



**Pricing American compound options with stochastic volatility and
correlated interest rates**

by

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SUMMARY

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A compound option is an option on an option. Accurate prices are generated using simulation approaches to price compound options, but considerable computing time is required, especially when early exercise opportunities are available. This dissertation explores the use of finite difference methods as an efficient alternative.

Stock prices, stock price volatilities, and interest rates were assumed to follow correlated stochastic differential equations. Six finite difference algorithms were used to price European and American compound options - an explicit and alternating-direction implicit (ADI) method for the one-state variable Black-Scholes model, the two-state variable Heston model, and the three-state variable Heston-Hull-White (HHW) and Heston-Cox-Ingersoll-Ross (HCIR) models. The source code of these finite difference methods used a vectorised approach instead of traditional and slower-performing nested *for* loops (common in option pricing literature).

Optimal grid sizes and optimal parameters for non-uniform grid generation were identified for each of the six finite difference algorithms, providing a

balance between acceptable relative errors and reasonable execution times.

While the explicit methods executed faster than the ADI methods under all four pricing models, ADI methods were unconditionally stable and produced more accurate results. The two- and three-state variable explicit schemes were conditionally stable on a sufficiently fine time grid. Non-uniform grids produced more accurate results than uniform grids for both the explicit and ADI schemes.

Typical relative errors of order $1E-4$ were observed for compound options on non-uniform grids under the Black–Scholes model, while typical relative errors of order $1E-3$ were observed for compound options on non-uniform grids under the Heston model. Coarser grids were used for the HHW and HCIR schemes, where typical relative errors of order $1E-2$ were observed for compound options on non-uniform grids.

Under all four pricing models, the execution times of the Douglas, Craig–Sneyd, modified Craig–Sneyd, and Hundsdorfer–Verwer schemes did not differ significantly. The Douglas and Craig–Sneyd schemes achieved marginally smaller errors on average than other ADI schemes. The Crank–Nicolson schemes performed much slower than other ADI schemes.

For cases where the Feller conditions were not met, ADI schemes remained stable, while the explicit schemes were mostly unstable.

Finite difference algorithms were found to be computationally advantageous over the simulation approach. Execution times were considerably shorter, and outputs for compound options were found to have errors of order between $1E-2$ and $1E-3$ relative to the simulation values.

A comprehensive set of reference prices was provided for European and American compound options under the four pricing models. American call-on-put compound options had the most significant overall price increases relative to their European variants. American put-on-put compound options had the most significant overall price decreases. American put-on-call compound options only showed marginal price increases and American call-on-call showed negligible price differences.

DECLARATION

I declare that the dissertation, which I hereby submit for the degree Master of Finance in Financial Engineering at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

ETHICS STATEMENT

The author, whose name appears on the title page of this dissertation/thesis, has obtained, for the research described in this work, the applicable research ethics approval.

The author declares that he has observed the ethical standards required in terms of the University of Pretoria's Code of Ethics for Researchers and the Policy guidelines for responsible research.

A handwritten signature in black ink, appearing to read 'Ruan Nel', enclosed within a hand-drawn oval.

Ruan Nel
December 2023

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List of Abbreviations

ADI	Alternating-direction implicit
BS	Black–Scholes
CIR	Cox–Ingersoll–Ross
CN	Crank–Nicolson
CS	Craig–Sneyd
Do	Douglas
HCIR	Heston–Cox–Ingersoll–Ross
HHW	Heston–Hull–White
HV	Hundsdorfer–Verwer
HW	Hull–White
LSMC	Least-squares Monte Carlo
MCS	Modified Craig–Sneyd
PDE	Partial differential equation

List of Symbols

ζ	Equidistant points for the non-uniform interest rate grid
η	Equidistant points for the non-uniform variance grid
θ	ADI scheme parameter
θ_r	Interest rate term structure
θ_v	Long-run average variance of the stock price
κ_r	Mean reversion rate of the interest rate
κ_v	Mean reversion rate of the stock price variance
λ_r	Price of interest rate risk
λ_v	Price of volatility risk
μ	Annual expected return
ξ	Equidistant points for the non-uniform stock price grid
$\rho_{S,r}$	Correlation between the stock price and the risk-free rate of interest
$\rho_{S,v}$	Correlation between the stock price and the variance of the stock price
$\rho_{v,r}$	Correlation between the variance of the stock price and the risk-free rate of interest
$\rho_{x,r}$	Correlation between the log-transformed stock price and the risk-free rate of interest
$\rho_{x,v}$	Correlation between the log-transformed stock price and the variance of the stock price
σ_r	Volatility of the interest rate
σ_S	Volatility of stock price
σ_v	Volatility of stock price variance
φ	Adapted column vector process
φ_r	Girsanov transformation kernel for the interest rate process
φ_S	Girsanov transformation kernel for the stock price process or market price of risk
φ_v	Girsanov transformation kernel for the variance process
$\Phi(\mu, \sigma^2)$	Normal density function with mean μ and variance σ^2
ϕ	Underlying option type (1 if call option and -1 if put option)
ψ	Contingent claim function

ω	Overlying option type (1 if call option and -1 if put option)
Ω	Sample space
a_u	Regression coefficients
A	Sparse matrix
C_C	Call-on-call option
C_P	Call-on-put option
D	Underlying (or daughter) option
E_R	Relative error
E_{SD}	Relative standard error
$\mathcal{F}$	Filtration process
K_D	Strike price of the underlying (or daughter) option
K_M	Strike price of the overlying (or mother) option
L	Likelihood process
L_u	Laguerre polynomial of degree u
m	Mean
M	Overlying (or mother) option
$n(\cdot)$	Standard normal probability density function
$N(\cdot)$	Standard normal cumulative distribution function
$N_2(\cdot, \cdot, \rho)$	Standard normal bivariate cumulative distribution function with correlation ρ
N_r	Number of interest rate intervals
N_S	Number of stock price intervals
N_t	Number of time intervals
N_v	Number of variance intervals
N_P	Number of simulation paths
$\mathbb{P}$	Physical probability measure
p	Distinguishing constant for the HHW and HCIR models
P_C	Put-on-call option
P_P	Put-on-put option
q	Annual continuous dividend rate
$\mathbb{Q}$	Risk-neutral probability measure
r	Interest rate
S	Stock price
S^*	Critical stock price

SD	Sample standard deviation
t	Time
T	Fixed time period
T_D	Maturity date of the underlying (or daughter) option
T_M	Maturity date of the overlying (or mother) option
u	True value
$\hat{u}$	Approximated value
$\bar{u}$	Sample mean of a set of approximated values
v	Instantaneous variance of the stock price
V	Continuation value
w	Standard deviation
W	Standard Brownian motion
W_r	Standard Brownian motion for the interest rate process
W_S	Standard Brownian motion for the stock price process
W_v	Standard Brownian motion for the variance process
x	Log-transformed stock price
X	Geometric Brownian motion
z	Random variable with standard normal distribution
z^*	Value of the random variable with standard normal distribution that corresponds with the critical stock price
Z_r	Correlated standard Brownian motion for the interest rate process
Z_S	Correlated standard Brownian motion for the stock price process
Z_v	Correlated standard Brownian motion for the variance process

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1 Introduction

Compound options are options on options that are often traded on currency and fixed-income markets [1]. The holder of the overlying option has the right, but not the obligation, to buy (long) or sell (short) an underlying option. The holder of the underlying option will, in turn, have the right, but not the obligation, to buy (long) or sell (short) a stock [2].

The overlying option, often called the *mother* option and denoted as M , has a strike price of K_M and is exercisable at time T_M . The underlying option, often called the *daughter* option and denoted as D , has a strike price of K_D and is exercisable at time T_D . Typically, $T_M \leq T_D$ [2].

Compound options are categorised as European or American compound options. In the case of a European compound option, the overlying and underlying options may only be exercised at their respective expiration dates. American-style variants of a compound option, on the other hand, allow for both the overlying and underlying options to be exercised at any optimal point before their respective expiration dates [2].

The four main compound option types that are reviewed include:

- a call option on a call option (call-on-call or C_C);
- a call option on a put option (call-on-put or C_P);
- a put option on a call option (put-on-call or P_C); and
- a put option on a put option (put-on-put or P_P).

Compound options are known to be sensitive to movements in volatility and interest rates [2]. When constant volatility and interest rates are assumed, one risks undervaluing a compound option [3].

It is assumed that the stock price, stock price volatility, and interest rate follow correlated stochastic differential equations. Pricing compound options under the Black–Scholes, Heston [4], Heston–Hull–White (HHW) [5][6], and Heston–Cox–Ingersol–Ross (HCIR) [7][6] pricing models are considered. The Heston model is a two-state variable model in which the stock price and stock price volatility follow stochastic processes, while the interest rate is constant. The HHW and HCIR models are three-state variable models in which the stock price, stock price volatility, and interest rate follow stochastic processes.

Accurate prices are generated using simulation approaches to price compound options, but considerable computing time is required, especially when early exercise opportunities are available [2]. This dissertation explores

using explicit and alternating-direction implicit (ADI) finite difference methods as an efficient alternative.

The American feature introduces some complexity to the valuation of compound options. As the stock's price evolves, it may become optimal to exercise the underlying option at some point before its expiration date. Consequently, the point at which it becomes optimal to exercise the overlying option (to unlock the underlying option) will be affected. For example, suppose the optimal exercise point of the underlying option lies before T_M . In that case, the overlying option should either be exercised at some optimal point before or on the intended exercise date of the underlying option.

Results for American compound options are not readily available. Chiarella and Kang (2013) presented results for an American put-on-put option using the HCIR model [2] with zero correlation between the stock price and interest rate and zero correlation between the stock price volatility and interest rate. Their numerical solutions were obtained by solving the HCIR pricing PDE using an implicit Crank–Nicolson numerical scheme on uniform grids. Their results were verified by using a least-squares Monte Carlo (LSMC) simulation method.

This dissertation aims to expand on the results presented by Chiarella and Kang by presenting prices for European and American compound options under each of the four pricing models mentioned above. Non-zero correlations are assumed between the stock price and interest rate and between the stock price volatility and interest rate. Non-uniform grids are used to improve price accuracy.

Six finite difference algorithms are presented to price compound options, namely:

- an explicit method to solve the one-state variable Black–Scholes pricing partial differential equation (PDE);
- an ADI method to solve the one-state variable Black–Scholes pricing PDE;
- an explicit method to solve the two-state variable Heston pricing PDE;
- an ADI method to solve the two-state variable Heston pricing PDE;
- an explicit method to solve the three-state variable HHW/HCIR pricing PDE; and
- an ADI method to solve the three-state variable HHW/HCIR pricing PDE.

For the ADI methods, five different time discretisation schemes, as given by Haentjens and Hout (2012) [8], are compared. These include the Crank–Nicolson, Douglas, Craig–Sneyd, modified Craig–Sneyd and Hundsdorfer–Verwer ADI schemes.

Each finite difference algorithm is evaluated based on accuracy, stability, and efficiency. The performance of the algorithms is assessed using both uniform and non-uniform spatial grids.

A three-state variable LSMC simulation approach is presented that can be used to price European or American compound options under any one of the four pricing models mentioned above. The simulation approach is used to verify the results of the finite difference algorithms where a closed-form or semi-closed-form pricing solution does not exist. Pricing solutions for the underlying option include the Black–Scholes closed-form solution [9], the Heston closed-form solution [4] and the HHW semi-closed-form solution [10]. The only pricing solution available for a compound option is the Geske Black–Scholes closed-form solution [1].

Non-zero correlation factors and constant mean-reversion levels are assumed for the two- and three-state variable models. The dissertation further explores cases where (and where not) the Feller condition is satisfied.

The source code used a vectorised approach instead of traditional and slower-performing nested *for* loops often found in option pricing literature.

2 Stochastic models

2.1 Stochastic processes and Brownian motions

The definitions of a stochastic process, standard Brownian motion, and geometric Brownian motion are widely known in financial mathematics and are given below as a preliminary.

Definition 2.1 (Stochastic Process [11]). *A stochastic process is a collection of random variables indexed against another variable. The indexing variable often refers to time. For these processes, the value of the random variable changes over time in an unpredictable way.*

In the case of a discrete-time stochastic process $X = \{X_n, n = 0, 1, 2, \dots\}$, the value of the variable changes at fixed points in time. In the case of a continuous-time stochastic process $X = \{X(t), 0 \leq t \leq \infty\}$, the variable's value can change at any time.

Definition 2.2 (Standard Brownian Motion [9]). *A stochastic process $W(t)$ is called a standard Brownian motion (or Wiener process) if the following conditions hold:*

1. $W(0) = 0$.

2. The process W has independent increments, i.e. if $r < s \leq t < u$, then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.
3. For $s < t$ the stochastic variable $W(t) - W(s) \sim \Phi[0, t - s]$ where $\Phi(m, w^2)$ is a normal distribution with mean m and variance w^2 .
4. W has continuous trajectories.

A practical corollary that stems from Definition 2.2 is that for $s < t$,

$$W(t) = W(s) + \sqrt{t - s} \cdot z \quad (2.1)$$

where $z \sim \Phi(0, 1)$.

A geometric Brownian motion is a type of standard Brownian motion that incorporates a deterministic drift term and volatility. This makes it a fundamental concept employed to model various factors in the financial markets, such as stock prices, volatility (or variance), and interest rates.

Definition 2.3 (Geometric Brownian Motion [9]). A stochastic process $X(t)$ is said to follow a geometric Brownian motion if it satisfies the stochastic differential equation

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t) \quad (2.2)$$

where α and σ are constants, and $W(t)$ is a standard Brownian Motion.

If the initial condition of a geometric Brownian motion is known, the solution can be expressed in closed form using stochastic calculus techniques, such as Itô's lemma.

Proposition 2.1 (Solution to a Geometric Brownian Motion [9]). The solution to the equation

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t), \quad (2.3)$$

$$X(0) = x_0, \quad (2.4)$$

is given by

$$X(t) = x_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}. \quad (2.5)$$

The expected value of $X(t)$ is given by

$$\mathbb{E}[X(t)|x_0] = x_0 e^{\alpha t} \quad (2.6)$$

and the variance is given by

$$\text{Var}[X(t)|x_0] = x_0^2 e^{2\alpha t} (e^{\sigma^2 t} - 1). \quad (2.7)$$

Proof. This proof is extensively available in the literature and can be found in Section 5.2 of [9]. $\square$

2.2 One-state variable Black–Scholes model

In the Black–Scholes environment, it is assumed that the stock price $S = \{S(t), t \geq 0\}$ evolves according to the stochastic differential equation (SDE) [11]

$$dS = \mu S dt + \sigma_S S dW_S^{\mathbb{P}}, \quad S(0) \geq 0 \quad (2.8)$$

where μ is the annual expected return, $\sigma_S > 0$ is the annual stock price volatility, $W_S^{\mathbb{P}} = \{W_S^{\mathbb{P}}(t), t \geq 0\}$ is a continuous standard Brownian motion with respect to the physical measure $\mathbb{P}$, and $S(0)$ is the initial stock price.

2.3 The two-state variable Heston stochastic volatility model

The widely celebrated Black–Scholes model provides a simple and elegant framework to derive closed-form solutions for many financial derivatives [4]. A shortcoming of this model often pointed out in the literature, is that it assumes constant volatility and a constant rate of interest [12]. The consequence of this assumption is that long-maturity financial derivatives that are sensitive to volatility and interest rate changes may be undervalued or overpriced [3]. Therefore, it becomes important to consider changes in volatility and interest rates when pricing models are chosen.

The assumption of constant volatility is often violated in financial markets, with literature demonstrating that volatility is time-dependent [4]. Stochastic volatility models were introduced in 1987 to address these violations [3]. Skewness in spot-asset returns and implied volatility smiles across a range of strike prices can be explained by considering the correlation between spot-asset returns and volatility [4].

The popular Heston stochastic volatility model assumes that the stock price $S = \{S(t), t \geq 0\}$ and stock price variance $v = \{v(t), t \geq 0\}$ evolve according to the set of stochastic differential equations [4]

$$dS = \mu S dt + \sqrt{v} S dW_S^{\mathbb{P}}, \quad S(0) \geq 0, \quad (2.9)$$

$$dv = \kappa_v(\theta_v - v)dt + \sigma_v \sqrt{v} dW_v^{\mathbb{P}}, \quad v(0) \geq 0, \quad (2.10)$$

where μ , $W_S^{\mathbb{P}}$ and $S(0)$ are as in (2.8). The parameter $\kappa_v > 0$ represents the mean rate at which v reverts to the long-run average variance, $\theta_v > 0$. The parameter $\sigma_v > 0$ represents the volatility of the variance, and $W_v^{\mathbb{P}} = \{W_v^{\mathbb{P}}(t), t \geq 0\}$ is a continuous standard Brownian motion with respect to the physical measure $\mathbb{P}$. The initial variance $v(0)$ is often chosen as the square of the initial stock price volatility, σ_S .

The two processes, $dW_S^{\mathbb{P}}$ and $dW_v^{\mathbb{P}}$, are correlated such that [4]

$$dW_S^{\mathbb{P}} dW_v^{\mathbb{P}} = \rho_{S,v} dt, \quad -1 < \rho_{S,v} < 1. \quad (2.11)$$

The correlation value often takes on a negative value in financial markets. A typical observation is that when stock prices decrease, volatility increases [13].

The process for v is non-centrally chi-square distributed, while the process for $\sqrt{v}$ is Rayleigh distributed.

2.4 Three-state variable HHW and HCIR hybrid models

When financial markets are in a state of crisis, stock prices drop, and a typical response is for investors to withdraw from the markets to minimise their losses [12]. Central banks, in turn, respond by decreasing interest rates to stimulate stagnant economies, which causes stock prices to increase [12]. Therefore, it is reasonable to assume that some correlation between stock prices and interest rates exists [12].

The 2008 financial crisis led to investors shifting their focus to long-maturity financial derivatives, which are typically less sensitive to short-term market fluctuations but more sensitive to market smiles and skews [14]. These types of derivatives require pricing models that incorporate the stochastic nature of interest rates [14].

Many hybrid models have been introduced to account for changes in volatility and interest rates over time. These hybrid models are typically expressed as a system of stochastic differential equations of several state variables with a full correlation matrix. Ideally, the model being used should allow a non-zero correlation structure to be set while providing an adequate fit to market-implied volatility skews and smiles. Hybrid models differ for equity, foreign exchange and interest rate derivatives [12].

The three-state variable HHW and HCIR hybrid models expand on the Heston stochastic volatility model, adding a stochastic interest rate component driven by the Hull–White [5] or Cox–Ingersoll–Ross (CIR) [7] interest rate process [6]. In these models, it is assumed that the stock price $S = \{S(t), t \geq 0\}$, stock price variance $v = \{v(t), t \geq 0\}$ and interest rate $r = \{r(t), t \geq 0\}$ evolve according to the set of stochastic differential equations

$$dS = \mu S dt + \sqrt{v} S dZ_S^{\mathbb{P}}, \quad S(0) \geq 0, \quad (2.12)$$

$$dv = \kappa_v(\theta_v - v)dt + \sigma_v \sqrt{v} dZ_v^{\mathbb{P}}, \quad v(0) \geq 0, \quad (2.13)$$

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r r^p dZ_r^{\mathbb{P}} \quad (2.14)$$

where μ and $S(0)$ are as in (2.9), and κ_v , θ_v , σ_v and $v(0)$ are as in (2.10). The parameter κ_r represents the mean rate at which r reverts to the time-dependent interest rate term structure, $\theta_r = \{\theta_r(t), t \geq 0\}$. The parameter $\sigma_r > 0$ represents the volatility of the interest rate. The constant p is introduced such that $p = 0$ for the HHW model, and $p = \frac{1}{2}$ for the HCIR model.

$Z_S^{\mathbb{P}} = \{Z_S^{\mathbb{P}}(t), t \geq 0\}$, $Z_v^{\mathbb{P}} = \{Z_v^{\mathbb{P}}(t), t \geq 0\}$, and $Z_r^{\mathbb{P}} = \{Z_r^{\mathbb{P}}(t), t \geq 0\}$ are standard Brownian motions with respect to the physical measure $\mathbb{P}$ and are correlated according to

$$dZ_S^{\mathbb{P}} dZ_v^{\mathbb{P}} = \rho_{S,v} dt, \quad -1 < \rho_{S,v} < 1, \quad (2.15)$$

$$dZ_S^{\mathbb{P}} dZ_r^{\mathbb{P}} = \rho_{S,r} dt, \quad -1 < \rho_{S,r} < 1, \quad (2.16)$$

$$dZ_v^{\mathbb{P}} dZ_r^{\mathbb{P}} = \rho_{v,r} dt, \quad -1 < \rho_{v,r} < 1. \quad (2.17)$$

$Z_S^{\mathbb{P}}$, $Z_v^{\mathbb{P}}$, and $Z_r^{\mathbb{P}}$ are determined using the set of equations

$$\begin{pmatrix} dZ_S^{\mathbb{P}} \\ dZ_v^{\mathbb{P}} \\ dZ_r^{\mathbb{P}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{S,v} & \sqrt{1 - \rho_{S,v}^2} & 0 \\ \rho_{S,r} & \frac{\rho_{v,r} - \rho_{S,v}\rho_{S,r}}{\sqrt{1 - \rho_{S,v}^2}} & \sqrt{1 - \rho_{S,r}^2 - \left(\frac{\rho_{v,r} - \rho_{S,v}\rho_{S,r}}{\sqrt{1 - \rho_{S,v}^2}}\right)^2} \end{pmatrix} \begin{pmatrix} dW_S^{\mathbb{P}} \\ dW_v^{\mathbb{P}} \\ dW_r^{\mathbb{P}} \end{pmatrix} \quad (2.18)$$

where $W_S^{\mathbb{P}} = \{W_S^{\mathbb{P}}(t), t \geq 0\}$, $W_v^{\mathbb{P}} = \{W_v^{\mathbb{P}}(t), t \geq 0\}$ and $W_r^{\mathbb{P}} = \{W_r^{\mathbb{P}}(t), t \geq 0\}$ are independent and uncorrelated standard Brownian motions.

The interest rate component for both the Hull–White and CIR models is an extension of the Vasicek interest rate model.

2.5 Girsanov transformation and the risk-free measure

As a preliminary to deriving the pricing PDE for the two- and three-state variable models presented above, it is necessary to transform the stock price, variance and interest rate processes from the physical measure $\mathbb{P}$ to the risk-free measure $\mathbb{Q}$. To do this, two important concepts, namely a martingale and martingale measure, are defined below.

Definition 2.4 (Martingale [9]). A discrete-time random process $X_n : n = 0, 1, \dots$ is a martingale if

$$\mathbb{E}[X_m | \mathcal{F}_n] = X_n, \quad \text{for all } n \leq m, \quad (2.19)$$

where $\mathcal{F}_n$ denotes the information available at time n .

Definition 2.5 (Martingale Measure [9]). Consider a financial market that consists of N different financial assets, each denoted as $S^i = \{S^i(t), t \geq 0\}$ for $i = 1, 2, \dots, N_t$ and $t \in \{0, 1\}$. Fix the asset S^1 as the numeraire asset. A probability measure $\mathbb{Q}$ defined on the sample space Ω is a martingale measure if it satisfies the following conditions:

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$. That is, both measures agree on the events with probability zero and the events with probability one.
2. For every $i = 1, \dots, N$, the normalised asset price process

$$Z_t^i = \frac{S_t^i}{S_t^1} \quad (2.20)$$

is a martingale under the measure $\mathbb{Q}$.

A martingale is thus a process that is constant in its conditional mean; that is, the future value is, on average, the same as the current value, regardless of past events. A martingale measure is a particular probability measure under which the discounted price of a financial asset is a martingale. These two concepts lead us to define an arbitrage-free or risk-free market.

Theorem 2.1 (Arbitrage-free Market [9]). Given a fixed numeraire, the market is arbitrage-free if and only if there exists a risk-free measure $\mathbb{Q}$, which is a martingale measure.

Proof. The proof for this theorem can be found in Section 3.3 in [9]. □

The Girsanov transformation is used to transform the physical measure to the risk-free measure and is given below.

Theorem 2.2 (The Girsanov Theorem [9]). Let $W^{\mathbb{P}} = \{W^{\mathbb{P}}(t), t \geq 0\}$ be a scalar d -dimensional $\mathbb{P}$ -Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \underline{\mathcal{F}})$ with sample space Ω , filtration $\mathcal{F} = \{\mathcal{F}(t), t \geq 0\}$ and probability measure $\mathbb{P}$. Let $\varphi = \{\varphi(t), t \geq 0\}$ be any d -dimensional adapted column vector process. For a fixed time T , define the likelihood process $L = \{L(t), t \in [0, T]\}$ as

$$dL = \varphi^* L dW^{\mathbb{P}}, \quad L(0) = 1, \quad (2.21)$$

with

$$L(t) = e^{\int_0^t \varphi^*(s) dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \|\varphi(s)\|^2 ds}. \quad (2.22)$$

Assume that the initial condition guarantees that

$$E^{\mathbb{P}}[L(T)] = 1. \quad (2.23)$$

If the new probability measure $\mathbb{Q}$ is defined as

$$d\mathbb{Q} = L(T)d\mathbb{P} \quad (2.24)$$

on $\mathcal{F}(T)$, then

$$dW^{\mathbb{Q}} = dW^{\mathbb{P}} - \varphi dt \quad (2.25)$$

where $W^{\mathbb{Q}}$ is a scalar d -dimensional $\mathbb{Q}$ -Wiener process.

Proof. The proof for this theorem can be found in Section 11.3 in [9]. $\square$

The process φ is often called the Girsanov kernel of the measure transformation.

2.5.1 Transformations of the stock price processes

The Girsanov transformation is applied to the stock price processes in the one-, two- and three-state variable models. For each of these processes,

$$dW_S^{\mathbb{Q}} = dW_S^{\mathbb{P}} - \varphi_S dt \quad (2.26)$$

where $\varphi_S = \{\varphi_S(t), t \geq 0\}$ is the Girsanov kernel of the stock price and is often called the market price of risk. Substituting (2.26) into (2.8) and rearranging yields

$$dS = (\mu + \sigma_S \varphi_S) S dt + \sigma_S S dW_S^{\mathbb{Q}}, \quad (2.27)$$

while substituting (2.26) into (2.9) or (2.12) and rearranging yields

$$dS = (\mu + \sigma_S \varphi_S) S dt + \sqrt{v} S dW_S^{\mathbb{Q}}. \quad (2.28)$$

For the induced measure $\mathbb{Q}$ to be a martingale measure, the local rate of return should equal the risk-free rate r minus the annual continuous dividend rate q [9], i.e.

$$\mu + \sigma_S \varphi_S = r - q. \quad (2.29)$$

By substituting (2.29) into (2.27), the $\mathbb{Q}$ -dynamics of S under the Black–Scholes model becomes

$$dS = (r - q) S dt + \sigma_S S dW_S^{\mathbb{Q}}. \quad (2.30)$$

Similarly, by substituting (2.29) into (2.28), the $\mathbb{Q}$ -dynamics of S under the Heston, HHW and HCIR models become

$$dS = (r - q) S dt + \sqrt{v} S dW_S^{\mathbb{Q}}. \quad (2.31)$$

2.5.2 Transformations of the variance process

The Girsanov transformation is applied to the variance processes in the two- and three-state variable models. For each of these processes,

$$dW_v^{\mathbb{Q}} = dW_v^{\mathbb{P}} - \varphi_v dt \quad (2.32)$$

where $\varphi_v = \{\varphi_v(t), t \geq 0\}$ is the Girsanov kernel of the stock price variance. Substituting (2.32) into (2.10) or (2.13) and rearranging yields

$$dv = \left(\kappa_v(\theta_v - v) - \sigma_v \sqrt{v} \varphi_v \right) dt + \left(\sigma_v \sqrt{v} \right) dW_v^{\mathbb{Q}}. \quad (2.33)$$

Since the expression for φ_v is unknown, the term $\sigma_v \sqrt{v} \varphi_v$ is often shortened to $\lambda_v(t, v)$ in the literature [4], and the $\mathbb{Q}$ -dynamics of v under the Heston, HHW and HCIR models becomes

$$dv = \left(\kappa_v(\theta_v - v) - \lambda_v(t, v) \right) dt + \sigma_v \sqrt{v} dW_v^{\mathbb{Q}}. \quad (2.34)$$

The term $\lambda_v(t, v)$ is referred to as the price of volatility risk and must be independent of the specific asset [4].

2.5.3 Transformations of the interest rate process

The Girsanov transformation is applied to the interest rate process in the three-state variable models. For this process,

$$dW_r^{\mathbb{Q}} = dW_r^{\mathbb{P}} - \varphi_r dt \quad (2.35)$$

where $\varphi_r = \{\varphi_r(t), t \geq 0\}$ is the Girsanov kernel of the interest rate. Substituting (2.35) into (2.14) and rearranging yields

$$dr = \left(\kappa_r(\theta_r - r) - \sigma_r r^p \varphi_r \right) dt + \left(\sigma_r r^p \right) dW_r^{\mathbb{Q}}. \quad (2.36)$$

Since the expression for φ_r is unknown, the term $\sigma_r r^p \varphi_r$ is often shortened to $\lambda_r(t, r)$ in the literature [2], and the $\mathbb{Q}$ -dynamics of r under the HHW and HCIR models becomes

$$dr = \left(\kappa_r(\theta_r - r) - \lambda_r(t, r) \right) dt + \sigma_r r^p dW_r^{\mathbb{Q}}. \quad (2.37)$$

2.6 Parameter calibration and methods of pricing

The parameters in the variance and interest rate processes are normally estimated through a calibration process. The calibration process typically

involves solving the parameter values repeatedly using live market data for European options [12].

Since closed-form solutions are not available for hybrid models where a full non-zero correlation matrix is assumed, simulations or numerical techniques are used to price financial derivatives. The choice of pricing method when using hybrid models is an important consideration [12].

When calibrating the parameters, speed and efficiency are crucial. When pricing a new derivative, robustness and accuracy of the pricing method are important [12].

2.7 Feller condition

The variance process in the Heston model contains a $\sqrt{v}$ term and the interest rate process in the HCIR model contains a $\sqrt{r}$ term. Since the square root of a negative number will lead to an undefined value, it is important to choose appropriate parameter values during calibration and simulation that will prevent the variance and interest rate processes from reaching negative values. The HHW model does not contain a $\sqrt{r}$ term and, therefore, will not produce an undefined value when negative interest rate values are reached.

The Feller condition states that if the condition

$$2\kappa_v\theta_v > \sigma_v^2 \quad (2.38)$$

is met, then v is always positive and cannot reach a value of zero. Similarly, if the condition

$$2\kappa_r\theta_r > \sigma_r^2 \quad (2.39)$$

is met, then r is always positive and cannot reach a value of zero [7].

Figure 2.1(a) shows four simulation paths for the variance process using the Heston model and Figure 2.1(b) shows four simulation paths for the interest rate using the HCIR model, while Figure 2.1(c) shows four simulation paths for the interest rate using the HHW model. A duration of $T = 1$ and 1,000 time steps were used. It was assumed that $\rho_{S,v} = \rho_{S,r} = \rho_{v,r} = 0$.

The starting point $v(0) = 0.1$ and long-run mean variance $\theta_v = 0.2$ is indicated on Figure 2.1(a). For each variance simulation path, $\sigma_v = 0.05$ was chosen. The κ_v values were varied such that three simulation paths met the Feller condition ($\kappa_v = 7, 4, 3$), while one simulation path does not meet the Feller condition ($\kappa_v = 0.005$).

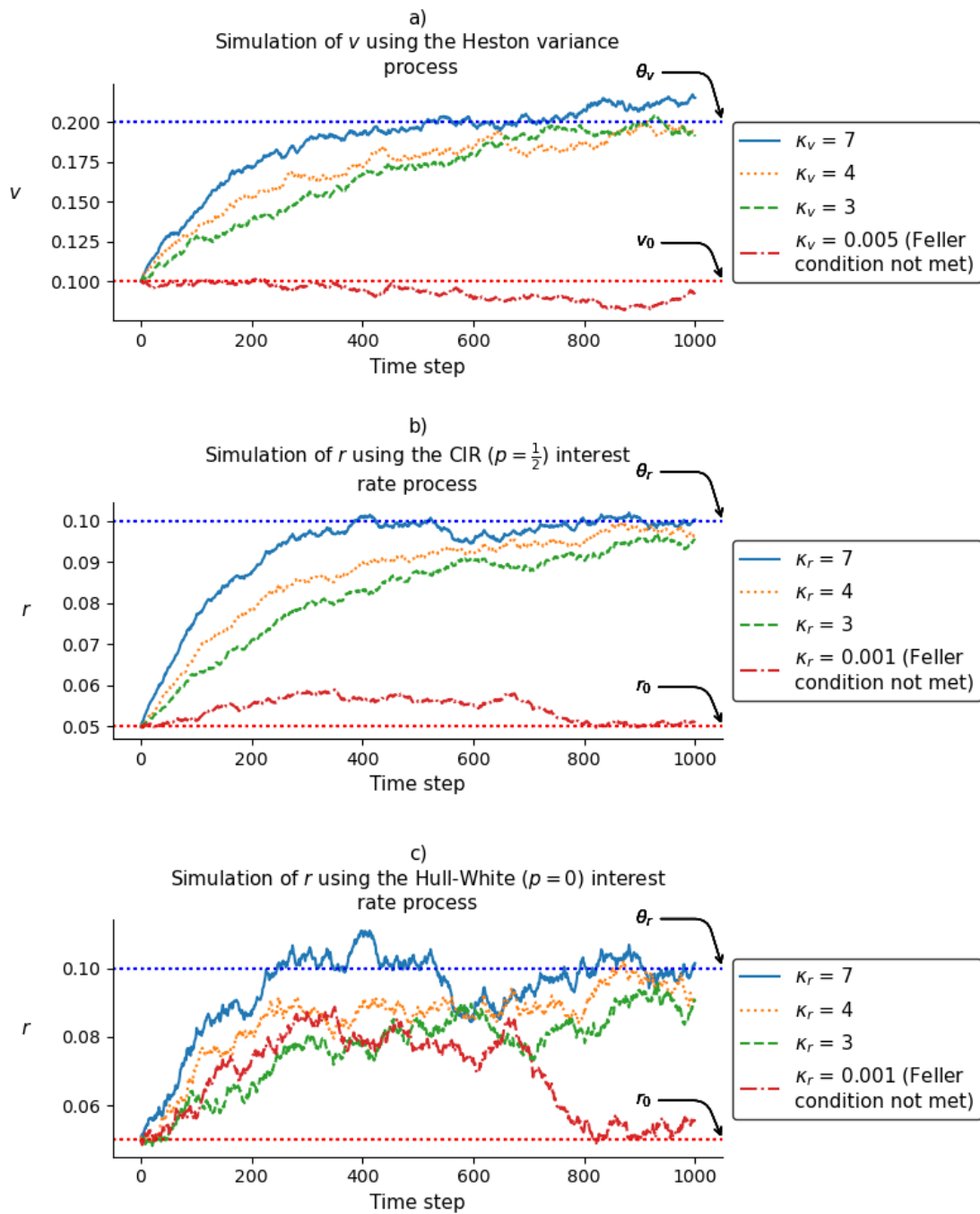


Figure 2.1: Simulated paths for the a) variance process of the Heston model; b) interest rate process of the CIR model; and c) interest rate process of the Hull-White model.

Similarly, the starting point $r(0) = 0.05$ and long-run mean interest rate $\theta_r = 0.1$ are indicated on Figure 2.1(b) and Figure 2.1(c). For each interest rate simulation path, $\sigma_r = 0.025$ was chosen. The κ_r values are varied such that three simulation paths meet the Feller condition ($\kappa_v = 7, 4, 3$), while one simulation path does not meet the Feller condition ($\kappa_r = 0.001$).

From Figure 2.1(a), it is clear that when the Feller condition is met, the variance process reaches the long-run mean variance faster as κ_v increases. When the Feller condition is not met, the variance remains low and does not reach the long-run mean average.

From Figure 2.1(b) and Figure 2.1(c), it is clear that when the Feller condition is met, the interest rate process reaches the interest rate term structure faster as κ_r increases. When the Feller condition is not met, the interest rate remains much lower than the long-run mean interest rate.

For the same values of κ_r , θ_r and σ_r , the HHW interest rate process behaves more erratically than the HCIR interest rate process. The $\sqrt{r}$ term in the HCIR model has a dampening effect on the volatility term when $r < 1$, while it has an intensifying effect on the volatility term when $r > 1$.

2.8 Affine processes and requirements for closed-form solutions

An important consideration when defining a hybrid model is whether it fits into the class of affine diffusion processes.

Definition 2.6 (Affine Diffusion Process [6]). *Consider the model*

$$d\mathbf{X}(t) = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t))d\mathbf{W}(t) \quad (2.40)$$

defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with sample space Ω , filtration $\mathcal{F} = \{\mathcal{F}(t), t \geq 0\}$ and risk-neutral probability measure $\mathbb{Q}$. $\mathbf{X}(t)$ is an n -dimensional state vector in some space $\mathcal{D} \subset \mathbb{R}^n$, $\mathbf{W}(t)$ is an $\mathcal{F}(t)$ -standard column vector of independent standard Brownian Motions in $\mathbb{R}^n$, $\mu(\mathbf{X}(t)) : \mathcal{D} \rightarrow \mathbb{R}^n$, and $\sigma(\mathbf{X}(t)) : \mathcal{D} \rightarrow \mathbb{R}^{n \times n}$. The model is said to be in affine form if

$$\mu(\mathbf{X}(t)) = a_0 + a_1 \mathbf{X}(t), \quad (2.41)$$

$$(\sigma(\mathbf{X}(t))\sigma(\mathbf{X}(t))^T)_{i,j} = (c_0)_{i,j} + (c_1)_{ij}^T \mathbf{X}(t), \quad (2.42)$$

$$r(\mathbf{X}(t)) = r_0 + r_1^T \mathbf{X}(t) \quad (2.43)$$

for $i, j = 1, \dots, n$, where $r_0 \in \mathbb{R}$, $(a_0, r_1) \in \mathbb{R}^n$, $(a_1, c_0) \in \mathbb{R}^{n \times n}$ and $c_1 \in \mathbb{R}^{n \times n \times n}$.

The conditions above imply that for the system of stochastic differential equations to be affine, each element of the model's drift $\mu(\mathbf{X}(t))$ (2.41), covariance matrix $\sigma(\mathbf{X}(t))\sigma(\mathbf{X}(t))^T$ (2.42), and the interest rate component $r(\mathbf{X}(t))$ (2.43) should be a linear function of the state variables. The condition given by (2.43) implies that the interest rate component can be constant or a stochastic process driven by one or more of the state variables.

The main advantage of an affine hybrid model is that it gives rise to the efficient determination of the model's characteristic function, which allows for closed-form solutions or analytical expressions to be derived. This, in turn, makes parameter calibration more efficient [12].

Given a model that is in affine form as per Definition 2.6, then for a state vector $\mathbf{X}(t)$, the discounted characteristic function takes on the form [6]

$$\begin{aligned} f(\mathbf{u}; \mathbf{X}(t), t, T) &= \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(s) ds + i \mathbf{u}^T \mathbf{X}(t)} \middle| \mathcal{F}(t) \right] \\ &= e^{A(\mathbf{u}; t, T) + \mathbf{B}^T(\mathbf{u}; t, T) \mathbf{X}(t)}. \end{aligned}$$

The coefficients $A(\mathbf{u}; t, T)$ and $\mathbf{B}(\mathbf{u}; t, T)$ have to satisfy the complex-valued ordinary differential equations [6]

$$\begin{cases} \frac{d}{dt} A(\mathbf{u}; t, T) = -r_0 + \mathbf{B}^T(\mathbf{u}; t, T) a_0 + \frac{1}{2} \mathbf{B}^T(\mathbf{u}; t, T) c_0 \mathbf{B}(\mathbf{u}; t, T), \\ \frac{d}{dt} \mathbf{B}(\mathbf{u}; t, T) = -r_1 + a_1^T \mathbf{B}(\mathbf{u}; t, T) + \frac{1}{2} \mathbf{B}^T(\mathbf{u}; t, T) c_1 \mathbf{B}(\mathbf{u}; t, T). \end{cases}$$

The Heston model (2.9) and (2.10) is not in affine form due to the non-linearity of the stock price process. More specifically, this is due to $\sqrt{v}S$ in the stock price volatility term. However, the Heston model is in the affine form under the log-transform of the stock price $x(t) = \ln(S(t))$ [6] for which the log-transformed stock price process becomes

$$dx = \left(r - \frac{1}{2}v \right) dt + \sqrt{v} dW_x^{\mathbb{Q}}, \quad x(0) = \ln(S(0)). \quad (2.44)$$

Heston (1993) used this transformation to derive an appropriate characteristic function for the Heston model, which has the form

$$f(u; x(t), v(t), t, T) = e^{A(u; t, T) + B(u; t, T)x(t) + C(u; t, T)v(t)}$$

where $A(u; t, T)$, $B(u; t, T)$ and $C(u; t, T)$ are available in closed form [4]. The closed-form solution presented by Heston was for a vanilla call option. A modified version of this closed-form solution is presented in Section 4.3.2 for an underlying put and underlying call option applicable to a compound option.

It is especially difficult to derive closed-form solutions for the three-state variable hybrid models for the cases when $\rho_{S,r} \neq 0$ and $\rho_{v,r} \neq 0$ [12].

The HHW and HCIR models are not in affine form [6]. The model under the log-transform of the stock price $x(t) = \ln(S(t))$ is also not in the affine form [6]

since the log-transformed instantaneous covariance matrix

$$\sigma(\mathbf{X}(t))\sigma(\mathbf{X}(t))^T = \begin{bmatrix} v & \rho_{x,v}\sigma_v v & \rho_{x,r}\sigma_r r^p \sqrt{v} \\ * & \sigma_v^2 v & \rho_{v,r}\sigma_v \sigma_r r^p \sqrt{v} \\ * & * & \sigma_r^2 r^{2p} \end{bmatrix} \quad (2.45)$$

is non-linear. By setting $\rho_{x,r} = \rho_{v,r} = 0$, the log-transformed model is of the affine form. In't Hout et al. in 2007 [10] used this transformation to derive an appropriate characteristic function for the HHW model that has the form

$$f(u; x(t), v(t), r(t), t, T) = e^{A(u;t,T) + B(u;t,T)x(t) + C(u;t,T)v(t) + D(u;t,T)r(t)}$$

where $A(u;t,T)$, $B(u;t,T)$, $C(u;t,T)$ and $D(u;t,T)$ are available in closed form. This semi-closed-form solution was presented for a vanilla call option. A modified version of this closed-form solution is presented in Section 4.3.3 for an underlying put and underlying call option applicable to a compound option.

Grzelak and Oosterlee (2011) presented two projection techniques to derive an affine approximated closed-form characteristic function for the HHW and HCIR hybrid models for the case when $\rho_{x,v} \neq 0$, $\rho_{x,r} \neq 0$ and $\rho_{v,r} = 0$ [6]. This was done by approximating the $\rho_{x,r}\sigma_r r^p \sqrt{v}$ term to linearise the covariance matrix (2.45). Their approximations included a deterministic and a stochastic approach.

Grzelak and Oosterlee (2011) also presented an affine approximated closed-form characteristic function for the HHW model for the case when $\rho_{x,v} \neq 0$, $\rho_{x,r} \neq 0$ and $\rho_{v,r} \neq 0$ [6]. This was done by approximating the $\rho_{x,r}\sigma_r r^p \sqrt{v}$ and $\rho_{v,r}\sigma_v \sigma_r r^p \sqrt{v}$ terms to linearise the covariance matrix (2.45).

3 Pricing partial differential equations

3.1 Multidimensional Itô's formula

This section presents the pricing PDEs for the one-, two- and three-state variable models defined in Section 2.2 through Section 2.4. The multidimensional Itô's formula is used.

Theorem 3.1 (Multidimensional Itô's Formula [9]). *Consider the n -dimensional vector process $\mathbf{X}(t) = [X_1, \dots, X_n]^T$, where the component $X_i = \{X_i(t), t \geq 0\}$ has a stochastic differential given by*

$$dX_i = \mu_i dt + \sigma_i dW_i \quad (3.1)$$

where $\mu_i = \{\mu_i(t), t \geq 0\}$ and $\sigma_i = \{\sigma_i(t), t \geq 0\}$ are adapted processes, and $W_i = \{W_i(t), t \geq 0\}$ are independent standard Brownian Motions for $i = 1, \dots, n$. Then the following holds:

1. The process $f(t, X_1, \dots, X_n)$ has a stochastic differential given by

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^n \mu_i \frac{\partial f}{\partial X_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j} \right\} dt + \sum_{i=1}^n \sigma_i \frac{\partial f}{\partial X_i} dW_i. \quad (3.2)$$

2. Alternatively, the differential is given by

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial X_i} dX_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial X_i \partial X_j} dX_i dX_j \quad (3.3)$$

with the formal multiplication table

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_i = 0, \\ (dW_i)^2 = dt, & i = 1, \dots, n, \\ dW_i \cdot dW_j = 0, & i \neq j. \end{cases} \quad (3.4)$$

3.2 Pricing PDE for the Black–Scholes model

Theorem 3.2 (Black–Scholes Pricing PDE [9]). Assume that the stock price S follows the geometric Brownian motion

$$dS = (r - q)S dt + \sigma_S S dW_S^{\mathbb{Q}} \quad (3.5)$$

with respect to the risk-free measure $\mathbb{Q}$ and that one wants to price a contingent claim of the form $\mathcal{X} = \psi(S(T))$, which has a price process of the form $\Pi(t; \mathcal{X})$. The only pricing function f of the form

$$\Pi(t, \mathcal{X}) = f(t, S),$$

which is consistent with the absence of arbitrage, is when f is the solution of the boundary value problem

$$\begin{aligned} \frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf &= 0, \\ f(T, S) &= \psi(S), \end{aligned} \quad (3.6)$$

in the domain $[0, T] \times \mathbb{R}_+$.

Proof. The proof is extensively available throughout the literature and can be found in [9]. $\square$

3.3 Pricing PDE for the Heston model

Theorem 3.3 (Heston Pricing PDE [4]). Assume that the stock price S and stock price variance v follows the geometric Brownian motions

$$dS = (r - q)Sdt + \sqrt{v}SdW_S^{\mathbb{Q}}, \quad (3.7)$$

$$dv = \left(\kappa_v(\theta_v - v) - \lambda_v(t, v) \right) dt + \sigma_v \sqrt{v} dW_v^{\mathbb{Q}} \quad (3.8)$$

with respect to the risk-free measure $\mathbb{Q}$. Assume also that one wants to price a contingent claim of the form $\mathcal{X} = \psi(S(T))$, which has a price process of the form $\Pi(t; \mathcal{X})$. The only pricing function f of the form

$$\Pi(t, \mathcal{X}) = f(t, S, v),$$

which is consistent with the absence of arbitrage, is when f is the solution of the boundary value problem

$$\begin{aligned} & \frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} \\ & + \frac{1}{2} v S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 f}{\partial v^2} + \rho_{S,v} \sigma_v v S \frac{\partial^2 f}{\partial S \partial v} - rf = 0, \end{aligned} \quad (3.9)$$

$$f(T, S, v) = \psi(S),$$

in the domain $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. Consider a simple contingent claim of the form $\mathcal{X} = \psi(S(T))$. Assume that this claim can be traded on a market and that its price process $\Pi(t; \mathcal{X})$ has the form

$$\Pi(t, \mathcal{X}) = f(t, S, v)$$

for some smooth function f . The aim is to find f for the market to be free of arbitrage possibilities.

By applying Itô's multidimensional formula (3.3), the change in the value of f with respect to t , S and v is given by the Taylor expansion

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial v} dv + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} dv^2 + \frac{\partial^2 f}{\partial S \partial v} dS dv. \quad (3.10)$$

If S and v evolve according to the processes in (3.7) and (3.8), respectively, then the terms dS^2 , dv^2 and $dS dv$ in (3.10) are given by

$$\begin{cases} dS^2 &= \left((r - q)Sdt + \sqrt{v}SdW_S^{\mathbb{Q}} \right)^2, \\ dv^2 &= \left((\kappa_v(\theta_v - v) - \lambda_v(t, v))dt + \sigma_v \sqrt{v}dW_v^{\mathbb{Q}} \right)^2, \\ dS dv &= \left((r - q)Sdt + \sqrt{v}SdW_S^{\mathbb{Q}} \right) \left((\kappa_v(\theta_v - v) - \lambda_v(t, v))dt + \sigma_v \sqrt{v}dW_v^{\mathbb{Q}} \right), \end{cases}$$

which, after applying the formal multiplication table (3.4), are simplified to

$$\begin{cases} dS^2 &= vS^2 dt, \\ dv^2 &= \sigma_v^2 v dt, \\ dS dv &= \rho_{S,v} \sigma_v v S dt. \end{cases} \quad (3.11)$$

Substituting (3.7), (3.8) and (3.11) into (3.10) yields the expression

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + (r - q)S \frac{\partial f}{\partial S} dt + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} dt \\ &\quad + \sqrt{v}S \frac{\partial f}{\partial S} dW_S^{\mathbb{Q}} + \sigma_v \sqrt{v} \frac{\partial f}{\partial v} dW_v^{\mathbb{Q}} + \frac{1}{2}vS^2 \frac{\partial^2 f}{\partial S^2} dt \\ &\quad + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} dt + \rho_{S,v} \sigma_v v S \frac{\partial^2 f}{\partial S \partial v} dt. \end{aligned} \quad (3.12)$$

Definition 2.2 states that standard Brownian motions have zero expectation; therefore, $\mathbb{E}[dW_S^{\mathbb{Q}}] = \mathbb{E}[dW_v^{\mathbb{Q}}] = 0$. The expectation of df becomes

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[df] &= \frac{\partial f}{\partial t} dt + (r - q)S \frac{\partial f}{\partial S} dt + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} dt \\ &\quad + \frac{1}{2}vS^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} dt \\ &\quad + \rho_{S,v} \sigma_v v S \frac{\partial^2 f}{\partial S \partial v} dt. \end{aligned} \quad (3.13)$$

For there not to be any arbitrage opportunities, the process f needs to earn the same instantaneous return as other short-term risk-free securities [11]. Hence,

$$\mathbb{E}^{\mathbb{Q}}[df] = rf dt. \quad (3.14)$$

Substituting (3.13) into (3.14) and dividing by dt yields (3.9). $\square$

3.4 Log-transformed pricing PDE for the Heston model

Corollary 3.1 (Log-transformed Heston Pricing PDE [4]). *The log-transform of (3.9) in terms of the log-stock price $x(t) = \ln(S(t))$ is given by*

$$\begin{aligned} \frac{\partial f}{\partial t} + (r - q - \frac{1}{2}v) \frac{\partial f}{\partial x} + (\kappa_v(\theta_v - v) - \lambda_v v) \frac{\partial f}{\partial v} \\ + \frac{1}{2}v \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} + \rho_{S,v} \sigma_v v \frac{\partial^2 f}{\partial x \partial v} - rf = 0. \end{aligned} \quad (3.15)$$

Proof. The first-order partial derivative of f with respect to S can be written in terms of x by applying the chain rule as follows

$$\frac{\partial f}{\partial S} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial S} = \frac{1}{S} \frac{\partial f}{\partial x}. \quad (3.16)$$

The second-order partial derivative of f with respect to S can be written in terms of x by applying the product rule as follows

$$\begin{aligned} \frac{\partial^2 f}{\partial S^2} &= \frac{\partial}{\partial S} \left(\frac{\partial f}{\partial S} \right) = \frac{\partial}{\partial S} \left(\frac{1}{S} \frac{\partial f}{\partial x} \right) = -\frac{1}{S^2} \frac{\partial f}{\partial x} + \frac{1}{S} \frac{\partial}{\partial S} \left(\frac{\partial f}{\partial x} \right) \\ &= -\frac{1}{S^2} \frac{\partial f}{\partial x} + \frac{1}{S} \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial S} = -\frac{1}{S^2} \frac{\partial f}{\partial x} + \frac{1}{S^2} \frac{\partial^2 f}{\partial x^2}. \end{aligned} \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.9) and simplifying yields the log-transformed Heston pricing PDE (3.15). The price of volatility risk is written as a linear function of the variance such that $\lambda_v(t, v) = \lambda_v v$. $\square$

3.5 Pricing PDE for the HHW/H CIR model

Theorem 3.4 (HHW and HCIR Pricing PDE [6]). Assume that the stock price S , stock price variance v and interest rate r follow the set of processes

$$dS = (r - q)S dt + \sqrt{v}S dZ_S^{\mathbb{Q}}, \quad (3.18)$$

$$dv = \left(\kappa_v(\theta_v - v) - \lambda_v(t, v) \right) dt + \sigma_v \sqrt{v} dZ_v^{\mathbb{Q}}, \quad (3.19)$$

$$dr = \left(\kappa_r(\theta_r - r) - \lambda_r(t, r) \right) dt + \sigma_r r^p dZ_r^{\mathbb{Q}}, \quad (3.20)$$

with respect to the risk-free measure $\mathbb{Q}$. Assume also that one wants to price a contingent claim of the form $\mathcal{X} = \psi(S(T))$, which has a price process of the form $\Pi(t; \mathcal{X})$. The only pricing function f of the form

$$\Pi(t, \mathcal{X}) = f(t, S, v, r),$$

which is consistent with the absence of arbitrage, is when f is the solution to the following boundary value problem

$$\begin{aligned} \frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} + (\kappa_r(\theta_r - r) - \lambda_r(t, r)) \frac{\partial f}{\partial r} \\ + \frac{1}{2} v S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2} \sigma_r^2 r^{2p} \frac{\partial^2 f}{\partial r^2} + \rho_{S,v} \sigma_v v S \frac{\partial^2 f}{\partial S \partial v} \\ + \rho_{S,r} \sigma_r \sqrt{v} r^p S \frac{\partial^2 f}{\partial S \partial r} + \rho_{v,r} \sigma_v \sigma_r \sqrt{v} r^p \frac{\partial^2 f}{\partial v \partial r} - r f = 0, \end{aligned} \quad (3.21)$$

$$f(T, S, v, r) = \psi(S),$$

in the domain $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$.

Proof. Consider a simple contingent claim of the form $\mathcal{X} = \psi(S(T))$. Assume that this claim can be traded on a market and that its price process $\Pi(t; \mathcal{X})$ has the form

$$\Pi(t, \mathcal{X}) = f(t, S, v, r)$$

for some smooth function f . The aim is to find f for the market to be free of arbitrage possibilities.

By applying Itô's multidimensional formula (3.3), the change in the value of f with respect to t , S , v and r is given by the Taylor expansion

$$\begin{aligned} df = & \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial v} dv + \frac{\partial f}{\partial r} dr \\ & + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} dv^2 + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} dr^2 \\ & + \frac{\partial^2 f}{\partial S \partial v} dS dv + \frac{\partial^2 f}{\partial S \partial r} dS dr + \frac{\partial^2 f}{\partial v \partial r} dv dr. \end{aligned} \quad (3.22)$$

If S , v and r evolve according to the processes (3.18), (3.19) and (3.20), respectively, then the terms dS^2 , dv^2 , dr^2 , $dS dv$, $dS dr$ and $dv dr$ in (3.22) are given by

$$\begin{cases} dS^2 = & \left((r - q)S dt + \sqrt{v}S dZ_S^{\mathbb{Q}} \right)^2, \\ dv^2 = & \left((\kappa_v(\theta_v - v) - \lambda_v(t, v)) dt + \sigma_v \sqrt{v} dZ_v^{\mathbb{Q}} \right)^2, \\ dr^2 = & \left((\kappa_r(\theta_r - r) - \lambda_r(t, r)) dt + \sigma_r r^p dZ_r^{\mathbb{Q}} \right)^2, \\ dS dv = & \left((r - q)S dt + \sqrt{v}S dZ_S^{\mathbb{Q}} \right) \left((\kappa_v(\theta_v - v) - \lambda_v(t, v)) dt + \sigma_v \sqrt{v} dZ_v^{\mathbb{Q}} \right), \\ dS dr = & \left((r - q)S dt + \sqrt{v}S dZ_S^{\mathbb{Q}} \right) \left((\kappa_r(\theta_r - r) - \lambda_r(t, r)) dt + \sigma_r r^p dZ_r^{\mathbb{Q}} \right), \\ dv dr = & \left((\kappa_v(\theta_v - v) - \lambda_v(t, v)) dt + \sigma_v \sqrt{v} dZ_v^{\mathbb{Q}} \right) \times \\ & \left((\kappa_r(\theta_r - r) - \lambda_r(t, r)) dt + \sigma_r r^p dZ_r^{\mathbb{Q}} \right), \end{cases}$$

which, after applying the formal multiplication table (3.4), are simplified to

$$\begin{cases} dS^2 & = vS^2 dt, \\ dv^2 & = \sigma_v^2 v dt, \\ dr^2 & = \sigma_r^2 r^{2p} dt, \\ dS dv & = \rho_{S,v} \sigma_v v S dt, \\ dS dr & = \rho_{S,r} \sigma_r \sqrt{v} r^p S dt, \\ dv dr & = \rho_{v,r} \sigma_v \sigma_r \sqrt{v} r^p dt. \end{cases} \quad (3.23)$$

Substituting (3.18), (3.19), (3.20) and (3.23) into (3.22) yields the expression

$$\begin{aligned}
df = & \frac{\partial f}{\partial t} dt + (r - q)S \frac{\partial f}{\partial S} dt + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} dt \\
& + (\kappa_r(\theta_r - r) - \lambda_r(t, r)) \frac{\partial f}{\partial r} dt + \sqrt{v}S \frac{\partial f}{\partial S} dZ_S^Q + \sigma_v \sqrt{v} \frac{\partial f}{\partial v} dZ_v^Q + \sigma_r r^p \frac{\partial f}{\partial r} dZ_r^Q \\
& + \frac{1}{2}vS^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} dt + \frac{1}{2}\sigma_r^2 r^{2p} \frac{\partial^2 f}{\partial r^2} dt \\
& + \rho_{S,v}\sigma_v vS \frac{\partial^2 f}{\partial S \partial v} dt + \rho_{S,r}\sigma_r \sqrt{v}r^p S \frac{\partial^2 f}{\partial S \partial r} dt \\
& + \rho_{v,r}\sigma_v \sigma_r \sqrt{v}r^p \frac{\partial^2 f}{\partial v \partial r} dt. \quad (3.24)
\end{aligned}$$

Definition 2.2 states that standard Brownian motions have zero expectation; therefore, $\mathbb{E}[dZ_S^Q] = \mathbb{E}[dZ_v^Q] = \mathbb{E}[dZ_r^Q] = 0$. The expectation of df becomes

$$\begin{aligned}
\mathbb{E}^Q[df] = & \frac{\partial f}{\partial t} dt + (r - q)S \frac{\partial f}{\partial S} dt + (\kappa_v(\theta_v - v) - \lambda_v(t, v)) \frac{\partial f}{\partial v} dt \\
& + (\kappa_r(\theta_r - r) - \lambda_r(t, r)) \frac{\partial f}{\partial r} dt + \frac{1}{2}vS^2 \frac{\partial^2 f}{\partial S^2} dt + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} dt \\
& + \frac{1}{2}\sigma_r^2 r^{2p} \frac{\partial^2 f}{\partial r^2} dt + \rho_{S,v}\sigma_v vS \frac{\partial^2 f}{\partial S \partial v} dt + \rho_{S,r}\sigma_r \sqrt{v}r^p S \frac{\partial^2 f}{\partial S \partial r} dt \\
& + \rho_{v,r}\sigma_v \sigma_r \sqrt{v}r^p \frac{\partial^2 f}{\partial v \partial r} dt. \quad (3.25)
\end{aligned}$$

For there not to be any arbitrage opportunities, the process f needs to earn the same instantaneous return as other short-term risk-free securities [11]. Hence,

$$\mathbb{E}^Q[df] = rf dt. \quad (3.26)$$

Substituting (3.25) into (3.26) and dividing by dt yields (3.21). $\square$

3.6 Log-transformed pricing PDE for the HHW/HCIR model

Corollary 3.2 (Log-transformed HHW and HCIR Pricing PDE [6]). *The log-transform of (3.21) in terms of the log-stock price $x(t) = \ln(S(t))$ is given by*

$$\begin{aligned}
\frac{\partial f}{\partial t} + (r - q - \frac{1}{2}v) \frac{\partial f}{\partial x} + (\kappa_v(\theta_v - v) - \lambda_v v) \frac{\partial f}{\partial v} + (\kappa_r(\theta_r - r) - \lambda_r r) \frac{\partial f}{\partial r} \\
+ \frac{1}{2}v \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2}\sigma_r^2 r^{2p} \frac{\partial^2 f}{\partial r^2} + \rho_{S,v}\sigma_v v \frac{\partial^2 f}{\partial x \partial v} \\
+ \rho_{S,r}\sigma_r \sqrt{v}r^p \frac{\partial^2 f}{\partial x \partial r} + \rho_{v,r}\sigma_v \sigma_r \sqrt{v}r^p \frac{\partial^2 f}{\partial v \partial r} - rf = 0. \quad (3.27)
\end{aligned}$$

Proof. Substituting the first-order partial derivative of f with respect to x (3.16) and the second-order partial derivative of f with respect to x (3.17) into (3.21) and simplifying yields the log-transformed HHW and HCIR pricing PDE (3.27). The price of volatility and interest rate risk are written as linear functions of volatility and interest rate, respectively, so that $\lambda_v(t, v) = \lambda_v v$ and $\lambda_r(t, r) = \lambda_r r$. $\square$

4 European compound options

4.1 Benefits and real-world applications

A compound option gives the investor some degree of exposure to the long-term underlying option without paying for the underlying option at the onset. It allows for high-leverage positions to be taken in an environment where the investor is uncertain about the protection capabilities of the underlying option. Should the investor decide not to exercise the underlying option due to adverse movements in the stock price, the investor would have paid less for the overlying option than they would have paid for the underlying option alone. Only once both overlying and underlying options are exercised will the total premium paid be more than that of only the underlying option. Compound options most often occur in currency and fixed-income markets [15].

The compound option model can be used to model many different phased investment undertakings where one acquisition phase is contingent on the preceding phases [16]. For example, suppose a company is tendering at T_0 on a contract for which it would require financing between T_M and T_D if successful. The company has based their prices on T_0 interest rates and is, therefore, at risk of an increase in interest rates between T_0 and T_M . The company could purchase an interest rate cap starting at T_M with maturity T_D to protect themselves. However, it would prove expensive should it not receive the contract. Alternatively, for a smaller upfront premium, the company could purchase a call option with a maturity T_M on an interest rate cap starting on T_M with maturity T_D . Should the company not receive the contract, it could let the call option lapse. If the company gets the contract, it could exercise the option and receive the interest rate cap. The company has, therefore, leveraged itself by taking a substantial position for a small upfront premium [2].

Since the time to expiration of the overlying option is often shorter than that of the underlying option, investors can use compound options to extend the life

of an underlying option position [17]. It allows one to create n -fold compound option models with multiple exercise dates while checking at each exercise date whether it is optimal to stop or to invest further in the underlying option [18].

The chooser option is a generalisation of the compound option in that it gives the holder the option to choose between a put or a call option as the underlying option when the overlying option is exercised. It is ideal for the undecided and protects the holder against a significant stock price jump in either direction [11].

4.2 Payoff functions

4.2.1 State vector

Throughout this dissertation, the notation $\mathbf{X}(t)$ is used to represent the state vector, which is defined as

$$\mathbf{X}(t) = \begin{cases} \{S(t)\} & \text{for the Black–Scholes model,} \\ \{S(t), v(t)\} & \text{for the Heston model,} \\ \{S(t), v(t), r(t)\} & \text{for the HHW/HCIR models.} \end{cases} \quad (4.1)$$

4.2.2 Payoff function of the underlying option

The payoff functions of an underlying call and put option can be written as a single expression if the variable ϕ is introduced such that

$$\phi = \begin{cases} +1 & \text{for an underlying call option,} \\ -1 & \text{for an underlying put option.} \end{cases} \quad (4.2)$$

The payoff function of the underlying option D at its maturity T_D is given by [11]

$$\max\left\{\phi\left(S(T_D) - K_D\right), 0\right\} \quad (4.3)$$

where $S(T_D)$ is the value of the stock price at T_D , and K_D is the strike price of the underlying option.

4.2.3 Payoff function of the overlying option

The payoff functions of the four main compound option types can be combined into a single expression if the variable ω is introduced such

that

$$\omega = \begin{cases} +1 & \text{for an overlying call option,} \\ -1 & \text{for an overlying put option.} \end{cases} \quad (4.4)$$

The payoff function of the compound (or overlying) option M at its maturity T_M is given by [19]

$$\max \left\{ \omega \left(D(\mathbf{X}(T_M), T_M, T_D, K_D, \phi) - K_M \right), 0 \right\} \quad (4.5)$$

where K_M is the strike price of the overlying option, and $D(\mathbf{X}(t), t, T_D, K_D, \phi)$ is the discounted price at t of the underlying option with maturity T_D and strike price K_D .

4.3 Underlying options under the Black–Scholes, Heston and HHW models

4.3.1 Closed-form solution for the Black–Scholes model

To calculate the value of $D(\mathbf{X}(t), t, T_D, K_D, \phi)$ in (4.5) under the Black–Scholes model if the underlying option is European, one can use the closed-form solution for a vanilla option given below.

Theorem 4.1 (Black–Scholes Closed-form Solution for a European Underlying Option [9]). *For the state vector $\mathbf{X}(t) = \{S(t)\}$, the price at t of a European underlying option with maturity T_D and strike price K_D is given by $\Pi(t) = D(S(t), t, T_D, K_D, \phi)$, where*

$$D(S(t), t, T_D, K_D, \phi) = \phi \left(S(t) e^{-q(T_D-t)} N(\phi d_1) - K_D e^{-r(T_D-t)} N(\phi d_2) \right). \quad (4.6)$$

Here

$$d_1 = \frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - t}} \quad (4.7)$$

and

$$d_2 = d_1 - \sigma_S \sqrt{T_D - t}. \quad (4.8)$$

$N(x)$ is the standard normal cumulative distribution function given by

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du. \quad (4.9)$$

Proof. The proof is extensively available in the literature and can be found in Section 7.5 of [9]. $\square$

4.3.2 Closed-form solution for the Heston model

To calculate the value of $D(\mathbf{X}(t), t, T_D, K_D, \phi)$ in (4.5) under the Heston model if the underlying option is European, one can use the Heston closed-form solution for a vanilla call option. The original solution in [4] did not include dividend-paying stocks. A modified version of the closed-form solution is provided below and makes provision for an underlying put option and dividend-paying stocks.

Theorem 4.2 (Heston Closed-form Solution for a European Underlying Option [4]). *For the state vector $\mathbf{X}(t) = \{S(t), v(t)\}^T$ with $x(t) = \ln(S(t))$, the price at t of a European underlying option with maturity T_D and strike price K_D is given by $\Pi(t) = D(\mathbf{X}(t), t, T_D, K_D, \phi)$, where*

$$D(\mathbf{X}(t), t, T_D, K_D, \phi) = S(t)e^{-q(T_D-t)} \left(P_1(\mathbf{X}(t), t, T_D) - \frac{1-\phi}{2} \right) - K_De^{-r(T_D-t)} \left(P_2(\mathbf{X}(t), t, T_D) - \frac{1-\phi}{2} \right). \quad (4.10)$$

The probabilities P_1 and P_2 can be calculated using the formula

$$P_j(\mathbf{X}(t), t, T_D) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Real} \left[\frac{e^{-\ln(K_D)ui} f_j(u; \mathbf{X}(t), t, T_D)}{ui} \right] du \quad (4.11)$$

for $j = 1, 2$, with i satisfying the equation $i^2 = -1$. The characteristic function $f_j(u; \mathbf{X}(t), t, T_D)$ is given by

$$f_j(u; \mathbf{X}(t), t, T_D) = e^{A_j(u; t, T_D) + B(u)x(t) + C_j(u; t, T_D)v(t)} \quad (4.12)$$

where

$$A_j(u; t, T_D) = (r - q)ui(T_D - t) + \frac{\kappa_v \theta_v}{\sigma_v^2} \times \left((b_j - \rho_{S,v} \sigma_v ui + d_j)(T_D - t) - 2 \ln \left[\frac{1 - g_i e^{d_j(T_D-t)}}{1 - g_j} \right] \right), \quad (4.13)$$

$$B(u) = ui, \quad (4.14)$$

and

$$C_j(u; t, T_D) = \frac{b_j - \rho_{S,v} \sigma_v ui + d_j}{\sigma_v^2} \left[\frac{1 - e^{d_j(T_D-t)}}{1 - g_j e^{d_j(T_D-t)}} \right] \quad (4.15)$$

with variables

$$b_j = \begin{cases} \kappa_v + \lambda_v - \rho_{S,v} \sigma_v & \text{if } j = 1, \\ \kappa_v + \lambda_v & \text{if } j = 2, \end{cases} \quad (4.16)$$

$$c_j = \begin{cases} \frac{1}{2} & \text{if } j = 1, \\ -\frac{1}{2} & \text{if } j = 2, \end{cases} \quad (4.17)$$

$$d_j = \sqrt{(\rho_{S,v}\sigma_v ui - b_j)^2 - \sigma_v^2(2c_j ui - u^2)}, \quad (4.18)$$

$$g_j = \frac{b_j - \rho_{S,v}\sigma_v ui + d_j}{b_j - \rho_{S,v}\sigma_v ui - d_j}. \quad (4.19)$$

Proof. The proof can be found in [4]. $\square$

4.3.3 Semi-closed-form solution for the HHW model

To calculate the value of $D(\mathbf{X}(t), t, T_D, K_D, \phi)$ in (4.5) under the HHW model if the underlying option is European, one can use the semi-closed-form solution derived by In't Hout et al. in 2007 [10] for a vanilla call option. The original solution in [10] did not include dividend-paying stocks. A modified version of the closed-form solution is provided below and makes provision for an underlying put option and dividend-paying stocks.

Theorem 4.3 (HHW Closed-form Solution for a European Underlying Option [10]). *For the state vector $\mathbf{X}(t) = \{S(t), v(t), r(t)\}$ with $x(t) = \ln(S(t))$, and $\rho_{S,r} = \rho_{v,r} = 0$, the price at t of a European underlying option with maturity T_D and strike price K_D is given by $\Pi(t) = D(\mathbf{X}(t), t, T_D, K_D, \phi)$, where*

$$D(\mathbf{X}(t), t, T_D, K_D, \phi) = S(t)e^{-q(T_D-t)} \left(P_1(\mathbf{X}(t), t, T_D) - \frac{1-\phi}{2} \right) - K_D e^{-h(r(t), t, T_D)} \left(P_2(\mathbf{X}(t), t, T_D) - \frac{1-\phi}{2} \right). \quad (4.20)$$

The probabilities P_1 and P_2 can be calculated using the formula

$$P_j(\mathbf{X}(t), t, T_D) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Real} \left[\frac{e^{-\ln(K_D)ui} f_j(u; \mathbf{X}(t), t, T_D)}{ui} \right] du, \quad (4.21)$$

for $j = 1, 2$, with i satisfying the equation $i^2 = -1$. The rate at which K_D is discounted, $h(r(t), t, T_D)$, is given by

$$h(r(t), t, T_D) = \frac{r(t)}{\kappa_r} (1 - e^{-\kappa_r(T_D-t)}) + \int_t^{T_D} \theta_r(\tau) (1 - e^{-\kappa_r(T_D-\tau)}) d\tau - \frac{\sigma_r^2}{2\kappa_r^2} \left((T_D - t) + \frac{2}{\kappa_r} e^{-\kappa_r(T_D-t)} - \frac{1}{2\kappa_r} e^{-2\kappa_r(T_D-t)} - \frac{3}{2\kappa_r} \right). \quad (4.22)$$

The characteristic function $f_j(u; \mathbf{X}(t), t, T_D)$ is given by

$$f_j(u; \mathbf{X}(t), t, T_D) = e^{A_j(u; t, T_D) + B(u)x(t) + C_j(u; t, T_D)v(t) + D_j(u; t, T_D)r(t) + e_j h(r(t), t, T_D)} \quad (4.23)$$

where

$$\begin{aligned}
 A_j(u; t, T_D) = & \frac{\kappa_v \theta_v}{\sigma_v^2} \left((b_j - \rho_{S,v} \sigma_v u i + d_j)(T_D - t) - 2 \ln \left[\frac{1 - g_j e^{d_j(T_D - t)}}{1 - g_j} \right] \right) \\
 & + (u i - e_j) \int_t^{T_D} \theta_r(\lambda) (1 - e^{-\kappa_r(T_D - \lambda)}) d\lambda + \frac{\sigma_r^2}{2\kappa_r^2} (u i - e_j)^2 \times \\
 & \left((T_D - t) + \frac{2}{\kappa_r} e^{-\kappa_r(T_D - t)} - \frac{1}{2\kappa_r} e^{-2\kappa_r(T_D - t)} - \frac{3}{2\kappa_r} \right) - q u i (T_D - t), \quad (4.24)
 \end{aligned}$$

$$B(u) = u i, \quad (4.25)$$

$$C_j(u; t, T_D) = \frac{b_j - \rho_{S,v} \sigma_v u i + d_j}{\sigma_v^2} \left[\frac{1 - e^{d_j(T_D - t)}}{1 - g_j e^{d_j(T_D - t)}} \right], \quad (4.26)$$

and

$$D_j(u; t, T_D) = \frac{u i - e_j}{\kappa_r} (1 - e^{-\kappa_r(T_D - t)}) \quad (4.27)$$

with variables

$$b_j = \begin{cases} \kappa_v + \lambda_v - \rho_{S,v} \sigma_v & \text{if } j = 1, \\ \kappa_v + \lambda_v & \text{if } j = 2, \end{cases} \quad (4.28)$$

$$c_j = \begin{cases} \frac{1}{2} & \text{if } j = 1, \\ -\frac{1}{2} & \text{if } j = 2, \end{cases} \quad (4.29)$$

$$d_j = \sqrt{(\rho_{S,v} \sigma_v u i - b_j)^2 - \sigma_v^2 (2c_j u i - u^2)}, \quad (4.30)$$

$$e_j = \begin{cases} 0 & \text{if } j = 1, \\ 1 & \text{if } j = 2, \end{cases} \quad (4.31)$$

$$g_j = \frac{b_j - \rho_{S,v} \sigma_v u i + d_j}{b_j - \rho_{S,v} \sigma_v u i - d_j}. \quad (4.32)$$

Proof. The proof can be found in [10]. □

4.4 Compound options under the Black–Scholes model

This section derives the closed-form solution for a European compound option in the Black–Scholes environment. The integral identity for a bivariate normal cumulative distribution below is used extensively throughout the derivation.

Theorem 4.4 (Integral Identity for a Bivariate Normal Cumulative Distribution [20]). Suppose that $n(x)$ is a standard normal probability density function of x given by

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad (4.33)$$

and $N(x)$ is the corresponding standard cumulative distribution function given by

$$\begin{aligned} N(x) &= \int_{-\infty}^x n(u) du \\ &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du. \end{aligned} \quad (4.34)$$

Suppose also that $N_2(x, y, \rho)$ is a standardised bivariate normal cumulative distribution function with correlation coefficient ρ , given by

$$N_2(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^x \int_{-\infty}^y e^{-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}} du dv. \quad (4.35)$$

The identity

$$\int_a^\infty n(x) N(bx + c) dx = N_2\left(-a, \frac{c}{\sqrt{1+b^2}}, \frac{-b}{\sqrt{1+b^2}}\right) \quad (4.36)$$

holds true for any $\{a, b, c\} \in \mathbb{R}$.

Proof. The proof for this theorem can be found in [20]. $\square$

The risk-neutral valuation theorem below, which stems from the famous Feynman-Kač theorem [9], is used.

Theorem 4.5 (Risk-neutral Valuation [9]). *The arbitrage-free price of a contingent claim $\psi(S(T))$ is given by $\Pi(t, \psi) = f(t, S(t))$, where*

$$f(t, S(t)) = e^{-r(T-t)} \mathbb{E}_{t, S(t)}^{\mathbb{Q}} [\psi(S(T))] \quad (4.37)$$

and the $\mathbb{Q}$ -dynamics of $S = \{S(t), t \geq 0\}$ are given by (2.30).

Proof. The proof for this theorem can be found in Section 7.4 of [9]. $\square$

4.4.1 Closed-form solution for the Black–Scholes model

The closed-form solution of a European call-on-call compound option using the Black–Scholes model was first developed by Geske in 1979 [1]. Hull gives a closed-form expression for each of the four compound option types in [11]. The four closed-form expressions given by Hull are combined into a single

expression in this dissertation using ϕ (Section 4.2.2) and ω (Section 4.2.3). The derivation of the combined closed-form expression is given below.

To the author's knowledge, no closed-form solution for European compound options exists under the Heston, HHW, or HCIR models.

Theorem 4.6 (Black–Scholes Closed-form Solution for a European Compound Option [1]). *The price at t of a European compound option with maturity T_M and strike price K_M on an underlying option with maturity T_D and strike price K_D , is given by $\Pi(t) = M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi)$, where*

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\ & \phi \omega S(t) e^{-q(T_D-t)} N_2 \left(\phi \omega a_1, \phi b_1, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right) \\ & - \phi \omega K_D e^{-r(T_D-t)} N_2 \left(\phi \omega a_2, \phi b_2, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right) \\ & - \omega e^{-r(T_M-t)} K_M N(\phi \omega a_2). \end{aligned} \quad (4.38)$$

Here,

$$a_1 = \frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right) (T_M - t)}{\sigma_S \sqrt{T_M - t}}, \quad (4.39)$$

$$a_2 = a_1 - \sigma_S \sqrt{T_M - t}, \quad (4.40)$$

$$b_1 = \frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right) (T_D - t)}{\sigma_S \sqrt{T_D - t}}, \quad (4.41)$$

$$b_2 = b_1 - \sigma_S \sqrt{T_D - t}. \quad (4.42)$$

$N_2(x, y, \rho)$ is a standard normal bivariate cumulative distribution function with correlation coefficient ρ , given by

$$N_2(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^x \int_{-\infty}^y e^{-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}} du dv \quad (4.43)$$

S^* is the critical stock price such that the underlying option price equals K_M at T_M . I.e.:

$$\begin{aligned} D(S^*, T_M, T_D, K_D, \phi) &= \phi \left(S^* e^{-q(T_D-T_M)} N(\phi d_1) - K_D e^{-r(T_D-T_M)} N(\phi d_2) \right) \\ &= K_M \end{aligned} \quad (4.44)$$

where

$$d_1 = \frac{\ln\left(\frac{S^*}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - T_M)}{\sigma_S \sqrt{T_D - T_M}} \quad (4.45)$$

and

$$d_2 = d_1 - \sigma_S \sqrt{T_D - T_M}. \quad (4.46)$$

Proof. Consider the following assumptions made about a risk-free market [1], [11]:

- The risk-free rate of the market, r , and the dividend yield of the stock, q , are known, constant and the same for all maturities.
- Short-selling of the stock with full use of proceeds is allowed.
- There are no transaction costs or taxes on the stock.
- Security markets are arbitrage-free, perfect and competitive.
- Security trading is continuous.

Suppose that the stock price S evolves according to (2.30). By applying Proposition 2.1, the price of the stock at T_M is given by

$$S(T_M) = S(t) e^{\left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t) + \sigma_S W_S^Q}. \quad (4.47)$$

Using the properties of a standard Brownian motion from Definition 2.2, $W_S^Q = \sqrt{T_M - t} \cdot z$ is substituted into (4.47) to obtain

$$S(T_M) = S(t) e^{\left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t) + \sigma_S \sqrt{T_M - t} \cdot z} \quad (4.48)$$

where z follows a standard normal distribution.

Rearranging (4.48) gives

$$z = \frac{\ln\left(\frac{S(T_M)}{S(t)}\right) - \left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}} \sim \Phi(0, 1). \quad (4.49)$$

It is further known from Proposition 2.1 that $u = \ln\left(\frac{S(T_M)}{S(t)}\right)$ has the distribution

$$u \sim \Phi\left(\left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t), \sigma_S^2(T_M - t)\right) \quad (4.50)$$

where $t \in [0, T_M]$ and $\Phi(m, w^2)$ is a normal distribution with mean m and variance w^2 [11]. The probability density function of the normal distribution $\Phi(m, w^2)$ is given by

$$g(x) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(x-m)^2}{2w^2}}, \quad (4.51)$$

so the probability density function of u is given by

$$g(u) = \frac{1}{\sigma_S \sqrt{2\pi(T_M - t)}} \exp \left\{ -\frac{\left(u - (r - q - \frac{\sigma_S^2}{2})(T_M - t) \right)^2}{2\sigma_S^2(T_M - t)} \right\}. \quad (4.52)$$

Using risk-neutral valuation (Theorem 4.5), the value of a compound option at time t is the discounted expected payoff of the compound option at T_M , given by

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= e^{-r(T_M - t)} \mathbb{E}[\max\{\omega(D(S(T_M), T_M, T_D, K_D, \phi) - K_M), 0\}] \\ &= e^{-r(T_M - t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ g(u) du. \end{aligned} \quad (4.53)$$

Since the stock price is a function of z , (4.53) has to be transformed to an integral over z . To do this, $u = \ln\left(\frac{S(T_M)}{S(t)}\right)$, its derivative with respect to z ,

$$du = \sigma_S \sqrt{T_M - t} dz, \quad (4.54)$$

and probability density function (4.52) are substituted into (4.53), which yields

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= e^{-r(T_M - t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ \frac{1}{\sigma_S \sqrt{2\pi(T_M - t)}} \times \\ &\quad \exp \left\{ -\frac{\left(\ln\left(\frac{S(T_M)}{S(t)}\right) - (r - q - \frac{\sigma_S^2}{2})(T_M - t) \right)^2}{2\sigma_S^2(T_M - t)} \right\} \sigma_S \sqrt{T_M - t} dz. \end{aligned} \quad (4.55)$$

Substituting (4.49) into (4.55) and simplifying, yields

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= e^{-r(T_M - t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= e^{-r(T_M - t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ n(z) dz \end{aligned} \quad (4.56)$$

where $n(x)$ is the standard normal probability density function given by (4.33).

Before evaluating the integral in (4.56), it is important to note that the payoff of a compound option will be non-negative only when

$$\omega D(S(T_M), T_M, T_D, K_D, \phi) \geq \omega K_M. \quad (4.57)$$

ω should not be cancelled out in this inequality since it can be either +1 or -1.

Let S^* be the critical stock price such that the price of the underlying option is equal to K_M at T_M . If $\phi S > \phi S^*$ at T_M , the underlying option will be exercised since it is in-the-money. Conversely, if $\phi S < \phi S^*$ at T_M , the underlying option is worthless at expiry since it is out-of-the-money. Therefore, considering a risk-neutral environment, an S^* is required such that the value of the underlying option at T_M is equal to K_M , i.e.

$$\begin{aligned} D(S^*, T_M, T_D, K_D, \phi) &= \phi \left(S^* e^{-q(T_D - T_M)} N(\phi d_1) - K_D e^{-r(T_D - T_M)} N(\phi d_2) \right) \\ &= K_M. \end{aligned} \quad (4.58)$$

Alternatively, a critical value z^* is required such that

$$D\left(S(t)e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)+\sigma_S\sqrt{T_M-t}z^*}, T_M, T_D, K_D, \phi\right) = K_M. \quad (4.59)$$

Substituting (4.59) into (4.57) yields

$$\begin{aligned} \omega D\left(S(t)e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)+\sigma_S\sqrt{T_M-t}z}, T_M, T_D, K_D, \phi\right) \\ \geq \omega D\left(S(t)e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)+\sigma_S\sqrt{T_M-t}z^*}, T_M, T_D, K_D, \phi\right). \end{aligned} \quad (4.60)$$

(4.60) is expanded using (4.6) as follows:

$$\begin{aligned} \omega \phi S(t) e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)+\sigma_S\sqrt{T_M-t}z} e^{-q(T_D-T_M)} N(\phi d_1) - \omega \phi K_D e^{-r(T_D-T_M)} N(\phi d_2) \\ \geq \omega \phi S(t) e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)+\sigma_S\sqrt{T_M-t}z^*} e^{-q(T_D-T_M)} N(\phi d_1) - \omega \phi K_D e^{-r(T_D-T_M)} N(\phi d_2). \end{aligned} \quad (4.61)$$

By testing the four combinations of ϕ and ω , while taking into account that D is strictly increasing when $\phi = 1$ and strictly decreasing when $\phi = -1$, it becomes clear that the inequality above will hold as long as

$$-\phi \omega z \leq -\phi \omega z^*. \quad (4.62)$$

Inequality (4.62) provides the region in which the payoff of a compound option is always positive and is incorporated into the integral in (4.56) by introducing the variable y such that

$$y = -\phi\omega z \leq -\phi\omega z^* = \phi\omega a_2 \quad (4.63)$$

where

$$a_2 = \frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q - \sigma_S^2/2\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}}. \quad (4.64)$$

To transform the integral in (4.56) to an integral over y , the following two cases are considered.

1. For $\phi\omega = -1$, the transformation is as follows:

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= e^{-r(T_M-t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ n(z) dz \\ &= -\phi\omega e^{-r(T_M-t)} \int_{-\infty}^{\phi\omega a_2} (\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M) n(y) dy \\ &= e^{-r(T_M-t)} \int_{-\phi\omega a_2}^{\infty} (\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M) n(y) dy \end{aligned} \quad (4.65)$$

2. For $\phi\omega = 1$, the transformation is as follows:

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= e^{-r(T_M-t)} \int_{-\infty}^{\infty} [\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M]^+ n(z) dz \\ &= -\phi\omega e^{-r(T_M-t)} \int_{\phi\omega a_2}^{-\infty} (\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M) n(y) dy \\ &= -e^{-r(T_M-t)} \int_{\phi\omega a_2}^{-\infty} (\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M) n(y) dy \\ &= e^{-r(T_M-t)} \int_{-\phi\omega a_2}^{\infty} (\omega D(S(T_M), T_M, T_D, K_D, \phi) - \omega K_M) n(y) dy \end{aligned} \quad (4.66)$$

As part of this transformation, $S(T_M)$ in (4.65) and (4.66) becomes

$$S(T_M) = S(t) e^{\left(r - q - \frac{\sigma_S^2}{2}\right)(T_M-t) - \sigma_S \sqrt{(T_M-t)} \phi \omega y}. \quad (4.67)$$

Since the same result is obtained for both cases, (4.65) will be used going forward. Substituting (4.6), (4.33), and (4.67) into (4.65) and expanding gives

$$\begin{aligned}
 M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= \phi \omega e^{-r(T_M-t)} \times \\
 &\int_{-\phi \omega a_2}^{\infty} S(t) e^{\left(r-q-\frac{\sigma_S^2}{2}\right)(T_M-t)-\sigma_S \sqrt{T_M-t} y \phi \omega} e^{-q(T_D-T_M)} N(\phi d_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
 &\quad - \phi \omega e^{-r(T_M-t)} \int_{-\phi \omega a_2}^{\infty} K_D e^{-r(T_D-T_M)} N(\phi d_2) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
 &\quad - \omega e^{-r(T_M-t)} \int_{-\phi \omega a_2}^{\infty} K_M \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \quad (4.68)
 \end{aligned}$$

The integral identity given in Theorem 4.4 is used to evaluate the integrals in (4.68). The constant values are rearranged first, after which the exponents are cancelled out to obtain

$$\begin{aligned}
 M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) &= \\
 &\phi \omega S(t) e^{-q(T_D-t)} \int_{-\phi \omega a_2}^{\infty} N(\phi d_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}-y\sigma_S \sqrt{T_M-t} \phi \omega - \frac{\sigma_S^2(T_M-t)}{2}} dy \\
 &\quad - \phi \omega K_D e^{-r(T_D-t)} \int_{-\phi \omega a_2}^{\infty} N(\phi d_2) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
 &\quad - \omega e^{-r(T_M-t)} K_M \int_{-\infty}^{\phi \omega a_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \quad (4.69)
 \end{aligned}$$

Because $\phi^2 \omega^2$ will always equal 1, the exponent term in the first part of (4.69) can conveniently be shortened by completing the square as follows

$$\begin{aligned}
 &-\frac{y^2}{2} - y\sigma_S \sqrt{T_M-t} \phi \omega - \frac{\sigma_S^2(T_M-t)}{2} \phi^2 \omega^2 \\
 &= -\frac{1}{2} \left(y^2 + 2\sigma_S \sqrt{T_M-t} y \phi \omega + \sigma_S^2(T_M-t) \phi^2 \omega^2 \right) \\
 &= -\frac{1}{2} \left(y + \sigma_S \sqrt{T_M-t} \phi \omega \right)^2. \quad (4.70)
 \end{aligned}$$

Substitute (4.70) into (4.69) and replace the part in the third integral that resembles the standard cumulative distribution function (4.34) with the notation $N(x)$ as follows

$$\begin{aligned}
M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\
& \phi \omega S(t) e^{-q(T_D-t)} \int_{-\phi \omega a_2}^{\infty} N(\phi d_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y + \sigma_S \sqrt{T_M-t} \phi \omega)^2} dy \\
& - \phi \omega K_D e^{-r(T_D-t)} \int_{-\phi \omega a_2}^{\infty} N(\phi d_2) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
& - \omega e^{-r(T_M-t)} K_M N(\phi \omega a_2). \quad (4.71)
\end{aligned}$$

The parts in the first and second integral that resemble the normal probability density function (4.33) are replaced with the notation $n(x)$. The equation is divided into three parts and denoted as such, giving

$$\begin{aligned}
M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\
& \underbrace{\phi \omega S(t) e^{-q(T_D-t)} \int_{-\phi \omega a_2}^{\infty} n(y + \sigma_S \sqrt{T_M-t} \phi \omega) N(\phi d_1) dy}_{\text{First part}} \\
& - \underbrace{\phi \omega K_D e^{-r(T_D-t)} \int_{-\phi \omega a_2}^{\infty} n(y) N(\phi d_2) dy}_{\text{Second part}} \\
& - \underbrace{\omega e^{-r(T_M-t)} K_M N(\phi \omega a_2)}_{\text{Third part}}. \quad (4.72)
\end{aligned}$$

The integrals in the first and second part of (4.72) are in a format that resembles the left-hand side of the integral identity given by (4.36). Therefore, the integral identity can be used to express these integrals using the standard bivariate normal cumulative distribution function notation, $N_2(x, y, \rho)$. To do this, the parameter in the term $N(\phi d_2)$ needs to be expanded as follows

$$\phi d_2 = by + c_2, \quad (4.73)$$

while the parameter in the term $N(\phi d_1)$ has to be expanded as follows

$$\phi d_1 = b(y + \sigma_S \phi \omega \sqrt{T_M-t}) + c_1, \quad (4.74)$$

where the values b , c_1 and c_2 need to be solved. Substituting (4.73) and (4.74) into (4.72) gives

$$\begin{aligned}
M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \underbrace{\phi \omega S(t) e^{-q(T_D-t)} \int_{-\phi \omega a_2}^{\infty} n(y + \sigma_S \sqrt{T_M-t} \phi \omega) N(b(y + \sigma_S \phi \omega \sqrt{T_M-t}) + c_1) dy}_{\text{First part}} \\
& \underbrace{-\phi \omega K_D e^{-r(T_D-t)} \int_{-\phi \omega a_2}^{\infty} n(y) N(b y + c_2) dy}_{\text{Second part}} \\
& \underbrace{-\omega e^{-r(T_M-t)} K_M N(\phi \omega a_2)}_{\text{Third part}}. \quad (4.75)
\end{aligned}$$

The subsequent steps are focussed on solving for b , c_1 and c_2 in (4.75). We refer back to (4.73) and (4.74) to do this.

Substitute d_2 from (4.8) and y from (4.63) into (4.73) to obtain

$$\phi \underbrace{\left(\frac{\ln\left(\frac{S^*}{K_D}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_D - T_M)}{\sigma_S \sqrt{T_D - T_M}} \right)}_{d_2} = b \underbrace{\left(\phi \omega \frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}} \right)}_y + c_2. \quad (4.76)$$

Similarly, substitute d_1 from (4.7) and y from (4.63) into (4.74) to obtain

$$\phi \underbrace{\left(\frac{\ln\left(\frac{S^*}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - T_M)}{\sigma_S \sqrt{T_D - T_M}} \right)}_{d_1} = b \underbrace{\left(\phi \omega \frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}} \right)}_y + \sigma_S \phi \omega \sqrt{T_M - t} + c_1. \quad (4.77)$$

Since there are three variables to solve and only two equations, the value of b is intuitively chosen as

$$b = -\omega \frac{\sqrt{T_M - t}}{\sqrt{T_D - T_M}} \quad (4.78)$$

such that it will cancel out ω in (4.76) and (4.77) while changing the denominator on the right-hand side to match the denominator on the left-hand side. The negative sign is added to transform the $\ln\left(\frac{S(t)}{S^*}\right)$ term on the right-hand side to match $\ln\left(\frac{S^*}{K_D}\right)$ on the left-hand side, assuming c_1 and c_2 will contain the necessary $\ln\left(\frac{S(t)}{K_D}\right)$ term.

By substituting (4.78) into (4.76) and (4.77), c_1 and c_2 are solved and given by

$$c_1 = \phi \left(\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - T_M}} \right) \quad (4.79)$$

and

$$c_2 = \phi \left(\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - T_M}} \right), \quad (4.80)$$

respectively.

With b , c_1 and c_2 now known, the parameters on the right-hand side of the integral identity given by (4.36) with respect to the integrals in the first and second part of (4.75) are calculated as follows.

1. The $-a$ parameter for the integral in the first part of (4.75) is given by

$$\begin{aligned} -a + \phi\omega\sigma_S\sqrt{T_M - t} &= -(-\phi\omega a_2) + \phi\omega\sigma_S\sqrt{T_M - t} \\ &= \phi\omega \left[\frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}} \right] + \phi\omega\sigma_S\sqrt{T_M - t} \\ &= \phi\omega \underbrace{\left[\frac{\ln\left(\frac{S(t)}{S^*}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_M - t)}{\sigma_S \sqrt{T_M - t}} \right]}_{a_1} \end{aligned} \quad (4.81)$$

where a_1 is as denoted in (4.81).

2. The $-a$ parameter for the integral in the second part of (4.75) is given by

$$\begin{aligned} -a &= -(-\phi\omega a_2) \\ &= \phi\omega a_2 \end{aligned} \quad (4.82)$$

where a_2 is as given by (4.64).

3. The $\frac{c}{\sqrt{1+b^2}}$ parameter for the integral in the first part of (4.75) is given by

$$\begin{aligned} \frac{c_1}{\sqrt{1+b^2}} &= \frac{\phi \left[\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - T_M}} \right]}{\sqrt{1 + \left[-\omega \frac{\sqrt{T_M - t}}{\sqrt{T_D - T_M}} \right]^2}} \\ &= \phi \underbrace{\left[\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q + \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - t}} \right]}_{b_1} \end{aligned} \quad (4.83)$$

where b_1 is as denoted in (4.83).

4. The $\frac{c}{\sqrt{1+b^2}}$ parameter for the integral in the second part of (4.75) is given by

$$\begin{aligned} \frac{c_2}{\sqrt{1+b^2}} &= \frac{\phi \left[\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - T_M}} \right]}{\sqrt{1 + \left[-\omega \frac{\sqrt{T_M - t}}{\sqrt{T_D - T_M}} \right]^2}} \\ &= \phi \underbrace{\left[\frac{\ln\left(\frac{S(t)}{K_D}\right) + \left(r - q - \frac{\sigma_S^2}{2}\right)(T_D - t)}{\sigma_S \sqrt{T_D - t}} \right]}_{b_2} \end{aligned} \quad (4.84)$$

where b_2 is as denoted in (4.84).

5. The $\frac{-b}{\sqrt{1+b^2}}$ parameter (or the correlation coefficient) for the integrals in both the first and second part of (4.75) is given by

$$\frac{-b}{\sqrt{1+b^2}} = \frac{-\left[-\omega \frac{\sqrt{T_M - t}}{\sqrt{T_D - T_M}} \right]}{\sqrt{1 + \left[-\omega \frac{\sqrt{T_M - t}}{\sqrt{T_D - T_M}} \right]^2}} = \omega \sqrt{\frac{T_M - t}{T_D - t}}. \quad (4.85)$$

Equations (4.81), (4.82), (4.83), (4.84) and (4.85) are used in conjunction with Theorem 4.4 to express the integrals in the first and second part of (4.75) in standard bivariate normal cumulative distribution function notation, with the result being the closed-form solution of a European compound option as given by (4.38). $\square$

The first term in (4.38) gives the risk-neutral expectation of the stock price, conditional on the joint probability that the overlying option is in-the-money,

i.e. $\phi\omega S(t) > \phi\omega S^*$, at T_M and that the underlying option is in-the-money, i.e. $\phi S(t) > \phi K_D$, at T_D . This joint probability is illustrated in Figure 4.1, where the x -axis represents the $\phi\omega a_1$ term in $N_2\left(\phi\omega a_1, \phi b_1, \omega\sqrt{\frac{T_M-t}{T_D-t}}\right)$ and the y -axis represents the ϕb_1 term.

The second term in the pricing equation gives the expected cash payment at the maturity of the underlying option, also conditional on the joint probability that both the overlying and underlying options are in-the-money. The third term gives the expected cash payment at the maturity of the overlying option conditional on the probability that the overlying option will be in-the-money, i.e. $\phi\omega S(t) > \phi\omega S^*$, at T_M .

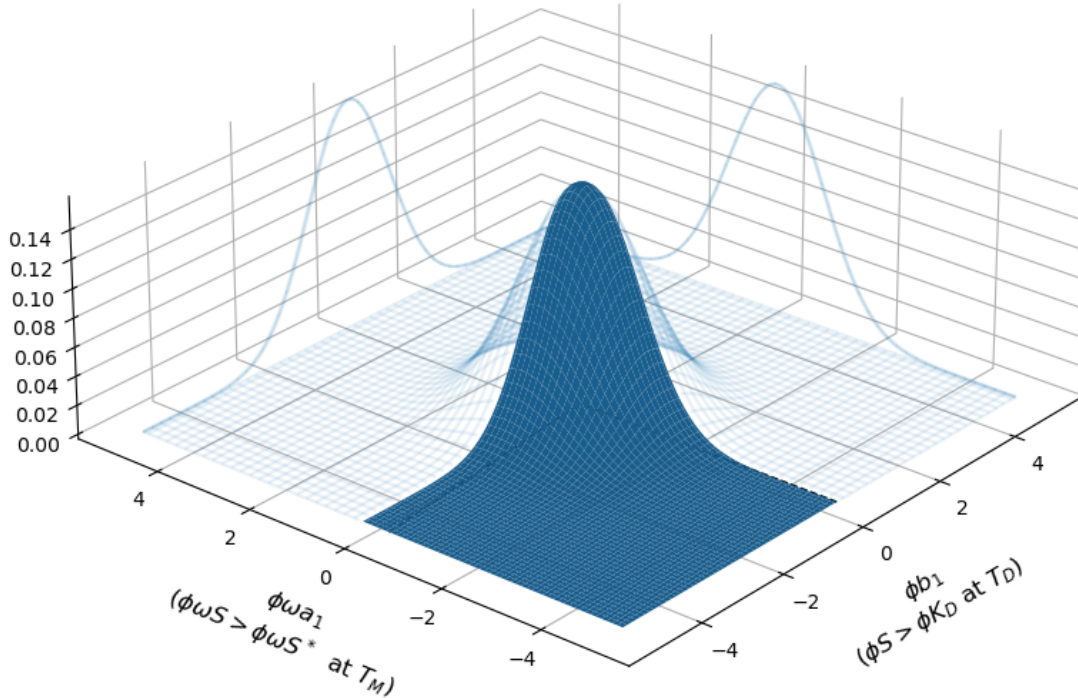


Figure 4.1: Cumulative joint probability that $S(t) > \phi\omega S^*$ at T_M (given by a_1), and $S(t) > \phi K_D$ at T_D (given by b_1), for a call-on-call option.

4.4.2 Solving the critical stock price

The Newton-Raphson method is used to solve the critical stock price in (4.58) [21]. Suppose that it is of interest to find the root of $f(x) = 0$ where $f(x)$ is a continuous and differentiable function. The Newton-Raphson method can be used to approximate the root close to the point $x = x_0$ by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, \dots \quad (4.86)$$

until a sufficiently accurate value is achieved. To approximate S^* , start by setting $S_0^* = S(0)$. S^* can be approximated by using

$$S_{n+1}^* = S_n^* + \frac{K_M - D(S_n^*, t, T_D, K_D, \phi)}{\frac{\partial}{\partial S} D(S_n^*, t, T_D, K_D, \phi)} \quad \text{for } n = 0, 1, \dots \quad (4.87)$$

where the delta of a vanilla option is given by

$$\frac{\partial}{\partial S} D(S(t), t, T_D, K_D, \phi) = e^{-q(T_D-t)} \left(N(d_1) + \frac{\phi-1}{2} \right). \quad (4.88)$$

In this dissertation, this process is repeated until $|S_{n+1}^* - S_n^*| \leq 1E-5$ is reached. The process, however, terminates if $n = 100$ is reached.

4.4.3 No-arbitrage bounds

The no-arbitrage upper and lower bounds on the valuation of European compound options in the Black–Scholes environment are given by Schilling (2001) [22].

Theorem 4.7 (No-arbitrage Lower Bound for European Compound Options under the Black–Scholes Model). *Given a European compound option with maturity T_M and strike price K_M on an underlying option with maturity T_D and strike price K_D . For $t \in [0, T_M)$, a compound option has the following lower bound:*

$$M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) > \max\{\omega \left(D(S(t), t, T_D, K_D, \phi) - e^{-r(T_M-t)} K_M \right), 0\}. \quad (4.89)$$

Proof. The proofs for the lower bound can be found in [22]. □

Theorem 4.8 (No-arbitrage Upper Bounds for European Compound Options under the Black–Scholes Model). *Given a European compound option with maturity T_M and strike price K_M on an underlying option with maturity T_D and strike price K_D . For $t \in [0, T_M)$,*

1. *A compound option with an overlying call option has the following tight and slack upper bounds:*

$$\begin{aligned} M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=1} &< \underbrace{D(S(t), t, T_D, K_D + \phi K_M e^{-r(T_D-T_M)}, \phi)}_{\text{tight upper bound}} \\ &< \underbrace{D(S(t), t, T_D, K_D, \phi)}_{\text{slack upper bound}}. \end{aligned} \quad (4.90)$$

2. A compound option with an overlying put option has the following tight and slack upper bounds:

$$\begin{aligned}
 M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=-1} &< \underbrace{D(S(t), t, T_D, K_D + \phi K_M e^{-r(T_D-T_M)}, \phi) - D(S(t), t, T_D, K_D, \phi) + K_M e^{-r(T_M-t)}}_{\text{tight upper bound}} \\
 &< \underbrace{e^{-r(T_M-t)} K_M}_{\text{slack upper bound}}. \tag{4.91}
 \end{aligned}$$

Proof. The proofs for the upper bounds can be found in [22]. $\square$

The price curves for European compound options and their no-arbitrage upper bounds (Theorem 4.8) and lower bounds (Theorem 4.7) are shown in Figure 4.2. Subplot a) shows the curves for compound options where the overlying is a call option, and subplot b) shows the curves for compound options where the overlying is a put option. The parameter values $r = 0.05$, $q = 0.01$, $\sigma_S = 0.15$, $K_M = 5$, $K_D = 100$, $T_M = 0.5$ and $T_D = 1$ were used, while $S(0)$ was varied.

Figure 4.2(a) and Figure 4.2(b) show that the compound option prices remain above their respective lower bounds while they remain below their respective tight and slack upper bounds. Moreover:

1. A compound option will always be worth more than its discounted intrinsic value.
2. When the overlying is a call option, the compound option will always be worth less than its underlying option.
3. When the overlying is a put option, the compound option will always be worth less than its discounted strike price.

Points 2) and 3) above support the statement made in Section 4.1 that an investor pays less for the compound option than they would otherwise have paid for the underlying option alone.

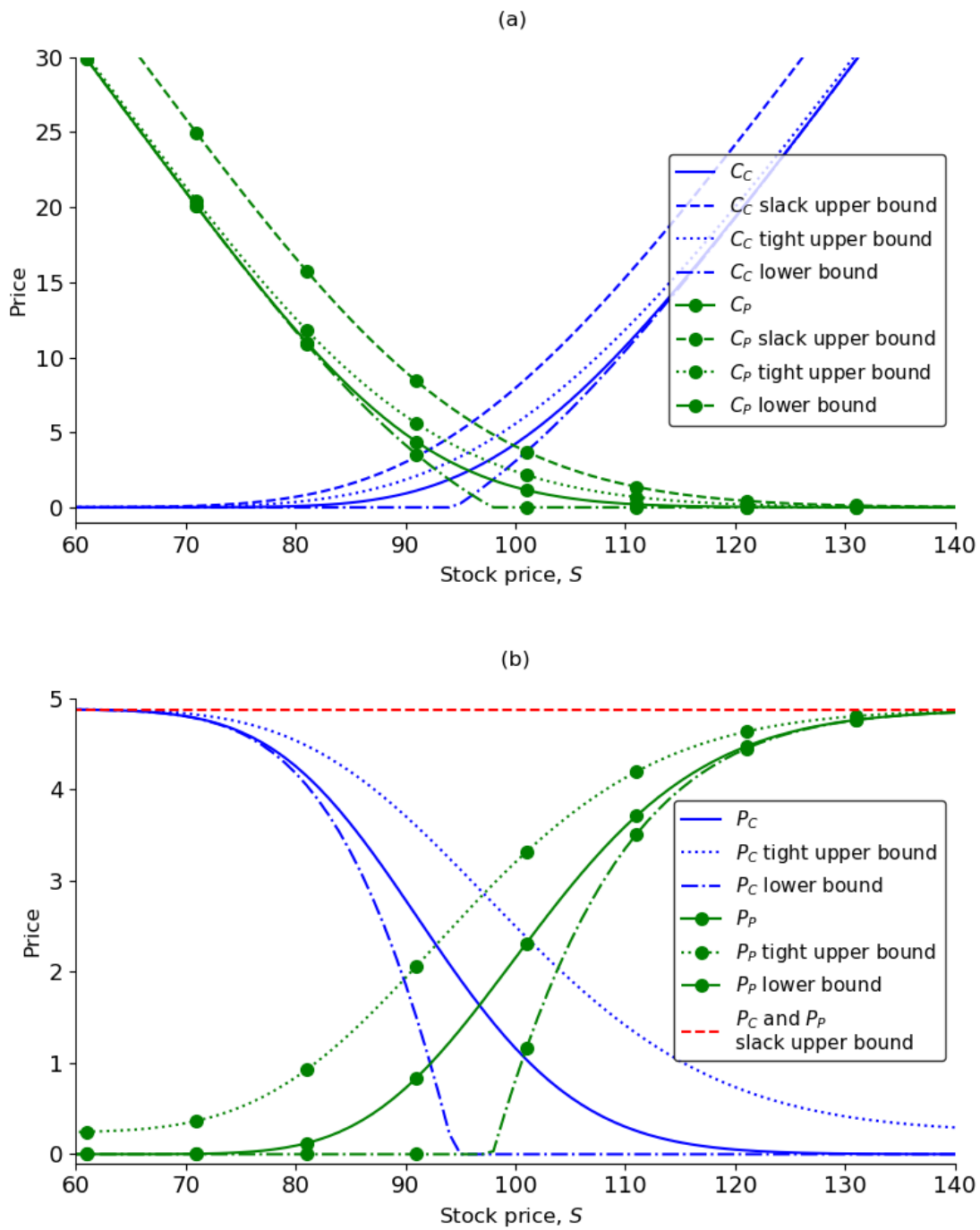


Figure 4.2: Price curves of European compound option vs. their lower and upper bounds under the Black-Scholes model.

4.4.4 Convergence to the tight upper bounds

Schilling (2001) showed that the value of European compound options in the Black-Scholes environment converge to their tight upper bound as T_D approaches T_M [22].

Theorem 4.9 (Convergence to the Upper No-arbitrage Bounds under the Black–Scholes Model). *Given a European compound option with maturity T_M and strike price K_M on an underlying option with maturity T_D and strike price K_D . For $t \in [0, T_M)$ the value of a compound option approaches its upper no-arbitrage bound as T_M approaches T_D , i.e.*

$$\lim_{T_D \rightarrow T_M} M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=1} = D(S(t), t, T_D, K_D + \phi K_M e^{-r(T_D - T_M)}, \phi) \quad (4.92)$$

and

$$\begin{aligned} \lim_{T_D \rightarrow T_M} M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=-1} = \\ D(S(t), t, T_D, K_D + \phi K_M e^{-r(T_D - T_M)}, \phi) \\ - D(S(t), t, T_D, K_D, \phi) + K_M e^{-r(T_M - t)}. \end{aligned} \quad (4.93)$$

Proof. The proof for these limits can be found in [22]. $\square$

The price curves for European compound options and their no-arbitrage tight upper bounds (Theorem 4.8) are shown in Figure 4.3. The parameter values $r = 0.05$, $q = 0.01$, $\sigma_S = 0.15$, $K_M = 5$, $K_D = 100$, $T_M = 0.5$ and $S(0) = 100$ were used, while T_D was varied.

Figure 4.3 shows that the price of each compound option type approaches its no-arbitrage tight upper bound as T_D approaches T_M . However, when $T_D = T_M$, the closed-form solution returns an error because S^* cannot be solved using (4.58) when $T_D - T_M = 0$. In such a case, the investor immediately exercises the underlying option when the overlying option is exercised. Therefore, the value of the underlying option at T_M will be its payoff value, which has the following implications:

1. When the overlying is a call option, the payoff of the compound option at T_M will be $\max\{\max\{\phi(S(T_M) - K_D), 0\} - K_M, 0\}$. This is equivalent to the payoff at T_D of an underlying option with a strike price of $K_D + \phi K_M$, which, when discounted under the risk-neutral measure, is precisely the value of the tight upper bound.
2. When the overlying is a put option, the payoff of the compound option at T_M will be $\max\{K_M - \max\{\phi(S(T_M) - K_D), 0\}, 0\}$. When discounted under the risk-neutral measure, then by the put-call parity relationship for a vanilla option, is precisely the value of the tight upper bound.

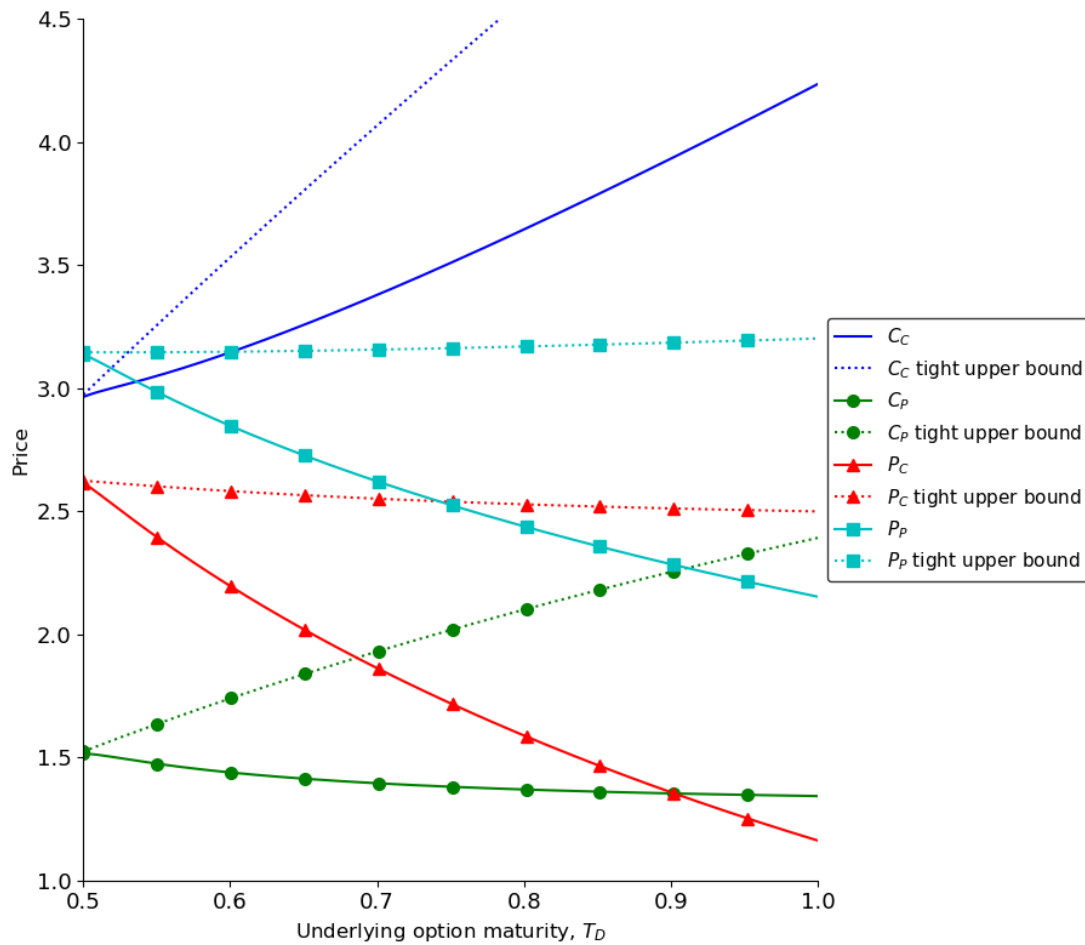


Figure 4.3: Convergence of European compound option prices to their tight upper bounds under the Black-Scholes model.

4.4.5 Convergence to the slack upper bounds

Surface plots of European compound option prices are shown in Figure 4.4 and Figure 4.5 with colour gradients, while surface plots of their corresponding slack upper bounds (as given in Theorem 4.8) are shown using grey meshes. The parameter values $r = 0.05$, $q = 0.01$, $\sigma_S = 0.15$ and $S(0) = 100$ were used. In Figure 4.4, $T_D = 1$ and $T_M = 0.5$ were used, while K_D and K_M were varied. In Figure 4.5, $T_D = 1$ and $K_D = 100$ were used, while the ratios T_M/T_D and K_M/K_D were varied. The following observations were made:

1. Figure 4.4(a) shows that the value of a call-on-call option increases non-linearly in the direction of a decreasing K_D and K_M . The price curve approaches its slack upper bound (the price of its underlying call option) as K_M becomes negligibly small.

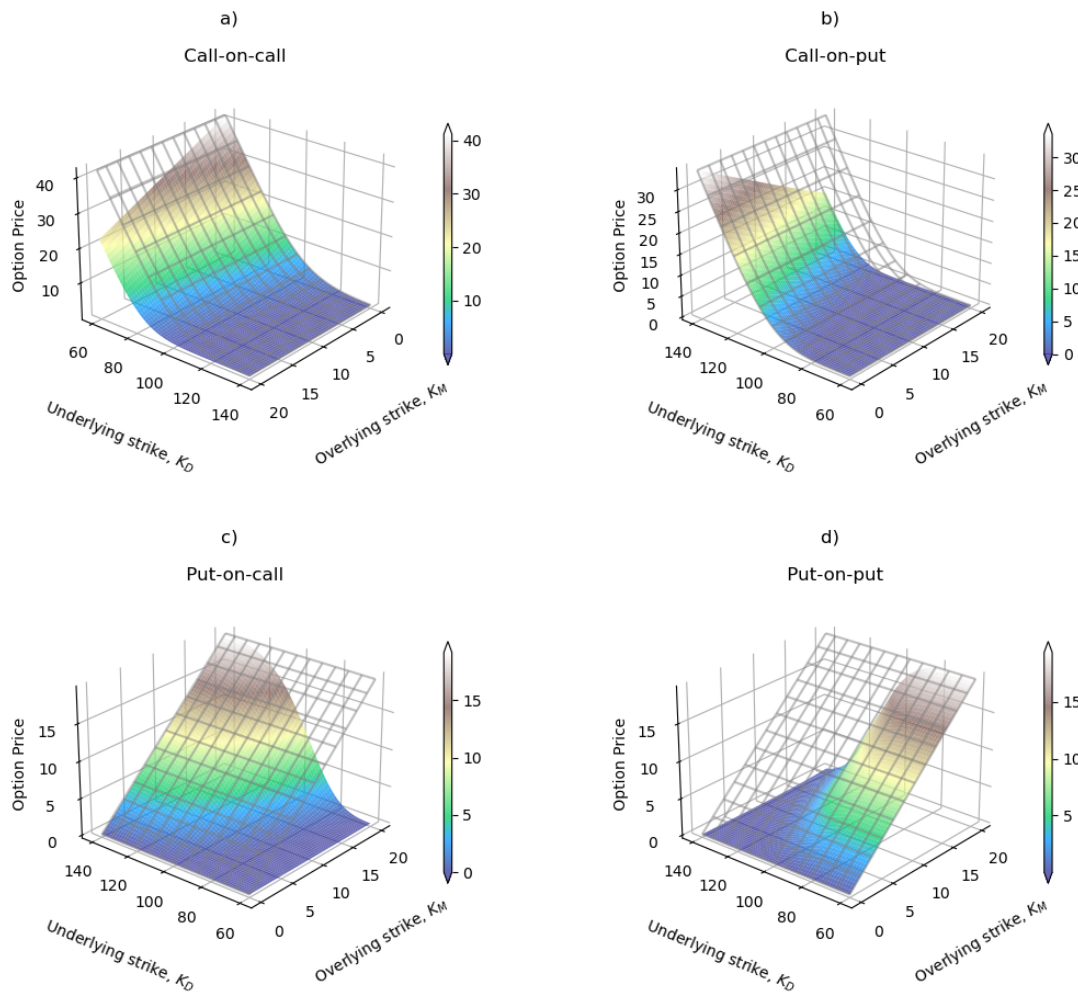


Figure 4.4: Convergence of European compound option prices to their slack upper bounds with xy -grids K_D and K_M under the Black–Scholes model.

2. Figure 4.4(b) shows that the value of a call-on-put option increases non-linearly in the direction of an increasing K_D and decreasing K_M . The price curve approaches its slack upper bound (the price of its underlying put option) as K_M becomes negligibly small.
3. Figure 4.4(c) shows that the value of a put-on-call option increases non-linearly in the direction of an increasing K_D and K_M . The price curve approaches its slack upper bound (the discounted overlying strike price) as K_D increases.
4. Figure 4.4(d) shows that the value of a put-on-put option increases non-linearly in the direction of a decreasing K_D and increasing K_M . The price curve approaches its slack upper bound (the discounted overlying strike price) as K_D decreases.

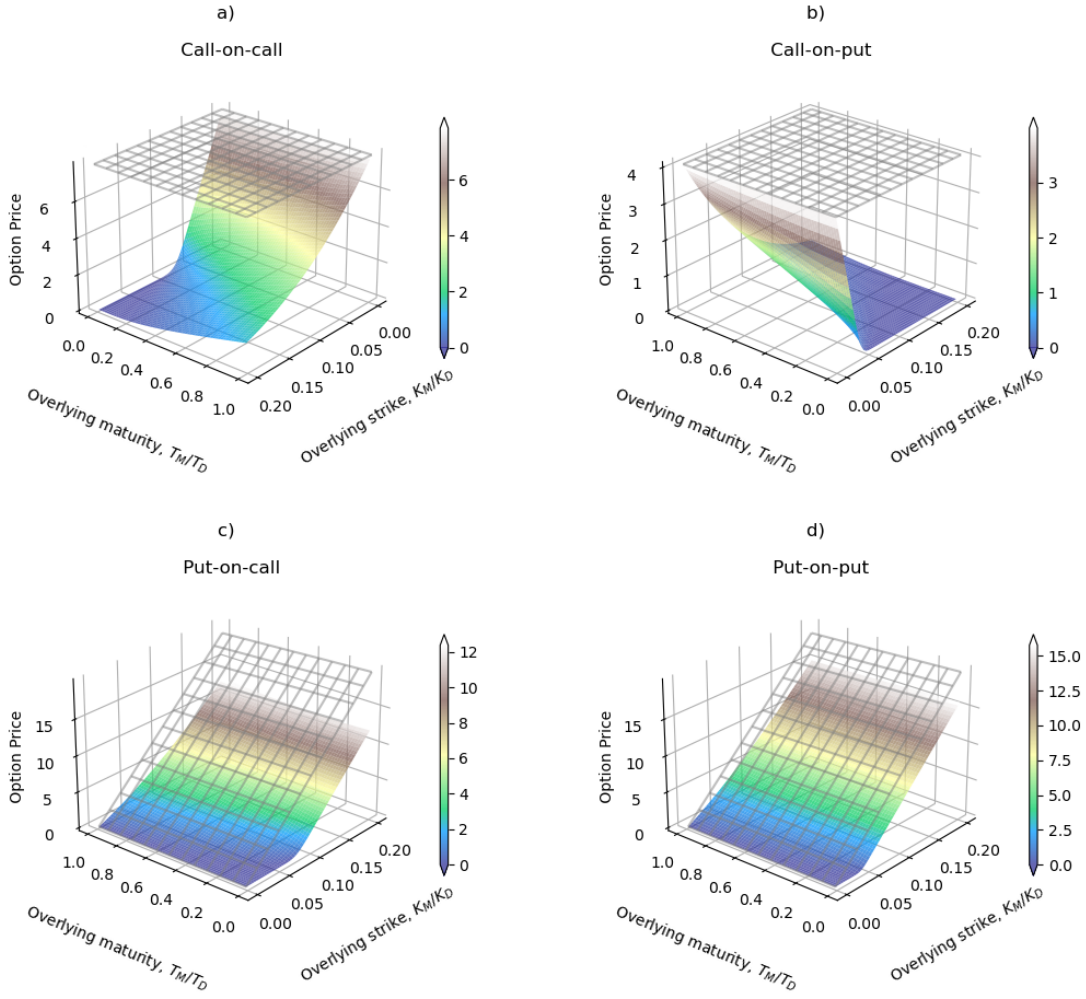


Figure 4.5: Convergence of European compound option prices to their slack upper bounds with xy -grids T_M/T_D and K_M/K_D under the Black–Scholes model.

5. Figure 4.5 shows that for all four types of compound options, the price curve approaches its slack upper bound as T_M approaches T_D and as K_M becomes negligibly small in proportion to K_D .

4.4.6 Put-call parity

The put-call parity relationship for European compound options in the Black–Scholes environment was given by Schilling in 2001 [22].

Theorem 4.10 (Put-call Parity for a European Compound Option under the Black–Scholes Model). *Given a European compound option with maturity T_M and strike price K_M on an underlying option with maturity T_D and strike price K_D , the following put-call parity holds for $t \in [0, T_M]$,*

$$M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=1} = M(S(t), t, T_M, T_D, K_M, K_D, \phi) \Big|_{\omega=-1} + D(S(t), t, T_D, K_D, \phi) - K_M e^{-r(T_M-t)}. \quad (4.94)$$

Proof. The proof for the parity can be found in [22]. □

The put-call parity is graphically represented in Figure 4.6(a) and Figure 4.6(b).

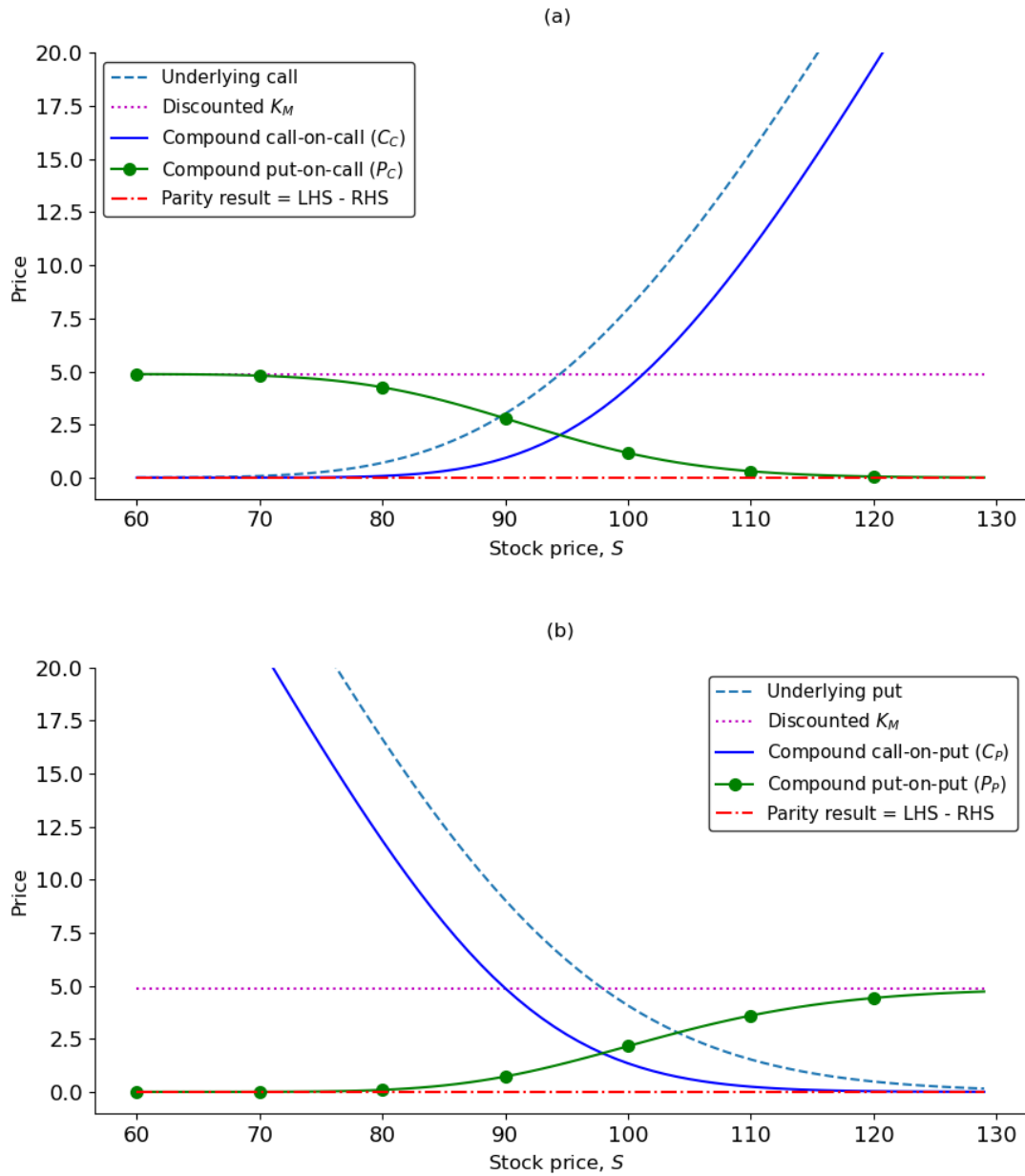


Figure 4.6: Put-call parity result for European compound options under the Black-Scholes model.

The parity result, which is the right-hand side of (4.94) subtracted from the left-hand side, is plotted in each graph. Parameter values $r = 0.05$, $q = 0.01$, $\sigma_S = 0.15$, $K_M = 5$, $K_D = 100$, $T_M = 0.5$ and $T_D = 1$ were used.

Figure 4.6(a) shows the parity relationship between compound options where the underlying is a call option. The price curves for a call-on-call option, a put-on-call option, an underlying call option and the discounted overlying strike price are plotted for varying values of $S(0)$. The parity relationship holds, with the parity result being consistently zero for all values of $S(0)$.

Figure 4.6(b) shows the parity relationship between compound options where the underlying is a put option. The price curves for a call-on-put option, a put-on-put option, an underlying put option and the discounted overlying strike price are plotted for varying values of $S(0)$. Again, the parity relationship holds, with the parity result being consistently zero.

4.4.7 Option price sensitivities

The option price sensitivities (or Greek risk measures) for European compound options in the Black–Scholes environment are given below. Their respective surface plots are shown in Appendix A.

Theorem 4.11 (European Compound Option Price Sensitivities under the Black–Scholes Model [19]). *The price sensitivities for a European compound option $M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi)$ for $t \in [0, T_M]$ are as follows.*

1. The delta of a compound option is given by

$$\frac{\partial}{\partial S} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = \phi \omega e^{-q(T_D-t)} N_2 \left(\phi \omega a_1, \phi b_1, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right). \quad (4.95)$$

2. The gamma of a compound option is given by

$$\frac{\partial^2}{\partial S^2} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = \frac{e^{-q(T_D-t)}}{S(t)\sigma_S} \left(\frac{1}{\sqrt{T_M-t}} n(-a_1) N(\phi d_1) + \frac{\omega}{\sqrt{T_D-t}} n(b_1) N(-\phi \omega c) \right) \quad (4.96)$$

where

$$c = \frac{-a_2 \sqrt{T_D-t} + b_2 \sqrt{T_M-t}}{\sqrt{T_D-T_M}}. \quad (4.97)$$

3. The theta of a compound option is given by

$$\begin{aligned} \frac{\partial}{\partial t} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\ & \phi \omega q S(t) e^{-q(T_D-t)} N_2 \left(\phi \omega a_1, \phi b_1, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right) \\ & - \phi \omega r K_D e^{-r(T_D-t)} N_2 \left(\phi \omega a_2, \phi b_2, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right) - \omega r e^{-r(T_M-t)} K_M N(\phi \omega a_2) \\ & - \frac{\sigma_S}{2} S(t) e^{-q(T_D-t)} \left(\frac{1}{\sqrt{T_M-t}} n(-a_1) N(\phi d_1) + \frac{\omega}{\sqrt{T_D-t}} n(b_1) N(-\phi \omega c) \right). \end{aligned} \quad (4.98)$$

4. The rho of a compound option is given by

$$\begin{aligned} \frac{\partial}{\partial r} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\ & \phi \omega (T_D - t) K_D e^{-r(T_D-t)} N_2 \left(\phi \omega a_2, \phi b_2, \omega \sqrt{\frac{T_M-t}{T_D-t}} \right) \\ & + \omega (T_M - t) e^{-r(T_M-t)} K_M N(\phi \omega a_2). \end{aligned} \quad (4.99)$$

5. The vega of a compound option is given by

$$\begin{aligned} \frac{\partial}{\partial \sigma_S} M(S(t), t, T_M, T_D, K_M, K_D, \omega, \phi) = & \\ S(t) e^{-q(T_D-t)} \left[\sqrt{T_M-t} n(-a_1) N(\phi d_1) + \omega \sqrt{T_D-t} n(b_1) N(-\phi \omega c) \right]. \end{aligned} \quad (4.100)$$

Proof. The proof for these sensitivities can be found in [19]. $\square$

The vega and rho values are divided by one hundred to get the resulting option price change for one percentage point change in volatility and interest rate, respectively. The theta values are divided by three hundred sixty-five to get the resulting option price change for one calendar day.

The plots show that compound options with overlying call options have surface profiles similar to their underlying options, mainly attributed to their payoff functions being directly proportional to the value of their underlying options. Compound options with overlying put options have surface profiles that differ substantially from their underlying options, which is expected since their payoff values increase as their underlying options decrease.

Most notably, the Vega surface plots show that compound options are more sensitive to changes in stock price volatility than their underlying options. For $r = 0.05$, $q = 0.01$, $\sigma = 0.15$, $S(0) = 100$, $K_M = 5$, $K_D = 100$, $T_M = 0.5$

and $T_D = 1$, a 0.01% increase in volatility will increase the price of a call-on-put option by 18.54%, while the price of the underlying put price increases by only 9.18%. Similarly, a 0.01% increase in volatility will increase the price of a call-on-call option by 8.30%, while the underlying call price increases by only 4.69%. Higher sensitivity to volatility is expected since a compound option has two constituents, namely the stock price and the underlying option, which are independently affected by changes in volatility. In contrast, an underlying option has one constituent, namely the stock price, which is independently affected by changes in volatility. Therefore, the impact of changes in volatility continues to increase as the number of folds in an n -fold compound option model increases.

5 Three-state variable simulation method to price European and American compound options

This section presents a Monte Carlo simulation method to price compound options. The method provides early exercise opportunities for both underlying and overlying options.

Chiarella and Kang [2] used an LSMC simulation method to determine reference values for an American put-on-put compound option under the HCIR model with $\rho_{S,r} = \rho_{v,r} = 0$. They reported that, although highly accurate, the execution time of the simulation method was slow.

Similarly, a three-state variable simulation method is used to determine reference values for all four compound options under the Heston, HHW and HCIR models in this dissertation, with non-zero correlation values, $\rho_{S,r}$ and $\rho_{v,r}$. A modified version of the method of least-squares by Longstaff and Schwartz (2011) [23] is used to account for early exercise opportunities.

The method involves simulating N_P individual stock price paths between T_0 and T_M . These stock price paths will depend on the evolution of the simulated variance and interest rate paths. At T_D , there will be a total of N_P stock price termination points, each point with a vector $\{S(T_M), v(T_M), r(T_M)\}$.

Each termination point is then used as a point of origin from which another N_P individual stock price paths between T_M and T_D are simulated. Again, these stock price paths will depend on the evolution of the simulated variance and interest rate values between T_M and T_D . At T_D , there will be a total of N_P^2 stock price termination points, each point with a vector $\{S(T_D), v(T_D), r(T_D)\}$. The simulated paths are then used to calculate the discounted prices of the

underlying options at T_M and the discounted prices of the overlying options at T_0 . This process is illustrated in Figure 5.1.

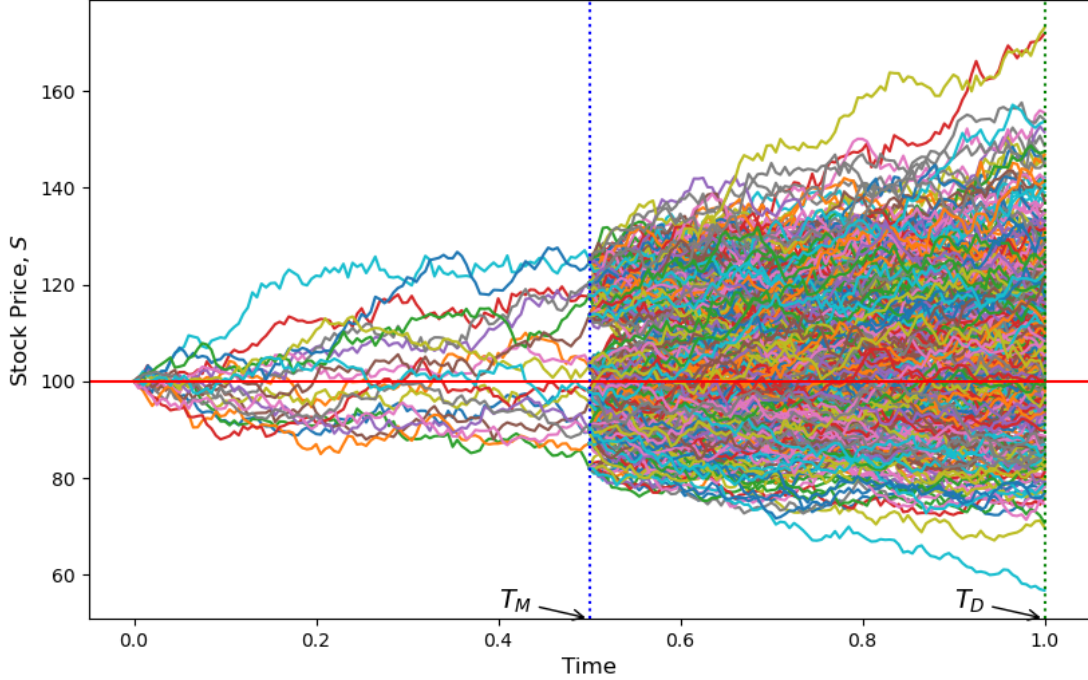


Figure 5.1: Stock price simulation paths to price compound options.

As a starting point, the duration T_0 to T_M is divided into N_t equally spaced time intervals, each with a distance

$$\Delta t_M = \frac{T_M - T_0}{N_t} \quad (5.1)$$

such that $T_0 < T_0 + \Delta t_M, T_0 + 2\Delta t_M, \dots, T_0 + (N_t - 1)\Delta t_M < T_0 + N_t\Delta t_M = T_M$. The duration T_M to T_D is divided into N_t equally spaced time intervals, each with a distance

$$\Delta t_D = \frac{T_D - T_M}{N_t} \quad (5.2)$$

such that $T_M < T_M + \Delta t_D, T_M + 2\Delta t_D, \dots, T_M + (N_t - 1)\Delta t_D < T_M + N_t\Delta t_D = T_D$.

5.1 Simulation paths from T_0 to T_M

For the period T_0 to T_M , let S_i^n , v_i^n and r_i^n represent the stock price, variance and interest rate values, respectively, at $t^n = T_0 + n\Delta t_M$ for $n = 0, 1, \dots, N_t$ of the i 'th simulation path for $i = 1, \dots, N_p$. The initial values at T_0 are $S_i^0 = S(0)$, $v_i^0 = v(0)$ and $r_i^0 = r(0)$ for all $i = 1, \dots, N_p$.

The i 'th stock price at the n 'th time step can be approximated using

$$S_i^n = S_{i,j}^{n-1} e^{(r_i^{n-1} - q - v_i^{n-1}/2)\Delta t_M + \sqrt{v_i^{n-1}\Delta t_M} Z_S}, \quad (5.3)$$

while the i 'th variance and interest rate values at the n 'th time step can be approximated using

$$v_i^n = v_{i,j}^{n-1} + \kappa_v(\theta_v - v_i^{n-1})\Delta t_M + \sigma_v \sqrt{v_i^{n-1}\Delta t_M} Z_v \quad (5.4)$$

and

$$r_i^n = r_{i,j}^{n-1} + \kappa_r(\theta_r - r_i^{n-1})\Delta t_M + \sigma_r (r_i^{n-1})^p \sqrt{\Delta t_M} Z_r, \quad (5.5)$$

respectively. The correlated standard Brownian motions Z_S , Z_v and Z_r are calculated using

$$Z_S = W_S, \quad (5.6)$$

$$Z_v = \rho_{S,v} W_S + \sqrt{1 - \rho_{S,v}^2} W_v, \text{ and} \quad (5.7)$$

$$Z_r = \rho_{S,r} W_S + \frac{\rho_{v,r} - \rho_{S,v}\rho_{S,r}}{\sqrt{1 - \rho_{S,v}^2}} W_v + \sqrt{1 - \rho_{S,r}^2 - \left(\frac{\rho_{v,r} - \rho_{S,v}\rho_{S,r}}{\sqrt{1 - \rho_{S,v}^2}} \right)^2} W_r \quad (5.8)$$

where W_S , W_v and W_r are independent standard Brownian motions.

5.2 Simulation paths from T_M to T_D

For the period T_M to T_D , let $S_{i,j}^n$, $v_{i,j}^n$ and $r_{i,j}^n$ represent the stock price, variance and interest rate values, respectively, at $t^n = T_M + n\Delta t_D$ for $n = 0, 1, \dots, N_t$ of the j 'th simulation path for $j = 1, \dots, N_P$ that branches off from the termination point $\{S_i^{N_t}, v_i^{N_t}, r_i^{N_t}\}$. The initial values at T_M are, therefore, $S_{i,j}^0 = S_i^{N_t}$, $v_{i,j}^0 = v_i^{N_t}$ and $r_{i,j}^0 = r_i^{N_t}$ for all $i = 1, \dots, N_P$ and $j = 1, \dots, N_P$.

The j 'th stock price value with respect to the i 'th initial stock price path at the n 'th time step can be approximated using

$$S_{i,j}^n = S_{i,j}^{n-1} e^{(r_{i,j}^{n-1} - q - v_{i,j}^{n-1}/2)\Delta t_D + \sqrt{v_{i,j}^{n-1}\Delta t_D} Z_S}, \quad (5.9)$$

while the j 'th variance and interest rate values with respect to the i 'th initial stock price path at the n 'th time step can be approximated using

$$v_{i,j}^n = v_{i,j}^{n-1} + \kappa_v(\theta_v - v_{i,j}^{n-1})\Delta t_D + \sigma_v \sqrt{v_{i,j}^{n-1}\Delta t_D} Z_v \quad (5.10)$$

and

$$r_{i,j}^n = r_{i,j}^{n-1} + \kappa_r(\theta_r - r_{i,j}^{n-1})\Delta t_D + \sigma_r (r_{i,j}^{n-1})^p \sqrt{\Delta t_D} Z_r, \quad (5.11)$$

respectively. The correlated standard Brownian motions Z_S , Z_v and Z_r are as given above.

5.3 Underlying option prices at T_M

Once all N_p^2 stock price paths have been simulated, the payoff values of the underlying option at T_D are calculated using

$$D_{i,j}^{N_t} = \max\{0, \phi(S_{i,j}^{N_t} - K_D)\} \quad (5.12)$$

for $i = 1, \dots, N_p$ and $j = 1, \dots, N_p$. The payoff values are then discounted back to T_M using the simulated interest rates, according to

$$D_{i,j}^{n-1} = e^{-r_{i,j}^n \Delta t_D} D_{i,j}^n \quad (5.13)$$

for $n = N_t, \dots, 1$.

The discounted values of the underlying options at T_M are taken as the mean of the discounted underlying values of each cluster group, i.e.

$$D_i^{N_t} = \frac{1}{N_p} \sum_{j=1}^{N_p} \left[\left(\prod_{n=1}^{N_t} e^{-r_{i,j}^n \Delta t_D} \right) D_{i,j}^{N_t} \right] \quad (5.14)$$

for $i = 1, \dots, N_p$.

5.4 Overlying option price at T_0

The payoff values of the overlying option at T_M are calculated using

$$M_i^{N_t} = \max\{0, \omega(D_i^{N_t} - K_M)\} \quad (5.15)$$

for $i = 1, \dots, N_p$. The payoff values are then discounted back to T_0 using the simulated interest rates, according to

$$M_i^{n-1} = e^{-r_i^n \Delta t_M} M_i^n \quad (5.16)$$

for $n = N_t, \dots, 1$.

The discounted value of the compound option at T_0 is taken as the mean of all discounted overlying values, i.e.

$$M^0 = \frac{1}{N_p} \sum_{i=1}^{N_p} \left[\left(\prod_{n=1}^{N_t} e^{-r_i^n \Delta t_M} \right) M_i^{N_t} \right]. \quad (5.17)$$

5.5 Should early exercise opportunities be available

5.5.1 Underlying option with early exercise opportunities

Consider first the price of an American underlying option with maturity $T_D - T_M$ at T_M given the starting points $S_i^{N_t}$, $v_i^{N_t}$, and $r_i^{N_t}$. Since early exercise

opportunities are available, it is of interest to find the expected discounted payoff at T_M that yields the greatest payoff for each initial stock price termination point. In particular,

$$D_i^{N_t} = \sup_{n \in [0, N_t]} \frac{1}{N_p} \sum_{j=1}^{N_p} \left[\left(\prod_{k=1}^n e^{-r_{i,j}^k \Delta t_D} \right) \max \{0, \phi(S_{i,j}^n - K_D)\} \right] \quad (5.18)$$

for $i = 1, \dots, N_p$. Alternatively, it is of interest to find the optimal stopping time t^{n^*} that yields the largest expected discounted payoff, i.e.

$$D_i^{N_t} = \frac{1}{N_p} \sum_{j=1}^{N_p} \left[\left(\prod_{k=1}^{n^*} e^{-r_{i,j}^k \Delta t_D} \right) \max \{0, \phi(S_{i,j}^{n^*} - K_D)\} \right] \quad (5.19)$$

for $i = 1, \dots, N_p$.

The method of least-squares developed by Longstaff and Schwartz is used to find the optimal stopping time [23]. The term *continuation value*, which represents the value of holding onto the option, is introduced. If the payoff value of the underlying option exceeds the continuation value, then the underlying option will be exercised immediately. For the optimal stopping time t^{n^*} , the continuation value at the n 'th time step for the i 'th cluster group of stock prices is defined as the conditional expectation

$$V_i^n = \mathbb{E} \left[\left(\prod_{k=n}^{n^*} e^{-r_{i,j}^k \Delta t_D} \right) \max \{0, \phi(S_{i,j}^{n^*} - K_D)\} \mid S_{i,j}^n \right] \quad (5.20)$$

for all $j = 1, \dots, N_p$.

The L^2 Hilbert space of square-integrable functions relative to some measure is a space which has a countable orthonormal basis. Since the conditional expectation is an element of the L^2 space, the conditional expectation can be represented as a linear function of the elements of the orthonormal basis. The theory behind this justification is discussed in detail in [23]. In this dissertation, the choice of basis functions onto which the discounted underlying payoff values will be projected is the set of Laguerre polynomials given by:

$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ L_2(x) &= \frac{1}{2}(2 - 4x + x^2), \\ &\vdots \\ L_u(x) &= \sum_{k=0}^u \frac{(-1)^k}{k!} \binom{u}{k} x^k. \end{aligned} \quad (5.21)$$

Given the set of all in-the-money paths in the i 'th cluster group of stock prices, the conditional expectation at the n 'th time step can be estimated as

$$\hat{V}_{i,j}^n = \sum_{u=0}^{\infty} \hat{a}_u L_u(S_{i,j}^n) \quad (5.22)$$

for all $j = 1, \dots, N_p$, where $\hat{a}_u$ are the estimated regression coefficients.

The regression coefficients are calculated by regressing the discounted underlying option values for all the in-the-money paths against the stock prices using regular least squares. Therefore, the aim is to estimate the regression coefficients that minimise the function

$$\sum_{j=1}^{N_p} \left(e^{-r_{i,j}^{n+1} \Delta t_D} D_{i,j}^{n+1} - \sum_{u=0}^{\infty} \hat{a}_u L_u(S_{i,j}^n) \right)^2 \Big|_{\phi(S_{i,j}^n - K_D) > 0}. \quad (5.23)$$

The continuation values for the in-the-money paths are then calculated using (5.22) once the regression coefficients have been estimated.

The values of $D_{i,j}^n$ are set equal to the payoff value of the underlying option for all paths where the payoff value exceeds the continuation value, while the underlying prices are discounted by one time step for the remaining paths - that is

$$D_{i,j}^n = \begin{cases} \phi(S_{i,j}^n - K_D) & \text{if } \phi(S_{i,j}^n - K_D) > 0 \text{ and } \phi(S_{i,j}^n - K_D) > \hat{V}_{i,j}^n, \\ e^{-r_{i,j}^{n+1} \Delta t_D} D_{i,j}^{n+1} & \text{otherwise.} \end{cases} \quad (5.24)$$

This is done at each time step for $n = N_t - 1, \dots, 1$. At $n = 0$ the values of the American underlying options are taken as the mean of each cluster group of underlying prices, i.e.

$$D_i^0 = \frac{1}{N_p} \sum_{j=1}^{N_p} e^{-r_{i,j}^1 \Delta t_D} D_{i,j}^1 \quad (5.25)$$

for $i = 1, \dots, N_p$.

5.5.2 Overlying option with early exercise opportunities

As one iterates backwards from T_M to T_0 , a new set of American underlying option prices needs to be calculated at each time step since the optimal stopping time of the underlying option may fall before T_M . This process is illustrated in Figure 5.2. Note the difference from Figure 5.1.

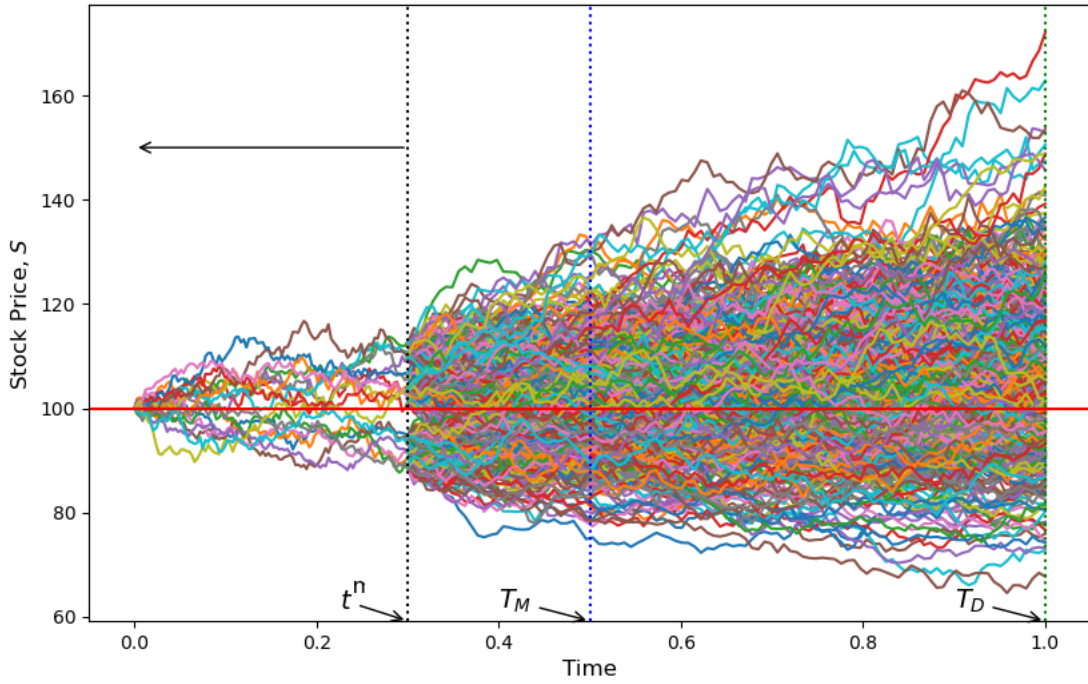


Figure 5.2: Stock price simulation paths to price compound options with early exercise opportunities.

Section 5.5.1 provides the strategy to price the American underlying options $D_i^{N_t}$ with maturity $T_D - T_M$ at T_M given the starting points $S_i^{N_t}$, $v_i^{N_t}$, and $r_i^{N_t}$. The same strategy is used to price the American underlying options D_i^n with maturity $T_D - t^n$ at $T_0 + n\Delta t_M$ for $n = N_t - 1, \dots, 1$ given the starting points S_i^n , v_i^n , and r_i^n .

Since early exercise opportunities are also available for the overlying option, it is of interest to find the expected discounted payoff at T_0 of the compound option that yields the greatest payoff. In particular,

$$M^0 = \sup_{n \in [0, N_t]} \frac{1}{N_p} \sum_{i=1}^{N_p} \left[\left(\prod_{k=1}^n e^{-r_i^k \Delta t_M} \right) \max \{0, \omega(D_i^n - K_M)\} \right]. \quad (5.26)$$

Alternatively, it is of interest to find the optimal stopping time t^{n^*} that yields the largest expected discounted payoff. I.e.

$$M^0 = \frac{1}{N_p} \sum_{i=1}^{N_p} \left[\left(\prod_{k=1}^{n^*} e^{-r_i^k \Delta t_M} \right) \max \{0, \omega(D_i^{n^*} - K_M)\} \right]. \quad (5.27)$$

If the payoff value of the underlying option exceeds the continuation value, then the overlying option will be exercised immediately. For the optimal stopping time t^{n^*} , the continuation value at the n 'th time step is defined as

the conditional expectation

$$V^n = \mathbb{E} \left[\left(\prod_{k=n}^{n^*} e^{-r_i^k \Delta t_M} \right) \max \{0, \omega(D_i^{n^*} - K_M)\} \mid D_i^n \right]. \quad (5.28)$$

Given the set of all in-the-money paths, the conditional expectation at the n 'th time step can be estimated as

$$\hat{V}_i^n = \sum_{u=0}^{\infty} \hat{a}_u L_u(D_i^n) \quad (5.29)$$

where $\hat{a}_u$ are the estimated regression coefficients.

The regression coefficients are calculated by regressing the discounted overlying option values for all the in-the-money paths against the underlying option prices using regular least squares. Therefore, the aim is to estimate the regression coefficients that minimise the function

$$\sum_{i=1}^{N_p} \left(e^{-r_i^{n+1} \Delta t_M} M_i^{n+1} - \sum_{u=0}^{\infty} \hat{a}_u L_u(D_i^n) \right)^2 \Big|_{\omega(D_i^n - K_M) > 0}. \quad (5.30)$$

The continuation values for the in-the-money paths are then calculated using (5.29) once the regression coefficients have been estimated.

The values of M_i^n are set equal to the payoff value of the overlying option for all paths where the payoff value exceeds the continuation value, while the overlying prices are discounted by one time step for the remaining paths - that is

$$M_i^n = \begin{cases} \omega(D_i^n - K_M) & \text{if } \omega(D_i^n - K_M) > 0 \text{ and } \omega(D_i^n - K_M) > \hat{V}_i^n, \\ e^{-r_i^{n+1} \Delta t_M} M_i^{n+1} & \text{otherwise.} \end{cases} \quad (5.31)$$

This is done at each time step for $n = N_t - 1, \dots, 1$. At $n = 0$ the value of the American compound option is taken as the mean of overlying prices, i.e.

$$M^0 = \frac{1}{N_p} \sum_{i=1}^{N_p} e^{-r_i^1 \Delta t_M} M_i^1. \quad (5.32)$$

5.6 Convergence of the simulation method in the literature

For their benchmark results, Longstaff and Schwartz [23] used 100,000 simulation paths and 50 exercisable points per year, sufficient to produce

results with standard errors ranging between 0.007 and 0.0024 of a dollar. Their results also indicate that, for a given number of simulation paths, increasing the number of basis functions improves convergence up to a certain point, after which the convergence rate does not improve enough to justify the increased computational time associated with a higher number of basis functions. For their results, three basis functions were sufficient to obtain accuracies of order $1E-3$.

6 Finite difference methods to price European and American compound options

This section presents six finite difference algorithms to price compound options using the models presented in Section 2. The algorithms provide early exercise opportunities for both underlying and overlying options. These algorithms are:

- an explicit method to solve the one-state variable Black–Scholes pricing PDE;
- an ADI method to solve the one-state variable Black–Scholes pricing PDE;
- an explicit method to solve the two-state variable Heston pricing PDE;
- an ADI method to solve the two-state variable Heston pricing PDE;
- an explicit method to solve the three-state variable HHW/HCIR pricing PDE; and
- an ADI method to solve the three-state variable HHW/HCIR pricing PDE.

The primary objective of these algorithms is to produce results that are sufficiently close to those obtained using the three-state variable simulation method presented in Section 5 but with significantly shorter execution times.

6.1 Explicit vs ADI methods

Explicit numerical schemes in options pricing use the solution values at the current time step to calculate solution values at an earlier time step. For an implicit numerical scheme, the solution values at an earlier time step are calculated by solving a system of equations which include the current and earlier time step solution values.

A numerical scheme is consistent if its truncation error approaches zero as the mesh grid is refined. The stability of a numerical scheme implies that the solution produced by the scheme is not too sensitive to small changes in initial

data and that any errors produced by the scheme diminish as one moves from one computation step to the next. The convergence of a numerical scheme implies that the solution produced by the scheme approaches the true solution as the mesh grid is refined [24].

For the $m \times m$ matrix A , and $m \times 1$ vectors F and K , an explicit numerical scheme of the form

$$F^{n-1} = AF^n + K^n, \quad (6.1)$$

which resembles the explicit option pricing problem, is stable if and only if the infinity norm of A conforms to the inequality

$$\|A\|_\infty \leq 1. \quad (6.2)$$

This condition is true under the Black–Scholes model for only certain combinations of r , q , σ_S and Δt , meaning that the explicit numerical scheme is conditionally stable [25], [26].

Similarly, for the $m \times m$ matrix B , an implicit numerical scheme of the form

$$BF^{n-1} = F^n + K^{n-1}, \quad (6.3)$$

which resembles the implicit option pricing problem, is stable if and only if the infinity norm of B^{-1} conforms to the inequality

$$\|B^{-1}\|_\infty \leq 1. \quad (6.4)$$

This condition is true under the Black–Scholes model for all values of r , q , σ_S and Δt , meaning that the implicit numerical scheme is unconditionally stable [25], [26].

The Lax equivalence theorem states that for a well-posed linear initial value problem, a consistent numerical scheme is convergent if and only if it is stable [24]. Therefore, since the implicit numerical scheme given above is consistent with the order $\mathcal{O}(\Delta t, \Delta^2 S)$, then by the Lax equivalence theorem, the Black–Scholes implicit numerical scheme will always converge for any value of r , q , σ_S and Δt .

Explicit schemes are generally less complicated and computationally inexpensive to solve. Implicit schemes, on the other hand, are generally computationally more demanding to solve since they involve calculating matrix inverses, which are known to take exponentially longer as the size of the matrix to be inverted increases.

Explicit schemes tend to be conditionally stable, and convergence depends greatly on the size of the time step. Implicit schemes tend to be unconditionally stable, and convergence is achieved even when relatively large time steps are used.

An ADI scheme is a type of implicit scheme that uses a two-step iteration process and is often used to solve Sylvester matrix equations of the form

$$AF + FB = C. \quad (6.5)$$

The two-step iteration process involves successively solving the above matrix equation in both time directions. For example, a single iteration involves solving for $F^{n-1/2}$ in

$$(A - \beta^{n-1}I)F^{n-1/2} = F^n(B - \beta^{n-1}) + C \quad (6.6)$$

after which F^{n-1} is solved in

$$F^{n-1}(B - \alpha^{n-1}I) = (A - \alpha^{n-1}I)F^{n-1/2} + C \quad (6.7)$$

where α^{n-1} and β^{n-1} are referred to as shift parameters, and I is an identity matrix. This method leads to a more efficient and stable solution than explicit alternatives. Convergence depends greatly on the selection of the shift parameters. Variations of this method can be found throughout option pricing literature. This dissertation focuses on the Crank–Nicolson, Douglas, Craig–Sneyd, modified Craig–Sneyd and Hundsdorfer–Verwer ADI methods.

ADI methods further allow the decomposition of the matrix equations into smaller sub-problems by splitting the spatial derivatives. This is especially ideal when large matrices are involved.

The Crank–Nicolson method

- is unconditionally stable;
- exhibits good converge properties; and
- achieves a second-order accuracy in time and space.

The Douglas method [27]

- is unconditionally stable; and
- achieves a first-order accuracy in time; however,
- converges slower than the Crank–Nicolson method.

The Craig–Sneyd method [28]

- is unconditionally stable;

- achieves a second-order accuracy in time and space;
- offers stability and convergence improvements over the Douglas scheme by using a weighted averaging approach; and
- was specifically designed to solve option pricing problems.

The modified Craig–Sneyd method [29]

- is unconditionally stable;
- offers better stability and convergence results than its predecessor by expanding on the weighted averaging approach; and
- maintains a second-order accuracy in time and space.

The Hundsdorfer–Verwer method [30]

- is unconditionally stable;
- achieves a second-order accuracy in time and space; and
- was specifically designed to handle stiff systems of PDEs relating to diffusion-type problems.

6.2 Vectorised approach

The finite difference algorithms were programmed in Python programming language. A vectorised approach was used to solve the numerical schemes instead of traditional and slower-performing nested *for* loops often encountered in option pricing literature.

Several studies have shown that a vectorised approach provides numerous performance advantages over nested *for* loops when it comes to computations involving large numerical arrays. This is primarily due to hardware-level optimisations and parallel processing capabilities that modern processors and graphics processing units offer. Parallel processing allows the same computations to be performed on different data elements simultaneously, while manually written nested *for* loops are restricted to computations in sequence [31].

Modern technologies that allow parallel processing include:

- Single instruction, multiple data (or SIMD) units are hardware components that perform the same computation on multiple data operands concurrently.
- Parallel processing is a method in computing that allows multicore processors and graphic processing units to handle different parts of one task on different cores.

- Vector-based libraries (such as NumPy and SciPy in Python) that are specifically written and optimised to perform multidimensional array calculations.

These parallel processing technologies force operations on an entire multi-dimensional array rather than individual elements. This leads to the reduced overhead of loop control and indexing. It improves cache memory utilisation by accessing contiguous memory locations, reducing cache misses.

6.3 Unbounded and bounded regions

The following unbounded regions should be considered when approaching an option pricing problem under stochastic stock prices, volatility, and interest rates [32].

1. The Black–Scholes model:

$$D(S, t) \in [0, +\infty] \times (0, T_D]; \text{ and} \\ M(S, t) \in [0, +\infty] \times (0, T_M].$$

2. The Heston model:

$$D(S, v, t) \in [0, +\infty] \times [0, +\infty] \times (0, T_D]; \text{ and} \\ M(S, v, t) \in [0, +\infty] \times [0, +\infty] \times (0, T_M].$$

3. The HHW model:

$$D(S, v, r, t) \in [0, +\infty] \times [0, +\infty] \times [-\infty, +\infty] \times (0, T_D]; \text{ and} \\ M(S, v, r, t) \in [0, +\infty] \times [0, +\infty] \times [-\infty, +\infty] \times (0, T_M].$$

4. The HCIR model:

$$D(S, v, r, t) \in [0, +\infty] \times [0, +\infty] \times [0, +\infty] \times (0, T_D]; \text{ and} \\ M(S, v, r, t) \in [0, +\infty] \times [0, +\infty] \times [0, +\infty] \times (0, T_M].$$

Note that the minimum interest rate under the HCIR model is zero because the $\sqrt{r}$ term will result in undefined values.

To solve the pricing PDEs numerically, these unbounded regions are restricted to workable bounded regions by choosing appropriate values for S_{max} , v_{max} , r_{max} , and r_{min} .

1. The Black–Scholes model:

$$D(S, t) \in [0, S_{max}] \times (0, T_D]; \text{ and} \\ M(S, t) \in [0, S_{max}] \times (0, T_M].$$

2. The Heston model:

$$D(S, v, t) \in [0, S_{max}] \times [0, v_{max}] \times (0, T_D]; \text{ and} \\ M(S, v, t) \in [0, S_{max}] \times [0, v_{max}] \times (0, T_M].$$

3. The HHW model:

$$D(S, v, r, t) \in [0, S_{max}] \times [0, v_{max}] \times [r_{min}, r_{max}] \times (0, T_D]; \text{ and} \\ M(S, v, r, t) \in [0, S_{max}] \times [0, v_{max}] \times [r_{min}, r_{max}] \times (0, T_M].$$

4. The HCIR model:

$$D(S, v, r, t) \in [0, S_{max}] \times [0, v_{max}] \times [0, r_{max}] \times (0, T_D]; \text{ and} \\ M(S, v, r, t) \in [0, S_{max}] \times [0, v_{max}] \times [0, r_{max}] \times (0, T_M].$$

The values for S_{max} , v_{max} , and r_{max} are sufficiently large positive constants such that the ranges are inclusive of the starting values $S_0 = S(0)$, $v_0 = v(0)$ and $r_0 = r(0)$, respectively [32].

6.4 Setting up the spatial and time grids

The S , v , r , and t dimensions are split into N_S , N_v , N_r and N_t intervals, respectively. Uniform and non-uniform methods are used to discretise the spatial dimensions.

6.4.1 Uniform time discretisation

For a uniformly spaced time grid, the value t_n is defined as

$$t_n = T_0 + n\Delta t \quad \text{for} \quad n = 0, 1, \dots, X_t, \dots, N_t$$

where

$$\Delta t = \frac{T_D - T_0}{N_t}. \quad (6.8)$$

The index along the time grid that corresponds with the exercise date of the overlying option is given by

$$X_t = \frac{T_M - T_0}{\Delta t}. \quad (6.9)$$

6.4.2 Uniform spatial discretisation

For a uniformly spaced spatial grid, the values S_i , v_j and r_k are defined as

$$\begin{aligned} S_i &= S_{min} + i\Delta S & \text{for } i &= 0, 1, \dots, N_S, \\ v_j &= v_{min} + j\Delta v & \text{for } j &= 0, 1, \dots, N_v, \text{ and} \\ r_k &= r_{min} + k\Delta r & \text{for } k &= 0, 1, \dots, N_r \end{aligned}$$

where

$$\Delta S = \frac{S_{max} - S_{min}}{N_S}, \quad (6.10)$$

$$\Delta v = \frac{v_{max} - v_{min}}{N_v}, \text{ and} \quad (6.11)$$

$$\Delta r = \frac{r_{max} - r_{min}}{N_r}. \quad (6.12)$$

6.4.3 Non-uniform spatial discretisation

For non-uniformly distributed spatial grids, the values t_n , S_i , v_j and r_k are calculated using the methods described below.

6.4.3.1 Stock price dimension

The non-uniform spatial discretisation of the stock price dimension involves setting up a uniformly distributed interval between S_{left} and S_{right} that contains S_0 , while all points outside this interval will be non-uniformly distributed – that is, points closer to the interval will be spaced closely together and will gradually move further apart from one another as they approach S_{min} and S_{max} .

The modified version of the method given in [8] is given below:

1. Choose an interval $[S_{left}, S_{right}]$ wherein S is uniform and non-uniform outside. S_0 should be contained within this interval. For this dissertation,

$$S_{left} = \min\{S_0, (1 - c)K_D\} \quad (6.13)$$

and

$$S_{right} = \max\{S_0, (1 + c)K_D\} \quad (6.14)$$

were chosen, where $c \in [0, 1]$.

2. Define ξ_{min} , ξ_{int} and ξ_{max} as

$$\xi_{min} = \sinh^{-1} \left(\frac{S_{min} - S_{left}}{d_1} \right), \quad (6.15)$$

$$\xi_{int} = \frac{S_{right} - S_{left}}{d_1}, \quad \text{and} \quad (6.16)$$

$$\xi_{max} = \xi_{int} + \sinh^{-1} \left(\frac{S_{max} - S_{right}}{d_1} \right), \quad (6.17)$$

respectively.

3. Calculate $\Delta\xi$ as

$$\Delta\xi = \frac{1}{N_S} (\xi_{max} - \xi_{min}). \quad (6.18)$$

4. Let the equidistant points $\xi_{min} = \xi_0 < \xi_1 < \dots < \xi_{N_S} = \xi_{max}$ be given by

$$\xi_i = \xi_{min} + i\Delta\xi \quad \text{for} \quad i = 0, 1, \dots, N_S. \quad (6.19)$$

5. Then, $S_i = \Psi(\xi_i)$ for $i = 0, \dots, N_S$ where

$$\Psi(\xi_i) = \begin{cases} S_{left} + d_1 \sinh(\xi_i) & \text{if } \xi_{min} \leq \xi_i < 0, \\ S_{left} + d_1 \xi_i & \text{if } 0 < \xi_i < \xi_{int}, \\ S_{right} + d_1 \sinh(\xi_i - \xi_{int}) & \text{if } \xi_{int} < \xi_i \leq \xi_{max}. \end{cases} \quad (6.20)$$

6.4.3.2 Variance dimension

The non-uniform spatial discretisation of the variance dimension involves setting up a non-uniformly distributed grid wherein points closer to v_{min} will be spaced closely together and will gradually move further apart from one another as they approach v_{max} .

The modified version of the method given in [8] is given below:

1. Let the equidistant points $\eta_0 < \eta_1 < \dots < \eta_{N_v}$ be given by

$$\eta_j = j\Delta\eta \quad \text{for} \quad j = 0, 1, \dots, N_v \quad (6.21)$$

where

$$\Delta\eta = \frac{1}{N_v} \sinh^{-1} \left(\frac{v_{max} - v_{min}}{d_2} \right). \quad (6.22)$$

2. The variance points are then given by

$$v_j = d_2 \sinh(\eta_j) \quad \text{for} \quad j = 0, 1, \dots, N_v. \quad (6.23)$$

6.4.3.3 Interest rate dimension

The non-uniform spatial discretisation of the interest rate dimension involves setting up a non-uniformly distributed grid wherein points closer to r_0 , on either side, will be spaced closely together and will gradually move further apart from one another as they approach $-r_{max}$ and r_{max} .

The modified version of the method given in [8] is given below:

1. Define ζ_{min} and ζ_{max} as

$$\zeta_{min} = \sinh^{-1} \left(\frac{r_{min} - r_0}{d_3} \right) \quad \text{and} \quad (6.24)$$

$$\zeta_{max} = \sinh^{-1} \left(\frac{r_{max} - r_0}{d_3} \right), \quad (6.25)$$

respectively.

2. Calculate $\Delta\zeta$ as

$$\Delta\zeta = \frac{1}{N_r} (\zeta_{max} - \zeta_{min}). \quad (6.26)$$

3. Let the equidistant points $\zeta_{min} = \zeta_0 < \zeta_1 < \dots < \zeta_{N_v} = \zeta_{max}$ be given by

$$\zeta_k = \zeta_{min} + k\Delta\zeta \quad \text{for} \quad k = 0, 1, \dots, N_r. \quad (6.27)$$

4. The interest rate points are then given by

$$r_k = r_0 + d_3 \sinh(\zeta_k) \quad \text{for} \quad k = 0, 1, \dots, N_r. \quad (6.28)$$

6.4.3.4 Non-uniform spatial grid samples

Samples of the non-uniform grids for the S , v , and r spatial dimensions are set up according to the algorithms in Section 6.4.3, and the results are shown in Figure 6.1. For these samples, $S_0 = K_D = 100$, $r_0 = 0.15$, $c = 0.1$, and $N_S = N_v = N_r = 60$. The mesh points along the S direction are concentrated around K_D , the mesh points along the v direction are concentrated around 0, and the mesh points along the r direction are concentrated around r_0 .

6.5 Setting up the grid arrays

A grid array is constructed for the underlying option denoted by D , while a separate grid array is constructed for the overlying option denoted by M . M will have the same size spatial dimensions as D but a shorter time dimension.

In the Black–Scholes model, the two-dimensional grid arrays D and M will have $(N_S + 1)(N_t + 1)$ and $(N_S + 1)(X_t + 1)$ total nodes, respectively. The points $D(S_i, t_n)$ and $M(S_i, t_n)$ will be represented using D_i^n and M_i^n , respectively.

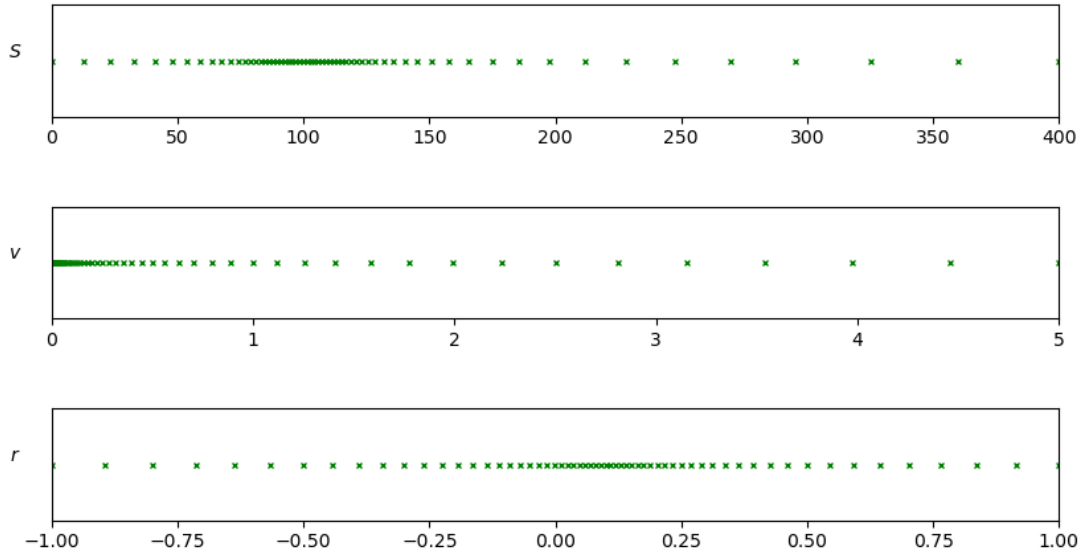


Figure 6.1: Non-uniform spatial grids for S , v and r , respectively.

In the Heston model, the three-dimensional grid arrays D and M will have $(N_S + 1)(N_v + 1)(N_t + 1)$ and $(N_S + 1)(N_v + 1)(X_t + 1)$ total nodes, respectively. The points $D(S_i, v_j, t_n)$ and $M(S_i, v_j, t_n)$ will be represented using $D_{i,j}^n$ and $M_{i,j}^n$, respectively.

In the HHW/HCIR models, the four-dimensional grid arrays D and M will have $(N_S + 1)(N_v + 1)(N_r + 1)(N_t + 1)$ and $(N_S + 1)(N_v + 1)(N_r + 1)(X_t + 1)$ total nodes, respectively. The points $D(S_i, v_j, r_k, t_n)$ and $M(S_i, v_j, r_k, t_n)$ will be represented using $D_{i,j,k}^n$ and $M_{i,j,k}^n$, respectively.

An example of a uniform grid array D for a call option at T_D using the HHW model is shown in Figure 6.2(a), while a non-uniform grid array is shown in Figure 6.2(b). For this sample, $T_D = 1$, $S_0 = 100$, $v_0 = 0$ and $r_0 = 0.15$ were used. The colour bars represent the scale of the payoff values, and to exaggerate the colour shading, $K_D = 10$ was used. From Figure 6.2(b), one can see that the grid points are concentrated around the point (S_0, v_0, r_0) .

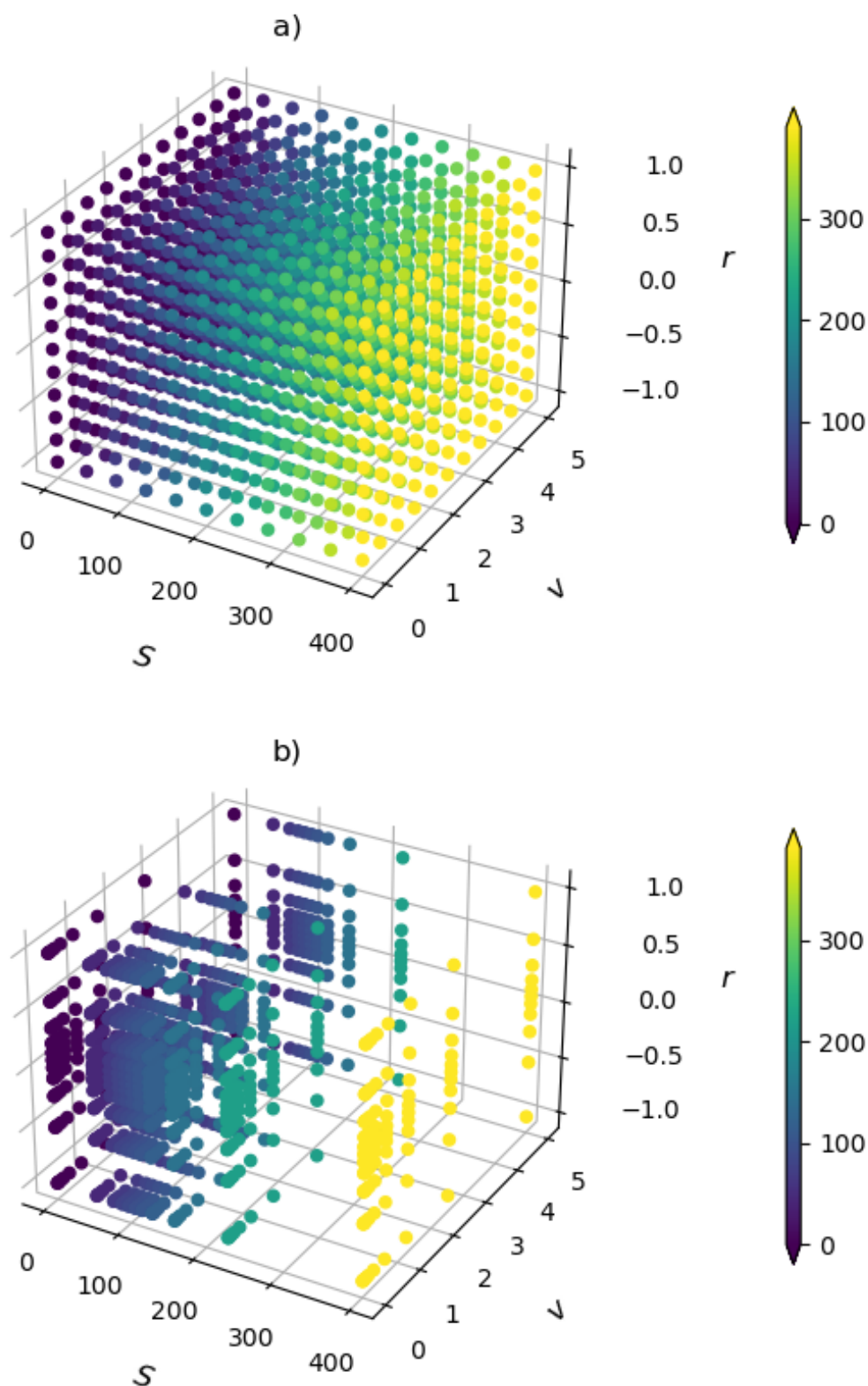


Figure 6.2: Uniform [subplot a)] and non-uniform [subplot b)] grid array D for an underlying call option using the HHW model.

6.6 Setting the terminal conditions

6.6.1 Underlying option

The terminal conditions for D at $t_{N_t} = T_D$ are calculated using (4.3).

1. For the Black–Scholes model, the following applies

$$D_i^{N_t} = \max\{\phi(S_i - K_D), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S. \quad (6.29)$$

2. For the Heston model, the following applies

$$D_{i,j}^{N_t} = \max\{\phi(S_i - K_D), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v. \quad (6.30)$$

3. For the HHW/H CIR models, the following applies

$$D_{i,j,k}^{N_t} = \max\{\phi(S_i - K_D), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r. \quad (6.31)$$

6.6.2 Overlying option

The terminal conditions for M at $t_{X_t} = T_M$ are calculated using (4.5).

1. For the Black–Scholes model, the following applies

$$M_i^{X_t} = \max\{\omega(D_i^{X_t} - K_M), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S. \quad (6.32)$$

2. For the Heston model, the following applies

$$M_{i,j}^{X_t} = \max\{\omega(D_{i,j}^{X_t} - K_M), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v. \quad (6.33)$$

3. For the HHW/H CIR models, the following applies

$$M_{i,j,k}^{X_t} = \max\{\omega(D_{i,j,k}^{X_t} - K_M), 0\} \quad \text{for} \quad i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r. \quad (6.34)$$

6.7 Explicit numerical methods

In the subsequent sections, F represents both D and M .

After the terminal values have been set at T_D for D and T_M for M , the strategy for the explicit methods is to iterate backwards in time until T_0 is reached. At

each time step for $t \in (T_D, T_0]$, the values of the boundary nodes are first set for D , after which the values of the internal nodes are calculated using the explicit schemes below. Similarly, at each time step for $t \in (T_M, T_0]$, the values of the boundary nodes are first set for M , after which the values of the internal nodes are calculated.

6.7.1 Black–Scholes explicit method

In the pricing PDE (3.6), the $\frac{\partial F}{\partial t}$ term is substituted with the first-order backward approximation

$$\frac{\partial F(t_n; S_i)}{\partial t} = \frac{F_i^n - F_i^{n-1}}{\Delta t}. \quad (6.35)$$

Further substitution of $S = S_i$ yields the numerical scheme

$$\begin{aligned} F_i^{n-1} = & F_i^n + (r - q)S_i\Delta t \frac{\partial F(t_n; S_i)}{\partial S} \\ & + \frac{1}{2}\sigma_S^2 S_i^2 \Delta t \frac{\partial^2 F(t_n; S_i)}{\partial S^2} \\ & - r\Delta t F_i^n. \end{aligned} \quad (6.36)$$

6.7.2 Heston explicit method

In the pricing PDE (3.9), the $\frac{\partial F}{\partial t}$ term is substituted with the first-order backward approximation

$$\frac{\partial F(t_n; S_i, v_j)}{\partial t} = \frac{F_{i,j}^n - F_{i,j}^{n-1}}{\Delta t}. \quad (6.37)$$

Further substitution of $S = S_i$ and $v = v_j$ yields the numerical scheme

$$\begin{aligned} F_{i,j}^{n-1} = & F_{i,j}^n + (r - q)S_i\Delta t \frac{\partial F(t_n; S_i, v_j)}{\partial S} \\ & + (\kappa_v(\theta_v - v_j) - \lambda_v(t_n, S_i, v_j))\Delta t \frac{\partial F(t_n; S_i, v_j)}{\partial v} \\ & + \frac{1}{2}v_j S_i^2 \Delta t \frac{\partial^2 F(t_n; S_i, v_j)}{\partial S^2} \\ & + \frac{1}{2}\sigma_v^2 v_j \Delta t \frac{\partial^2 F(t_n; S_i, v_j)}{\partial v^2} \\ & + \rho_{S,v}\sigma_v v_j S_i \Delta t \frac{\partial^2 F(t_n; S_i, v_j)}{\partial S \partial v} \\ & - r\Delta t F_{i,j}^n. \end{aligned} \quad (6.38)$$

6.7.3 HHW/HCIR explicit method

In the pricing PDE (3.21), the $\frac{\partial F}{\partial t}$ term is substituted with the first-order backward approximation

$$\frac{\partial F(t_n; S_i, v_j, r_k)}{\partial t} = \frac{F_{i,j,k}^n - F_{i,j,k}^{n-1}}{\Delta t}. \quad (6.39)$$

Further substitution of $S = S_i$, $v = v_j$ and $r = r_k$ yields the numerical scheme

$$\begin{aligned} F_{i,j,k}^{n-1} = & F_{i,j,k}^n + (r_k - q)S_i\Delta t \frac{\partial F(t_n; S_i, v_j, r_k)}{\partial S} \\ & + (\kappa_v(\theta_v - v_j) - \lambda_v(t_n, S_i, v_j, r_k))\Delta t \frac{\partial F(t_n; S_i, v_j, r_k)}{\partial v} \\ & + (\kappa_r(\theta_r - r_k) - \lambda_r(t_n, r_k))\Delta t \frac{\partial F(t_n; S_i, v_j, r_k)}{\partial r} \\ & + \frac{1}{2}v_j S_i^2 \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial S^2} \\ & + \frac{1}{2}\sigma_v^2 v_j \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial v^2} \\ & + \frac{1}{2}\sigma_r^2 r_k^{2p} \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial r^2} \\ & + \rho_{S,v}\sigma_v v_j S_i \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial S \partial v} \\ & + \rho_{S,r}\sigma_r \sqrt{v_j} r_k^p S_i \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial S \partial r} \\ & + \rho_{v,r}\sigma_v \sigma_r \sqrt{v_j} r_k^p \Delta t \frac{\partial^2 F(t_n; S_i, v_j, r_k)}{\partial v \partial r} \\ & - r_k \Delta t F_{i,j,k}^n. \end{aligned} \quad (6.40)$$

6.8 Implicit numerical methods

After the terminal conditions have been set at T_D for D and T_M for M , the strategy for the implicit methods is to iterate backwards in time until T_0 is reached. At each time step for $t \in (T_D, T_0]$, the boundary and internal nodes for D are solved simultaneously using the ADI schemes below. Similarly, at each time step for $t \in (T_M, T_0]$, the boundary and internal nodes for M are solved simultaneously.

Again, F represents both D and M .

6.8.1 Black–Scholes implicit method

The initial value problem with pricing PDE (3.6) takes on the form

$$F'(t) + AF(t) + g(t) = 0 \quad (6.41)$$

with terminal conditions as given in Section 6.6. Here, A is a time-independent $(N_S + 1) \times (N_S + 1)$ sparse matrix that contains the derivative terms

$$(r - q)S_i \frac{\partial}{\partial S} + \frac{1}{2}\sigma_S^2 S_i^2 \frac{\partial^2}{\partial S^2} - r \quad \text{for} \quad i = 1, 2, \dots, N_S - 1.$$

The boundary conditions are contained in the time-dependent $(N_S + 1) \times 1$ vector $g(t_n) = g^n$ where the point F_i^n corresponds with the point g_i^n . The change in boundary vector between time steps is calculated using $\Delta g^{n-1} = g^{n-1} - g^n$.

The five ADI time discretisation schemes discussed in Section 6.1 are used to solve (6.41) as follows:

1. The Crank–Nicolson scheme:

$$\begin{cases} Y_0 &= F^n + \Delta t(AF^n + g^n), \\ Y_1 &= Y_0 + \frac{1}{2}\Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ F^{n-1} &= Y_1. \end{cases} \quad (6.42)$$

2. The Douglas scheme [8]:

$$\begin{cases} Y_0 &= F^n + \Delta t(AF^n + g^n), \\ Y_1 &= Y_0 + \theta\Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ F^{n-1} &= Y_1. \end{cases} \quad (6.43)$$

3. The Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= F^n + \Delta t(AF^n + g^n), \\ Y_1 &= Y_0 + \theta\Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ \tilde{Y}_1 &= Y_1 + \theta\Delta t(A\tilde{Y}_1 - AF^n + \Delta g^{n-1}), \\ F^{n-1} &= \tilde{Y}_1. \end{cases} \quad (6.44)$$

4. The modified Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= F^n + \Delta t(AF^n + g^n), \\ Y_1 &= Y_0 + \theta\Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ \tilde{Y}_0 &= Y_0 + (\frac{1}{2} - \theta)\Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ \tilde{Y}_1 &= \tilde{Y}_0 + \theta\Delta t(A\tilde{Y}_1 - AF^n + \Delta g^{n-1}), \\ F^{n-1} &= \tilde{Y}_1. \end{cases} \quad (6.45)$$

5. The Hundsdorfer–Verwer scheme [8]:

$$\begin{cases} Y_0 &= F^n + \Delta t(AF^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ \widetilde{Y}_0 &= Y_0 + \frac{1}{2} \Delta t(AY_1 - AF^n + \Delta g^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A\widetilde{Y}_1 - AY_1 + \Delta g^{n-1}), \\ F^{n-1} &= \widetilde{Y}_1. \end{cases} \quad (6.46)$$

The value $\theta = 0.8$ was used for all ADI schemes.

6.8.2 Heston implicit method

For the Heston implicit numerical, $U(t_n) = U^n$ is introduced, which is the column-stacked reshaping of $F(t_n)$ into an $m \times 1$ vector where $m = (N_S + 1)(N_v + 1)$. The point $F_{i,j}^n$ corresponds with the point U_h^n where $h = (N_S + 1)(j - 1) + i$.

Thereafter, the initial value problem with pricing PDE (3.9) takes on the form

$$U'(t) + AU(t) + g(t) = 0 \quad (6.47)$$

with terminal conditions as given in Section 6.6. Here, A is a time-independent $m \times m$ sparse matrix that contains the derivative terms from (3.9). Four $m \times m$ sparse submatrices A_0 , A_1 , and A_2 are defined such that

$$A = A_0 + A_1 + A_2. \quad (6.48)$$

The boundary conditions are contained in the time-dependent $m \times 1$ vector $g(t_n) = g^n$ where the point $F_{i,j}^n$ corresponds with the point g_h^n . The three $m \times 1$ subvectors g_0^n , g_1^n , and g_2^n are defined analogously to that of A such that

$$g^n = g_0^n + g_1^n + g_2^n. \quad (6.49)$$

The change in boundary vector between time steps is calculated using $\Delta g^{n-1} = g^{n-1} - g^n$.

The derivative terms from (3.9) are split across these three sparse matrices as follows:

1. A_0 includes the mixed derivative terms

$$\rho_{S,v} \sigma_v v_j S_i \frac{\partial^2}{\partial S \partial v} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 1, 2, \dots, N_v - 1. \end{aligned}$$

2. A_1 includes the S -derivative terms

$$(r - q)S_i \frac{\partial}{\partial S} \quad \text{for} \quad \begin{array}{l} i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v; \end{array}$$

and

$$\frac{1}{2}v_j S_i^2 \frac{\partial^2}{\partial S^2} \quad \text{for} \quad \begin{array}{l} i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v; \end{array}$$

and

$$-\frac{1}{2}r \quad \text{for} \quad \begin{array}{l} i = 1, 2, \dots, N_S - 1; \\ j = 1, 2, \dots, N_v - 1. \end{array}$$

3. A_2 includes the v -derivative terms

$$(\kappa_v(\theta_v - v_j) - \lambda_v(t_n, S_i, v_j, r_k)) \frac{\partial}{\partial v} \quad \text{for} \quad \begin{array}{l} i = 0, 1, \dots, N_S; \\ j = 0, 1, \dots, N_v - 1; \end{array}$$

and

$$\frac{1}{2}\sigma_v^2 v_j \frac{\partial^2}{\partial v^2} \quad \text{for} \quad \begin{array}{l} i = 0, 1, \dots, N_S; \\ j = 1, 2, \dots, N_v - 1; \end{array}$$

and

$$-\frac{1}{2}r \quad \text{for} \quad \begin{array}{l} i = 1, 2, \dots, N_S - 1; \\ j = 1, 2, \dots, N_v - 1. \end{array}$$

4. The $-rF$ term is split between A_0 , A_1 and A_2 .

The five ADI time discretisation schemes discussed in Section 6.1 are used to solve (6.47) as follows:

1. The Crank–Nicolson scheme:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \frac{1}{2}\Delta t(A_1 Y_1 - AU^n + \Delta g^{n-1}), \\ U^{n-1} &= Y_1. \end{cases} \quad (6.50)$$

2. The Douglas scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ U^{n-1} &= Y_2. \end{cases} \quad (6.51)$$

3. The Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ \widetilde{Y}_0 &= Y_0 + \frac{1}{2} \Delta t(A_0 Y_2 - A_0 U^n + \Delta g_0^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 U^n + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 U^n + \Delta g_2^{n-1}), \\ U^{n-1} &= \widetilde{Y}_2. \end{cases} \quad (6.52)$$

4. The modified Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ \widehat{Y}_0 &= Y_0 + \theta \Delta t(A_0 Y_2 - A_0 U^n + \Delta g_0^{n-1}), \\ \widetilde{Y}_0 &= \widehat{Y}_0 + (\frac{1}{2} - \theta) \Delta t(A Y_2 - A U^n + \Delta g^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 U^n + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 U^n + \Delta g_2^{n-1}), \\ U^{n-1} &= \widetilde{Y}_2. \end{cases} \quad (6.53)$$

5. The Hundsdorfer–Verwer scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ \widetilde{Y}_0 &= Y_0 + \frac{1}{2} \Delta t(A Y_2 - A U^n + \Delta g^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 Y_2 + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 Y_2 + \Delta g_2^{n-1}), \\ U^{n-1} &= \widetilde{Y}_2. \end{cases} \quad (6.54)$$

The value $\theta = 0.8$ was used for all ADI schemes.

6.8.3 HHW/HCIR implicit method

For the HHW/HCIR implicit numerical scheme, $U(t_n) = U^n$ is introduced, which is the column-stacked reshaping of $F(t_n)$ into an $m \times 1$ vector where $m = (N_S + 1)(N_v + 1)(N_r + 1)$. The point $F_{i,j,k}^n$ corresponds with the point U_h^n where $h = (N_S + 1)(N_v + 1)(k - 1) + (N_S + 1)(j - 1) + i$.

Thereafter, the initial value problem with pricing PDE (3.21) takes on the form

$$U'(t) + AU(t) + g(t) = 0 \quad (6.55)$$

with terminal conditions as given in Section 6.6. Here, A is a time-independent $m \times m$ sparse matrix that contains the derivative terms from (3.21). Four $m \times m$ sparse submatrices A_0 , A_1 , A_2 and A_3 are defined such that

$$A = A_0 + A_1 + A_2 + A_3. \quad (6.56)$$

The boundary conditions are contained in the time-dependent $m \times 1$ vector $g(t_n) = g^n$ where the point $F_{i,j,k}^n$ corresponds with the point g_h^n . The four $m \times 1$ subvectors g_0^n , g_1^n , g_2^n , and g_3^n are defined analogously to that of A such that

$$g^n = g_0^n + g_1^n + g_2^n + g_3^n. \quad (6.57)$$

The change in boundary vector between time steps is calculated using $\Delta g^{n-1} = g^{n-1} - g^n$.

The derivative terms from (3.21) are split across these four sparse matrices as follows:

1. A_0 includes the mixed derivative terms

$$\rho_{S,v}\sigma_v v_j S_i \frac{\partial^2}{\partial S \partial v} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 1, 2, \dots, N_r - 1; \end{aligned}$$

and

$$\rho_{S,r}\sigma_r \sqrt{v_j} r_k^p S_i \frac{\partial^2}{\partial S \partial r} \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 1, 2, \dots, N_r - 1; \end{aligned}$$

and

$$\rho_{v,r}\sigma_v\sigma_r \sqrt{v_j} r_k^p \frac{\partial^2}{\partial v \partial r} \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 1, 2, \dots, N_r - 1. \end{aligned}$$

2. A_1 includes the S -derivative terms

$$(r_k - q)S_i \frac{\partial}{\partial S} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 0, 1, \dots, N_v; \\ k &= 1, 2, \dots, N_r - 1; \end{aligned}$$

and

$$\frac{1}{2}v_j S_i^2 \frac{\partial^2}{\partial S^2} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 0, 1, \dots, N_v; \\ k &= 1, 2, \dots, N_r - 1; \end{aligned}$$

and

$$-\frac{1}{2}r_k \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 1, 2, \dots, N_r - 1. \end{aligned}$$

3. A_2 includes the v -derivative terms

$$(\kappa_v(\theta_v - v_j) - \lambda_v(t_n, S_i, v_j, r_k)) \frac{\partial}{\partial v} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 0, 1, \dots, N_v - 1; \\ k &= 0, 1, \dots, N_r; \end{aligned}$$

and

$$\frac{1}{2}\sigma_v^2 v_j \frac{\partial^2}{\partial v^2} \quad \text{for} \quad \begin{aligned} i &= 0, 1, \dots, N_S; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 0, 1, \dots, N_r; \end{aligned}$$

and

$$-\frac{1}{2}r_k \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 1, 2, \dots, N_v - 1; \\ k &= 1, 2, \dots, N_r - 1. \end{aligned}$$

4. A_3 includes the r -derivative terms

$$(\kappa_r(\theta_r - r_k) - \lambda_r(t_n, r_k)) \frac{\partial}{\partial r} \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 0, 1, \dots, N_v; \\ k &= 0, 1, \dots, N_r; \end{aligned}$$

and

$$\frac{1}{2}\sigma_r^2 r_k^{2p} \frac{\partial^2}{\partial r^2} \quad \text{for} \quad \begin{aligned} i &= 1, 2, \dots, N_S - 1; \\ j &= 0, 1, \dots, N_v; \\ k &= 0, 1, \dots, N_r. \end{aligned}$$

5. The $-rF$ term is split between A_0 , A_1 , A_2 and A_3 .

The five ADI time discretisation schemes discussed in Section 6.1 are used to solve (6.55).

1. The Crank–Nicolson:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \frac{1}{2}\Delta t(A_1 Y_1 - AU^n + \Delta g^{n-1}), \\ U^{n-1} &= Y_1. \end{cases} \quad (6.58)$$

2. The Douglas scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ Y_3 &= Y_2 + \theta \Delta t(A_3 Y_3 - A_3 U^n + \Delta g_3^{n-1}), \\ U^{n-1} &= Y_3. \end{cases} \quad (6.59)$$

3. The Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ Y_3 &= Y_2 + \theta \Delta t(A_3 Y_3 - A_3 U^n + \Delta g_3^{n-1}), \\ \widetilde{Y}_0 &= Y_0 + \frac{1}{2}\Delta t(A_0 Y_3 - A_0 U^n + \Delta g_0^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 U^n + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 U^n + \Delta g_2^{n-1}), \\ \widetilde{Y}_3 &= \widetilde{Y}_2 + \theta \Delta t(A_3 \widetilde{Y}_3 - A_3 U^n + \Delta g_3^{n-1}), \\ U^{n-1} &= \widetilde{Y}_3. \end{cases} \quad (6.60)$$

4. The modified Craig–Sneyd scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ Y_3 &= Y_2 + \theta \Delta t(A_3 Y_3 - A_3 U^n + \Delta g_3^{n-1}), \\ \widehat{Y}_0 &= Y_0 + \theta \Delta t(A_0 Y_3 - A_0 U^n + \Delta g_0^{n-1}), \\ \widetilde{Y}_0 &= \widehat{Y}_0 + (\frac{1}{2} - \theta) \Delta t(A Y_3 - A U^n + \Delta g^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 U^n + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 U^n + \Delta g_2^{n-1}), \\ \widetilde{Y}_3 &= \widetilde{Y}_2 + \theta \Delta t(A_3 \widetilde{Y}_3 - A_3 U^n + \Delta g_3^{n-1}), \\ U^{n-1} &= \widetilde{Y}_3. \end{cases} \quad (6.61)$$

5. The Hundsdorfer–Verwer scheme [8]:

$$\begin{cases} Y_0 &= U^n + \Delta t(AU^n + g^n), \\ Y_1 &= Y_0 + \theta \Delta t(A_1 Y_1 - A_1 U^n + \Delta g_1^{n-1}), \\ Y_2 &= Y_1 + \theta \Delta t(A_2 Y_2 - A_2 U^n + \Delta g_2^{n-1}), \\ Y_3 &= Y_2 + \theta \Delta t(A_3 Y_3 - A_3 U^n + \Delta g_3^{n-1}), \\ \widetilde{Y}_0 &= Y_0 + \frac{1}{2} \Delta t(A Y_3 - A U^n + \Delta g^{n-1}), \\ \widetilde{Y}_1 &= \widetilde{Y}_0 + \theta \Delta t(A_1 \widetilde{Y}_1 - A_1 Y_3 + \Delta g_1^{n-1}), \\ \widetilde{Y}_2 &= \widetilde{Y}_1 + \theta \Delta t(A_2 \widetilde{Y}_2 - A_2 Y_3 + \Delta g_2^{n-1}), \\ \widetilde{Y}_3 &= \widetilde{Y}_2 + \theta \Delta t(A_3 \widetilde{Y}_3 - A_3 Y_3 + \Delta g_3^{n-1}), \\ U^{n-1} &= \widetilde{Y}_3. \end{cases} \quad (6.62)$$

The value $\theta = 0.8$ was used for all ADI schemes.

6.9 Boundary conditions at the minimum stock price

6.9.1 Underlying option

At $S = 0$, an underlying call option will be worthless, while an underlying put option will be the value of the discounted strike price. The boundary condition is given by

$$D(\mathbf{X}(t))|_{S=0} = \frac{1 - \phi}{2} K_D e^{-r(T_D - t)} \quad (6.63)$$

for $t \in [0, T_D)$.

Based on (6.63) for an underlying option, the boundary values at $S_{min} = 0$ are calculated using the equations below.

1. Black–Scholes explicit:

$$D_0^{n-1} = \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n+1)} \quad (6.64)$$

2. Heston explicit:

$$D_{0,j}^{n-1} = \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n+1)} \quad \text{for } j = 0, 1, \dots, N_v \quad (6.65)$$

3. HHW/HCIR explicit:

$$D_{0,j,k}^{n-1} = \left(\frac{1-\phi}{2} \right) K_D e^{-r_k\Delta t(N_t-n+1)} \quad \text{for } j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r \quad (6.66)$$

4. Black–Scholes ADI:

$$(g_1)_0^n = -r \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n)} \quad (6.67)$$

5. Heston ADI:

$$(g_1)_{0,j}^n = \left(-\frac{1}{2}r \right) \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n)} \quad \text{for } j = 0, 1, \dots, N_v \quad (6.68)$$

6. HHW/HCIR ADI:

$$(g_1)_{0,j,k}^n = \left(-\frac{1}{2}r_k \right) \left(\frac{1-\phi}{2} \right) K_D e^{-r_k\Delta t(N_t-n)} \quad \text{for } j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r \quad (6.69)$$

6.9.2 Overlying option

At $S = 0$, a call-on-call compound option will be worthless since the value of the underlying option will be worthless. A put-on-put compound option will also be worthless since the overlying strike price will be much less than the value of the underlying option (the discounted underlying strike price in this case).

On the other hand, the price of a put-on-call compound option will be the discounted overlying strike price, while the price of a call-on-put compound option will be the difference between the discounted underlying strike price and the discounted overlying strike price.

The boundary conditions for the overlying option at $S = 0$ are therefore given by

$$M(\mathbf{X}(t))|_{S=0} = \begin{cases} 0 & \text{if } \phi = 1 \text{ and } \omega = 1, \\ K_D e^{-r(T_D-t)} - K_M e^{-r(T_M-t)} & \text{if } \phi = -1 \text{ and } \omega = 1, \\ K_M e^{-r(T_M-t)} & \text{if } \phi = 1 \text{ and } \omega = -1, \\ 0 & \text{if } \phi = -1 \text{ and } \omega = -1, \end{cases} \quad (6.70)$$

for $t \in [0, T_M)$. The conditions above can conveniently be combined into a single expression given by

$$M(\mathbf{X}(t))|_{S=0} = \left(\frac{1-\omega\phi}{2} \right) \left(\frac{1+\omega}{2} \right) K_D e^{-r(T_D-t)} - \omega \left(\frac{1-\omega\phi}{2} \right) K_M e^{-r(T_M-t)} \quad (6.71)$$

for $t \in [0, T_M)$.

Based on (6.71) for an overlying option, the boundary values at $S_{min} = 0$ are calculated using the equations below.

1. Black–Scholes explicit:

$$M_0^{n-1} = \left(\frac{1-\omega\phi}{2} \right) \left(\frac{1+\omega}{2} \right) K_D e^{-r\Delta t(N_t-n+1)} - \omega \left(\frac{1-\omega\phi}{2} \right) K_M e^{-r\Delta t(X_t-n+1)} \quad (6.72)$$

2. Heston explicit:

$$M_{0,j}^{n-1} = \left(\frac{1-\omega\phi}{2} \right) \left(\frac{1+\omega}{2} \right) K_D e^{-r\Delta t(N_t-n+1)} - \omega \left(\frac{1-\omega\phi}{2} \right) K_M e^{-r\Delta t(X_t-n+1)} \quad \text{for } j = 0, 1, \dots, N_v \quad (6.73)$$

3. HHW/HCIR explicit:

$$M_{0,j,k}^{n-1} = \left(\frac{1-\omega\phi}{2} \right) \left(\frac{1+\omega}{2} \right) K_D e^{-r_k \Delta t(N_t-n+1)} - \omega \left(\frac{1-\omega\phi}{2} \right) K_M e^{-r_k \Delta t(X_t-n+1)} \quad \text{for } \begin{matrix} j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r \end{matrix} \quad (6.74)$$

4. Black–Scholes ADI:

$$(g_1)_0^n = -r \left(\left(\frac{1-\omega\phi}{2} \right) \left(\frac{1+\omega}{2} \right) K_D e^{-r\Delta t(N_t-n)} - \omega \left(\frac{1-\omega\phi}{2} \right) K_M e^{-r\Delta t(X_t-n)} \right) \quad (6.75)$$

5. Heston ADI:

$$(g_1)_{0,j}^n = \left(-\frac{1}{2}r\right) \left(\left(\frac{1-\omega\phi}{2}\right) \left(\frac{1+\omega}{2}\right) K_D e^{-r\Delta t(N_t-n)} - \omega \left(\frac{1-\omega\phi}{2}\right) K_M e^{-r\Delta t(X_t-n)} \right) \quad \text{for } j = 0, 1, \dots, N_v \quad (6.76)$$

6. HHW/HCIR ADI:

$$(g_1)_{0,j,k}^n = \left(-\frac{1}{2}r_k\right) \left(\left(\frac{1-\omega\phi}{2}\right) \left(\frac{1+\omega}{2}\right) K_D e^{-r_k\Delta t(N_t-n)} - \omega \left(\frac{1-\omega\phi}{2}\right) K_M e^{-r_k\Delta t(X_t-n)} \right) \quad \text{for } \begin{matrix} j = 0, 1, \dots, N_v; \\ k = 0, 1, \dots, N_r \end{matrix} \quad (6.77)$$

6.10 Boundary conditions at the maximum stock price

6.10.1 Underlying option

When $S \rightarrow \infty$, an underlying call option will certainly be exercised since the value of the underlying call option converges to its payoff value. Figure A.1 in Appendix A shows that the call option's delta approaches 1 (or $e^{-q(T_D-t)}$ for a dividend-paying stock) as $S \rightarrow \infty$ for all values of v , r and t .

Conversely, an underlying put option will remain worthless for any stock price above the underlying strike price. Figure A.1 shows that the put option's delta approaches 0 as $S \rightarrow \infty$ for all values of v , r and t .

The boundary condition becomes

$$\lim_{S \rightarrow \infty} \frac{\partial D(\mathbf{X}(t))}{\partial S} = \frac{1+\phi}{2} e^{-q(T_D-t)} \quad (6.78)$$

for $t \in [0, T_D)$.

Based on (6.78) for an underlying option, the boundary values at S_{max} are calculated using the equations below.

1. Black-Scholes explicit:

$$D_{N_S}^{n-1} = D_{N_S-1,j,k}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1+\phi}{2}\right) e^{-q\Delta t(N_t-n+1)} \quad (6.79)$$

2. Heston explicit:

$$D_{N_S,j}^{n-1} = D_{N_S-1,j}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1+\phi}{2}\right) e^{-q\Delta t(N_t-n+1)} \quad \text{for } j = 0, 1, \dots, N_v - 1 \quad (6.80)$$

3. HHW/HCIR explicit:

$$D_{N_S, j, k}^{n-1} = D_{N_S-1, j, k}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1+\phi}{2} \right) e^{-q\Delta t(N_t-n+1)} \quad \text{for } \begin{matrix} j = 0, 1, \dots, N_v - 1; \\ k = 1, 2, \dots, N_r - 1 \end{matrix} \quad (6.81)$$

4. Black–Scholes ADI:

$$(g_1)_{N_S}^n = (r - q) \left(\frac{1+\phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t-n)} \quad (6.82)$$

5. Heston ADI:

$$(g_1)_{N_S, j}^n = (r - q) \left(\frac{1+\phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t-n)} \quad \text{for } j = 0, 1, \dots, N_v - 1 \quad (6.83)$$

6. HHW/HCIR ADI:

$$(g_1)_{N_S, j, k}^n = (r_k - q) \left(\frac{1+\phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t-n)} \quad \text{for } \begin{matrix} j = 0, 1, \dots, N_v - 1; \\ k = 0, 1, \dots, N_r \end{matrix} \quad (6.84)$$

6.10.2 Overlying option

A call-on-call compound option will certainly be exercised when $S \rightarrow \infty$ since the value of the underlying call option will converge to its payoff value and, therefore, the value of the compound option will converge to its payoff value. Figure A.4 in Appendix A shows that a call-on-call option's delta approaches 1 (or $e^{-q(T_D-t)}$ for a dividend-paying stock) as $S \rightarrow \infty$ for all values of v , r and t .

Figures A.4 and Figure A.5 in Appendix A show that the delta of a call-on-put, put-on-call and put-on-put option approaches 0 as $S \rightarrow \infty$ for all values of v , r and t .

The boundary conditions for the overlying option at $S \rightarrow \infty$ are therefore given by

$$\lim_{S \rightarrow \infty} \frac{\partial M(\mathbf{X}(t))}{\partial S} = \begin{cases} e^{-q(T_D-t)} & \text{if } \phi = 1 \text{ and } \omega = 1, \\ 0 & \text{if } \phi = -1 \text{ and } \omega = 1, \\ 0 & \text{if } \phi = 1 \text{ and } \omega = -1, \\ 0 & \text{if } \phi = -1 \text{ and } \omega = -1, \end{cases} \quad (6.85)$$

which can conveniently be combined into a single expression given by

$$\lim_{S \rightarrow \infty} \frac{\partial M(\mathbf{X}(t))}{\partial S} = \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) e^{-q(T_D - t)} \quad (6.86)$$

for $t \in [0, T_M)$.

Based on (6.86) for an overlying option, the boundary values at S_{max} are calculated using the equations below.

1. Black–Scholes explicit:

$$M_{N_S}^{n-1} = M_{N_S-1}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) e^{-q\Delta t(N_t - n + 1)} \quad (6.87)$$

2. Heston explicit:

$$M_{N_S, j}^{n-1} = M_{N_S-1, j}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) e^{-q\Delta t(N_t - n + 1)} \\ \text{for } j = 0, 1, \dots, N_v - 1 \quad (6.88)$$

3. HHW/HCIR explicit:

$$M_{N_S, j, k}^{n-1} = M_{N_S-1, j, k}^{n-1} + (S_{N_S} - S_{N_S-1}) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) e^{-q\Delta t(N_t - n + 1)} \\ \text{for } j = 0, 1, \dots, N_v - 1; \\ k = 1, 2, \dots, N_r - 1 \quad (6.89)$$

4. Black–Scholes ADI:

$$(g_1)_{N_S}^n = (r - q) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t - n)} \quad (6.90)$$

5. Heston ADI:

$$(g_1)_{N_S, j}^n = (r - q) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t - n)} \quad \text{for } j = 0, 1, \dots, N_v - 1 \\ (6.91)$$

6. HHW/HCIR ADI:

$$(g_1)_{N_S, j, k}^n = (r_k - q) \left(\frac{1 + \phi\omega}{2} \right) \left(\frac{1 + \phi}{2} \right) S_{N_S} e^{-q\Delta t(N_t - n)} \quad \text{for } j = 0, 1, \dots, N_v - 1; \\ k = 0, 1, \dots, N_r \\ (6.92)$$

6.11 Boundary conditions at minimum stock price variance

6.11.1 Underlying option

Heston [4] proposes that the solution to the two-state variable pricing PDE (3.9) satisfies an inhomogeneous first-order PDE when $v = 0$, given by

$$0 = \frac{\partial D}{\partial t} + (r - q)S \frac{\partial D}{\partial S} + \kappa_v \theta_v \frac{\partial D}{\partial v} - rD \quad (6.93)$$

for $t \in [0, T_D)$. Similarly, the solution to the three-state variable pricing PDE (3.21) satisfies an inhomogeneous second-order PDE when $v = 0$, given by

$$0 = \frac{\partial D}{\partial t} + (r - q)S \frac{\partial D}{\partial S} + \kappa_v \theta_v \frac{\partial D}{\partial v} + (\kappa_r(\theta_r - r) - \lambda_r(t, r)) \frac{\partial D}{\partial r} + \frac{1}{2} \sigma_r^2 r^{2p} \frac{\partial^2 D}{\partial r^2} - rD \quad (6.94)$$

for $t \in [0, T_D)$.

The boundary values at $v_{min} = 0$ are calculated using (6.38) for the two-state variable model and (6.40) for the three-state-variable models and setting $v = 0$.

6.11.2 Overlying option

At $v = 0$, it is assumed that the same boundary conditions that apply to the underlying option also apply to the overlying option. That is

$$0 = \frac{\partial M}{\partial t} + (r - q)S \frac{\partial M}{\partial S} + \kappa_v \theta_v \frac{\partial M}{\partial v} - rD \quad (6.95)$$

for $t \in [0, T_M)$ for the two-state variable PDE (3.9) and

$$0 = \frac{\partial M}{\partial t} + (r - q)S \frac{\partial M}{\partial S} + \kappa_v \theta_v \frac{\partial M}{\partial v} + (\kappa_r(\theta_r - r) - \lambda_r(t, r)) \frac{\partial M}{\partial r} + \frac{1}{2} \sigma_r^2 r^{2p} \frac{\partial^2 M}{\partial r^2} - rD \quad (6.96)$$

for $t \in [0, T_M)$ for the three-state variable PDE (3.21).

The boundary values at $v_{min} = 0$ are calculated using (6.38) for the two-state variable model and (6.40) for the three-state-variable models and setting $v = 0$.

6.12 Boundary conditions at the maximum stock price variance

6.12.1 Underlying option

When $v \rightarrow \infty$, extreme jumps in the stock price become likely. Figure A.2 in Appendix A shows that underlying put and call options always have a positive vega, so the price of an underlying option increases as the variance increases. However, the price of an underlying option will not increase indefinitely and will be bounded from above.

An underlying call option will be, at most, the value of the stock price discounted by the dividend rate. On the other hand, an underlying put option will be, at most, the value of the discounted strike price.

One can combine the boundary conditions for an underlying call and put option as follows:

$$\lim_{v \rightarrow \infty} D(\mathbf{X}(t)) = \frac{1+\phi}{2} S e^{-q(T_D-t)} + \frac{1-\phi}{2} K_D e^{-r(T_D-t)} \quad (6.97)$$

for $t \in [0, T_D)$.

Based on (6.97) for an underlying option, the boundary values at v_{max} are calculated using the equations below.

1. Heston explicit:

$$D_{i,N_v}^{n-1} = \left(\frac{1+\phi}{2} \right) S_i e^{-q\Delta t(N_t-n+1)} + \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n+1)} \quad \text{for } i = 1, 2, \dots, N_S \quad (6.98)$$

2. HHW/HCIR explicit:

$$D_{i,N_v,k}^{n-1} = \left(\frac{1+\phi}{2} \right) S_i e^{-q\Delta t(N_t-n+1)} + \left(\frac{1-\phi}{2} \right) K_D e^{-r_k\Delta t(N_t-n+1)} \quad \text{for } \begin{matrix} i = 1, 2, \dots, N_S; \\ k = 0, 1, \dots, N_r \end{matrix} \quad (6.99)$$

3. Heston ADI:

$$(g_2)_{i,N_v}^n = \left(-\frac{1}{2}r \right) \left(\left(\frac{1+\phi}{2} \right) S_i e^{-q\Delta t(N_t-n)} + \left(\frac{1-\phi}{2} \right) K_D e^{-r\Delta t(N_t-n)} \right) \quad \text{for } i = 1, 2, \dots, N_S \quad (6.100)$$

4. HHW/HCIR ADI:

$$(g_2)_{i,N_v,k}^n = \left(-\frac{1}{2}r_k\right) \left(\left(\frac{1+\phi}{2}\right) S_i e^{-q\Delta t(N_t-n)} + \left(\frac{1-\phi}{2}\right) K_D e^{-r_k\Delta t(N_t-n)} \right) \\ \text{for } \begin{matrix} i = 1, 2, \dots, N_S; \\ k = 0, 1, \dots, N_r \end{matrix} \quad (6.101)$$

6.12.2 Overlying option

Again, when $v \rightarrow \infty$, extreme jumps in the stock price becomes likely. Figures A.6 and A.7 in Appendix A show that the vega of compound options approaches zero as the stock price volatility increases. The consequence is that the value of a compound option converges to an upper limit as the variance increases indefinitely.

The value of a call-on-call compound option will be, at most, the value of the stock price discounted by the dividend rate. The value of a put-on-call compound option will be the discounted overlying strike price. The value of a call-on-put compound option will be the difference between the discounted underlying strike price and the discounted overlying strike price. A put-on-put compound option will be worthless.

The boundary conditions for the overlying option at $v \rightarrow \infty$ are given by

$$\lim_{v \rightarrow \infty} M(\mathbf{X}(t)) = \begin{cases} S e^{-q(T_D-t)} & \text{if } \phi = 1 \text{ and } \omega = 1, \\ K_D e^{-r(T_D-t)} - K_M e^{-r(T_M-t)} & \text{if } \phi = -1 \text{ and } \omega = 1, \\ K_M e^{-r(T_M-t)} & \text{if } \phi = 1 \text{ and } \omega = -1, \\ 0 & \text{if } \phi = -1 \text{ and } \omega = -1, \end{cases} \quad (6.102)$$

for $t \in [0, T_M)$. The conditions above can conveniently be combined into a single expression given by

$$\lim_{v \rightarrow \infty} M(\mathbf{X}(t)) = \left(\frac{1+\phi}{2}\right) \left(\frac{1+\omega}{2}\right) S e^{-q(T_D-t)} + \left(\frac{1-\omega\phi}{2}\right) \left(\frac{1+\omega}{2}\right) K_D e^{-r(T_D-t)} \\ - \omega \left(\frac{1-\omega\phi}{2}\right) K_M e^{-r(T_M-t)} \quad (6.103)$$

for $t \in [0, T_M)$.

Based on (6.103) for an overlying option, the boundary values at v_{max} are calculated using the equations below.

1. Heston explicit:

$$M_{i,N_v}^{n-1} = \left(\frac{1+\phi}{2}\right)\left(\frac{1+\omega}{2}\right)S_i e^{-q\Delta t(N_t-n+1)} + \left(\frac{1-\omega\phi}{2}\right)\left(\frac{1+\omega}{2}\right)K_D e^{-r\Delta t(N_t-n+1)} - \omega\left(\frac{1-\omega\phi}{2}\right)K_M e^{-r\Delta t(X_t-n+1)} \quad \text{for } i = 1, 2, \dots, N_S \quad (6.104)$$

2. HHW/HCIR explicit:

$$M_{i,N_v,k}^{n-1} = \left(\frac{1+\phi}{2}\right)\left(\frac{1+\omega}{2}\right)S_i e^{-q\Delta t(N_t-n+1)} + \left(\frac{1-\omega\phi}{2}\right)\left(\frac{1+\omega}{2}\right)K_D e^{-r_k\Delta t(N_t-n+1)} - \omega\left(\frac{1-\omega\phi}{2}\right)K_M e^{-r_k\Delta t(X_t-n+1)} \quad \text{for } i = 1, 2, \dots, N_S; \quad k = 0, 1, \dots, N_r \quad (6.105)$$

3. Heston ADI:

$$(g_2)_{i,N_v}^n = \left(-\frac{1}{2}r\right)\left(\left(\frac{1+\phi}{2}\right)\left(\frac{1+\omega}{2}\right)S_i e^{-q\Delta t(N_t-n)} + \left(\frac{1-\omega\phi}{2}\right)\left(\frac{1+\omega}{2}\right)K_D e^{-r\Delta t(N_t-n)} - \omega\left(\frac{1-\omega\phi}{2}\right)K_M e^{-r\Delta t(X_t-n)}\right) \quad \text{for } i = 1, 2, \dots, N_S \quad (6.106)$$

4. HHW/HCIR ADI:

$$(g_2)_{i,N_v,k}^n = \left(-\frac{1}{2}r_k\right)\left(\left(\frac{1+\phi}{2}\right)\left(\frac{1+\omega}{2}\right)S_i e^{-q\Delta t(N_t-n)} + \left(\frac{1-\omega\phi}{2}\right)\left(\frac{1+\omega}{2}\right)K_D e^{-r_k\Delta t(N_t-n)} - \omega\left(\frac{1-\omega\phi}{2}\right)K_M e^{-r_k\Delta t(X_t-n)}\right) \quad \text{for } i = 1, 2, \dots, N_S; \quad k = 0, 1, \dots, N_r \quad (6.107)$$

6.13 Boundary conditions at the minimum interest rate

6.13.1 Underlying option

Figure A.3 in Appendix A shows that the rho of an underlying call option approaches zero as the interest rate approaches very large negative values. The rho of an underlying put option, on the other hand, decreases proportionally to the discounted underlying strike price and time to expiration.

The boundary condition as $r \rightarrow -\infty$ is given by the Neumann boundary condition

$$\lim_{r \rightarrow -\infty} \frac{\partial D(\mathbf{X}(t))}{\partial r} = \left(\frac{\phi - 1}{2} \right) K_D (T_D - t) e^{-r(T_D - t)} \quad (6.108)$$

for $t \in [0, T_D)$.

Based on (6.108) for an underlying option, the boundary values at r_{min} are calculated using the equations below.

1. HHW/HCIR explicit:

$$D_{i,j,0}^{n-1} = D_{i,j,1}^{n-1} + (r_0 - r_1) \left(\frac{\phi - 1}{2} \right) K_D \Delta t (N_t - n + 1) e^{-r_0 \Delta t (N_t - n + 1)} / 100$$

for $i = 1, 2, \dots, N_S;$
 $j = 0, 1, \dots, N_v - 1$ (6.109)

2. HHW/HCIR ADI:

$$(g_3)_{i,j,0}^n = (\kappa_r(\theta_r - r_0) - \lambda_r(t, r_0)) \left(\frac{\phi - 1}{2} \right) K_D \Delta t (N_t - n) e^{-r_0 \Delta t (N_t - n)} / 100$$

for $i = 0, 1, \dots, N_S;$
 $j = 0, 1, \dots, N_v$ (6.110)

6.13.2 Overlying option

Figures A.8 and A.9 in Appendix A shows that the rho of a call-on-call and put-on-put option approaches zero as the interest rate approaches very large negative values. The rho of a put-on-call option decreases proportionally to the product of the discounted underlying strike price and time to expiration. The rho of a call-on-put option decreases proportionally to the product of the discounted overlying strike price and time to expiration, subtracted from the product of the discounted underlying strike price.

The boundary condition for the overlying option as $r \rightarrow -\infty$ is given by the Neumann boundary condition

$$\lim_{r \rightarrow -\infty} \frac{\partial M(\mathbf{X}(t))}{\partial r} = \begin{cases} 0 & \text{if } \phi = 1 \text{ and } \omega = 1, \\ -K_D (T_D - t) e^{-r(T_D - t)} + K_M (T_M - t) e^{-r(T_M - t)} & \text{if } \phi = -1 \text{ and } \omega = 1, \\ -K_M (T_M - t) e^{-r(T_M - t)} & \text{if } \phi = 1 \text{ and } \omega = -1, \\ 0 & \text{if } \phi = -1 \text{ and } \omega = -1, \end{cases} \quad (6.111)$$

which can conveniently be combined into a single expression given by

$$\lim_{r \rightarrow -\infty} \frac{\partial M(\mathbf{X}(t))}{\partial r} = \left(\frac{\omega\phi - 1}{2} \right) \left(\frac{1 + \omega}{2} \right) K_D(T_D - t) e^{-r(T_D - t)} + \omega \left(\frac{1 - \omega\phi}{2} \right) K_M(T_M - t) e^{-r(T_M - t)} \quad (6.112)$$

for $t \in [0, T_M)$.

Based on (6.112) for an overlying option, the boundary values at r_{min} are calculated using the equations below.

1. HHW/HCIR explicit:

$$M_{i,j,0}^{n-1} = M_{i,j,1}^{n-1} + (r_0 - r_1) \left(\frac{\omega\phi - 1}{2} \right) \times \left(\left(\frac{1 + \omega}{2} \right) K_D \Delta t (N_t - n + 1) e^{-r_0 \Delta t (N_t - n + 1)} - \omega K_M \Delta t (X_t - n + 1) e^{-r_0 \Delta t (X_t - n + 1)} \right) / 100$$

for $i = 1, 2, \dots, N_S;$
 $j = 0, 1, \dots, N_v - 1$ (6.113)

2. HHW/HCIR ADI:

$$(g_3)_{i,j,0}^n = \left(\kappa_r(\theta_r - r_0) - \lambda_r(t, r_0) \right) \left(\frac{\omega\phi - 1}{2} \right) \times \left(\left(\frac{1 + \omega}{2} \right) K_D \Delta t (N_t - n) e^{-r_0 \Delta t (N_t - n)} - \omega K_M \Delta t (X_t - n) e^{-r_0 \Delta t (X_t - n)} \right) / 100$$

for $i = 0, 1, \dots, N_S;$
 $j = 0, 1, \dots, N_v$ (6.114)

6.14 Boundary values at the maximum interest rate

6.14.1 Underlying option

Figure A.3 in Appendix A shows that the rho of an underlying put and call option approaches zero as the interest rate approaches very large positive values. For the three-state variable pricing PDE (3.21), it is assumed that the same limit applies.

The boundary condition as $r \rightarrow \infty$ is given by the Neumann boundary condition

$$\lim_{r \rightarrow \infty} \frac{\partial D(\mathbf{X}(t))}{\partial r} = 0 \quad (6.115)$$

for $t \in [0, T_D)$.

Based on (6.115) for an underlying option, the boundary values at $r = r_{max}$ are calculated using the equations below.

1. HHW/HCIR explicit:

$$D_{i,j,N_r}^{n-1} = D_{i,j,N_r-1}^{n-1} \quad \text{for} \quad \begin{matrix} i = 1, 2, \dots, N_S; \\ j = 0, 1, \dots, N_v - 1 \end{matrix} \quad (6.116)$$

2. HHW/HCIR ADI: No part of g_3 is specified for r_{max} as this boundary condition is accounted for when solving the implicit ADI schemes.

6.14.2 Overlying option

Figures A.8 and A.9 in Appendix A show that the rho of all four compound option types approaches zero as the interest rate approaches very large positive values. For the three-state variable pricing PDE (3.21), it is assumed that the same limit applies.

The boundary condition for the overlying option as $r \rightarrow \infty$ is given by the Neumann boundary condition

$$\lim_{r \rightarrow \infty} \frac{\partial M(\mathbf{X}(t))}{\partial r} = 0 \quad (6.117)$$

for $t \in [0, T_M)$.

Based on (6.117) for an overlying option, the boundary values at r_{max} are calculated using the equations below.

1. HHW/HCIR explicit:

$$M_{i,j,N_r}^{n-1} = M_{i,j,N_r-1}^{n-1} \quad \text{for} \quad \begin{matrix} i = 1, 2, \dots, N_S; \\ j = 0, 1, \dots, N_v - 1 \end{matrix} \quad (6.118)$$

2. HHW/HCIR ADI: No part of g_3 is specified for r_{max} as this boundary condition is accounted for when solving the implicit ADI schemes.

6.15 Approximations for the first- and second-order derivatives

6.15.1 Derivatives with respect to the stock price

The first-order derivative approximation of F with respect to S_i for $i = 1, 2, \dots, N_S - 1$ is given by [8]

$$\frac{\partial F(t_n; S_i)}{\partial S} = \beta_{-1}F(t_n; S_{i-1}) + \beta_0F(t_n; S_i) + \beta_1F(t_n; S_{i+1}) \quad (6.119)$$

where

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta S_{i+1}}{\Delta S_i(\Delta S_i + \Delta S_{i+1})}, \\ \beta_0 &= \frac{\Delta S_{i+1} - \Delta S_i}{\Delta S_i \Delta S_{i+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta S_i}{\Delta S_{i+1}(\Delta S_i + \Delta S_{i+1})}.\end{aligned}$$

The second-order derivative approximation of F with respect to S_i for $i = 1, 2, \dots, N_S - 1$ is given by [8]

$$\frac{\partial^2 F(t_n; S_i)}{\partial S^2} = \delta_{-1}F(t_n; S_{i-1}) + \delta_0F(t_n; S_i) + \delta_1F(t_n; S_{i+1}) \quad (6.120)$$

where

$$\begin{aligned}\delta_{-1} &= \frac{2}{\Delta S_i(\Delta S_i + \Delta S_{i+1})}, \\ \delta_0 &= \frac{-2}{\Delta S_i \Delta S_{i+1}}, \text{ and} \\ \delta_1 &= \frac{2}{\Delta S_{i+1}(\Delta S_i + \Delta S_{i+1})}.\end{aligned}$$

6.15.2 Derivatives with respect to the stock price variance

The first-order derivative approximation of F with respect to v_j for $j = 0$ is given by [8]

$$\frac{\partial F(t_n; v_j)}{\partial v} = \gamma_0F(t_n; v_j) + \gamma_1F(t_n; v_{j+1}) + \gamma_2F(t_n; v_{j+2}) \quad (6.121)$$

where

$$\begin{aligned}\gamma_0 &= \frac{-2\Delta v_{j+1} - \Delta v_{j+2}}{\Delta v_{j+1}(\Delta v_{j+1} + \Delta v_{j+2})}, \\ \gamma_1 &= \frac{\Delta v_{j+1} + \Delta v_{j+2}}{\Delta v_{j+1} \Delta v_{j+2}}, \text{ and} \\ \gamma_2 &= \frac{-\Delta v_{j+1}}{\Delta v_{j+2}(\Delta v_{j+1} + \Delta v_{j+2})}.\end{aligned}$$

The first-order derivative approximation of F with respect to v_j for $j > 0$ and all $v_j < 1$ is given by [8]

$$\frac{\partial F(t_n; v_j)}{\partial v} = \beta_{-1}F(t_n; v_{j-1}) + \beta_0F(t_n; v_j) + \beta_1F(t_n; v_{j+1}) \quad (6.122)$$

where

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta v_{j+1}}{\Delta v_j(\Delta v_j + \Delta v_{j+1})}, \\ \beta_0 &= \frac{\Delta v_{j+1} - \Delta v_j}{\Delta v_j \Delta v_{j+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta v_j}{\Delta v_{j+1}(\Delta v_j + \Delta v_{j+1})}.\end{aligned}$$

The first-order derivative approximation of F with respect to v_j for $j = 1, 2, \dots, N_v - 2$ and for all $v_j \geq 1$ is given by [8]

$$\frac{\partial F(t_n; v_j)}{\partial v} = \alpha_{-2}F(t_n; v_{j-2}) + \alpha_{-1}F(t_n; v_{j-1}) + \alpha_0F(t_n; v_j) \quad (6.123)$$

where

$$\begin{aligned}\alpha_{-2} &= \frac{\Delta v_j}{\Delta v_{j-1}(\Delta v_{j-1} + \Delta v_j)}, \\ \alpha_{-1} &= \frac{-\Delta v_{j-1} - \Delta v_j}{\Delta v_{j-1} \Delta v_j}, \text{ and} \\ \alpha_0 &= \frac{\Delta v_{j-1} + 2\Delta v_j}{\Delta v_j(\Delta v_{j-1} + \Delta v_j)}.\end{aligned}$$

The second-order derivative approximation of F with respect to v_j for $j = 1, 2, \dots, N_v - 1$ is given by [8]

$$\frac{\partial^2 F(t_n; v_j)}{\partial v^2} = \delta_{-1}F(t_n; v_{j-1}) + \delta_0F(t_n; v_j) + \delta_1F(t_n; v_{j+1}) \quad (6.124)$$

where

$$\begin{aligned}\delta_{-1} &= \frac{2}{\Delta v_j(\Delta v_j + \Delta v_{j+1})}, \\ \delta_0 &= \frac{-2}{\Delta v_j \Delta v_{j+1}}, \text{ and} \\ \delta_1 &= \frac{2}{\Delta v_{j+1}(\Delta v_j + \Delta v_{j+1})}.\end{aligned}$$

6.15.3 Derivatives with respect to the interest rate

The first-order derivative approximation of F with respect to r_k for $k = 1, 2, \dots, N_r - 1$ is given by [8]

$$\frac{\partial F(t_n; r_k)}{\partial r} = \beta_{-1}F(t_n; r_{k-1}) + \beta_0F(t_n; r_k) + \beta_1F(t_n; r_{k+1}) \quad (6.125)$$

where

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta r_{k+1}}{\Delta r_k(\Delta r_k + \Delta r_{k+1})}, \\ \beta_0 &= \frac{\Delta r_{k+1} - \Delta r_k}{\Delta r_k \Delta r_{k+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta r_k}{\Delta r_{k+1}(\Delta r_k + \Delta r_{k+1})}.\end{aligned}$$

The second-order derivative approximation of F with respect to r_k for $i = 1, 2, \dots, N_r - 1$ is given by [8]

$$\frac{\partial^2 F(t_n; r_k)}{\partial r^2} = \delta_{-1} F(t_n; r_{k-1}) + \delta_0 F(t_n; r_k) + \delta_1 F(t_n; r_{k+1}) \quad (6.126)$$

where

$$\begin{aligned}\delta_{-1} &= \frac{2}{\Delta r_k(\Delta r_k + \Delta r_{k+1})}, \\ \delta_0 &= \frac{-2}{\Delta r_k \Delta r_{k+1}}, \text{ and} \\ \delta_1 &= \frac{2}{\Delta r_{k+1}(\Delta r_k + \Delta r_{k+1})}.\end{aligned}$$

6.15.4 Mixed derivative terms

The second-order mixed derivative approximation of F with respect to S_i and v_j for $i = 1, 2, \dots, N_S - 1$ and $j = 1, 2, \dots, N_v - 1$ is given by

$$\frac{\partial^2 F(t_n; S_i, v_j)}{\partial S \partial v} = \beta_{-1} \left(\frac{\partial F(t_n; S_i, v_{j-1})}{\partial S} \right) + \beta_0 \left(\frac{\partial F(t_n; S_i, v_j)}{\partial S} \right) + \beta_1 \left(\frac{\partial F(t_n; S_i, v_{j+1})}{\partial S} \right) \quad (6.127)$$

where the derivative approximations of F with respect to S are calculated using (6.119) and

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta v_{j+1}}{\Delta v_j(\Delta v_j + \Delta v_{j+1})}, \\ \beta_0 &= \frac{\Delta v_{j+1} - \Delta v_j}{\Delta v_j \Delta v_{j+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta v_j}{\Delta v_{j+1}(\Delta v_j + \Delta v_{j+1})}.\end{aligned}$$

The second-order mixed derivative approximation of F with respect to S_i and r_k for $i = 1, 2, \dots, N_S - 1$ and $k = 1, 2, \dots, N_r - 1$ is given by

$$\frac{\partial^2 F(t_n; S_i, r_k)}{\partial S \partial r} = \beta_{-1} \left(\frac{\partial F(t_n; S_i, r_{k-1})}{\partial S} \right) + \beta_0 \left(\frac{\partial F(t_n; S_i, r_k)}{\partial S} \right) + \beta_1 \left(\frac{\partial F(t_n; S_i, r_{k+1})}{\partial S} \right) \quad (6.128)$$

where the derivative approximations of F with respect to S are calculated using (6.119) and

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta r_{k+1}}{\Delta r_k(\Delta r_k + \Delta r_{k+1})}, \\ \beta_0 &= \frac{\Delta r_{k+1} - \Delta r_k}{\Delta r_k \Delta r_{k+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta r_k}{\Delta r_{k+1}(\Delta r_k + \Delta r_{k+1})}.\end{aligned}$$

The second-order mixed derivative approximation of F with respect to v_j and r_k for $j = 1, 2, \dots, N_v - 1$ and $k = 1, 2, \dots, N_r - 1$ is given by

$$\frac{\partial^2 F(t_n; v_j, r_k)}{\partial v \partial r} = \beta_{-1} \left(\frac{\partial F(t_n; v_j, r_{k-1})}{\partial v} \right) + \beta_0 \left(\frac{\partial F(t_n; v_j, r_k)}{\partial v} \right) + \beta_1 \left(\frac{\partial F(t_n; v_j, r_{k+1})}{\partial v} \right) \quad (6.129)$$

where the derivative approximations of F with respect to v are calculated using (6.122) and

$$\begin{aligned}\beta_{-1} &= \frac{-\Delta r_{k+1}}{\Delta r_k(\Delta r_k + \Delta r_{k+1})}, \\ \beta_0 &= \frac{\Delta r_{k+1} - \Delta r_k}{\Delta r_k \Delta r_{k+1}}, \text{ and} \\ \beta_1 &= \frac{\Delta r_k}{\Delta r_{k+1}(\Delta r_k + \Delta r_{k+1})}.\end{aligned}$$

6.16 Early exercise opportunities

For an American underlying option where early exercise opportunities are allowed, the value of each node in D is compared with the option's intrinsic value.

- For both the Black–Scholes explicit and implicit schemes, if $D_i^n < \phi(S_i - K_D)$, then D_i^n is set equal to $\phi(S_i - K_D)$.
- For both the Heston explicit and implicit schemes, if $D_{i,j}^n < \phi(S_i - K_D)$, then $D_{i,j}^n$ is set equal to $\phi(S_i - K_D)$.
- For both the HHW/H CIR explicit and implicit schemes, if $D_{i,j,k}^n < \phi(S_i - K_D)$, then $D_{i,j,k}^n$ is set equal to $\phi(S_i - K_D)$.

Similarly, for an American overlying option where early exercise opportunities are allowed, the value of each node in M is compared with the option's intrinsic value.

- For both the Black–Scholes explicit and implicit schemes, if $M_i^n < \omega(D_i^n - K_M)$, then M_i^n is set equal to $\omega(D_i^n - K_M)$.

- For both the Heston explicit and implicit schemes, if $M_{i,j}^n < \omega(D_{i,j}^n - K_M)$, then $M_{i,j}^n$ is set equal to $\omega(D_{i,j}^n - K_M)$.
- For both the HHW/H CIR explicit and implicit schemes, if $M_{i,j,k}^n < \omega(D_{i,j,k}^n - K_M)$, then $M_{i,j,k}^n$ is set equal to $\omega(D_{i,j,k}^n - K_M)$.

6.17 Grid interpolation

Once all the values of M have been calculated through backward iteration, the value of M that corresponds with

- S_0 and T_0 is taken as the present value of the compound option in the case of the Black–Scholes model,
- S_0, v_0 and T_0 is taken as the present value of the compound option in the case of the Heston model, and
- S_0, v_0, r_0 and T_0 is taken as the present value of the compound option in the case of the HHW/H CIR models.

This is achieved in Python programming using the `RegularGridInterpolator` function from the `SciPy` library.

7 Results

The source code to produce the results discussed in this section was implemented using Python 3.8 on a desktop computer running Windows 11 with an Intel Core i7 9700 K 4.90 GHz, 12 M cache processor and 32 GB 3600 MHz DDR4 RAM.

7.1 Selecting an optimal grid distributions

The point of departure for this section was to establish suitable stock price, variance and interest rate grid sizes and non-uniform grid generation parameters for the six finite difference methods that would produce sufficiently accurate results while being computationally efficient.

The reference values against which the accuracies were measured were the closed-form solutions for each model or the outcome of a sufficiently accurate Monte Carlo simulation run wherever the closed-form solution does not exist. A European put option and a European put-on-put option with model parameters listed in Table 7.1 were considered for these assessments. The underlying option was first considered in isolation for each dimension, followed by the overlying option. These assessment results were then used in subsequent sections to calculate reference compound option values.

General parameters		Heston parameters		HHW parameters	
Parameter	Value	Parameter	Value	Parameter	Value
q	0.005	κ_v	1.50	κ_r	0.30
S_0	100	θ_v	0.02	θ_r	0.04
v_0	0.04	σ_v	0.15	σ_r	0.10
r_0	0.04	λ_v	0.00	λ_r	0.00
T_D	1	$\rho_{S,v}$	-0.50	$\rho_{S,r}$	0.00
K_D	100			$\rho_{v,r}$	0.00
T_M	0.5			p	0
K_M	4				

Table 7.1: Parameters for Table 7.2 to Table 7.7.

Tables 7.2 to 7.7 present the relative errors for the explicit and five ADI finite difference schemes using uniform and non-uniform spatial grids. They also present the relative errors and relative standard errors for the three-state variable simulation approach.

The relative error for the finite difference methods was calculated using

$$E_R = \left| \frac{\hat{u} - u}{u} \right| \quad (7.1)$$

where u represented the true value and $\hat{u}$ represented the approximated value.

The relative error for the simulation method was calculated using

$$E_R = \left| \frac{\bar{u} - u}{u} \right| \quad (7.2)$$

where $\bar{u}$ represented the sample mean of the set of simulation runs. The relative error of the lowest and highest values within the set of simulations was also shown. The relative standard error was calculated using

$$E_{SD} = \frac{SD}{u} \quad (7.3)$$

with SD being the sample standard deviation of the set of simulation runs.

7.1.1 Selecting an optimal stock price grid distribution

The error results for a European underlying put option and a European put-on-put option using the Black–Scholes model are presented in Table 7.2 and

Table 7.3, respectively. This assessment aimed to find suitable values for N_S , N_t , d_1 , and c providing low relative errors at reasonably short execution times. To vary d_1 , the values $\tilde{d}_1$ were varied where $d_1 = K_D/\tilde{d}_1$.

The reference value was obtained using the closed-form solution (4.6) given by

$$D(\mathbf{X}(t), t, T_D, K_D) \Big|_{\phi=-1} = 6.196947.$$

The reference value is obtained using the Geske closed-form solution (4.38) given by

$$M(\mathbf{X}(t), t, T_M, T_D, K_M, K_D) \Big|_{\phi=-1, \omega=-1} = 1.081413.$$

Table 7.2 and Table 7.3 show that all six schemes produced smaller errors when non-uniform grids were used as opposed to uniform grids. At $N_S = 100$, typical relative errors of order $1\text{E}-3$ were observed on non-uniform grids for both the underlying and compound options.

The suitability of $\tilde{d}_1$ and c one chooses depends on the value of S_{max} . At the $N_S = 100$ level, the effects of varying values of $\tilde{d}_1$ and c on a non-uniform stock price grid were investigated further. $S_{min} = 0$ and $S_{max} = 14K_D$ were chosen. At each fixed level of N_t and $\tilde{d}_1$, the relative errors were observed to decrease as c decreased. However, it is important to consider that when $c = 0$, the non-uniform stock price grid will not have a uniform section around K_D . Since an option will not always be at-the-money, preserving the uniform section in the non-uniform grid becomes prudent. From the observed errors, $c = 0.1$ was selected as an appropriate value when moving to the Heston and HHW models.

The non-uniform stock price grid became more compressed around S_{left} and S_{right} as d_1 tended to zero and became less compressed around S_{left} and S_{right} as d_1 increased. The grid resembled a uniform grid as d_1 became infinitely large. This is why the error was observed to increase as $\tilde{d}_1$ decreased (i.e. d_1 increased). From the observed errors, $\tilde{d}_1 = 5$ was selected as an appropriate value when moving to the Heston and HHW models.

The errors decreased as larger values of N_S and N_t were used. Although one could achieve a relative error of order $1\text{E}-4$ over a non-uniform grid when $N_S > 300$, it became impractical to use a grid this fine when solving the Heston and HHW schemes. The execution time increased drastically if the grid array was too large. For example, let $N_S = 2m$ and $N_v = N_r = m$. For $m = 200$, the Heston grid array at each point along the time dimension consisted of $2m^2 = 80,000$ points, while the HHW grid array consisted of $2m^3 = 16,000,000$ points – a large set of points to solve for a standard computer. It was found that $m = 50$ (i.e. $2m^2 = 5,000$ and $2m^3 = 250,000$)

Black–Scholes finite difference with uniform grids									
N_S	N_t	$\tilde{d}_1$	c	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
75	10,000	–	–	0.108292	0.108282	0.108277	0.108277	0.108282	0.108282
100	10,000	–	–	0.011736	0.011753	0.011763	0.011763	0.011753	0.011753
200	10,000	–	–	0.011130	0.011115	0.011106	0.011106	0.011115	0.011115
300	10,000	–	–	0.007864	0.007848	0.007839	0.007839	0.007848	0.007848
400	10,000	–	–	0.004479	0.004463	0.004453	0.004453	0.004463	0.004463
Black–Scholes finite difference with non-uniform grids									
N_S	N_t	$\tilde{d}_1$	c	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
75	1,000	6	0.1	0.001143	0.001302	0.001398	0.001398	0.001302	0.001302
75	10,000	6	0.1	0.001286	0.001302	0.001312	0.001312	0.001302	0.001302
100	1,000	4	0.05	0.000830	0.000671	0.000576	0.000576	0.000671	0.000671
100	1,000	4	0.1	0.000446	0.000287	0.000192	0.000192	0.000287	0.000287
100	1,000	4	0.2	0.000891	0.001051	0.001147	0.001147	0.001051	0.001051
100	1,000	5	0.05	0.000589	0.000749	0.000844	0.000844	0.000749	0.000749
100	1,000	5	0.1	0.000991	0.000832	0.000737	0.000737	0.000832	0.000832
100	1,000	5	0.2	0.001276	0.001117	0.001022	0.001022	0.001117	0.001117
100	1,000	6	0.05	0.000279	0.000438	0.000534	0.000534	0.000438	0.000438
100	1,000	6	0.1	0.000724	0.000565	0.000470	0.000470	0.000565	0.000565
100	1,000	6	0.2	0.000533	0.000374	0.000279	0.000279	0.000374	0.000374
100	5,000	4	0.05	0.000703	0.000671	0.000652	0.000652	0.000671	0.000671
100	5,000	4	0.1	0.000319	0.000287	0.000268	0.000268	0.000287	0.000287
100	5,000	4	0.2	0.001019	0.001051	0.001070	0.001070	0.001051	0.001051
100	5,000	5	0.05	0.000717	0.000749	0.000768	0.000768	0.000749	0.000749
100	5,000	5	0.1	0.000864	0.000832	0.000813	0.000813	0.000832	0.000832
100	5,000	5	0.2	0.001149	0.001117	0.001098	0.001098	0.001117	0.001117
100	5,000	6	0.05	0.000406	0.000438	0.000457	0.000457	0.000438	0.000438
100	5,000	6	0.1	0.000597	0.000565	0.000546	0.000546	0.000565	0.000565
100	5,000	6	0.2	0.000406	0.000374	0.000355	0.000355	0.000374	0.000374
100	10,000	5	0.1	0.000848	0.000832	0.000822	0.000822	0.000832	0.000832
200	1,000	5	0.1	0.000135	0.000024	0.000119	0.000119	0.000024	0.000024
200	1,000	5	0.2	0.000204	0.000045	0.000050	0.000050	0.000045	0.000045
200	10,000	5	0.1	0.000008	0.000024	0.000033	0.000033	0.000024	0.000024
200	10,000	5	0.2	0.000061	0.000045	0.000035	0.000035	0.000045	0.000045
300	10,000	5	0.1	0.000067	0.000051	0.000041	0.000041	0.000051	0.000051
400	10,000	5	0.1	0.000052	0.000036	0.000026	0.000026	0.000036	0.000036
Monte Carlo simulation									
Runs	N_P	N_t		Lower E_R	Average E_R	Upper E_R	E_{SD}		
30	1,000	200		0.070011	0.015933	0.111104	0.046930		
30	10,000	200		0.031899	0.002299	0.033948	0.015398		
30	100,000	200		0.016010	0.000697	0.010212	0.005817		
30	1,000,000	200		0.002379	0.000097	0.002593	0.001389		

Legend: CN: Crank–Nicolson; Do: Douglas; CS: Craig–Sneyd; HV: Hundsdorfer–Verwer; MCS: Modified Craig–Sneyd.

Table 7.2: Relative errors for a European underlying put option using the Black–Scholes model.

Black–Scholes finite difference with uniform grids									
N_S	N_t	$\tilde{d}_1$	c	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
75	10,000	–	–	0.031007	0.031130	0.031131	0.031131	0.031130	0.031130
100	10,000	–	–	0.039591	0.039484	0.039502	0.039502	0.039484	0.039484
200	10,000	–	–	0.013511	0.013683	0.013676	0.013676	0.013683	0.013683
300	10,000	–	–	0.006088	0.005897	0.005902	0.005902	0.005897	0.005897
400	10,000	–	–	0.005917	0.005764	0.005776	0.005776	0.005764	0.005764
Black–Scholes finite difference with non-uniform grids									
N_S	N_t	$\tilde{d}_1$	c	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
75	1,000	6	0.1	0.000920	0.000912	0.000844	0.000844	0.000912	0.000912
75	10,000	6	0.1	0.000136	0.000319	0.000312	0.000312	0.000319	0.000319
100	1,000	4	0.05	0.000632	0.001140	0.001065	0.001065	0.001140	0.001140
100	1,000	4	0.1	0.000555	0.001089	0.000993	0.000993	0.001089	0.001089
100	1,000	4	0.2	0.002223	0.000375	0.000441	0.000441	0.000375	0.000375
100	1,000	5	0.05	0.000012	0.001662	0.001566	0.001566	0.001661	0.001661
100	1,000	5	0.1	0.000951	0.000816	0.000739	0.000739	0.000815	0.000815
100	1,000	5	0.2	0.000551	0.001196	0.001117	0.001117	0.001196	0.001196
100	1,000	6	0.05	0.000258	0.001936	0.001844	0.001844	0.001935	0.001935
100	1,000	6	0.1	0.002131	0.000333	0.000406	0.000406	0.000334	0.000334
100	1,000	6	0.2	0.002303	0.000492	0.000563	0.000563	0.000492	0.000492
100	5,000	4	0.05	0.000261	0.000616	0.000601	0.000601	0.000616	0.000616
100	5,000	4	0.1	0.000235	0.000564	0.000545	0.000545	0.000564	0.000564
100	5,000	4	0.2	0.001272	0.000903	0.000916	0.000916	0.000903	0.000903
100	5,000	5	0.05	0.000806	0.001136	0.001117	0.001117	0.001136	0.001136
100	5,000	5	0.1	0.000063	0.000291	0.000275	0.000275	0.000291	0.000291
100	5,000	5	0.2	0.000323	0.000672	0.000657	0.000657	0.000672	0.000672
100	5,000	6	0.05	0.001075	0.001410	0.001392	0.001392	0.001410	0.001410
100	5,000	6	0.1	0.001219	0.000859	0.000874	0.000874	0.000859	0.000859
100	5,000	6	0.2	0.001380	0.001018	0.001032	0.001032	0.001018	0.001018
100	10,000	5	0.1	0.000048	0.000225	0.000218	0.000218	0.000225	0.000225
200	1,000	5	0.1	0.000641	0.001073	0.000988	0.000988	0.001073	0.001073
200	1,000	5	0.2	0.000787	0.000902	0.000813	0.000813	0.000902	0.000902
200	10,000	5	0.1	0.000311	0.000482	0.000474	0.000474	0.000482	0.000482
200	10,000	5	0.2	0.000142	0.000311	0.000302	0.000302	0.000311	0.000311
300	10,000	5	0.1	0.000050	0.000120	0.000111	0.000111	0.000120	0.000120
400	10,000	5	0.1	0.000217	0.000048	0.000057	0.000057	0.000048	0.000048
Monte Carlo simulation									
Runs	N_p	N_t		Lower E_R	Average E_R	Upper E_R	E_{SD}		
30	1,000	200		0.059649	0.010194	0.080529	0.036209		
30	10,000	200		0.029588	0.001840	0.033797	0.012707		

Legend: CN: Crank–Nicolson; Do: Douglas; CS: Craig–Sneyd; HV: Hundsdorfer–Verwer; MCS: Modified Craig–Sneyd.

Table 7.3: Relative errors for a European put-on-put option using the Black–Scholes model.

provided a suitable balance between a relatively fast execution time and an acceptable error level.

The ADI schemes produced smaller errors on average than the explicit scheme for both underlying and compound options, although there were several cases where this was not observed. The relative errors of the five ADI schemes differed by only a small margin for both uniform and non-uniform grids. The Douglas and Craig–Sneyd schemes achieved smaller errors on average than other ADI schemes. The relative errors between the Crank–Nicolson, modified Craig–Sneyd and Hundsdorfer–Verwer schemes typically differed within the order $1\text{E}-7$.

For the underlying option, the simulation approach had a relative error of $9.7\text{E}-5$ and a relative standard error of $1.4\text{E}-3$ when $N_p = 1,000,000$ was used. For the compound option, the simulation method had a relative error of $1.8\text{E}-3$ and a relative standard error of $1.3\text{E}-2$ when $N_p = 10,000$ was used. Simulation runs with $N_p = 100,000$ and $N_p = 1,000,000$ were excluded for the compound option due to excessively long execution times.

7.1.2 Selecting an optimal variance grid distribution

The error results for a European underlying put option and a European put-on-put option using the Heston model are presented in Table 7.4 and Table 7.5, respectively. This assessment aimed to find suitable values for N_v , N_t and d_2 providing low relative errors at reasonably short execution times. To vary d_2 , the values $\tilde{d}_2$ were varied where $d_2 = V_{\max}/\tilde{d}_2$. The optimal values $N_S = 100$, $\tilde{d}_1 = 500$ and $c = 0.1$ from the preceding section were used.

The reference value for the European put option was obtained using the closed-form solution (4.10) and given by

$$D(\mathbf{X}(t), t, T_D, K_D) \Big|_{\phi=-1} = 5.184925.$$

The reference value for the European put-on-put option was obtained using the simulation approach with $N_p = 10,000$, $N_t = 200$ and 5 basis functions, which is given by

$$M(\mathbf{X}(t), t, T_M, T_D, K_M, K_D) \Big|_{\phi=-1, \omega=-1} = 1.471831.$$

The Heston explicit scheme was often unstable. To a large degree, the error stemmed from the $\frac{1}{2}vS^2\frac{\partial^2 f}{\partial S^2}$ term in (3.9) which produced large values when $v > 1$. This instability occurred less frequently when a non-uniform grid was used. A relative error with the ‘-’ symbol in the proceeding tables indicates the algorithm’s instability.

Heston finite difference with uniform grids								
N_v	N_t	$\bar{d}_2$	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
20	5,000	–	–	0.080747	0.080762	0.080762	0.080748	0.080747
20	10,000	–	0.080735	0.080747	0.080755	0.080755	0.080747	0.080747
30	5,000	–	–	0.038018	0.038036	0.038037	0.038018	0.038018
30	10,000	–	0.038003	0.038018	0.038027	0.038027	0.038018	0.038018
40	10,000	–	–	–	–	–	–	–
50	10,000	–	–	–	–	–	–	–
100	10,000	–	–	–	–	–	–	–
Heston finite difference with non-uniform grids								
N_v	N_t	$\bar{d}_2$	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
20	1,000	600	–	0.001048	0.001153	0.001149	0.001049	0.001049
20	5,000	600	0.001016	0.001049	0.001069	0.001069	0.001049	0.001049
20	10,000	600	0.001032	0.001049	0.001059	0.001059	0.001049	0.001049
30	1,000	600	–	0.000901	0.000796	0.000800	0.000901	0.000901
30	5,000	600	0.000933	0.000901	0.000880	0.000881	0.000901	0.000901
30	10,000	600	0.000917	0.000901	0.000890	0.000891	0.000901	0.000901
40	1,000	600	–	0.000525	0.000421	0.000425	0.000525	0.000525
40	5,000	500	0.000721	0.000689	0.000668	0.000669	0.000689	0.000689
40	5,000	600	0.000558	0.000525	0.000504	0.000505	0.000525	0.000525
40	5,000	700	0.000556	0.000523	0.000502	0.000503	0.000523	0.000523
40	5,000	800	0.000952	0.000919	0.000898	0.000899	0.000919	0.000919
40	10,000	500	0.000705	0.000689	0.000678	0.000679	0.000689	0.000689
40	10,000	600	0.000542	0.000525	0.000515	0.000515	0.000525	0.000525
40	10,000	700	0.000539	0.000523	0.000513	0.000513	0.000523	0.000523
40	10,000	800	0.000936	0.000919	0.000909	0.000909	0.000919	0.000919
50	1,000	600	–	0.000683	0.000579	0.000583	0.000683	0.000683
50	5,000	500	0.001022	0.000989	0.000968	0.000969	0.000989	0.000989
50	5,000	600	0.000716	0.000683	0.000662	0.000663	0.000683	0.000683
50	5,000	700	0.000875	0.000842	0.000821	0.000822	0.000842	0.000842
50	5,000	800	0.000662	0.000630	0.000609	0.000610	0.000630	0.000630
50	10,000	500	0.001005	0.000989	0.000979	0.000979	0.000989	0.000989
50	10,000	600	0.000700	0.000683	0.000673	0.000673	0.000683	0.000683
50	10,000	700	0.000858	0.000842	0.000831	0.000832	0.000842	0.000842
50	10,000	800	0.000646	0.000630	0.000619	0.000620	0.000630	0.000630
100	5,000	600	0.000988	0.000955	0.000934	0.000935	0.000955	0.000955
100	10,000	600	0.000972	0.000955	0.000945	0.000945	0.000955	0.000955
Monte Carlo simulation								
Runs	N_p	N_t	Lower E_R	Average E_R	Upper E_R	E_{SD}		
30	1,000	200	0.094275	0.002817	0.112035	0.053319		
30	10,000	200	0.025649	0.000532	0.035291	0.014538		
30	100,000	200	0.010210	0.000817	0.008658	0.004704		
30	1,000,000	200	0.002500	0.000521	0.002818	0.001235		

Legend: CN: Crank–Nicolson; Do: Douglas; CS: Craig–Sneyd; HV: Hundsdorfer–Verwer;

MCS: Modified Craig–Sneyd.

Table 7.4: Relative errors for a European underlying put option using the Heston model.

Heston finite difference with uniform grids								
N_v	N_t	$\tilde{d}_2$	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
20	5,000	600	–	0.127075	0.127044	0.127039	0.127075	0.127075
20	10,000	600	0.126967	0.127024	0.127009	0.127006	0.127024	0.127024
30	5,000	600	–	0.044668	0.044708	0.044713	0.044668	0.044668
30	10,000	600	0.044804	0.044738	0.044758	0.044761	0.044738	0.044738
40	10,000	600	–	–	–	–	–	–
50	10,000	600	–	–	–	–	–	–
100	10,000	600	–	–	–	–	–	–
Heston finite difference with non-uniform grids								
N_v	N_t	$\tilde{d}_2$	Explicit E_R	CN E_R	Do E_R	CS E_R	MCS E_R	HV E_R
20	1,000	600	–	0.003865	0.003880	0.003815	0.003865	0.003865
20	5,000	600	0.003169	0.003399	0.003402	0.003389	0.003399	0.003399
20	10,000	600	0.003225	0.003341	0.003342	0.003336	0.003341	0.003341
30	1,000	600	–	0.000643	0.000657	0.000592	0.000643	0.000643
30	5,000	600	0.000057	0.000173	0.000175	0.000163	0.000173	0.000173
30	10,000	600	0.000001	0.000114	0.000115	0.000109	0.000114	0.000114
40	1,000	600	–	0.000905	0.000921	0.000857	0.000905	0.000905
40	5,000	500	0.000014	0.000243	0.000246	0.000233	0.000243	0.000243
40	5,000	600	0.000203	0.000435	0.000439	0.000426	0.000435	0.000435
40	5,000	700	0.000180	0.000410	0.000412	0.000399	0.000410	0.000410
40	5,000	800	0.000363	0.000132	0.000129	0.000142	0.000132	0.000132
40	10,000	500	0.000070	0.000184	0.000186	0.000179	0.000184	0.000184
40	10,000	600	0.000261	0.000377	0.000378	0.000372	0.000377	0.000377
40	10,000	700	0.000236	0.000351	0.000352	0.000346	0.000351	0.000351
40	10,000	800	0.000306	0.000191	0.000189	0.000196	0.000191	0.000191
50	1,000	600	–	0.000603	0.000619	0.000554	0.000603	0.000603
50	5,000	500	0.000434	0.000205	0.000202	0.000215	0.000205	0.000205
50	5,000	600	0.000099	0.000133	0.000136	0.000123	0.000133	0.000133
50	5,000	700	0.000322	0.000093	0.000091	0.000104	0.000093	0.000093
50	5,000	800	0.000051	0.000181	0.000185	0.000172	0.000181	0.000181
50	10,000	500	0.000378	0.000264	0.000262	0.000269	0.000264	0.000264
50	10,000	600	0.000042	0.000074	0.000075	0.000069	0.000074	0.000074
50	10,000	700	0.000267	0.000152	0.000151	0.000158	0.000152	0.000152
50	10,000	800	0.000006	0.000122	0.000124	0.000118	0.000122	0.000122
100	5,000	600	0.000560	0.000330	0.000327	0.000340	0.000330	0.000330
100	10,000	600	0.000504	0.000389	0.000387	0.000394	0.000389	0.000389
Monte Carlo simulation								
Runs	N_P	N_t	Lower E_R	Average E_R	Upper E_R	E_{SD}		
30	1,000	200	0.061366	0.001418	0.051340	0.026802		
30	10,000	200	0.023168	0.000505	0.044141	0.013764		

Legend: CN: Crank–Nicolson; Do: Douglas; CS: Craig–Sneyd; HV: Hundsdorfer—Verwer;
MCS: Modified Craig—Sneyd.

Table 7.5: Relative errors for a European put-on-put option using the Heston model.

Table 7.4 and Table 7.5 show that all six schemes produced smaller errors when non-uniform grids were used as opposed to uniform grids. Typical relative errors of order $1\text{E}-3$ were observed on non-uniform grids for both the underlying and compound options.

The suitability of $\tilde{d}_2$ one chooses depends on the value of V_{max} . At the $N_v \in \{40, 50\}$ levels, the effects of varying values of $\tilde{d}_2$ on a non-uniform variance grid was investigated further. $V_{min} = 0$ and $V_{max} = 1.1$ were chosen. The non-uniform variance grid became more compressed towards V_{min} as d_2 tended to zero and stretched toward V_{max} as d_2 increased. The grid resembled a uniform grid as d_2 became infinitely large. This is why the error was observed to increase as $\tilde{d}_2$ decreased (i.e. d_2 increased). From the observed errors, $\tilde{d}_2 = 600$ was selected as an appropriate value when moving to the HHW model since the errors tended to trough at this point.

The ADI schemes produced smaller errors on average than the explicit scheme for both underlying and compound options, although there were several cases where this was not observed. The relative errors of the five ADI schemes differed by only a small margin for both uniform and non-uniform grids. The Douglas and Craig–Sneyd schemes achieved smaller errors on average than other ADI schemes.

For the underlying option, the simulation approach had a relative error of $5.2\text{E}-4$ and a relative standard error of $1.2\text{E}-3$ when $N_p = 1,000,000$ was used. For the compound option, the simulation method had a relative error of $5.1\text{E}-4$ and a relative standard error of $1.4\text{E}-2$ when $N_p = 10,000$ was used. Simulation runs with $N_p = 100,000$ and $N_p = 1,000,000$ were excluded for the compound option due to excessively long execution times.

7.1.3 Selecting an optimal interest rate grid distribution

The error results for a European underlying put option and a European put-on-put option using the HHW model are presented in Table 7.6 and Table 7.7, respectively. This assessment aimed to find suitable values for N_r , N_t and d_3 providing low relative errors at reasonably short execution times. To vary d_3 , the values $\tilde{d}_3$ were varied where $d_3 = R_{max}/\tilde{d}_3$. The optimal values $N_S = 100$, $\tilde{d}_1 = 500$, $c = 0.1$, $N_v = 50$ and $\tilde{d}_2 = 600$ from the preceding sections were used.

The reference value for the European put option was obtained using the closed-form solution (4.20) and given by

$$D(\mathbf{X}(t), t, T_D, K_D) \Big|_{\phi=-1} = 5.523341.$$

The reference value for the European put-on-put option was obtained using the simulation approach with $N_p = 10,000$, $N_t = 200$ and 5 basis functions, which is given by

$$M(\mathbf{X}(t), t, T_M, T_D, K_M, K_D) \Big|_{\phi=-1, \omega=-1} = 1.449435.$$

As observed with the Heston explicit scheme, the HHW explicit scheme was often unstable, which occurred less frequently when a non-uniform grid was used. To a large degree, the error stemmed from the $\frac{1}{2}vS^2\frac{\partial^2 f}{\partial S^2}$ term in (3.21) which produced large values when $v > 1$. A relative error with the ‘-’ symbol in the proceeding tables indicates the algorithm’s instability.

Table 7.6 and Table 7.7 show that all six schemes produced smaller errors when non-uniform grids were used as opposed to uniform grids. Typical relative errors of order $1\text{E}-3$ were observed on non-uniform grids for the underlying option, while typical relative errors of order $1\text{E}-2$ were observed for the compound option. This is an acceptable difference, given that the reference value was an approximation.

The suitability of $\tilde{d}_3$ one chooses depends on the values of R_{min} and R_{max} . At the $N_r \in \{30, 40\}$ levels, the effects of varying values of $\tilde{d}_3$ on a non-uniform interest rate grid was investigated further. $R_{min} = -1$ and $R_{max} = 1$ were chosen. The non-uniform interest rate grid became more compressed towards r_0 from both sides as d_3 tended to zero and stretched outward towards R_{min} and R_{max} as d_3 increased. The grid resembled a uniform grid as d_3 became infinitely large. This is why the error was observed to increase as $\tilde{d}_3$ decreased (i.e. d_3 increased). From the observed errors, $\tilde{d}_3 = 400$ was selected as an appropriate value to generate reference compound option prices with since the errors tended to trough at this point.

The relative errors of the five ADI schemes differed by only a small margin for both uniform and non-uniform grids. The Douglas and Craig–Sneyd schemes achieved smaller errors on average than other ADI schemes.

For the underlying option, the simulation approach had a relative error of $1.5\text{E}-4$ and a relative standard error of $1.7\text{E}-3$ when $N_p = 1,000,000$ was used. For the compound option, the simulation method had a relative error of $2.5\text{E}-3$ and a relative standard error of $1.8\text{E}-2$ when $N_p = 10,000$ was used. Simulation runs with $N_p = 100,000$ and $N_p = 1,000,000$ were excluded for the compound option due to excessively long execution times.

HHW finite difference with uniform grids							
N_r	N_t	$\tilde{d}_3$	Explicit E_R	Do E_R	CS E_R	MCS E_R	HV E_R
30	100	–	–	0.021920	0.021913	0.021571	0.021575
30	10,000	–	–	–	–	–	–
40	100	–	–	0.032143	0.032136	0.031792	0.031795
40	10,000	–	–	–	–	–	–
50	100	–	–	0.038365	0.038358	0.038012	0.038016
HHW finite difference with non-uniform grids							
N_r	N_t	$\tilde{d}_3$	Explicit E_R	Do E_R	CS E_R	MCS E_R	HV E_R
10	100	400	–	0.000090	0.000086	0.000295	0.000290
10	1,000	400	–	0.000251	0.000252	0.000290	0.000290
10	10,000	400	0.000562	0.000286	0.000286	0.000290	0.000290
20	100	300	–	0.000098	0.000097	0.000479	0.000474
20	100	400	–	0.000085	0.000084	0.000466	0.000461
20	100	500	–	0.000076	0.000074	0.000456	0.000451
20	1,000	300	–	0.000435	0.000435	0.000474	0.000474
20	1,000	400	–	0.000423	0.000422	0.000461	0.000461
20	1,000	500	–	0.000413	0.000413	0.000451	0.000451
20	10,000	400	0.000467	0.000457	0.000457	0.000461	0.000461
30	100	400	–	0.000161	0.000159	0.000539	0.000534
30	1,000	300	–	0.000502	0.000502	0.000540	0.000540
30	1,000	400	–	0.000496	0.000495	0.000534	0.000534
30	1,000	500	–	0.000491	0.000491	0.000529	0.000529
30	10,000	400	–	0.000530	0.000530	0.000534	0.000534
40	100	400	–	0.000187	0.000185	0.000564	0.000558
40	1,000	300	–	0.000524	0.000524	0.000562	0.000562
40	1,000	400	–	0.000521	0.000520	0.000559	0.000559
40	1,000	500	–	0.000518	0.000518	0.000556	0.000556
40	10,000	400	–	0.000555	0.000555	0.000559	0.000559
50	100	400	–	0.000199	0.000196	0.000575	0.000570
50	1,000	400	–	0.000532	0.000532	0.000570	0.000570
Monte Carlo simulation							
Runs	N_P	N_t	Lower E_R	Average E_R	Upper E_R	E_{SD}	
30	10,000	200	0.02677	0.001684	0.027149	0.014801	
30	100,000	200	0.015659	0.000187	0.01062	0.005926	
30	1,000,000	200	0.002886	0.000150	0.002647	0.001688	

Legend: Do: Douglas; CS: Craig—Sneyd; HV: Hundsdorfer—Verwer; MCS: Modified Craig—Sneyd.

Table 7.6: Relative errors for a European underlying put option using the HHW model.

HHW finite difference with uniform grids							
N_r	N_t	$\tilde{d}_3$	Explicit E_R	Do E_R	CS E_R	MCS E_R	HV E_R
30	100	400	–	0.013114	0.012780	0.015091	0.015115
30	10,000	400	–	–	–	–	–
40	100	400	–	0.009735	0.009403	0.011627	0.011650
40	10,000	400	–	–	–	–	–
50	100	400	–	0.002598	0.002275	0.004642	0.004666
HHW finite difference with non-uniform grids							
N_r	N_t	$\tilde{d}_3$	Explicit E_R	Do E_R	CS E_R	MCS E_R	HV E_R
10	100	400	–	0.008172	0.007532	0.008802	0.008813
10	1,000	400	–	0.003706	0.003641	0.003769	0.003769
10	10,000	400	0.003039	0.003254	0.003247	0.003260	0.003260
20	100	400	–	0.008222	0.007577	0.008880	0.008892
20	1,000	300	–	0.003805	0.003740	0.003871	0.003871
20	1,000	400	–	0.003803	0.003738	0.003868	0.003868
20	1,000	500	–	0.003775	0.003710	0.003840	0.003840
20	10,000	400	0.003259	0.003355	0.003348	0.003361	0.003361
30	100	300	–	0.008196	0.007550	0.008856	0.008868
30	100	400	–	0.008193	0.007547	0.008853	0.008865
30	100	500	–	0.008206	0.007560	0.008866	0.008878
30	1,000	300	–	0.003782	0.003717	0.003848	0.003848
30	1,000	400	–	0.003779	0.003714	0.003845	0.003845
30	1,000	500	–	0.003791	0.003726	0.003857	0.003857
30	10,000	400	–	0.003332	0.003325	0.003338	0.003338
40	100	300	–	0.008192	0.007546	0.008854	0.008866
40	100	400	–	0.008186	0.007540	0.008850	0.008862
40	100	500	–	0.008197	0.007551	0.008855	0.008868
40	1,000	400	–	0.003776	0.003711	0.003842	0.003843
40	10,000	400	–	0.003329	0.003323	0.003336	0.003336
50	100	400	–	0.008212	0.007566	0.008875	0.008887
50	1,000	400	–	0.003803	0.003738	0.003869	0.003869
Monte Carlo simulation							
Runs	N_p	N_t	Lower E_R	Average E_R	Upper E_R	E_{SD}	
30	1,000	200	0.044711	0.017297	0.057803	0.023559	
30	10,000	200	0.017758	0.002479	0.036393	0.018254	

Legend: Do: Douglas; CS: Craig—Sneyd; HV: Hundsdorfer—Verwer; MCS: Modified Craig—Sneyd.

Table 7.7: Relative errors for a European put-on-put option using the HHW model.

7.2 Execution times

In this section, the execution times of the six finite difference algorithms are compared with those of the simulation approach. For this assessment, the parameters listed in Table 7.1 were used; however, non-zero correlation parameters $\rho_{S,r} = 0.3$ and $\rho_{v,r} = 0.2$ were used. For the finite difference methods, $N_t = 10,000$ was chosen, while the grid factor m was varied such that $N_S = 2m$ and $N_v = N_r = m$. For the simulation method, $N_t = 200$ and five basis functions were chosen, while N_P was varied.

Figure 7.1 shows plots of the execution times to price a European put-on-put compound option, while Figure 7.2 shows plots of the execution times to price an American put-on-put compound option. The curves of the simulation approach were positioned relative to the curves of the finite difference methods such that the typical order of accuracies matched. The data points on the curves of the simulation approach where the execution times stretched beyond 188 hours were extrapolated values and are indicated with grey dots.

Most evident from these two figures was that the inclusion of early exercise opportunities increased the overall execution time of the simulation approach to a far greater extent than the finite difference methods. When 10, 100, 1,000, and 10,000 simulation paths were used, it took approximately 0.062, 0.77, 23, and 2,260 seconds, respectively, to price a European compound option, and approximately 52.2, 470, 10,460, and 678,600 seconds, respectively, to price an American compound option. The simulation approach offered no benefit beyond 10,000 paths and, by extrapolation, would take several weeks to match the order of accuracy that the finite difference methods offered.

The results presented by Chiarella and Kang showed that their simulation approach using 1,000,000 simulation paths and 50 time steps took up to 139 hours to price an American put-on-put compound option. This was long, considering their finite difference approach took up to 17 minutes to complete, depending on their grid size [2].

Including early exercise opportunities only marginally increased the overall execution times of the six finite difference algorithms.

Under the Black–Scholes model, the execution times of the explicit and all five ADI schemes to price both European and American compound options were very short. At $m = 100$ and $N_t = 10,000$, execution times did not typically exceed 10 seconds. The curves for the Black–Scholes schemes increased linearly as m increased.

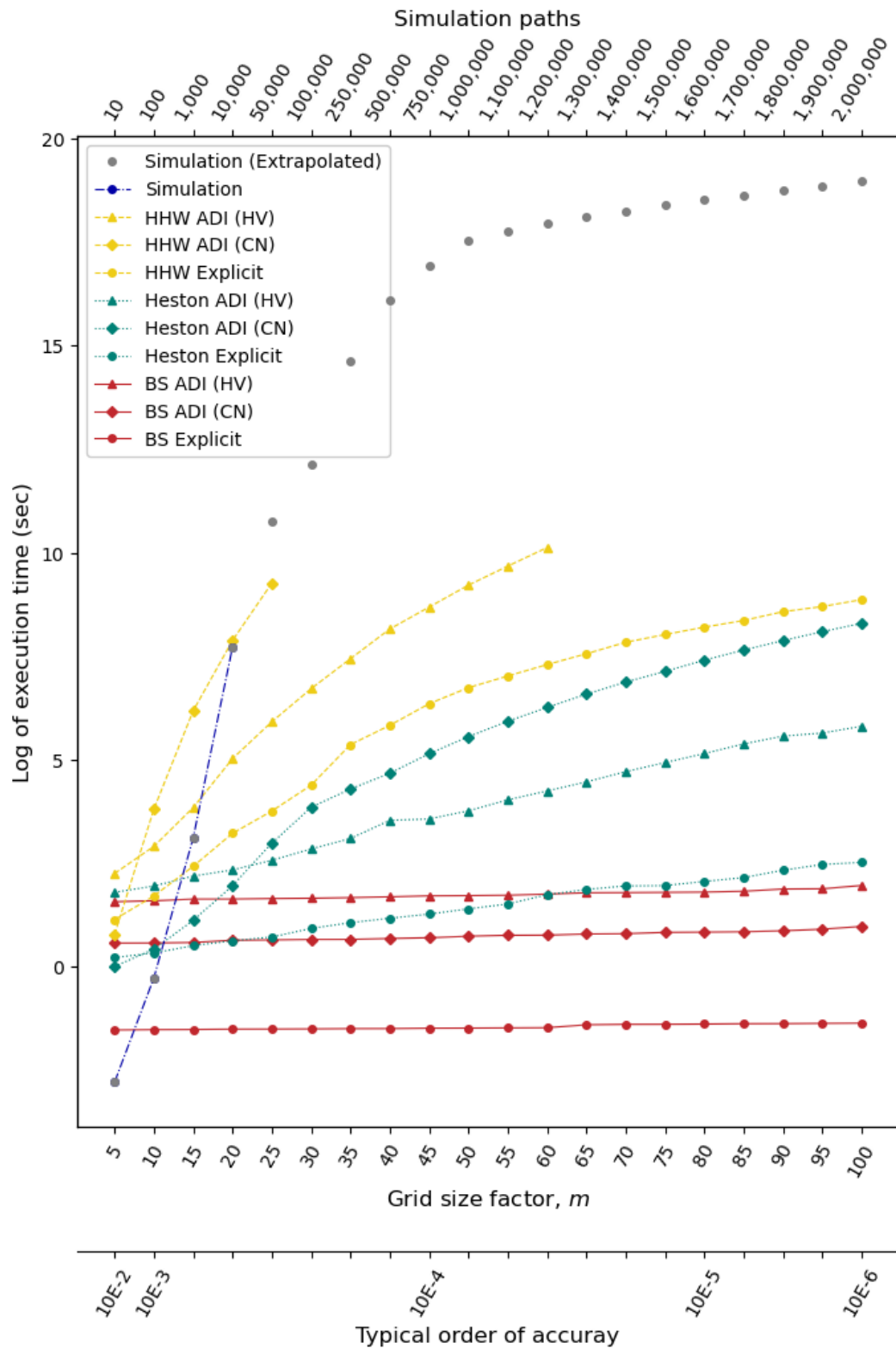


Figure 7.1: Execution times of the finite difference algorithms vs. the simulation method for a European put-on-put option.

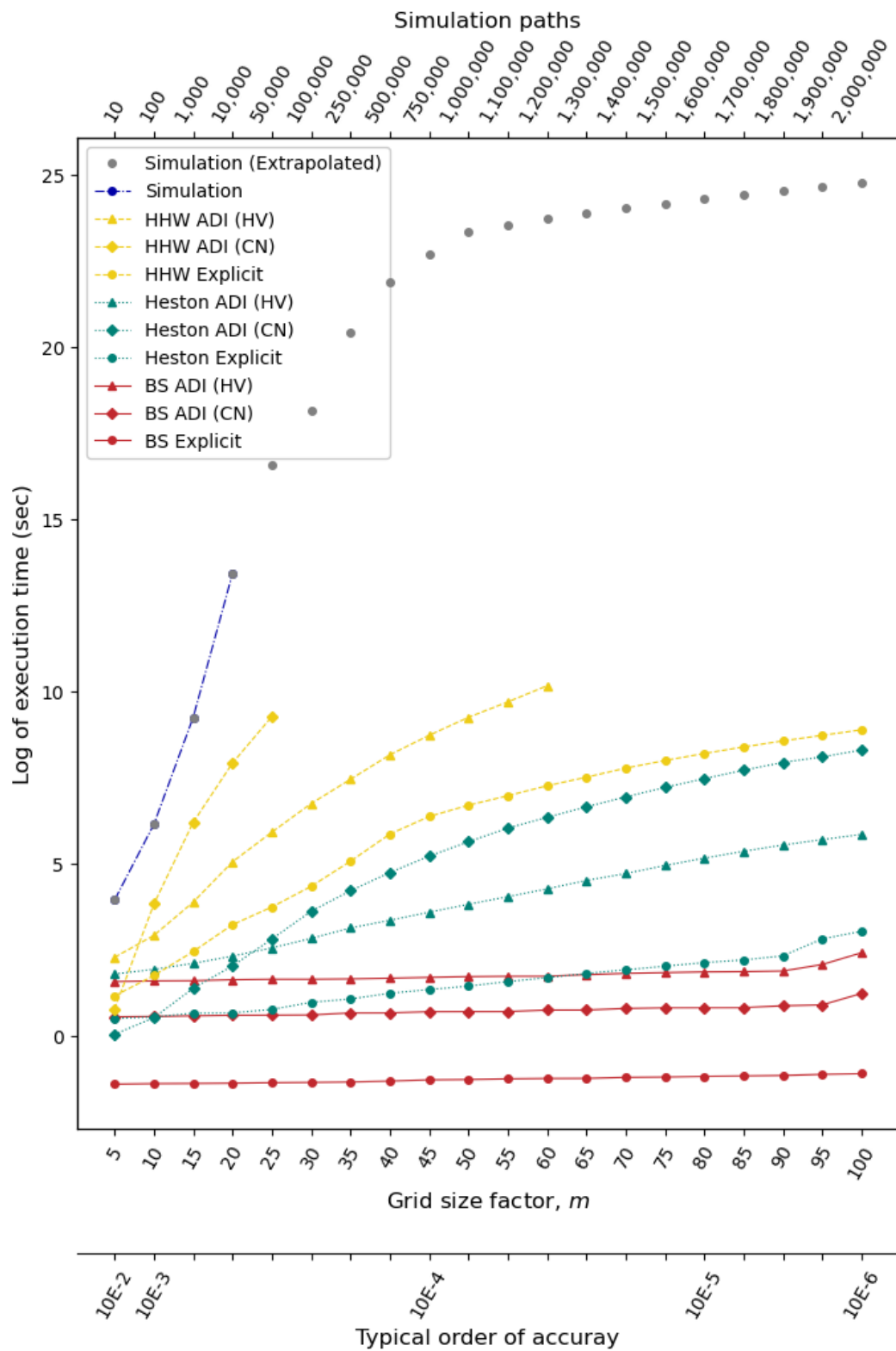


Figure 7.2: Execution times of the finite difference algorithms vs. the simulation method for an American put-on-put option.

The execution times of the Heston schemes were also reasonably short. At $m = 50$ and $N_t = 10,000$, the explicit scheme concluded in approximately 13 and 22 seconds for a European and American compound option, respectively, the Hundsdorfer–Verwer scheme concluded in approximately 330 and 350 seconds, respectively, while the Crank–Nicolson scheme concluded in approximately 4,020 and 4,060 seconds, respectively. In contrast with the observations for the Black–Scholes schemes, the Heston Crank–Nicolson scheme performed much slower than other Heston ADI schemes. The execution times of the Heston Douglas, Craig–Sneyd, and modified Craig–Sneyd schemes did not differ significantly from that of the Hundsdorfer–Verwer scheme.

The HHW schemes took substantially longer to execute than the Heston schemes. At $m = 50$ and $N_t = 10,000$, the explicit scheme was the fastest and concluded in approximately 800 and 850 seconds for a European and American compound option, respectively, while the Hundsdorfer–Verwer scheme concluded in approximately 10,102 and 10,472 seconds, respectively. At $m > 60$, it became computationally demanding to calculate the inverse of the decomposed matrices, and the HHW ADI schemes offered no practical benefit beyond this point.

The execution times of the HHW Douglas, Craig–Sneyd, and modified Craig–Sneyd schemes did not differ significantly from that of the Hundsdorfer–Verwer scheme. On the other hand, the HHW Crank–Nicolson scheme did not offer any practical benefits. Despite having taken much longer to execute than any other HHW ADI schemes, the application would run out of memory beyond $m > 25$.

7.3 Compound options with early exercise opportunities - simulation approach

Prices for American underlying and compound options using the three-state variable simulation approach were compared with prices from the literature. Prices for all four pricing models with non-zero correlation parameters were considered. The sample model parameters, prices, and relative errors are listed in Appendix C. The largest relative error amongst these samples was 0.0118.

7.4 Compound options with early exercise opportunities - finite difference methods

This section compares the prices of European compound options with their American variants. The model parameter Set A listed in Table 7.8 was used. Tables 7.9, 7.10, 7.11 and 7.12 show the prices for a call-on-call, put-on-call, call-on-put and put-on-put compound options, respectively, together with their underlying options.

General parameters		Heston parameters		HHW parameters	
Parameter	Value	Parameter	Value	Parameter	Value
q	0.005	κ_v	1.50	κ_r	0.30
S_0	100	θ_v	0.02	θ_r	0.04
v_0	0.04	σ_v	0.15	σ_r	0.10
r_0	0.04	λ_v	0.00	λ_r	0.00
T_D	1	$\rho_{S,v}$	-0.50	$\rho_{S,r}$	0.30
K_D	100			$\rho_{v,r}$	0.20
T_M	0.5				
K_M	4				

Bounded regions		Grid sizes		Non-uniform grid	
Parameter	Value	Parameter	Value	Parameter	Value
S_{min}	0	N_S	100	d_1	$K_D/5$
S_{max}	$14K_D$	N_v	50	c	0.1
V_{min}	0	N_r	50	d_2	$V_{max}/600$
V_{max}	1.1	N_t	10,000	$d_3(p=0)$	$R_{max}/400$
R_{max}	1			$d_3(p=\frac{1}{2})$	$R_{max}/100$
$R_{min}(p=0)$	-1				
$R_{min}(p=\frac{1}{2})$	0				

Simulation parameters	
Parameter	Value
N_P	100,000
N_t	200
Basis functions	5
Runs	30

Table 7.8: Parameter Set A.

HCIR parameters where $p = \frac{1}{2}$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound call-on-call	
					European	American	European	American
LSMC Simulation	–	–	–	–	8.711535	8.718425	5.698763	5.689781
HHW/HCIR Explicit	10,000	100	50	50	8.715653	8.715653	5.691627	5.691627
HHW/HCIR ADI (HV)	1,000	100	50	50	8.715567	8.715566	5.696531	5.696530
HHW parameters where $p = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound call-on-call	
					European	American	European	American
LSMC Simulation	–	–	–	–	9.273746	9.285961	6.340160	6.318666
HHW/HCIR Explicit	10,000	100	50	50	9.287893	9.393602	6.335476	6.431526
HHW/HCIR ADI (HV)	1,000	100	50	50	9.287821	9.393372	6.341332	6.437352
Heston parameters where $p = \kappa_r = \theta_r = \sigma_r = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound call-on-call	
					European	American	European	American
Heston Closed	–	–	–	–	8.607229	–	–	–
HHW Closed	–	–	–	–	8.607229	–	–	–
LSMC Simulation	–	–	–	–	8.615693	8.600600	5.568975	5.569830
Heston Explicit	10,000	100	50	–	8.610864	8.610864	5.572974	5.572974
Heston ADI (HV)	10,000	100	50	–	8.610772	8.610772	5.573038	5.573038
HHW/HCIR Explicit	10,000	100	50	50	8.610869	8.610869	5.572980	5.572980
HHW/HCIR ADI (HV)	1,000	100	50	50	8.610776	8.610776	5.577734	5.577734
Black–Scholes parameters where $p = \kappa_v = \theta_v = \sigma_v = \kappa_r = \theta_r = \sigma_r = \rho_{S,v} = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound call-on-call	
					European	American	European	American
BS Closed	–	–	–	–	9.619251	–	6.447996	–
Heston Closed	–	–	–	–	9.619238	–	–	–
HHW Closed	–	–	–	–	9.619238	–	–	–
LSMC Simulation	–	–	–	–	9.609998	9.585025	6.451994	6.446034
BS Explicit	10,000	400	–	–	9.619578	9.619578	6.448751	6.448751
BS ADI (HV)	10,000	400	–	–	9.619472	9.619472	6.448831	6.448831
Heston Explicit	10,000	100	50	–	9.620138	9.620138	6.450079	6.450079
Heston ADI (HV)	10,000	100	50	–	9.620032	9.620032	6.450155	6.450155
HHW/HCIR Explicit	10,000	100	50	50	9.620142	9.620142	6.450084	6.450084
HHW/HCIR ADI (HV)	1,000	100	50	50	9.620036	9.620036	6.455879	6.455879

Legend: BS: Black–Scholes; HV: Hundsdorfer–Verwer;

Table 7.9: Price comparison between the numerical methods for an underlying call and call-on-call option.

HCIR parameters where $p = \frac{1}{2}$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound call-on-put	
					European	American	European	American
LSMC Simulation	–	–	–	–	5.303409	5.600028	2.835259	3.151537
HHW/HCIR Explicit	10,000	100	50	50	5.298397	5.635060	2.838395	3.146406
HHW/HCIR ADI (HV)	1,000	100	50	50	5.298423	5.634696	2.840641	3.148679
HHW parameters where $p = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound call-on-put	
					European	American	European	American
LSMC Simulation	–	–	–	–	5.996729	6.001235	3.493539	3.495786
HHW/HCIR Explicit	10,000	100	50	50	5.994314	6.344105	3.486740	3.798245
HHW/HCIR ADI (HV)	1,000	100	50	50	5.994280	6.343671	3.489971	3.800784
Heston parameters where $p = \kappa_r = \theta_r = \sigma_r = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound call-on-put	
					European	American	European	American
Heston Closed	–	–	–	–	5.184925	–	–	–
HHW Closed	–	–	–	–	5.184925	–	–	–
LSMC Simulation	–	–	–	–	5.181338	5.556191	2.746446	3.115021
Heston Explicit	10,000	100	50	–	5.188553	5.602080	2.739639	3.117384
Heston ADI (HV)	10,000	100	50	–	5.188468	5.601938	2.739741	3.117432
HHW/HCIR Explicit	10,000	100	50	50	5.188560	5.602087	2.739646	3.117390
HHW/HCIR ADI (HV)	1,000	100	50	50	5.188476	5.601625	2.741675	3.119556
Black–Scholes parameters where $p = \kappa_v = \theta_v = \sigma_v = \kappa_r = \theta_r = \sigma_r = \rho_{S,v} = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound call-on-put	
					European	American	European	American
BS Closed	–	–	–	–	6.196947	–	3.357565	–
Heston Closed	–	–	–	–	6.196934	–	–	–
HHW Closed	–	–	–	–	6.196934	–	–	–
LSMC Simulation	–	–	–	–	6.205118	6.547956	3.352885	3.680464
BS Explicit	10,000	400	–	–	6.197266	6.550787	3.357871	3.689154
BS ADI (HV)	10,000	400	–	–	6.197168	6.550586	3.357971	3.689179
Heston Explicit	10,000	100	50	–	6.197827	6.549606	3.359436	3.690361
Heston ADI (HV)	10,000	100	50	–	6.197728	6.549426	3.359544	3.690390
HHW/HCIR Explicit	10,000	100	50	50	6.197834	6.549614	3.359442	3.690369
HHW/HCIR ADI (HV)	1,000	100	50	50	6.197735	6.549101	3.362313	3.693247

Legend: BS: Black–Scholes; HV: Hundsdorfer–Verwer;

Table 7.10: Price comparison between the numerical methods for an underlying put and call-on-put option.

HCIR parameters where $p = \frac{1}{2}$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound put-on-call	
					European	American	European	American
LSMC Simulation	–	–	–	–	8.705767	8.716564	0.901164	0.889990
HHW/HCIR Explicit	10,000	100	50	50	8.715653	8.715653	0.896336	0.898464
HHW/HCIR ADI (HV)	1,000	100	50	50	8.715567	8.715566	0.897023	0.900005
HHW parameters where $p = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound put-on-call	
					European	American	European	American
LSMC Simulation	–	–	–	–	9.284376	9.287647	0.972004	0.862534
HHW/HCIR Explicit	10,000	100	50	50	9.287893	9.393602	0.968560	0.960255
HHW/HCIR ADI (HV)	1,000	100	50	50	9.287821	9.393372	0.969393	0.961908
Heston parameters where $p = \kappa_r = \theta_r = \sigma_r = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound put-on-call	
					European	American	European	American
Heston Closed	–	–	–	–	8.607229	–	–	–
HHW Closed	–	–	–	–	8.607229	–	–	–
LSMC Simulation	–	–	–	–	8.605153	8.599663	0.887300	0.880127
Heston Explicit	10,000	100	50	–	8.610864	8.610864	0.882458	0.885141
Heston ADI (HV)	10,000	100	50	–	8.610772	8.610772	0.882599	0.885275
HHW/HCIR Explicit	10,000	100	50	50	8.610869	8.610869	0.882459	0.885141
HHW/HCIR ADI (HV)	1,000	100	50	50	8.610776	8.610776	0.883129	0.886668
Black–Scholes parameters where $p = \kappa_v = \theta_v = \sigma_v = \kappa_r = \theta_r = \sigma_r = \rho_{S,v} = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Call		Compound put-on-call	
					European	American	European	American
BS Closed	–	–	–	–	9.619251	–	0.749540	–
Heston Closed	–	–	–	–	9.619238	–	–	–
HHW Closed	–	–	–	–	9.619238	–	–	–
LSMC Simulation	–	–	–	–	9.630595	9.570144	0.747301	0.748849
BS Explicit	10,000	400	–	–	9.619578	9.619578	0.749412	0.751400
BS ADI (HV)	10,000	400	–	–	9.619472	9.619472	0.749582	0.751555
Heston Explicit	10,000	100	50	–	9.620138	9.620138	0.750181	0.751985
Heston ADI (HV)	10,000	100	50	–	9.620032	9.620032	0.750347	0.752146
HHW/HCIR Explicit	10,000	100	50	50	9.620142	9.620142	0.750181	0.751985
HHW/HCIR ADI (HV)	1,000	100	50	50	9.620036	9.620036	0.750931	0.753717

Legend: BS: Black–Scholes; HV: Hundsdorfer–Verwer;

Table 7.11: Price comparison between the numerical methods for an underlying call and put-on-call option.

HCIR parameters where $p = \frac{1}{2}$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound put-on-put	
					European	American	European	American
LSMC Simulation	–	–	–	–	5.295652	5.638533	1.464594	1.438155
HHW/HCIR Explicit	10,000	100	50	50	5.298397	5.635060	1.460694	1.443934
HHW/HCIR ADI (HV)	1,000	100	50	50	5.298423	5.634696	1.461608	1.445779
HHW parameters where $p = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound put-on-put	
					European	American	European	American
LSMC Simulation	–	–	–	–	5.982605	5.990428	1.418057	1.216492
HHW/HCIR Explicit	10,000	100	50	50	5.994314	6.344105	1.413701	1.393997
HHW/HCIR ADI (HV)	1,000	100	50	50	5.994280	6.343671	1.414564	1.395840
Heston parameters where $p = \kappa_r = \theta_r = \sigma_r = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound put-on-put	
					European	American	European	American
Heston Closed	–	–	–	–	5.184925	–	–	–
HHW Closed	–	–	–	–	5.184925	–	–	–
LSMC Simulation	–	–	–	–	5.182870	5.568407	1.477785	1.444343
Heston Explicit	10,000	100	50	–	5.188553	5.602080	1.471769	1.448964
Heston ADI (HV)	10,000	100	50	–	5.188468	5.601938	1.471939	1.449144
HHW/HCIR Explicit	10,000	100	50	50	5.188560	5.602087	1.471768	1.448963
HHW/HCIR ADI (HV)	1,000	100	50	50	5.188476	5.601625	1.472717	1.450849
Black–Scholes parameters where $p = \kappa_v = \theta_v = \sigma_v = \kappa_r = \theta_r = \sigma_r = \rho_{S,v} = \rho_{S,r} = \rho_{v,r} = 0$								
Model	N_t	N_S	N_v	N_r	Underlying Put		Compound put-on-put	
					European	American	European	American
BS Closed	–	–	–	–	6.196947	–	1.081413	–
Heston Closed	–	–	–	–	6.196934	–	–	–
HHW Closed	–	–	–	–	6.196934	–	–	–
LSMC Simulation	–	–	–	–	6.194309	6.553689	1.079125	1.067236
BS Explicit	10,000	400	–	–	6.197266	6.550787	1.081178	1.065549
BS ADI (HV)	10,000	400	–	–	6.197168	6.550586	1.081361	1.065748
Heston Explicit	10,000	100	50	–	6.197827	6.549606	1.082183	1.067921
Heston ADI (HV)	10,000	100	50	–	6.197728	6.549426	1.082374	1.068112
HHW/HCIR Explicit	10,000	100	50	50	6.197834	6.549614	1.082183	1.067921
HHW/HCIR ADI (HV)	1,000	100	50	50	6.197735	6.549101	1.083012	1.069801

Legend: BS: Black–Scholes; HV: Hundsdorfer–Verwer;

Table 7.12: Price comparison between the numerical methods for an underlying put and put-on-put option.

Each table is divided into four sections. The first and second sections show the prices using only the simulation approach and the three-state variable finite difference methods under the HCIR and HHW models, respectively. The third section shows the prices using the simulation approach as well as the two- and three-state variable finite difference methods with κ_r , θ_r , σ_r , and $\rho_{S,v}$ all being equal to zero. The last section compares the prices using all seven numerical methods with κ_v , θ_v , σ_v , κ_r , θ_r , σ_r , $\rho_{S,v}$, $\rho_{S,r}$, and $\rho_{v,r}$ all being equal to zero.

Throughout Tables 7.9, 7.10, 7.11 and 7.12, it was observed that the underlying option prices using the explicit and Hundsdorfer–Verwer schemes were generally within three to four decimal values from one another, while the compound option prices were generally within two to three decimal values from one another.

The finite difference values were generally within two to three decimal values from the mean simulation values. However, under the HHW model, the values of the American underlying and American overlying options were only within one decimal value from the mean simulation values. This suggests that the simulation approach has some drift occurring when $p = 0$ that needs further investigation.

The accuracies of these values provide confidence that the boundary values of the four main compound option types hold across all six finite difference algorithms. It further provides confidence that the HHW/HCIR finite difference algorithms will provide accurate results if only the Black–Scholes and Heston model parameters are specified and that the Heston finite difference algorithms will provide accurate results if only the Black–Scholes parameters are specified.

7.5 Reference values for compound options with early exercise opportunities

This section provides reference prices for European and American compound options under the one-, two- and three-state variable pricing models for various model parameters.

Table B.1 to Table B.12 in Appendix B show reference prices of the four main compound option types using the explicit and Hundsdorfer–Verwer ADI schemes and the difference in value between the schemes. The tables further show the approximate early exercise values for each scheme, which is the difference between the price of the American and European variants. This was done for varying values of S_0 , K_D , T_D , K_M , and T_M . The model

parameters were varied according to the six parameter sets listed in Table 7.8 and Table 7.13. The finite difference and simulation parameters in Table 7.8 were used, however, $N_S = 400$ was used for the Black–Scholes model and $N_r = 30$ was used for the HHW/HCIR ADI schemes.

Parameter	Set B	Set C	Set D	Set E	Set F
r_0	0.050	0.0550	0.101	0.103	0.040
v_0	0.100	0.0800	0.070	0.050	0.050
q	0.005	0.0050	0.005	0.005	0.005
κ_v	3.000	0.6067	2.500	0.500	0.300
θ_v	0.080	0.0707	0.060	0.040	0.040
σ_v	0.040	0.2928	0.500	1.000	0.900
λ_v	0.000	0.0000	0.000	0.000	0.000
κ_r	0.200	0.0500	0.150	0.080	0.160
θ_r	0.050	0.0550	0.101	0.103	0.040
σ_r	0.030	0.0600	0.100	0.090	0.030
λ_r	0.000	0.0000	0.000	0.000	0.000
$\rho_{S,v}$	0.600	-0.7571	-0.100	-0.900	-0.500
$\rho_{S,r}$	0.600	-0.3000	0.600	0.200	-0.500
$\rho_{v,r}$	-0.200	0.2000	-0.700	0.100	-0.200

Table 7.13: Parameter Sets B to F.

Note that the variance Feller condition is unmet in Set E and Set F. For these two sets, it was observed that the Heston and HHW/HCIR explicit algorithms were unstable and produced errors even with a sufficiently small time step ($N_T = 10,000$). The Heston and HHW/HCIR ADI schemes, on the other hand, remained stable.

7.5.1 Call-on-call options

Table B.1, Table B.2, and Table B.3 in Appendix B show that introducing early exercise opportunities into a call-on-call compound option had a negligible effect on the discounted price. Across the four pricing models, the maximum early exercise value observed as a percentage of the European price was only +0.06%.

This result was expected since the price of an American underlying call should be the same as its European variant [23]. Given the payoff function of the

overlying call, it would never be optimal to exercise the overlying call before its maturity date. Therefore, changing the compound option from a European to an American variant would not affect the overall price.

7.5.2 Call-on-put options

Table B.4, Table B.5, and Table B.6 in Appendix B show that introducing early exercise opportunities into a call-on-put compound option considerably impacted the discounted price - more so than any of the other three compound option types. The American variants were generally more expensive than the European variants. Across the four pricing models, the maximum early exercise value observed as a percentage of the European price was 45.66%.

The price increase was expected since an American underlying put is generally more expensive than its equivalent European variant [23] and will consequently, given the payoff function of the overlying call, increase the value of the compound option.

Suppose that an investor is looking to sell a certain number of a stock through a put option in the future since he believes that the stock price will drop. He is concerned that the price of the stock could increase instead. Therefore, he considers buying a call option on the put option, giving himself exposure to the put option for a smaller premium without having to pay the entire premium of the put option at the onset. The investor is also looking for the freedom of exercising the overlying call option at any point before its maturity, receiving the underlying put option, and then again being able to exercise the underlying put option at any point after receiving it if he believes that the stock price has dropped far enough. The compound option would ideally be set up as an American call-on-put option and would cost the investor slightly more.

7.5.3 Put-on-call options

Table B.7, Table B.8, and Table B.9 in Appendix B show that introducing early exercise opportunities into a put-on-call compound option only had a marginal effect on the discounted price. The American variants were generally only slightly more expensive than the European variants. The range of early exercise value as a percentage of the European price was higher for put-on-call options than for call-on-call options but less substantial than for call-on-put and put-on-put options.

A minor price increase was expected since the price of an American underlying call is the same as its European variant [23]; therefore, changing the underlying from a European to an American variant would not affect the price of the overlying put. The increase in the price of the overlying put was primarily attributed to the overlying put being American.

7.5.4 Put-on-put options

Table B.10, Table B.11, and Table B.12 in Appendix B show that introducing early exercise opportunities into a put-on-put compound option had a substantial effect on the discounted price. The American variants were generally less expensive than the European variants. Across the four pricing models, the maximum negative early exercise value observed as a percentage of the European price was -7.16% .

The price decrease was expected since an American underlying put is generally more expensive than its equivalent European variant [23] and will consequently, given the payoff function of the overlying put, decrease the value of the compound option.

8 Conclusion

The objective of this dissertation was to explore the use of explicit and ADI finite difference methods as an efficient alternative to simulation approaches to price compound options where early exercise opportunities are available. The source code for the finite difference methods used a vectorised approach instead of traditional and slower-performing nested *for* loops often encountered in option pricing literature.

Stock prices, stock price volatilities, and interest rates were assumed to follow correlated stochastic differential equations. Pricing compound options under the Black–Scholes, Heston, HHW, and HCIR pricing models were considered.

The objective at the onset of the results section was to identify optimal grid sizes and optimal parameters for non-uniform grid generation for each of the six finite difference algorithms, providing a balance between acceptable relative errors and reasonable execution times.

While the explicit methods executed faster than the ADI methods under all four pricing models, ADI methods were unconditionally stable and produced more accurate results. The two- and three-state variable explicit schemes were conditionally stable on a sufficiently fine time grid. Non-uniform grids

produced more accurate results than uniform grids for both the explicit and ADI schemes.

Finite difference algorithms were found to be computationally advantageous over the simulation approach. Execution times were considerably shorter, and outputs were typically within 2 or 3 decimal values of the simulation values.

At $N_S = 200$ and $N_t = 10,000$, typical relative errors of order $1E-4$ were observed for compound options on non-uniform grids under the Black–Scholes model. At these levels, execution times of the explicit and five ADI schemes did not typically exceed 10 seconds.

At $N_S = 100$, $N_v = 50$ and $N_t = 10,000$, typical relative errors of order $1E-3$ were observed for compound options on non-uniform grids under the Heston model. At these levels, the explicit scheme concluded in approximately 13 and 22 seconds for a European and American compound option, respectively, while the Hundsdorfer–Verwer scheme concluded in approximately 330 and 350 seconds, respectively.

At $N_S = 100$, $N_v = N_r = 50$ and $N_t = 10,000$, typical relative errors of order $1E-2$ were observed for compound options on non-uniform grids under the HHW and HCIR models. At these levels, the explicit scheme concluded in approximately 800 and 850 seconds for a European and American compound option, respectively, while the Hundsdorfer–Verwer scheme concluded in approximately 10,102 and 10,472 seconds, respectively.

Under the two- and three-state variable models, the Crank–Nicolson schemes performed much slower than other ADI schemes and offered little practical benefit. The execution times of the Douglas, Craig–Sneyd, modified Craig–Sneyd, and Hundsdorfer–Verwer schemes did not differ significantly. The Douglas and Craig–Sneyd schemes achieved marginally smaller errors on average than other ADI schemes.

ADI schemes remained stable for cases where the Feller conditions were not met, while the explicit schemes were mostly unstable.

An assessment of the similarities of values between the four pricing models showed that the HHW/HCIR finite difference methods provided accurate results when only the Black–Scholes and Heston model parameters were specified and that the Heston finite difference methods provided accurate results when only the Black–Scholes parameters were specified.

A comprehensive set of reference prices was provided for European and American compound options under the four pricing models. American

call-on-put compound options had the most significant overall price increases relative to their European variants. American put-on-put compound options had the most significant overall price decreases. American put-on-call compound options only showed marginal price increases and American call-on-call showed negligible price differences.

While considerable dedication has been directed towards the accuracy of the algorithms employed in this study, a critical examination reveals an opportune space for further enhancement of the HHW/H CIR ADI algorithm. Future research endeavours will focus on improving the execution time of this algorithm such that the reference compound option prices under the three-state variable models are obtained using more refined grids. Thank you for dedicating your time to reading this dissertation.

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A Greek surface plots for the Black–Scholes model

Figures A.1 to A.3 show option Greek surface plots for the underlying options, while Figures A.4 to A.9 show option Greek surface plots for the four compound options using the equations given by Theorem 4.11. For these surface plots, $r = 5\%$, $q = 1\%$, $\sigma_S = 15\%$, $S_0 = 100$, $K_D = 100$, $T_D = 1$, $K_M = 5$ and $T_M = 0.5$ were used.

A.1 Greek surface plots for underlying options

Figure A.1 shows delta surface plots for an underlying call [subplots a), c) and e)] and put [subplots b), d) and f)] option. Subplots a) and b) have xy-grids S vs. T_D , subplots c) and d) have xy-grids S vs. σ_S , and subplots e) and f) have xy-grids S vs. r .

Figure A.2 shows vega surface plots for an underlying call [subplots a), c) and e)] and put [subplots b), d) and f)] option. Subplots a) and b) have xy-grids σ_S vs. T_D , subplots c) and d) have xy-grids σ_S vs. S and subplots e) and f) have xy-grids σ_S vs. r .

Figure A.3 shows rho surface plots for an underlying call [subplots a), c) and e)] and put [subplots b), d) and f)] option. Subplots a) and b) have xy-grids r vs. T_D , subplots c) and d) have xy-grids r vs. S and subplots e) and f) have xy-grids r vs. σ_S .

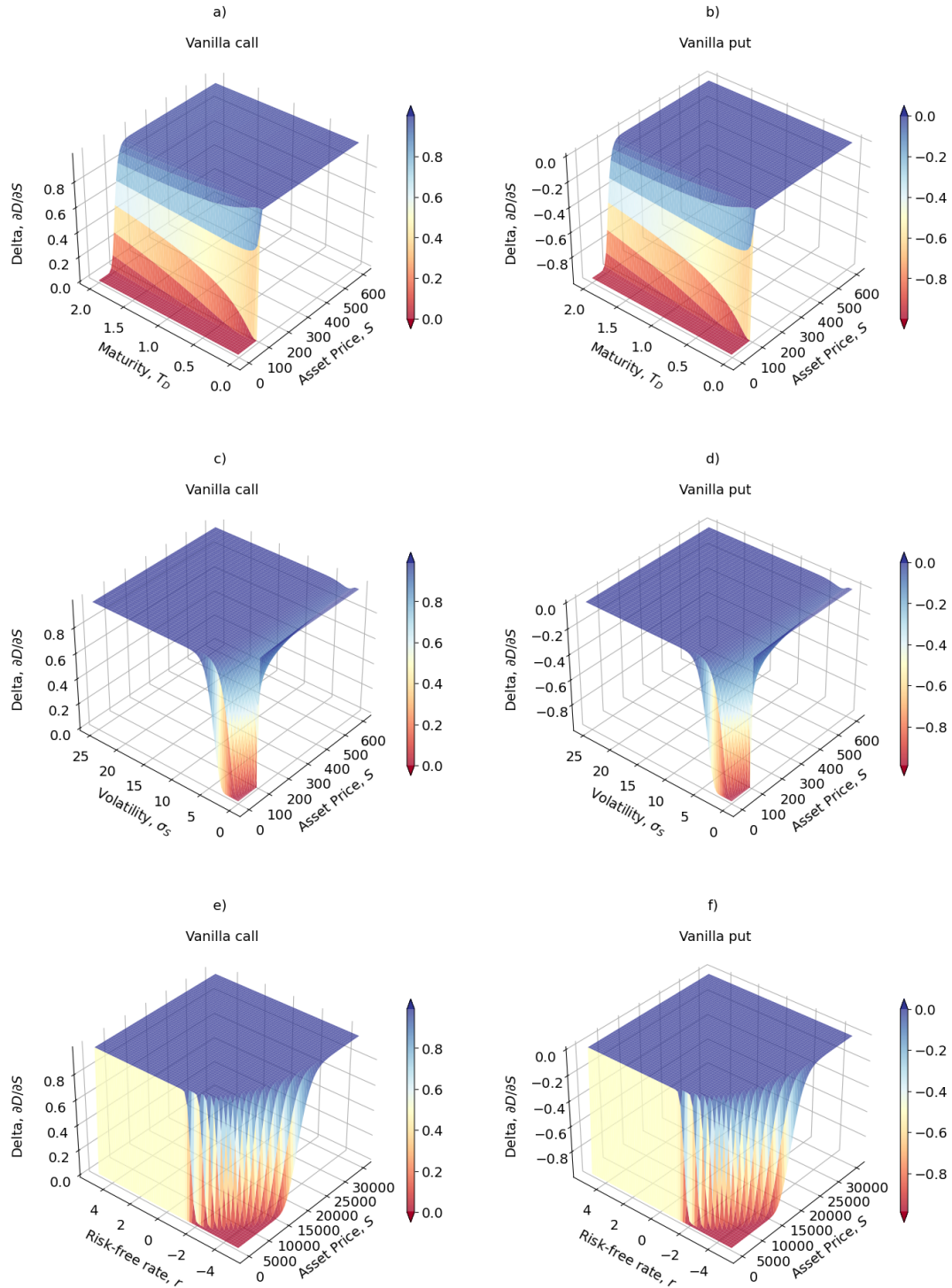


Figure A.1: Black–Scholes delta surface plots for an underlying call and put option.

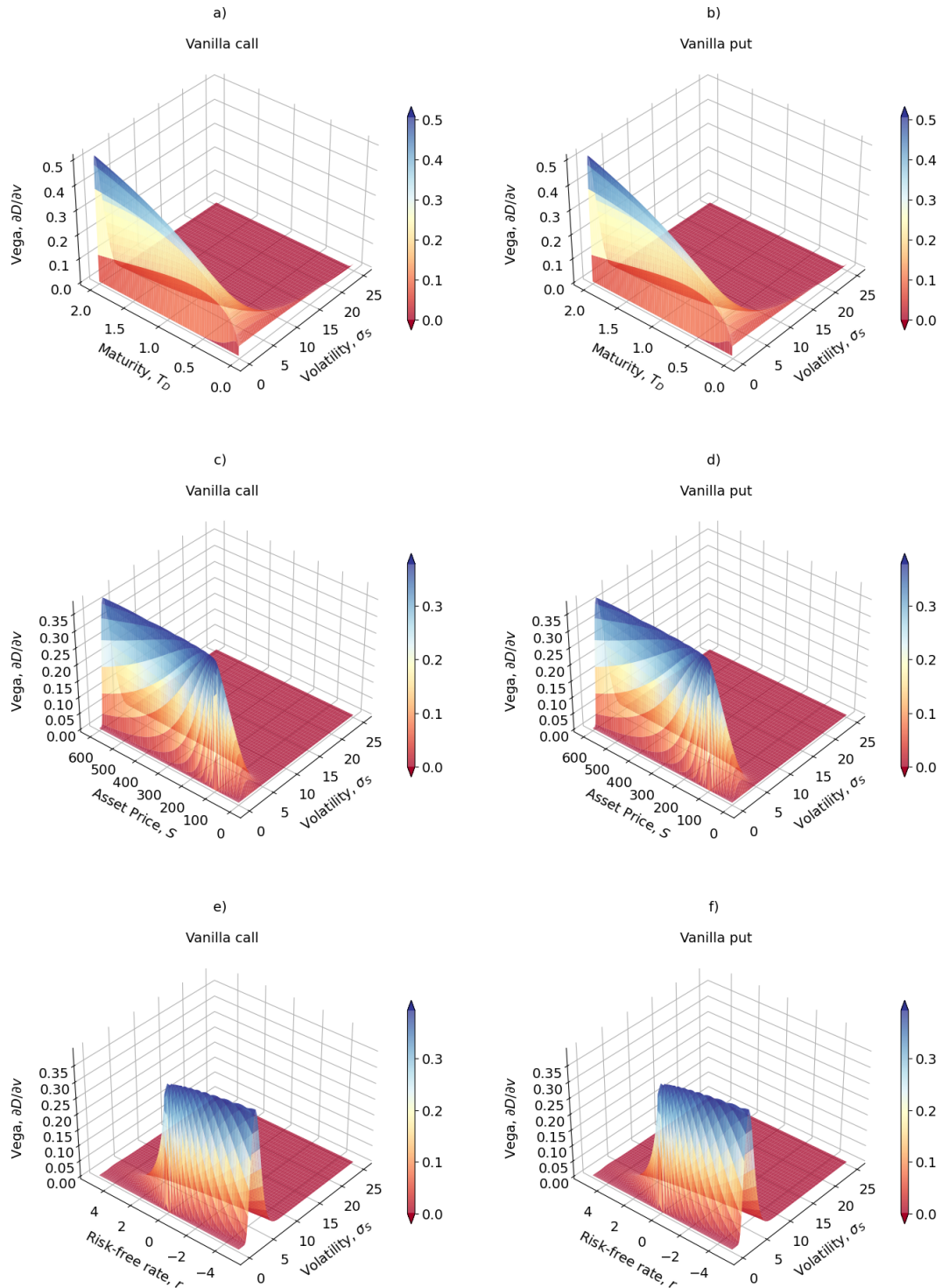


Figure A.2: Black–Scholes vega surface plots for an underlying call and put option.

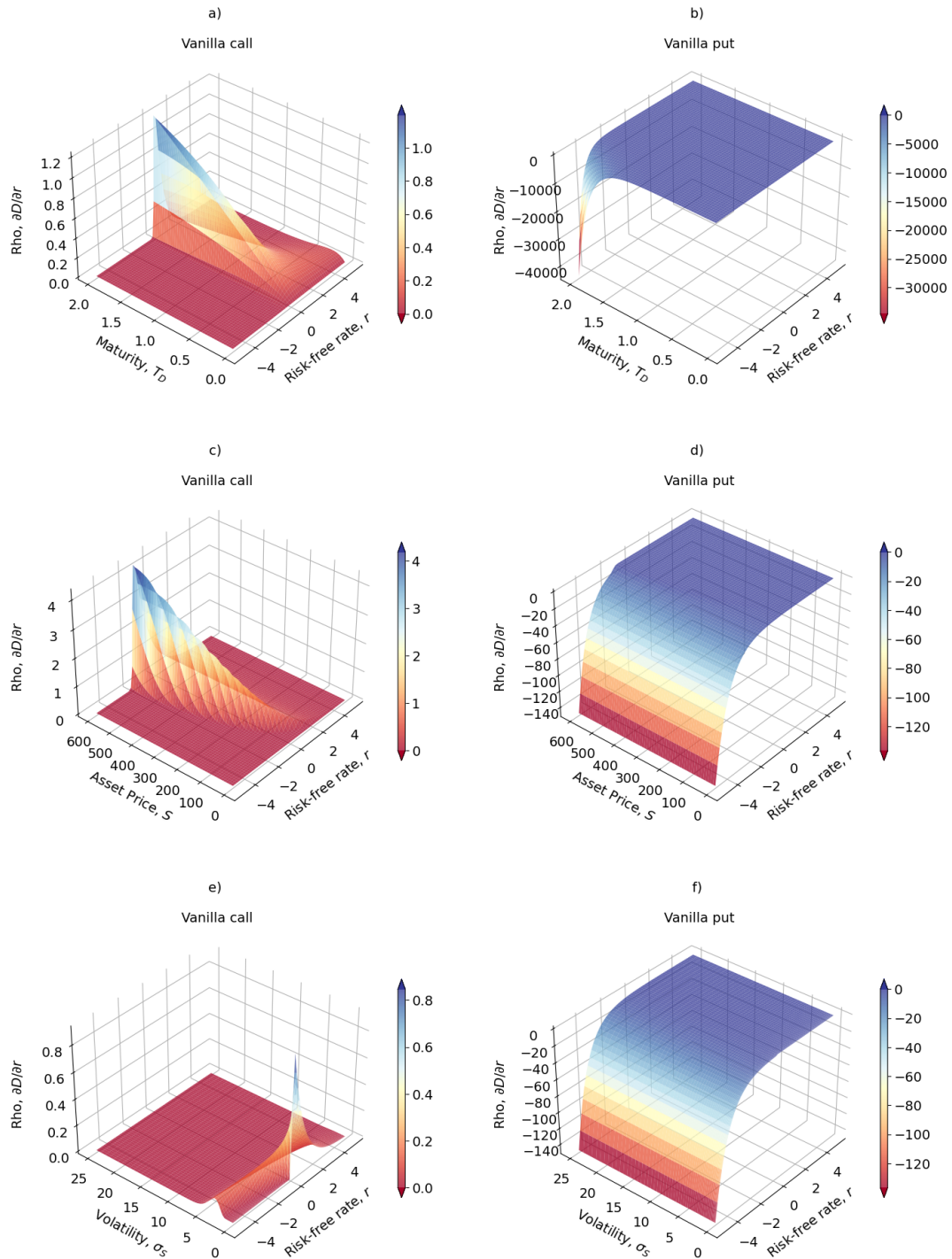


Figure A.3: Black–Scholes rho surface plots for an underlying call and put option.

A.2 Greek surface plots for compound options

Figure A.4 shows delta surface plots for a call-on-call [subplots a), c) and e)] and call-on-put [subplots b), d) and f)] option. Figure A.5 shows Delta surface plots for a put-on-call [subplots a), c) and e)] and put-on-put [subplots b), d) and f)] option. In both figures, subplots a) and b) have xy-grids S vs. T_D , subplots c) and d) have xy-grids S vs. σ_S and subplots e) and f) have xy-grids S vs. r .

Figure A.6 shows vega surface plots for a call-on-call [subplots a), c) and e)] and call-on-put [subplots b), d) and f)] option. Figure A.7 shows Vega surface plots for a put-on-call [subplots a), c) and e)] and put-on-put [subplots b), d) and f)] option. In both figures, subplots a) and b) have xy-grids σ_S vs. T_D , subplots c) and d) have xy-grids σ_S vs. S and subplots e) and f) have xy-grids σ_S vs. r .

Figure A.8 shows rho surface plots for a call-on-call [subplots a), c) and e)] and call-on-put [subplots b), d) and f)] option. Figure A.9 shows Rho surface plots for a put-on-call [subplots a), c) and e)] and put-on-put [subplots b), d), and f)] option. In both figures, subplots a) and b) have xy-grids r vs. T_D , subplots c) and d) have xy-grids r vs. S and subplots e) and f) have xy-grids r vs. σ_S .

For the plots where T_D was varied, T_M was taken as half of T_D .

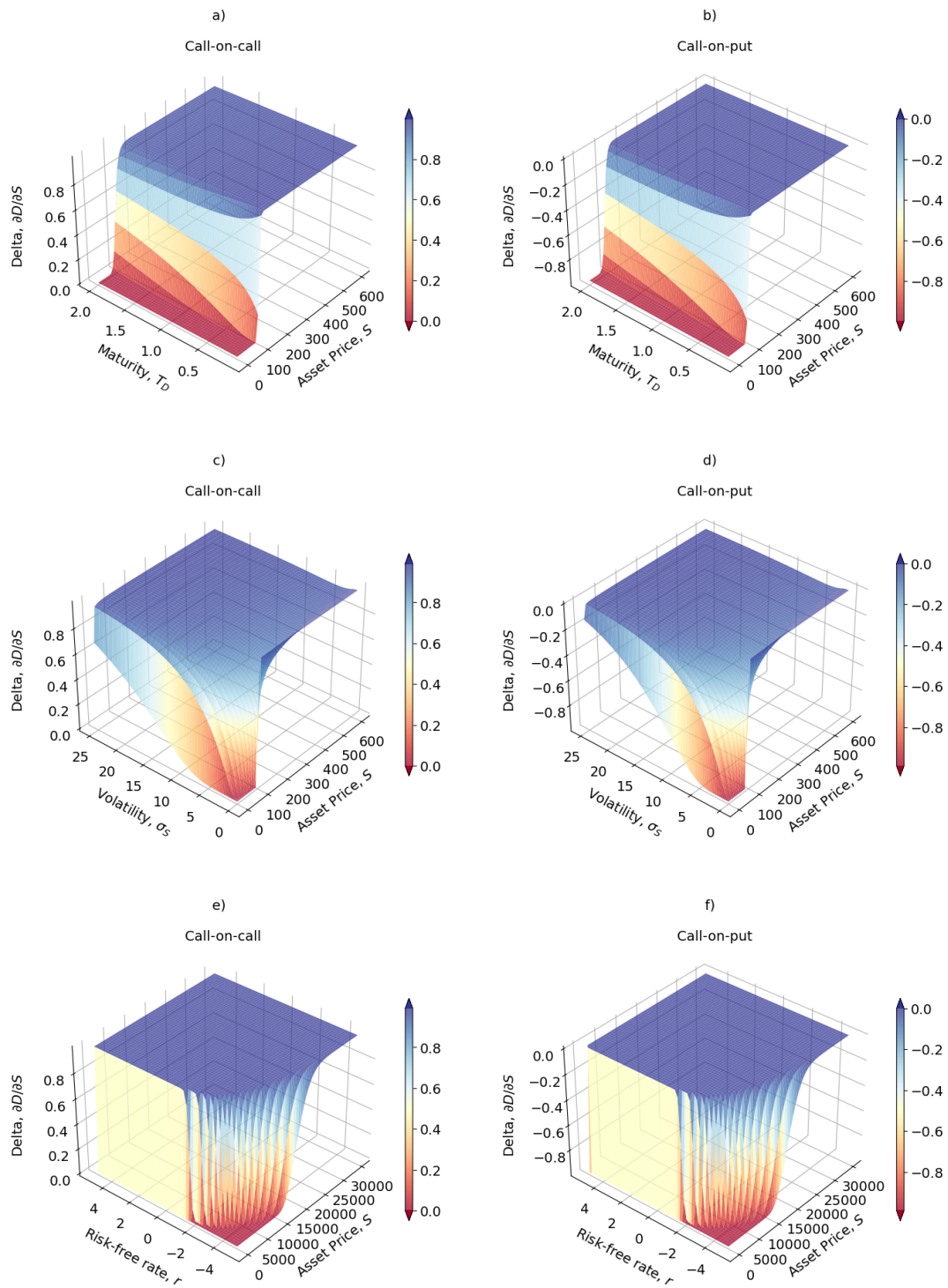


Figure A.4: Black–Scholes delta surface plots for a call-on-call and call-on-put option.

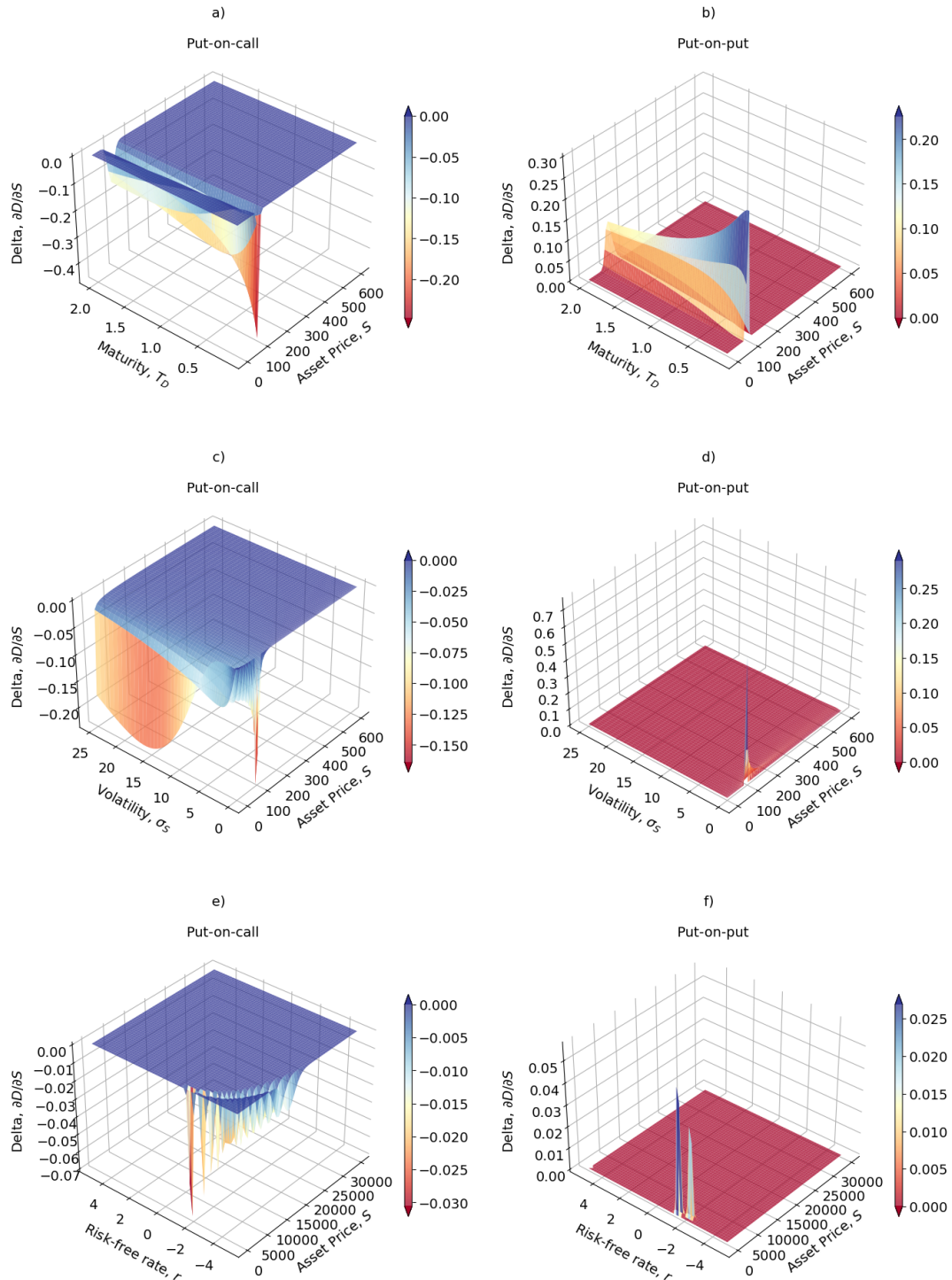


Figure A.5: Black–Scholes delta surface plots for a put-on-call and put-on-put option.

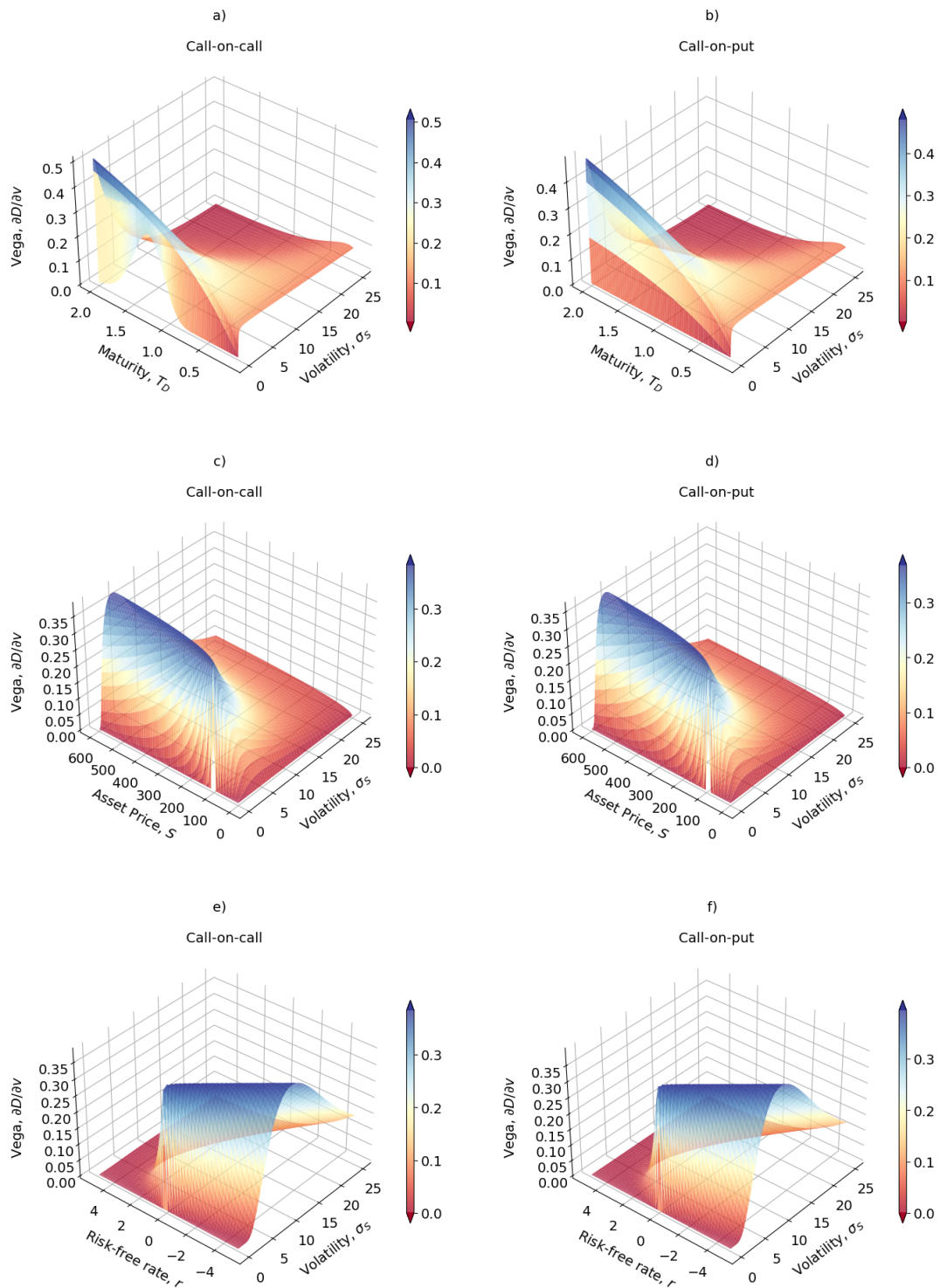


Figure A.6: Black–Scholes vega surface plots for a call-on-call and call-on-put option.

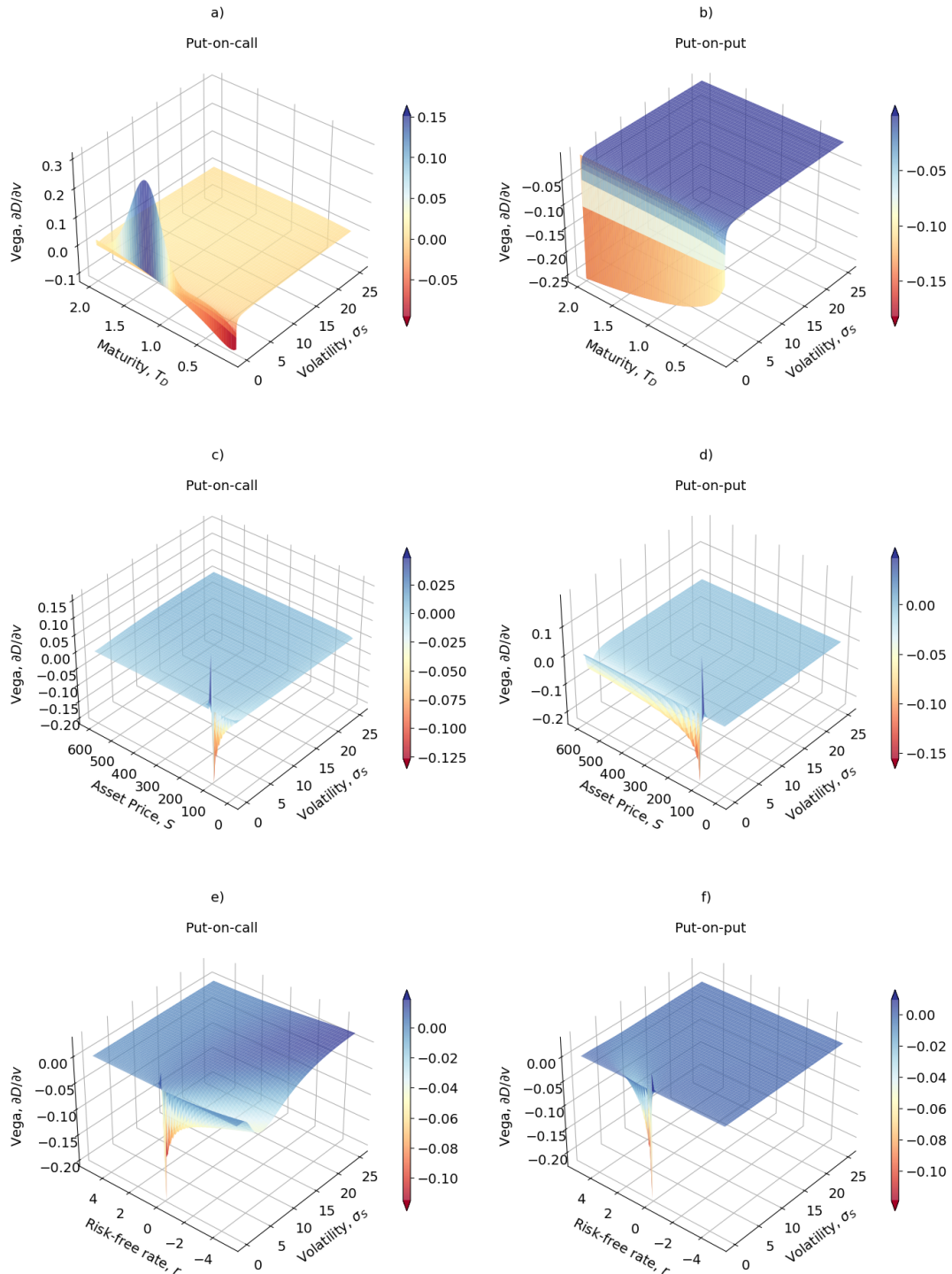


Figure A.7: Black–Scholes vega surface plots for a put-on-call and put-on-put option.

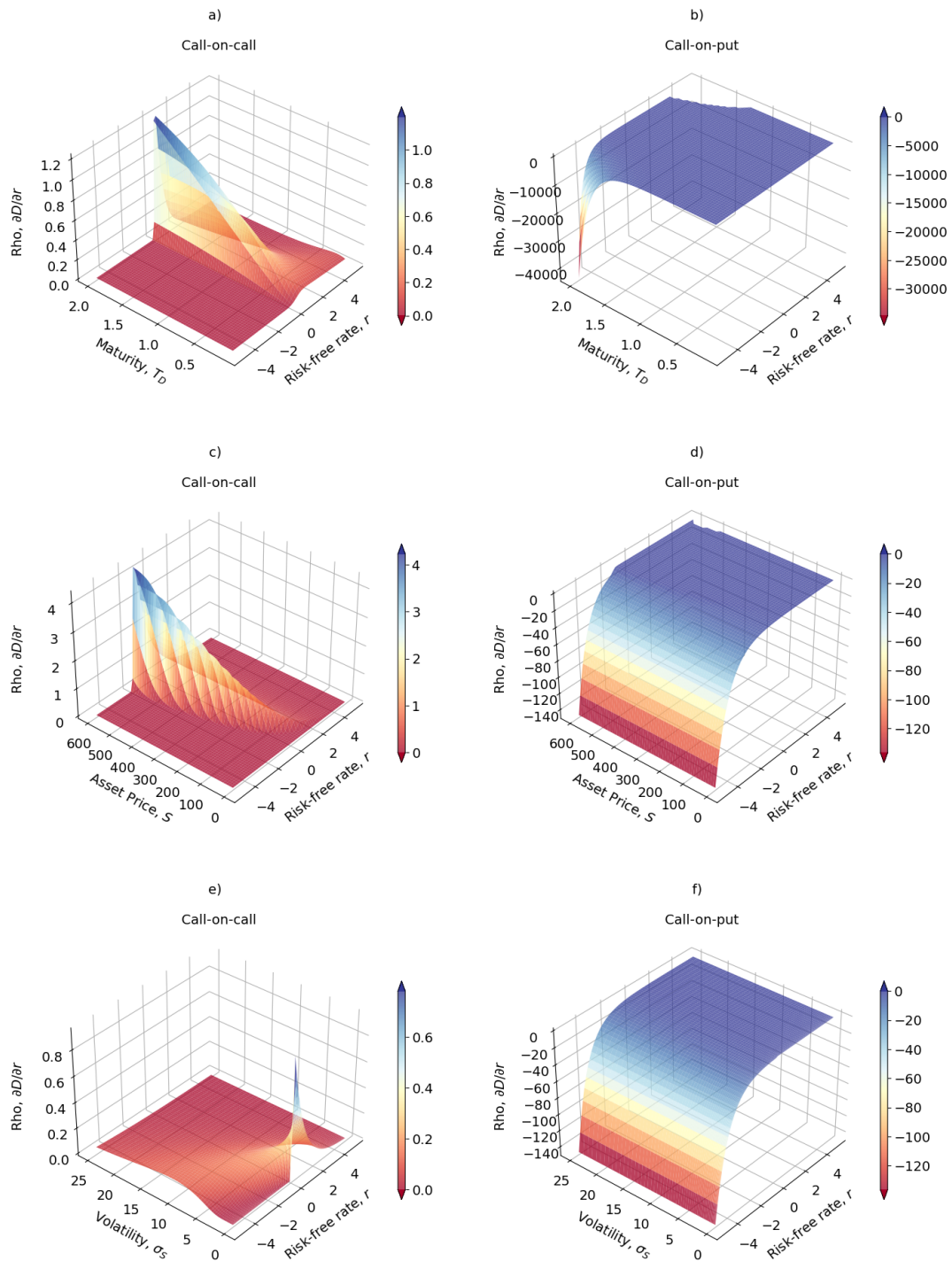


Figure A.8: Black–Scholes rho surface plots for a call-on-call and call-on-put option.

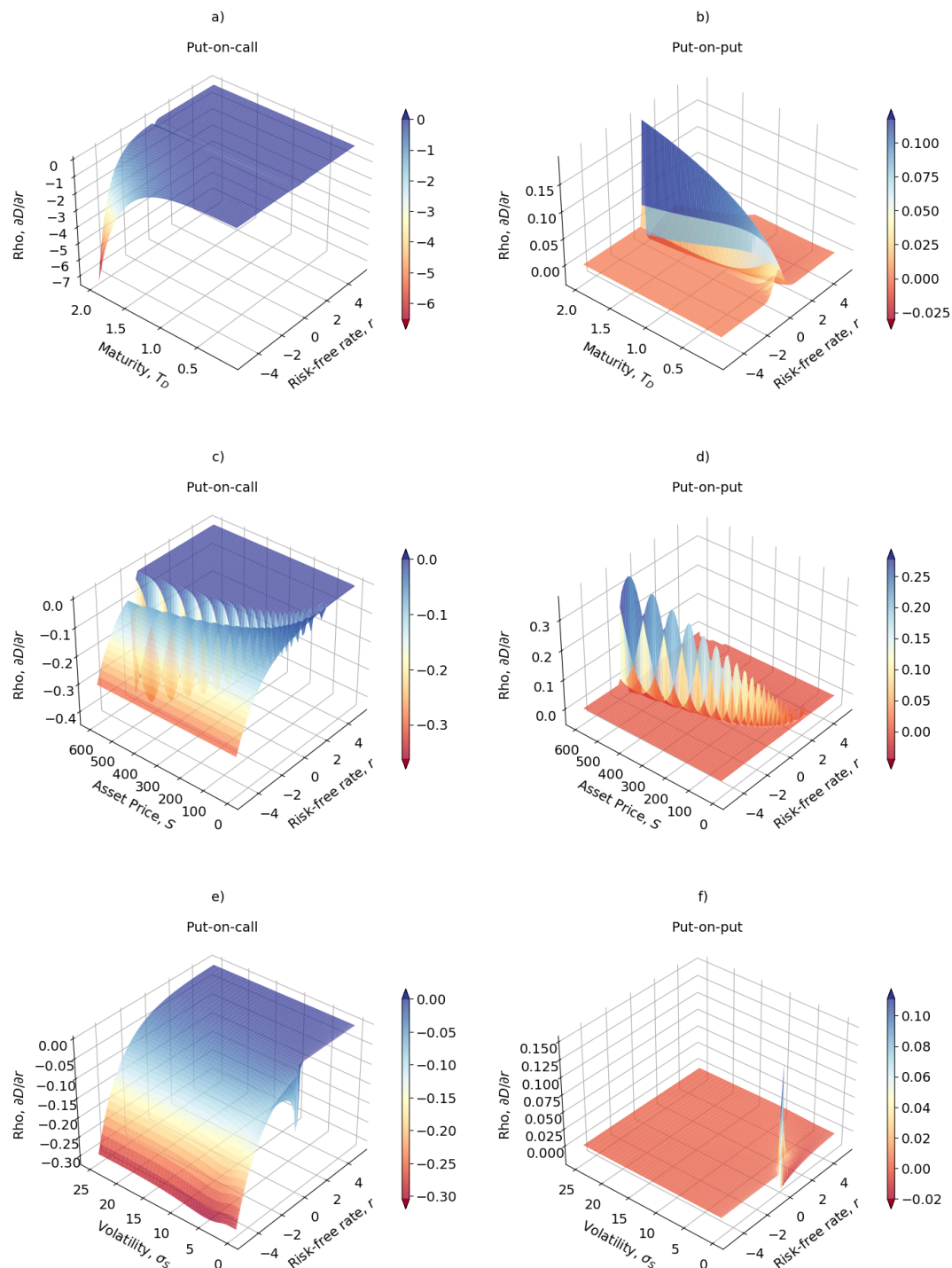


Figure A.9: Black–Scholes rho surface plots for a put-on-call and put-on-put option.

B Compound option reference values

B.1 Reference values for call-on-call compound options

The tables in the preceding pages are reference values for European and American call-on-call compound options for the Black–Scholes (Table [B.1](#)), Heston (Table [B.2](#)) and HHW/HCIR (Table [B.3](#)) models.

The parameters given in Table [7.8](#) and Table [7.13](#) are used.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Geske	Explicit	ADI (HV)	Explicit	ADI (HV)	Explicit	ADI (HV)
					closed	E_R	E_R	value	value	EE value	EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	6.4480	0.0001	0.0001	6.4488	6.4488	0.0000	0.0000
100	4	90	0.5	1.0	12.0007	0.0001	0.0001	12.0014	12.0014	0.0000	0.0000
90	4	100	0.5	1.0	2.3310	0.0003	0.0003	2.3316	2.3317	0.0000	0.0000
100	3	100	0.4	0.8	6.0086	0.0001	0.0001	6.0094	6.0094	0.0000	0.0000
100	6	100	0.5	1.2	5.9681	0.0001	0.0002	5.9688	5.9690	0.0000	0.0000
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	11.2430	0.0001	0.0001	11.2440	11.2440	0.0000	0.0000
100	4	90	0.5	1.0	16.2609	0.0000	0.0001	16.2617	16.2617	0.0000	0.0000
90	4	100	0.5	1.0	6.2311	0.0001	0.0002	6.2320	6.2321	0.0000	0.0000
100	3	100	0.4	0.8	10.3121	0.0001	0.0001	10.3131	10.3130	0.0000	0.0000
100	6	100	0.5	1.2	11.1092	0.0001	0.0001	11.1103	11.1104	0.0000	0.0000
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	10.2224	0.0001	0.0001	10.2233	10.2234	0.0000	0.0000
100	4	90	0.5	1.0	15.4478	0.0001	0.0001	15.4486	15.4487	0.0000	0.0000
90	4	100	0.5	1.0	5.2819	0.0002	0.0002	5.2828	5.2829	0.0000	0.0000
100	3	100	0.4	0.8	9.3679	0.0001	0.0001	9.3688	9.3688	0.0000	0.0000
100	6	100	0.5	1.2	10.0361	0.0001	0.0001	10.0372	10.0373	0.0000	0.0000
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	11.8075	0.0001	0.0001	11.8086	11.8086	0.0000	0.0000
100	4	90	0.5	1.0	17.5601	0.0001	0.0001	17.5611	17.5611	0.0000	0.0000
90	4	100	0.5	1.0	6.1723	0.0001	0.0002	6.1732	6.1733	0.0000	0.0000
100	3	100	0.4	0.8	10.5895	0.0001	0.0001	10.5906	10.5906	0.0000	0.0000
100	6	100	0.5	1.2	11.9505	0.0001	0.0001	11.9518	11.9519	0.0000	0.0000
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	10.4810	0.0001	0.0001	10.4820	10.4820	0.0000	0.0000
100	4	90	0.5	1.0	16.6158	0.0001	0.0001	16.6168	16.6168	0.0000	0.0000
90	4	100	0.5	1.0	4.8764	0.0002	0.0002	4.8773	4.8774	0.0000	0.0000
100	3	100	0.4	0.8	9.3636	0.0001	0.0001	9.3647	9.3647	0.0000	0.0000
100	6	100	0.5	1.2	10.5505	0.0001	0.0001	10.5518	10.5519	0.0000	0.0000
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	7.3242	0.0001	0.0001	7.3251	7.3251	0.0000	0.0000
100	4	90	0.5	1.0	12.7209	0.0001	0.0001	12.7216	12.7216	0.0000	0.0000
90	4	100	0.5	1.0	3.0079	0.0003	0.0003	3.0086	3.0088	0.0000	0.0000
100	3	100	0.4	0.8	6.8034	0.0001	0.0001	6.8043	6.8043	0.0000	0.0000
100	6	100	0.5	1.2	6.8927	0.0001	0.0002	6.8937	6.8938	0.0000	0.0000

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.1: Reference values for European and American call-on-call options using the Black–Scholes model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit value	ADI (HV) value	Differencee in value	Explicit value	ADI (HV) value	Explicit EE value	ADI (HV) EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	5.5730	5.5730	0.0001	5.5730	5.5730	0.0000	0.0000
100	4	90	0.5	1.0	11.5202	11.5202	0.0000	11.5202	11.5202	0.0000	0.0000
90	4	100	0.5	1.0	1.5352	1.5353	0.0001	1.5352	1.5353	0.0000	0.0000
100	3	100	0.4	0.8	5.3129	5.3129	0.0000	5.3129	5.3129	0.0000	0.0000
100	6	100	0.5	1.2	4.9910	4.9911	0.0001	4.9910	4.9911	0.0000	0.0000
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	10.4519	10.4524	0.0005	10.4524	10.4524	0.0005	-0.0000
100	4	90	0.5	1.0	15.5246	15.5253	0.0007	15.5252	15.5253	0.0006	-0.0000
90	4	100	0.5	1.0	5.5841	5.5844	0.0003	5.5843	5.5844	0.0002	-0.0000
100	3	100	0.4	0.8	9.6614	9.6613	0.0001	9.6613	9.6613	-0.0001	0.0000
100	6	100	0.5	1.2	10.2139	10.2208	0.0069	10.2202	10.2208	0.0063	-0.0000
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	9.8142	9.8142	0.0000	9.8142	9.8142	0.0000	0.0000
100	4	90	0.5	1.0	15.6474	15.6474	0.0000	15.6475	15.6474	0.0000	0.0000
90	4	100	0.5	1.0	4.3259	4.3260	0.0001	4.3260	4.3260	0.0000	0.0000
100	3	100	0.4	0.8	9.0554	9.0554	0.0000	9.0554	9.0554	0.0001	0.0000
100	6	100	0.5	1.2	9.5459	9.5460	0.0001	9.5459	9.5460	0.0001	0.0000
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	11.2115	11.2115	0.0000	11.2115	11.2115	0.0000	0.0000
100	4	90	0.5	1.0	17.2733	17.2733	0.0000	17.2733	17.2733	0.0000	0.0000
90	4	100	0.5	1.0	5.4987	5.4988	0.0001	5.4987	5.4988	0.0000	0.0000
100	3	100	0.4	0.8	10.0655	10.0655	0.0000	10.0655	10.0655	0.0000	0.0000
100	6	100	0.5	1.2	11.3161	11.3162	0.0001	11.3161	11.3162	0.0000	0.0000
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	-	9.8160	-	-	9.8192	-	0.0032
100	4	90	0.5	1.0	-	18.2198	-	-	18.2213	-	0.0014
90	4	100	0.5	1.0	-	1.5945	-	-	1.6425	-	0.0480
100	3	100	0.4	0.8	-	8.6077	-	-	8.6095	-	0.0018
100	6	100	0.5	1.2	-	10.0918	-	-	10.0949	-	0.0031
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	-	5.2121	-	-	5.2236	-	0.0115
100	4	90	0.5	1.0	-	12.5267	-	-	12.5282	-	0.0015
90	4	100	0.5	1.0	-	1.3844	-	-	1.3875	-	0.0031
100	3	100	0.4	0.8	5.4699	5.0457	0.4242	5.4749	5.0508	0.0050	0.0051
100	6	100	0.5	1.2	-	4.6291	-	-	4.6329	-	0.0038

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.2: Reference values for European and American call-on-call options using the Heston model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit value	ADI (HV) value	Differencee in value	Explicit value	ADI (HV) value	Explicit EE value	ADI (HV) EE value
Set A: $p = 0, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	6.3355	6.3352	0.0003	6.4315	6.4308	0.0960	0.0956
100	4	90	0.5	1.0	12.0006	12.0007	0.0001	12.2445	12.2440	0.2439	0.2434
90	4	100	0.5	1.0	2.1871	2.1869	0.0002	2.2064	2.2060	0.0193	0.0192
100	3	100	0.4	0.8	5.8466	5.8465	0.0001	5.9116	5.9112	0.0651	0.0648
100	6	100	0.5	1.2	5.9952	5.9946	0.0006	6.1180	6.1168	0.1228	0.1222
Set B: $p = 0, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	10.8009	10.8013	0.0004	10.8014	10.8013	0.0005	0.0000
100	4	90	0.5	1.0	15.8265	15.8270	0.0005	15.8272	15.8270	0.0007	0.0000
90	4	100	0.5	1.0	5.8947	5.8950	0.0003	5.8950	5.8950	0.0002	-0.0000
100	3	100	0.4	0.8	9.9143	9.9142	0.0001	9.9142	9.9142	-0.0001	-0.0000
100	6	100	0.5	1.2	10.6629	10.6695	0.0066	10.6693	10.6695	0.0065	0.0000
Set C: $p = 0, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055, \sigma_r = 0.06,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	9.5081	9.5084	0.0003	9.5938	9.5943	0.0857	0.0859
100	4	90	0.5	1.0	15.3891	15.3894	0.0003	15.4974	15.4980	0.1083	0.1086
90	4	100	0.5	1.0	4.0572	4.0575	0.0003	4.1067	4.1070	0.0495	0.0495
100	3	100	0.4	0.8	8.8297	8.8298	0.0001	8.8790	8.8793	0.0493	0.0495
100	6	100	0.5	1.2	9.1596	9.1603	0.0007	9.2887	9.2895	0.1291	0.1292
Set A: $p = 0.5, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	5.6916	5.6917	0.0000	5.6916	5.6917	0.0000	0.0000
100	4	90	0.5	1.0	11.5938	11.5937	0.0000	11.5938	11.5938	0.0000	0.0001
90	4	100	0.5	1.0	1.6375	1.6377	0.0001	1.6375	1.6377	0.0000	0.0000
100	3	100	0.4	0.8	5.3985	5.3986	0.0000	5.3985	5.3986	0.0000	0.0000
100	6	100	0.5	1.2	5.1440	5.1441	0.0001	5.1440	5.1441	0.0000	0.0000
Set B: $p = 0.5, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	10.5290	10.5293	0.0004	10.5294	10.5293	0.0005	-0.0000
100	4	90	0.5	1.0	15.5911	15.5916	0.0005	15.5917	15.5916	0.0006	-0.0000
90	4	100	0.5	1.0	5.6527	5.6529	0.0003	5.6529	5.6529	0.0002	-0.0000
100	3	100	0.4	0.8	9.7172	9.7171	0.0001	9.7172	9.7171	-0.0001	0.0000
100	6	100	0.5	1.2	10.3133	10.3198	0.0065	10.3196	10.3198	0.0063	-0.0000
Set C: $p = 0.5, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	9.7331	9.7335	0.0004	9.7331	9.7336	0.0000	0.0000
100	4	90	0.5	1.0	15.5824	15.5830	0.0005	15.5824	15.5830	0.0000	0.0000
90	4	100	0.5	1.0	4.2497	4.2501	0.0004	4.2498	4.2501	0.0001	0.0001
100	3	100	0.4	0.8	8.9969	8.9971	0.0002	8.9970	8.9972	0.0000	0.0000
100	6	100	0.5	1.2	9.4409	9.4417	0.0008	9.4410	9.4418	0.0001	0.0001
Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.											

Table B.3: Reference values for European and American call-on-call options using the HHW and HCIR models.

B.2 Reference values for call-on-put compound options

The tables in the preceding pages are reference values for European and American call-on-put compound options for the Black–Scholes (Table B.4), Heston (Table B.5) and HHW/HCIR (Table B.6) models.

The parameters given in Table 7.8 and Table 7.13 are used.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Geske	Explicit	ADI (HV)	Explicit	ADI (HV)	Explicit	ADI (HV)
					closed	E_R	E_R	value	value	EE value	EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	3.3576	0.0001	0.0001	3.6892	3.6892	0.3313	0.3312
100	4	90	0.5	1.0	0.8916	0.0003	0.0004	0.9846	0.9847	0.0928	0.0928
90	4	100	0.5	1.0	7.5766	0.0001	0.0001	8.3942	8.3942	0.8172	0.8171
100	3	100	0.4	0.8	3.4623	0.0001	0.0001	3.7235	3.7235	0.2608	0.2607
100	6	100	0.5	1.2	2.5394	0.0001	0.0002	2.9212	2.9213	0.3814	0.3813
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	6.9299	0.0001	0.0001	7.3916	7.3915	0.4613	0.4611
100	4	90	0.5	1.0	3.3300	0.0001	0.0001	3.5490	3.5490	0.2186	0.2185
90	4	100	0.5	1.0	11.0162	0.0000	0.0000	11.8200	11.8198	0.8033	0.8031
100	3	100	0.4	0.8	6.8262	0.0001	0.0001	7.1824	7.1823	0.3556	0.3555
100	6	100	0.5	1.2	6.0573	0.0001	0.0001	6.6194	6.6194	0.5616	0.5614
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	5.5687	0.0001	0.0001	6.0816	6.0815	0.5125	0.5124
100	4	90	0.5	1.0	2.3689	0.0001	0.0001	2.5871	2.5871	0.2179	0.2179
90	4	100	0.5	1.0	9.5783	0.0000	0.0000	10.5455	10.5453	0.9667	0.9665
100	3	100	0.4	0.8	5.5814	0.0001	0.0001	5.9804	5.9803	0.3985	0.3984
100	6	100	0.5	1.2	4.6612	0.0001	0.0002	5.2760	5.2761	0.6143	0.6142
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	3.4285	0.0000	0.0001	4.4305	4.4304	1.0018	1.0017
100	4	90	0.5	1.0	1.2230	0.0001	0.0002	1.5892	1.5892	0.3660	0.3660
90	4	100	0.5	1.0	6.6576	0.0000	0.0000	8.7768	8.7767	2.1191	2.1188
100	3	100	0.4	0.8	3.6779	0.0001	0.0001	4.4825	4.4825	0.8043	0.8041
100	6	100	0.5	1.2	2.4994	0.0001	0.0002	3.6407	3.6408	1.1410	1.1409
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	2.2317	0.0001	0.0002	3.1941	3.1941	0.9622	0.9621
100	4	90	0.5	1.0	0.5919	0.0002	0.0003	0.8639	0.8639	0.2719	0.2718
90	4	100	0.5	1.0	5.2127	0.0000	0.0001	7.6315	7.6314	2.4186	2.4183
100	3	100	0.4	0.8	2.5060	0.0001	0.0001	3.2970	3.2969	0.7907	0.7906
100	6	100	0.5	1.2	1.4439	0.0001	0.0003	2.4871	2.4871	1.0430	1.0429
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	4.1356	0.0001	0.0001	4.4715	4.4715	0.3354	0.3353
100	4	90	0.5	1.0	1.3302	0.0002	0.0003	1.4417	1.4418	0.1112	0.1112
90	4	100	0.5	1.0	8.3796	0.0001	0.0001	9.1257	9.1257	0.7456	0.7455
100	3	100	0.4	0.8	4.1948	0.0001	0.0001	4.4573	4.4573	0.2620	0.2619
100	6	100	0.5	1.2	3.2852	0.0002	0.0002	3.6797	3.6798	0.3939	0.3938

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.4: Reference values for European and American call-on-put options using the Black–Scholes model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit	ADI (HV)	Differencee	Explicit	ADI (HV)	Explicit	ADI (HV)
					value	value	in value	value	value	EE value	EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	2.7396	2.7397	0.0001	3.1174	3.1174	0.3777	0.3777
100	4	90	0.5	1.0	0.7533	0.7533	0.0001	0.8505	0.8505	0.0972	0.0972
90	4	100	0.5	1.0	6.5962	6.5963	0.0001	7.6551	7.6551	1.0589	1.0588
100	3	100	0.4	0.8	2.9287	2.9288	0.0001	3.2227	3.2227	0.2939	0.2939
100	6	100	0.5	1.2	1.9904	1.9906	0.0002	2.4223	2.4224	0.4318	0.4318
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	6.1926	6.1938	0.0012	6.6778	6.6778	0.4852	0.4840
100	4	90	0.5	1.0	2.7655	2.7666	0.0011	2.9816	2.9816	0.2160	0.2150
90	4	100	0.5	1.0	10.3063	10.3075	0.0011	11.1840	11.1839	0.8776	0.8764
100	3	100	0.4	0.8	6.2056	6.2056	0.0000	6.5810	6.5809	0.3754	0.3753
100	6	100	0.5	1.2	5.2631	5.2737	0.0106	5.8557	5.8562	0.5926	0.5825
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	5.4638	5.4639	0.0001	5.9472	5.9472	0.4834	0.4833
100	4	90	0.5	1.0	2.7710	2.7711	0.0001	2.9805	2.9805	0.2095	0.2094
90	4	100	0.5	1.0	8.7888	8.7889	0.0001	9.7918	9.7918	1.0030	1.0029
100	3	100	0.4	0.8	5.4699	5.4699	0.0000	5.8410	5.8410	0.3712	0.3711
100	6	100	0.5	1.2	4.6318	4.6320	0.0002	5.2178	5.2179	0.5860	0.5859
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	3.0208	3.0209	0.0001	4.0434	4.0434	1.0226	1.0225
100	4	90	0.5	1.0	1.1611	1.1612	0.0001	1.5240	1.5241	0.3630	0.3629
90	4	100	0.5	1.0	5.9923	5.9925	0.0002	8.3341	8.3340	2.3418	2.3415
100	3	100	0.4	0.8	3.2810	3.2811	0.0001	4.1159	4.1159	0.8349	0.8348
100	6	100	0.5	1.2	2.1843	2.1845	0.0002	3.3216	3.3217	1.1374	1.1372
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	–	2.3635	–	–	2.8874	–	0.5240
100	4	90	0.5	1.0	–	1.5696	–	–	1.7912	–	0.2216
90	4	100	0.5	1.0	–	3.2829	–	–	6.1178	–	2.8349
100	3	100	0.4	0.8	–	2.4594	–	18.1864	2.9066	–	0.4472
100	6	100	0.5	1.2	–	2.0044	–	–	2.5859	–	0.5815
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	–	3.3106	–	–	3.6093	–	0.2987
100	4	90	0.5	1.0	–	1.9190	–	–	2.0148	–	0.0958
90	4	100	0.5	1.0	–	6.1236	–	–	7.5457	–	1.4221
100	3	100	0.4	0.8	3.8181	3.3500	0.4682	4.0528	3.6140	0.2347	0.2640
100	6	100	0.5	1.2	–	2.9203	–	–	3.2236	–	0.3033

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.5: Reference values for European and American call-on-put options using the Heston model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit	ADI (HV)	Differencee	Explicit	ADI (HV)	Explicit	ADI (HV)
					value	value	in value	value	value	EE value	EE value
Set A: $p = 0, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	3.4867	3.4866	0.0001	3.7982	3.7960	0.3115	0.3094
100	4	90	0.5	1.0	1.1123	1.1126	0.0003	1.1855	1.1852	0.0732	0.0726
90	4	100	0.5	1.0	7.4966	7.4963	0.0003	8.4502	8.4457	0.9536	0.9494
100	3	100	0.4	0.8	3.4589	3.4589	0.0000	3.7019	3.7001	0.2430	0.2413
100	6	100	0.5	1.2	2.9105	2.9103	0.0002	3.2624	3.2600	0.3519	0.3496
Set B: $p = 0, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	6.5460	6.5473	0.0013	6.8253	6.8254	0.2793	0.2781
100	4	90	0.5	1.0	3.0391	3.0402	0.0011	3.1434	3.1434	0.1043	0.1032
90	4	100	0.5	1.0	10.6477	10.6489	0.0012	11.2395	11.2399	0.5918	0.5910
100	3	100	0.4	0.8	6.4620	6.4620	0.0000	6.6874	6.6875	0.2254	0.2255
100	6	100	0.5	1.2	5.7148	5.7253	0.0105	6.0473	6.0477	0.3325	0.3223
Set C: $p = 0, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055, \sigma_r = 0.06,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	5.2277	5.2279	0.0002	5.9540	5.9538	0.7263	0.7258
100	4	90	0.5	1.0	2.5976	2.5978	0.0001	2.9322	2.9322	0.3346	0.3344
90	4	100	0.5	1.0	8.5514	8.5517	0.0003	9.9241	9.9237	1.3726	1.3719
100	3	100	0.4	0.8	5.2814	5.2815	0.0001	5.8315	5.8314	0.5501	0.5498
100	6	100	0.5	1.2	4.3661	4.3665	0.0004	5.2473	5.2471	0.8812	0.8806
Set A: $p = 0.5, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	2.8384	2.8387	0.0003	3.1464	3.1466	0.3080	0.3079
100	4	90	0.5	1.0	0.7964	0.7966	0.0002	0.8714	0.8715	0.0749	0.0749
90	4	100	0.5	1.0	6.7218	6.7222	0.0004	7.6524	7.6525	0.9305	0.9303
100	3	100	0.4	0.8	3.0043	3.0044	0.0001	3.2460	3.2462	0.2418	0.2418
100	6	100	0.5	1.2	2.1015	2.1021	0.0006	2.4520	2.4524	0.3505	0.3504
Set B: $p = 0.5, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	6.2682	6.2695	0.0012	6.7029	6.7029	0.4347	0.4335
100	4	90	0.5	1.0	2.8235	2.8246	0.0011	3.0116	3.0117	0.1881	0.1871
90	4	100	0.5	1.0	10.3795	10.3807	0.0012	11.1884	11.1883	0.8088	0.8076
100	3	100	0.4	0.8	6.2610	6.2610	0.0000	6.6001	6.6000	0.3391	0.3390
100	6	100	0.5	1.2	5.3587	5.3693	0.0106	5.8869	5.8873	0.5282	0.5180
Set C: $p = 0.5, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	5.3920	5.3921	0.0001	5.9294	5.9294	0.5374	0.5373
100	4	90	0.5	1.0	2.7198	2.7199	0.0001	2.9578	2.9579	0.2380	0.2380
90	4	100	0.5	1.0	8.7124	8.7125	0.0001	9.7951	9.7950	1.0827	1.0825
100	3	100	0.4	0.8	5.4161	5.4162	0.0000	5.8263	5.8263	0.4102	0.4101
100	6	100	0.5	1.2	4.5448	4.5449	0.0002	5.1980	5.1981	0.6532	0.6531
Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.											

Table B.6: Reference values for European and American call-on-put options using the HHW and HCIR models.

B.3 Reference values for put-on-call compound options

The tables in the preceding pages are reference values for European and American put-on-call compound options for the Black–Scholes (Table [B.7](#)), Heston (Table [B.8](#)) and HHW/HCIR (Table [B.9](#)) models.

The parameters given in Table [7.8](#) and Table [7.13](#) are used.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Geske	Explicit	ADI (HV)	Explicit	ADI (HV)	Explicit	ADI (HV)
					closed	E_R	E_R	value	value	EE value	EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	0.7495	0.0002	0.0001	0.7514	0.7516	0.0020	0.0020
100	4	90	0.5	1.0	0.2562	0.0001	0.0003	0.2567	0.2568	0.0005	0.0005
90	4	100	0.5	1.0	1.6713	0.0001	0.0000	1.6777	1.6779	0.0067	0.0066
100	3	100	0.4	0.8	0.5001	0.0002	0.0001	0.5009	0.5011	0.0010	0.0010
100	6	100	0.5	1.2	1.1577	0.0002	0.0001	1.1613	1.1616	0.0038	0.0038
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	0.6070	0.0001	0.0002	0.6086	0.6088	0.0017	0.0017
100	4	90	0.5	1.0	0.3123	0.0003	0.0001	0.3129	0.3130	0.0008	0.0007
90	4	100	0.5	1.0	1.1089	0.0001	0.0001	1.1128	1.1130	0.0041	0.0040
100	3	100	0.4	0.8	0.4000	0.0000	0.0003	0.4009	0.4010	0.0008	0.0008
100	6	100	0.5	1.2	0.8792	0.0000	0.0003	0.8822	0.8825	0.0030	0.0030
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	0.6012	0.0001	0.0002	0.6031	0.6033	0.0020	0.0019
100	4	90	0.5	1.0	0.2810	0.0002	0.0002	0.2817	0.2818	0.0008	0.0007
90	4	100	0.5	1.0	1.1679	0.0001	0.0001	1.1729	1.1730	0.0050	0.0050
100	3	100	0.4	0.8	0.3990	0.0001	0.0002	0.4000	0.4001	0.0010	0.0009
100	6	100	0.5	1.2	0.8772	0.0000	0.0003	0.8807	0.8809	0.0035	0.0034
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	0.4178	0.0003	0.0000	0.4202	0.4203	0.0026	0.0025
100	4	90	0.5	1.0	0.1683	0.0003	0.0001	0.1691	0.1691	0.0008	0.0008
90	4	100	0.5	1.0	0.9215	0.0002	0.0001	0.9289	0.9290	0.0076	0.0076
100	3	100	0.4	0.8	0.2857	0.0000	0.0003	0.2870	0.2871	0.0013	0.0012
100	6	100	0.5	1.2	0.5795	0.0000	0.0003	0.5838	0.5840	0.0043	0.0043
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	0.3975	0.0003	0.0000	0.4000	0.4001	0.0026	0.0026
100	4	90	0.5	1.0	0.1291	0.0003	0.0002	0.1297	0.1298	0.0006	0.0006
90	4	100	0.5	1.0	1.0007	0.0002	0.0001	1.0097	1.0098	0.0092	0.0092
100	3	100	0.4	0.8	0.2758	0.0000	0.0003	0.2771	0.2771	0.0013	0.0013
100	6	100	0.5	1.2	0.5549	0.0001	0.0004	0.5594	0.5595	0.0044	0.0044
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	0.7225	0.0000	0.0002	0.7243	0.7245	0.0018	0.0018
100	4	90	0.5	1.0	0.2793	0.0000	0.0004	0.2799	0.2800	0.0006	0.0005
90	4	100	0.5	1.0	1.5263	0.0001	0.0001	1.5318	1.5319	0.0056	0.0055
100	3	100	0.4	0.8	0.4802	0.0000	0.0002	0.4811	0.4812	0.0009	0.0009
100	6	100	0.5	1.2	1.1011	0.0000	0.0002	1.1046	1.1049	0.0035	0.0034
Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer--Verwer.											

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdoerfer–Verwer.

Table B.7: Reference values for European and American put-on-call options using the Black–Scholes model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American				
					Explicit value	ADI (HV) value	Differencee in value	Explicit value	ADI (HV) value	Explicit EE value	ADI (HV) EE value	
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$												
100	4	100	0.5	1.0	0.8825	0.8826	0.0001	0.8851	0.8853	0.0027	0.0027	
100	4	90	0.5	1.0	0.2901	0.2902	0.0001	0.2907	0.2908	0.0006	0.0006	
	90	4	100	0.5	1.0	2.0364	2.0366	0.0001	2.0470	2.0471	0.0105	0.0105
100	3	100	0.4	0.8	0.5895	0.5896	0.0001	0.5907	0.5908	0.0012	0.0012	
100	6	100	0.5	1.2	1.4003	1.4005	0.0002	1.4057	1.4059	0.0054	0.0054	
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$												
100	4	100	0.5	1.0	0.6668	0.6664	0.0004	0.6681	0.6682	0.0013	0.0019	
100	4	90	0.5	1.0	0.3324	0.3322	0.0003	0.3328	0.3329	0.0004	0.0008	
	90	4	100	0.5	1.0	1.2259	1.2253	0.0006	1.2297	1.2299	0.0038	0.0046
100	3	100	0.4	0.8	0.4435	0.4436	0.0001	0.4444	0.4445	0.0009	0.0009	
100	6	100	0.5	1.2	0.9842	0.9791	0.0051	0.9826	0.9825	-0.0017	0.0034	
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$												
100	4	100	0.5	1.0	0.5799	0.5800	0.0001	0.5822	0.5823	0.0023	0.0023	
100	4	90	0.5	1.0	0.2689	0.2690	0.0001	0.2697	0.2698	0.0008	0.0008	
	90	4	100	0.5	1.0	1.2229	1.2230	0.0002	1.2296	1.2298	0.0068	0.0067
100	3	100	0.4	0.8	0.3828	0.3829	0.0001	0.3839	0.3840	0.0011	0.0011	
100	6	100	0.5	1.2	0.8619	0.8621	0.0002	0.8658	0.8660	0.0039	0.0039	
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$												
100	4	100	0.5	1.0	0.4478	0.4479	0.0001	0.4506	0.4507	0.0028	0.0028	
100	4	90	0.5	1.0	0.1849	0.1850	0.0001	0.1859	0.1859	0.0009	0.0009	
	90	4	100	0.5	1.0	1.0183	1.0184	0.0002	1.0268	1.0269	0.0085	0.0085
100	3	100	0.4	0.8	0.3139	0.3140	0.0001	0.3153	0.3154	0.0014	0.0014	
100	6	100	0.5	1.2	0.6263	0.6265	0.0002	0.6312	0.6313	0.0048	0.0048	
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$												
100	4	100	0.5	1.0	-	0.2921	-	-	0.2940	-	0.0019	
100	4	90	0.5	1.0	-	0.1316	-	-	0.1313	-	-0.0003	
	90	4	100	0.5	1.0	-	1.3224	-	1.3354	-	0.0131	
100	3	100	0.4	0.8	-	0.2162	-	-	0.2176	-	0.0014	
100	6	100	0.5	1.2	-	0.4112	-	-	0.4143	-	0.0032	
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$												
100	4	100	0.5	1.0	-	0.9073	-	-	0.9078	-	0.0006	
100	4	90	0.5	1.0	-	0.2712	-	-	0.2713	-	0.0001	
	90	4	100	0.5	1.0	-	2.8309	-	2.8330	-	0.0021	
100	3	100	0.4	0.8	0.6062	0.6585	0.0523	0.6067	0.6589	0.0004	0.0004	
100	6	100	0.5	1.2	-	1.4342	-	-	1.4355	-	0.0012	

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.8: Reference values for European and American put-on-call options using the Heston model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American				
					Explicit	ADI (HV)	Differencee	Explicit	ADI (HV)	Explicit	ADI (HV)	
					value	value	in value	value	value	EE value	EE value	
Set A: $p = 0, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$												
100	4	100	0.5	1.0	0.9686	0.9686	0.0000	0.9603	0.9602	-0.0083	-0.0083	
100	4	90	0.5	1.0	0.3778	0.3778	0.0001	0.3687	0.3687	-0.0091	-0.0091	
	90	4	100	0.5	1.0	1.9721	1.9723	0.0001	1.9716	1.9717	-0.0006	-0.0006
100	3	100	0.4	0.8	0.6420	0.6421	0.0001	0.6382	0.6382	-0.0039	-0.0039	
100	6	100	0.5	1.2	1.5158	1.5157	0.0001	1.4957	1.4956	-0.0201	-0.0201	
Set B: $p = 0, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$												
100	4	100	0.5	1.0	0.6844	0.6839	0.0004	0.6849	0.6850	0.0005	0.0011	
100	4	90	0.5	1.0	0.3535	0.3533	0.0003	0.3535	0.3537	0.0000	0.0004	
	90	4	100	0.5	1.0	1.2267	1.2261	0.0006	1.2288	1.2289	0.0021	0.0028
100	3	100	0.4	0.8	0.4544	0.4545	0.0001	0.4549	0.4551	0.0005	0.0005	
100	6	100	0.5	1.2	1.0068	1.0019	0.0048	1.0038	1.0038	-0.0030	0.0019	
Set C: $p = 0, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055, \sigma_r = 0.06,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$												
100	4	100	0.5	1.0	0.5673	0.5674	0.0001	0.5681	0.5682	0.0009	0.0008	
100	4	90	0.5	1.0	0.2550	0.2551	0.0001	0.2552	0.2553	0.0002	0.0002	
	90	4	100	0.5	1.0	1.2331	1.2332	0.0002	1.2370	1.2372	0.0039	0.0039
100	3	100	0.4	0.8	0.3736	0.3738	0.0001	0.3743	0.3744	0.0007	0.0007	
100	6	100	0.5	1.2	0.8506	0.8509	0.0003	0.8491	0.8494	-0.0014	-0.0015	
Set A: $p = 0.5, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$												
100	4	100	0.5	1.0	0.8963	0.8965	0.0001	0.8985	0.8986	0.0021	0.0021	
100	4	90	0.5	1.0	0.3024	0.3025	0.0001	0.3029	0.3030	0.0005	0.0005	
	90	4	100	0.5	1.0	2.0255	2.0257	0.0001	2.0342	2.0343	0.0086	0.0086
100	3	100	0.4	0.8	0.5981	0.5982	0.0001	0.5990	0.5991	0.0010	0.0010	
100	6	100	0.5	1.2	1.4196	1.4198	0.0002	1.4239	1.4241	0.0042	0.0042	
Set B: $p = 0.5, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$												
100	4	100	0.5	1.0	0.6706	0.6702	0.0004	0.6718	0.6719	0.0011	0.0017	
100	4	90	0.5	1.0	0.3370	0.3367	0.0003	0.3373	0.3374	0.0003	0.0007	
	90	4	100	0.5	1.0	1.2260	1.2254	0.0006	1.2294	1.2296	0.0034	0.0042
100	3	100	0.4	0.8	0.4459	0.4460	0.0001	0.4467	0.4468	0.0008	0.0008	
100	6	100	0.5	1.2	0.9893	0.9843	0.0050	0.9874	0.9873	-0.0020	0.0030	
Set C: $p = 0.5, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$												
100	4	100	0.5	1.0	0.5748	0.5749	0.0001	0.5773	0.5774	0.0025	0.0025	
100	4	90	0.5	1.0	0.2644	0.2645	0.0001	0.2653	0.2654	0.0009	0.0009	
	90	4	100	0.5	1.0	1.2227	1.2228	0.0002	1.2300	1.2301	0.0073	0.0073
100	3	100	0.4	0.8	0.3797	0.3798	0.0001	0.3809	0.3810	0.0012	0.0012	
100	6	100	0.5	1.2	0.8550	0.8552	0.0002	0.8593	0.8595	0.0043	0.0043	
Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.												

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.9: Reference values for European and American put-on-call options using the HHW and HCIR models.

B.4 Reference values for put-on-put compound options

The tables in the preceding pages are reference values for European and American put-on-put compound options for the Black–Scholes (Table [B.10](#)), Heston (Table [B.11](#)) and HHW/HCIR (Table [B.12](#)) models.

The parameters given in Table [7.8](#) and Table [7.13](#) are used.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Geske	Explicit	ADI (HV)	Explicit	ADI (HV)	Explicit	ADI (HV)
					closed	E_R	E_R	value	value	EE value	EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	1.0814	0.0002	0.0000	1.0655	1.0657	-0.0156	-0.0156
100	4	90	0.5	1.0	2.1773	0.0001	0.0000	2.1671	2.1673	-0.0100	-0.0100
90	4	100	0.5	1.0	0.3891	0.0002	0.0001	0.3792	0.3794	-0.0098	-0.0098
100	3	100	0.4	0.8	0.7039	0.0002	0.0000	0.6950	0.6951	-0.0087	-0.0087
100	6	100	0.5	1.2	1.8174	0.0001	0.0001	1.7727	1.7730	-0.0445	-0.0445
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	0.6721	0.0001	0.0001	0.6646	0.6648	-0.0075	-0.0075
100	4	90	0.5	1.0	1.2720	0.0001	0.0001	1.2637	1.2639	-0.0081	-0.0081
90	4	100	0.5	1.0	0.3221	0.0001	0.0003	0.3170	0.3171	-0.0051	-0.0051
100	3	100	0.4	0.8	0.4360	0.0001	0.0002	0.4317	0.4319	-0.0042	-0.0042
100	6	100	0.5	1.2	1.0527	0.0000	0.0002	1.0311	1.0314	-0.0215	-0.0215
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	0.8003	0.0002	0.0001	0.7885	0.7887	-0.0117	-0.0117
100	4	90	0.5	1.0	1.5196	0.0001	0.0000	1.5081	1.5083	-0.0114	-0.0114
90	4	100	0.5	1.0	0.3669	0.0002	0.0002	0.3589	0.3590	-0.0080	-0.0080
100	3	100	0.4	0.8	0.5180	0.0002	0.0000	0.5114	0.5116	-0.0064	-0.0064
100	6	100	0.5	1.2	1.2910	0.0000	0.0002	1.2573	1.2576	-0.0337	-0.0337
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	1.1466	0.0001	0.0001	1.1081	1.1083	-0.0384	-0.0384
100	4	90	0.5	1.0	1.9785	0.0000	0.0000	1.9511	1.9512	-0.0274	-0.0273
90	4	100	0.5	1.0	0.5647	0.0001	0.0002	0.5344	0.5345	-0.0303	-0.0302
100	3	100	0.4	0.8	0.7371	0.0000	0.0002	0.7153	0.7155	-0.0218	-0.0218
100	6	100	0.5	1.2	1.9446	0.0000	0.0001	1.8325	1.8328	-0.1121	-0.1121
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	1.4368	0.0000	0.0001	1.3780	1.3782	-0.0588	-0.0588
100	4	90	0.5	1.0	2.4150	0.0000	0.0000	2.3880	2.3882	-0.0269	-0.0268
90	4	100	0.5	1.0	0.6754	0.0001	0.0003	0.6264	0.6266	-0.0490	-0.0490
100	3	100	0.4	0.8	0.9286	0.0000	0.0001	0.8945	0.8947	-0.0340	-0.0340
100	6	100	0.5	1.2	2.4768	0.0000	0.0001	2.3106	2.3109	-0.1661	-0.1660
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	0.9563	0.0001	0.0001	0.9441	0.9443	-0.0122	-0.0121
100	4	90	0.5	1.0	1.9189	0.0001	0.0000	1.9092	1.9094	-0.0095	-0.0095
90	4	100	0.5	1.0	0.3702	0.0001	0.0004	0.3625	0.3627	-0.0077	-0.0077
100	3	100	0.4	0.8	0.6217	0.0000	0.0002	0.6148	0.6149	-0.0069	-0.0069
100	6	100	0.5	1.2	1.5821	0.0000	0.0002	1.5474	1.5477	-0.0348	-0.0347

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdoerfer–Verwer.

Table B.10: Reference values for European and American put-on-put options using the Black–Scholes model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit value	ADI (HV) value	Differencee in value	Explicit value	ADI (HV) value	Explicit EE value	ADI (HV) EE value
Set A: $q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	1.4718	1.4719	0.0002	1.4490	1.4491	-0.0228	-0.0228
100	4	90	0.5	1.0	2.5536	2.5538	0.0001	2.5461	2.5463	-0.0075	-0.0075
90	4	100	0.5	1.0	0.5700	0.5701	0.0001	0.5501	0.5503	-0.0199	-0.0199
100	3	100	0.4	0.8	0.9558	0.9559	0.0001	0.9436	0.9437	-0.0122	-0.0122
100	6	100	0.5	1.2	2.4887	2.4889	0.0002	2.4232	2.4234	-0.0655	-0.0655
Set B: $q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \rho_{S,v} = 0.6$											
100	4	100	0.5	1.0	0.7860	0.7863	0.0002	0.7771	0.7772	-0.0090	-0.0090
100	4	90	0.5	1.0	1.4642	1.4643	0.0001	1.4549	1.4550	-0.0093	-0.0092
90	4	100	0.5	1.0	0.3767	0.3769	0.0002	0.3704	0.3705	-0.0063	-0.0064
100	3	100	0.4	0.8	0.5100	0.5101	0.0002	0.5048	0.5050	-0.0052	-0.0052
100	6	100	0.5	1.2	1.2589	1.2575	0.0014	1.2306	1.2308	-0.0283	-0.0267
Set C: $q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \rho_{S,v} = -0.7571$											
100	4	100	0.5	1.0	1.0832	1.0834	0.0002	1.0718	1.0720	-0.0114	-0.0114
100	4	90	0.5	1.0	1.7111	1.7113	0.0002	1.7051	1.7053	-0.0060	-0.0060
90	4	100	0.5	1.0	0.5892	0.5893	0.0002	0.5767	0.5768	-0.0125	-0.0125
100	3	100	0.4	0.8	0.7033	0.7034	0.0002	0.6966	0.6968	-0.0066	-0.0066
100	6	100	0.5	1.2	1.7379	1.7382	0.0003	1.7052	1.7055	-0.0327	-0.0327
Set D: $q = 0.005, \kappa_v = 2.5, \theta_v = 0.06, \sigma_v = 0.5, \rho_{S,v} = -0.1$											
100	4	100	0.5	1.0	1.3663	1.3664	0.0002	1.3156	1.3158	-0.0507	-0.0507
100	4	90	0.5	1.0	2.2214	2.2216	0.0001	2.1941	2.1942	-0.0274	-0.0274
90	4	100	0.5	1.0	0.6706	0.6708	0.0002	0.6266	0.6268	-0.0440	-0.0440
100	3	100	0.4	0.8	0.8933	0.8935	0.0001	0.8642	0.8644	-0.0291	-0.0291
100	6	100	0.5	1.2	2.3123	2.3125	0.0002	2.1722	2.1725	-0.1401	-0.1400
Set E: $q = 0.005, \kappa_v = 0.5, \theta_v = 0.04, \sigma_v = 1, \rho_{S,v} = -0.9$											
100	4	100	0.5	1.0	–	2.9386	–	–	2.8923	–	-0.0463
100	4	90	0.5	1.0	–	3.1389	–	–	3.1309	–	-0.0080
90	4	100	0.5	1.0	–	2.5413	–	–	1.6906	–	-0.8507
100	3	100	0.4	0.8	–	2.1128	–	42.6577	2.0836	–	-0.0292
100	6	100	0.5	1.2	–	4.4924	–	–	4.4065	–	-0.0858
Set F: $q = 0.005, \kappa_v = 0.3, \theta_v = 0.04, \sigma_v = 0.9, \rho_{S,v} = -0.5$											
100	4	100	0.5	1.0	–	2.5685	–	–	2.4829	–	-0.0856
100	4	90	0.5	1.0	–	3.0547	–	–	3.0379	–	-0.0168
90	4	100	0.5	1.0	–	0.9480	–	–	0.7468	–	-0.2012
100	3	100	0.4	0.8	1.8012	1.8073	0.0061	1.7479	1.7537	-0.0533	-0.0536
100	6	100	0.5	1.2	–	4.0083	–	–	3.8533	–	-0.1550

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.11: Reference values for European and American put-on-put options using the Heston model.

S_0	K_M	K_D	T_M	T_D	European on European			American on American			
					Explicit value	ADI (HV) value	Difference in value	Explicit value	ADI (HV) value	Explicit EE value	ADI (HV) EE value
Set A: $p = 0, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	1.4137	1.4138	0.0001	1.3940	1.3943	-0.0197	-0.0195
100	4	90	0.5	1.0	2.4041	2.4043	0.0003	2.4100	2.4104	0.0059	0.0061
90	4	100	0.5	1.0	0.6254	0.6254	0.0000	0.5929	0.5930	-0.0325	-0.0324
100	3	100	0.4	0.8	0.9354	0.9356	0.0001	0.9236	0.9238	-0.0119	-0.0118
100	6	100	0.5	1.2	2.3083	2.3085	0.0002	2.2539	2.2545	-0.0544	-0.0540
Set B: $p = 0, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	0.7959	0.7961	0.0002	0.7918	0.7920	-0.0041	-0.0042
100	4	90	0.5	1.0	1.4460	1.4461	0.0001	1.4467	1.4469	0.0007	0.0007
90	4	100	0.5	1.0	0.3959	0.3961	0.0002	0.3912	0.3913	-0.0047	-0.0048
100	3	100	0.4	0.8	0.5178	0.5179	0.0001	0.5150	0.5152	-0.0027	-0.0028
100	6	100	0.5	1.2	1.2640	1.2625	0.0015	1.2495	1.2497	-0.0145	-0.0128
Set C: $p = 0, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055, \sigma_r = 0.06,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	1.0853	1.0855	0.0002	1.0622	1.0624	-0.0231	-0.0231
100	4	90	0.5	1.0	1.7322	1.7324	0.0002	1.7129	1.7131	-0.0193	-0.0193
90	4	100	0.5	1.0	0.5757	0.5759	0.0002	0.5540	0.5542	-0.0217	-0.0217
100	3	100	0.4	0.8	0.7026	0.7028	0.0002	0.6905	0.6906	-0.0121	-0.0121
100	6	100	0.5	1.2	1.7534	1.7537	0.0003	1.6866	1.6869	-0.0669	-0.0669
Set A: $p = 0.5, q = 0.005, \kappa_v = 1.5, \theta_v = 0.02, \sigma_v = 0.15, \kappa_r = 0.3, \theta_r = 0.04,$ $\sigma_r = 0.1, \rho_{S,v} = -0.5, \rho_{S,r} = 0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	1.4607	1.4608	0.0001	1.4439	1.4441	-0.0168	-0.0167
100	4	90	0.5	1.0	2.5310	2.5311	0.0001	2.5299	2.5300	-0.0011	-0.0010
90	4	100	0.5	1.0	0.5773	0.5774	0.0001	0.5595	0.5596	-0.0178	-0.0178
100	3	100	0.4	0.8	0.9514	0.9516	0.0001	0.9419	0.9420	-0.0096	-0.0096
100	6	100	0.5	1.2	2.4577	2.4577	0.0000	2.4095	2.4095	-0.0482	-0.0482
Set B: $p = 0.5, q = 0.005, \kappa_v = 3, \theta_v = 0.08, \sigma_v = 0.04, \kappa_r = 0.2, \theta_r = 0.05,$ $\sigma_r = 0.03, \rho_{S,v} = 0.6, \rho_{S,r} = 0.6, \rho_{v,r} = -0.2$											
100	4	100	0.5	1.0	0.7880	0.7883	0.0002	0.7803	0.7805	-0.0077	-0.0078
100	4	90	0.5	1.0	1.4598	1.4600	0.0001	1.4531	1.4532	-0.0068	-0.0067
90	4	100	0.5	1.0	0.3809	0.3811	0.0002	0.3749	0.3750	-0.0060	-0.0060
100	3	100	0.4	0.8	0.5116	0.5117	0.0002	0.5070	0.5072	-0.0045	-0.0046
100	6	100	0.5	1.2	1.2596	1.2581	0.0014	1.2351	1.2353	-0.0245	-0.0229
Set C: $p = 0.5, q = 0.005, \kappa_v = 0.6067, \theta_v = 0.0707, \sigma_v = 0.2928, \kappa_r = 0.05, \theta_r = 0.055,$ $\sigma_r = 0.06, \rho_{S,v} = -0.7571, \rho_{S,r} = -0.3, \rho_{v,r} = 0.2$											
100	4	100	0.5	1.0	1.0839	1.0841	0.0002	1.0704	1.0706	-0.0134	-0.0134
100	4	90	0.5	1.0	1.7171	1.7173	0.0002	1.7081	1.7083	-0.0090	-0.0090
90	4	100	0.5	1.0	0.5854	0.5856	0.0002	0.5717	0.5718	-0.0138	-0.0138
100	3	100	0.4	0.8	0.7031	0.7032	0.0002	0.6956	0.6957	-0.0075	-0.0075
100	6	100	0.5	1.2	1.7430	1.7433	0.0003	1.7038	1.7041	-0.0392	-0.0392

Legend: ADI: Alternating-direction Implicit; EE: Early exercise; HV: Hundsdorfer–Verwer.

Table B.12: Reference values for European and American put-on-put options using the HHW and HCIR models.

C Comparison of simulation values with values from the literature

Prices for European and American underlying and compound options using the three-state variable simulation approach are compared with values obtained from the literature.

Sample 1: Put option using the Black–Scholes model with $r = 0.06$, $q = 0$, $\sigma_S = 0.4$, $S_0 = 40$, $K_D = 40$, $T_D = 1$, $N_P = 1,000,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Longstaff and Schwartz (2011) [23]: 5.060
- Three-factor simulation approach: 5.058
- E_R : 0.0004

2. American variant:

- Longstaff and Schwartz (2011) [23]: 5.312
- Three-factor simulation approach: 5.318
- E_R : 0.001

Sample 2: Put option using the Heston model with $r_0 = 0.05$, $q = 0$, $v_0 = 0.04$, $S_0 = 100$, $K_D = 100$, $T_D = 0.5$, $\kappa_v = 3$, $\theta_v = 0.04$, $\sigma_v = 0.1$, $\rho_{S,v} = -0.7$, $N_P = 1,000,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Ruckdeschel, Sayer, and Szimayer (2011) [33]: 4.4312
- Three-factor simulation approach: 4.4343
- E_R : 0.0007

2. American variant:

- Ruckdeschel, Sayer, and Szimayer (2011) [33]: 4.6636
- Three-factor simulation approach: 4.6712
- E_R : 0.0016

Sample 3: Call option using the HHW model with all non-zero correlations with $r_0 = 0.1$, $q = 0$, $v_0 = 0.04$, $S_0 = 100$, $K_D = 100$, $T_D = 1$, $\kappa_v = 3$, $\theta_v = 0.12$, $\sigma_v = 0.8$, $\kappa_r = 0.2$, $\theta_r = 0.05$, $\sigma_r = 0.03$, $\rho_{S,v} = 0.6$, $\rho_{S,r} = 0.2$, $\rho_{v,r} = 0.4$, $N_P = 1,000,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Ullah (2019) [32]: 16.176
- Three-factor simulation approach: 16.134
- E_R : 0.0026

Sample 4: Put option using the HCIR model with $r_0 = 0.04$, $q = 0$, $v_0 = 0.01$, $S_0 = 100$, $K_D = 100$, $T_D = 0.5$, $\kappa_v = 1.5$, $\theta_v = 0.02$, $\sigma_v = 0.15$, $\kappa_r = 0.3$, $\theta_r = 0.04$, $\sigma_r = 0.1$, $\rho_{S,v} = 0.1$, $\rho_{S,r} = 0$, $\rho_{v,r} = 0$, $N_P = 1,000,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Medvedev and Scaillet (2009) [34]: 2.2254
- Three-factor simulation approach: 2.2236
- E_R : 0.0008

2. American variant

- Medvedev and Scaillet (2009) [34]: 2.3951
- Three-factor simulation approach: 2.3711
- E_R : 0.0100

Sample 5: Put option using the HCIR model with $r_0 = 0.04$, $q = 0$, $v_0 = 0.04$, $S_0 = 100$, $K_D = 100$, $T_D = 1$, $\kappa_v = 1.5$, $\theta_v = 0.02$, $\sigma_v = 0.15$, $\kappa_r = 0.3$, $\theta_r = 0.04$, $\sigma_r = 0.1$, $\rho_{S,v} = -0.5$, $\rho_{S,r} = 0$, $\rho_{v,r} = 0$, $N_P = 1,000,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Chiarella and Kang (2013) [2]: 5.0225
- Three-factor simulation approach: 5.0253
- E_R : 0.0006

2. American variant

- Chiarella and Kang (2013) [2]: 5.4899
- Three-factor simulation approach: 5.4832
- E_R : 0.012

Sample 5: Put-on-put option using the HCIR model with $r_0 = 0.04$, $q = 0$, $v_0 = 0.04$, $S_0 = 100$, $K_D = 100$, $T_D = 1$, $K_M = 4$, $T_M = 0.5$, $\kappa_v = 1.5$, $\theta_v = 0.02$, $\sigma_v = 0.15$, $\kappa_r = 0.3$, $\theta_r = 0.04$, $\sigma_r = 0.1$, $\rho_{S,v} = -0.5$, $\rho_{S,r} = 0$, $\rho_{v,r} = 0$, $N_P = 10,000$, $N_t = 200$, and 5 basis functions.

1. European variant:

- Chiarella and Kang (2013) [2]: 1.5228
- Three-factor simulation approach: 1.5295
- E_R : 0.0044

2. American variant

- Chiarella and Kang (2013) [2]: 1.5618
- Three-factor simulation approach: 1.5434
- E_R : 0.0118

D Python application

To run the application for this dissertation, open the main file called 'WTW 894.py' in a Python integrated development environment (IDE) such as Spyder or PyCharm running Python 3.8 or higher and select 'Run'. The structure of the application folder is shown below. The main file will automatically import the Python files in the subfolders 'Graphs', 'Libraries', and 'Tables'.

The application allows the user to replicate all the figures and tables presented in this dissertation, including the samples from the literature in Appendix C. The folder structure is as follows:

Application

