The matrix $2M + 2\rho_1\delta t^2 K$ is invertible since

$$[2M + 2\rho_1\delta t^2 K]\bar{x} \cdot \bar{x} = 2M\bar{x} \cdot \bar{x} + 2\rho_1\delta t^2 K\bar{x} \cdot \bar{x} > 0,$$

for $\bar{x} \neq \bar{0}$. (In other notation $X^t[2M + 2\rho_1\delta t^2 K]X > 0$ for $X \neq 0$.)

This implies that $\bar{u}_1$ is uniquely determined. Now consider $\bar{u}_{k+1}$:

$$[M + \frac{\delta t}{2}\, C + \delta t^2\rho_1\, K]\, \bar{u}_{k+1} = 2M\bar{u}_k - M\bar{u}_{k-1} + \frac{\delta t}{2}\, C\, \bar{u}_{k-1}$$
$$-\quad \delta t^2\, K\, [\, \rho_0\bar{u}_k + \rho_1\bar{u}_{k-1}\,] + \delta t^2\, [\rho_1 F(t_{k+1}) + \rho_1 F(t_k) + \rho_0 F(t_{k-1})]\,.$$

Again,

$$[M + \frac{\delta t}{2}\, C + \delta t^2\rho_1\, K]\bar{x} \cdot \bar{x} > 0,$$

for $\bar{x} \neq \bar{0}$ and the inverse of $[M + \frac{\delta t}{2}\, C + \delta t^2\rho_1\, K]$ exists. Therefore $\bar{u}_{k+1}$ is uniquely determined. □

Convergence for the system of ordinary differential equations

As mentioned before, Chapter 9 in [H] is devoted to various algorithms including Newmark schemes. We find the following statement in Section 9.1: "Again, stability plus consistency implies convergence." A rather general approach to Newmark schemes is followed leaving a number of proofs as exercises. In Exercises 2 and 3 on p.495 the central difference average acceleration algorithm is derived from the general Newmark scheme. Actually these are examples, leaving only elementary calculations to the reader. It is also stated in Section 9.1 that the central difference average acceleration algorithm is unconditionally stable. No proof is given but three references are given.

7.4.2 The error

Our aim is to prove convergence of the solution of the fully discrete problem to the solution of the the general second order problem and derive error estimates. Consider the error

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h].$$

Estimates for the error $u(t_k) - u_h(t_k)$ were obtained in Chapters 2 to 5 and estimates for the error $\bar{u}(t_k) - \bar{u}_k$ are available in [H, Chapter 9]. But what

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is required, is an estimate for $\|u_h(t_k) - u_k^h\|_W$ and there is no indication in [H, Chapter 9] how to obtain this. The natural approach is to consider the fully discrete problem in variational form as in [D73].

Problem G^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for each k,

$$\begin{aligned} & c(\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\delta t)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ + \ & b(\rho_1 u_{k+1}^h + \rho_0 u_k^h + \rho_1 u_{k-1}^h, v) = (\rho_1 f(t_{k+1}) + \rho_0 f(t_k) + \rho_1 f(t_{k-1}), v)_X, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\delta t v^h$.

The algorithm for the wave equation without damping is the scheme (3.2) [D73, p.886] and (3.10) [D73, p.888] for boundary damping. Problem G^h-D above is the general version. This approach was followed by others e.g. [B76] and [Z-PhD]. However, in [Z-PhD] an estimate for $u_h(t_k) - u_k^h$ is obtained while in [D73] (and others) the error $u(t_k) - u_k^h$ is estimated directly. In this section we follow the approach in [Z-PhD] and discuss the "direct approach" in Section 7.5.

It is convenient to introduce the following transformation. For any $\bar{x} \in R_n$, let

$$T_h \bar{x} = \sum_{i=1}^{n} x_i \phi_i \in S^h.$$

The mapping T_h is obviously a linear bijection.

Proposition 7.4.2 *The sequence $\{\bar{u}_k\}$ is a solution of Problem G-ODE-D if and only if $\{u_k^h\} = \{T_h \bar{x}\}$ is a solution of Problem G^h-D.*

Proof The proof is similar to the proof of Proposition 1 in Section 7.1 (only easier). □

Corollary 7.4.1 *Problem G^h-D has a unique solution for any pair of functions d^h and v^h in S^h.*

Remark
We consider Problem G^h-D for theory but Problem G-ODE-D is used for computation.

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The objective is to prove that $\|u_h(t_k) - u_k^h\|_W \leq K_1 \delta t^2$ for some constant K_1. Then, since $\|u(t_k) - u_h(t_k)\|_W \leq K_2 h^\kappa$ for some constant K_2 and some positive integer κ, the estimate below follows.

Error estimate

$$\begin{aligned} \|u(t_k) - u_k^h\|_W &\leq \|u(t_k) - u_h(t_k)\|_W + \|u_h(t_k) - u_k^h\|_W \\ &\leq K_2 h^\kappa + K_1 \delta t^2. \end{aligned}$$

7.4.3 The local error

If $f \in C^2(J)$, then $u_h \in C^4(J)$ and optimal error estimates can be obtained. The first step is to substitute the solution of problem G^h into Problem G^h-D and estimate the truncation error. The procedure, using Taylor expansions, is standard. Note that $\rho_0 = \frac{1}{4}$ and $\rho_1 = \frac{1}{2}$ for the average acceleration method.

Proposition 7.4.3 *If $u_h \in C^4(J)$ is the solution of Problem G^h, then*

$$\begin{aligned} & 2c(u_h(\delta t), v) + \frac{\delta t^2}{2} b(u_h(\delta t), v) = 2c(d^h, v) - \frac{\delta t^2}{2} b(d^h, v) \\ - \; & 2\delta t c(v^h, v) - \delta t^2 \; a(v^h, v) + \frac{\delta t^3}{2} \; b(v^h, v) + \frac{\delta t^2}{2}(f(\delta t) + f(0), v)_X \\ + \; & c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \leq \max_J \|u_h^{(4)}(t)\|_W \delta t^3$ and $\|r_2^h\|_E \leq \max_J \|u_h^{(4)}(t)\|_E \delta t^3$.

Proposition 7.4.4 *If $u_h \in C^4(J)$ is the solution of Problem G^h, then*

$$\begin{aligned} & c(u_h(t + \delta t) - 2u_h(t) + u_h(t - \delta t), v) \\ + \; & \frac{\delta t}{2} a(u_h(t + \delta t) - u_h(t - \delta t), v) \\ + \; & \frac{\delta t^2}{4} b(u_h(t + \delta t) + 2u_h(t) + u_h(t - \delta t, v) \\ = \; & \frac{\delta t^2}{4}(f(t + \delta t) + 2f(t) + f(t - \delta t), v)_X \\ + \; & c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \leq \max_J \|u_h^{(4)}(t)\|_W \delta t^3$ and $\|r_2^h\|_E \leq \max_J \|u_h^{(4)}(t)\|_E \delta t^3$.

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Lemma 7.4.1 *If $u_h \in C^4(J)$ is the solution of Problem G^h and $\{u_k^h\}$ the solution of Problem G^h-D, then*

$$\|u_h(\delta t) - u_1^h\|_W \le \|e_1^h\|_W + \delta t \|e_2^h\|_E \|u_h(\delta t) - u_1^h\|_E \|u_h(\delta t) - u_1^h\|_W^{-1}.$$

Proof Recall that

$$\begin{aligned} & c(u_1^h - 2u_0^h + u_{-1}^h, v) + \frac{\delta t}{2} a(u_1^h - u_{-1}^h, v) \\ + \; & \frac{\delta t^2}{4}\, b(u_1^h + 2u_0^h + u_{-1}^h, v) = \frac{\delta t^2}{4}\, (f(t_1) + 2f(t_0) + f(t_{-1}), v)_X, \end{aligned}$$

for each $v \in S^h$, while $u_0^h = d^h$, $u_1^h - u_{-1}^h = 2\delta t v^h$ and $f(t_{-1}) = f(t_1)$. It follows that

$$\begin{aligned} & c(2u_1^h, v) + \frac{\delta t^2}{2}\, b(u_1^h, v) = 2c(d^h, v) - \frac{\delta t^2}{2}\, b(d^h, v) \\ + \; & 2\delta t c(v^h, v) - \delta t^2\, a(v^h, v) + \frac{\delta t^3}{2}\, b(v^h, v) + \frac{\delta t^2}{2}\, (f(t_1) + f(t_0), v)_X, \end{aligned}$$

for each $v \in S^h$. By Proposition 7.4.3

$$\begin{aligned} & 2c(u_h(\delta t), v) + \frac{\delta t^2}{2} b(u_h(\delta t), v) = 2c(d^h, v) - \frac{\delta t^2}{2} b(d^h, v) \\ - \; & 2\delta t c(v^h, v) - \delta t^2\, a(v^h, v) + \frac{\delta t^3}{2}\, b(v^h, v) + \frac{\delta t^2}{2} (f(\delta t) + f(0), v)_X \\ + \; & c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \le K\delta t^3$ and $\|r_2^h\|_E \le K\delta t^3$. Consequently

$$2c(u_h(\delta t) - u_1^h, v) + \frac{\delta t^2}{2} b(u_h(\delta t) - u_1^h, v) = c(r_1^h, v) + a(r_2^h, v),$$

for each $v \in S^h$. Therefore

$$\begin{aligned} \|u_h(\delta t) - u_1^h\|_W^2 \;&\le\; \frac{1}{2}|c(e_1^h, u_h(\delta t) - u_1^h)| + \frac{\delta t}{2}|a(e_2^h, u_h(\delta t) - u_1^h)| \\ &\le\; \|e_1^h\|_W \|u_h(\delta t) - u_1^h\|_W + \delta t \|e_2^h\|_E \|u_h(\delta t) - u_1^h\|_E \end{aligned}$$

and the desired estimate follows. □

Corollary 7.4.2 *For δt sufficiently small,*

$$\|u_h(\delta t) - u_1^h\|_W \le K \max_J \|u_h^{(4)}(t)\|_W \delta t^3.$$

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Lemma 7.4.2 *Suppose* $u_h \in C^4(J)$ *is the solution of Problem* G^h *and* $u^+ = u^+(u_h, t, \delta t)$ *is defined by*

$$\begin{aligned} & c(u^+ - 2u_h(t) + u_h(t-\delta t), v) \\ + & \frac{\delta t}{2} a(u^+ - u_h(t-\delta t), v) \\ + & \delta t^2 b(\rho_1 u^+ + \rho_0 u_h(t) + \rho_1 u_h(t-\delta t, v) \\ = & \delta t^2 (\rho_1 f(t+\delta t) + \rho_0 f(t) + \rho_1 f(t-\delta t), v)_X, \end{aligned}$$

for each $v \in S^h$*. If* $\Delta u = u_h(t+\delta t) - u^+$*, then*

$$\|\Delta u\|_W \leq \|e_1^h\|_W + \delta t \|e_2^h\|_E \|\Delta u\|_E \|\Delta u\|_W^{-1}.$$

Proof Using the result of Proposition 7.4.4, we have

$$c(\Delta u, v) + \frac{\delta t}{2} a(\Delta u, v) + \rho_1 \delta t^2 b(\Delta u, v) = c(e_1^h, v) + a(e_2^h, v),$$

for each $v \in S^h$. Therefore

$$\begin{aligned} \|\Delta u\|_W^2 & \leq |c(e_1^h, \Delta u)| + \delta t |a(e_2^h, \Delta u)| \\ & \leq \|e_1^h\|_W \|\Delta u\|_W + \delta t \|e_2^h\|_E \|\Delta u\|_E. \end{aligned}$$

This yields the desired estimate. □

Corollary 7.4.3 *For* δt *sufficiently small,*

$$\|\Delta u\|_W \leq K \max_J \|u_h^{(4)}(t)\|_W \delta t^3.$$

7.4.4 Stability and convergence

To obtain an estimate for the global error, we need a a stability result. Following [Z-PhD], we prove that the error growth is bounded by some constant times the number of time steps. It then follows from the estimate for the local error that

$$\|u_h(t_k) - u_k^h\|_W \leq K_1 \delta t^2.$$

Suppose $\{u_k^h\}$ and $\{w_k^h\}$ are solutions of Problem G^h-D with different initial values. Let $e_k = u_k^h - w_k^h$, then e_k is also a solution of Problem G^h-D but with $f = 0$:

$$\begin{aligned} & c(e_{k+1} - 2e_k + e_{k-1}, v) + \frac{\delta t}{2} a(e_{k+1} - e_{k-1}, v) \\ + & \delta t^2 b(\rho_1 e_{k+1} + \rho_0 e_k + \rho_1 e_{k-1}, v) = 0, \end{aligned} \quad (7.4.1)$$

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for each $v \in S^h$.

Following [Z-PhD] we assume modal damping. It is then possible to consider each mode individually using eigenvectors.

Eigenvalue problem

The vector w^h is an eigenvector with corresponding eigenvalue λ if

$$b(w^h, v) = \lambda c(w^h, v) \text{ for each } v \in S^h. \tag{7.4.2}$$

Let $w^h = T_h \bar{w}$, then w^h is an eigenvector with corresponding eigenvalue λ if and only if $K\bar{w} = \lambda M\bar{w}$. Since K and M are symmetric, there exist n mutually orthogonal eigenvectors for the matrix eigenvalue problem and it follows that there are n linearly independent eigenvectors for (7.4.2). But eigenvectors corresponding to different eigenvalues are orthogonal with respect to the inner product c. Therefore we may assume the existence of a set of orthonormal eigenvectors $\{w_1^h,\ w_2^h, \ldots,\ w_n^h\}$ which forms a basis for S^h.

Consider a solution $\{e_k\}$ of (7.4.1). For $k = 1$ to N,

$$e_k = \sum_{j=1}^{n} r_{k,j}\ w_j^h \text{ for } j = 1,\ 2, \ldots,\ n.$$

If we substitute for e_k in (7.4.1), we find that

$$\begin{aligned} & \sum_{j=1}^{n} (r_{k+1,j} - 2r_{k,j} + r_{k-1,j}) c(w_j^h, v) \\ + \quad & \frac{\delta t}{2} \sum_{j=1}^{n} (r_{k+1,j} - r_{k-1,j}) a(w_j^h, v) \\ + \quad & \delta t^2 \sum_{j=1}^{n} (\rho_1 r_{k+1,j} + \rho_0 r_{k,j} + \rho_1 r_{k-1}, j) b(w_j^h, v) = 0, \end{aligned}$$

for each $v \in S^h$. But for each $v \in S^h$ we have

$$\begin{aligned} b(w_j^h, v) &= \lambda_j c(w_j^h, v), \\ a(w_j^h, v) &= \mu b(w_j^h, v) + kc(w_j^h, v) = (\mu\lambda_j + k)c(w_j^h, v). \end{aligned}$$

Consequently, for each mode,

$$\begin{aligned} & (r_{k+1} - 2r_k + r_{k-1}) + (\mu\lambda + k)\frac{\delta t}{2}(r_{k+1} - r_{k-1}) \\ + \quad & \lambda\delta t^2(\rho_1 r_{k+1} + \rho_0 r_k + \rho_1 r_{k-1})) = 0, \end{aligned}$$

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with the subscript j dropped.

Consider the equation

$$\alpha r_{k+1} + \beta r_k + \gamma r_{k-1} = 0,$$

where

$$\begin{aligned}
\alpha &= 1 + (\mu\lambda + k)\frac{\delta t}{2} + \lambda \delta t^2 \rho_1, \\
\beta &= -2 + \lambda \delta t^2 \rho_0, \\
\gamma &= 1 - (\mu\lambda + k)\frac{\delta t}{2} + \lambda \delta t^2 \rho_1.
\end{aligned}$$

It has a solution $r_k = r^k$ where

$$\alpha r^2 + \beta r + \gamma = 0.$$

If $\rho_0 \leq 2\rho_1$, it follows by direct calculation that $|r| \leq 1$ for real or imaginary roots. Consequently r_k is bounded by $|r_1|$ and $|r_0|$. Therefore we have the following stability result.

Lemma 7.4.3 *If $\rho_0 \leq 2\rho_1$, then there exists a $K > 0$ such that*

$$\|e_k\|_W \leq K(\|e_1\|_W + \|e_0\|_W).$$

Remarks

1. For the average acceleration method $\rho_0 = \frac{1}{2}$ and $\rho_1 = \frac{1}{4}$. Therefore the method is unconditionally stable.
2. A different approach is followed in [D73] and [B76], see the next section.
3. Recall that the problem without damping is a special case of modal damping.

7.5 Direct convergence result for the fully discrete approximation

In [D73] Dupont does not estimate $\|u_h(t_k) - u_k^h\|_W$, but $\|u(t_k) - u_k^h\|_W$ directly. The disadvantage of this approach is that he assumes the existence of a 4-th order time derivative for the exact solution. This is a very restrictive

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assumption as one can see with the one-dimensional wave equation using d'Alembert's method. On the other hand, it is quite reasonable to assume that the solution u_h of Problem G^h has a derivative of order four (as was done in Section 7.4). Note that only the wave equation is considered in [D73] where the cases with no damping and boundary damping are considered separately.

For the undamped case Scheme (3.2) in [D73, p.886] is used:

$$\begin{aligned} & (\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + \frac{1}{4}\, b(\theta u_{k+1}^h + (1-2\theta)u_k^h + \theta u_{k-1}^h, v) \\ = \; & \frac{1}{4}\, (\theta f(t_{k+1}) + (1-2\theta) f(t_k) + \theta f(t_{k-1}), v) \end{aligned}$$

for each $v \in S^h$. Note that $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

We find the following paragraph at the bottom of p.886: "This is a second order correct in Δt approximation to (2.5). The reason for the choice $\theta = 1/4$ is that for $\theta \geq 1/4$ the analogues of (3.2) are stable independently of Δt and S^h, and the time truncation is minimized over this class by taking $\theta = 1/4$." No references are given.

For boundary damping Scheme (3.10) in [D73, p.888] with $\theta = 1/4$, is used:

$$\begin{aligned} & (\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + (2\delta t)^{-1}\langle u_{k+1}^h - u_{k-1}^h, v\rangle \\ + \; & \frac{1}{4}\, b(\theta u_{k+1}^h + (1-2\theta)u_k^h + \theta u_{k-1}^h, v) \\ = \; & \frac{1}{4}\, (\theta f(t_{k+1}) + (1-2\theta) f(t_k) + \theta f(t_{k-1}), v) \\ + \; & \frac{1}{4}\, \langle(\theta g(t_{k+1}) + (1-2\theta) g(t_k) + \theta g(t_{k-1}), v\rangle, \end{aligned}$$

for each $v \in S^h$. Stability is not even mentioned.

The convergence results are Theorem 3 in [D73, p.888] for the undamped case and Theorem 4 in [D73, p.889] for boundary damping. It needs to be emphasized that in both cases the existence of a 4-th order time derivative for the exact solution is assumed.

The book [OR] appeared in 1976. Section 9.6 deals with "Hyperbolic equations of second order." However, only the undamped case is considered. For the fully discrete approximation central differences is used. The authors also consider the direct approach as in [D73]. In Remark 1 on p.415 we read: "The techniques used here represent adaptations of Dupont [D73] to an explicit fully discrete scheme ...". The convergence result suffers from the

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same limitation as the results in [D73]: " ... following Dupont [D73], we assume that the (exact) solution is such that $\partial^4 u/\partial t^4 \in L^2(L^2(\Omega))$."

Following [D73], Baker in [B76] also derive a direct convergence result and with the same limitation. Recall that the convergence result for the semi-discrete problem in [B76] has the advantage that less regularity is assumed. But this advantage is lost when convergence for the fully discrete case is treated. Stability is not mentioned but it is implicit in Lemma 4.2. on p.570. The proof starts on p.571 and continues to the middle of p.575. But the problem in [B76] is for the wave equation without damping which may be treated the same as modal damping, and the assumption regarding a 4-th order time derivative for the exact solution is clearly not necessary.

The idea to estimate $\|u(t_k) - u_k^h\|_W$ directly (as in [D73]) is used in other articles, e.g.[S94], [FXX99], [Wu03], [Wu05] and [Wu06]. In all these papers the existence of a fourth order time derivative for the exact solution is assumed.

7.6 An explicit method for large systems

As mentioned in Section 7.2, any finite element method tends to be implicit due to the presence of the mass matrix M, but that it was changed to a diagonal matrix by so called mass lumping. The idea was based on intuition and was without mathematical justification. However, the numerical results were good and stability improved (as mentioned before). In stead of

$$\delta t^{-2}\ M[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] = K\bar{u}_k + F(t_k),$$

the system

$$\delta t^{-2}\ D[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] = K\bar{u}_k + F(t_k)$$

is considered. The matrix M is replaced by the diagonal matrix D. The system is now essentially explicit since it is a trivial matter to compute D^{-1}. This explicit central difference method is of course not unconditionally stable and it is necessary to consider stability criteria. However, it has advantages for large systems that arise in engineering applications.

In the introduction to [Wu06] the importance of the explicit method is stressed. "The explicit finite element method has been extensively developed for the transient dynamic analysis to meet the increasing demand of engineering application." As mentioned above, a diagonal mass matrix is needed

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for an explicit method. In Section 3 of the paper, piecewise linear and bilinear basis functions are considered for one-, two- and three-dimensional cases. The equivalent diagonal mass matrices are derived and the errors estimated in Lemma 3.3. Using this result it is proved in Theorem 4.1 that the rate of convergence is the same as for using consistent mass matrices.

The procedure followed by Wu is best explained in [Wu03]. In the abstract it is formulated: "The explicit finite element method for transient dynamics of linear elasticity is formulated by using Galerkin method for space and the central difference method for time." The fully discrete problem is formulated as follows.

$$\begin{aligned}
\bar{u}_n &= \bar{u}_{n-1} + \delta_t \bar{v}_{n-\frac{1}{2}}, \\
F_n &= K\bar{u}_n + F(t_n), \\
\bar{a}_n &= D^{-1}F_n, \\
\bar{v}_{n+\frac{1}{2}} &= \bar{v}_{n-\frac{1}{2}} + \delta_t \bar{a}_n.
\end{aligned}$$

The same procedure is used in [Wu05] for the Reissner-Mindlin plate.

To prove convergence the error $e^h = u(t_n) - u_n^h$ is split similar to the semi-discrete case as

$$e^h = e_p + e = \langle \bar{w}, \bar{\beta} \rangle + \langle \hat{w}, \hat{\beta} \rangle.$$

The term on the right is the notation of [Wu05]. Similar to the semi-discrete case the challenge is to estimate e. This is done in Theorem 4.1 [Wu05]. Halfway through the proof a stability assumption (4.11) is introduced. The disadvantage of Wu's approach to the theory is discussed in the previous section.

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Chapter 8

Conclusion

8.1 Galerkin method

Despite the decision to focus on Galerkin's method for this dissertation, the project turned out to be rather ambitious and it was not possible to cover the material envisaged. Numerical integration is important for reasons already mentioned but is not discussed in this dissertation. The articles [BD76], [S02] and [SPC03] were mentioned but a detailed study could be a report on its own. However, enough results have been studied to conclude that the theory is not complete. It may be stated that

1. Except for the theory in [IKS91], the general results are not suitable to cover all linear vibration models;

2. Assumptions in the theory are too restrictive.

The contribution of this dissertation to the theory may be summarized as follows.

1. The proofs in Sections 2.3, 2.4 and 2.5 are from [Z-PhD]. The auxiliary results are formulated in a different way to make the presentation more readable. Also, the assumptions are numbered for easy reference.

2. The notation for errors is standardized in this dissertation to make comparison between different articles easier.

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3. The similarities and differences between [Z-PhD, Section 5.3] and the semi-discrete case in [D73] are explained.

4. A generalized version of the assumptions in [D73] and [B76] are formulated in Section 2.4. This makes it possible to consider vibrating structures and not only the wave equation.

5. It is shown in Section 2.6 that the convergence result of [B76] for the time continuous approximation can be generalized. It holds for any linear vibration problem without damping.

6. In the general theory it is often necessary to assume that $(Pu)''$ exists. A sufficient condition is $u \in C^2([0,T],V)$ but it is not necessary and this last assumption is too restrictive. In this regard the experiments in Section 7.3 are illuminating. Further investigation is necessary.

7. In Chapter 6, Theorem 2.1 in [IKS91] is modified and it is shown that it may be applied to the general linear vibration problem.

8. In Chapter 4, the method used in [D73] is applied to the Timoshenko beam with boundary damping.

9. The presentation of the theory in this dissertation provides a basis for future research. It is clear where better theoretical results are desired.

8.2 Mixed finite element methods

To illustrate the basic idea behind the mixed finite element method, we consider the vibration of a membrane. The first equation of motion for a plate is also valid for a membrane; it is Equation (5.1.1):

$$\rho h \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q.$$

The constitutive equation for a tightly stretched membrane is $\boldsymbol{Q} = k\nabla w$ (see [I, Section 6.6]).

For the mixed finite element method a different variational form is used. The constitutive equation is not substituted into the variational equation:

$$(\rho h \partial_t^2 w, v) = (\operatorname{div} \boldsymbol{Q}, v) + (q, v) \tag{8.2.1}$$

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(for each test function v). Instead an additional variational equation

$$(\boldsymbol{Q}, z)_{0,2} = -(w, \operatorname{div} z), \tag{8.2.2}$$

is used for each $z \in H(\Omega, \operatorname{div})$. This space is the subspace of $\mathcal{L}^2(\Omega)^2$ consisting of vector valued functions z with $\operatorname{div} z \in \mathcal{L}^2(\Omega)$. In the mixed finite element method the aim is to find a pair $\langle w, z \rangle$ that satisfies 8.2.1 and 8.2.2.

In [CDW90] the authors consider the undamped n-dimensional wave equation. It is the problem described above but not restricted to $n = 2$. The advantages of the method are mentioned briefly: displacements and stresses are approximated simultaneously and a higher order approximation of the stresses is obtained.

In [CDW96] Problem D73 without boundary forcing is considered. It may be seen as a follow up article of [CDW90]. Boundary damping (or feedback control) is introduced and error estimates are derived for continuous and fully discrete approximations.

The mixed finite element method is also used for the vibration of a Timoshenko beam in [S94] and [FXX99] (see Chapter 4). In both articles the authors state that their reason for using this method is to overcome locking. The locking effect becomes severe when the dimensionless thickness $d \ll 1$. (Alternatively this happens when the parameter α, defined in section 4.1.1 is extremely large.) In [S94] it is stated that a reduced integration approach is equivalent to the mixed finite element method. Reference to the work of Arnold is given.

8.3 Discontinuous Galerkin method

We briefly considered a recent paper (2006) on the discontinuous Galerkin method. In this article [GSS06]:
"The symmetric interior penalty discontinuous Galerkin finite element method is presented for the numerical discretization of the second-order wave equation."

In the introduction the authors discuss difficulties encountered with the continuous Galerkin method.
"To avoid these difficulties, we consider instead discontinuous Galerkin (DG) methods. Based on discontinuous finite element spaces, these methods easily handle elements of various types and shapes, irregular nonmatching grids,

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and even locally varying polynomial order; thus, they are ideally suited for hp-adaptivity. Here continuity is weakly enforced across mesh interfaces by adding suitable bilinear forms, so-called numerical fluxes, to standard variational formulations. These fluxes are easily included within an existing conforming finite element code."

Only the undamped wave equation (Problem B76) is considered. Also, error estimates are derived only for the semi-discrete problem - based on the method in [B76]. For the fully discrete problem, the authors refer to [H].

It should be noted that the continuous Galerkin method is still used and in [LH09] the theory is based on [D73] and [B76]. In this dissertation we focussed on the continuous Galerkin method and cannot compare it to the discontinuous Galerkin method.

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Appendix A

Sobolev spaces

A.1 The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$ and denote its closure by $\bar{\Omega}$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω.

Theorem A.1.1 *The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with inner product*

$$(f, g) = \int_\Omega fg = \int_\Omega fg \, d\mu,$$

where μ is the n-dimensional Lebesgue measure.

Proof See [Ru, Theorem 3.11, p.69].

Notation Unless otherwise stated the norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$.

Definition Set $S_f = \text{closure}\{\, x \in \Omega \mid f(x) \neq 0 \,\}$. Then $C_0^\infty(\Omega) = \{\, f \in C^\infty(\Omega) \mid S_f \subset \Omega \,\}$.

Remark
If $f \in C_0^\infty(\Omega)$, then the distance between S_f and the boundary of Ω is positive.

Theorem A.1.2 *$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.*

Proof See [Ad, Theorem 2.13, p.28].

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A.2 Sobolev spaces

The one-dimensional case

Suppose Ω is a bounded open interval. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition For f, g in $H^m(\Omega)$,

$$[f,g]_m = (f^{(m)}, g^{(m)}) \quad \text{and}$$
$$|f|_m = \sqrt{[f,f]_m} \quad \text{for} \quad m = 0,\ 1,\ \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

Definition Inner product
For f and g in $H^m(\Omega)$, $(f,g)_m = \sum_{k=0}^{m}(f^{(k)}, g^{(k)})$ for $m = 0,\ 1,\ \ldots$

The bilinear form $(\cdot,\cdot)_m$ has all the properties of an inner product.

Definition Norm
For f in $H^m(\Omega)$, $\|f\|_m = \sqrt{(f,f)_m}$ for $m = 0,\ 1,\ \ldots$

The higher dimensional case

Suppose Ω is an open subset of $\mathbb{R}^n$. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Notation Let $\alpha = (\alpha_1,\ \alpha_2,\ \ldots \alpha_n)$, then $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 \cdots \alpha_n$.

Remark $|\alpha|$ denotes the order of the derivative.

Suppose Ω is a bounded open interval or a bounded open convex subset of R^2.

Definition For f, g in $H^m(\Omega)$,

$$[f,g]_m = \sum_{|\alpha|=m} \partial^\alpha f \partial^\alpha g \quad \text{and}$$
$$|f|_m = \sqrt{[f,f]_m}.$$

Definition The inner product for $H^m(\Omega)$ is defined by

$$(f,g)_m = \sum_{k=0}^{m} [f,g]_k \quad \text{for} \quad m = 0,\ 1,\ \ldots$$

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Definition The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f,g)_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Sobolev spaces of vector valued functions

Definition

$$u \in \mathcal{L}^2(\Omega)^2 \;\; \text{if} \;\; u_i \in \mathcal{L}^2(\Omega) \;\; \text{for} \;\; i = 1,\, 2.$$

$$u \in \mathcal{L}^2(\Gamma)^2 \;\; \text{if} \;\; u_i \in \mathcal{L}^2(\Gamma) \;\; \text{for} \;\; i = 1,\, 2.$$

$$u \in H^k(\Omega)^2 \;\; \text{if} \;\; u_i \in H^k(\Omega) \;\; \text{for} \;\; i = 1,\, 2.$$

$$[u,v]_{m,2} = [u_1,v_1]_m + [u_2,v_2]_m \;\; \text{for} \;\; u \in H^m(\Omega)^2 \;\; \text{and} \;\; v \in H^m(\Omega)^2\,.$$

$$|u|_{m,2} = \sqrt{[u,u]_{m,2}} \;\; \text{for} \;\; u \in \mathcal{L}^2(\Omega)^2\,.$$

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|^{\Omega}_{m,2}$ and $\|\cdot\|^{\Gamma}_{m,2}$.

Definition The inner product for $H^m(\Omega)^2$ is defined by

$$(f,g)_{m,2} = \sum_{k=0}^{m} [f,g]_{k,2} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f,g)_{m,2}} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

A.3 Fundamental properties of Sobolev spaces

Assumption Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

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Remark
It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$.

Theorem A.3.1 *The space $H^m(\Omega)$ is complete.*

Proof See [Ad, Theorem 3.2, p.45].

Theorem A.3.2 *$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.*

Proof See [OR, Theorem 2.10, p.53].

Remark
A function in $H^m(\Omega)$ can be approximated by a function in $C^m(\bar{\Omega})$: if $u \in H^m(\Omega)$ then for any $\varepsilon > 0$ there exists a $\phi \in C^m(\bar{\Omega})$ such that $\|u - \phi\|_m < \varepsilon$.

Theorem A.3.3 *Sobolev's lemma*
Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > m + n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \leq \|u\|_p \quad \textit{for} \quad |\alpha| \leq m.$$

Proof See [OR, Theorem 3.10, p.80].

Remark One-dimensional case
If $n = 1$ in Theorem 3, then $u \in C^{p-1}(\bar{\Omega})$ and $\|u^{(k)}\|_{\sup} \leq \|u\|_p$ for $k \leq m-1$.

A.4 Inequalities

The one-dimensional case

Proposition A.4.1 *Consider any $u \in C^1[0, \ell]$. For any two points x and y in $[0, \ell]$,*

$$|u(x)| \leq \sqrt{\ell}\|u'\| + |u(y)|.$$

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Proof For any f and $g \in \mathcal{L}^2(\Omega)$ we have the Cauchy-Schwartz inequality

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right)$$

for $x > y$. By choosing $g = 1$ we find that

$$\left(\int_y^x f\right)^2 \leq \left(\int_y^x f^2\right)(x-y) \leq \ell\|f\|^2. \tag{A.4.1}$$

Hence $\left|\int_y^x f\right| \leq \sqrt{\ell}\|f\|$ for each $f \in \mathcal{L}^2(0,\ell)$. Since $u(x) - u(y) = \int_y^x u'$,

$$\begin{aligned} |u(x)| &\leq \left|\int_y^x u'\right| + |u(y)| \\ &\leq \sqrt{\ell}\|u'\| + |u(y)|, \end{aligned}$$

from Equation (A.4.1). □

Proposition A.4.2 *For any* $u \in C^1[0,\ell]$ *with a zero in* $[0,\ell]$ *we have*

$$\|u\|_{\sup} \leq \sqrt{\ell}\|u'\|.$$

Proof Suppose $u(y) = 0$, then $|u(x)| \leq \sqrt{\ell}\|u'\|$ by Proposition A.4.1. The result follows from the fact that $\sqrt{\ell}\|u'\|$ is an upper bound for $|u|$. □

Proposition A.4.3 *For any* $u \in C^1[0,\ell]$ *with a zero in* $[0,\ell]$ *we have*

$$\|u\| \leq \ell\|u'\|.$$

Proof We use Proposition A.4.2,

$$\|u\|^2 = \int_0^\ell \big(u(x)\big)^2 dx \leq \ell\|u\|_{\sup}^2 \leq \ell^2\|u'\|^2.$$ □

Proposition A.4.4 *Let* $T[0,\ell] = \{u \in C^1[0,\ell] \mid u(0) = 0\}$. *Denote the closure of* $T[0,\ell]$ *in* $H^1(0,\ell)$ *by* $V(0,\ell)$. *For any* $u \in V(0,\ell)$,

$$\|u\| \leq \ell\|u'\|.$$

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Proof There exists a sequence $\{u_n\} \in C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Since $\big|\|u_n\|_1 - \|u\|_1\big| \leq \|u_n - u\|_1$, $\|u_n\|_1 \to \|u\|_1$ as $n \to \infty$. Consequently $\|u_n\| \to \|u\|$ and $\|u_n'\| \to \|u'\|$ as $n \to \infty$.

The result follows from Proposition A.4.3 by taking the limits. □

Proposition A.4.5 *For any $u \in C^1[0,\ell]$ there exists a constant $K_\ell > 0$ such that*

$$|u(\ell)| \leq K_\ell \|u\|_1.$$

Proof If u has a zero in $[0,\ell]$, then $|u(\ell)| \leq \sqrt{\ell}\|u'\|$ by Proposition A.4.1. If u does not have a zero, suppose $u > 0$ on $[0,\ell]$ and let m be the minimum of u on $[0,\ell]$. Let $w(x) = u(x) - m$, then

$$|u(\ell)| \leq |w(\ell)| + m \leq \sqrt{\ell}\|w'\| + m, \tag{A.4.2}$$

by Proposition A.4.2. But $w' = u'$ and $\|u\|^2 = \int_0^\ell u^2 \geq m^2\ell$.

Therefore, from Equation (A.4.2),

$$|u(\ell)| \leq \sqrt{\ell}\|u'\| + \frac{1}{\sqrt{\ell}}\|u\|$$

which implies the result. If $u < 0$, then $-u > 0$ and we have the result. □

A.5 Trace

The one-dimensional case

From Theorem A.3.2 we have that for each $u \in H^1(0,\ell)$, there exists a sequence $\{u_n\} \subset C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

Theorem A.5.1 *Suppose $u \in H^1(0,\ell)$. Then, there exists a unique real number u_ℓ such that for any sequence $\{u_n\} \subset C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$, $\lim_{n\to\infty} u_n(\ell) = u_\ell$.*

Proof Let $\{u_n\}$ be any sequence in $C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. The sequence $\{u_n\}$ is Cauchy in $H^1(0,\ell)$ so by Proposition A.4.5

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$\{u_n(\ell)\}$ is Cauchy in $\mathbb{R}$. Thus $\lim_{n\to\infty} u_n(\ell)$ exists. This limit is independent of the sequence $\{u_n\}$. Indeed, if $\{v_n\}$ is another sequence in $C^1[0,\ell]$ such that $\|v_n - u\|_1 \to 0$ as $n \to \infty$ then

$$\|v_n - u_n\|_1 \leq \|v_n - u\|_1 + \|u - u_n\|_1 \to 0,$$

as $n \to \infty$. So by Proposition 2.4.4 we have that $\left|v_n(\ell) - u_n(\ell)\right| \to 0$ as $n \to \infty$. □

Definition Trace operator Γ_ℓ
For each $u \in H^1(0,\ell)$ let

$$\Gamma_\ell\, u = u_\ell.$$

Theorem A.5.2 *The mapping Γ_ℓ is linear and bounded. In fact*

$$|\Gamma_\ell\, u| \leq K_\ell\, \|u\|_1.$$

Proof The linearity follows from the properties of limits. We have that,

$$\frac{|u_n(\ell)|}{\|u_n\|_1} \leq K_\ell \text{ for all } n,$$

and by considering the limits we have that

$$|\Gamma_\ell\, u| \leq K_\ell \|u\|_1.$$

This proves that Γ_ℓ is bounded. □

Remark
A trace operator Γ_0 can be defined similarly at zero.

The two-dimensional and three-dimensional cases

Suppose Ω is an open convex subset of $\mathbb{R}_n$ with a piecewise smooth boundary $\partial\Omega$ and $\Sigma \subset \partial\Omega$ is connected. It is then possible to define $\mathcal{L}^2(\Sigma)$.

Theorem A.5.3 *Suppose $u \in H^1(\Omega)$. Then there exists a unique $u_\Sigma \in \mathcal{L}^2(\Sigma)$ such that for any sequence $\{u_n\} \subset C^1(\bar{\Omega})$ where $\|u_n - u\|_1 \to 0$ as $n \to \infty$, $\lim_{n\to\infty} u_n = u_\Sigma$.*

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Proof [OR, pp.141-142]

Definition The trace operator Γ
For each $u \in H^1(\Omega)$ let

$$\Gamma u = u_\Sigma.$$

Theorem A.5.4 *The mapping* Γ *is linear and bounded. In fact*

$$\|\Gamma\ u\|_\Sigma \leq C_s \|u\|_1.$$

Proof See [OR, pp.141-142]

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Appendix B

Interpolation

The results in this appendix are simplified versions of the theory in [OR, Chapter 6], [OC, Chapter 4] and [SF, Chapter 5].

B.1 The one dimensional case

The interval $I = [0, \ell]$ is partitioned into n elements referred to as a finite element mesh. Using this mesh a finite dimensional space S^h is defined. An interpolation operator Π is a transformation from $H^k(I)$ to S^h.

It is first necessary to define interpolation on an element.

Definition Interpolation on the element $I_e = [a, b]$.
Suppose there are n basis functions that are nonzero on the element I_e, denoted by ψ_i. Let $(x_1,\ x_2,\ \ldots,\ x_n)$ be an ordered n-tuple of points in I_e – not necessarily distinct. Suppose $(\beta_1,\ \beta_2,\ \ldots,\ \beta_n)$ is an ordered n-tuple of nonnegative integers – each less than or equal to m. For a function in $u \in C^m(I_e)$ we define the interpolant Π_e by

$$\Pi_e u = \sum_{i=1}^{n} D^{\beta_i} u(x_i)\psi_i.$$

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Examples

1. Hermite cubic basis functions:
   There are four basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = x_3 = a, \;\; x_2 = x_4 = b, \;\; \beta_1 = \beta_2 = 0$ and $\beta_3 = \beta_4 = 1$.

$$\begin{aligned} \Pi_e u &= \sum_{i=1}^{4} D^{\beta_i} u(x_i)\psi_i \\ &= u(a)\psi_1 + u(b)\psi_2 + u'(a)\psi_3 + u'(b)\psi_4. \end{aligned}$$

2. Piecewise linear basis functions:
   There are two basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = a, \;\; x_2 = b$ and $\beta_1 = \beta_2 = 0$.

$$\Pi_e u = \sum_{i=1}^{2} D^{\beta_i} u(x_i)\psi_i = u(a)\psi_1 + u(b)\psi_2.$$

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element I_e is denoted by $[f]_{I_e}$.

Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{I_e} = \Pi_e [u]_{I_e} \quad \text{for each element} \quad I_e.$$

Notation

$\mathcal{P}_j(I_e)$: the set of polynomials on I_e of degree at most j is denoted by $\mathcal{P}_j(I_e)$.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e.

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e.

Remark
If $k \geq s(\Pi_e) + 1$, then the interpolation operator Π_e is defined on $H^k(I_e)$.

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Recall that $|\cdot|_k$ denotes the seminorm of order k, i.e.

$$|u|_k = \|u^{(k)}\|\,.$$

(See Appendix A.)

The interpolation error

Theorem B.1.1 below is formulated as a special case of a general result. This result may be found in [SF, p.144], [OC, p.76] and [OR, p.279].

We will use $\widehat{C}$ to denote a generic constant.

Theorem B.1.1 *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1,$$

for the interpolation operator Π. Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(I)$ we have

$$|\Pi u - u|_{m,I} \leq \widehat{C} h^{k-m} |u|_{k,I} \quad \textit{for} \quad m = 0,\ 1,\ \ldots, k.$$

The interpolation operator is denoted by Π_L for piecewise linear basis functions and by Π_c for Hermite cubics.

Corollary B.1.1 *Hermite cubic basis functions.*
There exists a constant $\widehat{C}_c$ such that if $u \in H^k(I)$ for

a) $2 \leq k \leq 4$, *then*

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

b) $k > 4$, *then*

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof It is clear that $s(\Pi_c) = 1$ and it can be shown that $r(\Pi_c) = 3$. Consequently Theorem B.1.1 is applicable with $k = 2,\ 3$ or 4. □

Corollary B.1.2 *Piecewise linear basis functions*
There exists a constant $\widehat{C}_L$ such that if $u \in H^k(0, \ell)$ for $k \geq 2$, then

$$\|\Pi_L u - u\|_1 \leq \widehat{C}_L h |u|_2.$$

Proof It is clear that $s(\Pi_L) = 1$ and it can be shown that $r(\Pi_L) = 1$. Consequently Theorem B.1.1 is applicable with $k = 2$. □

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B.2 The two-dimensional case

Recall that the domain Ω is partitioned into n elements. An interpolation operator Π is a transformation from $H^k(\Omega)$ to S^h, the finite dimensional subspace.

It is first necessary to define interpolation on an element.

Definition Interpolation on the element Ω_e.
Suppose there are n basis functions that are nonzero on the element Ω_e, denoted by ψ_i. Let $(x_1,\ x_2,\ \ldots,\ x_n)$ be an ordered n-tuple of points in Ω_e – not necessarily distinct. Suppose $(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$ is an ordered n-tuple of multi-indices with $|\alpha_i| \leq m$. For a function in $u \in C^m(\Omega_e)$ we define the interpolant Π_e by

$$\Pi_e u = \sum_{i=1}^{n} \partial^{\alpha_i} u(x_i)\psi_i.$$

Examples

1. For piecewise linear basis functions on the triangle Ω_e:

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3,$$

   where a, b and c are the vertices of the triangle.

2. For piecewise bilinear basis functions on a quadrilateral Ω_e:

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3 + w(d)\psi_4,$$

   where a, b c and d are the vertices of Ω_e.

3. For Hermite cubic basis functions on a triangle Ω_e:
   There are 10 basis functions which are not zero on Ω_e. If a, b and c are the vertices of Ω_e and d the centroid, let $x_i = a$ for $i = 1, 2, 3$, $x_i = b$ for $i = 4, 5, 6$, $x_i = c$ for $i = 7, 8, 9$ and $x_{10} = d$. The multi-indices are $\alpha_i = (0, 0)$ for $i = 1, 4, 7, 10$, $\alpha_i = (1, 0)$ for $i = 2, 5, 8$ and $\alpha_i = (0, 1)$ for $i = 3, 6, 9$.

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element Ω_e is denoted by $[f]_{\Omega_e}$.

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Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{\Omega_e} = \Pi_e[u]_{\Omega_e} \quad \text{for each element } \Omega_e.$$

Notation

$\mathcal{P}_j(\Omega_e)$: the set of polynomials on Ω_e of degree at most j.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e.

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e.

Recall that for a two-dimensional domain Ω, $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix A.)

Theorem B.2.1 below is formulated as a special case of a general result. This result may be found in [SF, p.144], [OC, p.76] and [OR, p.279]. In the theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e.

Theorem B.2.1 *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 2 \leq k \leq r(\Pi_e) + 1,$$

for the interpolation operator Π. Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have

$$|\Pi u - u|_{m,\Omega} \leq \widehat{C} h^{k-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ \ldots, k.$$

Remark
The constant $\widehat{C}$ depends on the shape of the elements in the finite element mesh.

Corollary B.2.1 *Piecewise linear basis functions on triangle elements. The interpolation operator is denoted by $\Pi_\triangle$. If $k \geq 2$, then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_\triangle u - u|_{m,\Omega} \leq \widehat{C} h^{2-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ 2.$$

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Corollary B.2.2 *Piecewise bilinear basis functions on rectangle elements. The interpolation operator is denoted by Π_b. If $k \geq 2$, then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_b u - u|_{m,\Omega} \leq \widehat{C} h^{2-m} |u|_{k,\Omega} \quad \textit{for} \quad m = 0,\ 1,\ 2.$$

Corollary B.2.3 *Hermite cubic basis functions. The interpolation operator is denoted by Π_c. If $k \geq 2$, then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_c u - u|_{m,\Omega} \leq \widehat{C} h^{2-m} |u|_{k,\Omega} \quad \textit{for} \quad m = 0,\ 1,\ 2.$$

B.3 Vector-valued functions

If an interpolation operator Π is defined on $H^k(\Omega)$ we may define one on $H^k(\Omega)^2$. For $u = \langle u_1,\ u_2 \rangle \in H^k(\Omega)^2$, we define

$$\Pi_2\, u = \langle \Pi\, u_1,\ \Pi\, u_2 \rangle\, .$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2\, .$$

(See Appendix A.)

Lemma B.3.1 *If $\|\Pi\ v - v\|_m \leq \widehat{C} h^{k-m} |v|_k$ for $v \in H^k(\Omega)$, then*

$$\|u - \Pi_2 u\|_{m,2} \leq \widehat{C} h^{k-m} |u|_{k,2} \textit{ for } u \in H^k(\Omega)^2.$$

Proof

$$|\, u - \Pi_2\, u|\,_{m,2}^{\,2} = |\, u_1 - \Pi\, u_1|\,_m^{\,2} + |\, u_2 - \Pi\, u_2|\,_m^{\,2}.$$

□

For piecewise bilinear basis functions on rectangles, let $\Pi_B\, u = \langle \Pi_b\, u_1, \Pi_b\, u_2 \rangle$.

Corollary B.3.1 *If $k \geq 2$, then there exists a constant $\widehat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$|\, u - \Pi_B\, u|\,_{m,2} \leq \widehat{C} h |u|_{2,2} \textit{ for } m = 0, 1, 2.$$

Proof The result follows from Corollary B.2.2 and Lemma B.3.1. □

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Appendix C

Assumptions in Chapter 2

Assumption A1 (Subsection 2.3.1)
If u is a solution of Problem G, then

$$u \in C^1(J; V) \cap C^2(J; W).$$

Assumption A2 (Subsection 2.3.1)
Assume that $u \in C(J, V)$ has the property that $(Pu) \in C^2(J)$.

Assumption A3 (Subsection 2.4.2)
There exists a subspace $H(V, k)$ of V, a positive constant $\widehat{C}$ and a positive integer α such that for any $u \in H(V, k)$

$$\inf_{v \in S^h} \|u - v\|_E \leq \widehat{C} h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V, k)$.

Assumption A4 (Subsection 2.4.2)
The solution u of Problem G satisfies

$$u \in C^2(J, V).$$

Assumption A5 (Section 2.5)
There exists a subspace $H(V, k)$ of V, an interpolation operator Π and constants C_Π and α such that for $u \in H(V, k)$:

$$\|u - \Pi u\|_E \leq C_\Pi h^\alpha \|u\|_{H(V,k)}.$$

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Assumption A6 (Section 2.5)
$H(V,k)$ is a dense subset of V.

Assumption A7 (Section 2.7)
$E_b \subset H(V,k)$ and for any $u \in E_b$ we have $\|u\|_{H(V,k)} \leq \widehat{c}_b \|y\|_W$ where

$$b(u,v) = (y,v) \ \forall v \in V.$$

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Appendix D

Summary of abstract spaces

The properties of the spaces V, W and X are of critical importance in the theory. For convenience we present a summary here.

Note that $V \subset W \subset X$.

Space	Inner product	Norm
Energy space V	$b(\cdot, \cdot)$	Energy norm $\|\cdot\|_E$
Inertia space W	$c(\cdot, \cdot)$	Inertia norm $\|\cdot\|_W$
X	$(\cdot, \cdot)_X$	$\|\cdot\|_X$

Estimates

There exist constants C_b and C_c such that

$$\begin{aligned} \|u\|_W &\leq C_b\|u\|_E \text{ for all } u \in V, \\ \|u\|_X &\leq C_c\|u\|_W \text{ for all } u \in W. \end{aligned}$$

Topological property

V is dense in W with respect to the inertia norm.

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Appendix E

Gronwall and Young's inequalities

Proposition E.0.1 *If there exists a constant $c > 0$ such that $\phi' \leq c\phi$, then*

$$\phi(t) \leq e^{ct}\phi(0), \quad \textit{for} \quad t > 0.$$

Proof

$$\frac{d}{dt}(\phi(t)\, e^{-ct}) = e^{-ct}\,(\phi'(t) - c\phi(t)) \leq 0.$$

Therefore $\phi(t)\, e^{-ct} - \phi(0) \leq 0$ and the result follows. □

Lemma E.0.2 ***Gronwall's inequality***
Suppose $\psi \in C^1[0,T]$. If there exist positive constants c and K such that

$$\psi(t) \leq c\int_0^t \psi + K,$$

then

$$\psi(t) \leq Ke^{ct}, \quad \textit{for each } t \in [0,T].$$

Proof Let $\phi(t) = c\int_0^t \psi + K$, then $\phi' = c\psi$ and $\phi(0) = K$. Since $\phi' \leq c\phi$, we have

$$\phi(t) \leq e^{ct}K \quad \text{from Proposition E.0.1.}$$

The result follows from the fact that $\psi \leq \phi$. □

The following simple inequality is very useful.

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Lemma E.0.3 ***Young's inequality***

$$ab \leq \frac{1}{2}\left(\varepsilon^2 a^2 + \varepsilon^{-2} b^2\right).$$

Proof $(\varepsilon a - \varepsilon^{-1} b)^2 \geq 0$ implies that

$$\varepsilon^2 a^2 + \varepsilon^{-2} b^2 \geq 2ab.$$

□

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