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A CHARACTERIZATION OF SOME FINITE QUASISIMPLE GROUPS USING THEIR CHARACTER CODEGREES

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Abstract. Let G be a finite group. For an irreducible character χ of G, define its codegree by $\mathrm{cod}(\chi) = |G : \ker\chi|/\chi(1)$. Furthermore, define $\mathrm{cod}(G) = \{\mathrm{cod}(\chi) : \chi \in \mathrm{Irr}(G)\}$. A recent conjecture of Hung and Moretó states that if $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$ and H is a finite non-abelian simple group, then $G \cong H$. They verified this conjecture for sporadic groups, alternating groups of degree at least five and many simple groups of Lie type with small Lie rank. We propose an extension of this conjecture as follows: If $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$ and H is a finite quasisimple group, then $G \cong H/N$ where $1 \leqslant N \leqslant Z(G)$ and $\mathrm{cod}(G) = \mathrm{cod}(H)$ if and only if $G \cong H$. We show that the conjecture holds when $H \cong \mathrm{SL}_2(q)$, $q \geqslant 5$ or $\mathrm{SL}_3(q)$, $q \geqslant 2$.

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1. Introduction.

Let G be a finite group. Let $\mathrm{Irr}(G)$ denote the set of complex irreducible characters of G and $\mathrm{cd}(G)$ the set of character degrees of G. In the early 1990s ([7, 8]), the codegree of a character $\chi \in \mathrm{Irr}(G)$ was studied and it was defined to be $\mathrm{cod}(\chi) = |G|/\chi(1)$. To avoid ambiguity, Qian, Wang and Wei introduced a modified definition which is now in use. The modified version is as follows: For an irreducible character χ of G, define its codegree by

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$$\operatorname{cod}(\chi) = \frac{|G : \ker \chi|}{\chi(1)}.$$

Indeed the definition for $\operatorname{cod}(\chi)$ when $\chi \in \operatorname{Irr}(G/N)$ and $\chi \in \operatorname{Irr}(G)$ coincide for any normal subgroup N of G, unlike in the first version. The sets $\operatorname{cd}(G)$ and $\operatorname{cod}(G)$ are known to encode a great deal of information about the structure of a finite group G. We shall first discuss some results on $\operatorname{cd}(G)$. Huppert [18] conjectured that if H is a finite simple group and G is a finite group such that $\operatorname{cd}(G) = \operatorname{cd}(H)$, then $G \cong H \times A$ for some abelian group A. This, if true, points to the fact that non-abelian finite simple groups are in essence determined by their character degrees. Huppert's conjecture has been shown to hold for all sporadic simple groups, alternating groups of degree at least 5 and some simple groups of Lie type with low Lie rank (see references in [15] for most of the results). A recent article of Tong-Viet [26] verified the conjecture for all the remaining simple exceptional groups of Lie type.

Hung, Majozi, Tong-Viet and Wakefield [15] proposed an extension Huppert's conjecture to quasisimple groups: If G is a finite group and H is a finite quasisimple group with $M(H/Z(H))$, the Schur multiplier of $H/Z(H)$, cyclic, then $\operatorname{cd}(G) = \operatorname{cd}(H)$ if and only if $G \cong H \circ A$, a central product of H and an abelian group A. The extra condition is necessary since $G = 2^2{\cdot}\Omega_8^+(2)$ and $H = G/Z$ for Z a central subgroup of order 2 is such that $\operatorname{cd}(G) = \operatorname{cd}(H)$ but G is not isomorphic to a central product of H with any abelian group. This conjecture has been verified for all sporadic quasisimple groups (with the exception of $2{\cdot}M_{12}$), $\mathrm{SL}_2(q)$ and $\mathrm{SL}_3(q)$ (see [15, 24]).

We now turn to character codegrees. In [21, Question 20.27], Qian proposed an analogue of Huppert's conjecture:

CONJECTURE 1.1. (Qian) *Let H be a non-abelian finite simple group and G be a finite group. Then $\operatorname{cod}(G) = \operatorname{cod}(H)$ if and only if $G \cong H$.*

This conjecture has been shown to hold for all sporadic simple groups, alternating groups of degree at least 5 and many simple groups of Lie type with low Lie rank (see [1, 4, 10, 11, 13, 14, 22]), all projective special linear groups and all finite non-abelian simple exceptional groups of Lie type [25]. Hung and Moretó [16] showed that non-abelian simple groups can be determined by the character codegrees together with their multiplicities up to isomorphism. In a follow up article in [17], the authors proposed a stronger conjecture to the one proposed by Qian, Conjecture 1.1:

CONJECTURE 1.2. (Hung and Moretó) *Let G be a finite group and let H be a non-abelian finite simple group. If $\operatorname{cod}(G) \subseteq \operatorname{cod}(H)$, then $G \cong H$.*

Hung and Moretó verified this for all sporadic simple groups, alternating groups of degree at least 5, ${}^2B_2$, ${}^2G_2$, ${}^2F_4$, G_2, ${}^3D_4$, ${}^2E_6$, $A_{\leqslant 3}$, ${}^2A_{\leqslant 6}$, C_2, C_3, and B_3. Note that we use the notation above as a once off for convenience. Motivated by all conjectures mentioned above, we propose an extension of Conjecture 1.2:

CONJECTURE 1.3. *Let G be a finite group and let H be a finite quasisimple group. If $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, then $G \cong H/Z$ for some normal subgroup Z of H such that $1 \leqslant Z \leqslant Z(H)$. Moreover, $\mathrm{cod}(G) = \mathrm{cod}(H)$ if and only if $G \cong H$.*

We confirm this Conjecture 1.3 for $\mathrm{SL}_2(q)$ and $\mathrm{SL}_3(q)$.

THEOREM A. *Let G be a finite group and let $H \cong \mathrm{SL}_2(q)$, $q \geqslant 5$ or $\mathrm{SL}_3(q)$, $q \geqslant 2$. If $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, then either $G \cong H/Z(H)$ or $G \cong H$. Moreover, $\mathrm{cod}(G) = \mathrm{cod}(H)$ if and only if $G \cong H$.*

We describe our strategy to prove Theorem A. We think that this strategy will work for quasisimple groups with a centre of prime order. For a more general case, some other approach is needed.

Suppose that G is a finite group and H is a finite quasisimple group. Assume that $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$ and $|Z(H)| = s$, where s is a prime.

- *Step 1* : If N is a unique minimal normal subgroup of G, $G/N \cong H$ and $N \nleqslant Z(G)$, then $|N|^2$ divides $|H|$. Furthermore, if $H \cong \mathrm{SL}_2(q)$ or $H \cong \mathrm{SL}_3(q)$, then $|N| = r$ for some prime r. For the last part of this, we make use of the indices of maximal subgroups of H.
- *Step 2* : If G is simple, then $G \cong H/Z(H)$. This implies that if N is a maximal normal subgroup of G, then $G/N \cong H/Z(H)$.
- *Step 3* : If G has a unique normal subgroup, then $G \cong H$.
- *Step 4* : If N is a normal subgroup of G (possibly trivial) and $G/N \cong H$, then $N = 1$.
- *Step 5* : Using all this information, we show that the conclusion of the theorem holds.

We remark that in our proof, we consider $\mathrm{SL}_2(5)$ and $\mathrm{SL}_3(4)$ as special cases within our arguments. This is because $\mathrm{SL}_2(5)$ has less character codegrees than any $\mathrm{SL}_2(q)$, where $q > 5$ is odd whilst $\mathrm{SL}_3(4)$ has more character codegrees than $\mathrm{SL}_3(q)$, $q \geqslant 7$.

Our notation follows that in [19] and [9] with some exceptions. Note that G denotes a finite group, S is a non-abelian simple group and unless specified otherwise. We denote a cyclic group of order n by C_n. However, in some exceptional cases in the tables, we opt to denote C_n by n.

2. Preliminary results.

We start off this section with a number theoretic result.

LEMMA 2.1. ([23]) *Consider integers $x, y > 0$ and $m, n > 1$. For the following equation*

$$x^m - y^n = 1,$$

the only solution is $x = 3, m = 2$ and $y = 2, n = 3$.

We collect some results on irreducible characters of finite groups.

PROPOSITION 2.2. *(Burnside)* ([19, Theorem 3.15]) *Let* $\chi \in \mathrm{Irr}(G)$ *be non-linear, then* $\chi(g) = 0$ *for some* $\chi \in \mathrm{Irr}(G)$.

LEMMA 2.3. *Let* S *be a finite non-abelian simple group. Then there exists a non-principal irreducible character of* S *that extends to* $\mathrm{Aut}(S)$.

Proof. Follows, for example, from [5, Theorems 2, 3, 4]. □

LEMMA 2.4. ([5, Lemma 5]) *Let* G *be a finite group, and* $N = S_1 \times \cdots \times S_k$ *be a minimal normal subgroup of* G, *where every* S_i *is isomorphic to a non-abelian simple group* S. *If* $\theta \in \mathrm{Irr}(S)$ *extends to* $\mathrm{Aut}(S)$, *then* $\varphi = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ *extends to* G.

Recall that if G is a finite group with a normal subgroup $N \subseteq Z(G)$ such that G/N is isomorphic to a group Q, then we call G a central extension of N by Q.

LEMMA 2.5. ([19, Corollary 11.20]) *Let* G *be a central extension of* N *by* Q. *If* $N \subseteq G'$, *then* N *is isomorphic to a subgroup of* $M(Q)$.

We shall need some information on the Schur multipliers of some quasisimple groups.

PROPOSITION 2.6. ([20, Theorem 7.1.1]) *The Schur multiplier of* $\mathrm{SL}_n(q)$ *is trivial with the following exceptions:*

(a) *If* $G \in \{\mathrm{SL}_n(4), \mathrm{SL}_3(2), \mathrm{SL}_3(3)\}$, *then* $M(G) = C_2$.

(b) $M(\mathrm{SL}_2(9)) = C_3$.

(c) $M(\mathrm{SL}_3(4)) = C_4 \times C_4$.

If $\mathrm{PSL}_n(q)$ *is not one of the groups listed below, then* $\mathrm{SL}_n(q)$ *is a unique Schur cover of* $\mathrm{PSL}_n(q)$ *and* $M(\mathrm{PSL}_n(q)) = C_m$ *where* $m = \gcd(q-1, n)$.

(d) *If* $G \in \{\mathrm{PSL}_3(2), \mathrm{PSL}_3(3)\}$, *then* $M(G) = C_2$.

(e) $M(\mathrm{PSL}_2(9)) = C_6$.

(f) $M(\mathrm{PSL}_3(4)) = C_4 \times C_{12}$.

LEMMA 2.7. *Every cyclic normal subgroup of a perfect group is central.*

Proof. Let G be perfect and let $N \trianglelefteq G$ be cyclic. An application of the Normalizer-Centralizer theorem implies that $G/C_G(N)$ can be embedded in the abelian group $\mathrm{Aut}(N)$. This implies that $C_G(N)$ contains $G' = G$, that is, $C_G(N) = G$. □

We now turn to character codegrees. The following result on character codegrees of perfect groups is quite useful to restrict our study to perfect groups.

Lemma 2.8. ([16, Theorem 2.3]) *A finite non-trivial group G is perfect if and only if G does not have any prime character codegree.*

One of the results of Hung and Moretó in their quest to prove that if $\operatorname{cod}(G) \subseteq \operatorname{cod}(H)$ where G and H are simple groups, then $G \cong H$, is the following:

Lemma 2.9. ([16, Theorem 3.1]) *Let S be a non-abelian finite simple group and let G be a finite group. If $\operatorname{cod}(S) \subseteq \operatorname{cod}(G)$, then $|S|$ divides $|G|$.*

Lemma 2.10. *If $|\operatorname{cod}(G)| \leq 3$, then G is solvable.*

Proof. This follows from [2, Lemma 3.1 and Theorem 3.4]. □

By Lemma 2.10, we have that if G is non-solvable, then $|\operatorname{cod}(G)| \geq 4$. This leads us to a result on non-abelian simple groups with a few character codegrees. Let us consider the size of the set of codegrees of the direct products of simple groups.

Lemma 2.11. *Let S be a non-abelian simple group. Then $|\operatorname{cod}(S \times S)| \geqslant 10$. If $|\operatorname{cod}(S)| \geqslant 5$, then $|\operatorname{cod}(S \times S)| \geqslant 14$.*

Proof. Let $\chi \times \theta \in \operatorname{Irr}(S \times S)$, where $\chi \in \operatorname{Irr}(S)$ is non-principal and $\theta \in \operatorname{Irr}(S)$ is principal. Since $\ker(\chi \times \theta) = S$, we have that $\operatorname{cod}(\chi \times \theta) = \operatorname{cod}(\chi)$. Now consider when $\theta \in \operatorname{Irr}(S)$ is non-principal. Since $\chi \times \theta$ is faithful, $\operatorname{cod}(\chi \times \theta) = \operatorname{cod}(\chi) \cdot \operatorname{cod}(\theta)$. The first statement of the lemma follows since $|\operatorname{cod}(G)| \geq 4$ and the second also follows by counting all the possible codegrees. □

The following result is derived from the fact that all non-principal irreducible characters of S are faithful, which means that $|\operatorname{cd}(S)| = |\operatorname{cod}(S)|$ when S is simple. Simple groups with few character degrees were classified in [3]. The result below then follows:

Theorem 2.12. ([28, Lemma 2.1]) *Let S be a finite non-abelian simple group. If $|\operatorname{cod}(S)| \leq 10$, then one of the following holds:*

(a) $|\operatorname{cod}(S)| = 4$ *and* $S \cong \mathrm{PSL}_2(k)$ *for* $k = 2^f$, $f \geq 2$;

(b) $|\operatorname{cod}(S)| = 5$ *and* $S \cong \mathrm{PSL}_2(k)$ *for* $p \neq 2$, $k = p^f > 5$;

(c) $|\operatorname{cod}(S)| = 6$ *and* $S \cong {}^2B_2(k)$ *for* $k = 2^{2f+1}$, $f \geq 1$ *or* $S \cong \mathrm{PSL}_3(4)$;

(d) $|\operatorname{cod}(S)| = 7$ *and S is isomorphic to either* $\mathrm{PSL}_3(3), A_7, M_{11}$, *or* J_1;

(e) $|\operatorname{cod}(S)| = 8$ *and S is isomorphic to either* $\mathrm{PSL}_3(k)$ *where* $4 < k \not\equiv 1 \pmod 3$, $\mathrm{PSU}_3(k)$ *where* $4 < k \not\equiv -1 \pmod 3$, *or* $G_2(2)'$;

(f) $|\operatorname{cod}(S)| = 9$ *and S is isomorphic to either* $\mathrm{PSL}_3(k)$ *where* $4 < k \equiv 1 \pmod 3$ *or* $\mathrm{PSU}_3(k)$ *where* $4 < k \equiv -1 \pmod 3$;

(g) $|\mathrm{cod}(S)| = 10$ *and* $S \cong M_{22}$.

THEOREM 2.13. ([28, Lemma 2.4]) *If* S *is a non-abelian simple group isomorphic to either* $\mathrm{PSL}_3(3), A_7, M_{11}, J_1, M_{22}, \mathrm{PSL}_2(k)$, $\mathrm{PSL}_3(k)$, $\mathrm{PSU}_3(k)$ *or* ${}^2B_2(k)$, *then the codegree set of* S *can be found in* Table 1.

Take note that some of the classical groups in Theorem 2.13 have particular conditions which are *Table 1.* We recall that D_{2n}, for $2n \geq 4$ denotes the dihedral group of order $2n$. We denote the generalized quaternion group of order $4n$ by Q_{4n}. This is the group generated by elements a, b where $a^n = b^2$, $a^{2n} = 1$ and $b^{-1}ab = a^{-1}$.

Table 1: Codegrees of some simple groups

Simple group S	Codegree set of S
$\mathrm{PSL}_2(k), k = 2^t \geq 4$	$\{1, k(k-1), k(k+1), k^2-1\}$
$\mathrm{PSL}_2(k), k > 5$ odd, $\epsilon = (-1)^{(k-1)/2}$	$\{1, \frac{k(k-1)}{2}, \frac{k(k+1)}{2}, \frac{k^2-1}{2}, k(k-\epsilon)\}$
${}^2B_2(k), k = 2^{2f+1}$ $f \geq 1, r = 2^f$	$\{1, (k-1)(k^2+1), k^2(k-1, 2^{3f+2}(k^2+1),$ $k^2(k-2r+1), k^2(k+2r+1)\}$
$\mathrm{PSL}_3(3)$	$\{1, 2^4 \cdot 3^2 \cdot 13, 2^4 \cdot 3^3, 3^3 \cdot 13, 2^3 \cdot 3^3, 2^4 \cdot 13, 2^4 \cdot 3^2\}$
A_7	$\{1, 2^2 \cdot 3 \cdot 5 \cdot 7, 2^2 \cdot 3^2 \cdot 7, 2^2 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 7,$ $2^3 \cdot 3 \cdot 5, 2^3 \cdot 3^2\}$
M_{11}	$\{1, 2^3 \cdot 3^2 \cdot 11, 2^4 \cdot 3^2 \cdot 5, 3^2 \cdot 5 \cdot 11, 2^2 \cdot 3^2 \cdot 5,$ $2^4 \cdot 11, 2^4 \cdot 3^2\}$
J_1	$\{1, 3 \cdot 5 \cdot 11 \cdot 19, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, 2^3 \cdot 3 \cdot 5 \cdot 19,$ $7 \cdot 11 \cdot 19, 2^3 \cdot 3 \cdot 5 \cdot 11, 2^3 \cdot 3 \cdot 5 \cdot 7\}$
$\mathrm{PSL}_3(k)$ $4 < k \not\equiv 1 \pmod 3$	$\{1, (k^2+k+1)(k+1)(k-1)^2, k^2(k^2+k+1)(k-1)^2,$ $k^3(k^2+k+1), k^2(k+1)(k-1)^2, k^3(k-1)(k+1),$ $k^3(k+1)(k-1)^2, k^3(k-1)^2\}$
$\mathrm{PSL}_3(k)$ $4 < k \equiv 1 \pmod 3$	$\{1, \frac{1}{3}(k^2+k+1)(k+1)(k-1)^2, \frac{1}{3}k^2(k^2+k+1)(k-1)^2,$ $\frac{1}{3}k^3(k^2+k+1), \frac{1}{3}k^2(k+1)(k-1)^2, \frac{1}{3}k^3(k-1)(k+1),$ $\frac{1}{3}k^3(k+1)(k-1)^2, \frac{1}{3}k^3(k-1)^2, k^3(k-1)^2\}$
$\mathrm{PSU}_3(k)$ $4 < k \not\equiv -1 \pmod 3$	$\{1, (k^2-k+1)(k+1)^2(k-1), k^2(k^2-k+1)(k+1)^2,$ $k^3(k^2-k+1), k^2(k+1)^2(k-1), k^3(k-1)(k+1),$ $k^3(k+1)^2(k-1), k^3(k+1)^2\}$
$\mathrm{PSU}_3(k)$ $4 < k \equiv -1 \pmod 3$	$\{1, \frac{1}{3}(k^2-k+1)(k+1)^2(k-1), \frac{1}{3}k^2(k^2-k+1)(k+1)^2,$ $\frac{1}{3}k^3(k^2-k+1), \frac{1}{3}k^2(k+1)^2(k-1), \frac{1}{3}k^3(k-1)(k+1),$ $\frac{1}{3}k^3(k+1)^2(k-1), \frac{1}{3}k^3(k+1)^2, k^3(k+1)^2\}$
M_{22}	$\{1, 2^7 \cdot 3 \cdot 5 \cdot 11, 2^7 \cdot 7 \cdot 11, 2^7 \cdot 3^2 \cdot 7, 2^7 \cdot 5 \cdot 7, 2^6 \cdot 3^2 \cdot 5,$ $2^6 \cdot 3 \cdot 11, 2^7 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 11, 2^7 \cdot 3^2\}$

Table 2: Maximal subgroups of $\mathrm{SL}_2(q)$

Structure	Condition	Index
$p^n : (q-1)$		$q+1$
$Q_{2(q-1)}$	$q \neq 5, 7, 9, 11$	$q(q+1)/2$
$Q_{2(q+1)}$	$q \neq 7, 9$	$q(q-1)/2$
$\mathrm{SL}_2(q_0).2$	$q = q_0^2$	$q_0(q+1)/2$
$\mathrm{SL}_2(q_0)$	$q = q_0^s$ where s is an odd prime	$\frac{q_0^{s-1}(q-1)(q+1)}{(q_0-1)(q_0+1)}$
$2_{-}^{1+2}.S_3$	$q = p \equiv \pm 1 \pmod 8$	$\frac{q(q-1)(q+1)}{48}$
$2^{\cdot}A_4$	$q = p \equiv \pm 3, 5, \pm 13 \pmod{40}$	$\frac{q(q-1)(q+1)}{24}$
$2^{\cdot}A_5$	$q = p \equiv \pm 1 \pmod{10}$	$\frac{q(q-1)(q+1)}{120}$
	$q = p^2$ with $p \equiv \pm 3 \pmod{10}$	

Theorem 2.14. ([6, Table 8.1]) *Let $q = p^n > 5$ be an odd prime power, then the maximal subgroups of* $\mathrm{SL}_2(q)$ *are as listed and described in* Table 2.

Theorem 2.15. ([6, Table 8.1]) *Let $7 \leq p^n = q \equiv 1 \pmod 3$ where p is prime and n is an integer. The maximal subgroups of* $\mathrm{SL}_3(q)$ *are listed and described in* Table 3.

Table 3: Maximal subgroups of $\mathrm{SL}_3(q)$

Structure	Condition	Index
$(p^n \times p^n) : \mathrm{GL}_2(q)$		q^2+q+1
$(q-1)^2 : S_3$	$q \geq 5$	$q^3(q^2+q+1)(q+1)/6$
$(q^2+q+1) : 3$	$q \neq 4$	$q^3(q+1)(q-1)^2/3$
$\mathrm{SL}_3(q_0).\,\gcd(\frac{q-1}{q_0-1}, 3)$	$q = q_0^r$, r is prime	$\frac{q^3(q^3-1)(q^2-1)}{\gcd(\frac{q-1}{q_0-1},3)q_0(q_0-1)(q_0+1)}$
$3_{+}^{1+2} : Q_8.\frac{\gcd(q-1,9)}{3}$	$q = p \equiv 1 \pmod 3$	$\frac{q^3(q^3-1)(q^2-1))}{\gcd(q-1,9)\cdot 2^3\cdot 3^2}$
$C_3 \times \mathrm{SO}_3(q)$	q is odd	$q^2(p^3-1)/3$
$\gcd(q_0-1,3) \times \mathrm{SU}_3(q_0)$	$q = q_0^2$	$\frac{q^3(q^3-1)(q^2-1)}{\gcd(q_0-1,3)q_0^3(q_0^3+1)(q-1)}$
$C_3 \times \mathrm{PSL}_2(7)$	$q = p \equiv 1, 2, 4 \pmod 7$, $q \neq 2$	$q^3(q^3-1)(q^2-1)/504$
$3^{\cdot}A_6$	$q = p \equiv 1, 4 \pmod{15}$	$q^3(q^3-1)(q^2-1)/1080$
	$q = p^2$, $p \equiv 2, 3 \pmod 5$, $p \neq 3$	

Maximal subgroups of $\mathrm{SL}_3(4)$ are given in the following table:

Table 4: Maximal subgroups of $\mathrm{SL}_3(4)$

Structure	Index
$(2^2 \times 2^2) : \mathrm{GL}_2(4)$	21
$\mathrm{SL}_3(2).C_3$	120
$\mathrm{SU}_3(2)$	280
$3^{\cdot}A_6$	56

The following result is an extension of part of [25, Lemma 2.3]. This is Step 1 of our strategy outlined in the introduction.

Lemma 2.16. *Let $q = p^n$ be a prime power and let N be a unique minimal normal proper subgroup of a finite group G such that N is an elementary abelian r-group where r is some prime and N is not central in G. If $\mathrm{cod}(G) \subseteq \mathrm{cod}(G/N)$, then $|N|^2$ divides $|G : N|$. Moreover, if $G/N \cong H$ where either $H = \mathrm{SL}_2(q)$ with $q > 5$ odd or $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$, then $|N| = r$.*

Proof. Consider the action of G on $\mathrm{Irr}(N) \cong N$ by conjugation. Denote the G-orbits on $\mathrm{Irr}(G)$ by $\mathcal{O}_i$ for integers $0 \leq i \leq m$, with $\mathcal{O}_0 = \{1_N\}$ and $\theta_i \in \mathcal{O}_i$. Since N is not central in G, it follows that $\mathcal{O}_0$ is the only orbit of size 1. If all orbits of size greater than 1 have a size divisible by r, then by the isomorphism $\mathrm{Irr}(N) \cong N$, we conclude that $r \mid |N| - 1$, which is a contradiction. Hence, choose an orbit $\mathcal{O}_{i_0} = \mathcal{O}$, where $i_0 > 0$, with the representative $\theta_{i_0} = \theta$ such that $r \nmid |\mathcal{O}|$.

Set $T = I_G(\theta)$ and let $\theta^T = \sum_i \psi_i(1)\psi_i$, where $\psi_i \in \mathrm{Irr}(T \mid \theta)$. By Clifford Theory (see [19, Chapter 6]), we can write $\theta^G = \sum_i \psi_i(1)\chi_i$, where each $\chi_i \in \mathrm{Irr}(G \mid \theta)$ is faithful and $\psi_i^G = \chi_i$. The degree of χ_i is then given by $\chi_i(1) = |G : T|\, \psi_i(1) = |\mathcal{O}|\, \psi_i(1)$. We can choose a character $\vartheta_i \in \mathrm{Irr}(G/N)$ such that

$$\mathrm{cod}(\chi_i) = \frac{|G|}{\chi_i(1)} = \frac{|G/N|}{|\ker \vartheta_i|\, \vartheta_i(1)} = \mathrm{cod}(\vartheta_i).$$

Therefore, the degree of χ_i is given by $\chi_i(1) = |\ker \vartheta_i|\, |N|\, \vartheta_i(1)$ and so $|\mathcal{O}|\, \psi_i(1) = |\ker \vartheta_i|\, |N|\, \vartheta_i(1)$. Since we chose $\mathcal{O}$ to have size non-divisible by r, it follows that $\psi_i(1)$ is divisible by $|N|$; that is, $\psi_i(1) = c_i\, |N|$ for a positive integer c_i. This yields the equality below:

$$\theta^T(1) = |T : N|\, \theta(1) = |T : N| = \sum_i \psi_i(1)^2 = |N|^2 \sum_i c_i^2.$$

Thus $|N|^2$ divides $|G : N|$.

Let $|N| = r^t$ for some integer t. For a contradiction, we assume may that $t > 1$.

Suppose $H = \mathrm{SL}_2(q)$ where $q = p^n > 5$ is odd and p a prime. Since $r^{2t} \mid |G/N| = q(q-1)(q+1)$, we show that $r^t \leq \sqrt{2}\sqrt{q+1}$.

Let $q - 1 = 2^m$. It follows that $q + 1$ is divisible by 2 and an odd prime; moreover, $(q+1)/2$ is odd. So $\gcd(q-1, q+1) = 2$. Hence 2^{m+1} is the largest prime power divisor of $q(q-1)(q+1)$ and so $r^t \leq \sqrt{2}\sqrt{q-1} < \sqrt{2}\sqrt{q+1}$.

Let $q+1=2^m$. It follows that $q-1$ is divisible by 2 and an odd prime; also, $(q-1)/2$ is odd. Thus $\gcd(q-1,q+1)=2$. Hence 2^{m+1} is the largest prime power divisor of $q(q-1)(q+1)$ and so $r^t \leq \sqrt{2}\sqrt{q+1}$.

If $q-1 \neq 2^m$ and $q+1 \neq 2^m$, then $q-1$ and $q+1$ are both divisible by 2 and an odd prime. Furthermore, $\gcd(q-1,q+1)=2$ and so the power of 2 dividing $(q-1)(q+1)$ must be less than or equal to q. Hence every prime power dividing $q(q-1)(q+1)$ must be less than or equal to q, which implies that $r^t \leq \sqrt{q} < \sqrt{2}\sqrt{q+1}$.

We now show that $|N| > \sqrt{2}\sqrt{q+1}$. Let $1_N \neq \phi \in \mathrm{Irr}(N)$ and denote $I_G(\phi)$ by K. Now $|N| = |\mathrm{Irr}(N)| > |G:K|$ as $\mathrm{Irr}(G)$ contains all the conjugates of ϕ in G (there are $|G:K|$ of them) and $1_G \in \mathrm{Irr}(G)$.

Choose a maximal subgroup L/N of $\mathrm{SL}_2(q)$ containing K/N. Therefore, $|G:K| \geq |G:L| = |G/N:L/N|$. From Theorem 2.14 and Table 2, it follows that $|N| = |G/N:L/N| > \sqrt{2}\sqrt{q+1}$ where $|G/N:L/N|$ is the index of the maximal subgroup in $\mathrm{SL}_2(q)$, a contradiction.

Let $H=\mathrm{SL}_3(q)$ where $7 \leq q \equiv 1 \pmod 3$. We have that $r^{2t} \mid |\mathrm{SL}_3(q)| = q^3(q^3-1)(q^2-1)$. Note that $q^3(q^3-1)(q^2-1) = q^3(q^2+q+1)(q+1)(q-1)^2$. Furthermore, $\gcd(q^2+q+1,q-1) = \gcd(q^2+q+1,q+1) = 1$ and either $\gcd(q+1,q-1)=1$ or $\gcd(q+1,q-1)=2$. Hence we see that q^3 is the largest prime power divisor of $|\mathrm{SL}_3(q)|$. This implies that $|N| = r^t \leq \sqrt{q^3}$.

Let $1_N \neq \vartheta \in \mathrm{Irr}(N)$ and denote $I_G(\vartheta)$ by K. Then $|N| = |\mathrm{Irr}(N)| > |G:K|$. Now let L/N be a maximal subgroup of $\mathrm{SL}_3(q)$ containing K/N. It follows that $|G:K| = |G/N:K/N| \geq |G/N:L/N|$. By Theorem 2.15 and Table 3, we have that $|N| > \sqrt{q^3}$, a contradiction.

We therefore conclude that $t=1$ and the result follows. □

3. Proof of Theorem A.

We prove the statement in Step 2 that is mentioned in the introduction. We prove this in Theorems 3.2 and 3.3. We first collect information on the character codegrees of $\mathrm{SL}_2(q)$ using character degrees of $\mathrm{PSL}_2(q)$ and $\mathrm{SL}_2(q)$.

LEMMA 3.1. *Let $G=\mathrm{SL}_2(q)$ where $q>3$ is odd. Then* $\mathrm{cod}(G)$ *is given by the following:*

(a) $\mathrm{cod}(\mathrm{SL}_2(5)) = \{1, 2^2\cdot 3\cdot 5, 2\cdot 3\cdot 5, 2^2\cdot 5, 2^2\cdot 3, 3\cdot 5\}$.

(b) *If* $4 \mid (q-1)$ *and* $q>5$, *then*

$\mathrm{cod}(G) = \{1, \frac{q(q-1)}{2}, \frac{q(q+1)}{2}, \frac{(q-1)(q+1)}{2}, q(q-1), q(q+1), 2q(q+1)\}$.

(c) *If* $4 \mid (q+1)$, *then* $\mathrm{cod}(G) = \{1, \frac{q(q-1)}{2}, \frac{q(q+1)}{2}, \frac{(q-1)(q+1)}{2}, q(q-1), q(q+1), 2q(q-1)\}$.

THEOREM 3.2. *Let $q>3$ be an odd prime power and let S be a non-abelian finite simple group. If* $\mathrm{cod}(S) \subseteq \mathrm{cod}(\mathrm{SL}_2(q))$, *then* $S \cong \mathrm{PSL}_2(q)$.

Proof. We first consider the case when $q = 5$. Since $4 \leq |\mathrm{cod}(S)| \leq 6$, we can apply Theorem 2.12.

If $|\mathrm{cod}(S)| = 5$, then $S \cong \mathrm{PSL}_2(k)$, with $k > 5$ odd. Note that $|\mathrm{SL}_2(5)| = 120$. Given that $|\mathrm{PSL}_2(k)| > 120$ for $k > 5$, Lemma 2.9 gives a contradiction. We arrive at a similar contradiction when $|\mathrm{cod}(S)| = 6$. It follows that $S \cong \mathrm{PSL}_2(k)$, where $k = 2^f \geq 4$. By Lemma 2.9, we have that $k = 4$.

Now suppose that $q > 5$. By Lemma 2.10, we conclude that $4 \leq |\mathrm{cod}(S)| \leq 7$. Lemma 3.1 implies that

$$\mathrm{cod}(\mathrm{SL}_2(q)) = \{1, \frac{q(q-1)}{2}, \frac{q(q+1)}{2}, \frac{(q-1)(q+1)}{2}, q(q-1), q(q+1), 2q(q+\epsilon)\},$$

where $\epsilon = (-1)^{(q-1)/2}$.

(a) Suppose that $|\mathrm{cod}(S)| = 7$. Then using Theorem 2.12(d), we obtain that S is isomorphic to either $\mathrm{PSL}_3(3), A_7, M_{11}$, or J_1. By Theorem 2.13 and Table 1, $\mathrm{cod}(A_7)$ contains only even numbers. The codegree set $\mathrm{cod}(J_1)$ contains 3 odd numbers, hence S cannot be isomorphic to any of these groups. If $S \cong M_{11}$ or $S \cong \mathrm{PSL}_3(3)$, then $3^2 \cdot 5 \cdot 11$ and $3^3 \cdot 13$ are the only non-trivial odd number in $\mathrm{cod}(M_{11})$ and $\mathrm{cod}(\mathrm{PSL}_3(3))$, respectively. It follows that $2 \cdot 3^2 \cdot 5 \cdot 11 = q(q \pm 1)$ or $2 \cdot 3^3 \cdot 13 = q(q \pm 1)$. In both cases, no such q exists.

(b) Suppose that $|\mathrm{cod}(S)| = 6$. Then by Theorem 2.12(c), either S is isomorphic to ${}^2B_2(k)$, where $k = 2^{2f+1}$ and $f \geq 1$, or S is isomorphic to $\mathrm{PSL}_3(4)$.

Let $S \cong \mathrm{PSL}_3(4)$. Since $3^2 \cdot 5 \cdot 7$ is the only non-trivial odd number in $\mathrm{cod}(\mathrm{PSL}_3(4))$, it follows that $q(q \pm 1) = 2 \cdot 3^2 \cdot 5 \cdot 7$ which is a contradiction since no such q exists.

Now let $S \cong {}^2B_2(k), k = 2^{2f+1}$ and $f \geq 1$. It follows that

$\mathrm{cod}(S) = \{1, (k-1)(k^2+1), k^2(k-1), 2^{3f+2}(k^2+1), k^2(k-2r+1), k^2(k+2r+1)\}$, which is contained in $\mathrm{cod}(\mathrm{SL}_2(q))$ where $r = 2^f$. Now $2^{3f+2}(k^2+1)$ is the largest codegree in $\mathrm{cod}(S)$ which is not divisible by another codegree in $\mathrm{cod}(S)$, thus $\mathrm{cod}(S) = \mathrm{cod}(\mathrm{SL}_2(q)) \setminus \{2q(q+\epsilon)\}$. Hence $\mathrm{cod}(S)$ must have $q(q+1)$ as its largest codegree which is a contradiction since it is divisible by one other codegree, namely $q(q+1)/2$.

(c) Suppose that $|\mathrm{cod}(S)| = 5$. Then by Theorem 2.12(b), S is isomorphic to $\mathrm{PSL}_2(k), k > 5$. Moreover,

$$\mathrm{cod}(S) = \{1, \frac{k(k-1)}{2}, \frac{k(k+1)}{2}, \frac{k^2-1}{2}, k(k-\epsilon_1)\} \subseteq \mathrm{cod}(\mathrm{SL}_2(q)).$$

Consider $k(k-\epsilon_1) \in \mathrm{cod}(S)$, the largest codegree in $\mathrm{cod}(S)$. The codegree $2q(q+\epsilon)$ in $\mathrm{cod}(\mathrm{SL}_2(q))$ is divisible by two non-trivial codegrees in $\mathrm{cod}(\mathrm{SL}_2(q)$. If $2q(q+\epsilon) = k(k-\epsilon_1)$, it would have to follow that either $\mathrm{cod}(S) \subseteq \mathrm{cod}(\mathrm{SL}_2(q)) \setminus \{q(q+\epsilon)\}$ or $\mathrm{cod}(S) \subseteq \mathrm{cod}(\mathrm{SL}_2(q)) \setminus \{q(q+\epsilon)/2\}$ since $k(k-\epsilon_1)$ is divisible by only one codegree in $\mathrm{cod}(S)$. In the first case, $q(q+\epsilon)/2 \in \operatorname{cod}(S)$ which is a contradiction, as $k(k-\epsilon_1)$ is only twice as large as the non-trivial codegree which divides it. In the second case, $q(q+\epsilon) \in \operatorname{cod}(S)$ and $q(q+\epsilon) = k(k-\epsilon_1)/2$. When $\epsilon = \epsilon_1 = 1$, $q(q+1) = k(k-1)/2$. Since $\gcd(q, q+1) = 1$, either $k \mid q$ or $k \mid q+1$. In the former case, it follows that $q+1 \mid (k-1)/2$ and $k < (k-1)/2$, which is a contradiction, while in the latter case it follows that $q \mid (k-1)/2$ and $k-(k-1)/2 < 1$, leading to another contradiction. On the other hand, $\epsilon = \epsilon_1 = -1$ implies $k(k+1)/2 = q(q-1)$. Again, either $k \mid q$ or $k \mid q-1$. If $k \mid q$, then $q-1 \mid (k+1)/2$ and $q-1 \leq (k+1)/2 < k \leq q$ which is a contradiction since $k-(k+1)/2 > 1$. So $k \mid q-1$ and $q \mid (k+1)/2$. This is a contradiction since $(k+1)/2 < k$.

We now consider the cases where ϵ and ϵ_1 are distinct. When $\epsilon = 1$ and $\epsilon_1 = -1$, we have that $q(q-1) = k(k+1)/2$. If $k \mid q$, then $1 > k-(k+1)/2$, which is contradictory. If $k \mid q-1$, then $k < (k+1)/2$, again a contradiction. Now when $\epsilon = -1$ and $\epsilon_1 = 1$, we have $q(q+1) = k(k-1)/2$. If $k \mid q$, then $q+1 < q$, a contradiction. If $k \mid q+1$, then $1 > k-(k-1)/2$, another contradiction.

We have thus shown that $\operatorname{cod}(S) \subseteq \operatorname{cod}(\mathrm{SL}_2(q)) \setminus \{2q(q+\epsilon)\}$. Therefore $k(k-\epsilon_1)$ is either $q(q-1)$ or $q(q+1)$.

(i) Let $\epsilon = -1$.
Case 1: $k(k+1) = q(q-1)$.
If $q \mid k$, then $k+1 \mid q-1$. Therefore $q \leq k < k+1 \leq q-1$, a contradiction.
If $q \mid k+1$, then $k \mid q-1$. Thus $q \leq k+1 \leq q$ and so $k = q-1$. In particular, this means that $\frac{q(q-2)}{2} \in \operatorname{cod}(\mathrm{SL}_2(q))$, a contradiction.
Case 2: $k(k+1) = q(q+1)$.
If $q \mid k$, then $k+1 \mid q+1$. Moreover, $q \leq k$ and $k+1 \leq q+1$ which gives $k = q$.
If $q \mid k+1$, then $k \mid q+1$. This implies that $q \leq k+q \leq q+2$, that is, $q-1 \leq k \leq q+2$. Hence k is either $q, q+1$ or $q-1$. Now $k \neq q$ as q can not divide $q+1$ for $q > 5$, and if $k = q-1$, then $q-1 \mid q+1$ which is a contradiction for $q > 5$. Lastly if $k = q+1$, then $q \mid q+2$, leading to a contradiction when $q > 5$.

(ii) Let $\epsilon = 1$.
Case 1: $k(k-1) = q(q+1)$.
If $q \mid k$, then $k-1 \mid q+1$. This implies that $q \leq k$ and $k-1 \leq q+1$, so $q-1 \leq k-1 \leq q+1$. Hence $q \leq k \leq q+2$. We therefore have that $k = q$, $k = q+1$, or $k = q+2$. Since $q \mid k$ and $\gcd(q, q+1) = 1$, it follows that $k \neq q+1$. Also, $k \neq q+2$ since this would imply that $q = aq-2$, a contradiction as $q > 5$ and $a \geq 0$.
If $k = q$, then $q-1 \mid q+1$, a contradiction since $q > 5$.
If $q \mid k-1$, then $k \mid q+1$. Thus $q \leq k-1$ and $k \leq q+1$ which implies that $q+1 \leq k \leq q+1$, hence $k = q+1$. This is a contradiction since this would mean $\frac{q(q+2)}{2} \in \operatorname{cod}(\mathrm{SL}_2(q))$, but $q(q+2)$ is odd so $\frac{q(q+2)}{2} \notin \mathbb{N}$.

Case 2: $k(k-1) = q(q-1)$.

If $q \mid k-1$, then $k \mid q-1$, which implies that $q \leq k-1 \leq k \leq q-1$, a contradiction.

However, if $q \mid k$, then $k-1 \mid q-1$. Hence $q \leq k$ and $k-1 \leq q-1$, implying that $k = q$. Therefore, $S \cong \mathrm{PSL}_2(q)$.

(d) Suppose that $|\mathrm{cod}(S)| = 4$. Then by Theorem 2.12(a), S is isomorphic to $\mathrm{PSL}_2(k)$ with $k = 2^t \geq 4$. Furthermore, $\mathrm{cod}(S) = \{1, k(k-1), k(k+1), k^2 - 1\} \subseteq \mathrm{cod}(\mathrm{SL}_2(q))$.

  (i) Let $4 \mid q-1$.

  Since k^2-1 is odd, we must have that $(k-1)(k+1) = \frac{q(q+1)}{2}$. Note that $\gcd(k-1, k+1) = 1$, so either $q \mid k-1$ or $q \mid k+1$. If $q \mid k-1$, then $k+1 \mid \frac{q+1}{2}$. But $\frac{q+1}{2} < q$ (since $q > 5$), hence $k+1 < k-1$ which is a contradiction. If $q \mid k+1$, then $k-1 \mid \frac{q+1}{2}$. Therefore $k-1 \leq \frac{q+1}{2} \leq q \leq k+1$, but $q - \frac{q+1}{2} > 2$ (since $q > 5$), thus it follows that $k+1-(k-1) > 2$, which is a contradiction.

  (ii) Let $4 \mid q+1$.

  Since k^2-1 is odd, we must have that $(k-1)(k+1) = \frac{q(q-1)}{2}$. Note that $\gcd(k-1, k+1) = 1$, so either $q \mid k-1$ or $q \mid k+1$. If $q \mid k-1$, then $k+1 \mid \frac{q-1}{2}$. But $\frac{q-1}{2} < q$, hence $k+1 < k-1$, which is a contradiction. If $q \mid k+1$, then $k-1 \mid \frac{q-1}{2}$. Therefore $k-1 \leq \frac{q-1}{2} \leq q \leq k+1$, but $q - \frac{q-1}{2} > 2$ (since $q > 5$), thus it follows that $k+1-(k-1) > 2$, which is a contradiction.

This concludes our proof. □

Given that the center of $\mathrm{SL}_3(q)$ has an order of $\gcd(3, q-1) = 1$ when $q \not\equiv 1 \pmod 3$ (q is a prime power), it follows that $\mathrm{PSL}_3(q) = \mathrm{SL}_3(q)$. This was dealt with by Yang and Liu in [22].

Suppose that $4 < q \equiv 1 \pmod 3$. The character degree set of $\mathrm{PSL}_3(q)$ is given by (see [27]):

$$\begin{aligned}\mathrm{cd}(\mathrm{PSL}_3(q)) = \{&1, q^3, q(q+1), (q-1)^2(q+1), q(q^2+q+1), (q-1)(q^2+q+1),\\ &q^2+q+1, (q+1)(q^2+q+1), (q+1)(q^2+q+1)/3\}.\end{aligned}$$

From [15], the character degree set of $\mathrm{SL}_3(q)$ is

$$\begin{aligned}\mathrm{cd}(\mathrm{SL}_3(q)) = \{&1, q^3, q(q+1), (q-1)^2(q+1), q(q^2+q+1), (q-1)(q^2+q+1),\\ &q^2+q+1, (q+1)(q^2+q+1), (q+1)(q^2+q+1)/3, (q-1)^2(q+1)/3\},\end{aligned}$$

We see that $(q-1)^2(q+1)/3$ is the only degree of $\mathrm{SL}_3(q)$ which comes from a faithful irreducible character and so, with the aid of [22, Table 1], when $7 \leq q \equiv 1$

(mod 3), the codegrees set of $\mathrm{SL}_3(q)$ is

$$\begin{aligned}&\mathrm{cod}(\mathrm{SL}_3(q))\\&\quad=\{1, \frac{1}{3}(q^2+q+1)(q+1)(q-1)^2, \frac{1}{3}q^2(q^2+q+1)(q-1)^2, \frac{1}{3}q^3(q^2+q+1),\\&\quad\frac{1}{3}q^2(q+1)(q-1)^2, \frac{1}{3}q^3(q-1)(q+1), \frac{1}{3}q^3(q+1)(q-1)^2, \frac{1}{3}q^3(q-1)^2,\\&\quad q^3(q-1)^2, 3q^3(q^2+q+1)\}.\end{aligned}$$

For $\mathrm{SL}_3(4)$, we have that

$$\begin{aligned}\mathrm{cod}(\mathrm{SL}_3(4)) =&\{1, 2^6\cdot 3^2\cdot 7, 2^6\cdot 3^2\cdot 5, 2^6\cdot 3\cdot 7, 2^4\cdot 3^2\cdot 7, 2^6\cdot 3\cdot 5,\\&2^4\cdot 3^2\cdot 5, 2^6\cdot 3^2, 2^6\cdot 7, 2^6\cdot 5, 3^2\cdot 5\cdot 7\}.\end{aligned}$$

For a prime p, we denote the p-part of an integer n by n_p.

THEOREM 3.3. *Let* $4 \le q \equiv 1 \pmod 3$ *where* q *is a prime power and let* S *be a non-abelian finite simple group. If* $\mathrm{cod}(S) \subseteq \mathrm{cod}(\mathrm{SL}_3(q))$, *then* $S \cong \mathrm{PSL}_3(q)$.

Proof. Suppose that $q = 4$. Since S is non-abelian finite simple group such that $\mathrm{cod}(S) \subseteq \mathrm{cod}(\mathrm{SL}_3(4))$, we have that $4 \le |\mathrm{cod}(S)| \le 11$. Note that $|\mathrm{SL}_3(4)| = 60480$. By Lemma 2.9, S cannot be isomorphic to $\mathrm{PSL}_3(3), M_{11}, J_1, M_{22}, M_{12}, M_{23}, \mathrm{PSL}_3(k)$ where either $4 < k \equiv 1 \pmod 3$ or $4 < k \not\equiv 1 \pmod 3$,$\mathrm{PSU}_3(k)$ where either $4 < k \equiv -1 \pmod 3$ or $4 < k \not\equiv -1 \pmod 3$, ${}^2G_2(k)$ where $k = 3^{2f+1}$ with $f \ge 1$, ${}^2B_2(k)$ where $k = 3^{2f+1}$ with $f \ge 1$. Furthermore, S cannot be isomorphic to $A_7, G_2(2)'$, or $\mathrm{PSL}_4(2)$ as these groups contain a codegree not in $\mathrm{cod}(\mathrm{SL}_3(4))$. We now apply Theorem 2.12. Note that we only have to consider $|\mathrm{cod}(S)| = 4, 5$ and 6.

(a) Let $|\mathrm{cod}(S)| = 4$. Then $S \cong \mathrm{PSL}_2(k)$ where $k = 2^f \ge 4$. It follows that $k^2 - 1 = 3^2\cdot 5\cdot 7$. This is a contradiction as the equation has no integer solution.

(b) Let $|\mathrm{cod}(S)| = 5$. Then $S \cong \mathrm{PSL}_2(k)$ where $k > 5$ is odd. It follows that $k(k\pm 1) = 2\cdot 3^3\cdot 5\cdot 7$. These equations have no integer solution, a contradiction.

(c) Hence $|\mathrm{cod}(S)| = 6$ and $S \cong \mathrm{PSL}_3(4)$ as required.

Now suppose that $q \ge 7$. Since S is a non-abelian finite simple group and $\mathrm{cod}(S) \subseteq \mathrm{SL}_3(q)$, we see that $4 \le |\mathrm{cod}(S)| \le 10$.

We appeal to Theorem 2.12 and Theorem 2.13.

(a) Let $|\mathrm{cod}(S)| = 4$. Then $S \cong \mathrm{PSL}_2(k)$ where $k = 2^f \ge 4$. Furthermore,

$$\mathrm{cod}(\mathrm{PSL}_2(k)) = \{1, k(k-1, k(k+1), k(k+1), k^2-1)\}.$$

Let q be even and consider $k(k-1)$. Since $k(k-1)$ is even, it follows that $k(k-1) \in \mathrm{cod}(\mathrm{SL}_3(q)) \setminus \{\frac{1}{3}(q^2+q+1)(q+1)(q-1)^2\}$. If we first assume that $k(k-1) = \frac{1}{3}q^2(q^2+q+1)(q-1)^2$, then we can see that $k = (k(k-1))_2 =$

$(\frac{1}{3}q^2(q^2+q+1)(q-1)^2)_2 = q^2$. The rest of the remaining cases are handled similarly; implying that either $k = q^2$ or $k = q^3$. However, this implies that either $q^4 - 1$ or $q^6 - 1$ are in $\mathrm{cod}(\mathrm{SL}_3(q))$, a contradiction.

Now let q be odd. In this case, the only odd codegrees of $\mathrm{SL}_3(q)$ are $\frac{1}{3}q^3(q^2+q+1)$ and $3q^3(q^2+q+1)$. We first assume that $k^2 - 1 = (k-1)(k+1) = \frac{1}{3}q^3(q^2+q+1)$. Since $\gcd(k+1, k-1) = 1$, we can see that either $q^3 \mid (k+1)$ or $q^3 \mid (k-1)$. If $q^3 \mid (k+1)$, then $3(k-1) \mid (q^2+q+1)$. This results in the inequality $q^3 \leq k+1 < 3(k-1) \leq q^2+q+1$, a contradiction since $q \geq 7$. Therefore $q^3 \mid (k-1)$, which implies that $3(k+1) \mid (q^2+q+1)$. This again yields the inequality $q^3 < q^2+q+1$, a contradiction. Hence it follows that $k^2 - 1 = (k-1)(k+1) = 3q^3(q^2+q+1)$. So either $q^3 \mid (k+1)$ or $q^3 \mid (k-1)$. If $q^3 \mid (k+1)$, then $(k-1) \mid 3(q^2+q+1)$. This gives the inequality $k - 1 \leq 3(q^2+q+1) < q^3 \leq k+1$; implying that $k+1-(k-1) = 2 > q^3 - 3(q^2+q+1) \geq 172$ for $q \geq 7$, a contradiction. Therefore $q^3 \mid (k-1)$ and so $(k+1) \mid 3(q^2+q+1)$. This implies that $k - 1 > k + 1$, a contradiction.

(b) Let $|\mathrm{cod}(S)| = 5$. Then $S \cong \mathrm{PSL}_2(k)$ where $k > 5$ is odd. However this implies that $\mathrm{cod}(\mathrm{SL}_3(q))$ contains two codegrees such that one is half the other, this is a contradiction.

(c) Let $|\mathrm{cod}(S)| = 6$. Then either $S \cong {}^2B_2(k)$ where $k = 2^{2f+1}$ and $f \geq 1$, or $S \cong \mathrm{PSL}_3(4)$.

Let $S \cong \mathrm{PSL}_3(4)$. If q is even, then $3^3 \cdot 5 \cdot 7 = (q^2+q+1)(q+1)(q-1)^2$. This equation results in the solution $q = 4$, a contradiction since we must have that $q \geq 7$. Hence it follows that q is odd. If $3^3 \cdot 5 \cdot 7 = q^3(q^2+q+1)$, then we have a contradiction since the equation has no integer solution. Thus it follows that $3 \cdot 5 \cdot 7 = q^3(q^2+q+1)$ which also results in no integer solution, a contradiction.

Therefore we can assume that $S \cong {}^2B_2(k)$ where $k = 2^{2f+1}$ and $m = 2^f$.

Let q be even. Note that the only two non-trivial 2-parts of codegrees in $\mathrm{SL}_3(q)$ are q^2 and q^3, this implies that $k^2 = q^3$ and $2^{2f+1} = q^2$. We therefore see that $q = 2^f$. This implies that $q^3 = 2^{3f} = k^2$, which is a contradiction.

Let q be odd. Then codegree set of $\mathrm{SL}_3(q)$ has exactly 3 codegress with the same 2-part; namely $((q+1)(q-1)^2)_2$. Similarly, ${}^2B_2(k)$ has exactly 3 codegrees with the same 2-part k^2. It then follows that $\frac{1}{3}q^3(q+1)(q-1)^2 = k^2(q+2m+1)$, $\frac{1}{3}q^2(q+1)(q-1)^2 = k^2(k-2m+1)$, and $\frac{1}{3}(q^2+q+1)(q+1)(q-1)^2 = k^2(k-2)$. This leads to the conclusion that $(q+2m+1) = q(2-2m+1)$, a contradiction.

(d) Let $|\mathrm{cod}(S)| = 7$. Then $S \cong \mathrm{PSL}_3(3), A_7, J_1$ or M_{11}.

Let $S \cong \mathrm{PSL}_3(3)$. Note that $3^3 \cdot 13$ is the only odd codegree of $\mathrm{PSL}_3(3)$. Let q be even. It then follows that $3^4 \cdot 13 = (q^2+q+1)(q+1)(q-1)^2$, a contradiction since this equation has no integer solution. If q is odd, then either $3^4 \cdot 13 = q^3(q^2+q+1)$ or $3^2 \cdot 13 = q^3(q^2+q+1)$. Both equations have no integer solution.

Let $S \cong A_7$. If q is even, then every codegree of $\mathrm{SL}_3(q)$ has a 2-part q^2 or q^3. However every codegree of A_7 has a 2-part which is either 2^2 or 2^3. This implies that $q = 2$, a contradiction. If q is odd, then $((q-1)^2(q+1))_2 = 2^3$. Hence we can see that $(q-1)_2 = (q+1)_2$, a contradiction.

Let $S \cong J_1$. Note that J_1 has two non-trivial odd codegrees. This implies that q is odd. However this would force $3 \cdot 5 \cdot 11 \cdot 19 = \frac{1}{9}(7 \cdot 11 \cdot 19)$ since the two non-trivial odd codegrees of $\mathrm{SL}_3(q)$ are $\frac{1}{3}q^3(q^2+q+1)$ and $3q^3(q^2+q+1)$.

So we have that $S \cong M_{11}$. If q is even, then every codegree of $\mathrm{SL}_3(q)$ has a 2-part which is either q^2 or q^3. This is a contradiction as every codegree of M_{11} must have a 2-part which is either $2^2, 2^3$ or 2^4. If q is odd, then either $3^3 \cdot 5 \cdot 11 = q^3(q^2+q+1)$ or $3 \cdot 5 \cdot 11 = q^3(q^2+q+1)$. Both equations have no integer solution, a contradiction.

(e) Let $|\mathrm{cod}(S)| = 8$. Then $S \cong \mathrm{PSL}_3(k)$ where $4 < k \not\equiv 1 \pmod 3$, $S \cong \mathrm{PSU}_3(k)$ where $4 < k \not\equiv -1 \pmod 3$, or $S \cong G_2(2)'$.

Let $S \cong \mathrm{PSL}_3(k)$ where $4 < k \not\equiv 1 \pmod 3$. We can see that $\frac{1}{3}q^3(q+1)(q-1)^2 = k^3(k+1)(k-1)^2$ as there is only one codegree divided by three non-trivial codegrees in $\mathrm{cod}(\mathrm{SL}_3(q))$ and $\mathrm{cod}(\mathrm{PSL}_3(q))$. Equating the smallest of the codegrees which divide the single codegree implies that $\frac{1}{3}q^3(q-1)^2 = k^3(k-1)^2$. We therefore have that $q+1 = k+1$; that is, $q = k$. This is a contradiction since $q \equiv 1 \pmod 3$ but $k \not\equiv 1 \pmod 3$.

Let $S \cong \mathrm{PSU}_3(k)$ where $4 < k \not\equiv -1 \pmod 3$. Similar to the above paragraph, we can see that $\frac{1}{3}q^3(q+1)(q-1)^2 = k^3(k+1)^2(k-1)$ and $\frac{1}{3}q^3(q-1)^2 = k^3(k+1)(k-1)$. Which implies that $q = k$, a contradiction.

Lastly, let $S \cong G_2(2)'$. If q is even, it follows that $3^4 \cdot 7 = (q^2+q+1)(q+1)(q-1)^2$ which has no integer solution. If q is odd, then either $3^2 \cdot 7 = q^3(q^2+q+1)$ or $3^4 \cdot 7 = q^3(q^2+q+1)$. Both of which have no integer solution, a contradiction.

(f) Hence $|\mathrm{cod}(S)| = 9$. Then either $S \cong \mathrm{PSL}_3(k)$ where $4 < k \equiv 1 \pmod 3$ or $S \cong \mathrm{PSU}_3(k)$ where $4 < k \equiv -1 \pmod 3$.

If $S \cong \mathrm{PSU}_3(k)$ where $4 < k \equiv -1 \pmod 3$. Then we can see that $\frac{1}{3}q^3(q+1)(q-1)^2 = \frac{1}{3}k^3(k+1)^2(k-1)$ as there is only one codegree divided by three non-trivial codegrees in both $\mathrm{cod}(\mathrm{SL}_3(q))$ and $\mathrm{cod}(\mathrm{PSU}_3(q))$. Moreover, we have that $\frac{1}{3}q^3(q-1)^2 = \frac{1}{3}k^3(k+1)(k-1)$ after comparing the smallest of the codegrees dividing the single codegree; implying that $q = k$, a contradiction.

Hence it must follow that $S \cong \mathrm{PSL}_3(k)$ where $4 < k \equiv 1 \pmod 3$. A similar analysis to the above paragraph yields that $7 \leq k = q$.

(g) Let $|\mathrm{cod}(S)| = 10$. Then $S \cong M_{22}$ and $\mathrm{cod}(M_{22}) = \mathrm{cod}(\mathrm{SL}_3(q))$. This is a contradiction since M_{22} has only even codegrees. The proof is now complete. □

COROLLARY 3.4. *Let q be a prime power and let N be a maximal normal subgroup of a finite group G. If $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, then G is perfect and either $G/N \cong$*

$\mathrm{PSL}_2(q)$ *when* $H = \mathrm{SL}_2(q)$ *with* $q > 3$ *odd or* $G/N \cong \mathrm{PSL}_3(q)$ *when* $H = \mathrm{SL}_3(q)$ *with* $4 \leq q \equiv 1 \pmod 3$.

Proof. By Lemma 2.8, it follows that G is perfect and G/N is a non-abelian finite simple group. Therefore, by Theorems 3.2 and 3.3, $G/N \cong H/Z(H)$. □

THEOREM 3.5. *Let* q *be a prime power and suppose that either* $H = \mathrm{SL}_2(q)$ *with* $q > 3$ *odd or* $H = \mathrm{SL}_3(q)$ *with* $q \equiv 1 \pmod 3$ *and* $q \geqslant 4$. *Let* G *be a finite group containing a unique normal subgroup. If* $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, *then* $G \cong H$.

Proof. Let N be the unique normal subgroup of G. It follows, from Corollary 3.4, that G is perfect and $G/N \cong H/Z(H)$.

(a) $|N| \,|\, |G/N|$.

Choose $1_N \neq \vartheta \in \mathrm{Irr}(N)$. Let $\psi \in \mathrm{Irr}(T|\vartheta)$, where $T = I_G(\vartheta)$. Set $\chi = \psi^G \in \mathrm{Irr}(G)$ by Clifford Theory. Since χ lies over ϑ, it follows that $\chi \in \mathrm{Irr}(G|N)$ and χ is faithful. Furthermore, $\chi(1) = |G : T|\,\psi(1)$, and

$$\mathrm{cod}(\chi) = \frac{|G|}{|G:T|\,\psi(1)} = \frac{|T|}{\psi(1)} = |T:N|\,\frac{|N|}{\psi(1)}.$$

However, $\psi(1) |\, |T : N|$ by [19, Theorem 6.15], and so $|N|\, |\mathrm{cod}(\chi)$; moreover, every member of $\mathrm{cod}(G/N)$ divides $|G/N|$.

(i) $H = \mathrm{SL}_2(5)$. Note that $2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 5 \in \mathrm{cod}(G|N)$ both divide $2^2 \cdot 3 \cdot 5 = |\mathrm{PSL}_2(5)|$.

(ii) $H = \mathrm{SL}_2(q)$ with $q > 5$ odd. It follows that either $q(q+1), 2q(q+1) \in \mathrm{cod}(G|N)$ when $4 \mid (q-1)$ or $q(q-1), 2q(q-1) \in \mathrm{cod}(G|N)$ when $4 \mid (q+1)$. In either case, both codegrees divide $|\mathrm{PSL}_2(q)| = q(q-1)(q+1)/2 = |G/N|$.

(iii) $H = \mathrm{SL}_3(4)$. We see that $2^6 \cdot 3^2 \cdot 5 \cdot 7 = |\mathrm{PSL}_3(4)|$ is divisible by every member of $\mathrm{cod}(G|N) = \{2^6 \cdot 3^2 \cdot 7, 2^6 \cdot 3^2 \cdot 5, 2^6 \cdot 3 \cdot 5, 2^6 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5\}$.

(iv) $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$. It follows that $3q^3(q^3+q+1) \in \mathrm{cod}(G|N)$ divides $\frac{1}{3}q^3(q^3-1)(q^2-1) = \frac{1}{3}q^3(q^2+q+1)(q+1)(q-1)^2 = |\mathrm{PSL}_3(q)| = |G/N|$ since $7 \leq q \equiv 1 \pmod 3$.

Thus, every member of $\mathrm{cod}(G)$ divides $|G/N|$ and so $|N| \mid |G/N|$.

(c) $N = Z(G)$.

Since $N \trianglelefteq G$, we have that $C_G(N) \trianglelefteq G$. From the fact that N is abelian and maximal normal in G, it follows that either $C_G(N) = G$ or $C_G(N) = N$. For a contradiction, suppose that $C_G(N) = N$. Now we know that $|N| = r^t$ for some prime r and positive integer t as N is elementary abelian. But if $t = 1$, then by the Normalizer-Centralizer theorem

$$N_G(N)/C_G(N) = G/N \cong U,$$

where U is some subgroup of $\mathrm{Aut}(N) \cong C_{r-1}$. This is impossible since G/N is non-abelian. Hence $t > 1$.

(i) $H = \mathrm{SL}_3(4)$. By (b), $|N|$ is either $2^6, 2^5, \ldots, 2^2$ or 3^2. With an application of the Normalizer-Centralizer theorem, we can embed $\mathrm{PSL}_3(4)$ into $\mathrm{Aut}(N) \cong \mathrm{GL}_t(r)$. After considering if $|N| \mid |\mathrm{GL}_t(r)|$ for each possible order of N, we see that the only possible orders of N are $2^4, 2^5$ and 2^6.

Let $1_N \neq \vartheta \in \mathrm{Irr}(N)$ and denote $I_G(\vartheta)$ by T. We have that $|N| = |\mathrm{Irr}(N)| > |G : T|$. Let $Z = Z(\mathrm{SL}_3(4))$ and let K be a maximal subgroup of $\mathrm{SL}_3(4)$ containing Z such that T/N is isomorphic to a subgroup of K/Z. It follows that $|G : T| = |G/N : T/N| \geq |G/N : K/Z|$ since K/Z must be maximal in G/N. If $K \cong (2^2 \times 2^2) : \mathrm{GL}_2(4)$, then $|N| > 21$. If $K \cong \mathrm{SL}_3(2).C_3$, then $|N| > 120$. If $K \cong \mathrm{SU}_3(2)$, then $|N| > 280$. If $K \cong 3 \cdot A_6$, then $|N| > 56$. Therefore, $|N|$ is either 2^5 or 2^6.

Using [12], we can see that the group $\mathrm{GL}_5(2)$ contains exactly one subgroup of order $|\mathrm{PSL}_3(4)|$ up to conjugacy; namely, A_8. Thus we cannot embed $\mathrm{PSL}_3(4)$ into $\mathrm{GL}_5(2)$. Similarly, we cannot embed $\mathrm{PSL}_3(4)$ into $\mathrm{GL}_6(2)$.

(ii) $H = \mathrm{SL}_2(5)$. By (b), $|N| = 2^2$, which is a contradiction since we cannot embed A_5 into S_3.

(iii) $H = \mathrm{SL}_2(q)$ with $q > 5$ odd. Note that $r^t \mid |\mathrm{PSL}_2(q)| = \frac{1}{2}q(q-1)(q+1)$ by (b). We show that $r^t \leq q+1$. Let $q - 1 = 2^m$ for some integer m. It follows that 2 and an odd prime divide $q+1$. Hence every prime power factor of $q+1$ is less than or equal to q. Furthermore, $(q+1)/2$ is odd which implies that $\gcd(q-1, (q+1)/2) = 1$. So q is the largest prime power divisor of $\frac{1}{2}q(q-1)(q+1)$. Meaning that $r^t \leq q$.

Assume q is a perfect power. If $q + 1 = 2^m$, then Theorem 2.1 yields a contradiction. Therefore $q+1$ is divisible by 2 and an odd prime. Since $\gcd(q+1, (q-1)/2) = 1$, it follows that $\frac{1}{2}q(q-1)(q+1)$ has q as its largest prime power divisor. If q is prime and q is not a Mersenne prime ($q+1 \neq 2^m$ for every integer m), then $q+1$ must be divisible by 2 and a prime greater than equal to 3 by the assumptions on q. It again follows that q is still the largest prime power divisor of $\frac{1}{2}q(q-1)(q+1)$. Thus, q is the largest prime power divisor of $|\mathrm{PSL}_2(q)|$ and so $r^t \leq q$ when q is not a Mersenne prime.

In the Mersenne prime case, it follows that $q + 1 = 2^m$ is the largest prime power divisor of $|\mathrm{PSL}_2(q)|$ for some integer m, meaning that $r^t \leq q+1$.

We have that $|N| = |\mathrm{Irr}(N)| > |G : T|$, where $T = I_G(\vartheta)$ with $\vartheta \neq 1_N$. We now show that $|N| > q+1$. Choose a maximal subgroup K of $\mathrm{SL}_2(q)$ containing $Z = Z(G)$ such that T/N is isomorphic to a subgroup of K/Z. Then $|G : T| = |G/N : T/N| \geq |G/N : K/Z|$ as K/Z is a maximal subgroup of G/N containing T/N. In the case that $K \cong 3^{\cdot}A_5$ with $q = 11$, it follows that $|N| > |G/N : K/Z| = q(q-1)(q+1)/120 = q$. This implies that $|N| = 12$ (since $|N| \leq q+1$) which is impossible since N is elementary abelian. Furthermore, if $K \cong 2_{-}^{1+2} {}^{\cdot} S_3$ with $q = 7$, then $|N| > |G/N : K/Z| = q(q-1)(q+1)/48 = q$. Hence $|N| = 8$ and so G is a group of order 1344. Using GAP's Small Group library, we can filter through all 11720 groups of order 1344 leaving us with exactly two groups with a unique minimal normal subgroup. These are $2^3 {}^{\cdot}\mathrm{PSL}_2(7)$ and $2^3 : \mathrm{PSL}_2(7)$ (`SmallGroup(1344,814)` and `SmallGroup(1344, 11686)`, respectively). However, if G is isomorphic to either of these two groups, then $\mathrm{cod}(G) = \{1, 21, 24, 28, 56, 64, 96, 192\} \nsubseteq \mathrm{cod}(\mathrm{SL}_2(7))$, a contradiction. Otherwise, from Table 2, $|N| > q + 1$, a contradiction.

(iv) $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$. By (b), we have that $r^t \mid \frac{1}{3}q^3(q^3 - 1)(q^2 - 1) = |\mathrm{PSL}_3(q)|$. Note that $\frac{1}{3}q^3(q^3 - 1)(q^2 - 1) = \frac{1}{3}q^3(q^2 + q + 1)(q + 1)(q - 1)^2$. Futhermore, $\gcd(q^2 + q + 1, q - 1) = 1$, $\gcd(q^2 + q + 1, q + 1) = 1$, and either $\gcd(q+1, q-1) = 1$ or $\gcd(q+1, q-1) = 2$. Hence we see that q^3 is the largest prime power divisor of $|\mathrm{PSL}_3(q)|$. This implies that $|N| = r^t \leq q^3$.

Let $1_N \neq \vartheta \in \mathrm{Irr}(N)$ and denote $I_G(\vartheta)$ by T. Now $|N| = |\mathrm{Irr}(N)| > |G : T|$. Let $Z = Z(\mathrm{SL}_3(q))$ and let K be a maximal subgroup of $\mathrm{SL}_3(q)$ containing Z such that T/N is isomorphic to a subgroup of K/Z. It follows that $|G : T| = |G/N : T/N| \geq |G/N : K/Z|$ since K/Z is maximal in G/N. The following argument is from the proof of [22, Theorem 3.3], we include it for completeness. If K is not of type $(p^n \times p^n) : \mathrm{GL}_2(q)$, then $|N| > q^3$, which is a contradiction. Hence K must be of type $(p^n \times p^n) : \mathrm{GL}_2(q)$, implying that $|N| > |G : T| \geq |G/N : K/Z| = q^2+q+1$. Now if $r = p$ (recall that $|N| = r^t$), then $q^2 < |N| \leq q^3$. When $r \neq p$, it follows that $r = 2$ and $|N| = 2(q - 1)^2$ where $q - 1$ is a power of 2. Note that by (b), it follows that $|T|/\theta(1)$ is a codegree of G for all $\theta \in \mathrm{Irr}(T \mid \vartheta)$. If either $|N| = q^3$ where $r = p$ or $|N| = 2(q - 1)^2$ where $q - 1$ is a 2-power, we have $|T/N|/\theta(1)$ is relatively prime to r since $|N|$ is the r-part of $|G/N|$. Let a be the r-part of $|T/N|$, then the r-part of $\theta(1)$ is also a. Clifford theory (see [19, Chapter 6]) implies that $|T/N|$ is the sum of $\theta(1)^2$ for $\theta \in \mathrm{Irr}(T \mid \vartheta)$. However, it follows that $a^2 \mid a$ and so $a = 1$; that is, $|T/N|$ is relatively prime to r. If $r = 2$ and $q - 1$ is a power of 2, then $|T/N| \leq q^3(q + 1)$. If $r = p$, then T/N is isomorphic to a subgroup of $\mathrm{GL}_2(q)/Z$ which means that $|T/N| \leq (q - 1)^2(q + 1)/\gcd(3, q - 1)$. This is a contradiction since $|G : T| > q^3$. Now assume that $r = p$ and $q^2 < |N| < q^3$. It can be checked that q^3 is the p-part of $|T|/\theta(1)$ and that the p-part of $|T/N|/\theta(1)$ is less that q, or equivalently, $|T/N|_p/q < \theta(1)_p$ for all $\theta \in \mathrm{Irr}(T \mid \vartheta)$. Note that $|T/N|$ is the sum of $\theta(1)^2$ for all $\theta \in \mathrm{Irr}(T \mid \vartheta)$. So $|T/N|_p < q^2$ and $|N| > |G : T| > q(q^2 + q + 1)$, which is a contradiction.

Therefore, we see that $C_G(N) = G$.

(d) $G \cong H$.

We therefore assume that $N \subseteq G' \cap Z(G)$. The group N is therefore embedded into $M(G/N)$.

(i) $H = \mathrm{SL}_3(4)$. It follows that $M(G/N) = C_4 \times C_{12}$. We thus have that G is isomorphic to either $2 \cdot \mathrm{PSL}_3(4), 2' \cdot \mathrm{PSL}_3(4), 2'' \cdot \mathrm{PSL}_3(4)$, or $3 \cdot \mathrm{PSL}_3(4)$ (see [9, page 23]). However $\mathrm{cod}(2 \cdot \mathrm{PSL}_3(4)) = \mathrm{cod}(\mathrm{PSL}_3(4)) \cup \{4030, 1440, 1120, 630, 576, 448\}$, $\mathrm{cod}(2' \cdot \mathrm{PSL}_3(4)) = \mathrm{cod}(\mathrm{PSL}_3(4)) \cup \{5040, 720, 630, 504\}$, and $\mathrm{cod}(2'' \cdot \mathrm{PSL}_3(4)) = \mathrm{cod}(\mathrm{PSL}_3(4)) \cup \{2016, 1440, 1120, 630, 504\}$. Thus $G \cong 3 \cdot \mathrm{PSL}_3(4) \cong \mathrm{SL}_3(4)$.

(ii) $H = \mathrm{SL}_2(5)$. It follows that $M(G/N) = M(\mathrm{PSL}_2(5)) = C_2$ and so $N \cong C_2$. Therefore $G \cong \mathrm{SL}_2(5)$.

(iii) $H = \mathrm{SL}_2(q)$ with $q > 5$ odd. When $q \neq 9$, given that $M(\mathrm{PSL}_2(q)) = C_2$, it follows from Lemma 2.5 that $N \cong C_2$ and $G \cong \mathrm{SL}_2(q)$. However, $M(\mathrm{PSL}_2(9)) = C_6$ and so either $N \cong C_2$ or $N \cong C_3$. This means that $G \cong \mathrm{SL}_2(q)$ or $G \cong 3 \cdot A_6$. It follows that $G \cong \mathrm{SL}_2(q)$ since $\mathrm{cod}(3 \cdot A_6) = \{360, 180, 120, 72, 45, 40, 36, 1\}$.

(iv) $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$. Hence N can be embedded in $M(\mathrm{PSL}_3(q)) = C_3$, implying that $N \cong C_3$. Therefore $G \cong \mathrm{SL}_3(q)$. □

Theorem 3.6. *Let q be a prime power and suppose that either $H = \mathrm{SL}_2(q)$ with $q > 3$ odd or $H = \mathrm{SL}_3(q)$ with $q \equiv 1 \pmod 3$ and $q \geqslant 4$. Let N be a normal subgroup (possibly trivial) of a finite group G. If $G/N \cong H$ and $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, then $N = 1$.*

Proof. Suppose that $1 < N$. Let $M \leq N$ be normal in G such that N/M is a chief factor of G. Without loss of generality, we may assume that N is minimal normal in G.

(a) N is a unique minimal normal subgroup of G.

Let $S \neq N$ be a minimal normal subgroup of G. We assume $S \subseteq K$ is a maximal normal subgroup of G. In fact, it follows that $S, N \subseteq K$.

Suppose that $N \nsubseteq K$. Then we have a direct product $N \times K$. However, since K is maximal normal in G, it follows that $G = N \times K$ and $N \cong \mathrm{PSL}_2(q)$ by Corollary 3.4. Furthermore, $K \cong \mathrm{SL}_2(q)$ and so $G \cong \mathrm{PSL}_2(q) \times \mathrm{SL}_2(q)$ (respectively, $G \cong \mathrm{PSL}_3(q) \times \mathrm{SL}_3(q)$). This means that G has a factor group isomorphic group to $\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)$ (respectively, $\mathrm{PSL}_3(q) \times \mathrm{PSL}_3(q)$) and by Lemma 2.11, we have that $|\mathrm{cod}(G)| \geqslant 10 > |\mathrm{cod}(\mathrm{SL}_2(q))|$ ($|\mathrm{cod}(G)| \geqslant 14 > |\mathrm{cod}(\mathrm{SL}_3(q))|$), a contradiction.

Note that $N \subseteq NS \subseteq K$ but K/N is a proper normal subgroup of G/N. Also, $S \cong NS/N$ is a proper normal subgroup of G/N, this forces $K/N = NS/N \cong S \cong Z(H)$ and so $K = N \times S$. If there exists $1 < U/S < K/S$ normal in G/S, we have a normal subgroup U of G such that $S < U < K$. This means $U \cap N > 1$ and $U = K = N \times S$ by minimality of N. This is a contradiction. Since K/S is also a maximal normal subgroup of G/S, it follows that $K/S \cong N$ is the unique normal subgroup of G/S. An application of Theorem 3.5 yields $G/S \cong H$.

We can therefore assume that $N \cong S \cong Z(H)$. By Lemma 2.7, it follows that $K \subseteq Z(G) \cap G'$. If $H = \mathrm{SL}_2(q)$, then $Z(H) = C_2$. However, since $M(\mathrm{PSL}_2(q)) = C_2$ when $q \neq 9$ and $M(\mathrm{PSL}_2(9)) = C_6$, Lemma 2.5 gives a contradiction. If $H = \mathrm{SL}_3(q)$, then $Z(H) = C_3$. Then since $M(\mathrm{PSL}_3(q)) = C_3$, we again have a contradiction since K cannot be embedded into C_3. Hence the result follows.

(b) N is elementary abelian.

Suppose that N is non-abelian. Then since N is minimal normal in G, it follows that $N = S^n$ where S is a non-abelian simple group. Now choose $\chi \in \mathrm{Irr}(G)$ which extends $1_N \neq \psi \in \mathrm{Irr}(N)$. If we assume $\ker\chi > 1$, then it must contain N. This implies that $\chi(1) > 1$ since ψ is not principal. But then $\psi(n) = \chi_N(n) = \chi(1) = \psi(1)$ for all $n \in N$, which is a contradiction by Proposition 2.2. Hence, χ must be faithful.

This yields the codegree given by

$$\mathrm{cod}(\chi) = \frac{|G|}{\chi(1)} = |G:N|\,\frac{|N|}{\psi(1)} = |H|\,\frac{|N|}{\psi(1)},$$

which implies that $|H| \mid \mathrm{cod}(\chi)$, a contradiction. Thus N is abelian and so N is elementary abelian.

(c) $|N| \mid |G/N|$.

Choose $1_N \neq \vartheta \in \mathrm{Irr}(N)$. Let $\psi \in \mathrm{Irr}(T|\vartheta)$ where $T = I_G(\vartheta)$. Set $\chi = \psi^G \in \mathrm{Irr}(G)$ by Clifford Theory. Since χ lies over ϑ, it follows that $\chi \in \mathrm{Irr}(G|N)$ and χ is faithful. Furthermore, $\chi(1) = |G:T|\,\psi(1)$, and

$$\mathrm{cod}(\chi) = \frac{|G|}{|G:T|\,\psi(1)} = \frac{|T|}{\psi(1)} = |T:N|\,\frac{|N|}{\psi(1)}.$$

However, $\psi(1) \mid |T:N|$ by [19, Theorem 6.15], and so $|N| \mid \mathrm{cod}(\chi)$. The result follows since $\mathrm{cod}(\chi) \mid |G/N|$ as $\mathrm{cod}(G) \subseteq \mathrm{cod}(G/N)$.

(d) N is not self centralizing.

For a contradiction, assume that $C_G(N) = N$.

(i) $H = \mathrm{SL}_2(5)$. Assume that $|N| = r^t$ where r is prime and t is a positive integer. The Normalizer-Centralizer theorem implies that $t > 1$. By (c), $|N| = 2^2, 2^3$, a contradiction since $|\mathrm{SL}_2(5)| \nmid |\mathrm{GL}_2(2)|$ and $|\mathrm{SL}_2(5)| \nmid |\mathrm{GL}_3(2)|$.

(ii) $H = \mathrm{SL}_3(4)$. Assume that $|N| = r^t$ where r is prime and t is a positive integer. The Normalizer-Centralizer theorem implies that $t > 1$. By (c), $|N|$ is either $2^6, 2^5, \ldots, 2^2$, 3^2, 3^3. With an application of the Normalizer-Centralizer theorem, we can embed $\mathrm{SL}_3(4)$ into $\mathrm{Aut}(N) \cong \mathrm{GL}_t(r)$. After considering if $|N| \mid |\mathrm{GL}_t(r)|$ for each possible order of N, we see that the only possible order of N is 2^6. Let $1_N \neq \vartheta \in \mathrm{Irr}(N)$ and denote $I_G(\vartheta)$ by T. It follows that $|N| = |\mathrm{Irr}(N)| > |G:T|$. Moreover, given that L/N is a maximal subgroup containing T/N, then either $L/N \cong (2^2 \times 2^2) : \mathrm{GL}_2(4)$ or $L/N \cong 3 \cdot A_6$. Now set $\chi = \psi^G \in \mathrm{Irr}(G \,|\, \vartheta)$ where $\psi \in \mathrm{Irr}(T \,|\, \vartheta)$. As χ is faithful, we have that $\mathrm{cod}(\chi) = |T|/\psi(1)$. However, $\psi(1) = |G:T|\,\epsilon_\chi$ where $\epsilon_\chi = [\chi_N, \vartheta]$. This yields that

$$\mathrm{cod}(\chi) = \frac{|T|}{|G:T|\,\epsilon_\chi}.$$

Let $L/N \cong (2^2 \times 2^2) : \mathrm{GL}_2(4)$. We note that $|T| \mid |N|\,|L:N| = 2^{12} \cdot 3^2 \cdot 5$ and so $7 \nmid |T|$. On the other hand, $|G:L| = 3 \cdot 7 \mid |G:T|$. This implies that, $\mathrm{cod}(\chi)$ is not an integer, a contradiction. Now let $L/N \cong 3 \cdot A_6$. We then have that $|T| \mid |N|\,|L:N| = 2^5 \cdot 3^3 \cdot 5$ and that $|G:L| = 2^3 \cdot 7 \mid |G:T|$. Again implying that $\mathrm{cod}(\chi)$ is not an integer, a contradiction.

(iii) $H = \mathrm{SL}_2(q)$ with $q > 5$ odd or $H \cong \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$.

An application of Lemma 2.16 implies that $|N| = r$ for some prime r. However, the Normalizer-Centralizer theorem yields $N_G(N)/C_G(N) = G/N \cong R$, where R is some subgroup of $\mathrm{Aut}(N) \cong C_{r-1}$ which is impossible since G/N is non-abelian.

(e) The final contraction.

We can assume N is a proper subgroup of $W = C_G(N)$, which is normal in G.

Case 1: $N \not\subseteq Z(G)$.

Consider $H = \mathrm{SL}_3(4)$. Since W/N is normal in G/N, it follows that $W/N \cong C_3$. Furthermore, $G/W \cong (G/N)/(W/N) \cong \mathrm{PSL}_3(4)$. But we cannot embed $\mathrm{PSL}_3(4)$ into $\mathrm{GL}_t(r)$ for $r^t = 2^2, \dots, 2^6, 3^2$, or 3^3, which is a contradiction (see Theorem 3.5(c)). If we consider $H = \mathrm{SL}_2(5)$, we also get a contradiction since we cannot embed $\mathrm{PSL}_2(5)$ into $\mathrm{GL}_t(r)$ for $r^t = 2^2$ or 2^3. If $H = \mathrm{SL}_2(q)$ with $q > 5$ odd or $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$, then an appeal to Lemma 2.16 yields N is a cyclic group of prime order r. Since W/N is normal in G/N, we see that $W/N \cong Z(H)$. Furthermore, $G/W \cong (G/N)/(W/N)$ is isomorphic to $H/Z(H)$; that is, W is maximal normal in G. We arrive at a contradiction by the Normalizer-Centralizer theorem since we cannot embed $H/Z(H)$ into $\mathrm{Aut}(N) \cong C_{r-1}$.

Case 2: $N \subseteq Z(G)$.

Since G is perfect, $N \subseteq Z(G) \cap G'$. It follows that N can be embedded in $M(H)$. Let $H = \mathrm{SL}_2(q)$ with $q > 5$ odd. When $q \neq 9$, the Schur multiplier of $\mathrm{SL}_2(q)$ is trivial which is a contradiction. On the other hand, $M(\mathrm{SL}_2(9)) = C_3$. This forces $N \cong C_3$, and so $G \cong 6 \cdot A_6$. But $\mathrm{cod}(6 \cdot A_6) = \{360, 180, 120, 90, 72, 45, 40, 36, 1\} \not\subseteq \mathrm{cod}(\mathrm{SL}_2(9))$, a contradiction. Now let $H = \mathrm{SL}_3(q)$ with $7 \leq q \equiv 1 \pmod 3$. Then N can be embedded into $M(\mathrm{SL}_3(q)) = 1$, a contradiction. When $H = \mathrm{SL}_2(5)$, we have that $M(H) = 1$, a contradiction. On the other hand, when $H = \mathrm{SL}_3(4)$, it follows that $M(H) = C_4 \times C_4$. Hence $N \cong C_2$. For any maximal normal subgroup U of G, we must have that $G/U \cong \mathrm{PSL}_3(4)$ by Corollary 3.4. This implies that $U = N \times V$ with $V \cong C_3$, a contradiction since N must be a unique minimal normal subgroup. □

THEOREM 3.7. *Let q be a prime power and suppose that either $H = \mathrm{SL}_2(q)$ with $q > 3$ odd or $H = \mathrm{SL}_3(q)$ with $4 \leqslant q \equiv 1 \pmod 3$. Let G be a finite group. If $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$, then either $G \cong H/Z(H)$ or $G \cong H$.*

Proof. In the case that G is a non-abelian simple group, it follows by Theorem 3.2 and Theorem 3.3 that $G \cong H/Z(H)$. Hence we let $1 < N$ be a maximal normal subgroup of G. By Corollary 3.4, we have that $G/N \cong H/Z(H)$. Now choose $M < N$ such that N/M is a chief factor of G. That is, N/M is minimal normal in G/M. Let $G/M = \overline{G}$ and let $N/M = \overline{N}$. Note that $\overline{N}$ is also maximal normal in $\overline{G}$ since $\overline{G}/\overline{N} \cong G/N$ is simple.

Suppose, for a contradiction, that $\overline{K}$ is a minimal normal subgroup of $\overline{G}$ distinct from $\overline{N}$. Since $\overline{N} \not\subseteq \overline{K}$, $\overline{N}$ is maximal normal in $\overline{G}$ and $\overline{N} \subseteq \overline{N} \times \overline{K}$, it follows that $\overline{G} = \overline{N} \times \overline{K}$. By Lemma 2.11, we have that $|\mathrm{cod}(G)| \geqslant 10 > |\mathrm{cod}(\mathrm{SL}_2(q))|$ and $|\mathrm{cod}(G)| \geqslant 14 > |\mathrm{cod}(\mathrm{SL}_3(q))|$, a contradiction.

This yields that $\overline{N}$ is unique minimal normal in $\overline{G}$. It follows that $\overline{N}$ is the unique normal subgroup of $\overline{G}$ such that $\mathrm{cod}(\overline{G}) \subseteq \mathrm{cod}(H)$. By Theorem 3.5, it follows that $\overline{G} \cong H$. However, by Theorem 3.6, it follows that $M = 1$ and so $G \cong H$. □

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