)$ and $v_0 = u_h'(0)$.

Theorem 7.1.5. *Suppose*

1. $u_0^h = \Pi u_0$ *and* $u_1^h = \Pi u_1$,
2. $u''(t) \in \mathcal{L}^2([0,T], H^k \cap V)$,
3. $f \in C^2([0,T], X)$ *and*
4. *the sequence* $\{u_k^h\}$ *is a solution of Problem Plate-W^h-FD.*

Then

$$\begin{aligned}
&\|u(t_k) - u_k^h\|_V \\
&\quad \leq \widehat{C}h^\alpha \left(\|u(t)\|_{H^2} + C_b\|u'(t)\|_{H^2}\right) + \sqrt{24e^{3t}}\widehat{C}h^\alpha \left[\int_0^T C_b\|u''(\cdot)\|_{H^2}\right. \\
&\qquad +3K_a \max\|u'(t)\|_{H^2} + 3K_a \int_0^T \|u''(\cdot)\|_{H^2} \\
&\qquad \left. +2C_b\|u_1\|_{H^2} + 2\sqrt{1+K_a}\|u_0\|_{H^2} + \sqrt{K_a}\|u_1\|_{H^2}\right] \\
&\qquad + C_T\left(\tau^2 \max\left\|u_h^{(4)}(s)\right\|_W + \tau^2 \max\|u_h'''(s)\|_W\right),
\end{aligned}$$

where C^T is a constant depending on T.

The approach taken by Wu

In the introduction to [Wu06] the importance of the explicit method is stressed. "The explicit finite element method has been extensively developed for the transient dynamic analysis to meet the increasing demand of engineering application."

The fully discrete problem is formulated in Section 3 and the explicit scheme explained. The algorithm is vitually the same as central differences. Therefore the theory is a special case of the theory in Section 4.6 of this dissertation.

In Section 4 the projection is defined and the decomposition of the errors introduced as in Section 2.3 of this dissertation. The notation in [Wu05] is the author's own and differs from [Dup73] and [Bak76].

7.2 Multi-dimensional elasto-dynamics

Consider the linear vibration problem for a three-dimensional solid in Section 1.4. To derive the variational form start with the equation of motion. Multiply the equation by an arbitrary vector valued function v and integrate.

We make use of the following Green's formula which follows from the divergence theorem:

$$\int_\Omega \text{div}\, T \cdot \mathbf{v}\, dV \;=\; -\int_\Omega \text{tr}\,(TV)\, dV + \int_{\partial\Omega} T\mathbf{v} \cdot \mathbf{n}\, dS$$

where

$$V = \nabla \mathbf{v} = \begin{bmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{bmatrix}$$

and "tr" denotes the trace of the matrix. Note that we used the symmetry of T.

Recall from Section 1.4 that the boundary of Ω consists of two parts, Σ where the displacements are specified and Γ where the traction is specified.

Since T is symmetric,

$$\int_\Omega \text{div}\, T \cdot \mathbf{v}\, dV \;=\; -\int_\Omega \text{tr}\,(TV)\, dV + \int_{\partial\Omega} T\mathbf{n} \cdot \mathbf{v}\, dS. \qquad (7.2.1)$$

Test functions

$$T(\Omega) = \{\mathbf{v} \in C^1(\overline{\Omega})^3 \mid \mathbf{v} = 0 \text{ on } \Sigma\}$$

We derive the variational form from the equation of motion, Equation (7.2.1) and Hooke's law.

The **variational form** of the vibration problem is to find a displacement $\mathbf{u}$ such that the boundary condition on Σ is satisfied and

$$\begin{aligned}
&\int_\Omega \rho \partial_t^2 \mathbf{u} \cdot \mathbf{v} \, dV \\
= &\int_\Omega c \text{ tr } (\boldsymbol{\mathcal{E}} V) + k \text{ tr}(\boldsymbol{\mathcal{E}})\text{tr}(V) \, dV + \int_\Omega \rho \mathbf{b} \cdot \mathbf{v} \, dV + \int_\Gamma \mathbf{T} n \cdot \mathbf{v} \, dS,
\end{aligned}$$

for each $\mathbf{v}$ in $T(\Omega)$.

Remark

The presence of the product $T\mathbf{n} \cdot \mathbf{v}$ in the boundary integral suggests that either $T\mathbf{n}$ or $\mathbf{v}$ must be prescribed on the boundary. The two possible boundary conditions are referred to as complementary boundary conditions.

Let

$$b(\mathbf{u}, \mathbf{v}) = \int_\Omega c \text{ tr } (\boldsymbol{\mathcal{E}} V) + k \text{ tr}(\boldsymbol{\mathcal{E}})\text{tr}(V) \, dV,$$

Problem EDV

Find $\mathbf{u}$ such that for each $t > 0$, $u(t) \in T(\Omega)$ and

$$\left(\rho \partial_t^2 \mathbf{u}, \mathbf{v}\right) = b(u, v) + \int_\Omega \rho \mathbf{b} \cdot \mathbf{v} \, dV + \int_\Gamma \mathbf{T} n \cdot \mathbf{v} \, dS \quad \text{for each } v \in T(\Omega).$$

Weak variational form and existence of solutions

The two-dimensional vibration problem is similar to Problem Plate-W (Reissner-Mindlin plate) while the three-dimensional problem differs slightly. The abstract formulation is the same and the theory depends on the estimate in Korn's inequality (Theorem B.3 in Appendix B). The finite element approximation for the problem is considered in [Wu03] and [Wu06].

Chapter 8

Conclusion

8.1 Overview

In the research for the Masters degree, principally a literature study, the focus was on the Finite Element Method applied to vibrating systems: vibrating beams, plates, multi-dimensional elastic solids, hybrid models and linked vibrating solids. The emphasis of this dissertation was on the Finite Element Method applied to these model problems.

An important aspect was the comparison of different models for the same structure. The chosen method to compare models is numerical simulation. Therefore convergence of the finite element approximation is important. Specific attention was given to:

- determining whether error estimates are available in general and any conditions necessary for convergence;
- error analysis and convergence of solutions;
- existence of solutions.

A search through the literature revealed that a substantial number of publications deals with the finite element method applied to the multi-dimensional wave equation but less on vibrating beams, plates and multi-dimensional elastic solids. Hardly anything has been published on error estimates and finite element approximations for hybrid models and structures, linked systems or structures with boundary damping or damping in joints.

In Chapter 1 we present an introduction to the theory of vibrating beams, plates and multi-dimensional elastic solids. For beams we start with the most general linear model. In particular we looked at the differences between the Euler Bernoulli, Rayleigh and Timoshenko models for the transverse vibration of a beam. We show how the Reissner-Mindlin plate model is derived from the equations of motion and how the classical plate model follows by making additional assumptions.

Also in Chapter 1 we mention briefly comparison results in the literature concerning the Euler Bernoulli, Rayleigh and Timoshenko models for the transverse vibration of a beam. In [SP06] and [LVV09a] the Timoshenko theory is compared to a multi-dimensional model. It was found that the Timoshenko theory is an excellent approximation in the case of beam applications, i.e. for transverse loads. In the articles [VV06] and [LVV09a] the Timoshenko model is compared to Rayleigh or Euler-Bernoulli models. The Rayleigh and Euler-Bernoulli models can be useful when the parameter α is large. A value of α of anything between 2000 and 5000 may be sufficient, but it depends on the initial data or the manner of excitation. This means that it is not possible to determine beforehand whether the Euler-Bernoulli or Rayleigh model is suitable for a given application (unless α is extremely large). The Euler-Bernoulli and Rayleigh models are compared in [VV06]. The values for the first 5 eigenvalues indicate that the difference between the models is probably not significant.

In Chapter 2 we present general existence results for hyperbolic equations which may be applied to the model problems. These results are from the article [VV02]. In this article the existence theorems are given in a convenient form for application to the finite element method. Unfortunately only weak solutions are guaranteed. For higher regularity properties one may consult [Eva98] but the model problem does not include damping and only homogenous Dirichlet boundary conditions. Strict compatibility conditions must be imposed on the initial data and regularity conditions on the forcing function to yield these higher regularity properties.

In Chapter 3 we study the available convergence theory for hyperbolic equations to determine whether error estimates are available in general and any conditions necessary for convergence. A general theory is required that can be applied to any linear vibrating system. We considered only [OR76], [BV13], [BV14] and [Bas14] since other important articles (e.g. [Bak76] and [Dup73]) are already considered in these publications. These results were then applied to each of the different model problems in Chapters 4 to 7.

In the article [BV13] convergence is proved but with weaker assumptions than the other articles considered. This is achieved by splitting the error into semi-discrete and fully discrete errors. In the article semi-discrete and fully discrete error estimates for the Galerkin approximation of a general linear second order hyperbolic problem are derived. Viscous type damping is also incorporated and so the results in [BV13] could be applied to problems like the Reissner-Mindlin plate model. The results and proofs in the article [BV13] are mostly given in sufficient detail in the article, hence the focus in this dissertation was to state the results and apply them. The results can not be applied to problems with boundary damping.

Hybrid beam models with boundary damping are considered in Chapters 4 and 5 (where the theory in [BV13] is not applicable). The model problem in Chapter 4 was introduced in [ZVV04] where the finite element method is used to compute natural frequencies and modes of vibration. In [Bas14, Section 8.2] the model problem is considered briefly. Aspects covered are existence and convergence of the finite element approximation. The idea in this dissertation was to apply the general theory for convergence but this is only available for the semi-discrete approximation in [BV14]. We present a complete derivation for the error estimate for the fully discrete approximation using [OR76] and [Bas14].

In Chapter 5 we study [BDV14] which is based on the work in [AS02]. In the article [AS02] the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. The model is more realistic than other articles: the fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account. The Euler-Bernoulli theory for a beam with Kelvin-Voigt damping is used. The beam is clamped at one end and the tip body is attached to the other end. The existence of a unique solution for the model problem is established. In [BDV14] the authors present a complete analysis of the model. Firstly they derive a correct linear approximation for the model problem. Next they also prove the existence of a unique solution. Their approach differs from that in [AS02]: they write the model problem in variational form and use results from [VV02] where general linear vibration models in variational form are considered. We show that it makes virtually no difference if the Rayleigh model is used instead of the Euler-Bernoulli model.

We observe that in [ZVV04] the Timoshenko beam theory is used which is a better model but the interface condition for the tip body is more realistic in

[BDV14].

Vertical structures are the focus of Chapter 6. Model problems for high rise buildings, towers and other vertical "beams" e.g. industrial chimneys are analyzed. Beam models appear to be popular but in the articles considered no justification was offered. In this chapter we compare Timoshenko, Rayleigh and Euler-Bernoulli models for vertical structures. Beam models for chimneys seem reasonable but beam models for buildings are questionable. (Tall buildings are often modeled as vertical beams, see e.g. [HV07] and [WFH01] and the references in [WFH01].)

Recall that our objective is to compare models. To start, we consider the most general linear model for a vertical beam where gravity is taken into account.

Simulation is necessary to determine displacements, acceleration and stresses in structures subjected to earthquake loads or wind loads.

In Section 6.2 we compare beam models for buildings. Simulation of Timoshenko and Rayleigh models were done. We did not consider existence of solutions and convergence since this is done in Section 6.4 for a more complex model.

In Section 6.3 a vertical Euler-Bernoulli model but with concentrated masses to represent the floors in [WFH01] is mentioned. We consider multiple beam models with concentrated masses to represent the floors, which we believe is a better model.

A vertical slender structure with resilient seating is the topic in Section 6.4. We consider hybrid Timoshenko and Rayleigh beam models for the structure. A Rayleigh model is in [LVV05], where the origin of the problem is discussed and references are given. Models that correspond to Newland's lumped parameter system models [New89, p 129-132] and [New84] are presented. The structure is modelled as a beam mounted vertically and gravity is taken into account. Modeling the behavior of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations. The modeling is done in great detail in [Lab06] where the finite element method is used for a comprehensive modal analysis, and convergence of finite element approximations for eigenvalue problems is considered. Existence results can be found in [LVV09b].

In Chapter 7 plate models and model problems for elastic solids are considered.

From the introduction to the article [Wu05], we conclude that this is the first result on convergence of the finite element method for the vibration of a Reissner-Mindlin plate. With no damping or viscous damping the theory in [BV13] can be applied. In [Wu05] the fully discrete problem is formulated in Section 3 and the explicit scheme explained. The algorithm is virtually the same as central differences. Therefore the theory is a special case of the theory in Section 4.6 of this dissertation. In [LVV09c] a plate-beam system is considered. In out opinion the notation is preferable and we followed their presentation for the plate problem in [Wu05].

Linear elasto-dynamics [Wu03] is also briefly discussed in Section 7.3. The theory is almost the same as for the plate problem.

8.2 Results

As this is a literature study, there are few results that are really new. The main conclusion is that there is scope for research in modeling and finite element analysis and implementation.

Recall that one objective was to compare models. In Chapter 6 vertical beam models with the effect of gravity were compared. To start, we considered the most general linear model and then simplified to obtain workable Timoshenko and Rayleigh models. It turned out that there is a lot more to do than anticipated. It now seems that a complete dissertation can be written on beam models for vertical structures. The work done in Section 6.3 is promising and is worth following up. The comparison between the Timoshenko and Rayleigh beams is only a start. Also, multi-beam models should be compared to single beam models.

The error estimates obtained in [BV13] require weaker regularity assumptions than other articles but the assumptions are still too restrictive. Consequently convergence results for even classical solutions are not available. Note that the existence theorems in Chapter 2, which are convenient to apply, guarantee only weak solutions.

8.3 Further research

The comparison of the Timoshenko an Rayleigh models in Section 6.2 is an obvious topic for further research. Systematic experiments are required to reach a conclusion. Simulation of earthquakes is another possibility since the excitation is different from that caused by wind.

Another topic which should definitely be investigated is the comparison of multi-beam models with single beam models for buildings. A vertical Euler-Bernoulli model but with concentrated masses to represent the floors, is used in [WFH01]. Our proposed multiple beam model should be compared to the model in [WFH01] and single beam models.

Clearly it is essential to have reliable approximations for solutions, and the problems mentioned with error estimates should be addressed. For a start a continued search of the literature is indicated.

Appendix A

Sobolev Spaces

The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $I\!R^n$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω. The first result is well known.

Theorem A.1. *The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with* **inner product**

$$(f,g) = \int_\Omega fg = \int_\Omega fg \; d\mu$$

where μ is the n-dimensional Lebesgue measure.

Theorem A.2. *The space $\mathcal{L}^2(\Omega)$ is separable. (See [Ada75, Th 2.15, p 28]).*

Theorem A.3. *$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$. (See [Ada75, Th 2.13, p 28]).*

Suppose Ω is a **bounded open** interval in $I\!R$ (bounded open subset of $I\!R^n$ when higher dimensions are required). The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition

For f and g in $H^m(\Omega)$,

$$[f,g]_m = (f^{(m)}, g^{(m)}) \quad \text{for} \quad m = 0,\ 1,\ \ldots$$

For $m \geq 1$, the bilinear form $[\cdot,\cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f,f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f, f]_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

Suppose Ω is a **bounded open convex** subset of $\mathbb{R}^2$. The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Remark

It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Definition

For f and g in $H^m(\Omega)$,

$$[f, g]_m = \sum_{i+j=m} (\partial_1^i \partial_2^j f, \partial_1^i \partial_2^j g) \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

For $m \geq 1$ the bilinear form $[\cdot, \cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f, f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f, f]_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

The boundary

Recall that a curve is called **smooth** if its parametrization has a continuous derivative. The boundary of Ω is called **piecewise smooth** if it consists of a finite number of smooth curves.

For a vector valued function r such that $r_i \in C^1[a, b]$ for $i = 1,\, 2$, the range $\boldsymbol{C}$ of r defines a smooth curve in the plane.

Suppose that $\boldsymbol{C}$ is a part of the boundary of Ω. A function f is Lebesgue integrable on $\boldsymbol{C}$ if $f \circ r \sqrt{(r_1')^2 + (r_2')^2}$ is Lebesgue integrable on the interval $[a, b]$.

A function f is in $\mathcal{L}^2(\boldsymbol{C})$ if f^2 is Lebesgue integrable over $\boldsymbol{C}$. The inner product for $\mathcal{L}^2(\boldsymbol{C})$ is defined by

$$(f, g)_{\boldsymbol{C}} = \int_{\boldsymbol{C}} fg \; ds = \int_a^b (f \circ r)\,(g \circ r)\, \sqrt{(r_1')^2 + (r_2')^2}\;.$$

When necessary, we use the **notation** $(f, g)_\Omega$ and $(f, g)_\Gamma$ to avoid confusion.

Vector valued functions

Definition

$$u \in \mathcal{L}^2(\Omega)^2 \;\text{ if }\; u_i \in \mathcal{L}^2(\Omega) \;\text{ for }\; i = 1,\, 2.$$

$$u \in \mathcal{L}^2(\Gamma)^2 \;\text{ if }\; u_i \in \mathcal{L}^2(\Gamma) \;\text{ for }\; i = 1,\, 2.$$

$$u \in H^k\Omega)^2 \;\text{ if }\; u_i \in H^k(\Omega) \;\text{ for }\; i = 1,\, 2.$$

$$[u, v]_{m,2} = [u_1, v_1]_m + [u_2, v_2]_m \;\text{ for }\; u \in \mathcal{L}^2(\Omega)^2 \;\text{ and }\; v \in \mathcal{L}^2(\Omega)^2\,.$$

$$|u|_{m,2} = \sqrt{[u, u]_{m,2}} \;\text{ for }\; u \in \mathcal{L}^2(\Omega)^2\,.$$

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|_{m,2}^{\Omega}$ and $\|\cdot\|_{m,2}^{\Gamma}$.

Definitions

Suppose Ω is a **bounded open interval** or a **bounded open convex** subset of $I\!R^2$.

Notation

$H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

Definition

The inner product for $H^m(\Omega)$ is defined by

$$(f, g)_m = \sum_{k=0}^{m} [f, g]_k \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f,g)_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The inner product for $H^m(\Omega)^2$ is defined by

$$(f,g)_{m,2} = \sum_{k=0}^{m} \, [f,g]_{k,2} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f,g)_{m,2}} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Theorem A.4. *The space $H^m(\Omega)$ is complete (See [Ada75, Th 3.2, p 45]).*

Theorem A.5. *$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.*
(See [OR76, Th 2.10, p 53].)

Appendix B

Inequalities

Proposition B.1. *Consider any $u \in C^1[0,1]$. For any two points x and y in* $[0,1]$,

$$|u(x)| \leq \|u'\| + |u(y)|.$$

Proof. Assuming that $x > y$ (without loss of generality), we have

$$u(x) = \int_y^x u' + u(y).$$

But $|\int_y^x f| \leq \|f\|$ for any $f \in \mathcal{L}^2(0,1)$. This follows from the Cauchy-Schwartz inequality

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right)$$

by choosing $g = 1$. The rest is obvious. □

Theorem B.1. *For any $u \in C^1[0,1]$ with a zero in $[0,1]$ we have*

$$\|u\| \leq \|u'\|.$$

Proof. Use by Proposition B.1. □

Proposition B.2. *For any $u \in C^1[0,1]$,* $\quad |u(0)| \leq \sqrt{2}\ \|u\|_1$.

Proof. Let $g(x) = 1 - x$ and $v = gu$ and consider the fact that

$$u(0) = v(0) = -\int_0^1 v' + v(1)\,.$$

Since $v(1) = 0$,

$$|u(0)| = \left| \int_0^1 (u'g + ug') \right| \leq \|u'\|\|g\| + \|u\|\|g'\| \leq \|u'\| + \|u\|.$$

Now use Young's inequality. □

Suppose Ω is a bounded open convex subset of $I\!R^2$ with a piecewise smooth boundary. The following result is referred to as the Poincare-Friedrichs inequality.

Theorem B.2. *Suppose Σ is a part of the boundary of Ω with nonzero length. Denote the set*

$$\{u \in C^1(\bar{\Omega}) \,|\, u = 0 \text{ on } \Sigma\}$$

by $F(\Omega)$. There exists a constant c_F such that, for each $u \in F(\Omega)$,

$$\|u\| \leq c_F |u|_1.$$

Proof. See e.g. [Bra01, p 30]. □

Corollary B.1. *Suppose Σ_1 and Σ_2 are parts of the boundary of Ω with nonzero length. Denote the set*

$$\{u \in C^1(\bar{\Omega})^2 \,|\, u_1 = 0 \text{ on } \Sigma_1 \text{ and } u_2 = 0 \text{ on } \Sigma_2\}$$

by $F(\Omega)^2$. There exists a constant c_F such that for each $u \in F(\Omega)^2$,

$$\|u\|_{0,2} \leq c_F |u|_{1,2}\,.$$

Theorem B.3. *Korn's inequality*

Suppose b_B is the bilinear form for the Reissner-Mindlin plate. There exists a constant c_Ω such that

$$|u|_{1,2}^2 \leq c_\Omega\, b_B(u,u)$$

for each $u \in V$.

Proof. See e.g. [Bra01, p 288-289]. □

Theorem B.4. *Gronwall's inequality*

Suppose $\psi \in C^1[0,T]$. *If there exist positive constants* c *and* K *such that*

$$\psi \leq c \int_0^t \psi + K$$

then

$$\psi(t) \leq K e^{ct} \text{ for each } t \in [0,T].$$

Proof. Let $\phi(t) = c \int_0^t \psi + K$, then $\phi' = c\psi$ and $\phi(0) = K$. Since $\phi' \leq c\phi$, we have

$$\phi(t) \leq e^{ct} K \quad \text{from Proposition ??}.$$

The result follows from the fact that $\psi \leq \phi$. $\square$

Theorem B.5. *Discrete version of Gronwall's inequality*

If $\phi(t)$ *and* $\psi(t)$ *are nonnegative functions with* $\psi(t)$ *nondecreasing and*

$$\phi(R\tau) \leq \psi(R\tau) + C\tau \sum_{n=0}^{R-1} \phi(n\tau)$$

then

$$\phi(R\tau) \leq \psi(R\tau) e^{CR\tau} \text{ where } C \geq 0.$$

Appendix C

Trace

Definition (Trace operator Γ)

For $u \in C(\bar{\Omega})$, the function Γu is the restriction of the function u to Γ.

Theorem C.1. *The trace operator Γ can be extended to a bounded linear operator mapping $H^1(\Omega)$ onto $L^2(\partial\Omega)$ and $\|\Gamma u\|_{\partial\Omega} \leq K\|u\|_1^\Omega$.*

Proof. This result is a special case of results in [OR76, p 141-142]. $\square$

Definition

For $u \in H^1(\Omega)^2$, we define Γu by

$$\Gamma u = \langle \Gamma u_1, \Gamma u_2 \rangle.$$

Appendix D

Hermite cubics

Hermite cubics

The well-known Hermite piecewise cubics (see [SF73]) are successfully used as basis functions for the Galerkin approximation in beam problems.

The construction and properties of Hermite cubics are treated in detail in the book of Strang and Fix ([SF73, p 55-59]). Divide the interval $[a, b]$ into n subintervals by a partitioning

$$a \;=\; x_0 \;< x_1 \;<\; \cdots \;<\; x_n \;=\; b.$$

This yields n elements, $\Omega_i = [x_{i-1}, x_i]$, each of length h_i, for $i = 1, 2, \ldots, n$.

For $i = 0, 1, \ldots, n$, we have two piecewise cubics denoted by $\delta_i^{(j)}$ with $j = 1$ or $j = 2$ with the following properties:

1. For $k = 0, 1, \ldots, n,\quad i = 0, 1, \ldots, n$ and $j = 1, 2,$ the restriction of $\delta_k^{(j)}$ to any Ω_i is either a cubic polynomial or zero.
2. $\delta_i^{(j)} \in C^1[a, b]$ and $D^2\delta_i^{(j)}$ is piecewise continuous with possible discontinuities at the nodes.
3. $\delta_i^{(1)}(x_i) = 1,\quad D\delta_i^{(1)}(x_i) = 0,\quad \delta_i^{(2)}(x_i) = 0,\quad D\delta_i^{(2)}(x_i) = 1.$
4. $\delta_i^{(1)}(x_k) = 0,\quad D\delta_i^{(1)}(x_k) = 0,\quad \delta_i^{(2)}(x_k) = 0,\quad D\delta_i^{(2)}(x_k) = 0$ if $k \neq i$.
5. $\delta_i^{(j)}$ is zero on any element Ω_k with $k \neq i$ or $i + 1$.

We refer to these two types of functions as Type 1 ($j = 1$) or Type 2 ($j = 2$) functions. Typical graphs of $\delta_i^{(1)}$ and $\delta_i^{(2)}$ are shown in Figures 1 and 2.

Figure 1: Type 1 Hermite piecewise cubic

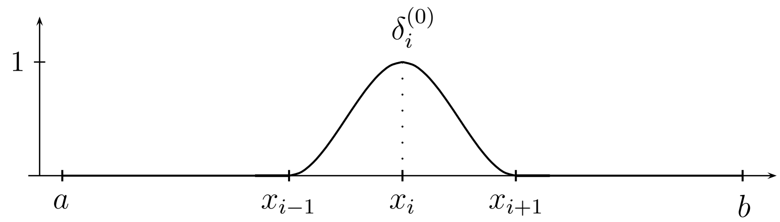
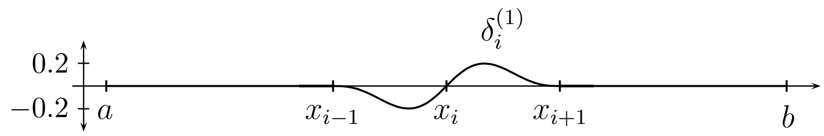


Figure 2: Type 2 Hermite piecewise cubic



Ordering of basis functions

The basis functions are ordered such that $\delta_1, \delta_2, \cdots, \delta_{n+1}$ are Type 1 basis functions and $\delta_{n+2}, \delta_{n+3}, \cdots, \delta_{2n+2}$ are Type 2 basis functions.

Definition Matrices M, K, and L.

For $i, j = 1, 2, \cdots, n$ the matrices are defined as

$$M_{ij} = \int_0^\ell \delta_j \delta_i,$$
$$K_{ij} = \int_0^\ell \delta_j' \delta_i',$$
$$L_{ij} = \int_0^\ell \delta_j \delta_i'.$$

Appendix E

Interpolation

The interval $I = [0, \ell]$ is partitioned into n elements referred to as a finite element mesh. Using this mesh a finite dimensional space S^h is defined. An interpolation operator Π is a transformation from $H^k(I)$ to S^h.

It is first necessary to define interpolation on an element.

Definitions

See [SF73], [OC83] or [OR76].

Notation

$\mathcal{P}_j(I_e)$: the set of polynomials on I_e of degree at most j is denoted by $\mathcal{P}_j(I_e)$.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e.

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e.

Remark

If $k \geq s(\Pi_e) + 1$, then the interpolation operator Π_e is defined on $H^k(I_e)$.

Recall that $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k = \|u^{(k)}\|\,.$$

(See Appendix A.)

The interpolation error

Theorem E.1 below is formulated as a special case of a general result. This result may be found in [SF73, p.144], [OC83, p.76] and [OR76, p.279].

We will use $\widehat{C}$ to denote a generic constant.

Theorem E.1. *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1$$

for the interpolation operator Π. Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(I)$ we have

$$|\Pi u - u|_{m,I} \leq \widehat{C} h^{k-m} |u|_{k,I} \quad \textit{for} \quad m = 0,\ 1,\ \ldots, k.$$

The interpolation operator is denoted by Π_L for piecewise linear basis functions and by Π_c for Hermite cubics.

Corollary E.1. *Hermite cubic basis functions.*
There exists a constant $\widehat{C}_c$ such that if $u \in H^k(I)$ for

a) $2 \leq k \leq 4$, *then*

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

b) $k > 4$, *then*

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof. It is clear that $s(\Pi_c) = 1$ and it can be shown that $r(\Pi_c) = 3$. Consequently Theorem E.1 is applicable with $k = 2,\ 3$ or 4. $\square$

Corollary E.2. *Piecewise linear basis functions*
There exists a constant $\widehat{C}_L$ such that if $u \in H^k(I)$ for $k \geq 2$, then

$$\|\Pi_L u - u\|_1 \leq \widehat{C}_L h |u|_2$$

Proof. It is clear that $s(\Pi_L) = 1$ and it can be shown that $r(\Pi_L) = 1$. Consequently Theorem E.1 is applicable with $k = 2$. $\square$

The two-dimensional case

Recall that the domain Ω is partitioned into n elements. An interpolation operator Π is a transformation from $H^k(\Omega)$ to S^h, the finite dimensional subspace.

Definitions

See [SF73], [OC83] or [OR76].

Notation

$\mathcal{P}_j(\Omega_e)$: the set of polynomials on Ω_e of degree at most j.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e.

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e.

Recall that for a two-dimensional domain Ω, $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix A.)

Theorem E.2 below is formulated as a special case of a general result. As mentioned before, this result may be found in [SF73, p.144], [OC83, p.76] and [OR76, p.279]. In the theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e.

Theorem E.2. *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 2 \leq k \leq r(\Pi_e) + 1$$

for the interpolation operator Π. *Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi u - u|_{m,\Omega} \leq \widehat{C} h^{k-m} |u|_{k,\Omega} \quad \textit{for} \quad m = 0,\ 1,\ \ldots, k.$$

Remark

The constant $\widehat{C}$ depends on the shape of the elements in the finite element mesh.

Corollary E.3. *Piecewise linear basis functions on triangle elements.*
The interpolation operator is denoted by $\Pi_\triangle$*. If* $k \geq 2$*, then there exists a constant* $\widehat{C}_\triangle$ *such that for any* $u \in H^k(\Omega)$ *we have*

$$|\Pi_\triangle u - u|_{m,\Omega} \leq \widehat{C}_\triangle h^{2-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ 2.$$

Corollary E.4. *Piecewise bilinear basis functions on rectangle elements.*

The interpolation operator is denoted by Π_b*. If* $k \geq 2$*, then there exists a constant* $\widehat{C}_b$ *such that for any* $u \in H^k(\Omega)$ *we have*

$$|\Pi_b u - u|_{m,\Omega} \leq \widehat{C}_b h^{2-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ 2.$$

Corollary E.5. *Hermite cubic basis functions.*

The interpolation operator is denoted by Π_c*. If* $k \geq 2$*, then there exists a constant* $\widehat{C}_c$ *such that for any* $u \in H^k(\Omega)$ *we have*

$$|\Pi_c u - u|_{m,\Omega} \leq \widehat{C}_c h^{2-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ 2.$$

Vector-valued functions

If an interpolation operator Π is defined on $H^k(\Omega)$ we may define one on $H^k(\Omega)^2$. For $u = \langle u_1,\, u_2 \rangle \in H^k(\Omega)^2$, we define

$$\Pi_2\, u = \langle \Pi\, u_1,\, \Pi\, u_2 \rangle\,.$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2\,.$$

(See Appendix A.)

Lemma E.1. *If* $\|\Pi\ v - v\|_m \leq \widehat{C} h^{k-m} |v|_k$ *for* $v \in H^k(\Omega)$*, then*

$$\|u - \Pi_2 u\|_{m,2} \leq \widehat{C} h^{k-m} |u|_{k,2} \text{ for } u \in H^k(\Omega)^2.$$

Proof.

$$|\, u - \Pi_2\, u|_{m,2}^2 = |\, u_1 - \Pi\, u_1|_m^2 + |\, u_2 - \Pi\, u_2|_m^2$$

□

For piecewise bilinear basis functions on rectangles, let $\Pi_B\, u = \langle \Pi_b\, u_1, \Pi_b\, u_2 \rangle$.

Corollary E.6. *If $k \geq 2$, then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$| \, u - \Pi_B \, u | _{m,2} \leq \widehat{C} h |u|_{2,2} \text{ for } m = 0, 1, 2.$$

Proof. The result follows from Corollary 4 and Lemma 1. □

For piecewise bicubic basis functions on rectangles, let $\Pi_{Bc}\, u = \langle \Pi_{bc}\, u_1, \Pi_{bc}\, u_2 \rangle$.

Corollary E.7. *If $k \geq 2$, then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$| \, u - \Pi_{Bc} \, u | _{m,2} \leq \widehat{C} h |u|_{2,2} \text{ for } m = 0, 1, 2.$$

Appendix F

Motion of a slender solid

Consider a solid with reference configuration the region $\mathcal{B}$ and any point $\bar{x}$ in $\mathcal{B}$. Suppose the position of the point at time t is $\bar{R}(\bar{x}, t)$. The velocity of the point at time t is then $\bar{v}(\bar{x}, t) = \partial_t \bar{R}(\bar{x}, t)$.

It is necessary to consider the conservation laws for each part of the solid. Suppose $\mathcal{R}$ is an arbitrary part of $\mathcal{B}$ with boundary Σ. Let $\bar{t}_{\mathcal{R}}$ denote the traction on $\mathcal{R}$ and $\bar{b}$ the body force (density).

Conservation of momentum

$$\frac{d}{dt} \int_{\mathcal{R}} \rho \bar{v} \; dV = \int_{\Sigma} \bar{t}_{\mathcal{R}} \; dS + \int_{\mathcal{R}} \bar{b} \; dV.$$

Conservation of angular momentum

$$\frac{d}{dt} \int_{\mathcal{R}} (\bar{r} - \bar{p}) \times \rho \bar{v} \; dV = \int_{\Sigma} (\bar{r} - \bar{p}) \times \bar{t}_{\mathcal{R}} \; dS + \int_{\mathcal{R}} (\bar{r} - \bar{p}) \times \bar{b} \; dV,$$

where $\bar{p}$ is any fixed point. We may assume that $\bar{p} = 0$ without loss of generality.

In this appendix we consider the motion of a slender solid $\mathcal{B}$ that is straight in its undeformed state. For the reference configuration we assume the existence of a straight line in the solid – referred to as the axis – such that every cross-section perpendicular to this line has its centroid on the line. Stated another way: the straight line joins all the centroids. We may choose coordinates for the reference configuration in such a way that the axis is the line $y = z = 0$ and the cross-sections are parallel to the $yz-$plane.

We assume that the cross-sections are all the same. Such a body is referred to as prismatic. The length of the axis is the length of the body and the "diameter" of a cross-section is assumed to be substantially less than the length for a slender body.

Motion

It is assumed that every cross-section executes a rigid motion. To be specific, we assume that the position of a point (x, y, z) in $\mathcal{B}$ at time t is given by

$$\bar{R}(x, y, z, t) = \bar{r}_0(x, t) + y\bar{e}_y(x, t) + z\bar{e}_z(x, t),$$

where $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_z$ move with the cross-section. This means that a cross-section remains plane during the motion.

Clearly $\bar{r}_0(x, t)$ is the position of the centroid $(x, 0, 0)$ of a cross section at time t and the position of (x, y, z) relative to the centroid is $\bar{r} = y\bar{e}_y + z\bar{e}_z$.

In the conservation laws formulated above, $\mathcal{R} \subset \Omega$ may be arbitrary. We now consider a special case where $\mathcal{R}$ is the part of the solid between the cross sections where $x = a$ and $x = b$. The velocity of the point (x, y, z) at time t is

$$\bar{v} = \partial_t\bar{r}_0 + \partial_t\bar{r} = \partial_t\bar{r}_0 + y\partial_t\bar{e}_y + z\partial_t\bar{e}_z.$$

Proposition F.1. Momentum

If the density ρ is constant, then the momentum of $\mathcal{R}$ is

$$\int_{\mathcal{R}} \rho\bar{v}(\cdot, t)\ dV = \rho A \int_a^b \partial_t\bar{r}_0(x, t)\ dx,$$

where A is the area of the cross-section.

Proof.

$$\begin{aligned}\int_{\mathcal{R}} \rho\bar{v}dV &= \rho \int_a^b \int_{\mathcal{D}} dA\ \partial_t\bar{r}_0(x, t)dx \\ &+ \rho \int_a^b \int_{\mathcal{D}} ydA\ \partial_t\bar{e}_y(x, t)dx \\ &+ \rho \int_a^b \int_{\mathcal{D}} zdA\ \partial_t\bar{e}_z(x, t)dx.\end{aligned}$$

Note that $\int_{\mathcal{D}} dA = A$ and $\int_{\mathcal{D}} ydA = \int_{\mathcal{D}} zdA = 0$. □

Proposition F.2. Angular momentum

If the density ρ is constant, then the angular momentum of $\mathcal{R}$ about $\bar{0}$ is

$$\begin{aligned}\int_{\mathcal{R}} \bar{R} \times \rho \bar{v} dV &= \rho A \int_a^b \bar{r}_0(x,t) \times \partial_t \bar{r}_0(x,t)\ dx \\ &+ \rho \int_a^b \int_{\mathcal{D}} \bar{r}(\cdot,t) \times \partial_t \bar{r}(\cdot,t)\ dAdx,\end{aligned}$$

where $\mathcal{D}$ denotes the relevant cross-section.

Proof. Note that $\bar{R} \times \rho \bar{v} = \bar{r}_0 \times \partial_t \bar{r}_0 + \bar{r}_0 \times \partial_t \bar{r} + \bar{r} \times \partial_t \bar{r}_0 + \bar{r} \times \partial_t \bar{r}$. If we integrate to calculate the angular momentum, two terms vanish. Consider

$$\begin{aligned}&\int_a^b \bar{r}_0(x,t) \times \int_{\mathcal{D}} \partial_t \bar{r}(\cdot,t)\ dAdx \\ =\ &\int_a^b \bar{r}_0(x,t) \times \left(\int_{\mathcal{D}} ydA\ \partial_t \bar{e}_y(x,t)dx + \int_{\mathcal{D}} zdA\ \partial_t \bar{e}_z(x,t)dx \right).\end{aligned}$$

As in Proposition F.1, note that $\int_{\mathcal{D}} ydA = \int_{\mathcal{D}} zdA = 0$ and the result is zero. Similarly,

$$\int_a^b \int_{\mathcal{D}} \bar{r}(x,t)\ dA \times \partial_t \bar{r}_0(x,t)dx = \bar{0}.$$

□

Traction

The boundary Σ of $\mathcal{R}$ consists of three parts: the cross-sections at $x = a$ and $x = b$ and part of the surface of the solid between $x = a$ and $x = b$ – which we denote by Σ_{ab}.

Force and couple on a cross-section

The traction on a cross-section is equivalent to the force $\bar{F}(x,t)$ acting at the centroid and a couple $\bar{M}(x,t)$. Consequently the forces exerted **on** $\mathcal{R}$ are $\bar{F}(b,t)$ and $-\bar{F}(a,t)$ and the couples are $\bar{M}(b,t)$ and $-\bar{M}(a,t)$.

Traction on the surface

We assume that the traction on the surface Σ_{ab} is zero. Due to the assumption above

$$\int_{\Sigma} \bar{t}_{\mathcal{R}}\ dS = \bar{F}(b,t) - \bar{F}(a,t)$$

and

$$\int_\Sigma \bar{R} \times \bar{t}_\mathcal{R} \; dS = \bar{r}_0(b,t) \times \bar{F}(b,t) - \bar{r}_0(a,t) \times \bar{F}(a,t) + \bar{M}(b,t) - \bar{M}(a,t).$$

Equations of motion

From a mathematical point of view we now have a one-dimensional model for the slender solid and the reference configuration is $[0, \ell]$. Due to Propositions 1 and 2 the *conservation laws* may be reformulated in a more convenient form.

It is useful to introduce the following notation

$$\bar{H}(x,t) = \rho \int_\mathcal{D} \bar{r} \times \partial_t \bar{r} dA.$$

The quantity $\bar{H}(x,t)$ is referred to as the angular momentum density (about the centroid).

Conservation of momentum

$$\frac{d}{dt} \int_a^b \rho A \partial_t \bar{r}_0(x,t) \; dx = \bar{F}(b,t) - \bar{F}(a,t) + \int_a^b \bar{P}(x,t) \; dx.$$

Conservation of angular momentum

$$\begin{aligned}
&\frac{d}{dt} \int_a^b \rho A \bar{r}_0(x,t) \times \partial_t \bar{r}_0(x,t) dx + \frac{d}{dt} \int_a^b \bar{H}(x,t) \; dx \\
&= \bar{r}_0(b,t) \times \bar{F}(b,t) - \bar{r}_0(a,t) \times \bar{F}(a,t) + \bar{M}(b,t) - \bar{M}(a,t) \\
&+ \int_a^b \bar{r}_0(b,t) \times \bar{P}(x,t) \; dx..
\end{aligned}$$

The *equations of motion* follow from the conservation laws. By a standard argument we have the first result.

Proposition F.3.

$$\rho A \partial_t^2 \bar{r}_0 = \partial_x \bar{F} + \bar{P}.$$

Proposition F.4.

$$\partial_t \bar{H} = \partial_x \bar{r}_0 \times \bar{F} + \partial_x \bar{M}.$$

Proof. We first prove, using the standard argument that

$$\rho A \bar{r}_0 \times \partial_t^2 \bar{r}_0 + \partial_t \bar{H} = \partial_x(\bar{r}_0 \times \bar{F}) + \partial_x \bar{M}.$$

Now, $\partial_x(\bar{r}_0 \times \bar{F}) = \partial_x \bar{r}_0 \times \bar{F} + \bar{r}_0 \times \partial_x \bar{F}$ and hence the result follows from Proposition F.3. $\square$

Bibliography

[Ada75] R A Adams. *Sobolev spaces*. Series 3. Academic Press, New York, San Francisco, London, 1975.

[APBS78] F B Seely A P Boresi, O M Sidebottom and J O Smith. *Advanced mechanics of materials*. John Wiley & sons, New York, third edition edition, 1978.

[AS02] K T Andrews and M Shillor. Vibrations of a beam with a damping tip body. *Math. Comput. Modelling*, 35:1033–1042, 2002.

[Bak76] G A Baker. Error estimates for finite element methods for second order hyperbolic equations. *SIAM Journal Numerical Analysis*, 13(4):564–576, 1976.

[Bas11] M Basson. Finite element approximation of general second order hyperbolic equations. Master's thesis, University of Pretoria, 2011.

[Bas14] M Basson. *Error analysis for Galerkin finite element approximation of general second order hyperbolic equations*. PhD thesis, University of Pretoria, 2014.

[BDV14] M Basson, M De Villiers, and NFJ Van Rensburg. Solvability of a model for the vibration of a beam with a damping tip body. *Journal of Applied Mathematics*, 2014.

[Bra01] D Braess. *Finite elements*. Cambridge university press, Cambridge, ii edition, 2001.

[BV13] M Basson and NFJ Van Rensburg. Galerkin finite element approximation of general linear second order hyperbolic equations. *Numerical Functional Analysis and Optimization*, 34(9):976–1000, 2013.

[BV14] M Basson and NFJ Van Rensburg. Galerkin finite element approximation of the general linear second order hyperbolic equation. Technical Report UPWT(16), 2014.

[Cow66] G R Cowper. *the shear coefficient in timoshenko's beam theory.* Journal of Applied Mechanics*, 33:335–340, 1966.*

[Dup73] T Dupont. $\mathcal{L}^2$*-estimates for galerkin methods for second order hyperbolic equations.* SIAM Journal of Numerical Analysis*, 10(5):880–889, October 1973.*

[Eva98] L C Evans. Partial Differential Equations*. American Mathematical Society, 1998.*

[FL10] A Flaga and T Lipecki. Code approaches to vortex shedding and own model. Engineering structures*, 32:1530–1536, 2010.*

[Fun65] Y C Fung. Foundations of solid mechanics*. Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1965.*

[HV07] J W Hijmissen and W T Van Horssen. On aspects of damping for a vertical beam with a tuned mass damper at the top. Nonlinear Dyn*, 50:169–190, 2007.*

[Inm94] D J Inman. Engineering vibration*. Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1994.*

[Lab06] A Labuschagne. Finite element analysis of plate and beam models*. PhD thesis, University of Pretoria, 2006.*

[LVV05] A Labuschagne, N F J Van Rensburg, and A J Van der Merwe. Distributed parameter models for a vertical slender structure on a resilient seating. Mathematical and Computer Modelling*, 41:1021–1033, 2005.*

[LVV09a] A Labuschagne, N F J Van Rensburg, and A J Van der Merwe. Comparison of linear beam theories. Mathematical and Computer Modelling*, 49:20–30, 2009.*

[LVV09b] A Labuschagne, N F J Van Rensburg, and A J Van der Merwe. Solvability of a hybrid model for a vertical slender structure. JNAIAM J. Numer. Anal. Ind. Appl. Math.*, 4(3-4):181–191, 2009.*

[LVV09c] *A Labuschagne, N F J Van Rensburg, and A J Van der Merwe. Vibration of a Reissner-Mindlin-Timoshenko plate-beam system.* Mathematical and Computer Modelling*, 50:1033–1044, 2009.*

[Min51] *R D Mindlin. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates.* Journal of Applied Mechanics*, 18(1):31–38, 1951.*

[Myk66] *N. O. Myklestad. Statics of deformable bodies. The Macmillan Company, New York, 1966.*

[New84] *D E Newland. Calculation of the effect of a resilient seating on the vibration characteristic of slender structures.* Engng Struc*, 6:307–314, 1984.*

[New89] *D E Newland.* Mechanical vibration analysis and computation*. Longman, Essex, 1989.*

[OC83] *J T Oden and F Carey.* Finite elements: Mathematical aspects*. Prentice-Hall International, London, 1983.*

[OR76] *J T Oden and J N Reddy.* An introduction to the mathematical theory of finite elements*. John Wiley & Sons, New York-London-Sydney-Toronto, 1976.*

[Paz83] *A Pazy.* Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York-London-Sydney-Toronto, 1983.*

[Rei88] *H Reismann.* Elastic Plates: Theory and Application*. John Wiley & Sons, New York, 1988.*

[SF73] *G Strang and G J Fix.* An Analysis of the Finite Element Method*. Prentice-Hall, New Jersey, 1973.*

[SP06] *N G Stephen and S Puchegger. On the validity range of timoshenko beam theory.* J. Sound Vib.*, 297:1082–1087, 2006.*

[Tim37] *S Timoshenko.* Vibration problems in Engineering*. D van Nostrand Company, Inc., New-York, 2 edition, 1937.*

[VV02] *N F J Van Rensburg and A J Van der Merwe. Analysis of the solvability of linear vibration models.* Applicable Analysis*, 81(5):1143–1159, 2002.*

[VV06] *N F J Van Rensburg and A J Van der Merwe. Natural frequencies and modes of a timoshenko beam.* Wave Motion*, 44:58–69, 2006.*

[VVR10] *N F J Van Rensburg, A J Van der Merwe, and A Roux. Waves in a vibrating solid with boundary damping.* Wave motion*, 47:663–675, 2010.*

[VZV09] *N F J Van Rensburg, L Zietsman, and A J Van der Merwe. Solvability of a Reissner-Mindlin-Timoshenko plate-beam vibration model.* IMA Journal of Applied Mathematics*, 74(1):149–162, 2009.*

[WFH01] *A P Wang, R F Fung, and S C Huang. Dynamic analysis of a tall building with a tuned-mass-damper device subjected to earthquake excitations.* Journal of Sound and Vibration*, 244(1):123–136, 2001.*

[Wu03] *S R Wu. A priori error estimates for explicit finite element for linear elasto-dynamics by Galerkin method and central difference method.* Comput. Methods Appl. Mech. Engrg.*, 192(51-52):5329–5353, 2003.*

[Wu05] *S R Wu. A priori error estimation of a four-node Reissner-Mindlin plate element for elasto-dynamics.* Comput. Methods Appl. Mech. Engrg.*, 194(18-20):2257–2281, 2005.*

[Wu06] *S R Wu. Lumped mass matrix in explicit finite element method for transient dynamics of elasticity.* Comput. Methods Appl. Mech. Engrg.*, 195(44-47):5983–5994, 2006.*

[Zie00] *L Zietsman.* Finite element analysis of vibration models with interface conditions. *PhD thesis, University of Pretoria, 2000.*

[ZVV04] *L Zietsman, N F J Van Rensburg, and A J Van der Merwe. A timoshenko beam with tip body and boundary damping.* Wave Motion*, 39(3):199211, 2004.*