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Master's Degree Dissertation

On groups with few p'-character degrees

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Declaration

I declare that the dissertation, which I hereby submit for the degree Master of Science at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature: ____________ **Date:** 2023/03/23

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Abstract

Seitz's theorem asserts that a finite group has exactly one non-linear irreducible character of degree greater than one if and only if the group is either an extraspecial 2-group or the group is isomorphic to a one-dimensional affine group over some field. An extension of Seitz's theorem is Thompson's celebrated theorem which states if the degrees of all non-linear irreducible characters of a group are divisible by a fixed prime p, then the group contains a normal p-complement. More recently, in 2020, as an extension to Thompson's theorem, Giannelli, Rizo, and Schaeffer Fry showed that if the character degree set of a group G contains only two p'-character degrees (where $p > 3$ is a prime), then G contains a normal subgroup N such that N has a normal p-complement and G/N has a normal p-complement. Moreover, G is solvable. In this dissertation, we explore a variation of Thompson's Theorem. We explore the structure of finite groups that have exactly one non-linear irreducible character whose degree is non-divisible by a fixed prime p. We call such groups $(*)$-groups (p divides the order of the group). In 1998, Kazarin and Berkovich characterized the structure of $(*)$-groups. We give a detailed proof of their work for solvable groups. Moreover, we produce a classification of $(*)$-groups of order less than or equal to 100.

Nomenclature

$\mathbb{Z}$, $\mathbb{Z}_+$	set of integers, set of positive integers
$a \mid b$	a divides b where a and b are integers
G	a finite group
$\|G\|$	order of the finite group G
$o(g)$ where $g \in G$	order of the element g
C_n	cyclic group of order n
$G \cong H$	G is isomorphic to the group H
$\mathrm{Sym}(\Omega)$, S_n	the symmetric group on Ω, the symmetric group on n letters
A_n	alternating group on n letters
$\mathrm{Aut}(G)$	the automorphism group of G
$F, F^\times$	a field, the multiplicative group $F - \{0\}$
Q_8	quaternion group of order eight
D_{2n}	dihedral group of order $2n$
Dic_n	dicyclic group of order $4n$
$\mathrm{GL}(n, F)$	the general linear group of degree n over a field F
$\mathrm{GL}(n, p^n)$	the general linear group of degree n over a field of order p^n
$H \leq G, H < G$	H is a subgroup of G, H is a proper subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\|G : H\|$	index of H in G
$\langle X \rangle$	the subgroup of generated by a set X
G', G'', G'''	first, second, and third derived subgroup of G, respectively
$G^{(n)}$	nth derived subgroup of G
$\mathrm{dl}(G)$	the derived length of G
$H \operatorname{char} G$	H is characteristic in G
$\mathbf{Z}(G)$, $\mathbf{N}_H(G)$	center of G, normalizer of H in G
G/H	factor group of G by H
$\mathrm{Syl}_p(G)$, n_p	set of Sylow p-subgroups of G, number of Sylow p-subgroups
$H \times K, H \rtimes K$	direct product of H and K, semidirect product of H by K
$\mathrm{Orb}(x) = \mathcal{O}_x$	the orbit of x
$\mathrm{Stab}(x) = G_x$	the stabilizer of x in G
$\mathrm{AGL}_1(p^n)$	one-dimensional affine group over a field of order p^n

Nomenclature

g^G where $g \in G$	conjugacy class of g in G
$F[G]$	a group algebra
1_G	principal character of G
$[\phi, \psi]$	$\frac{1}{\lvert G\rvert}\sum_{g\in G}\phi(g)\overline{\psi(g)}$
$\mathrm{Irr}(G)$	set of complex irreducible characters of G
χ_H	the restriction of χ to H
ϕ^G	class function ϕ induced on G
$\mathrm{cd}(G)$	$\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$
$\mathrm{cd}_p(G)$	$\{\chi(1) : \chi \in \mathrm{Irr}(G)$ and $p \mid \chi(1)\}$
$\mathrm{cd}_{p'}(G)$	$\{\chi(1) : p \nmid \chi(1)$ and $\chi \in \mathrm{Irr}(G)\}$
$o(\chi)$	the determinantal order of χ
$\hat{G}$	group of linear characters of G
$\mathbf{O}^p(G)$	smallest normal subgroup N of G such that G/N is a p-group
$\mathrm{Irr}_1(G)$	set of non-linear irreducible characters of G
$\mathrm{Irr}_1(G, p')$	set of non-linear irreducible characters of G with p'-character degree
$\mathrm{ES}(m, p)$	extraspecial p-group of order p^{2m+1}
$\mathrm{Deg}(G)$	character degree sequence of a group G

Contents

Introduction

A *group* is defined as a set together with an associative binary operation such that every element has an inverse and an identity exists within the set. Groups are defined very abstractly, and it can sometimes be problematic when trying to prove statements regarding them. Representation theory and character theory aim to remedy this problem by representing the elements of groups as familiar objects–matrices. More specifically, a homomorphism, called a *representation* of G, is defined from a group G to the group $\mathrm{GL}(n, F)$ (the group of $n \times n$ invertible matrices over a field F under matrix multiplication). The information within a representation is condensed by defining a function that maps each group element to the trace of its associated matrix under the representation. These functions are called *characters* of the group.

Representation and character theory, respectively, have proven useful in answering group-theoretic questions. An example of this is Burnside's $p^\alpha q^\beta$ theorem; this is a purely group-theoretic statement that asserts that a group of order $p^\alpha q^\beta$ is solvable. It took nearly 70 years to find a proof of this theorem that does not involve representation theory [18, pg 71]. Characters, and even more so, *irreducible* characters, which are a basis for the vector space of class functions (characters are also class functions), play an essential role in determining the structure of a group. The most trivial example is that a group G is abelian if and only if every irreducible character of G is of degree one—the *degree* of a character being the number produced when evaluating the character at the identity. More closely related to this dissertation is the famous Ito-Michler theorem which asserts that the degrees of all non-linear (degree greater than 1) irreducible characters of a group G are non-divisible by a prime p if and only if the group contains a normal abelian Sylow p-subgroup. Note from the above-stated theorems that the relationships established are between purely group-theoretic properties and purely character-theoretic properties.

The celebrated Thompson's theorem states if the degrees of all non-linear irreducible characters of a group are divisible by a prime p (p is a fixed prime), then the group contains a normal p-complement. In this dissertation, we explore a variation of Thompson's Theorem. *We explore the structure of groups that have exactly one non-linear irreducible character whose degree is non-divisible by a fixed prime p.* We call such groups $(*)$*-groups* (p divides the order of the group). In 1998, Kazarin and Berkovich [12] characterized the structure of $(*)$-groups. We give a detailed proof of their work for solvable groups (Chapter 5). More recently, in 2020, Giannelli, Rizo, and Schaeffer Fry [2] showed that if the condition of Kazarin and Berkovich is relaxed, that is, $\left|\mathrm{cd}_{p'}(G)\right| = |\{\chi(1) : \chi \in \mathrm{Irr}(G) \text{ and } p \nmid \chi(1)\}| = 1$ where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G, then G contains a normal subgroup N such that N has a normal p-complement and G/N has a normal p-complement. Moreover, G is solvable.

This dissertation is divided into two parts, namely the preliminaries and the main work regarding $(*)$-groups.

Part I, Preliminaries: In *Chapter 1*, we provide some general group theoretic results and introduce groups of interest in this dissertation—solvable, nilpotent, and Frobenius groups. *Chapter 2* gives an overview of character-theoretic results. We define modules, representations, and characters (irreducible characters). We state some essential theorems for Part 2, including Frobenius reciprocity, Clifford's theorem and Ito's theorem.

Part II, Groups with few p'-character degrees: *Chapter 3* is devoted to proving Thompson's theorem. The normal p-complement of a group is defined here. We define the p-residue of a group G, $\mathbf{O}^p(G)$, and show how this is used to characterize a group containing a normal p-complement. Seitz [17], in 1968, showed that, loosely speaking, a group has exactly one non-linear irreducible character if and only if either the group is a 2-group or it is isomorphic to some specific Frobenius group. We use his result to show that nilpotent $(*)$-groups do not exist (it is essential to realize that for a group to be called a $(*)$-group, we require p to divide the order of the group!). In *Chapter 5*, we prove the result[12, Theorem A] given by Kazarin and Berkovich; that is, we give a characterization of $(*)$-groups (the solvable case). We then give examples of $(*)$-groups in *Chapter 5*. We find all $(*)$-groups of order up to 100, applying the result by Kazarin and Berkovich.

Part I

Preliminaries

1 Groups

Mathematics is the art of giving the same name to different things.

– Henri Poincaré

1.1 Finite groups

This chapter gives a basic overview of the required group theory notions. We introduce, briefly, solvable, nilpotent and Frobenius groups. Well-known and readily accessible results will not be proved but referenced. Most of these results can be found in [10], [6] and [9].

A binary operation on a set G is a function that assigns each ordered pair of elements of G an element in G.

Definition 1.1.1. *A group G is a set together with a binary operation, which we will denote ab for $a, b \in G$, such that:*

1. $(ab)c = a(bc)$ *for all* $a, b, c \in G$*;*

2. *There exists an element in G, denoted by 1 (called the identity), such that $a1 = 1a = a$ for all $a \in G$;*

3. *For all $a \in G$ there exists $b \in G$ (called the inverse of a) such that $ab = ba = 1$ (the inverse of a is usually denoted by a^{-1}).*

Remark 1.1.2.

1. *The cardinality of a group G is called the order of G and is denoted by $|G|$. If the order of a group is finite, we call the group a finite group. We will henceforth only deal with finite groups. Thus when reading statements such as "let G be a group", "consider the group G, etc., we can* always assume the group to be finite*. Further, the letter G, without context, is always assumed to be a finite group.*

2. *If $g \in G$, the order of g, denoted by $o(g)$, is the smallest positive integer n such that $g^n = 1$. Further, if n is the smallest positive integer such that $g^n = 1$* for all *$g \in G$, we call n the exponent of G. An abelian group ($gh = hg$ for all g, h in the group) of prime exponent (for some prime p we have that $g^p = 1$ for all elements g in the group) is said to be elementary abelian.*

Definition 1.1.3. *Given groups G and H. A function $\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$, is called a homomorphism. A surjective homomorphism is called an epimorphism. A bijective homomorphism is called an isomorphism. Moreover, if there exists an isomorphism from G to H, then G and H are said to be isomorphic, denoted by $G \cong H$.*

Example 1.1.4. *The following are some examples of groups:*

- *If Ω is a set, then the set of bijections from Ω to Ω, under the composition of functions, forms a group, denoted by $\mathrm{Sym}(\Omega)$, called the the symmetric group on Ω. If $\Omega = \{1, \dots, n\}$ we denote $\mathrm{Sym}(\Omega)$ by S_n.*
- *The set of isomorphisms from G to G, under the composition of functions, forms a group, denoted by $\mathrm{Aut}(G)$, called the automorphism group of G.*
- *The set of $n \times n$ invertible matrices over a field F, under matrix multiplication, forms a group called the general linear group of degree n over F. This group is denoted by $\mathrm{GL}(n, F)$.*

A subset H within a group G such that H under the same operation of G is itself a group, is called a subgroup of G. If H is a subgroup of G, we will write $H \leq G$ and $H < G$ if $H \neq G$ (*a proper subgroup* of G). This subgroup will be called a normal subgroup of G, written $H \trianglelefteq G$, if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. Lagrange's theorem [10, Theorem 2.23] states that the subgroup's order always divides the group's order. Moreover, if $H \leq G$, the integer $|G| / |H|$ is denoted by $|G : H|$ and called the index of H in G.

Remark 1.1.5. *If G is a group, the set $\{1\}$ is a subgroup of G called the trivial subgroup. We will denote it as 1. From the context, whether 1 refers to the identity within a group or the trivial subgroup should be clear.*

Now it is easily shown that if $H, K \leq G$, then $H \cap K \leq G$. With this in mind, we can define the following.

Definition 1.1.6. *Let $X \subseteq G$. The subgroup of G generated by X, denoted by $\langle X \rangle$, is defined by*

$$\langle X \rangle = \bigcap_{H \in \mathcal{H}} H,$$

where $\mathcal{H}$ is the set of all subgroups of G containing X. Note that $\mathcal{H} \neq \emptyset$ since $G \in \mathcal{H}$.

Lemma 1.1.7. *[10, Lemma 2.4] Let $X \subseteq G$. Then $\langle X \rangle$ is the set of all finite products*

$$x_1 x_2 \cdots x_k,$$

where either x_i or x_i^{-1} is in X for all i.

Definition 1.1.8. *Let G be a group. The conjugacy class of $g \in G$ in G, denoted by g^G ($\mathbf{cl}(g)$ when convenient) is the set*

$$g^G = \mathbf{cl}(g) = \left\{ x \in G : x = hgh^{-1} \text{ for some } h \in G \right\}.$$

Remark 1.1.9. *Conjugacy classes of G partition G.*

Proposition 1.1.10. *[11, Proposition 12.19] A subgroup N of a group G is normal in G if and only if N is the union of conjugacy classes of G.*

Definition 1.1.11. *Let G be a group. The elements of G of the form $aba^{-1}b^{-1}$, denoted by $[a,b]$, are called commutators of G. Further, if X is the set of all commutators of G, then $\langle X \rangle$ is called the derived subgroup of G and is denoted by G'.*

Given that $H, K \leq G$, then $[H, K]$, called the commutator of H and K, is defined by

$$[H, K] = \langle [h, k] \,:\, h \in H, k \in K \rangle.$$

Note that $[G, G] = G'$.

Remark 1.1.12.

- *The notation $[*, *]$ will be used frequently for different objects. Depending on the object/-context, the meaning should be clear.*
- *Instead of writing $(G')'$, we write G'' or $G^{(2)}$. Moreover, $((G')')' = G''' = G^{(3)}$ and so on. Lastly, $G^{(n)}$ where $n \in \mathbb{N}$ is called the nth derived subgroup of G.*

Definition 1.1.13. *If $H \leq G$, we say H is characteristic in G, denoted by $H \operatorname{char} G$, if $\theta(H) = H$ for all $\theta \in \operatorname{Aut}(G)$.*

Remark 1.1.14. *Loosely speaking, if G is a group, then subgroups that can be described as "the something" are characteristic in G. However, it is essential for there to be no ambiguity with this description. For example, the set of all elements $g \in G$ such that $gx = xg$ for all $x \in G$, denoted by $\mathbf{Z}(G)$ and called* the *center of G, is a characteristic subgroup of G.* The *derived subgroup of G, G', is characteristic in G. However, given $H \leq G$,* the *normalizer of H in G (denoted by $\mathrm{N}_G(H)$), that is, the set of all $g \in G$ such that $H^g = \left\{ ghg^{-1} \,:\, h \in H \right\} = H$, is not characteristic in G. This is because the normalizer is dependent on the subgroup H. Similarly, the centralizer of an element $g \in G$, defined $\mathrm{C}_G(g) = \{x \in G \,:\, xg = gx\}$, or the centralizer of a subgroup H in G, defined $\mathrm{C}_G(H) = \{x \in G \,:\, xg = gx \text{ for all } g \in H\}$, are not characteristic in G. Finally, if $H \operatorname{char} G$, then $H \trianglelefteq G$.*

Lemma 1.1.15. *[10, Lemma 2.16] If $H \operatorname{char} N \trianglelefteq G$, then $H \trianglelefteq G$.*

Definition 1.1.16. *For $H \leq G$, the set containing elements gh, where $g \in G$ is fixed and $h \in H$, is called a coset of H in G, and is denoted by gH. Further, if $H \trianglelefteq G$, the set of cosets of H in G, denoted by G/H, is called the factor/quotient* group *of G by H. The operation of elements in G/H is defined:*

$$aHbH = abH \text{ for } aH, bH \in G/H.$$

Theorem 1.1.17. *[10, Theorem 3.10] If $H \trianglelefteq G$, then G/H is abelian if and only if $G' \subseteq H$.*

Theorem 1.1.18. *(First isomorphism theorem)[10, Theorem 3.3] Let G and H be groups such that $\phi : G \to H$ is a homomorphism. Then*

$$G/\ker\phi \cong \phi(G).$$

Remark 1.1.19. *Given the homomorphism ϕ in the theorem above. The normal subgroup $\ker\phi \trianglelefteq G$ is defined by $\ker\phi = \{g \in G : \phi(g) = 1\}$.*

Theorem 1.1.20. *(Correspondence)[10, Theorem 3.7] Suppose that $\phi : G \to H$ is a surjective homomorphism and $N = \ker\phi$. Given $\mathcal{S} = \{K \leq G : N \leq K\}$ and $\mathcal{T} = \{V : V \leq H\}$, then the mapping $\phi(\ )$ and $\phi^{-1}(\ )$ define bijections from $\mathcal{S}$ to $\mathcal{T}$ and $\mathcal{T}$ to $\mathcal{S}$, respectively. Moreover, these mapping respects containment, normality, indices, and factor groups.*

Theorem 1.1.21. *(Diamond)[10, Theorem 3.6] Let $N \trianglelefteq G$ and $H \leqslant G$. Then $H \cap N \trianglelefteq H$ and*

$$H/(H \cap N) \cong NH/N.$$

Theorem 1.1.22. *(Third isomorphism theorem)[10, Corollary 3.9] Let $N \subseteq M \trianglelefteq G$ where $N \trianglelefteq G$. Then*

$$(G/N)/(M/N) \cong G/M.$$

1.2 Solvable Groups

Definition 1.2.1. *For a group G, a subnormal series of G is a chain of subgroups*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G.$$

If $G_i \trianglelefteq G$ for all i, the series is called normal. Each G_{i+1}/G_i is called a factor of the series.

Definition 1.2.2. *If a group has a subnormal series with abelian factors, then it is called a solvable group.*

Example 1.2.3. *Consider the series*

$$S_3 > A_3 > 1.$$

Note that $|S_3/A_3| = 2$, thus $S_3/A_3 \cong C_2$ (see [3, Chapter 5] for a definition of A_n). Similarly $A_3/1 \cong C_3$. We have produced a subnormal series with abelian factors and so S_3 is solvable.

Remark 1.2.4. *We will denote cyclic groups of order n by C_n (see [10, page 16] for a definition of a cyclic group).*

Theorem 1.2.5. *[10, Theorem 8.3] If G is a group, then G is solvable if and only if there exists an integer $n \in \mathbb{Z}_+$ such that $G^{(n)} = 1$.*

If a group G is solvable, then $n \in \mathbb{Z}_+$ exists, such that $G^{(n)} = 1$. The smallest possible such integer is called the derived length of G, denoted by $\mathrm{dl}(G)$.

1.3 Nilpotent groups

If π is a set of primes, an integer is called a π-number if all of its prime factors are in π. A group (subgroup) whose order is a π-number is called a π-group (π-subgroup). If none of the prime factors within the group's (subgroup's) order are in π, we call the group (subgroup) a π'-group (π'-subgroup). Furthermore, if $H \leq G$ and H is a π-subgroup such that $\gcd(|H|, |G : H|) = 1$, then H is called a Hall π-subgroup of G (Hall subgroup if we need not specify the set of primes π).

Given $\pi = \{p\}$ for a prime p, we opt for the terms p-group (p-subgroup) and p'-group (p'-subgroup). Every group contains a p-subgroup (for $p \mid |G|$). A maximal p-subgroup of G is called a Sylow p-subgroup.

For reference, the following is a more formal definition of a p-group and Sylow p-subgroup.

Definition 1.3.1. *Let G be a group and p a prime. We call G a p-group if the group has an order which is of power p. Now if we let $|G| = p^k m$ where $p \nmid m$, then any subgroup of order p^k is called a Sylow p-subgroup. The set of all Sylow p-subgroups of G is denoted by $\mathrm{Syl}_p(G)$. It is well known that $n_p = \left|\mathrm{Syl}_p(G)\right| = |G : \mathrm{N}_G(P)|$ where $P \in \mathrm{Syl}_p(G)$ (see [10, Corolary 5.9]).*

Remark 1.3.2. *We note that if G is not finite, then G is defined to be a p-group if the order of every element (if it exists) is of prime power. When the group is finite, this definition is a characterization of the one we gave above.*

Theorem 1.3.3. *(The Sylow theorems) Let G be a group of order $p^k m$ where $p \nmid m$ and $k \in \mathbb{Z}_+$. The following then holds:*

1. *G has a subgroup of order p^k;*
2. *If $P, Q \in \mathrm{Syl}_p(G)$, then there exists $g \in G$ such that $P = Q^g$;*
3. *n_p (the number of Sylow p-subgroups) satisfies the following:*

$$n_p \mid m \text{ and } n_p \equiv 1 \mod p.$$

Definition 1.3.4. *A group G is said to be nilpotent if there exists a normal series*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

such that $G_{i+1}/G_i \subseteq \mathbf{Z}(G/G_i)$ for all i.

Example 1.3.5. *The following are some examples of nilpotent groups.*

- *From the definition, it is obvious that all abelian groups are nilpotent.*

- *Consider the quaternion group of order eight (see [10, Problems 1.9] for a definition)*

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}.$$

Note that $Z(Q_8) = \{1, -1\} = \langle -1 \rangle$ *and so we have the series*

$$1 \trianglelefteq \langle -1 \rangle \trianglelefteq Q_8.$$

Clearly, $\langle -1 \rangle / \{1\}$ *is contained* $Z(Q_8/\{1\})$ *since* $\langle -1 \rangle$ *is the centre of* Q_8*.*

We can show that

$$Q_8/\langle -1 \rangle = \{\langle -1 \rangle, i\langle -1 \rangle, j\langle -1 \rangle, k\langle -1 \rangle\} \cong V_4,$$

which is an abelian group. Thus

$$Q_8/\langle -1 \rangle = Z(Q_8/\langle -1 \rangle),$$

and we have a non-abelian nilpotent group.

- p*-groups are nilpotent groups [10, Corollary 8.14].*

Remark 1.3.6. *The Klein-4 group, denoted by* V_4*, is one of the two groups of order* 4*, the other being a cyclic group.*

Theorem 1.3.7. *[10, Theorem 8.19] Let* G *be a group, then the following are equivalent:*

1. G *is nilpotent;*
2. *If* $H < G$*, then* $H < \mathbf{N}_G(H)$*;*
3. *Every Sylow subgroup of* G *is normal;*
4. G *is a direct product of Sylow subgroups.*

1.4 Frobenius Groups

Definition 1.4.1. *Let* G *be a group,* $N \trianglelefteq G$ *and* $H \leq G$*. We say* G *is the semidirect product of* N *by* H*, which we denote by* $G = N \rtimes H$*, if*

$$G = NH \text{ and } N \cap H = 1.$$

Remark 1.4.2. *If, additionally,* $H \trianglelefteq G$*, then we call the semidirect product a direct product of* N *and* H*, denoted by* $G = N \times H$*.*

Definition 1.4.3. *Let G be a group and Ω a non-empty set. If for each $g \in G$ and $x \in \Omega$, we define an operation, denoted $g \cdot x$, such that for $g, h \in G$ and $x \in \Omega$*

1. $1 \cdot x = x$;
2. $(gh) \cdot x = g \cdot (h \cdot x)$;

we then call G a permutation group on Ω. The operation $\cdot$ is called the action of G on Ω.

Remark 1.4.4. *We say the action is faithful if $g = 1 \in G$ is the only element such that $g \cdot x = x$ for all $x \in \Omega$.*

If G is a permutation group on Ω. The orbit of $x \in \Omega$ is defined to be the set

$$\mathrm{Orb}(x) = \mathcal{O}_x = \{g \cdot x \,:\, g \in G\}.$$

The stabilizer of x in G is defined to be the subgroup of G given by

$$\mathrm{Stab}(x) = G_x = \{g \in G \,:\, g \cdot x = x\}.$$

If $\Omega = \mathcal{O}_x$, we call G a transitive permutation group or say that G acts transitively on Ω.

The following well-known theorem is of fundamental importance in finite group theory and we state it here for reference:

Theorem 1.4.5. *(Orbit-Stabilizer theorem) Let G act on Ω and $x \in \Omega$. Then the following holds* $|\mathcal{O}_x| = |G : G_x|$.

Definition 1.4.6. *A transitive permutation group G on a set Ω (with $|\Omega| > 1$) is called a Frobenius group if the following holds:*

1. $G_x \neq 1$ *for all* $x \in \Omega$;
2. $G_x \cap G_y = 1$ *for all* $x \neq y$ *in* Ω.

Remark 1.4.7. *By definition, Frobenius groups are non-trivial.*

If G is a Frobenius group on Ω and $x \in \Omega$, then G_x is called a Frobenius complement of G. The Frobenius kernel N of G, is a normal subgroup of G [6, Corollary 9.1.5] which contains all points which do not fix any other point in Ω together with the identity of G.

Lemma 1.4.8. *A Frobenius complement is a proper subgroup of a Frobenius group.*

Proof. Consider G to be a Frobenius group on Ω and $x, y \in \Omega$ where $x \neq y$. Now if $G_x = G$, since $G_x, G_y \neq 1$ by definition, we have that $G_x \cap G_y = G_y \neq 1$, a contradiction. Thus Frobenius complements are proper subgroups of G. □

Proposition 1.4.9. *[6, Proposition 9.1.3] Let G be a Frobenius group with kernel N and complement H, then $|N| = |G : H| > 1$.*

Theorem 1.4.10. *[6, Corollary 9.1.6] Let G be a Frobenius group with kernel N and complement H, then*

$$G = N \rtimes H.$$

Proposition 1.4.11. *[6, Proposition 9.1.8] Let G be a Frobenius group with kernel N and complement H, then*

$$|H| \mid (|N| - 1).$$

Proposition 1.4.12. *[6, Corollary 9.1.2.] If G is a Frobenius group with a Frobenius complement H, then*

$$\mathbf{N}_G(H) = H.$$

Lemma 1.4.13. *Frobenius groups are not nilpotent.*

Proof. Suppose G is a Frobenius group on Ω. If $H = G_x$ for some $x \in \Omega$, then $H < G$ by *Lemma 1.4.8*. For a contradiction, let G be nilpotent. Thus $\mathbf{N}_G(H) < H$ by Theorem 1.3.7 (2). But $\mathbf{N}_G(H) = H$, since G is Frobenius, a contradiction. □

Theorem 1.4.14. *[6, Proposition 9.1.1] A group G is Frobenius if and only if it has a non-trivial proper subgroup H such that*

$$H \cap H^g = 1 \text{ for all } g \in G - H.$$

Remark 1.4.15. *The H in the theorem above is a complement of the Frobenius group G.*

Theorem 1.4.16. *[8, Problem 7.1] Let $G = N \rtimes H$. The following are equivalent:*

1. *G is a Frobenius group with a complement H and kernel N;*
2. *$\mathbf{C}_G(n) \subseteq N$ for all $1 \neq n \in N$;*
3. *$\mathbf{C}_H(n) = 1$ for all $1 \neq n \in N$.*

Proposition 1.4.17. *If G is a Frobenius group, then $\mathbf{Z}(G) = 1$.*

Proof. Let G be Frobenius with kernel N and complement H. For any $1 \neq g \in N$ we have

$$\mathbf{Z}(G) \subseteq \mathbf{C}_G(g) \subseteq N.$$

For a contradiction, choose $1 \neq x \in Z(G)$. Note that $1 \neq g \notin H$, thus we must have

$$H \cap H^g = H \cap H = H \neq 1,$$

this contradicts *Theorem 1.4.14*. Thus $g = 1$ and $\mathbf{Z}(G) = 1$. □

We offer some examples of Frobenius groups.

Example 1.4.18.

- $S_3 = \{(1), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$ *(see [3, Chapter 5] for a definition of the "cycle notation" used here) is an example of a Frobenius group. The stabilizers are given by*

$$G_1 = \{(1), (2\,3)\}, G_2 = \{(1), (1\,3)\} \text{ and } G_3 = \{(1), (1, 2)\}.$$

We easily see that the two properties in our definition of a Frobenius group are satisfied.

Moreover, since Frobenius groups have trivial centers and all non-trivial groups with order less than six are abelian, S_3 *is the smallest example of a Frobenius group.*

- *We now show that* A_4 *is a Frobenius group. The conjugacy classes of* A_4 *(see [11, Examples 12.8]) are given by:*

$$1, (1\,2)(3\,4)^{A_4}, (1\,2\,3)^{A_4} \text{ and } (1\,3\,2)^{A_4}.$$

The Klein 4*-group can be written as* $V_4 = \{1, a, b, c\}$ *where* $a = (1\,2)(3\,4)$, $b = (1\,3)(2\,4)$ *and* $c = (1\,4)(2\,3)$. *Note that* $V_4 \trianglelefteq A_4$ *since it is the union of the conjugacy classes* 1 *and* $(1\,2)(3\,4)^{A_4}$. *Now set* $H = \langle(1, 2, 3)\rangle = \{1, (1\,2\,3), (1\,3, 2)\}$. *We see that* H *is not normal in* A_4 *since it is not the union of conjugacy classes of* A_4.

Now $V_4 \cap H = 1$, *and*

$$|V_4H| = \frac{|V_4|\,|H|}{|V_4 \cap H|} = \frac{4 \cdot 3}{1} = 12 = |A_4|\,.$$

Since $V_4H \subseteq A_4$, *it follows* $A_4 = V_4H$. *Thus we have shown that,*

$$A_4 = V_4 \rtimes H.$$

Now by the Third Sylow theorem *(*H *is a Sylow 3-subgroup),* $n_3 \,|\, 4$ *and* $n_3 \equiv 1 \mod 3$. *Thus* $n_3 \in \{1, 4\}$. *Now* $n_3 \neq 1$ *since* H *is not normal in* A_4. *Thus* $n_3 = 4$ *and* $|A_4 : \mathbf{N}_{A_4}(H)| = 4$, *or* $\mathbf{N}_{A_4}(H) = H$ *since* $H \subseteq \mathbf{N}_{A_4}(H)$.

That is, $H^g = H$ *if and only if* $g \in H$. *Hence for any* $g \in G - H$, *we must have*

$$H^g \cap H = 1,$$

since the only subgroups of their intersection must be either of order one or three; but their intersection can not be of order three since $H^g \neq H$. *By* Theorem 1.4.14, A_4 *is Frobenius.*

Proposition 1.4.19. *[5, Proposition 3.7] Let* F *be a field. Any finite subgroup of the group* $F^\times$ *(multiplicative group* $F - \{0\}$*) is cyclic*

Example 1.4.20. *Consider a subgroup* G *of* $\mathrm{Sym}(F)$, *the symmetric group on a field* F *(the field being of order* $p^n > 2$ *where* p *is prime and* n *a positive integer). We define* G *to contain mappings*

$$T_{a,b} : x \mapsto ax + b,$$

where $a, b \in F$ and $a \neq 0$.

Now let

$$N = \left\{T_{1,b} : b \in F\right\} \text{ and } H = \left\{T_{a,0} \in G : 0 \neq a \in F\right\}.$$

It follows that N and H are abelian (F is abelian) subgroups of order p^n and $p^n - 1$, respectively. We now state some easily confirmed facts about elements in G. For any $T_{a,b}, T_{c,d} \in G$, the following hold:

- $T_{a,b}T_{c,d} = T_{ac,bc+d}$;
- $T_{a,b}^{-1} = T_{a^{-1},-a^{-1}b}$.

Since, $T_{a,b} = T_{1,b}T_{a,0}$, we see that $G = NH$. Further, $N \trianglelefteq G$ since

$$T_{a,b}T_{1,d}T_{a^{-1},-a^{-1}b} = T_{1,a^{-1}b+da^{-1}-a^{-1}b} = T_{1,a^{-1}d} \in N.$$

Moreover, $N \cap H = 1$; that is,

$$G = N \rtimes H.$$

If we choose $T_{1,0} \neq T_{1,d} \in N$ and $T_{a,b} \in \mathbf{C}_G(T_{1,d})$, then

$$T_{a,b}T_{1,d}T_{a^{-1},-a^{-1}b} = T_{1,a^{-1}d} = T_{1,d},$$

which implies $a^{-1}d = d$ and so $a = 1$. Thus $T_{a,b} \in N$ and $\mathbf{C}_G(T_{1,d}) \subseteq N$. By Theorem 1.4.16, *$G$ is Frobenius group with kernel N and complement H.*

Consider a mapping $\phi : H \to F^\times$ defined by

$$T_{a,0} \mapsto a.$$

This mapping is clearly a homomorphism and $T_{a,0} \in \ker\phi$ if and only if $T_{a,0} = T_{1,0}$. Thus by the First isomorphism theorem,

$$H \cong H/\ker\phi \cong \phi(H) \leq F^\times,$$

and so, by Proposition 1.4.19, *H is cyclic group of order $p^n - 1$.*

This group G is usually denoted by $\mathrm{AGL}_1(p^n)$ and called the one-dimensional affine group over F. In conclusion, the one-dimensional affine group over a finite field F (of order p^n) is a Frobenius group. This group is a semidirect product of the abelian subgroup N, of order p^n, by a cyclic group H, of order $p^n - 1$.

Remark 1.4.21. *For a Frobenius group with complement H and kernel N, $N \subseteq G'$. Thus if the complement H is abelian, then $G/N \cong H$ is abelian and by* Theorem 1.1.17 *$N = G'$. In particular, if we look at* Example 1.4.20, *where $\mathrm{AGL}_1(p^n) = N \rtimes H$, we have $N = G'$. That is, the derived subgroup is just the subgroup of all* translations *$T_{1,b}$.*

2 Character Theory

Give me a fruitful error anytime, full of seeds, bursting with its own corrections. You can keep your sterile truth for yourself.

– Vilfredo Pareto

In this chapter, we introduce modules, representations and characters. We give an overview of some essential character theory and prove those results that do not detract from the dissertation.

As alluded to in the introduction of this dissertation, a representation contains information on the structure of a group. However, the problem with representations is that they contain "too much" information. For instance, *similarity* of representations (and, in turn, an *isomorphism* between modules) encapsulates the idea of "sameness" between representations (and modules). Some of the data produced by representations distinguish between representations within the same similarity class and thus can be ignored [8, page 14]. Character theory aims to throw out most of this data and only keep enough that may still influence the structure of a group.

2.1 Modules and Representations

Definition 2.1.1. *Let F be a field. An F-algebra A is a vector space over F (F-space) which is also a ring with unity 1, such that the following holds:*

$$\lambda(xy) = (\lambda x)y = x(\lambda y) \text{ for all } x, y \in A \text{ and } \lambda \in F.$$

Example 2.1.2. *Let F be a field, G a group, and let $F[G]$ denote a set of formal sums*

$$\sum_{g \in G} \lambda_g g.$$

Given $u = \sum_{g\in G} \lambda_g g$ and $v = \sum_{h \in G} \mu_h h$, we define addition as:

$$u + v = \sum_{g \in G} (\lambda_g + \mu_g) g,$$

scalar multiplication as:

$$\lambda u = \sum_{g \in G} (\lambda \lambda_g) g \text{ for } \lambda \in F,$$

and multiplication as:

$$uv = \sum_{g,h \in G} (\lambda_g \mu_h)(gh).$$

$F[G]$ with the operations above can be easily shown to be an F-algebra. This algebra is called a group algebra.

Definition 2.1.3. *Let V be a finite dimensional vector space over F and A an F-algebra. We define a multiplication vx for $x \in A$ and $v \in V$. If the following hold:*

1. $vx \in V$,
2. $v(x + y) = vx + vy$ ($y \in A$),
3. $(v + u)x = vx + ux$ ($u \in V$),
4. $v(xy) = (vx)y$,
5. $\lambda(vx) = (\lambda v)x = v(\lambda x)$ ($\lambda \in F$),
6. $v1 = v$ ($1 \in A$),

then we call V an A-module.

Remark 2.1.4. *If A is an F-algebra and V is an A-module, a subalgebra of A is a subset which is also an F-algebra under the same operations as that in A. A submodule of V is defined similarly.*

Lemma 2.1.5. *[8, page 3] Let V be an A-module and $x \in A$. The mapping $x_V : V \to V$, defined by*

$$x_V(v) = vx,$$

is an endomorphism (linear transformation from V to V).

Remark 2.1.6. *Since x_V is a an endomorphism, we can consider its matrix (see [1, Definition 3.32]), which we will denote $[x]_{\mathcal{B}}$, where $\mathcal{B}$ is a basis of V.*

Definition 2.1.7. *Let V and W be A-modules. A linear transformation $\phi : V \to W$ such that*

$$\phi(va) = \phi(v)a \text{ for all } a \in A \text{ and } v \in V,$$

is called an A-homomorphism. If ϕ is a bijection, then ϕ is called an A-isomorphism and we say V and W are isomorphic and denote by $V \cong W$.

Definition 2.1.8. *Let V be an A-module. We say V is irreducible if its only submodules are $\{0\}$ and V. Otherwise, V is said to be reducible.*

Definition 2.1.9. *Let V be an A-module. If for every submodule U of V there exists a submodule W such that*

$$V = U \oplus W,$$

then we say that V is completely reducible.

Remark 2.1.10. *Here $V = U \oplus W$ is taken to mean V is a* direct sum *of U and V (see [1, Definition 1.40]).*

Theorem 2.1.11. *(*Maschke*)[8, Theorem 1.9] Let V be an $F[G]$-module where F is a field whose characteristic does not divide $|G|$, then V is completely reducible.*

Theorem 2.1.12. *[8, Theorem 1.10 and Lemma 1.11] Let V be an A-module. V is completely irreducible if and only if it is a direct sum of irreducible submodules.*

The two theorems above are, to a large extent, why there is a deep interest in irreducible modules (and, by extension, irreducible representations, and irreducible characters). We are able to reduce the study of $F[G]$-modules (F being an appropriate field, of course) to that of irreducible $F[G]$-modules.

Definition 2.1.13. *Let A be an F-algebra. A representation of A is a homomorphism $\mathfrak{X} : A \to M_n(F)$. The integer n is called the degree of the representation. If $\mathfrak{Y} : A \to M_n(F)$ is a representation of A, we say $\mathfrak{X}$ is similar to $\mathfrak{Y}$ if there exists a non-singular matrix P such that*

$$\mathfrak{X}(a) = P^{-1}\mathfrak{Y}(a)P \text{ for all } a \in A.$$

Let V be an A-module with a basis $\mathcal{B}$. If $x \in A$, then the mapping $\mathfrak{X} : x \mapsto [x]_{\mathcal{B}}$ defines a representation (see [6, Theorem 4.12]). Further, if $\mathcal{B}'$ is another basis of V, then the representation $x \mapsto [x]_{\mathcal{B}'}$ is similar to $\mathfrak{X}$ (see [6, Theorem 4.12]).

Conversely, suppose $\mathfrak{X} : A \to M_n(F)$ is a representation and V is the n-dimensional F-space F^n. Define $vx = v\mathfrak{X}(x)$ for $v \in V$ and $x \in A$. Then V defines an A-module.

If we start with a representation $\mathfrak{X}$ and construct a module V as above; we can construct the original representation $\mathfrak{X}$ from the module V with the choice of an appropriate basis.

Finally, let V and W be A-modules with basis $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then $V \cong W$ if and only if the representations

$$x \mapsto [x]_{\mathcal{B}} \text{ and } x \mapsto [x]_{\mathcal{B}'}$$

are similar [11, Theorem 7.6]. This offers some insight as to why representations and modules are considered "one and the same". Ideas in character theory can be presented from a module or representation-theoretic point of view.

Definition 2.1.14. *Let A be an F-algebra and let $\mathfrak{X} : A \to M_n(F)$ be a representation. We say $\mathfrak{X}$ is irreducible(reducible) if its corresponding A-module is irreducible(reducible).*

Let G be a group and F a field. Consider a representation $\mathfrak{X}$ (of degree n) of the group algebra $F[G]$. For each $g \in G \subseteq F[G]$, $\mathfrak{X}(g)$ is non-singular. Thus the restriction of $\mathfrak{X}$ to G is a group homomorphism from G into $\mathrm{GL}(n, F)$. We therefore define the following:

Definition 2.1.15. *Let G be a group and F a field. A representation of G is a homomorphism $\mathfrak{X} : G \to \mathrm{GL}(n, F)$. The integer n is called the degree of the representation.*

Now if $\mathfrak{X}_0 : G \to \mathrm{GL}(n, F)$ is a representation of G, then we can obtain a corresponding representation of $F[G]$ by defining

$$\mathfrak{X}\left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} \lambda_g \mathfrak{X}_0(g).$$

The adjectives "similar", "reducible" and "irreducible" are used on the representation of G as if they were used on the corresponding representation.

Theorem 2.1.16. *Let $N \trianglelefteq G$. Suppose that $\mathcal{N}$ is a set of representations of G/N and let $\mathcal{M}$ be a set of representations of G such that the representation $\mathfrak{X}$ of G is in $\mathcal{M}$ if and only if $N \subseteq \ker \mathfrak{X}$. Then the mapping $\phi : \overline{\mathfrak{X}} \mapsto \mathfrak{X}$, where $\overline{\mathfrak{X}} \in \mathcal{N}$ and $\mathfrak{X}$ is defined by*

$$\mathfrak{X} : g \mapsto \overline{\mathfrak{X}}(gN),$$

defines a bijective correspondence from $\mathcal{N}$ to $\mathcal{M}$.

Proof. Define a mapping $\phi : \overline{\mathfrak{X}} \mapsto \mathfrak{X}$ where $\mathfrak{X}$ is given by

$$\mathfrak{X} : g \mapsto \overline{\mathfrak{X}}(gN)$$

and $\overline{\mathfrak{X}}$ is in $\mathcal{N}$. Clearly, $\mathfrak{X}$ is a representation of G. Further, if $g \in N$, then $\mathfrak{X}(g) = \overline{\mathfrak{X}}(gN) = \overline{\mathfrak{X}}(1N) = \mathfrak{X}(1) = I$. That is, $N \subseteq \ker \mathfrak{X}$. Thus ϕ is indeed a mapping from $\mathcal{N}$ to $\mathcal{M}$.

Surjectivity: Consider $\mathfrak{X} \in \mathcal{M}$. Choose $\overline{\mathfrak{X}}$, defined by

$$\overline{\mathfrak{X}} : gN \mapsto \mathfrak{X}(g).$$

Now since $N \subseteq \ker \mathfrak{X}$, $\overline{\mathfrak{X}}$ is well-defined. Further, $\overline{\mathfrak{X}}$ is easily shown to be a representation of G/N (since $\mathfrak{X}$ is a representation of G); that is, $\overline{\mathfrak{X}} \in \mathcal{N}$. Finally, if $g \in G$, then

$$\phi(\overline{\mathfrak{X}})(g) = \overline{\mathfrak{X}}(gN) = \mathfrak{X}(g) \text{ by definition.}$$

Thus $\phi(\overline{\mathfrak{X}}) = \mathfrak{X}$.

Injectivity: If $\overline{\mathfrak{X}}, \overline{\mathfrak{N}} \in \mathcal{N}$, suppose $\mathfrak{X} = \phi(\overline{\mathfrak{X}}) = \phi(\overline{\mathfrak{N}}) = \mathfrak{N}$. Thus $\overline{\mathfrak{X}}(gN) = \mathfrak{X}(g) = \mathfrak{N}(g) = \overline{\mathfrak{N}}(g)$ for all $g \in G$, and $\overline{\mathfrak{X}} = \overline{\mathfrak{N}}$. Hence ϕ is a bijection and the proof is complete. □

Remark 2.1.17. *The representation $\overline{\mathfrak{X}}$ in $\mathcal{N}$ is irreducible if and only if its corresponding representation $\mathfrak{X}$ is irreducible (see [11, page 170]).*

2.2 On characters of a group

Definition 2.2.1. *Let $\mathfrak{X} : G \to GL(n, \mathbb{C})$ be a representation of G. Then a character χ of G afforded by $\mathfrak{X}$ is a function from G to $\mathbb{C}$ defined by*

$$\chi(g) = \operatorname{tr} \mathfrak{X}(g) \text{ for all } g \in G.$$

Further, χ is said to be irreducible (reducible) if and only if $\mathfrak{X}$ is irreducible (reducible). We denote the set of irreducible characters of G as $\operatorname{Irr}(G)$.

Remark 2.2.2.

1. *Alternatively, we can define characters as follows. Let V be a $\mathbb{C}[G]$-module with a basis $\mathcal{B}$. A character χ of G afforded by V is a mapping defined by*

$$\chi(g) = \operatorname{tr}\,[g]_{\mathcal{B}} \text{ for all } g \in G.$$

2. *The value $\chi(1)$, is called the degree of χ. If $\chi(1) = 1$, then χ is called a linear character; otherwise χ is called a non-linear character. In particular, linear characters form homomorphisms from G to $\mathbb{C}^{\times}$.*

3. *For each group G, the mapping $\mathfrak{X} : G \to \mathrm{GL}(1, F)$, defined by $g \mapsto (1)$, defines a representation of G. The character afforded by $\mathfrak{X}$, denoted by 1_G, is called the principal character of G.*

4. *A class function on a group G is a function from G to $\mathbb{C}$ which is constant on the conjugacy classes of G. Characters of a group are class functions (see [8, Lemma 2.3]).*

5. *Every class function of G can be expressed as a linear combination of irreducible characters. Further, a linear combination of irreducible characters of G is a character of G if and only if its coefficients are positive integers (see [8, Theorem 2.8]).*

Theorem 2.2.3. *[8, Theorem 3.11] Let G be a group and $\chi \in \operatorname{Irr}(G)$, then*

$$\chi(1) \mid |G| \,.$$

Definition 2.2.4. *Let χ be a character of G. The kernel of χ, denoted by $\ker \chi$, is a normal subgroup of G (see [8, Lemma 2.19]), defined by*

$$\ker \chi = \{g \in G : \chi(g) = \chi(1)\}\,.$$

Further, we define $\mathbf{Z}(\chi)$ to be the set

$$\mathbf{Z}(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}\,.$$

Remark 2.2.5. *Given $\chi \in \mathrm{Irr}(G)$, it can be shown that $\chi(1) \mid |G : \mathbf{Z}(\chi)|$ (see [8, Theorem 3.12]).*

Lemma 2.2.6. *[8, Corollary 2.7] A group G has k conjugacy classes if and only if $|\mathrm{Irr}(G)| = k$. Furthermore, given $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$, it follows that*

$$|G| = \sum_{i=1}^{k} \chi_i(1)^2.$$

Corollary 2.2.7. *A group G is abelian if and only if all of its irreducible characters are linear.*

Proof. ($\Longrightarrow$) Suppose that k is the number of conjugacy classes of G and let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. If G is abelian then $g^a = g$ for all $a, g \in G$; that is, $g^G = \{g\}$. Thus $|G| = k$. Therefore $k = \sum_{i=1}^{k} \chi_i(1)^2$. But $\chi_i(1) \geq 1$ for all i, thus $\chi_i(1) = 1$ for all i.

($\Longleftarrow$) Conversely, suppose that $\chi_i(1) = 1$ for all i. For a contradiction, suppose G is not abelian. Thus there exists a conjugacy class of G, say g^G for $g \in G$, of size greater than 1. But $k = \sum_{i=1}^{k} \chi_i(1)^2 = |G|$. Further, G must have k conjugacy classes. Thus

$$|G| = k < \overbrace{\sum_{h \in G - \{g\}} \left|h^G\right|}^{k-1 \text{ terms}} + \left|g^G\right|,$$

since $\left|g^G\right| > 1$, a contradiction. Thus each conjugacy class of G must be of size 1 and G is abelian. □

If $N \trianglelefteq G$, then *Theorem 2.1.16* can be given in terms of characters and so we get the following:

Lemma 2.2.8. *Let $N \trianglelefteq G$.*

1. *If χ is a character of G with $N \subseteq \ker \chi$, then χ is constant on the cosets of N in G and the function $\overline{\chi}$, defined by*

   $$\overline{\chi}(gN) = \chi(g) \text{ for all } gN \in G/N,$$

   is a character of G/N.

2. *If $\overline{\chi}$ is a character of G/N, then the function χ defined by*

   $$\chi(g) = \overline{\chi}(gN) \text{ for all } g \in G,$$

   is a character of G.

3. *In both (1) and (2), $\chi \in \mathrm{Irr}(G)$ if and only if $\overline{\chi} \in \mathrm{Irr}(G/N)$.*

Remark 2.2.9. *We often do not distinguish between the character χ of G where $N \subseteq \ker \chi$ and the corresponding character $\overline{\chi}$.*

For each group G there is an associated "invertible matrix", called the character table of G. Loosely speaking, the rows of this matrix correspond to irreducible characters of G and the columns correspond to the conjugacy class representatives of G. The following example is used to illustrate the concept expressed in *Lemma 2.2.8* through the character table of S_4.

Example 2.2.10. *Consider the character table of S_4 given below.*

Table 2.1: Character table of S_4 (see [11, Chapter 18 (18.1)]):

	(1)	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
1_{S_4}	1	1	1	1	1
λ	1	-1	1	1	-1
χ_1	2	0	-1	2	0
χ_2	3	1	0	-1	-1
χ_3	3	-1	0	-1	1

Let $N = V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ which is a normal subgroup of S_4. It is easily shown that $G/N \cong S_3$. Note that all of the elements of N, excluding (1), belong in the conjugacy class $(1\,2)(3\,4)^{S_4}$ and the column corresponding to $(1\,2)(3\,4)$ evaluate to the degree (for the first three characters). Thus N lies in the kernel of $1_{S_4}, \lambda$ and χ_1, respectively. So we expect the first three characters in Table 2.1 *to correspond to the character of S_3.*

The following table confirms this to be the case.

Table 2.2: Character table of S_3 (see [11, Examples 16.3]):

	(1)	(1 2)	(1 2 3)
1_{S_3}	1	1	1
λ	1	-1	1
χ	2	0	-1

Proposition 2.2.11. *[8, Corollary 2.3] If G is a group with the derived subgroup G', then there are $|G : G'|$ linear characters of G.*

Let ϕ, ψ be class functions of G. We define an inner product (see [1, Definition 6.3]) of the class functions by

$$[\phi, \psi] = \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

If ψ is a character of G and χ an irreducible character of G, we say χ is an irreducible constituent of ψ, if $[\psi, \chi] \neq 0$.

Proposition 2.2.12. *[8, Corollary 2.17] If χ and ψ are characters of G, then $[\chi, \psi]$ is a non-negative integer. Moreover, $\chi \in \mathrm{Irr}(G)$ if and only if $[\chi, \chi] = 1$.*

2.3 Characters of direct products

For characters χ and ψ of G, we define $\chi\psi$ to be the function from G to $\mathbb{C}$ given by

$$\chi\psi : g \mapsto \chi(g)\psi(g) \text{ for all } g \in G.$$

Proposition 2.3.1. *Let χ be a character of G and λ a linear character of G. Then, $\lambda\chi$ is a character of G. Moreover, $\lambda\chi \in \mathrm{Irr}(G)$ if and only if $\chi \in \mathrm{Irr}(G)$.*

Proof. Let χ be afforded by $\mathfrak{X}$. Define $\lambda\mathfrak{X} : G \to \mathrm{GL}(n, F)$ by

$$\lambda\mathfrak{X} : g \mapsto \lambda(g)\mathfrak{X}(g).$$

Since linear characters are homomorphisms and $\mathfrak{X}$ is a homomorphism, $\lambda\mathfrak{X}$ is a homomorphism. Moreover, $\mathrm{tr}\,\lambda(g)\mathfrak{X}(g) = \lambda(g)\,\mathrm{tr}\,\mathfrak{X}(g) = \lambda(g)\chi(g)$. Thus $\lambda\chi$ is a character of G afforded by the representation $\lambda\mathfrak{X}$. Now

$$\begin{aligned}
[\lambda\chi, \lambda\chi] &= \frac{1}{|G|}\sum_{g\in G} \lambda(g)\chi(g)\overline{\lambda(g)\chi(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\chi(g)}\lambda(g)\overline{\lambda(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\chi(g)} \text{ (note } \lambda(g) \text{ is a root of unity)} \\
&= [\chi, \chi].
\end{aligned}$$

Thus, by *Proposition 2.2.12*, $\lambda\chi \in \mathrm{Irr}(G)$ if and only if $\chi \in \mathrm{Irr}(G)$.

□

Proposition 2.3.2. *[8, Corollary 4.2] Let χ and ψ be characters of G, then $\chi\psi$ is a character of G.*

Definition 2.3.3. *Let $G = H \times K$ and χ, ψ be characters of H and K, respectively. The direct product of χ and ψ, denoted by $\chi \times \psi$, is defined by*

$$(\chi \times \psi)(hk) = \chi(h)\psi(k),$$

where $hk \in G$.

Lemma 2.3.4. *Let $G = H \times K$ and χ, ψ be characters of H and K, respectively. Then $\chi \times \psi$ is a character of G.*

Proof. Now $G/H \cong K$ and $G/K \cong H$, thus χ and ψ are characters of G (see *Remark 2.2.9*). Thus, by *Proposition 2.3.2*, $\chi \times \psi$ is a character of G. □

Theorem 2.3.5. *Let $G = H \times K$. Then the characters of the form $\chi \times \psi$ for $\chi \in \mathrm{Irr}(H)$ and $\psi \in \mathrm{Irr}(K)$ are exactly the irreducible characters of G.*

Proof. Let ϕ, ϕ_1 be distinct in $\mathrm{Irr}(H)$ and ψ, ψ_1 be distinct in $\mathrm{Irr}(K)$. Further, let $\chi = \phi \times \psi$ and $\chi_1 = \phi_1 \times \psi_1$. Then

$$\begin{aligned}[\chi, \chi_1] &= \frac{1}{|H|\,|K|} \sum_{g \in G} \phi(g)\psi(g)\overline{\phi_1(g)\psi_1(g)} \\ &= \frac{1}{|H|} \sum_{g \in G} \phi(g)\overline{\phi_1(g)} \frac{1}{|K|} \sum_{g \in G} \psi(g)\overline{\psi_1(g)} \\ &= [\phi, \phi_1]\,[\psi, \psi_1]\,.\end{aligned}$$

Similarly, $[\chi, \chi] = [\phi, \phi]\,[\phi_1, \phi_1]$ and $[\chi_1, \chi_1] = [\psi, \psi]\,[\psi_1, \psi_1]$. Thus, by *Proposition 2.2.12*, we see that $\chi \times \psi$ for $\chi \in \mathrm{Irr}(H)$ and $\psi \in \mathrm{Irr}(K)$ form a set of distinct irreducible characters of G. Further,

$$\sum_{\chi \in \mathrm{Irr}(H), \psi \in \mathrm{Irr}(K)} (\chi \times \psi)(1)^2 = \sum_{\chi, \psi} \chi(1)^2 \psi(1)^2 = \sum_{\chi} \chi(1)^2 \sum_{\psi} \psi(1)^2 = |H|\,|K| = |G|\,.$$

Thus $\{\chi \times \psi \ : \ \text{for } \chi \in \mathrm{Irr}(H) \text{ and } \psi \in \mathrm{Irr}(K)\}$ forms the set of all irreducible characters of G □

2.4 Normal subgroups

In this section, we introduce the ideas of induction and restriction of characters. We present results showing how we can obtain more information on the restricted character when the subgroup is normal. The situation is more complicated when restricting to a subgroup which is not normal. Most, if not all, of these results are well known in literature and are readily proved in [8]. A select few are proven here.

2.4.1 Restriction and induction

Let $H \leq G$ and suppose that χ is a character of G. The restriction of χ to H (χ is evaluated only on H), denoted by χ_H, is a character of H. Note that if χ is afforded by $\mathfrak{X}$, then $\mathfrak{X}$ restricted to H, is a homomorphism of H; this homomorphism affords χ_H.

Definition 2.4.1. *Let $H \leq G$ and let ϕ be a class function of H. We define ϕ^G, called the induced class function on G by ϕ, by*

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^{\circ}(g^{-1}xg),$$

where $\phi^\circ(h) = \phi(h)$ *for all* $h \in H$ *and* $\phi^\circ(x) = 0$ *for all* $x \in G - H$.

Remark 2.4.2. *Note that* $\phi^G(1) = |G : H|\,\phi(1)$.

Theorem 2.4.3. *(Frobenius reciprocity)[8, Lemma 5.2] Let* $H \leq G$ *and let* ϕ *be a class function on* H *and* θ *a class function on* G*, then*

$$\left[\phi^G, \theta\right] = [\phi, \theta_H]\,.$$

Proof. By definition,

$$\begin{aligned}\left[\phi^G, \theta\right] &= \frac{1}{|G|}\sum_{g\in G}\phi^G(g)\overline{\theta(g)}\\ &= \frac{1}{|G|}\sum_{g\in G}\left(\frac{1}{|H|}\sum_{x\in G}\phi^\circ(g^x)\right)\overline{\theta(g)}\\ &= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\phi^\circ(g^x)\overline{\theta(g)}\end{aligned}$$

But since ϕ° and θ are class functions evaluating on the same conjugacy class g^G for each $x \in G$, while g varies in G, we can rewrite this equation as:

$$\left[\phi^G, \theta\right] = \frac{1}{|G|}\frac{1}{|H|}\sum_{y\in G}\sum_{x\in G}\phi^\circ(y)\overline{\theta(y)}.$$

But

$$\sum_{x\in G}\phi^\circ(y)\overline{\theta(y)} = |G|\,\phi^\circ(y)\overline{\theta(y)},$$

and so

$$\begin{aligned}\left[\phi^G, \theta\right] &= \frac{1}{|H|}\sum_{y\in G}\phi^\circ(y)\overline{\theta(y)}\\ &= \frac{1}{|H|}\sum_{y\in G-H}\phi^\circ(y)\overline{\theta(y)} + \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\theta(y)} \text{ (recall that } \phi^\circ \text{ evaluates to 0 on } G-H)\\ &= 0 + \frac{1}{|H|}\sum_{y\in H}\phi(y)\overline{\theta(y)} = [\phi, \theta_H]\,.\end{aligned}$$

□

Corollary 2.4.4. *Let* $H \leq G$ *and let* ϕ *be a character of* H*, then* ϕ^G *is a character of* G.

2.4.2 Clifford theory

Definition 2.4.5. *Let $N \trianglelefteq G$ and suppose ψ is a character of N, then the conjugate of ψ in G by $g \in G$, denoted by ψ^g, is defined by*

$$\psi^g(x) = \psi(x^g) \text{ for } x \in N.$$

The following properties follow easily.

Theorem 2.4.6. *[8, Lemma 6.1] Let $N \trianglelefteq G$ and suppose ϕ, ψ are characters of N. If $a, b \in G$, then*

1. *ϕ^a is a character of N;*
2. *$\phi^{ab} = (\phi^a)^b$;*
3. *$[\phi^a, \psi^a] = [\phi, \psi]$;*
4. *$[\chi_N, \psi^a] = [\chi_N, \psi]$ for a character χ of G.*

From Theorem 2.4.6 (3), we see that if $\psi \in \mathrm{Irr}(N)$, then $\psi^a \in Irr(N)$. Furthermore, we note that the degrees of ψ and ψ^a are equal.

Theorem 2.4.7. *(Clifford)[8, Theorem 6.2] Let $N \trianglelefteq G$, $\chi \in \mathrm{Irr}(G)$ and ψ a constituent of χ_N such that $\psi = \psi_1, \psi_2, \dots, \psi_n$ are all distinct conjugates of ψ in G. Then,*

$$\chi_N = c\sum_{i=1}^{n} \psi_i$$

where $c = [\chi_N, \psi]$.

Proof. We consider $(\psi^G)_N$. Now for $a \in N$, we have

$$\psi^G(a) = \frac{1}{|N|}\sum_{x \in G} \psi^\circ(a^x) = \frac{1}{|N|}\sum_{x \in G} \psi(a^x) = \frac{1}{|N|}\sum_{x \in G} \psi^x(a)$$

since $N \trianglelefteq G$. Thus $|N|\,(\psi^G)_N = \sum_{x \in G} \psi^x$. Thus if $\phi \in \mathrm{Irr}(N)$ is such that ϕ is not conjugate to ψ, then $[(\psi^G)_N, \phi] = 0$. But by assumption and $[\chi_N, \psi] = [\chi, \psi^G] \neq 0$. It follows, $[\chi_N, \phi] = 0$; that is, the only irreducible constituents of χ_N are $\psi = \psi_1, \psi_2, \dots, \psi_n$. Further, if $c = [\chi_N, \psi]$, then by *Theorem 2.4.6 (4)* we have

$$\chi_N = c\sum \psi_i.$$

□

Definition 2.4.8. *Let $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$. The inertia group of ψ in G is defined by*

$$\mathrm{I}_G(\psi) = \{g \in G : \psi^g = \psi\}.$$

Note that $N \subseteq \mathrm{I}_G(\psi)$ since $\psi^n = \psi$ for $n \in N$ (ψ is a class function).

Remark 2.4.9. *Let* $\Omega = \mathrm{Irr}(N)$ *and* G *act on elements of* Ω *by conjugation (see* Definition 2.4.5*). By* Theorem 2.4.6 (2) *and noting that the conjugate of any* $\phi \in \mathrm{Irr}(N)$ *by* $1 \in G$ *is equal to* ϕ*, we see that we have indeed defined an action. Further, by definition of the* inertia group*, it follows for* $\psi \in \Omega$*, that* $G_\psi = \mathbf{I}_G(\psi)$ *and so the inertia group is a subgroup of* G*. Moreover, the* Orbit-Stabilizer *theorem states that*

$$\left|\mathcal{O}_\psi\right| = \left|G : \mathbf{I}_G(\psi)\right|,$$

where $\mathcal{O}_\psi$ *is the set of* distinct *conjugates of* ψ*. Thus if* n *and* c *are given as in* Theorem 2.4.7*, we see that*

$$n = \left|G : \mathbf{I}_G(\psi)\right|.$$

Moreover, $n \mid |G : N|$ *since* $|G : N| = |G : \mathbf{I}_G(\psi)|\,|\mathbf{I}_G(\psi) : N|$*. Further, it can be shown that* $c \mid |G : N|$*.*

Lemma 2.4.10. *[8, Theorem 6.11] Let* $N \trianglelefteq G$ *and* $\psi \in \mathrm{Irr}(N)$*. Suppose* $H = \mathbf{I}_G(\psi)$ $(N \subseteq H \subseteq G)$*,* $\mathcal{A}$ *is the set of all irreducible characters* ϕ *of* H *such that* ϕ_N *has* ψ *as an irreducible constituent, and* $\mathcal{B}$ *is the set of all irreducible characters* χ *of* G *such that* χ_N *has* ψ *as an irreducible constituent. Then the following holds:*

1. $\phi^G \in \mathrm{Irr}(G)$ *for* $\phi \in \mathcal{A}$*;*
2. *The mapping* $f : \mathcal{A} \to \mathcal{B}$ *defined by* $\psi \mapsto \psi^G$ *is a bijection;*
3. *If* $\chi = \phi^G$ *where* $\phi \in \mathcal{A}$*, then* χ_H *has* ϕ *as its unique irreducible constituent;*
4. *If* $\chi = \phi^G$ *where* $\phi \in \mathcal{A}$*, then* $[\chi_N, \psi] = [\phi_N, \psi]$*.*

Theorem 2.4.11. *(Ito)[8, Theorem 6.15] Let* $N \trianglelefteq G$ *is abelian, then* $\chi(1) \mid |G : N|$ *for all* $\chi \in \mathrm{Irr}(G)$*.*

Proof. Let $\chi \in \mathrm{Irr}(G)$. Now let λ be an irreducible constituent of χ_N. Note that λ is linear since N is abelian. Set $H = \mathbf{I}_G(\lambda)$. By *Lemma 2.4.10 (2)*, there exists $\psi \in \mathrm{Irr}(H)$ (where $[\psi_N, \lambda] = [\chi_N, \lambda] \neq 0$ by Lemma 2.4.10 (4)) such that $\psi^G = \chi$.

Now, by *Theorem 2.4.7*,

$$\psi_N = c \sum \lambda_i,$$

where $[\lambda, \psi_N] \neq 0$ and $\lambda = \lambda_1, \dots, \lambda_n$ are the distinct conjugates of λ *in H*. So $\lambda^h = \lambda$ for all $h \in H$ by definition. Thus we can write

$$\psi_N = c\lambda.$$

Thus, if $a \in N$, then $|\psi(a)| = |\psi_N(a)| = |e\lambda(a)| = e(1) = e\lambda(1) = \psi(1)$. That is, $a \in \mathbf{Z}(\psi)$ and $N \subseteq \mathbf{Z}(\psi)$. If we consider the fact that $|H : N| = |H : \mathbf{Z}(\psi)|\,|\mathbf{Z}(\psi) : N|$ and $\psi(1) \mid |H : \mathbf{Z}(\psi)|$ (see *Remark 2.2.5*), then $\psi(1) \mid |H : N|$; and so $\psi(1) \mid |G : N|$ ($|G : N| = |G : H|\,|H : N|$). Now

$\chi(1) = \psi^G(1) = |G : H|\psi(1)$, where $|G : H|$ and $\psi(1)$ both divide $|G : N|$; thus $\chi(1) \mid |G : N|$. The proof is complete. □

Theorem 2.4.12. *(Gallagher)[8, Theorem 6.17] Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ such that $\chi_N \in \mathrm{Irr}(N)$, then the characters $\mu\chi \in \mathrm{Irr}(G)$ are distinct for each $\mu \in \mathrm{Irr}(G/N)$. Further, each $\mu\chi$ is an irreducible constituent of $(\chi_N)^G$.*

Recall that in a group G, a chief series is a *normal series*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

such that for each $0 \leq i < n$, there exists no *normal subgroup H of G* with $G_i < H < G_{i+1}$. The factors of this series are called chief factors. It can be shown that *every normal subgroup is a term in some chief series!*

Theorem 2.4.13. *[8, Theorem 6.18] Let N/M be an abelian chief factor of G and $\chi \in \mathrm{Irr}(N)$ be invariant in G. Then one of the following holds:*

1. $\chi_M \in \mathrm{Irr}(M)$;
2. $\chi_M = c\psi$ *where* $\psi \in \mathrm{Irr}(M)$ *and* $c^2 = |N : M|$;
3. $\chi_M = \sum_{i=1}^{t} \psi_i$ *where* $\psi_i \in \mathrm{Irr}(M)$ *are distinct and* $t = |N : M|$.

Corollary 2.4.14. *[8, Corollary 6.19] Let $N \trianglelefteq G$, $\chi \in \mathrm{Irr}(G)$ and $|G : N| = p$, where p is a prime. Then one of the following holds:*

1. $\chi_N \in \mathrm{Irr}(N)$;
2. $\chi_N = \sum_{i=1}^{p} \psi_i$ *where* $\psi_i \in \mathrm{Irr}(N)$ *are distinct.*

Proof. Since G/N is of prime order, it follows that G/N is simple; that is, there exists no non-trivial normal proper subgroup H/N of G/N. Or by the *Correspondence theorem*, there exists no normal proper subgroup H of G which contains N. It follows that G/N is an abelian (cyclic) chief factor of G. We note that the prime $p = |G : N|$ can not be a square number, thus by *Theorem 2.4.13*, the result follows. □

Part II

Groups with few p'-character degrees

3 Thompson's theorem

The definition of a good mathematical problem is the mathematics it generates rather than the problem itself.

– Andrew Wiles

In this chapter, we introduce and prove a theorem that perhaps lies at the heart of this dissertation—Thompson's theorem. This remarkable theorem was first proved by John Griggs Thompson, in 1970, in a paper[19] titled "Normal p-complements and irreducible characters." We first introduce some terms and notation. If $G = N \rtimes P$ where $P \in \mathrm{Syl}_p(G)$ and $N \trianglelefteq G$, we call N a normal p-complement or, say G is p-nilpotent. The character degree set of G is defined to be the set $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. Further, $\mathrm{cd}_p(G) = \{\chi(1) : p \mid \chi(1) \text{ and } \chi \in \mathrm{Irr}(G)\}$ and $\mathrm{cd}_{p'}(G) = \{\chi(1) : p \nmid \chi(1) \text{ and } \chi \in \mathrm{Irr}(G)\}$. Thus, the statement of Thompson's theorem is as follows:

Theorem. *(Thompson) Let G be a group and p a fixed prime. If $\left|\mathrm{cd}_{p'}(G)\right| = 1$, then G contains a normal p-complement.*

Before we prove Thompson's theorem, we have to go through some preliminaries. Let χ be a character of a group G afforded by the representation $\mathfrak{X}$. Now if $\mathfrak{N}(g) = \det \mathfrak{X}(g)$, then $\mathfrak{N}(gh) = \det \mathfrak{X}(gh) = \det(\mathfrak{X}(g)\mathfrak{X}(h)) = \det \mathfrak{X}(g) \det \mathfrak{X}(h) = \mathfrak{N}(g)\mathfrak{N}(h)$. Thus $\mathfrak{N}$ defines a representation of G. The character afforded by the representation $\mathfrak{N}$ is denoted $\det \chi$. This character is linear since $\det \chi(1) = \det \mathfrak{X}(1) = 1$.

Proposition 3.1. *The set of linear characters of a group G, denoted by $\hat{G}$, forms a group.*

Proof. If χ and ψ are in $\mathrm{Irr}(G)$, we define $\chi\psi$ by

$$\chi\psi : g \mapsto \chi(g)\psi(g).$$

Set $\hat{G} = \{\lambda \in \mathrm{Irr}(G) : \lambda \text{ is linear}\}$. For $\lambda_1, \lambda_2 \in \hat{G}$, $\lambda_1\lambda_2$ is a character of G by *Proposition 2.3.2*. Further, $\lambda_1(1)\lambda_2(1) = 1$ thus $\lambda_1\lambda_2 \in \hat{G}$. For $\lambda \in \hat{G}$, clearly $\lambda 1_G = \lambda$, thus an identity exists in $\hat{G}$. Finally, for $\lambda \in \hat{G}$ afforded by the representation $\mathfrak{X}$, consider the mapping $\mathfrak{N} : g \mapsto \overline{\mathfrak{X}(g)}$. Now $\mathfrak{N}(gh) = \overline{\mathfrak{X}(gh)} = \overline{\mathfrak{X}(g)\mathfrak{X}(h)} = \mathfrak{N}(g)\mathfrak{N}(h)$, thus $\mathfrak{N}$ is a representation of G. Let $\overline{\lambda}$ be the character afforded by $\mathfrak{N}$. We see that $\overline{\lambda(1)} = \mathrm{tr}\,\mathfrak{N}(1) = \mathrm{tr}\,\overline{\mathfrak{X}(1)} = \overline{1} = 1$, thus $\overline{\lambda} \in \hat{G}$. Clearly, $\lambda : g \mapsto \overline{\lambda(g)}$. So $\lambda(g)\overline{\lambda(g)} = \lambda(g)\lambda(g^{-1}) = \lambda(gg^{-1}) = \lambda(1) = 1$ (see *Remark 2.2.2 (2)*). Thus $\lambda\overline{\lambda} = 1_G$. In conclusion, $\hat{G}$ forms a group. □

Given that $\hat{G}$ is a group, we can now define the following.

Definition 3.2. *Let G be a group and χ a character of G. If $\det \chi = \lambda$ (which is in $\hat{G}$) we define the determinantal order of χ as $o(\chi) = o(\lambda)$.*

Proposition 3.3. *Let G be a group. If $\chi \in \mathrm{Irr}(G)$, then $o(\chi) = |G : \ker \lambda|$ where $\det \chi = \lambda$.*

Proof. Let $\det\chi = \lambda$. Since λ is linear, $\lambda : G \mapsto \mathbb{C}^\times$ is a homomorphism. So, by the *First isomorphism theorem*, $G/\ker\lambda \cong \lambda(G)$. So $\lambda(G)$ is a finite subgroup of $\mathbb{C}^\times$; thus, by *Proposition 1.4.19*, $\lambda(G)$ is cyclic ($\lambda(G) = \langle\lambda(g_0)\rangle$) of order $|G : \ker\lambda| = n$. Thus, for all $g \in G$, $\lambda(g)^n = 1$; that is, $\lambda^n = 1_G$. Now if $k \leq n$ is a positive integer, such that $\lambda^k = 1_G$, then $\lambda(g_0)^k = 1$. But $o(\lambda(g_0)) = n$, implies $n \leq k$ and so $k = n$. Thus $o(\lambda) = n$ and the proof is complete. □

Remark 3.4. *When χ is linear (afforded by $\mathfrak{X}$), then*

$$\det\chi(g) = \det\mathfrak{X}(g) = \operatorname{tr}\mathfrak{X}(g) = \chi(g) \text{ for all } g \in G.$$

So $o(\chi) = |G : \ker\chi|$.

Let G be a group and π a set of primes. The π-residue of G, denoted by $\mathbf{O}^\pi(G)$, is the smallest normal subgroup of G such that $G/\mathbf{O}^\pi(G)$ is a π-group. If $\pi = \{p\}$, we write $\mathbf{O}^p(G)$ and call it the p-residue of G. The π-residue of G is a characteristic subgroup of G.

Theorem 3.5. *(Cauchy's theorem) If G is a group such that p is a prime which divides the order of G, then G contains an element of order p.*

Proposition 3.6. *Let G be a group and π a set of primes. If $X \subseteq G$ contains all the elements of G whose orders do not have prime divisors in π, then $\mathbf{O}^\pi(G) = \langle X\rangle$.*

Proof. Let $N = \langle X\rangle$. Now note that $N \operatorname{char} G$. We now show that G/N is a π-group. For a contradiction, let q be a prime such that $q \mid |G : N|$ and $q \notin \pi$. Then G/N contains an element $[g]$ (we use this notation instead of aN) of order q (*Cauchy's theorem*). Thus $g^q \in N$ but $g \notin N$ (note that if $g \in N$, then the order of $[g]$ would be less than q). It follows $o(g) = q^k m$ where $q \nmid m$; that is, there exists integers a, b such that $1 = aq + bm$. So we can write $g^{aq+bm} \notin N$. Further, $g^m \notin N$ (as if it was, then $g^{aq+bm} \in N$). By order of g, $(g^m)^{q^k} = 1$ and there exists no smaller integer t such that $(g^m)^t = 1$ since this would mean $o(g) \leq tm < q^k m$, a contradiction. Thus $o(g^m) = q^k$, a contradiction since N must contain all elements whose orders do not have prime divisors in π. Consequently $|G : N|$ is a π-number and G/N a π-group. So $\mathbf{O}^\pi(G) \subseteq N$.

Now consider $g \in G$ such that $o(g)$ does not contain any prime divisors in π. Further, consider $[g] \in G/\mathbf{O}^\pi(G)$ (here $[g]$ is the coset $g\mathbf{O}^\pi(G)$). Now $o([g]) \mid o(g)$, but $o([g])$ must be a π-number since $G/\mathbf{O}^\pi(G)$ is a π-group. So $o([g]) = 1$ and $g \in \mathbf{O}^\pi(G)$. Therefore $N \subseteq \mathbf{O}^\pi(G)$ and $N = \mathbf{O}^\pi(G)$.

□

By the notation $\mathbf{O}^{pp'}(G)$, we mean $\mathbf{O}^{p'}(\mathbf{O}^p(G))$.

Lemma 3.7. *A group G contains a normal p-complement if and only if $\mathbf{O}^{pp'}(G) = 1$.*

Proof. Suppose that G contains a normal p-complement, then $G = N \rtimes P$ where $P \in \operatorname{Syl}_p(G)$ and $N \trianglelefteq G$. Now since $G/N \cong P$ (*Diamond isomorphism theorem*), where P is a p-group, then $\mathbf{O}^p(G) \subseteq N$. If, for a contradiction $\mathbf{O}^p(G) \neq N$, then $|G : \mathbf{O}^p(G)| > |G : N| = |P|$ where

$|G : \mathbf{O}^p(G)|$ is a larger p-power (larger than $|P|$) dividing the order of G, a contradiction since P is a Sylow p-subgroup. Thus $\mathbf{O}^p(G) = N$. Further, $N/1$ is a p'-group, thus $\mathbf{O}^{p'}(N) = 1$. That is, $\mathbf{O}^{pp'}(G) = 1$.

Conversely, suppose that $\mathbf{O}^{pp'}(G) = 1$. Let $N = \mathbf{O}^p(G)$. It follows that $\mathbf{O}^{p'}(N) = 1$, with $N \trianglelefteq G$. Now, by definition of the p-residue, G/N is a p-group where N is a p'-group (N/1 is a p'-group) and $|G : N|\,|N| = G$. So $G/N \cong P$ where $P \in \mathrm{Syl}_p(G)$. Therefore $G = NP$ and N is a normal p-complement of G. □

Corollary 3.8. *For a group G, if $p \nmid |\mathbf{O}^p(G)|$, then G contains a normal p-complement.*

Proof. Note $p \nmid |\mathbf{O}^p(G)|$ implies $\mathbf{O}^p(G)$ contains only elements whose orders are not divisible by a prime p. Thus, by *Proposition 3.6 and Lemma 3.7*, $\mathbf{O}^{pp'}(G) = \langle 1 \rangle = 1$ and G contains a normal p-complement. □

Lemma 3.9. *[8, Theorem 12.1] Let p be a fixed prime. Define $\mathcal{X}(G) = \{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)$ and $p \nmid o(\chi)\}$ and $\mathcal{S}(G) = \sum_{\chi \in \mathcal{X}(G)} \chi(1)^2$, then*

$$|\mathbf{O}^p(G)| \equiv \mathcal{S}(G) \mod p.$$

Theorem 3.10. *Let G be a group and p a fixed prime. If $\left|\mathrm{cd}_{p'}(G)\right| = 1$, then G contains a normal p-complement.*

Proof. Let $\mathcal{X}(G) = \{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)$ and $p \nmid o(\chi)\}$ and $\mathcal{S}(G) = \sum_{\chi \in \mathcal{X}(G)} \chi(1)^2$ and $N = \mathbf{O}^{p'}(G)$. We first show that $\mathcal{X}(G) = \mathrm{Irr}(G/G'N)$.

If $\lambda \in \mathrm{Irr}(G/G'N)$, then $G' \subseteq G'N \subseteq \ker\lambda$ (see *Proposition 2.2.8*). Thus λ is linear. Further, if, for a contradiction, $p \mid o(\lambda) = |G : \ker\lambda|$. Then $p \mid |G : N| = |G : \ker\lambda|\,|\ker\lambda : N|$ since $N \subseteq G'N \subseteq \ker\lambda$—a contradiction ($G/N$ is a p'-group). Thus $p \nmid o(\lambda)$ and it follows $\lambda \in \mathcal{X}(G)$.

If $\lambda \in \mathcal{X}(G)$, then λ is linear by assumption (all non-linear irreducible characters have degrees divisible by p). Further, $p \nmid o(\lambda) = |G : \ker\lambda|$. That is, $G/\ker\lambda$ is a p'-group and so $N \subseteq \ker\lambda$. Thus $G', N \subseteq \ker\lambda$ and so $G'N \subseteq \ker\lambda$. Consequently $\lambda \in \mathrm{Irr}(G/G'N)$,

$$\mathcal{X}(G) = \mathrm{Irr}(G/G'N),$$

and $\mathcal{S}(G) = |G : G'N|$ (see *Proposition 2.2.6*). Now since $|G : N| = |G : G'N|\,|G'N : N|$ is a p'-number, $p \nmid \mathcal{S}(G) = |G : G'N|$. Thus, by *Lemma 3.9*, $p \nmid |\mathbf{O}^p(G)|$ and G contains a normal p-complement by *Corollary 3.8*. □

Now let p be a fixed prime. What happens if we change the degree of precisely one non-linear irreducible character in the hypothesis of Thompson's theorem? More specifically, suppose G is a group that contains exactly one non-linear irreducible character whose degree is non-divisible by a prime p. What is the structure of such groups? We investigate the question in the following chapters.

4 Seitz's theorem

I like friends who have independent minds because they tend to make you see problems from all angles.

– Nelson Mandela

Consider a finite group G. We define $\mathrm{Irr}_1(G)$ as the set of non-linear irreducible characters of G. In particular, for a fixed prime p, we define $\mathrm{Irr}_1(G, p')$ as the set of non-linear irreducible characters of G whose degrees are non-divisible by p. The primary groups of interest in this dissertation are groups G such that $|\mathrm{Irr}_1(G, p')| = 1$; these groups will henceforth be called $(*)$-groups. In particular, when we use phrases like "let G be a $(*)$-group", it should be understood that there is a fixed prime p such that $|\mathrm{Irr}_1(G, p')| = 1$.

We now address the existence of nilpotent $(*)$-groups; but before this is done, we prove a key theorem.

4.1 Seitz's theorem

We first provide some definitions and theorems which we will use in the proof of Seitz's theorem.

Definition 4.1.1. *A p-group G is called an extraspecial p-group, if $\mathbf{Z}(G)$ is of order p and $G/\mathbf{Z}(G)$ is a non-trivial elementary abelian p-group.*

Remark 4.1.2. *Note that extraspecial p-groups have order p^{2m+1} (see [14, Lemma 2.2.9]). We will when convenient, use $\mathrm{ES}(m, p)$ to denote an extraspecial group of order p^{2m+1}.*

Lemma 4.1.3. *Given a group G and $g \in G$, it follows that $g^G \subseteq gG'$.*

Proof. If we choose $a \in G$, then $g^a = a^{-1}(g)a = g(g^{-1}a^{-1}ga) = g[g^{-1}, a^{-1}] \in gG'$.

□

Lemma 4.1.4. *If $N \trianglelefteq G$ and $|N| = 2$, then $N \subseteq \mathbf{Z}(G)$.*

Proof. Suppose $\{1, a\} = N \trianglelefteq G$. We know $1 \in \mathbf{Z}(G)$. Now for any $g \in G$, by normality, either $a^g = 1$ or $a^g = a$. If $a^g = a$, then $g \in \mathbf{Z}(G)$ and we are done. But if $a^g = 1$, then $a = 1$, a contradiction. Thus $N \subseteq \mathbf{Z}(G)$. □

Lemma 4.1.5. *[16, Theorem 5.2.10] If $G/\mathbf{Z}(G)$ is nilpotent, then G is nilpotent.*

Below we state a well known theorem for reference.

Theorem 4.1.6. *(Frattini argument) Let $N \trianglelefteq G$ where $P \in \mathrm{Syl}_p(N)$. Then $G = N\,\mathbf{N}_G(P)$.*

In the following theorem, by a minimal normal subgroup of G, we mean a proper normal subgroup N of G such that there exists no $1 < H < N$ where $H \trianglelefteq G$.

Lemma 4.1.7. *If $N \trianglelefteq G$ and G acts transitively on $N - 1$ by conjugation, then N is an elementary abelian group of order p^r for some prime p. Moreover, N is a minimal normal subgroup of G.*

Proof. Since G is transitive on $N - 1$, if we choose two elements $x, y \in N - 1$, then there exists $g \in G$ such that $x^g = y$ for some $g \in G$. That is, x and y have the same order (isomorphisms preserve orders and $x \mapsto x^g$ is an isomorphism). This shows that all non-trivial element of N have the same order $n \neq 1$. Now if p, q are distinct prime factors of n, since $n \mid |N|$, we have $p, q \mid |N|$. Thus, by *Cauchy's theorem*, N contains elements of order p and q, respectively, a contradiction. So only one prime can divide n, say p. Now if $n = p^k$, where $k \geq 2$, then $1 \neq g \in N$ is an element of order p^k. But, similarly, by *Cauchy's theorem*, N contains an element of order p, a contradiction. Consequently every non-trivial element in N is of order p. Thus N is a non-trivial p-group.

Since N is a non-trivial p-group, $\mathbf{Z}(N) > 1$. Further, note that $\mathbf{Z}(N) \operatorname{char} N \trianglelefteq G$, thus $\mathbf{Z}(N) \trianglelefteq G$. Now we suppose, for a contradiction, that $Z(N) < N$. Choose a $x \in N - \mathbf{Z}(N)$ and $y \in \mathbf{Z}(N)$. By transitivity, there exists $g \in G$, such that $x = y^g \in \mathbf{Z}(N)$ ($\mathbf{Z}(N) \trianglelefteq G$), a contradiction. It follows $N = \mathbf{Z}(N)$. Therefore, we have shown that N is an abelian group of exponent p, i.e N is an elementary abelian p-group.

We now show that N is a minimal normal subgroup of G. Suppose, for a contradiction, that there exists $1 < H < N$ with $N \trianglelefteq G$. Now choose $1 \neq x \in H$ and $y \in N - H$. By transitivity, there exists $g \in G$ such that $x^g = y$. But $H \trianglelefteq G$ and so $y \in H$, a contradiction. Thus no such H can exist and N is a minimal normal subgroup of G.

□

Theorem 4.1.8. *(Seitz [17]) A group G has exactly one non-linear irreducible character if and only if one of the following holds:*

1. $|G| = 2^k, |G'| = 2$, *and* $\mathbf{Z}(G) = G'$ *where k is odd;*

2. $G \cong \mathrm{AGL}_1(p^n)$.

Proof. ($\Longrightarrow$) Since the number of irreducible characters of G equals the number of conjugacy classes of G, G has exactly one non-linear irreducible character if and only if G has $|G : G'| + 1$ conjugacy classes (see *Proposition 2.2.6* and *Proposition 2.2.11*). If we let $C = \{g_0G', \dots, g_tG'\}$ be the set of cosets of G', where $g_0 = 1$. Then, since $|C| = |G : G'| = t + 1$, we have $t + 2 = |G : G'| + 1$ is the number of conjugacy classes of G.

By *Lemma 4.1.3*, $1 = g_0^G, g_1^G, \dots, g_t^G$ are distinct conjugacy classes of G (distinct cosets of are disjoint). Now $G' \trianglelefteq G$ and so must be the union of conjugacy classes of G (distinct from $g_1^G, \dots, g_t^G$ since distinct cosets are disjoint). If G' is the union of more than two distinct conjugacy classes (including 1), then there are at least $a + t > t + 2$ conjugacy classes of G ($a > 2$), a contradiction. The only possibility is that 1 and $G' - 1$ are the conjugacy classes which form the union of G'. So 1, $G' - 1$, $g_1^G, \dots, g_t^G$ are the $t + 2$ conjugacy classes of G. But since the union of all conjugacy classes of G must equal to G, we must have $g_i^G = g_iG'$ for $i \geq 1$.

We can now say G has $|G : G'| + 1$ conjugacy classes if and only if $g_1G', \dots, g_tG'$ are conjugacy classes of G and G' is the union of the conjugacy classes 1 and $G' - 1$.

Now if G has exactly one non-linear irreducible character, then we have shown that G' is the union of conjugacy classes 1 and $G' - 1$. It follows that G acts transitively, by conjugation, on $G' - 1$. Thus, by *Lemma 4.1.7*, G' is an elementary abelian group (of order p^k) and a minimal normal subgroup of G. Also, since G' is abelian, $G'' = 1$ and thus G is solvable.

If we choose $1 \neq z \in G' - 1$, then $G' \subseteq \mathbf{C}_G(z) \trianglelefteq G$ since G' is abelian. It is easily confirmed that for any $h \in G$, $\mathbf{C}_G(z)^h = \mathbf{C}_G(z^h)$. By normality, we have $\mathbf{C}_G(z) = \mathbf{C}_G(z)^h = \mathbf{C}_G(z^h)$ for all $h \in G$. That is, all elements in G commute with z if and only if they commute with its conjugates. So $\mathbf{C}_G(z) = \mathbf{C}_G(G')$. Since $G' - 1 = z^G$, by the *Orbit-Stabilizer theorem*,

$$|G - 1| = \left|z^G\right| = |G : \mathbf{C}_G(z)| = \left|G : \mathbf{C}_G(G')\right| = p^k - 1.$$

Case 1: Suppose $|G' - 1| = |G : \mathbf{C}_G(G')| = 1$. So $|G'| = 2$, then $G' \subseteq \mathbf{Z}(G)$ and G is nilpotent (see *Lemma 4.1.4 and 4.1.5*). Now since $\mathbf{Z}(G) \trianglelefteq G$, $\mathbf{Z}(G)$ is the union of conjugacy classes of G. If $g_iG' = g_i^G \subseteq \mathbf{Z}(G)$ $(i \geq 1)$, then $g_i^g = g_i$ for all $g \in G$. That is, $g_iG' = \{g_i\}$, a contradiction. In general, all conjugacy classes contained in $\mathbf{Z}(G)$ are the ones of size one. Thus $\mathbf{Z}(G) = G'$.

Now choose $x \in G$ and $y^2 \in G$. Since G' of order two and every commutator is in $\mathbf{Z}(G)$, we have

$$\begin{aligned} 1 = [x, y]^2 &= xyx^{-1}y^{-1}xyx^{-1}y^{-1} = xy\left[x^{-1}, y^{-1}\right]x^{-1}y^{-1} = x\left[x^{-1}, y^{-1}\right]yx^{-1}y^{-1} \\ &= y^{-1}xyyx^{-1}y^{-1} = y^{-1}xy^2x^{-1}y^{-1}(y^{-1}y) = y\left[x, y^2, y\right] = y^{-1}y\left[x, y^2\right] = \left[x, y^2\right]. \end{aligned}$$

That is, y^2 commutes with an arbitrary x in G. So $y^2 \in \mathbf{Z}(G) = G'$. Thus if we consider an arbitrary $yG' \in G/G'$. Then

$$(yG')^2 = y^2G' = G'.$$

This means that G/G' is of exponent two. Since $|G'| = 2$, we have shown that G is a 2-group whose center is cyclic of order two and $G/\mathbf{Z}(G)$ is an elementary abelian 2-group. That is, G is an extraspecial 2-group. Therefore $|G|$ is 2^k where k is odd, $\mathbf{Z}(G) = G'$ and $|G'| = 2$.

Case 2: Now suppose $|G' - 1| = |G : \mathbf{C}_G(G')| = p^k - 1 > 1$. That is, $|G'| = p^k > 2$. Thus all non-trivial conjugacy classes must be of size greater than one. It follows $\mathbf{Z}(G) = 1$ since 1 is the only conjugacy class which can be contained in $\mathbf{Z}(G)$.

Now let q be a prime divisor of $p^k - 1$ and $Q \in \mathrm{Syl}_q(G)$. Note if $Q \subseteq \mathbf{C}_G(G')$, then

$$|G : Q| = \left|G : \mathbf{C}_G(G')\right|\left|\mathbf{C}_G(G') : Q\right|,$$

this would imply $q \mid |G : Q|$, which is impossible, thus $Q \nsubseteq \mathbf{C}_G(G')$. Now $G'Q \trianglelefteq G$ since it contains the derived subgroup. Thus, by the *Frattini argument*,

$$G = G'Q\,\mathbf{N}_G(Q) = G'\,\mathbf{N}_G(Q).$$

Now note that if $\mathbf{N}_G(Q) = G$, then $Q \trianglelefteq G$. Thus for $[g,h] = h^g h^{-1}$ in $[G', Q]$, we have

$$[g,h] \in G' \cap Q,$$

That is, $[G', Q] \subseteq G' \cap Q$. But we know G' has p-power order and q is not of p-power since it is a factor of $p^k - 1$; thus $G' \cap Q = 1$ and $[G', Q] = 1$. But $[G', Q] = 1$ means all the elements in Q commute with all the element in G', i.e $Q \subseteq \mathbf{C}_G(G')$—a contradiction. Thus $\mathbf{N}_G(Q) < G$.

If $h \in G' \cap \mathbf{N}_G(Q)$, then for any $g = ab \in G' \, \mathbf{N}_G(Q) = G$, we have

$$h^g = (h^a)^b = h^b \in \mathbf{N}_G(Q),$$

since G' is abelian. But $h^b h^{-1} = [b,h] \in G'$, so $[b,h]\, h = h^b = h^g \in G'$. That is, $G' \cap \mathbf{N}_G(Q) \trianglelefteq G$. Moreover, $G' \cap \mathbf{N}_G(Q) < G'$ since $Q \subseteq \mathbf{N}_G(Q) \subseteq G' \subseteq \mathbf{C}_G(G')$ is impossible. But we know that G' is a minimal normal subgroup, thus $G' \cap \mathbf{N}_G(Q) = 1$. So we have shown

$$G = G' \rtimes H,$$

where $H = \mathbf{N}_G(Q)$ is abelian (consider G/G' and see the *Diamond isomorphism theorem*).

Define $N = \mathbf{C}_G(G') \cap H$. We note that for arbitrary $y \in N$ and $g = ab = G' \rtimes H = G$, we have

$$\begin{aligned} y(ab) &= (ay)b \text{ (y commutes with any element in G')} \\ &= a(by) = (ab)y \text{ (H is abelian),} \end{aligned}$$

that is, $N \subseteq \mathbf{Z}(G) = 1$. That is, $N = 1$.

We now show that $\mathbf{C}_G(G') = G'$. We already know that$G' \subseteq \mathbf{C}_G(G')$.

Now choose $g \in \mathbf{C}_G(G')$. Note that $G' \subseteq \mathbf{C}_G(G)$. If $g \in G'$, then we are done. We may suppose that $g \notin G'$. Now $g \in G' \rtimes H$, that is, $g \in H^a$ for some $a \in G$. Thus $a^{-1}ga \in H$, but $\mathbf{C}_G(G') \trianglelefteq G$ so $a^{-1}ga \in \mathbf{C}_G(G')$. Therefore $a^{-1}ga \in N \subseteq \mathbf{Z}(G)$. This means $a^{-1}ga = 1$ consequently $g = 1$, a contradiction. Therefore $\mathbf{C}_G(G') = G'$.

We thus have $|G : G'| = |G : \mathbf{C}_G(G')| = p^k - 1$; that is, $|G| = |G : G'|\,|G'| = (p^k - 1)p^k$. Further, if d is the degree of the only non-linear irreducible character, then

$$\begin{aligned} &|G| = p^k(p^k - 1) = \left|G : G'\right| + d^2 = (p^k - 1) + d^2 \text{ which means that} \\ &(p^k - 1)p^k - (p^k - 1) = d^2 \text{ which implies that } d = p^k - 1. \end{aligned}$$

It follows that the degree of the only non-linear irreducible character is $p^k - 1$.

Thus we have shown that,

$$G = G' \rtimes H, |G| = p^k(p^k - 1), \left|G'\right| = p^k, \text{ and } |H| = d = \left|G : G'\right| = p^k - 1,$$

where G', H are abelian and d is the degree of the only non-linear irreducible character. This group is isomorphic to $\mathrm{AGL}_1(p^k)$ (see Remark after [7, Theorem 7.10])

($\Longleftarrow$) *Case 1:* We first suppose

$$|G'| = 2, \mathbf{Z}(G) = G', \text{ and } |G| = 2^k,$$

where k is odd. We again consider the cosets $g_0G', g_1G', \dots, g_tG'$ ($g_0 = 1$). Since distinct cosets are disjoint, and $g_i^G \subseteq g_iG'$, we see that $g_0^G, g_1^G, \dots, g_t^G$ are distinct conjugacy classes of G. Since $|G'| = 2$ and G' is the union of conjugacy classes (distinct from g_i^G for $i > 0$ since distinct cosets are disjoint), we have $G' - 1 = g^G$, for some $g_i \neq g \in G$, is a conjugacy class of G. It follows that $G' = \mathbf{Z}(G) = \{1, g\}$. Now if $|g_i^G| = 1$ for $i > 0$, then we would have $g_ih = hg_i$ for all $h \in G$. That is, $1, g \neq g_i \in \mathbf{Z}(G)$, a contradiction. Thus, $g_iG' = g_i^G$ for all $i > 0$. We have shown that $1, G' - 1 = g^G, g_1^G, \dots, g_t^G$ are all conjugacy classes of G, that is, G has $|G : G'| + 1$ irreducible characters. Therefore G has exactly one non-linear irreducible character.

Case 2: We now suppose that $G \cong \mathrm{AGL}_1(p^n) = N \rtimes H$ (we borrow the notation in *Example 1.4.20*). We have shown that $N = G'$ (see *Remark 1.4.21*). We now state some easily confirmed facts about elements in $\mathrm{AGL}_1(p^n)$. For any $T_{a,b}, T_{c,d} \in \mathrm{AGL}_1(p^n)$, the following holds:

- $T_{a,b}T_{c,d} = T_{ac,bc+d}$;
- $T_{a,b}^{-1} = T_{a^{-1},-a^{-1}b}$.

We show that G' contains only two conjugacy classes; namely 1 and $G' - 1$. Consider two arbitrary elements in $G' - 1$; say, $T_{1,a}$ and $T_{1,b}$ ($a, b \neq 0$). Choose $T_{ab^{-1},0}$ in $\mathrm{AGL}_1(p^n)$. Now

$$T_{ab^{-1},0}T_{1,a}T_{ab^{-1},0}^{-1} = T_{ab^{-1},0}T_{1,a}T_{a^{-1}b,0} = T_{ab^{-1},a}T_{a^{-1}b,0} = T_{1,b},$$

and so $T_{1,a}$ and $T_{1,b}$ are conjugate in $G' - 1$; that is, $G' - 1$ is a single conjugacy class.

Now consider, $\mathbf{cl}(T_{a,0})$ and $\mathbf{cl}(T_{b,0})$ with $a, b \notin \{1, 0\}$ and $a \neq b$ ($\mathbf{cl}(g)$ defines the conjugacy class of and element g in a group). If we suppose, for a contradiction, $T_{a,0}$ and $T_{b,0}$ are conjugate. Then there exists $T_{c,d}$ such that

$$T_{c,d}T_{a,0}T_{c,d}^{-1} = T_{c,d}T_{a,0}T_{c^{-1},-c^{-1}d} = T_{ac,ad}T_{c^{-1},-c^{-1}d} = T_{a,a(c^{-1}d)-c^{-1}d} = T_{a,c^{-1}d(a-1)} = T_{b,0},$$

but $a \neq b$ so we have a contradiction. Thus $\mathbf{cl}(T_{a,0})$ and $\mathbf{cl}(T_{b,0})$, with $a, b \notin \{1, 0\}$ and $a \neq b$, are distinct conjugacy classes. Further, note that $\mathbf{cl}(T_{a,0})$ is disjoint from G', since $a \notin \{1, 0\}$. We have shown that $1, G' - 1$, and all $\mathbf{cl}(T_{a,0})$ where a goes over $F - \{1, 0\}$, are distinct conjugacy classes of $\mathrm{AGL}_1(p^n)$. The union of these is obviously in $\mathrm{AGL}_1(p^n)$. But if we choose $T_{a,b} \in \mathrm{AGL}_1(p^n)$. If $a = 1$, then $T_{a,b} = T_{1,b} \in G'$. But if $a \neq 1$, then consider $T_{a-1,b}$. We have

$$T_{a-1,b}T_{a,0}T_{a-1,b}^{-1} = T_{a(a-1),ab}T_{(a-1)^{-1},-(a-1)^{-1}b} = T_{a,ab(a-1)^{-1}-(a-1)^{-1}b} = T_{a,(b(a-1)^{-1})(a-1)} = T_{a,b}.$$

Thus $T_{a,b} \in \mathbf{cl}(T_{a,0})$, and the union of all conjugacy classes $1, G' - 1$, and all $\mathbf{cl}(T_{a,0})$ where a goes over $F - \{1, 0\}$ is $\mathrm{AGL}_1(p^n)$. Consequently $1, G' - 1$, and all $\mathbf{cl}(T_{a,0})$ where a goes over

$F - \{1, 0\}$ are all the conjugacy classes of $\mathrm{AGL}_1(p^n)$. But there are $p^n - 2$ conjugacy classes of the form $\mathbf{cl}(T_{a,0})$ where a goes over $F - \{1, 0\}$. Therefore $\mathrm{AGL}_1(p^n)$ has p^n conjugacy classes. We know that $|G : G'| = p^n - 1$ since $|G'| = p^n$ and $|\mathrm{AGL}_1(p^n)| = p^n(p^n - 1)$. Thus $\mathrm{AGL}_1(p^n)$ has $|G : G'| + 1$ irreducible characters (conjugacy classes). That is, $\mathrm{AGL}_1(p^n)$ has exactly one non-linear irreducible character. But $G \cong \mathrm{AGL}_1(p^n)$, so G has exactly one non-linear irreducible character. The proof is complete.

□

Remark 4.1.9. *Consider a group G such that $|G'| = 2, \mathbf{Z}(G) = G'$, and $|G| = 2^k$ where k is odd. Thus G is a p-group where $\mathbf{Z}(G)$ is of order p ($p = 2$). If $G/Z(G)$ was trivial, then $G = \mathbf{Z}(G)$ which is impossible since $G' \neq 1$. Thus $G/\mathbf{Z}(G)$ is non-trivial. Further, similar to* "($\implies$) Case 1" in Theorem 4.1.8*, $G/\mathbf{Z}(G)$ is of exponent two; that is, $G/\mathbf{Z}(G)$ is an elementary abelian p-group. Thus Seitz's theorem can be stated as below:*

Theorem 4.1.10. *(Seitz) A group G has exactly one non-linear irreducible character if and only if one of the following holds:*

1. *G is an extraspecial 2-group;*
2. *$G \cong \mathrm{AGL}_1(p^n)$.*

Let p be a fixed prime. Consider $(*)$-group G such that $p \nmid |G|$. All non-linear irreducible characters of G must be of p'-degree since if it was not the case, *Theorem 2.2.3* would force $p \mid |G|$. Thus, $|\mathrm{Irr}_1(G)| = 1$. Now by *Seitz* (*Theorem 4.1.8*), it follows that either G is an extraspecial 2-group or G is isomorphic to $\mathrm{AGL}_1(p_0^n)$. We see that when $p \nmid |G|$, $(*)$-groups have already been classified. Therefore *the convention is to assume $p \mid |G|$ when considering a $(*)$-group G.*

4.2 On nilpotent groups

Proposition 4.2.1. *Let $\phi : G_1 \to G_2$ be an isomorphism and $H \leq G_1$. Then*

$$\phi(N_{G_1}(H)) = N_{G_2}(\phi(H)).$$

Proof. We let $\phi : G_1 \to G_2$ be an isomorphism and $H \leq G_1$. Suppose $y \in \phi(N_{G_1}(H))$. Then there exists $x \in N_{G_1}(H)$ such that $y = \phi(x)$. This implies $H^x = H$ and so $\phi(H^x) = \phi(H)$ which gives $\phi(H)^{\phi(x)} = \phi(H)$ since ϕ is a homomorphism. Therefore $\phi(H)^y = \phi(H)$ and $y \in N_{G_2}(\phi(H))$. We have now shown that $\phi(N_{G_1}(H)) \subseteq N_{G_2}(\phi(H))$.

Now we let $y \in N_{G_2}(\phi(H))$. This implies $\phi(H)^y = \phi(H)$. But since ϕ is surjective, there exists $x \in G_1$ such that $\phi(x) = y$. Thus we have $\phi(H^x) = \phi(H)^{\phi(x)} = \phi(H)$ since ϕ is a homomorphism. But ϕ is an isomorphism, thus $H^x = H$ and $x \in N_{G_1}(H)$. This then implies $y = \phi(x) \in \phi(N_{G_1}(H))$, so $N_{G_2}(\phi(H)) \subseteq \phi(N_{G_1}(H))$. It follows that $\phi(N_{G_1}(H)) = N_{G_2}(\phi(H))$. □

Lemma 4.2.2. *Let p, q be fixed primes and let G be a q-group with $q \neq p$. Then $|\mathrm{Irr}_1(G, p')| = 1$ if and only if G is an extra-special 2-group. Moreover, $p \neq 2$.*

Proof. ($\Longrightarrow$) Since all non-linear irreducible character of G must have q-power degrees (see *Theorem 2.2.3)*. Further, G is a group which has exactly one non-linear irreducible character since if other non-linear irreducible characters existed they would be of p'-degree. Thus G has exactly one non-linear irreducible character. By *Seitz's theorem* (*Theorem 4.1.8*), either (1) or (2) holds. Suppose, for a contradiction, that (2) holds. We have $G \cong K$ where K is a Frobenius group. Suppose $\phi : K \to G$ is an isomorphism and H a Frobenius complement of K. We know by *Lemma 1.4.8* that $H < K$. Thus it follows that $\phi(H) < G$; but G is nilpotent, since G is a q-group, thus $\phi(H) < \mathbf{N}_G(\phi(H))$ by *Theorem 1.3.7 (2)*. That is, $\phi(H) < \phi(N_K(H))$ by *Proposition 4.2.1*. It follows that $H < N_K(H) = H$ by *Proposition 1.4.12*, a contradiction. Thus (1) must hold and $G = \mathbf{ES}(m, 2)$. Moreover, $p \neq q = 2$.

($\Longleftarrow$) The converse holds trivially by *Theorem 4.1.8.* □

Remark 4.2.3.

- *We note that from the lemma above, q-groups are not $(*)$-groups since the fixed prime p does not divide the order of the group.*
- *It might be observed that* Lemma 4.2.2 *also follows from* Lemma 1.4.13. *However, our proof does not rely on us proving that if $G \cong K$ and K is Frobenius, then G is Frobenius.*

Lemma 4.2.4. *If $G = P \times Q$ is a nilpotent group, where P and Q are both non-trivial Sylow subgroups, then G is not a $(*)$-group.*

Proof. Suppose, for a contradiction, that G is a $(*)$-group. So $p \mid |G|$. Without loss of generality, let P be a Sylow p-subgroup. Since P is nilpotent, it is a non-trivial solvable group. Thus choose a linear character of P, say $\theta \neq 1_P$ (see *Proposition 2.2.11*). If Q is abelian, then all non-linear irreducible characters of G, which are of the form

$$\chi \times \psi \ (\chi \in \mathrm{Irr}(P) \text{ and } \psi \in \mathrm{Irr}(Q)),$$

would be of p-power (see *Definition 2.3.3 and Theorem 2.3.5*), a contradiction. Thus Q must be non-abelian. Then choose $\psi \in \mathrm{Irr}(Q)$ which is non-linear and has a degree non-divisible by p (see *Theorem 2.2.3*). Thus

$$1_P \times \psi \text{ and } \theta \times \psi,$$

are non-linear irreducible characters of G whose degrees are non-divisible by p, a contradiction.

□

Theorem 4.2.5. *If G is a nilpotent group, then G is not a $(*)$-group.*

Proof. Consider the nilpotent group $G = P_1 \times \cdots \times P_k$, where each P_i is a Sylow p_i-subgroup. For a contradiction, let G be a $(*)$-group. Without loss of generality, let P_1, P_2 be non-trivial. From *Lemma 4.2.4*, we can choose two distinct non-linear irreducible characters of $P_1 \times P_2$, say $\chi_1 \times \psi_1$ and $\chi_2 \times \psi_2$ with $\chi_i \in \text{Irr}(P_1)$ and $\psi_i \in \text{Irr}(P_2)$, which have degrees non-divisible by p. It follows that $\chi_1 \times \psi_1 \times 1_{P_3} \times \cdots \times 1_{P_k}$ and $\chi_2 \times \psi_2 \times 1_{P_3} \times \cdots \times 1_{P_k}$ are distinct non-linear irreducible characters of G whose degrees are non-divisible by p, a contradiction. Thus G must have exactly one non-trivial factor, say P_1. That is, $G = P_1 \times 1 \times \cdots \times 1 \cong P_1$. Now, P_1 cannot be a p-group since all irreducible characters of G

$$\chi \times 1_{P_2} \times \cdots \times 1_{P_k} \ (\chi \in \text{Irr}(P_1))$$

would be p-powers, a contradiction. Thus P_1 must be a non-trivial q-group ($q \neq p$) and $p \nmid |G|$, a contradiction since $(*)$-groups require $p \mid |G|$. □

5 On general $(*)$-groups

If I have seen further it is by standing on the shoulders of giants.

– Isaac Newton

This chapter will describe the structure of general $(*)$-groups. Upon our investigation $(*)$-groups, we discovered that Kazarin and Berkovich [12] definitively found (and proved) the general structure of these groups. Their result generalize Thompson and Seitz's theorems which we proved in the prior chapters. In this chapter, we give a proof of their findings. We present and prove the structure of $(*)$-groups. Our proof assumes solvability.

We first introduce some terminology and notation for convenience. Let p be a fixed prime. A group G is called a Thompson group if $\left|\mathrm{cd}_{p'}(G)\right| = 1$. Recall that we call a group G a $(*)$-*group* if $|\mathrm{Irr}_1(G, p')| = 1$. The definition of a $(*)$-group requires that $p \mid |G|$. We use the ordered triple (G, χ, p) to specify a group G, $\chi \in \mathrm{Irr}_1(G)$ and a fixed prime p. More formally, when referring to (G, χ, p) as a $(*)$-group, this means $\mathrm{Irr}_1(G, p') = \{\chi\}$. In the following chapter, we will frequently say (G, p) is a $(*)$-group instead of (G, χ, p) is a $(*)$-group when we do not need to reference that χ is the only non-linear irreducible character of G with p'-degree.

We prove the following theorem:

Theorem 5.1. *[12, Theorem A] Let p be a fixed prime such that $p \mid |G|$. Then (G, χ, p) is a $(*)$-group if and only if the following hold:*

1. *G' is a Thompson group;*
2. *G/G' is a cyclic group of order $p^n - 1$;*
3. *$G'' = \ker \chi$;*
4. *$G/G'' \cong \mathrm{AGL}_1(p^n)$.*

Theorem 5.2. *Let p be a fixed prime such that $p \mid |G|$. If (G, χ, p) satisfies condition (1) - (4) in* Theorem 5.1*, then (G, χ, p) is a $(*)$-group.*

Proof. Let (G, χ, p) satisfy condition (1) - (4) in *Theorem 5.1.*

Case 1: Suppose that $\ker \chi = 1$. By assumption $\mathrm{AGL}_1(p^n) \cong G/\ker \chi = G/1 \cong G$. Thus, by *Theorem 4.1.8*, G has exactly one non-linear irreducible character. If χ is linear, then $G' \subseteq \ker \chi = G'' \subseteq G'$; that is, $G'' = G'$. This means that $G/G' \cong \mathrm{AGL}_1(p^n)$, a contradiction since G/G' is abelian. Thus $\mathrm{Irr}_1(G) = \{\chi\}$. If $p \mid \chi(1)$, then

$$\begin{aligned} |G| &= \left|G : G'\right| + \chi(1)^2 \text{ (by } \textit{Lemma 2.2.6}) \\ &= p^n - 1 + pk \text{ for some integer } k \text{ } (\chi(1)^2 \text{ is divisible by } p) \\ &= p(p^{n-1} + k) - 1. \end{aligned}$$

That is, $p \nmid |G|$, a contradiction. Therefore, $p \nmid \chi(1)$ and $\mathrm{Irr}_1(G, p') = \{\chi\}$.

Case 2: Suppose that $\ker\chi > 1$. Now choose $\psi \in \mathrm{Irr}_1(G) - \{\chi\}$. Let ϕ be an irreducible constituent of $\psi_{G'}$. Since $G' \trianglelefteq G$, by *Theorem 2.4.7*,

$$\psi_{G'} = c\sum_{i=1}^{t} \phi_i \; (c = [\psi_{G'}, \phi]),$$

where $\phi = \phi_1, \ldots, \phi_t$ are the distinct conjugates of ϕ. If ϕ_i is linear, then $G'' \subseteq \ker\phi_i$ for all i. Now $\psi(1) = \psi_{G'}(1) = ct$. Further, if we choose $g \in G''$, then $\psi(g) = \psi_{G'}(g) = c\sum\phi_i(g) = ct = \psi(1)$ since $G'' \subseteq \ker\phi_i$ for all i. Thus $g \in \ker\psi$ and $\ker\chi = G'' \subseteq \ker\psi$. That is, $\chi \neq \psi \in \mathrm{Irr}(G/\ker\chi)$. But $G/\ker\chi \cong \mathrm{AGL}_1(p^n)$ and *Theorem 4.1.8* asserts $G/\ker\chi$ can only have exactly one non-linear irreducible character. This is a contradiction. Thus ϕ_i is a non-linear irreducible character of G'. Now G' is a Thompson group, therefore $p \mid \phi_i(1)$ for all i. Since $\psi(1) = \psi_{G'}(1)$, it follows $p \mid \psi(1)$.

We now show that χ is in $\mathrm{Irr}_1(G)$ such that $p \nmid \chi(1)$. Similar to the argument in *Case 1*, χ is non-linear. If $p \mid \chi(1)$, then

$$\begin{aligned}|G| &= |G : G'| + \chi(1)^2 + \sum \psi_i(1)^2 \text{ (where } \psi_i \text{ are non-linear such that } p \mid \psi_i(1)^2) \\ &= p^n - 1 + pk \text{ for some integer } k \; (\chi(1)^2 \text{ and } \sum\psi_i(1)^2 \text{ are divisible by } p) \\ &= p(p^{n-1} + k) - 1.\end{aligned}$$

That is, $p \nmid |G|$, a contradiction. Therefore, $p \nmid \chi(1)$ and $\mathrm{Irr}_1(G, p') = \{\chi\}$. □

Lemma 5.3. *[10, Corollary 5.23] Let G be a non-trivial p-group. Then G contains a normal subgroup of index p.*

Corollary 5.4. *Let p be a prime dividing the order of a group G. If G is nilpotent, then G contains a normal subgroup of index p.*

Proof. Let $G = P_1 \times \cdots \times P_k$ where each P_i is a Sylow p_i-subgroup (see *Theorem 1.3.7*). Without loss of generality, let P_1 be a Sylow p-subgroup, then by *Lemma 5.3*, P_1 contains a normal subgroup H of index p.

Set $N = H \times P_2 \times \cdots \times P_k$. Since $H \trianglelefteq P_1$ and each $P_i \trianglelefteq P_i$ for $i \geq 2$, it follows that $N \trianglelefteq G$. Moreover,

$$|G : N| = \frac{|P_1 \times \cdots \times P_k|}{|H \times P_2 \times \cdots \times P_k|} = |P_1| \,/\, |H| = p.$$

The proof is complete. □

If χ is a character of a group G, we say χ vanishes on $A \subseteq G$ if $\chi(g) = 0$ for all $g \in A$.

Lemma 5.5. *Let p be a fixed prime such that $p \mid |G|$. If (G, χ, p) is a $(*)$-group and $G' < G$, then*

1. *χ vanishes on $G - G'$. Moreover, $\ker\chi < G'$;*
2. *$p \nmid |G : G'|$.*

Proof. 1. Since $G' < G$, G has a non-principal linear character (see *Proposition 2.2.11*). Let $g \in G - G'$. Now we can choose λ, a non-principal linear character of G, such that $g \notin \ker \lambda$. This means that $\lambda(g) \neq 1$. Since $\chi \in \mathrm{Irr}(G)$, it follows $\lambda\chi \in \mathrm{Irr}(G)$ (by *Lemma 2.3.1*). Further, $\lambda\chi(1) = \lambda(1)\chi(1) = \chi(1)$, thus $\lambda\chi = \chi$ since χ is the only non-linear irreducible character of p'-degree. Consequently $\lambda\chi - \chi = 0$ and thus $(\lambda - 1_G)\chi$ vanishes on G. Therefore $(\lambda(g) - 1_G(g))\chi(g) = (\lambda(g) - 1)\chi(g) = 0$; since $\lambda(g) \neq 1$, this implies $\chi(g) = 0$. Hence χ vanishes on $G - G'$.

   Moreover, if $\chi(g) = \chi(1)$ for some $g \in G$, then $g \in G'$; that is; $\ker \chi \subseteq G'$. Note that since χ is not a linear character, $G' \nsubseteq \ker \chi$, so $\ker \chi < G'$ as required.

2. For a contradiction, suppose that $p \mid |G : G'|$. Since G/G' is nilpotent, then by *Corollary 5.4*, the factor group G/G' contains a normal subgroup H/G' of index p and by the *Correspondence* theorem, $G' \leq H \trianglelefteq G$. In particular, $|G/G' : H/G'| = |G : H| = p$. Thus, by *Corollary 2.4.14*, either $\chi_H \in \mathrm{Irr}(H)$ or $\chi_H = \psi_1 + \cdots + \psi_p$ where the ψ_i's are distinct irreducible characters of H. But if $\chi_H = \sum_{i=1}^{p} \psi_i$, then since *Theorem 2.4.7* asserts all irreducible constituents of χ_H must have the same degree, it follows $\chi(1) = \chi_H(1) = p\psi_1(1)$, a contradiction since χ must be of p'-degree. Thus $\chi_H \in \mathrm{Irr}(H)$. Now, by *Theorem 2.4.12*, all of $(\chi_H)^G$'s distinct irreducible constituents are given by $\lambda\chi$ where $\lambda \in \mathrm{Irr}(G/H)$ (a non-trivial abelian group). Hence if we choose χ and $1_{G/H} \neq \lambda \in \mathrm{Irr}(G/H)$, then χ and $\lambda\chi$ are distinct non-linear irreducible characters of G with p'-degree, a contradiction. Therefore $p \nmid |G : G'|$.

□

Consider a group G. For an arbitrary commutator $[a, b] \in G$, we have that

$$[a, b]\, G'' = \left[aG'', bG''\right]. \tag{5.1}$$

Thus it follows that, $(G/G'')' \subseteq G'/G''$. Now if xG'' is an arbitrary element in G'/G''. Then

$$xG'' = x_1 \cdots x_r G'' = x_1 G'' x_2 G'' \cdots x_r G'',$$

where x_i is a commutator in G (see *Lemma 1.1.7*). But each $x_i G' \in (G/G'')'$ by equation (5.1) above. That is, $xG'' \in (G/G'')'$. Thus we have the following lemma:

Lemma 5.6. *For a group G, the derived subgroup of G/G'' is given by G'/G''.*

Theorem 5.7. *Let p be a fixed prime where $p \mid |G|$. If (G, χ, p) is a $(*)$-group and $G'' < G'$, then condition (1) – (4) of* Theorem 5.1 *are satisfied.*

Proof. From assumption, $G' < G$ since $G' = G$ would mean $G'' = (G')' = G'$, a contradiction. That is, (G, χ, p) is a $(*)$-group such that $G' < G$. It follows from *Lemma 5.5 (2)* that $p \nmid |G : G'|$. Further, G/G'' is abelian if and only if $G' \subseteq G''$ and $G'' \subseteq G'$ (see *Theorem 1.1.17*). That is, G/G'' is abelian if and only if $G'' = G'$. But $G'' \neq G'$, thus G/G'' is non-abelian. Now choose $\psi \in \mathrm{Irr}_1(G/G'')$. Note that $G'/G'' \trianglelefteq G/G''$ where G'/G'' is abelian. Thus, by *Theorem 2.4.11*,

$$\psi(1) \mid \left|G/G'' : G'/G''\right| = \left|G : G'\right| \text{ (\textit{Third isomorphism theorem})}.$$

Hence $p \nmid \psi(1)$. *Thus, all non-linear irreducible characters of G/G'' are of p'-degree.* Moreover, if $\psi \neq \phi \in \mathrm{Irr}_1(G/G'')$, then ψ and ϕ are distinct non-linear irreducible characters of G of p'-degrees, a contradiction. *Consequently G/G'' has exactly one non-linear irreducible character.* Thus by *Theorem 4.1.8*, either G/G'' is $\mathbf{ES}(m, 2)$ or is isomorphic to $\mathrm{AGL}_1(q^n)$.

Case 1: Let $G'' = 1$. This implies $G \cong G/1 = G/G''$. Further, $\mathrm{Irr}_1(G) = \{\chi\}$ and $G = \mathbf{ES}(m, 2)$ or $G \cong \mathrm{AGL}_1(q^n)$.

If, for a contradiction, $G = \mathbf{ES}(m, 2)$, then $p = 2$ since $p \mid |G|$. Moreover, $\chi(1) \mid |G|$ and $\chi(1)$ is a 2-power, a contradiction.

It follows that

$$G \cong \mathrm{AGL}_1(q^n) = G'/G'' \rtimes H,$$

where G'/G'' "is the subgroup of translations" with order q^n (see *Lemma 5.6* and *Remark 1.4.21*) and $H \leq G/G''$ is a cyclic subgroup of order $q^n - 1$. But

$$H \cong \frac{G/G''}{G'/G''} \cong G/G'.$$

Thus G/G' is cyclic of order $q^n - 1$. In particular, $p \nmid |G : G'| = q^n - 1$ so $p \mid q^n$ ($|G| = q^n(q^n - 1)$) which forces $p = q$ and $G \cong \mathrm{AGL}_1(p^n)$.

We see that G' is Thompson since $|G'| = p^n$ (degrees of all non-linear irreducible characters must divide p^n) and G/G' is cyclic of order $p^n - 1$ (follows from paragraph above and $p = q$).

If, for a contradiction, $1 < \ker\chi$, it follows that $|G| > |G : \ker\chi|$. But since all linear characters contain $G' > \ker\chi$ (see *Lemma 5.5 (1)*), we must have $|G| > |G : \ker\chi| = |G : G'| + \chi(1)^2 + C$ where C is a non-negative integer, a contradiction. Thus $\ker\chi = 1 = G''$.

Case 2: Let $G'' > 1$. We first show that $G'' = \ker\chi$. Now G/G'' is non-abelian such that all non-linear irreducible characters are of p'-degree. Choose $\psi \in \mathrm{Irr}_1(G/G'')$. Hence $\psi \in \mathrm{Irr}(G)$ such that $G'' \subseteq \ker\psi$. But $\mathrm{Irr}_1(G, p') = \{\chi\}$, thus $G'' \subseteq \ker\psi = \ker\chi$ ($\psi = \chi$). If, for a contradiction, $G'' < \ker\chi$, we consider G/G'' and $G/\ker\chi$. Consequently $|G : G''| > |G : \ker\chi|$. But since all linear characters of G contain $G' > \ker\chi$, we must have $|G : G'| + \chi(1)^2 > |G : \ker\chi| = |G : G'| + \chi(1)^2 + C$ where C is a non-negative integer, this is a contradiction. Thus $\ker\chi = G''$.

We now show that G' is a Thompson group. Suppose that $\psi \in \mathrm{Irr}_1(G')$ and let $\phi \in \mathrm{Irr}(G)$ be an irreducible constituent of ψ^G. By *Frobenius reciprocity*,

$$0 \neq \left[\phi, \psi^G\right] = \left[\phi_{G'}, \psi\right].$$

That is, ψ is an irreducible constituent of $\phi_{G'}$. But $\phi(1) = \phi_H(1) \geq \psi(1) > 1$, thus ϕ is non-linear. Therefore $\phi_{G'}$ has a non-linear irreducible constituent, namely ψ. Now, by *Theorem*

2.4.7, it follows

$$\chi_{G'} = e\sum_{i=1}^{t}\sigma_i,$$

where $0 \neq [\chi_{G'}, \sigma]$ and $\sigma = \sigma_1, \dots, \sigma_t$ are distinct conjugates of $\sigma \in \mathrm{Irr}(G')$. If σ is non-linear (all σ_i are non-linear), then $G'' \nsubseteq \ker\sigma_i$ for all i. Thus we can choose $g_0 \in G'' - \bigcap\ker\sigma_i$ (note $\bigcap\ker\sigma_i \subseteq G'' = \ker\chi$). Hence for some $i_1, \dots, i_r \in \{1, \dots, t\}$ we have $\sigma_{i_1}(g_0), \dots, \sigma_{i_r}(g_0) \neq \sigma(1)$ and

$$\begin{aligned}\chi(1) &= \chi_{G'}(g_0)\ (\ker\chi = G'')\\ &= e\sum\sigma_i(1) + e\sum_{s=1}^{r}\sigma_{i_s}(g_0),\end{aligned}$$

where $i \notin \{i_1, \dots, i_r\}$. But for any $g \in G'' = \ker\chi$ (including g_0)

$$\chi(1) = \chi_{G'}(g) = e\sum\sigma_i(1) + e\sum_{s=1}^{r}\sigma_{i_s}(1) = e\sum\sigma_i(1) + e\sum_{s=1}^{r}\sigma(1),$$

a contradiction. This implies all irreducible constituent of $\chi_{G'}$ must be linear. So we have $\chi \neq \phi$. Now by *Theorem 2.4.7* and *Frobenius reciprocity*,

$$\phi_{G'} = c\sum_{i=1}^{t_1}\psi_i \text{ where } \left[\psi^G, \phi\right] = [\psi, \phi_{G'}] \neq 0,$$

and $\psi = \psi_1, \dots, \psi_{t_1}$ are the distinct conjugates of ϕ. Therefore $\phi(1) = ct_1\psi(1)$ where $c, t_1 \mid |G : G'|$. Thus $p \nmid c, t_1$ since $p \nmid |G : G'|$; that is $p \nmid \phi(1)$. This is a contradiction since G is a $(*)$-group. Thus all irreducible characters of G' are divisible by p and G' is a Thompson group.

We now show that $G/G'' \cong \mathrm{AGL}_1(p^n)$.

Suppose, for a contradiction, that $p > 2$ and $G/G'' = \mathbf{ES}(m, 2)$. Now $G''/G''' \trianglelefteq G/G'''$ is abelian, thus by *Theorem 2.4.11*, $\tau(1) \mid |G : G''|$ for all $\tau \in \mathrm{Irr}(G/G''')$. But $|G : G''|$ is a 2-power and $p > 2$, so $p \nmid \tau(1)$ for all $\tau \in \mathrm{Irr}(G/G''')$. That is, all non-linear irreducible characters of G/G''' are of p'-degree. Similar to the first paragraph of the proof (using *Theorem 4.1.8*), since G is a $(*)$-group, G/G''' is either $\mathbf{ES}(m_1, 2)$ or $\mathrm{AGL}_1(q_1^{n_1})$, a contradiction.

We may assume that $p = 2$ and $G/G'' = \mathbf{ES}(m, 2)$. We already know that $\chi \in \mathrm{Irr}(G/G'')$ where χ is of p'-degree. But $\chi(1) \mid |G/G''|$; that is, $\chi(1)$ is even and so is divisible by p, a contradiction.

Thus $G/G'' \cong \mathrm{AGL}_1(q^n)$. Now if we assume $q \neq p$. By, *Lemma 5.5 (2)*, $p \nmid |G : G''|$. Similar to the argument above (using *Theorem 4.1.8*), $G/G''' \cong \mathrm{AGL}_1(q^{n_2})$; but $\mathrm{AGL}_1(p^n)$ is not a proper epimorphic image of $\mathrm{AGL}_1(q^{n_2})$ thus we have a contradiction. So $q = p$ and $\mathrm{AGL}_1(p^n) \cong G/G''$.

Lastly, we show that G/G' is cyclic of order $p^n - 1$. Now since $G/G'' \cong \mathrm{AGL}_1(p^n)$, we have

$$G/G'' = G'/G'' \rtimes H,$$

where G'/G'' "is the subgroup of translations" with order p^n and $H \leq G/G''$ is a cyclic subgroup of order $p^n - 1$. But

$$H \cong \frac{G/G''}{G'/G''} \cong G/G'.$$

Thus G/G' is cyclic of order $p^n - 1$. The proof is complete.

□

Now note that *Theorem 5.2* is the converse of *Theorem 5.1*. For the other direction we have the extra premise that $G'' < G'$. If G is a solvable $(*)$-group, then $G^{(n)} = 1$ for some $n \in \mathbb{Z}_+$. Thus $G' < G$ since if $G' = G$, we have that $G^{(n)} = G > 1$ for all n, a contradiction. Similarly, $G'' < G'$. Thus if G is a solvable $(*)$-group, it follows that $G'' < G'$ and thus by *Theorem 5.7*, condition (1) – (4) of *Theorem 5.1* are satisfied.

So we have proved *Theorem 5.1* when G is solvable. Kazarin and Berkovich [12] showed that *Theorem 5.1* holds in general. In fact, they have shown that $(*)$-groups are solvable. Gianelli, Rizo, and Schaeffer Fry [2] showed the following:

Theorem 5.8. *(GRS 2020) [2, Theorem A] If G be a finite group and $p > 3$ be a prime. Suppose that $|\operatorname{cd}_{p'}(G)| = 2$, then G is solvable and there exists $N \trianglelefteq G$ such that N contains a normal p-complement and G/N contains a normal p-complement.*

Theorem 5.8 is a generalization of *Theorem 5.1*. Assuming $p > 3$ for *Theorem 5.1*, the distinction is that, in *Theorem 5.8*, the possibility for the existence of distinct non-linear irreducible characters of G with p'-degree is allowed. The only restriction is that there can only be one p'-number in the character degree set of G (excluding the degrees of linear characters). Furthermore, we could also use *Theorem 5.8* to conclude that $(*)$-groups are solvable (assuming $p > 3$)!

6 Examples of $(*)$-groups

Example is the school of mankind, and they will learn at no other.

– Edmund Burke

In this chapter, we provide some examples of $(*)$-groups. We give a couple of expected examples of $(*)$-groups; then we take a brute force approach; that is, we test if any group of order less than or equal to 100 is a $(*)$-group.

We first present some character theory of Frobenius groups.

Lemma 6.0.1. *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. If 1_N is an irreducible constituent of χ_N, then $N \subseteq \ker \chi$.*

Proof. By *Theorem 2.4.7*, we have

$$\chi_N = c \sum_{i=1}^{n} \lambda_i,$$

where $0 \neq c = [\chi_N, 1_N]$ and $1_N = \lambda_1, \ldots, \lambda_n$ are conjugates of 1_N in G. But $(1_N)^g = 1_N$ for all $g \in G$,

$$\chi_N = c1_N.$$

Hence $\chi(1) = \chi_N(1) = c1_N(1) = c$; that is,

$$\chi_N = \chi(1)1_N.$$

Now for each $n \in N$, $\chi(n) = \chi(1)1_N(n) = \chi(1)$ so $n \in \ker \chi$ and $N \subseteq \ker \chi$. □

Proposition 6.0.2. *Let $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$. Then $\psi^G \in \mathrm{Irr}(G)$ if and only if $\mathbf{I}_G(\psi) = N$.*

Proof. ($\Longrightarrow$) Suppose $\psi^G \in \mathrm{Irr}(G)$. Now by *Frobenius reciprocity*, it follows that

$$\left[(\psi^G)_N, \psi\right] = \left[\psi^G, \psi^G\right] = 1.$$

Hence, by *Theorem 2.4.7*, we can write

$$(\psi^G)_N = \sum_{i=1}^{n} \psi_i,$$

where $\psi = \psi_1, \ldots, \psi_n$ are conjugates of ψ in G. Thus $\psi^G(1) = \psi(1)\,|G : N| = n\psi(1)$ (see Remark 2.4.2) implying that $n = |G : N|$. It follows that,

$$|G : \mathbf{I}_G(\psi)| = n = |G : N|,$$

which implies $N = \mathbf{I}_G(\psi)$ since $N \subseteq I_G(\psi)$.

($\Longleftarrow$) Conversely, suppose $N = \mathbf{I}_G(\psi)$. Now let χ be an irreducible constituent of ψ^G. It follow, by *Frobenius reciprocity,* that

$$0 \neq c = \left[\psi^G, \chi\right] = [\psi, \chi_N],$$

and ψ is an irreducible constituent of χ_N. Thus by *Theorem 2.4.7*,

$$\chi_N = c \sum_{i=1}^{|G:N|} \psi_i,$$

where $\psi = \psi_1, \ldots, \psi_{|G:N|}$ are conjugates of ψ in G. Thus,

$$c\,|G : N|\,\psi(1) = \chi_N(1) = \chi(1) \leq \psi^G(1) = |G : N|\,\psi(1),$$

This implies that $c = 1$; moreover, $\chi(1) = \psi^G(1)$ and so $\psi^G = \chi \in \mathrm{Irr}(G)$.

□

Proposition 6.0.3. *[6, Proposition 9.1.15] If G is a Frobenius group with kernel N and $1_N \neq \psi \in \mathrm{Irr}(N)$, then*

$$\mathbf{I}_G(\psi) = N.$$

From *Proposition 6.0.3*, we see that if G is a Frobenius group with kernel N and $1_N \neq \psi \in \mathrm{Irr}(N)$, then given a non-principal irreducible character of N, say ψ, then $\psi^G \in \mathrm{Irr}(G)$ by *Proposition 6.0.2*.

Theorem 6.0.4. *Let G be a Frobenius group with kernel N and complement H. If $\chi \in \mathrm{Irr}(G)$, then either $N \subseteq \ker \chi$ or $\chi = \psi^G$ for some $\psi \neq 1_N$ in $\mathrm{Irr}(N)$.*

Proof.

Let $\chi \in \mathrm{Irr}(G)$. Now suppose ψ is a irreducible constituent of χ_N.

If $\psi \neq 1_N$, then by *Frobenius reciprocity,*

$$[\chi_N, \psi] = \left[\chi, \psi^G\right] \neq 0.$$

But $\psi^G \in \mathrm{Irr}(G)$, thus $\chi = \psi^G$.

Now if $\psi = 1_N$, then by *Lemma 6.0.1*, $N \subseteq \ker \chi$.

□

Remark 6.0.5. *We know that $G/N \cong H$. Therefore the irreducible characters of G which contain N in their kernel can just be viewed as irreducible characters of H. From this point of view, we can write $\mathrm{Irr}(H) \subseteq \mathrm{Irr}(G)$. Thus we rewrite* Theorem 6.0.4 *as*

$$\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \left\{\psi^G : 1_N \neq \psi \in \mathrm{Irr}(N)\right\}.$$

Example 6.0.6. *We let* $G = \mathrm{AGL}_1(p^n) = N \rtimes H = G' \rtimes H$ *as in* Example 1.4.20*. We recall that* $|G'| = p^n$ *and* $|H| = p^n - 1$.

By Theorem 4.1.8*,* G *has precisely one non-linear irreducible character, namely* $\mathrm{Irr}_1(G) = \{\chi\}$*. Now* $|G| = |G : G'| + \chi(1)^2 = p^n(p^n - 1)$*. But we know that* $|G : G'| = p^n - 1$*. It follows that* $\chi(1)^2 = p^n(p^n - 1) - (p^n - 1) = (p^n - 1)^2$*. Thus* $\chi(1) = p^n - 1$*.*

In conclusion, $\mathrm{Irr}_1(G, p') = \{\chi\}$ *and* $p \mid |G|$*, thus* G *is a* $(*)$*-group as we expected.*

Definition 6.0.7. *Let* G *be a group. An automorphism of* G *is called fixed-point-free (abbreviated f.p.f as in [6, page 181]) if it only fixes* $1 \in G$*.*

Lemma 6.0.8. *Let* N *be a group and* $H = \mathrm{Aut}(N)$ *such that any non-trival automorphism in* H *is f.p.f, then* $G = N \rtimes H$ *is a Frobenius group with kernel* N *and complement* H*.*

Proof. Choose $1 \neq \sigma \in H$. Since σ is a f.p.f it follows that $n^\sigma \neq n$ for any $1 \neq n \in N$. That is, $\mathrm{C}_H(n) = 1$, so by *Theorem 1.4.16*, $G = N \rtimes H$ is a Frobenius group with kernel N and complement H. □

Lemma 6.0.9. *[3, Theorem 6.4] For a prime* p*,* $\mathrm{Aut}(C_p) \cong C_{p-1}$*.*

Remark 6.0.10. *If* $C_p = \langle a \rangle$*, then the group* $\mathrm{Aut}(C_p)$ *contains the mappings* $\sigma_1, \ldots, \sigma_{p-1}$ *defined by* $\sigma_k : a \mapsto a^k$ *for* $k = 1, \ldots, p-1$*.*

Lemma 6.0.11. *The group* $G = C_p \rtimes C_{p-1}$ *is a Frobenius group where* p *is a prime.*

Proof. Suppose $C_p = \langle a \rangle$. We know that $\mathrm{Aut}(C_p) \cong C_{p-1}$. Now $\sigma_1 = 1$ (in $\mathrm{Aut}(C_p)$) since $\sigma_1(a) = a$ by definition, implying $\sigma_1(a^i) = \sigma_1(a)^i = a^i$. Now consider $\sigma_k \in \mathrm{Aut}(C_p)$ for $k = 2, \ldots, p-1$ (non-trivial element). It follows, for $1 \neq a^i \in C_p$ we have $\sigma_k(a^i) = \sigma_k(a)^i = a^{ki}$ and $a^i \neq (a^i)^k$ since σ_k is non-trivial. Thus σ_k is f.p.f and by *Lemma 6.0.8*, G is Frobenius. □

Example 6.0.12. *We now show that the Frobenius group* $G = C_p \rtimes C_{p-1}$ *is a* $(*)$*-group. Since* C_{p-1} *is abelian, it follows that* $G' = C_p$ *(see* Remark 1.4.21*). Thus* $G = G' \rtimes C_{p-1}$*. Now let* $1_{G'} \neq \psi \in \mathrm{Irr}(G')$*. It follows that* $\chi = \psi^G \in \mathrm{Irr}(G)$ *and* $\chi(1) = \psi^G(1) = |G : G'|\psi(1) = |C_{p-1}|\psi(1) = p - 1$ *since* C_p *is abelian and* $G/G' \cong C_{p-1}$*.*

Consider $1_{G'} \neq \theta \in \mathrm{Irr}(G')$*. Similarly,* $\sigma = \theta^G \in \mathrm{Irr}(G)$ *and* $\sigma(1) = p - 1$*.*

If, for a contradiction, $\chi \neq \sigma$*, then*

$$\begin{aligned} |G| = p(p-1) = \left|G : G'\right| + \chi(1)^2 + \sigma(1)^2 + C \ (C \text{ is non-negative}) \\ \implies p(p-1) = p - 1 + (p-1)^2 + (p-1)^2 + C \\ \implies p(p-1) = p(p-1) + (p-1)^2 + C, \end{aligned}$$

this is a contradiction. Thus $\chi = \sigma$*. Hence by* Theorem 6.0.4*,*

$$\mathrm{Irr}(G) = \mathrm{Irr}(C_{p-1}) \cup \{\chi\},$$

where each character in $\mathrm{Irr}(C_{p-1})$ *is linear since* C_{p-1} *is abelian and* χ *is of* p'*-degree; that is,* G *is a* $(*)$*-group.*

6.1 Dihedral (*)-groups

By D_{2n}, we mean the Dihedral group of order $2n$; that is, the group given by $\langle a, b \,:\, a^n = b^2 = 1 \text{ and } bab = a^{-1}\rangle$.

Lemma 6.1.1. *Let G be the dihedral group $D_{2n} = \langle a, b \,:\, a^n = b^2 = 1 \text{ and } bab = a^{-1}\rangle$, then $G' = \langle a^2\rangle$.*

Proof. Now $[b, a^k] = ba^{-k}ba^k = (ba^{-k}b)a^k = (a^{n-1})^k a^k = a^{2k}$. Thus $\langle a^2\rangle \subseteq G'$. Further, $[a^i, b^j] = 1$, $[a^i, a^j b] = a^i a^j b a^{-i} b a^{-j} = a^{i+j} a^i b b a^{-j} = a^{2i}$ and $[a^i b, a^j b] = a^i b a^j b b a^{-i} b a^{-j} = a^{i-j} b b a^{i-j} = a^{2(i-j)}$. That is, every commutator of G is in $\langle a^2\rangle$, but the derived subgroup is the smallest subgroup containing the commutators of G, thus $G' \subseteq \langle a^2\rangle$ and so $G' = \langle a^2\rangle$. □

Lemma 6.1.2. *[11, pg 108 (12.2)] The dihedral group D_{2n} where $n = 2m$ (n even) has $m + 3$ conjugacy classes given by:*

$$1, \{a^m\}, \left\{a.a^{-1}\right\}, \cdot, \left\{a^{m-1}, a^{-m-1}\right\} \text{ and } \left\{a^{2j} + 1b \,:\, j = 0, 1, \ldots, m-1\right\}, \left\{a^{2j} b \,:\, j = 0, 1, \ldots, m-1\right\}.$$

Lemma 6.1.3. *[3, Theorem 4.2] Let $a \in G$ such that $o(a) = n$ and $k \in \mathbb{Z}_+$. Then*

$$o(a^k) = \frac{n}{\gcd(n, k)}.$$

Corollary 6.1.4. *[3, Chapter 4, Corollary 3] Let $a \in G$ such that $o(a) = n$, then $o(a) = o(a^k)$ if and only if $\gcd(n, k) = 1$.*

Example 6.1.5. *Consider S_3. Now $D_6 \cong S_3 = \{(1), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$ and its character table is given below:*

Table 6.1: Character table of S_3:

	(1)	(1 2)	(1 2 3)
1_{S_3}	1	1	1
λ	1	-1	1
χ	2	0	-1

If $p = 3$, note $p \mid |S_3| = 6$. Further, χ is the only non-linear irreducible character of p'-degree, thus (S_3, p) is a $(*)$-group.

Proposition 6.1.6. *Let $q > 3$ be a prime. The dihedral group $G = D_{2q} = \langle a, b \,:\, a^q = b^2 = 1 \text{ and } bab = a^{-1}\rangle$ is not a $(*)$-group.*

Proof. Let p be a fixed prime. For a contradiction, let (G, p) be a $(*)$-group. By *Lemma 6.1.1*, $G' = \langle a^2 \rangle$. But $\gcd(2, q) = 1$ (see *Corollary 6.1.4*), thus $G' = \langle a \rangle$. Now $|G : G'| = 2$. Thus, by *Theorem 5.1*, $p^n - 1 = 2$; that is, $p = 3$ and $n = 1$ ($3^1 - 1 = 2$). But $p = 3 \nmid 2q = |G|$, a contradiction. Thus G is not a $(*)$-group. □

We are able to generalize the proposition above to get the following:

Proposition 6.1.7. *The only dihedral $(*)$-group of the form $G = D_{2n}$ where n is odd is D_6.*

Proof. Suppose p is a fixed prime and let (G, p) be a $(*)$-group. Now $G' = \langle a^2 \rangle = \langle a \rangle$ since $\gcd(2, n) = 1$ and n is odd (see *Corollary 6.1.4*). Thus, by *Theorem 5.1*, we have that $|G : G'| = 2 = p^n - 1$ and so $p = 3$. Further, G' is abelian and thus $G'' = 1$. Thus, by *Theorem 5.1*, G $\cong G/1 \cong \mathrm{AGL}_1(3) \cong D_6$. □

We naturally ask the same question for an even number: Is the dihedral group D_{2n} (n is even) a $(*)$-group? The following proposition addresses this.

Proposition 6.1.8. *No dihedral of the form D_{2n} (n is even) is a $(*)$-group.*

Proof. Consider a dihedral $(*)$-group of the form $G = D_{2n}$ ($n = 2m$) ((G, p) is a $(*)$-group for a fixed prime p). Now $G' = \langle a^2 \rangle$, so by *Lemma 6.1.3*, we can write

$$|G'| = \frac{2m}{\gcd(2, 2m)} = m.$$

It follows that $|G : G'| = 2(2m)/m = 4$. But by *Theorem 5.1*, $p^n - 1 = 4$. That is, $p = 5$ and $n = 1$ ($5^1 - 1 = 4$). But, G' is abelian and so $G'' = 1$. Thus, by *Theorem 5.1*, $G \cong G/1 = G/G'' \cong \mathrm{AGL}_1(5)$. This implies that $|G| = 5(5-1) = 20 = 2 \cdot 10$ ($m = p = 5$). Consequently $G = D_{20}$. Now by *Proposition 6.1.2*, G has $5+3 = 8$ conjugacy classes, thus G has 8 irreducible characters (see *Proposition 2.2.6*), 4 of them linear since $|G : G'| = 4$. Let $\chi_1, \cdots \chi_4$ be the distinct non-linear irreducible characters of G, then $|G| = 20 = 4 + \chi_1(1)^2 + \cdots + \chi_4(1)^2$ this implies that $16 = \chi_1(1)^2 \cdots + \chi_4(1)^2$. Since each $\chi_i(1) > 1$, this implies $\chi_i(1) = 2$ for all i. Which means all non-linear irreducible characters are of degree 2. That is, all non-linear irreducible character of G are of p'-degree. Thus D_{20} is not a $(*)$-group. □

In conclusion, we have shown that the only dihedral $(*)$-group is D_6. Now, it is well known that $D_6 \cong S_3$, thus we could, in turn, ask when is S_n a $(*)$-group?

6.2 The symmetric group on n letters

Example 6.2.1. *Now consider $G = S_4$. We know that $|G : G'| = 2$ ($G' = A_4$), so if we suppose (G, p) is a $(*)$-group (p is a fixed prime), then* Theorem 5.1 *implies $|G : G'| = 2 = p^n - 1$ which implies $p = 3$. Now G has 5 conjugacy classes ([11, Examples 12.16]) and so has 5 irreducible characters, two of them being linear since $|G : G'| = 2$. Thus $22 = \chi_1(1)^2 + \chi_2(1)^2 +$*

$\chi_3(1)^2$ (*see* Lemma 2.2.6) *where* $\chi_i \in \mathrm{Irr}_1(G)$, *for all* i, *are distinct. By inspection we see the only positive-integer solution (each integer is greater than 1) is* $\chi_1(1) = 2, \chi_2(1) = 3$, *and* $\chi_3(1) = 3$. *Thus we see that* S_4 *is a* $(*)$*-group since* χ_1 *is the only non-linear irreducible character of* p'*-degree.*

Example 6.2.2. *Let* $G = A_4 = V_4 \rtimes \langle(1\,2\,3)\rangle$. *Now by* Remark 1.4.21, $G' = V_4$. *It follows that* G *must have* $|G : G'| = 3$ *linear characters (see Proposition 2.2.11). It is easily shown that* G *has four conjugacy classes (see [11, Examples 12.18]), thus* G *must have four irreducible character. Hence* $12 = 1_{A_4}(1)^2 + \lambda_1(1)^2 + \lambda_2(1)^2 + \chi(1)^2 = 1^2 + 1^2 + 1^2 + \chi(1)^2$ *(see* Lemma 2.2.6*), were* $1_{A_4}, \lambda_1$ *and* λ_3 *are the linear characters of* G *and* χ *the only non-linear irreducible character. Thus* $\chi(1)^2 = 9$ *and* $\chi(1) = 3$. *If* $p = 2$, *then* $p \mid |G| = 12$ *and* χ *is the only non-linear irreducible character of* p'*-degree. Thus* G *is a* $(*)$*-group.*

We now state some well known facts on the symmetric group.

Proposition 6.2.3. *[10, Corollary 6.19] For* $n \geq 5$, *the only non-trivial proper normal subgroup of* S_n *is* A_n.

Proposition 6.2.4. *[11, Example 17.12] Let* $G = S_n$, *then* $G' = A_n$.

Proposition 6.2.5. *[13, Theorem 5.5] If* $n \geq 5$, *then* A_n *is simple.*

Note, for $n \geq 4$ A_n is non-abelian, since $A_4 \leq A_n$ and $(1\,2\,3)(2\,3\,4) = (1\,2)(3\,4)$ whereas $(2\,3\,4)(1\,2\,3) = (1\,3)(2\,4)$.

Proposition 6.2.6. *For* $n \geq 5$, $G = S_n$ *is not a* $(*)$*-group.*

Proof. Now $G' = A_n$ by *Proposition 6.2.4.* If (G, p) is a $(*)$-group for some fixed prime p, then $p \nmid |G : G'|$ and so $p \mid |G'|$ (*Lemma 5.5*). Further, by *Theorem 5.1*, G' is a Thompson group and so G' contains a normal p-complement (see *Theorem 3.10*). That is, $G' = N \rtimes P$ for some $P \in \mathrm{Syl}_p(G')$. Also, $N < G'$ since $p \mid |G'|$ which implies that $P \neq 1$. By simplicity of G' (*Proposition 6.2.5*), it follows that $N = 1$. Thus $G' = P$ and G' is a p-group. Now G' is simple, but G' is a non-trivial p-group which implies $1 < \mathbf{Z}(G') \trianglelefteq G'$; thus $G' = \mathbf{Z}(G')$ by simplicity, this is a contradiction since $G' = A_n$ is non-abelian for $n \geq 4$. This proves the theorem. □

Remark 6.2.7. *Note the conclusion of the proposition above could be reached by observing that all* $(*)$*-groups are solvable and* S_n *is non-solvable for* $n \geq 5$. *However, the proof demonstrates the utility of the established theorems in* Chapter 3 and 5.

Proposition 6.2.8. *The only symmetric groups on n letters which are* $(*)$*-groups are* S_3 *and* S_4.

We note that, S_3 and A_4 are Frobenius groups.

Now consider S_4. The normal subgroups of S_4 are V_4, S_4, A_4 and 1. If G is Frobenius group, then the either A_4 or V_4 must be its kernel since the kernel must be a non-trivial normal proper subgroup (note $V_4 \trianglelefteq S_4$ since $V_4 \trianglelefteq A_4$ char S_4 where $(S_4)' = A_4$). However for a Frobenius group with kernel N and complement H we must have $|H| \mid (|N|-1)$. If S_4 is Frobenius and has the kernel A_4, then the order of any complement must divide $|A_4| - 1 = 11$, a contradiction. Further, if the kernel is V_4, then the order of any complement must divide $|V_4| - 1 = 3$, a contradiction since the order of S_4 would be too large. Thus S_4 is not Frobenius! We see that not all (∗)-groups are Frobenius.

6.3 (∗)-groups of small order

As mentioned in the introduction, we now take the brute force approach. We test if each group of order up to 100 is a (∗)-group. This process is somewhat cumbersome, so we need to develop some methods to pre-emptively rule out some groups so that we do not have to go through all groups of order up to 100. The abelian group is the most prominent class of groups we can immediately rule out. This follows from the fact that (∗)-groups have a non-linear irreducible character. Nilpotent groups are also not (∗)-groups as shown in *Chapter 4*. The following lemma gives us a necessary condition that all (∗)-groups (up to an order of 100) must satisfy.

Lemma 6.3.1. *If G is a (∗)-group of order up to 100, then $|G|$ must be divisible by either of the following numbers,*

$$2 \cdot 3, 2^2 \cdot 3, 2^2 \cdot 5, 2 \cdot 3 \cdot 7, 2^3 \cdot 7, \textit{ or } 2^3 \cdot 3^2.$$

In particular, $|G : G''|$ must be exactly one of these numbers.

Proof. Let p be a fixed prime dividing the order of G and suppose (G, p) is a (∗)-group. By *Theorem 5.1*, we have that $G/G'' \cong \mathrm{AGL}_1(p^n)$ and so G must be divisible by $|G : G''| = p^n(p^n - 1)$.

Case 1: Let $p = 2$. If n is 2 or 3, then $p^n(p^n - 1)$ is 12 or 56, respectively (note that $n \neq 1$ since the underlying field of an affine linear group must contain more than two elements). But, if $n \geq 4$, then $p^n(p^n - 1) \geq 240 > 100$, a contradiction. In conclusion, when $p = 2$, then $|G|$ is divisible by either of the following,

$$2^2 \cdot 3, \text{ or } 2^3 \cdot 7.$$

Case 2: Let $p = 3$. Similar to *Case 1*, it follows that $|G|$ is divisible by either of the following,

$$2 \cdot 3, \text{ or } 2^3 \cdot 3^2.$$

Case 3: Let $p = 5$. Similar to *Case 1*, it follows that $|G|$ is divisible by $2^2 \cdot 5$.

Case 4: Let $p = 7$. Similar to *Case 1*, it follows that $|G|$ is divisible by $2 \cdot 3 \cdot 7$.

Case 5: Let p be a prime greater than 7, then for each positive integer n it follows that $p^n(p^n - 1) > 100$, this is a contradiction. Hence the result then follows.

□

Remark 6.3.2. *We see that a $(*)$-group of order up to 100 must be divisible by the numbers,*

$$2 \cdot 3, 3 \cdot 4, 4 \cdot 5, 6 \cdot 7, 7 \cdot 8, \textit{ or } 8 \cdot 9.$$

Thus $(*)$-groups of order up to 100 must be divisible by two consecutive numbers.

Note a corollary of *Lemma 6.3.1* would be: If G is a group of odd order under 100, then G is not a $(*)$-group. Moreover, if (G, p) is a general $(*)$-groupfor some prime p, then by *Theorem 5.1*, $|G : G'| = |\mathrm{AGL}_1(p^n)| = p^n(p^n - 1)$. Hence the following lemma holds:

Lemma 6.3.3. *There are no $(*)$-groups of odd order.*

Corollary 6.3.4. *Let p and q be primes such that $pq \leq 100$. With the exception of S_3, no non-abelian group of order pq is a $(*)$-group.*

Proof. Let $G \ncong S_3$; furthermore, let G be $(*)$-group such that G is a non-abelian group of order $pq \leq 100$. Since the order of G cannot be divisible by more than three primes, then by *Lemma 6.3.1*, it follows that $2 \cdot 3 \mid pq$. But this implies $|G| = 2 \cdot 3$ and since G is non-abelian, it follows that $G \cong S_3$. This is a contradiction. □

Remark 6.3.5. *Let $2 < q < p$ be primes. It can be shown that any non-abelian group of order pq is Frobenius with the kernel of order p and a complement of order q. We now let G be a Frobenius group with kernel P of order p and complement Q of order q. Since Q is abelian, we have that $G' = P$ (see* Remark 1.4.21*). If we let (G, p) be a $(*)$-group, then by* Theorem 5.1*, it follows that $|G : P| = q = p^n - 1$. But for any positive integer n, we have that $p^n - 1 > q$, this is a contradiction. Similarly, (G, q) cannot be a $(*)$-group.*

In conclusion, Corollary 6.3.4 *can be generalized by removing the condition that $pq \leq 100$.*

Proposition 6.3.6. *Let $G = H \times K$ where H and K are non-trivial, then G is not a $(*)$-group.*

Proof. Let p be a fixed prime dividing the order of G and let $(G, \chi \times \psi, p)$ be a $(*)$-group where $\chi \in \mathrm{Irr}(H)$ and $\psi \in \mathrm{Irr}(K)$ (see *Theorem 2.3.5*). Hence $(\chi \times \psi)(1)$ is a p'-number. This implies that $\chi(1)$ and $\psi(1)$ are p'-numbers. Moreover, either, $\chi(1) > 1$ or $\psi(1) > 1$. Without loss of generality, suppose $\chi(1) > 1$. Since K is non-trivial, choose $1_K \neq \theta \in \mathrm{Irr}(K)$, then

$$\chi \times 1_N \text{ and } \chi \times \theta,$$

are distinct non-linear irreducible character of G with p'-degree, this is a contradiction. □

Corollary 6.3.7. *A group which is a direct product of at least two non-trivial groups is not a $(*)$-group.*

For a positive integer n, the dicyclic group of order $4n$ is a group (non-abelian for $n > 1$) with the presentation given by

$$\mathrm{Dic}_n = \left\langle a, b \,:\, a^{2n} = 1, b^2 = a^n \text{ and } b^{-1}ab = a^{-1} \right\rangle.$$

In some texts, this group is called the *generalized quaternion group* and is denoted by Q_{4n}. This alternative name is justified by the fact that when $n = 2$, it follows that Dic_n is isomorphic to the quaternion group. It is well known that the dicyclic group Dic_n is indeed of order $4n$ and contains elements of the form $a^i b^j$ where $1 \leq i < 2n$ and $i = 0, 1$. Furthermore, it can be shown that $o(a^k) = 2n/k$ for $1 < k \leq 2n$.

Proposition 6.3.8. *Let $G = Dic_n$, then $G' = \langle a^2 \rangle$. Moreover, $G/G' \cong V_4$.*

Proof. For a positive integer $n > 1$, consider $G = \mathrm{Dic}_n = \langle a, b \,:\, a^{2n} = 1, b^2 = a^n \text{ and } b^{-1}ab = a^{-1} \rangle$. Let $N = \langle a^2 \rangle$. Note $N \trianglelefteq G$ as $b^{-1}a^{2k}b = \overbrace{(b^{-1}ab)\cdots(b^{-1}ab)}^{2k \text{ factors}} = (a^2)^{-k} \in N$. Since $[a, b] = a^{-1}b^{-1}ab = a^{-1}a^{-1} = (a^{-1})^2 \in N$, it follows that $N \subseteq G'$. But by *Proposition 6.1.3*, it follows that $|N| = n$ and so

$$|G/N| = \frac{4n}{n} = 4.$$

This implies that G/N is abelian and so by *Theorem 1.1.17*, it follows that $G' \subseteq N$ and $G' = N$.

Furthermore, G/N is of order 4 where aN and bN are distinct elements in G/N of order 2, thus $G/N \cong V_4$. □

Corollary 6.3.9. *The dicyclic group Dic_n is not a (∗)-group.*

Proof. Let $G = \mathrm{Dic}_n = \langle a, b \,:\, a^{2n} = 1, b^2 = a^n \text{ and } b^{-1}ab = a^{-1} \rangle$ and let p be a prime. Suppose (G, p) is a (∗)-group. Then by *Theorem 5.1*, we have that $G/G' \cong C_4$; contradicting *Proposition 6.3.8*. The proof is complete. □

We now go through every group of an order less than 100 (a list of these groups can be found in [15]). We will not consider abelian groups, nilpotent group (namely p-groups), or groups of an odd order as these are not (∗)-groups. Moreover, *Lemma 6.3.1* implies we can only consider groups whose orders are multiples of $2 \cdot 3, 2^2 \cdot 3, 2^2 \cdot 5, 2 \cdot 3 \cdot 7, 2^3 \cdot 7$, or $2^3 \cdot 3^2$.

We use GAP (see [4] for details on the GAP system) to find the information needed to assess whether the group under consideration is a (∗)-group or not. The list of "small groups" found in [15] conveniently has the unique identifiers given to each "small group" in the GAP system. The group S_4 has the identifier [24,12], and so the following code allows us to find its derived subgroup easily (where $G = S_4$ and $H = G'$):

```
gap> G := SmallGroup(24,12);
gap> H := DerivedSubgroup(G);
```

Notes on the notation used for these groups can be found in [15, page 23], with the exception of groups isomorphic to the *dicyclic* group of order $4n$, which we denote by Dic_n (when convenient) or the one-dimensional affine group over a field of p^n elements, which we denote by $\mathrm{AGL}_1(p^n)$ (when convenient). For a group G, at certain points in our arguments we will consider *all* degrees of the characters in $\mathrm{Irr}(G)$. The notation $\mathrm{Deg}(G)$ will be a sequence of the degrees of characters in $\mathrm{Irr}(G)$ and will be called the character degree sequence of G. As an example $\mathrm{Deg}(S_3) = (1, 1, 2)$. The GAP system can also be used to easily find $\mathrm{Deg}(G)$ for any group G. For instance, the code below can be used to find the components of $\mathrm{Deg}(S_4)$ by viewing the character table of S_4.

```
gap> G := SmallGroup(24,12);
gap> T = CharacterTable(G);
gap> Display(T);
```

- ($|G| = 6$): There is only one non-abelian group of order 6, that is S_3. This group is a $(*)$-group (see *Example 6.1.5*).
- ($|G| = 12$) There are three non-abelian groups of order 12. These are,

  $$C_3 \rtimes_\phi C_4, D_{12} \text{ and } A_4.$$

  The group A_4 has been shown to be a $(*)$-group in *Example 6.2.2*, and by *Proposition 6.1.8*, D_{12} is not a $(*)$-group.

  Consider $G = C_3 \rtimes_\phi C_4$. The derived subgroup of G is of order 3 and so $|G : G'| = 4 = 2^2$. If p is a fixed prime dividing $|G|$, then by *Theorem 5.1*, if (G, p) is a $(*)$-group, then $|G : G'| = 4 = p^n - 1$. This implies that $p = 5$, a contradiction since $5 \nmid 12$. Thus $C_3 \rtimes_\phi C_4$ is not a $(*)$-group.
- ($|G| = 18$) There are three non-abelian groups of order 18, given by:

  $$D_{18}, C_3 \times S_3 \text{ and } (C_3 \times C_3) \rtimes_\phi C_2.$$

  By *Proposition 6.3.6* and *Proposition 6.1.7* it follows that D_{18} and $C_3 \times S_3$ are not $(*)$-groups.

  Consider $G = (C_3 \times C_3) \rtimes_\phi C_2$. By *Theorem 1.1.17*, it follows that $G' \subseteq C_3 \times C_3$. But $C_3 \times C_3$ is abelian, so $G'' = 1$. Hence

  $$|G| = \left|G : G''\right| = 2 \cdot 3^2,$$

  Thus by *Lemma 6.3.1*, G is not a $(*)$-group.
- ($|G| = 20$) There are three non-abelian groups of order 20 given by:

  $$\mathrm{Dic}_5, C_5 \rtimes_\phi C_4 \text{ and } D_{20}.$$

  By *Example 6.0.12*, $C_5 \rtimes_\phi C_4 \cong \mathrm{AGL}_1(5)$ is a $(*)$-group. Furthermore by *Proposition 6.1.8* and *Proposition 6.3.9*, the groups D_{20} and Dic_5 are not $(*)$-groups.

- ($|G| = 24$) There are 12 non-abelian groups of order 24, 2 of these 12 groups are nilpotent and thus we do not consider them. The groups under consideration are:

  1. $C_3 \rtimes_\phi C_8$
  2. $\mathrm{SL}(2,3)$
  3. Dic_6
  4. $S_3 \times C_4$
  5. D_{24}
  6. $\mathrm{Dic}_3 \times C_2$
  7. $C_3 \rtimes_\phi D_8$
  8. S_4
  9. $A_4 \times C_2$
  10. $D_{12} \times C_2$

  By *Proposition 6.3.6* and *Proposition 6.1.8*, the only possible $(*)$-groups of order 24 are

  $$C_3 \rtimes_\phi C_8, \mathrm{SL}(2,3), \mathrm{Dic}_6, C_3 \rtimes_\phi D_8 \text{ and } S_4.$$

  Moreover, the group Dic_6 is not a $(*)$-group by *Proposition 6.3.9*, this leaves the groups

  $$C_3 \rtimes_\phi C_8, \mathrm{SL}(2,3), C_3 \rtimes_\phi D_8 \text{ and } S_4.$$

  With reference to *Example 6.2.1*, S_4 is a $(*)$-group.

  Consider $G = C_3 \rtimes_\phi C_8$. Let p be a fixed prime and suppose (G, p) is a $(*)$-group. The index of G' in G is $|G : G'| = 8$. Thus by *Theorem 5.1*, it follows that $|G : G'| = 8 = p^n - 1$. Hence $p = 3$ ($n = 2$). The second derived subgroup of G is trivial, thus by *Theorem 5.1*, it follows that $G \cong G/1 \cong \mathrm{AGL}_1(3^2)$, this is a contradiction. Thus G is not a $(*)$-group.

  Now let $G = \mathrm{SL}(2,3)$. Further, let p be a prime dividing the order of G and suppose (G, p) is a $(*)$-group. The index of G' in G is given by $|G : G'| = 3$. By *Theorem 5.1*, $|G : G'| = 3 = p^n - 1$ which implies $p = 2\,(n = 2)$. The character degree sequence of G is $\mathrm{Deg}(G) = (1,1,1,2,2,2,3)$. Thus we see that $\mathrm{SL}(2,3)$ is a $(*)$-group.

  Suppose $G = C_3 \rtimes_\phi D_8$ and let p be a fixed prime. Similarly, if (G, p) is a $(*)$-group, then $p = 5$. This is a contradiction since $5 \nmid 24$.

- ($|G| = 30$) There are 3 non-abelian groups of order 30. These are given by

  $$C_5 \times S_3, C_3 \times D_{10} \text{ and } D_{30}.$$

  By *Propositions 6.1.7 and 6.3.6*, these are not $(*)$-groups.

- ($|G| = 36$) There are 14 groups of order 36. Most of these are direct products and one is dihedral and thus are these not $(*)$-groups. We consider the groups

  $$C_9 \rtimes_\phi C_4, (C_2 \times C_2) \rtimes_\phi C_9, (C_3 \times C_3) \rtimes_\phi C_4 \text{ and } (C_3 \times C_3) \rtimes_\phi C_4.$$

  Let $G = C_9 \rtimes_\phi C_4$. Since C_9 and C_4 are abelian, it follows that $G'' = 1$ (see *Theorem*

1.1.17). That is, $|G| = |G : G''| = 2^2 \cdot 3^2$. Hence by *Proposition 6.3.1*, G is not a (∗)-group. Similarly, none of the other considered groups are (∗)-groups.

- ($|G| = 40$) There are 11 non-abelian groups of order 40. Of these 11 groups, 7 of them are direct products and and one of them is a dihedral group and hence are not (∗)-groups. The groups which will be under consideration are given by:

$$C_5 \rtimes_\phi C_8, C_5 \rtimes_\phi C_8, C_5 \rtimes_\phi Q_8 \text{ and } (C_{10} \times C_2) \rtimes_\phi C_2.$$

  Note $40 = 2^3 \times 5$. Similar to the argument used when $|G| = 36$; for each of these groups, the second derived subgroups is always trivial. Thus by *Lemma 6.3.1*, these groups are not (∗)-groups.

- ($|G| = 42$) There are 5 non-abelian groups of order 42. One is a dihedral group and 3 are direct products and so these are not (∗)-groups. The only candidate is the group $\mathrm{AGL}_1(7)$ which is a (∗)-group by *Example 6.0.6*.

- ($|G| = 48$) There are a total of 52 groups of order 48, 29 of these groups are direct products and are thus not (∗)-groups. The list of groups under consideration, excluding D_{48} and Dic_{12} as these are not (∗)-groups, is as follows:

  1. $C_3 \rtimes_\phi C_{16}$
  2. $(C_4 \times C_4) \rtimes_\phi C_3$
  3. $C_{24} \rtimes_\phi C_2$
  4. $C_{24} \rtimes_\phi C_2$
  5. $(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$
  6. $(C_3 \rtimes_\phi C_4) \rtimes_\phi C_4$
  7. $C_{12} \rtimes_\phi C_4$
  8. $(C_{12} \times C_2) \rtimes_\phi C_4$
  9. $(C_2 \times D_8) \rtimes_\phi C_2$
  10. $(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$
  11. $(C_3 \times Q_8) \rtimes_\phi C_2$
  12. $C_3 \rtimes_\phi \mathrm{Dic}_4$
  13. $(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
  14. $\mathrm{SL}(2,3).C_2$
  15. $\mathrm{GL}(2,3)$
  16. $A_4 \rtimes_\phi C_4$
  17. $\mathrm{SL}(2,3) \rtimes_\phi C_2$
  18. $(C_{12} \times C_2) \rtimes_\phi C_2$
  19. $(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
  20. $(C_4 \times S_3) \rtimes_\phi C_2$
  21. $(C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3$

  Consider the groups numbered 1,2,3, 4,7,8,18 and 21. Similar to the groups considered in groups of order 40, the second derived subgroup of these groups is trivial. In particular, the index of the second derived subgroups in each of these groups is the order of the group itself. Thus by *Lemma 6.3.1*, the groups numbered 1,2,3, 4,5,6,7,8,10,18 and 21 are not (∗)-groups.

  We are then left with the groups given by:

1. $(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$
2. $(C_3 \rtimes_\phi C_4) \rtimes_\phi C_4$
3. $(C_2 \times D_8) \rtimes_\phi C_2$
4. $(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$
5. $(C_3 \times Q_8) \rtimes_\phi C_2$
6. $C_3 \rtimes_\phi \mathrm{Dic}_4$
7. $(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
8. $\mathrm{SL}(2,3).C_2$
9. $\mathrm{GL}(2,3)$
10. $A_4 \rtimes_\phi C_4$
11. $\mathrm{SL}(2,3) \rtimes_\phi C_2$
12. $(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
13. $(C_4 \times S_3) \rtimes_\phi C_2$

Let $G = \mathrm{GL}(2,3)$. The derived subgroup of G is given by $G' = \mathrm{SL}(2,3)$ and so $|G : G'| = 2$. Let p be a fixed prime and let (G, p) be a $(*)$-group. Then by *Theorem 5.1*, it follows that $p^n - 1 = 2$, thus $p = 3$ and $n = 1$. But $\mathrm{Deg}(G) = (1,1,2,2,2,3,3,4)$. Hence, G is not a $(*)$-group.

Consider *Table 6.2* below. This table has four columns which contain the GAP ID of the group G under consideration, group name, the prime p such that $|G : G'| = p^n - 1$ and the character degree sequence of the group G.

Table 6.2: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[48,10]	$(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2)$
[48,12]	$(C_3 \rtimes_\phi C_4) \rtimes_\phi C_4$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2)$
[48,15]	$(C_3 \times D_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,4)$
[48,16]	$(C_3 \rtimes_\phi C_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,4)$
[48,17]	$(C_3 \times Q_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,4)$
[48,18]	$C_3 \rtimes_\phi \mathrm{Dic}_4$	5	$(1,1,1,1,2,2,2,2,2,2,2,4)$
[48,19]	$(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2)$
[48,28]	$\mathrm{SL}(2,3).C_2$	3	$(1,1,2,2,2,3,3,4)$
[48,29]	$\mathrm{GL}(2,3)$	3	$(1,1,2,2,2,3,3,4)$
[48,30]	$A_4 \rtimes_\phi C_4$	5	$(1,1,1,1,2,2,3,3,3,3)$
[48,33]	$\mathrm{SL}(2,3) \rtimes_\phi C_2$	7	$(1,1,1,1,1,1,2,2,2,2,2,2,3,3)$
[48,39]	$(C_2 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,4)$
[48,41]	$(C_4 \times S_3) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,4)$

In conclusion, we see that no group of order 48 is a $(*)$-group.

- ($|G| = 54$) There are a total of 15 groups of order 54, 9 of these are direct products and hence are not $(*)$-groups. Further, the dihedral group D_{54} is not a $(*)$-group. Thus the groups under consideration are as follows:

1. $((C_3 \times C_3) \rtimes_\phi C_3) \rtimes_\phi C_2$
2. $(C_9 \rtimes_\phi C_3) \rtimes_\phi C_2$
3. $(C_9 \times C_3) \rtimes_\phi C_2$
4. $((C_3 \times C_3) \rtimes_\phi C_3) \rtimes_\phi C_2$
5. $(C_3 \times C_3 \times C_3) \rtimes_\phi C_2$

Since $54 = 2 \cdot 3^3$, similar to the argument used for groups of order 36, the group $(C_3 \times C_3 \times C_3) \rtimes_\phi C_2$ is not a $(*)$-group.

Constructing a table similar to *Table 6.2* we obtain the following:

Table 6.3: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[54,5]	$((C_3 \times C_3) \rtimes_\phi C_3) \rtimes_\phi C_2$	7	$(1,1,1,1,1,1,2,2,2,6)$
[54,6]	$(C_9 \rtimes_\phi C_3) \rtimes_\phi C_2$	7	$(1,1,1,1,1,1,2,2,2,6)$
[54,7]	$(C_9 \times C_3) \rtimes_\phi C_2$	3	$(1,1,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[54,8]	$((C_3 \times C_3) \rtimes_\phi C_3) \rtimes_\phi C_2$	3	$(1,1,2,2,2,2,3,3,3,3)$

Therefore, we see that there are no $(*)$-groups of order 54.

To quicken the process of looking for $(*)$-groups, we will henceforth rule out groups which are quickly identified not to be $(*)$-groups and use a table similar to *Table 6.2*.

- $(|G| = 56)$ There are a total of 13 groups of order 56, 8 of these groups are direct products and one is D_{56}, thus these groups are not $(*)$-groups.

  Hence we consider the following groups:

  1. $C_7 \rtimes_\phi C_8$
  2. $C_7 \rtimes_\phi C_8$
  3. $(C_{14} \times C_2) \rtimes_\phi C_2$
  4. $\mathrm{AGL}_1(2^3)$

  By *Example 6.0.6*, $\mathrm{AGL}_1(2^3)$ is a $(*)$-group. Below, *Table 6.4* displays that the groups number 1,2 and 3 are not $(*)$-groups.

Table 6.4: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[56,1]	$C_7 \rtimes_\phi C_8$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2)$
[56,3]	$C_7 \rtimes_\phi C_8$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[56, 7]	$(C_{14} \times C_2) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2)$

In conclusion, $\mathrm{AGL}_1(2^3)$ is the only (∗)-group of order 56.

- ($|G| = 60$) A total of 9 of the 13 groups of order 60 are not (∗)-groups since they are direct products. Furthermore, we know that D_{60} is not a (∗)-group.

  If $G = C_{15} \rtimes_\phi C_4$ (there are two non-isomorphic groups with this structure), then $G'' = 1$ and so similar to the argument for groups of order 36, G is not a (∗)-group.

  Finally, if $G = A_5$, then $G/G'' \cong 1$ since $G' = A_5$. If we let (G, p) be a (∗)-group for some prime p, then by *Theorem 5.1*, it follows that G/G'' is non-abelian, this is a contradiction. Hence, G is not a (∗)-group.

  Therefore, there are no (∗)-groups of order 60.

- ($|G| = 66$) A total of 3 of the 4 groups of order 66 are direct products, the other being D_{66}. Hence there are no (∗)-groups of order 66.

- ($|G| = 72$) Of the 50 groups of order 72, we consider the 18 (excluding D_{72}) that are not direct products. These 18 groups are listed below:

  1. $C_9 \rtimes_\phi C_8$
  2. $Q_8 \rtimes_\phi C_9$
  3. $C_9 \rtimes_\phi Q_8$
  4. $(C_{18} \times C_2) \rtimes_\phi C_2$
  5. $(C_3 \times C_3) \rtimes_\phi C_8$
  6. $((C_2 \times C_2) \rtimes_\phi C_9) \rtimes_\phi C_2$
  7. $(C_3 \times C_3) \rtimes_\phi C_8$
  8. $(C_3 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
  9. $(C_6 \times S_3) \rtimes_\phi C_2$
  10. $(C_6 \times S_3) \rtimes_\phi C_2$
  11. $(C_3 \times C_3) \rtimes_\phi Q_8$
  12. $(C_3 \times C_3) \rtimes_\phi Q_8$
  13. $(C_{12} \times C_3) \rtimes_\phi C_2$
  14. $(C_6 \times C_6) \rtimes_\phi C_2$
  15. $\mathrm{AGL}_1(3^2)$
  16. $(S_3 \times S_3) \rtimes_\phi C_2$
  17. $(C_3 \times C_3) \rtimes_\phi Q_8$
  18. $(C_3 \times A_4) \rtimes_\phi C_2$

  By *Example 6.0.6*, we have that $\mathrm{AGL}_1(3^2)$ is a (∗)-group.

  Now the groups numbered 1,4,5,7,13 and 14 have a derived length of 2 since they are a semidirect product of an abelian group by an abelian group (see *Theorem 1.1.17*). If these groups where (∗)-groups then by *Theorem 5.1*, these groups would have to be isomorphic to $\mathrm{AGL}_1(3^2)$ as it must hold that

$$72 = |G| = |G : 1| = \left|G : G''\right| = |\mathrm{AGL}_1(p^n)|,$$

  which forces $p = 3$ and $n = 2$, where G is each of these groups (1,4,5,7,13 or 14). But none of these groups are isomorphic to $\mathrm{AGL}_1(3^2)$, hence the groups numbered 1,4,5,7,13 and 14 can not be (∗)-groups.

  Thus the remaining groups are given by:

1. $Q_8 \rtimes_\phi C_9$
2. $C_9 \rtimes_\phi Q_8$
3. $((C_2 \times C_2) \rtimes_\phi C_9) \rtimes_\phi C_2$
4. $(C_3 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$
5. $(C_6 \times S_3) \rtimes_\phi C_2$
6. $(C_6 \times S_3) \rtimes_\phi C_2$
7. $(C_3 \times C_3) \rtimes_\phi Q_8$
8. $(C_3 \times C_3) \rtimes_\phi Q_8$
9. $(S_3 \times S_3) \rtimes_\phi C_2$
10. $(C_3 \times C_3) \rtimes_\phi Q_8$
11. $(C_3 \times A_4) \rtimes_\phi C_2$

Table 6.5: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[72,1]	$Q_8 \rtimes_\phi C_9$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[72,3]	$C_9 \rtimes_\phi Q_8$	5	$(1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,3,3,3)$
[72,15]	$((C_2 \times C_2) \rtimes_\phi C_9) \rtimes_\phi C_2$	3	$(1,1,2,2,2,2,3,3,6)$
[72,21]	$(C_3 \times (C_3 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,4,4)$
[72,22]	$(C_6 \times S_3) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,4,4)$
[72,23]	$(C_6 \times S_3) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,4,4)$
[72,24]	$(C_3 \times C_3) \rtimes_\phi Q_8$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,4,4)$
[72,31]	$(C_3 \times C_3) \rtimes_\phi Q_8$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[72,40]	$(S_3 \times S_3) \rtimes_\phi C_2$	5	$(1,1,1,1,2,4,4,4,4)$
[72,41]	$(C_3 \times C_3) \rtimes_\phi Q_8$	5	$(1,1,1,1,2,8)$
[72,43]	$(C_3 \times A_4) \rtimes_\phi C_2$	3	$(1,1,2,2,2,2,3,3,6)$

Hence we see that the only (*)-group of order 72 is $\mathrm{AGL}_1(3^2)$.

- ($|G| = 78$) There are a total of 6 group of order 78, 4 of these group are direct products and hence no (*)-groups. The group D_{78} is also not a (*)-group.

  For the group $G = (C_{13} \rtimes_\phi C_3) \rtimes_\phi C_2$, we have $|G : G'| = 6$. Hence if (G, p) is a (*)-group for some prime p, it follows by *Theorem 5.1*, that $p = 7$ since $7^1 - 1 = 6 = |G : G'|$. This is a contradiction since $7 \nmid 78$. Thus G is not a (*)-group.

  In conclusion, there are no (*)-groups of order 78.

- ($|G| = 80$) Of the 52 total groups of order 80, 28 are direct products and so are not (*)-groups. The group D_{80} is also not a (*)-group. The list of the remaining groups is given below:

1. $C_5 \rtimes_\phi C_{16}$
2. $C_5 \rtimes_\phi C_{16}$
3. $C_{40} \rtimes_\phi C_2$
4. $C_{40} \rtimes_\phi C_2$
5. $C_5 \rtimes_\phi C_{16}$
6. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
7. $(C_5 \rtimes_\phi C_4) \rtimes_\phi C_4$
8. $C_{20} \rtimes_\phi C_4$
9. $(C_{20} \times C_2) \rtimes_\phi C_2$
10. $(C_5 \times D_8) \rtimes_\phi C_2$
11. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
12. $(C_5 \times Q_8) \rtimes_\phi C_2$
13. $C_5 \rtimes_\phi \mathrm{Dic}_4$
14. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
15. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
16. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
17. $C_{20} \rtimes_\phi C_4$
18. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
19. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
20. $(C_{20} \times C_2) \rtimes_\phi C_2$
21. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
22. $(C_4 \times D_{10}) \rtimes_\phi C_2$
23. $(C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_5$

The groups numbered 1,2,3,4,5,8,9,17,20 and 23 are of derived length 2, hence since $80 = 2^4 \times 5$ it follows by *Lemma 6.3.1*, that these groups are not (∗)-groups. The remaining group are listed below:

1. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
2. $(C_5 \rtimes_\phi C_4) \rtimes_\phi C_4$
3. $(C_5 \times D_8) \rtimes_\phi C_2$
4. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
5. $(C_5 \times Q_8) \rtimes_\phi C_2$
6. $C_5 \rtimes_\phi \mathrm{Dic}_4$
7. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
8. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
9. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
10. $(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$
11. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
12. $(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$
13. $(C_4 \times D_{10}) \rtimes_\phi C_2$

For these groups, consider the following table:

Table 6.6: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[80,10]	$(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[80,12]	$(C_5 \rtimes_\phi C_4) \rtimes_\phi C_4$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[80,15]	$(C_5 \times D_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,4,4)$
[80,16]	$(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,4,4)$
[80,17]	$(C_5 \times Q_8) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,4,4)$
[80,18]	$C_5 \rtimes_\phi \mathrm{Dic}_4$	5	$(1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,4,4)$
[80,19]	$(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2)$
[80,28]	$(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$	17	$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4,4,4,4)$
[80,29]	$(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,4,4,4,4)$
[80,33]	$(C_5 \rtimes_\phi C_8) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,4,4,4,4)$
[80,34]	$(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,4,4,4,4)$
[80,40]	$(C_2 \times (C_5 \rtimes_\phi C_4)) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,4,4)$
[80,42]	$(C_4 \times D_{10}) \rtimes_\phi C_2$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,4,4)$

Thus since none of the groups in *Table 6.6* are (∗)-groups, it follows that there are no (∗)-groups of order 80.

- ($|G| = 84$) Of the 15 total groups of order 84, 11 of them are direct products and one is D_{84}; hence these 12 groups are not (∗)-groups. The remaining groups are given by

  1. $(C_7 \rtimes_\phi C_4) \rtimes_\phi C_3$
  2. $C_{21} \rtimes_\phi C_4$
  3. $(C_{14} \times C_2) \rtimes_\phi C_3$

  The groups numbered 2 and 3 are of derived length 2 and so are not (∗)-groups by *Lemma 6.3.1* ($84 = 2^2 \cdot 3 \cdot 7$).

  Furthermore, we have that $\mathrm{Deg}((C_7 \rtimes_\phi C_4) \rtimes_\phi C_3) = (1,1,1,1,1,1,1,1,1,1,1,1,6,6)$, thus since the non-linear irreducible characters of $(C_7 \rtimes_\phi C_4) \rtimes_\phi C_3$ have the same degrees, this group can not be a (∗)-group.

  Therefore, there are no (∗)-groups of order 84.

- ($|G| = 90$) Excluding D_{90}, which is not a (∗)-group, 8 of the 10 groups of order 90 are direct products and hence not (∗)-groups. Now $(C_{15} \times C_3) \rtimes_\phi C_2$ is of derived length 2 and so cannot be a (∗)-group by *Lemma 6.3.1* ($90 = 2 \cdot 3^2 \cdot 5$).

  Thus there are no (∗)-groups of order 90.

- ($|G| = 96$) There are 231 groups of order 96. Due to the sheer number of groups we will not list groups which are we can readily rule out. These groups include groups which are direct products and groups of derived length 2 (note that $100 = 2^5{\times}3$ and see *Lemma 6.3.1*) as these can not be (*)-groups. We thus have the following remaining groups:

  1. $((C_4 \times C_2) \rtimes_\phi C_4) \rtimes_\phi C_3$
  2. $((C_4 \times C_4) \rtimes_\phi C_3) \rtimes_\phi C_2$
  3. $A_4 \rtimes_\phi C_8$
  4. $\mathrm{SL}(2,3) \rtimes_\phi C_4$
  5. $\mathrm{SL}(2,3) \rtimes_\phi C_4$
  6. $((C_8 \times C_2) \rtimes_\phi C_2) \rtimes_\phi C_3$
  7. $A_4 \rtimes_\phi Q_8$
  8. $(C_2 \times S_4) \rtimes_\phi C_2$
  9. $(C_2 \times \mathrm{SL}(2,3)) \rtimes_\phi C_2$
  10. $(\mathrm{SL}(2,3).C_2) \rtimes_\phi C_2$
  11. $(\mathrm{SL}(2,3).C_2) \rtimes_\phi C_2$
  12. $(\mathrm{SL}(2,3) \rtimes_\phi C_2) \rtimes_\phi C_2$
  13. $(C_2 \times C_2 \times A_4) \rtimes_\phi C_2$
  14. $(\mathrm{SL}(2,3) \rtimes_\phi C_2) \rtimes_\phi C_2$
  15. $(C_2 \times \mathrm{SL}(2,3)) \rtimes_\phi C_2$
  16. $(C_2 \times C_2 \times Q_8) \rtimes_\phi C_3$
  17. $((C_2 \times D_8) \rtimes_\phi C_2) \rtimes_\phi C_3$
  18. $((C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3) \rtimes_\phi C_2$

  Now consider the table below:

Table 6.7: Table with character degree sequence:

GAP ID	Group name	p	Deg(G)
[96,3]	$((C_4 \times C_2) \rtimes_\phi C_4) \rtimes_\phi C_3$	2	$(1,1,1,2,2,2,3,3,3,3,3,6)$
[96,64]	$((C_4 \times C_4) \rtimes_\phi C_3) \rtimes_\phi C_2$	3	$(1,1,2,3,3,3,3,3,3,6)$
[96,65]	$A_4 \rtimes_\phi C_8$	3	$(1,1,1,1,1,1,1,1,2,2,2,2,3,3,3,3,3,3,3,3)$
[96,66]	$\mathrm{SL}(2,3) \rtimes_\phi C_4$	5	$(1,1,1,1,2,2,2,2,2,2,3,3,3,3,4,4)$
[96,67]	$\mathrm{SL}(2,3) \rtimes_\phi C_4$	5	$(1,1,1,1,2,2,2,2,2,2,3,3,3,3,4,4)$
[96,74]	$((C_8 \times C_2) \rtimes_\phi C_2) \rtimes_\phi C_3$	13	$(1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3)$
[96,185]	$A_4 \rtimes_\phi Q_8$	5	$(1,1,1,1,2,2,2,2,2,3,3,3,3,6)$
[96,187]	$(C_2 \times S_4) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,3,3,3,3,6)$
[96,190]	$(C_2 \times \mathrm{SL}(2,3)) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,3,3,3,3,4,4,4)$
[96,191]	$(\mathrm{SL}(2,3).C_2) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,3,3,3,3,4,4,4)$
[96,192]	$(\mathrm{SL}(2,3).C_2) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,2,3,3,3,3,4,4)$
[96,193]	$(\mathrm{SL}(2,3) \rtimes_\phi C_2) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,3,3,3,3,4,4,4)$
[96,195]	$(C_2 \times C_2 \times A_4) \rtimes_\phi C_2$	5	$(1,1,1,1,2,2,2,2,2,3,3,3,3,6)$
[96,201]	$(\mathrm{SL}(2,3) \rtimes_\phi C_2) \rtimes_\phi C_2$	13	$(1,1,1,1,1,1,1,1,1,1,1,1,3,3,3,3,4,4,4)$
[96,202]	$(C_2 \times \mathrm{SL}(2,3)) \rtimes_\phi C_2$	13	$(1,1,1,1,1,1,1,1,1,1,1,1,3,3,3,3,4,4,4)$
[96,203]	$(C_2 \times C_2 \times Q_8) \rtimes_\phi C_3$	2	$(1,1,1,2,2,2,3,3,3,3,3,6)$
[96,204]	$((C_2 \times D_8) \rtimes_\phi C_2) \rtimes_\phi C_3$	2	$(1,1,1,3,3,3,3,3,4,4,4)$
[96,227]	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3) \rtimes_\phi C_2$	3	$(1,1,2,3,3,3,3,3,3,6)$

Thus from the table on the previous page, the groups $((C_4 \times C_4) \rtimes_\phi C_3) \rtimes_\phi C_2$ and $((C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3) \rtimes_\phi C_2$ are $(*)$-groups.

We confirm, using *Theorem 5.1*, that $G = ((C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3) \rtimes_\phi C_2$ is a $(*)$-group. A similar analysis can be carried out for $((C_4 \times C_4) \rtimes_\phi C_3) \rtimes_\phi C_2$.

Using GAP, we are able to get the following information on the group G. The derived subgroup of G is given by $G' = (C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3$. This implies that $G/G' \cong C_2$; that is, we have that G/G' is cyclic of order $2 = 3^1 - 1$. Moreover, the character degree sequence of G' is $\text{Deg}(G') = (1, 1, 1, 3, 3, 3, 3, 3)$. Hence G' is a Thompson group.

The second derived subgroup of G is given by $G'' = C_2 \times C_2 \times C_2 \times C_2$. This implies that $|G : G''| = 6$. If G/G'' is abelian, then by *Theorem 1.1.17* it follows that $G' = (C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3 = C_2 \times C_2 \times C_2 \times C_2 = G''$, which is a contradiction. Thus it follows that $G/G'' \cong S_3 \cong \text{AGL}_1(3)$.

Let $\text{Irr}_1(G, 3') = \{\chi\}$. Using the following GAP commands:

```
gap> G := SmallGroup(96,227);
gap> x := Irr(G);
gap> K := Kernel(x[3]);
gap> StructureDescription(K);
```

we are able to find that $\ker \chi = C_2 \times C_2 \times C_2 \times C_2 = G''$. From the facts established for the group G, *Theorem 5.1* also confirms G to be a $(*)$-group.

- ($|G| = 100$) There are a total 16 group of order 100. Of these 16 groups 9 are direct products and one is D_{100} and hence are not $(*)$-groups. The remaining groups are shown below:

  1. $C_{25} \rtimes_\phi C_4$
  2. $C_{25} \rtimes_\phi C_4$
  3. $(C_5 \times C_5) \rtimes_\phi C_4$
  4. $(C_5 \times C_5) \rtimes_\phi C_4$
  5. $(C_5 \times C_5) \rtimes_\phi C_4$
  6. $(C_5 \times C_5) \rtimes_\phi C_4$
  7. $(C_5 \times C_5) \rtimes_\phi C_4$

These numbered groups are of derived length 2 and so by *Lemma 6.3.1*, they are not $(*)$-groups.

We have thus shown that there are no $(*)$-groups of order 100.

Therefore, we have the following theorem:

Theorem 6.3.10. *Let G be a $(*)$-group such that $|G| \leq 100$, then G is one of the following groups:*

GAP ID	Group name
[6,1]	S_3
[12,3]	A_4
[20,3]	$\mathrm{AGL}_1(5)$
[24,3]	$SL(2,3)$
[24,12]	S_4
[42,1]	$\mathrm{AGL}_1(7)$
[56,11]	$\mathrm{AGL}_1(2^3)$
[72,39]	$\mathrm{AGL}_1(3^2)$
[96,64]	$((C_4 \times C_4) \rtimes_\phi C_3) \rtimes_\phi C_2$
[96, 227]	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes_\phi C_3) \rtimes_\phi C_2$

Closing remarks

We have seen that not all $(*)$-groups are Frobenius. We could ask, what is the general structure of Frobenius $(*)$-groups? Are they always a semidirect product of a group of its derived subgroup of order p^n by some abelian group of order $p^n - 1$? Can we classify such groups? A *metacyclic group* is defined as a group G which has a normal cyclic subgroup N such that G/N is cyclic. A *supersolvable* group is a group that contains a normal series with cyclic factors. A Metacyclic group can be shown to be supersolvable. The Frobenius group $C_p \rtimes C_{p-1}$ is metacyclic and thus supersolvable. Thus we have an example of an infinite family of supersolvable $(*)$-groups. Since we have an example of a supersolvable $(*)$-group and know that all $(*)$-groups are solvable; could we classify supersolvable $(*)$-groups? Is the class of supersolvable $(*)$-groups strictly smaller than the class of general $(*)$-groups? (Yes, consider the $(*)$-group S_4). Regrettably, these are worthwhile questions I could not incorporate into this dissertation.

Groups having one irreducible character of p'-degree have a fascinating structure. We see a strong interplay between Thompson groups, that is, groups having characters whose degrees are of p-power and these $(*)$-groups. The derived subgroup of these groups is Thompson. Moreover, $(*)$-group can be separated into a “top part” which has exactly one non-linear irreducible character (which is of p' degree), namely G/G'', and a “bottom part” G''. We see how Seitz’s theorem begins to materialise here. Considering all these ideas together is far from alien in hindsight, but translating these ideas into a concrete mathematical fact is a sight to behold. The work by Kazarin and Berkovich is remarkable.

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