Belief aggregation for representative agent models*

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Abstract

Representative agent models pin down equilibrium asset prices of an underlying heterogeneous agents economy through the utility maximization problem of a representative agent evaluated at aggregate endowment levels. This paper considers a complete markets asset exchange economy in which all economic agents are expected utility maximizers who share the same risk preferences but may have heterogeneous endowments and beliefs. For arbitrary well-behaved Bernoulli utility functions we derive belief aggregation formulas that characterize the beliefs of an expected utility maximizing representative agent.

Keywords: Arrow-Debreu economy; Heterogeneous beliefs; Representative agent
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1 Introduction

"Heterogeneous beliefs are a fact of life." Xiong (2013, p.677)

Aggregation models pin down the equilibrium prices of an underlying economy through the optimal zero net-trade decision of a single agent. Aggregation is analytically convenient because solving for the equilibrium prices for the underlying economy directly would require, firstly, to solve multiple utility maximization problems for the net-trade functions of all economic agents and, secondly, to find the market-clearing prices for all these net-trade functions. Representative agent models are specific aggregation models for which the single, i.e., representative, agent shares relevant preference features with the economic agents in the underlying economy. Beyond the analytical convenience of single agent models, representative agent models therefore provide important structural insights about the underlying economies which can be used for policy recommendations, economic interpretation, and empirical testing.[1] Aggregation works the better, the more homogeneous the economic agents are. Lucas (1978) interprets the representative agent as a ""stand in" for a large number of identical consumers" (p.1430). In reality, however, different economic agents have different risk-preferences, different beliefs, and different endowments.

This paper derives aggregation results for complete markets asset exchange economies under the assumption that the expected utility maximizing agents have identical risk preferences–expressed through a well-behaved Bernoulli utility function–but hold different beliefs and endowments.[2] The aim of our aggregation analysis is to describe technically convenient representative agent models in the following sense.

We speak of a representative agent model for the underlying economy whenever the equilibrium price ratios are equivalently pinned down by the optimal zero-trade of an expected utility maximizing (representative) agent who has the same Bernoulli util-

[1]For instance, whenever the representative agent closely resembles the economic agents, economic policy recommendations can separate efficiency from income distribution considerations as the preference maximization for this representative agent ensures efficient outcomes for all possible income distributions. This economic policy aspect of representative agent models goes back to the "unique community indifference maps" analysis of Gorman (1953).

[2]A well-behaved Bernoulli utility function is strictly increasing, strictly concave, and continuously differentiable on the strictly positive real line. We will also show how our results generalize to situations in which agents have different well-behaved Bernoulli utility functions (cf. Remarks 3 and 7). We also mention how our aggregation analysis could, in principle, be extended to an incomplete markets environment (however, such extension would come with a significant loss of analytical convenience, cf. Section 3.1.1).

ity function as the economic agents and who is endowed with either the average or the aggregate endowments of the underlying economy.

Since the commonly shared Bernoulli utility function as well as the aggregate endowments are fixed for the underlying economy, our technical challenge is to come up with belief aggregation formulas that transform the economic agents' heterogeneous beliefs into a probability measure which stands for the representative agent's belief.

To be specific, we consider a underlying complete markets asset exchange economy with S states of the world ω_s, $s \in \{1, ..., S\}$, and n economic agents $i \in \{1, ..., n\}$. Each economic agent i holds a subjective belief–given as an additive probability measure π_i on $(\Omega, 2^\Omega)$–and has endowments e_{is} in Arrow-Debreu securities $s \in \{1, ..., S\}$. The representative agent $\rho \in \{r, R\}$ has endowments $e_{\rho s}$ whereby we distinguish between an average, i.e., $e_{rs} = \frac{1}{n} \sum_{i=1}^{n} e_{is}$, and an aggregate, i.e., $e_{Rs} = \sum_{i=1}^{n} e_{is}$, representative agent model. As our main results, we derive the following belief aggregation formulas for arbitrary well-behaved Bernoulli utility functions (cf. Theorems 1 and 2).

General belief aggregation formulas. *Fix an equilibrium of the underlying economy and denote by c_{is}^* the equilibrium consumption of agent i in state ω_s. There exists a representative agent model for the underlying economy such that the representative agent's aggregate belief is given as the following additive probability measure π_ρ on $(\Omega, 2^\Omega)$:*

$$\pi_\rho(\omega_s) = \frac{\frac{1}{u'(e_{\rho s})} \prod_{i=1}^{n} \left[u'\left(c_{is}^*\right) \pi_i(\omega_s) \right]^{\frac{\nu_i}{\sum_{i=1}^{n} \nu_i}}}{\sum_{s'=1}^{S} \frac{1}{u'\left(e_{\rho s'}\right)} \prod_{i=1}^{n} \left[u'\left(c_{is'}^*\right) \pi_i(\omega_{s'}) \right]^{\frac{\nu_i}{\sum_{i=1}^{n} \nu_i}}} \text{ for all } \omega_s \in \Omega$$

for arbitrary agent coefficients $\nu_i > 0$, $i \in \{1, ..., n\}$. Mathematically equivalently, we have

$$\pi_\rho(\omega_s) = \frac{\frac{1}{u'(e_{\rho s})} \sum_{i=1}^{n} \mu_i u'\left(c_{is}^*\right) \pi_i(\omega_s)}{\sum_{s'=1}^{S} \frac{1}{u'\left(e_{\rho s'}\right)} \sum_{i=1}^{n} \mu_i u'\left(c_{is'}^*\right) \pi_i(\omega_{s'})} \text{ for all } \omega_s \in \Omega$$

for arbitrary agent weights $\mu_i > 0$, $i \in \{1, ..., n\}$.

For the special cases of CARA (constant absolute risk aversion) and CRRA (constant relative risk aversion) Bernoulli utility functions, we derive the following specifications of the above general belief aggregation formulas.

Belief aggregation for CARA Bernoulli utility. *There exists an average endowment representative agent model for the underlying CARA economy such that the representative agent's $\rho = r$ aggregate belief is given as*

$$\pi_r(\omega_s) = \frac{\left(\prod_{i=1}^{n} \pi_i(\omega_s)\right)^{\frac{1}{n}}}{\sum_{s'=1}^{S}\left(\prod_{i=1}^{n} \pi_i(\omega_{s'})\right)^{\frac{1}{n}}} \text{ for all } \omega_s \in \Omega.$$

Belief aggregation for CRRA Bernoulli utility with risk aversion coefficient $\gamma > 0$. *There exists a representative agent model for the underlying CRRA economy such that the representative agent's $\rho \in \{r, R\}$ aggregate belief is given as*

$$\pi_\rho(\omega_s) = \frac{\left(\sum_{i=1}^{n} \left(\mu_i^* \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\sum_{s'=1}^{S}\left(\sum_{i=1}^{n} \left(\mu_i^* \pi_i(\omega_{s'})\right)^{\frac{1}{\gamma}}\right)^{\gamma}} \text{ for all } \omega_s \in \Omega \tag{1}$$

with Pareto agent weights[3] *given as*

$$\mu_i^* = \frac{\prod_{k \neq i} \left(c_{ks^*}^*\right)^{-\gamma} \pi_k(\omega_{s^*})}{\sum_{j=1}^{n} \prod_{k \neq j} \left(c_{ks^*}^*\right)^{-\gamma} \pi_k(\omega_{s^*})}$$

for an arbitrarily chosen $s^* \in \{1, ..., S\}$.

[3] I speak of 'Pareto' agent weights whenever these weights would enter into the utilitarian welfare function that is maximized at the equilibrium allocation of the underlying economy, (which shows directly the Pareto optimality of the equilibrium in our complete markets economy).

Belief aggregation for logarithmic CRRA Bernoulli utility, i.e., $\gamma = 1$. *The aggregate belief (1) becomes the μ_i^*-weighted convex combination of the economic agents' beliefs*

$$\pi_\rho(\omega_s) = \sum_{i=1}^{n} \mu_i^* \pi_i(\omega_s) \text{ for all } \omega_s \in \Omega$$

with Pareto agent weights given as

$$\mu_i^* = \frac{\prod_{k \neq i} \left(\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} e_{ks'} \right) + e_{ks^*} \right)^{-1}}{\sum_{h=1}^{n} \prod_{k \neq h} \left(\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} e_{ks'} \right) + e_{ks^*} \right)^{-1}}$$

for an arbitrarily chosen $s^* \in \{1, ..., S\}$.

The remainder of our analysis is structured as follows. Section 2 introduces the underlying complete markets economy in which agents with heterogeneous beliefs exchange Arrow-Debreu securities. Section 3 presents belief aggregation formulas for arbitrary well-behaved Bernoulli utility functions. Section 4 uses these general formulas to derive belief aggregation results for CARA, CRRA, and logarithmic Bernoulli utility functions, respectively. Section 5 concludes with a brief discussion of related literature and an outlook on multiperiod extensions of our model. Detailed mathematical proofs are relegated to the Appendix.

2 Preliminaries: The underlying economy

We consider a static asset exchange economy in which the economic agents can freely exchange Arrow-Debreu securities. This complete markets economy consists of n agents $i \in \{1..., n\}$, S states of the world $\omega_s \in \{\omega_1, ..., \omega_S\} = \Omega$, and S corresponding Arrow-Debreu securities $s \in \{1, ..., S\}$ with payoffs

$$X_s(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_s \\ 0 & \text{else.} \end{cases}$$

The endowment and net-trade vectors of agent i in Arrow-Debreu securities are denoted by $e_i = (e_{i1}, ..., e_{in}) \in \mathbb{R}^S_{>0}$ and $\theta_i = (\theta_{i1}, ..., \theta_{in}) \in \mathbb{R}^S$, respectively. We write

$$e_s = \sum_{i=1}^{n} e_{is}$$

for the aggregate endowment in Arrow-Debreu security s and

$$c_s = \sum_{i=1}^{n} c_{is} = \sum_{i=1}^{n} \theta_{is} + e_{is}$$

for the aggregate consumption in state ω_s.

The agents are expected utility maximizers who share the same Bernoulli utility function $u : [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$ which is *well-behaved* in the following sense: u is strictly increasing, strictly concave, and continuously differentiable on $(0, \infty)$ whereby we simply set $u(0) = -\infty$ for the worst-case scenario of zero consumption. Agent i's belief is given as a subjective additive probability measure π_i on the space $(\Omega, 2^{\Omega})$ with full support on Ω. Let p_s denote the price of Arrow-Debreu security s, i.e., the state-price for a unit of consumption in state ω_s. Given the state-price vector $p = (p_1, ..., p_S)$ agent i chooses a net-trade vector from her budget-feasible set

$$B_i(p, e_i) = \left\{ \theta_i \in \mathbb{R}^S \mid \sum_{s=1}^{S} p_s \theta_{is} = 0,\ \theta_i + e_i \in \mathbb{R}^S_{\geq 0} \right\}$$

which maximizes her subjective expected utility over consumption $c_i = \theta_i + e_i \in \mathbb{R}^S_{\geq 0}$

$$E_{\pi_i} u(c_i) = \sum_{s=1}^{S} u(\theta_{is} + e_{is}) \pi_i(\omega_s).$$

We call the above economy with heterogeneous agents the *underlying economy.*

Definition 1. *A competitive equilibrium of the underlying economy, denoted* $(p^*, \theta_1^*, ..., \theta_n^*)$*, is characterized by the following two properties:*

(i) *Expected utility maximization at equilibrium prices: for* $i \in \{1, ..., n\}$

$$\theta_i^* \in \arg \max_{\theta_i \in B_i(p^*, e_i)} E_{\pi_i} u(\theta_i + e_i).$$

(ii) *Market clearing in all states: for* $s \in \{1, ..., S\}$

$$\sum_{i=1}^{n} \theta_{is}^* = 0.$$

Remark 1. While our formal analysis restricts attention to Arrow-Debreu securities, our framework naturally extends to complete markets with arbitrary assets. If there is any other asset in the economy–characterized by the payoff function $X : \Omega \to \mathbb{R}_{\geq 0}$–the equilibrium price of this asset, denoted p_X^*, is identical to the price of the payoff-equivalent portfolio of Arrow-Debreu securities, i.e.,

$$p_X^* = \sum_{s=1}^{S} X(\omega_s) p_s^*.$$

This pricing formula guarantees that there are no arbitrage-opportunities in the static economy whenever we consider additional assets beyond Arrow-Debreu securities.

3 General belief aggregation analysis

A representative agent model in our sense generates the equilibrium prices of the underlying economy at the zero net-trade whereby the representative agent is an expected utility maximizer who has the same Bernoulli utility function as the economic agents in the underlying economy. We distinguish between *average* and *aggregate endowment* representative agent models, which we denote by r and R, respectively. The respective initial endowments of the representative agent in both models are given as

$$e_{rs} = \frac{1}{n} e_s \text{ and } e_{Rs} = e_s \tag{2}$$

for all $s \in \{1, ..., S\}$.

Definition 2. *Fix an equilibrium $(p^*, \theta_1^*, ..., \theta_n^*) \in \mathbb{R}_{>0}^S \times \mathbb{R}^{nS}$ of the underlying economy. The single-agent economy with $\rho \in \{r, R\}$ is a 'representative agent model' for the underlying economy if and only if the aggregate belief π_ρ on $(\Omega, 2^\Omega)$ satisfies*

$$(0, ..., 0) \in \arg \max_{\left\{ (\theta_{\rho 1}, ..., \theta_{\rho S}) \in \mathbb{R}^S | \sum_{s=1}^{S} p_s^* \theta_{\rho s} = 0 \right\}} E_{\pi_\rho} u(\theta_{\rho s} + e_{\rho s}).$$

For well-behaved Bernoulli utility functions, any competitive equilibrium $(p^*, \theta_1^*, ..., \theta_n^*)$ must be an interior equilibrium that satisfies the following first-order conditions: for all $i \in \{1, ..., n\}$ and all $s \in \{1, ..., S\}$,

$$u'(\theta_{is}^* + e_{is}) \pi_i(\omega_s) = \lambda_i p_s^*. \tag{3}$$

Here $\lambda_i > 0$ denotes an agent-specific Lagrange multiplier which stands for the agent i's marginal utility from having an incremental unit of wealth available to be invested. The equilibrium price-ratios for any Arrow-Debreu securities $s, s' \in \{1, ..., S\}$ are determined by the MRS (=marginal rate of substitution) conditions

$$\frac{p_s^*}{p_{s'}^*} = \frac{u'\left(c_{is}^*\right)}{u'\left(c_{is'}^*\right)} \frac{\pi_i\left(\omega_s\right)}{\pi_i\left(\omega_{s'}\right)} \text{ for all } i \in \{1, ..., n\}. \tag{4}$$

A representative agent model for this underlying economy is characterized by

$$\frac{p_s^*}{p_{s'}^*} = \frac{u'\left(e_{\rho s}\right)}{u'\left(e_{\rho s'}\right)} \frac{\pi_\rho\left(\omega_s\right)}{\pi_\rho\left(\omega_{s'}\right)} \tag{5}$$

for all $s, s' \in \{1, ..., S\}$. Consequently, the aggregate belief π_ρ of a representative agent $\rho \in \{r, R\}$ is any additive probability measure on $\left(\Omega, 2^\Omega\right)$ that satisfies for all $s, s' \in \{1, ..., S\}$ equation (5) provided that the n equations (4) also hold for all $s, s' \in \{1, ..., S\}$. In what follows we derive different–but mathematically equivalent–expressions for π_ρ through mathematically equivalent transformations of these systems of equations.

3.1 A first belief aggregation formula for well-behaved Bernoulli utility functions

Given arbitrary agent coefficients $\nu_i > 0$ for $i \in \{1, ..., n\}$, transform (4) equivalently to

$$\left(\frac{p_s^*}{p_{s'}^*}\right)^{\nu_i} = \left(\frac{u'\left(c_{is}^*\right)}{u'\left(c_{is'}^*\right)} \frac{\pi_i\left(\omega_s\right)}{\pi_i\left(\omega_{s'}\right)}\right)^{\nu_i}.$$

Multiplying over all agents and using (5) yields the following implicit characterization of an aggregate belief.

Observation 1. *There exists a representative agent model for the underlying economy with well-behaved Bernoulli utility function whenever π_ρ is an additive probability measure on $\left(\Omega, 2^\Omega\right)$ that satisfies, for all $\omega_s, \omega_{s'} \in \Omega$,*

$$\frac{\prod_{i=1}^{n}\left[u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right)\right]^{\nu_i}}{\prod_{i=1}^{n}\left[u'\left(c_{is'}^*\right)\pi_i\left(\omega_{s'}\right)\right]^{\nu_i}} = \left(\frac{p_s^*}{p_{s'}^*}\right)^{\sum_{i=1}^n \nu_i} = \left(\frac{u'\left(e_{\rho s}\right)\pi_\rho\left(\omega_s\right)}{u'\left(e_{\rho s'}\right)\pi_\rho\left(\omega_{s'}\right)}\right)^{\sum_{i=1}^n \nu_i} \tag{6}$$

whereby the agent coefficients $\nu_i > 0$, $i \in \{1, ..., n\}$, are arbitrary.

By Observation 1, we have

$$\frac{\pi_\rho(\omega_s)}{\pi_\rho(\omega_{s'})} = \frac{\frac{1}{u'(e_{\rho s})}}{\frac{1}{u'(e_{\rho s'})}} \frac{\prod_{i=1}^{n} \left[u'(c_{is}^*)\,\pi_i(\omega_s)\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}}{\prod_{i=1}^{n} \left[u'(c_{is'}^*)\,\pi_i(\omega_{s'})\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}}. \tag{7}$$

Normalization $\sum_{s=1}^{S} \pi_\rho(\omega_s) = 1$ yields the following belief aggregation result for well-behaved Bernoulli utility functions.

Theorem 1. *For any given equilibrium* $(p^*, \theta_1^*, ..., \theta_n^*) \in \mathbb{R}_{>0}^S \times \mathbb{R}^{nS}$ *of the underlying economy, there exists a representative agent model such that the representative agent's aggregate belief is given as the additive probability measure* π_ρ *on* $\left(\Omega, 2^\Omega\right)$ *satisfying*

$$\pi_\rho(\omega_s) = \frac{\frac{1}{u'(e_{\rho s})} \prod_{i=1}^{n} \left[u'(c_{is}^*)\,\pi_i(\omega_s)\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}}{\sum_{s'=1}^{S} \frac{1}{u'(e_{\rho s'})} \prod_{i=1}^{n} \left[u'(c_{is'}^*)\,\pi_i(\omega_{s'})\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}} \text{ for all } \omega_s \in \Omega \tag{8}$$

whereby the agent coefficients $\nu_i > 0$, $i \in \{1, ..., n\}$, *are arbitrary.*

Remark 2. By the belief aggregation formula (8), the two concepts of representative agent models $\rho \in \{r, R\}$ coincide if and only if

$$\begin{aligned} \pi_r &= \pi_R \\ &\Leftrightarrow \\ \sum_{s'=1}^{S} \frac{u'(e_{rs})}{u'(e_{rs'})} &= \sum_{s'=1}^{S} \frac{u'(e_{Rs})}{u'(e_{Rs'})} \text{ for all } s \\ &\Leftarrow \\ \frac{u'\left(\frac{1}{n}\sum_{i=1}^n e_{is}\right)}{u'\left(\frac{1}{n}\sum_{i=1}^n e_{is'}\right)} &= \frac{u'\left(\sum_{i=1}^n e_{is}\right)}{u'\left(\sum_{i=1}^n e_{is'}\right)} \text{ for all } s, s'. \end{aligned} \tag{9}$$

The equivalence condition (9) is, e.g., satisfied for CRRA but not for CARA Bernoulli utility functions whenever $n > 1$.

Remark 3. Observe that Theorem 1 easily generalizes to the case where the economic agents have different risk-attitudes in the form of heterogeneous well-behaved Bernoulli utility functions u_i, $i \in \{1, ..., n\}$. Namely, fix some well-behaved Bernoulli utility function u_ρ for the representative agent and note that (8) becomes

$$\pi_\rho(\omega_s) = \frac{\frac{1}{u'_\rho(e_{\rho s})} \prod_{i=1}^{n} \left[u'_i(c^*_{is})\,\pi_i(\omega_s)\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}}{\sum_{s'=1}^{S} \frac{1}{u'_\rho(e_{\rho s'})} \prod_{i=1}^{n} \left[u'_i(c^*_{is'})\,\pi_i(\omega_{s'})\right]^{\frac{\nu_i}{\sum_{i=1}^n \nu_i}}} \text{ for all } \omega_s \in \Omega. \tag{10}$$

3.1.1 Discussion: Incomplete markets

The belief aggregation formula (8) applies to all economies in which the economic agents with heterogeneous beliefs can freely exchange Arrow-Debreu securities. Starting with Harrison and Kreps (1978) there exists an interest in combining heterogeneous beliefs with *incomplete* asset markets, i.e., markets where institutional arrangements (e.g., short-selling constraints) result in constraints that keep economic agents from freely exchanging Arrow-Debreu securities. The question thus arises in how far our belief aggregation analysis for complete markets could be carried over to incomplete markets.[4]

Consider the canonical example of an incomplete markets economy in which the economic agents (i) can only trade their endowments in two different assets $X : \Omega \to \mathbb{R}_{\geq 0}$ and $Y : \Omega \to \mathbb{R}_{\geq 0}$ whereby (ii) the asset payoff matrix does not span $\mathbb{R}^S$ as there are $S \geq 3$ different states of the world. If there exists a competitive equilibrium $(p^*, \theta^*_1, ..., \theta^*_n) \in \mathbb{R}^2_{>0} \times \mathbb{R}^{n2}$ characterized by FOCs, we must have

$$\frac{p^*_X}{p^*_Y} = \frac{\sum_{s=1}^{S} u'\left(X(\omega_s)\left(e_{iX} + \theta^*_{iX}\right)\right) X(\omega_s)\,\pi_i(\omega_s)}{\sum_{s=1}^{S} u'\left(Y(\omega_s)\left(e_{iY} + \theta^*_{iY}\right)\right) Y(\omega_s)\,\pi_i(\omega_s)} \text{ for all } i \in \{1, ..., n\}.$$

Analogously to the characterization of aggregate beliefs (6) in Observation 1 (with $\nu_i = 1$ for all i), there would exist a representative agent model which generates the equilibrium $(p^*, \theta^*_1, ..., \theta^*_n)$ of the incomplete markets economy whenever there are aggregate beliefs π_ρ

[4] Krueger and Lustig (2010) investigate for a continuum of agents with *homogeneous* beliefs in how far aggregation results for complete markets carry over to incomplete asset markets.

that satisfy the equation

$$\frac{\prod_{i=1}^{n}\sum_{s=1}^{S}u'\left(X\left(\omega_s\right)\left(e_{iX}+\theta_{iX}^*\right)\right)X\left(\omega_s\right)\pi_i\left(\omega_s\right)}{\prod_{i=1}^{n}\sum_{s=1}^{S}u'\left(Y\left(\omega_s\right)\left(e_{iY}+\theta_{iY}^*\right)\right)Y\left(\omega_s\right)\pi_i\left(\omega_s\right)} = \left(\frac{p_X^*}{p_Y^*}\right)^n \tag{11}$$

$$= \left(\frac{\sum_{s=1}^{S}u'\left(X\left(\omega_s\right)e_{\rho X}\right)X\left(\omega_s\right)\pi_\rho\left(\omega_s\right)}{\sum_{s=1}^{S}u'\left(Y\left(\omega_s\right)e_{\rho Y}\right)Y\left(\omega_s\right)\pi_\rho\left(\omega_s\right)}\right)^n.$$

In contrast to our complete markets economy, however, this equation is in $S-1\geq 2$ unknown probabilities $\pi_\rho\left(\omega_s\right)$ so that we cannot pin down a unique additive probability measure π_ρ as we did in Theorem 1. Moreover, while we might be able to pin down the set of all aggregate beliefs π_ρ that are a solution to (11), this solution will depend on the specifics of this incomplete markets economy, i.e., on the specific payoff functions X and Y. That is, whereas belief aggregation formulas for complete markets are always the same–because Arrow-Debreu securities can always be traded on complete markets–belief aggregation formulas for incomplete markets are bound to be different for different incomplete markets.

To conclude: Although it is, in principle, possible to come up with belief aggregation results for incomplete markets economies, such models would lack the analytical convenience of a uniquely pinned down aggregate belief as well as the generality of belief aggregation formulas for complete markets.

3.2 An equivalent belief aggregation formula with agent weights

Given arbitrary agent weights $\mu_i>0$ for $i\in\{1,...,n\}$, transform (3) equivalently to

$$\mu_i u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right)=\mu_i\lambda_i p_s^* \text{ for all } s\in\{1,...,S\}. \tag{12}$$

Because of

$$\sum_{i=1}^{n}\mu_i u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right) = p_s^*\sum_{i=1}^{n}\mu_i\lambda_i \text{ for all } s\in\{1,...,S\}$$

$$\Rightarrow$$

$$\frac{\sum_{i=1}^{n}\mu_i u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right)}{\sum_{i=1}^{n}\mu_i u'\left(c_{is'}^*\right)\pi_i\left(\omega_{s'}\right)} = \frac{p_s^*}{p_{s'}^*},$$

we obtain:

Observation 2. *There exists a representative agent model for the underlying economy with well-behaved Bernoulli utility function whenever the aggregate belief π_ρ satisfies*

$$\frac{\sum_{i=1}^{n} \mu_i u'\left(c_{is}^*\right) \pi_i\left(\omega_s\right)}{\sum_{i=1}^{n} \mu_i u'\left(c_{is'}^*\right) \pi_i\left(\omega_{s'}\right)} = \frac{p_s^*}{p_{s'}^*} = \frac{u'\left(e_{\rho s}\right) \pi_\rho\left(\omega_s\right)}{u'\left(e_{\rho s'}\right) \pi_\rho\left(\omega_{s'}\right)}$$

for all $\omega_s, \omega_{s'} \in \Omega$ whereby the agent weights $\mu_i > 0$, $i \in \{1, ..., n\}$, are arbitrary.

Straightforward transformation and normalization yields the next result.

Theorem 2. *The aggregate belief (8) is mathematically equivalently given as*

$$\pi_\rho\left(\omega_s\right) = \frac{\frac{1}{u'(e_{\rho s})} \sum_{i=1}^{n} \mu_i u'\left(c_{is}^*\right) \pi_i\left(\omega_s\right)}{\sum_{s'=1}^{S} \frac{1}{u'\left(e_{\rho s'}\right)} \sum_{i=1}^{n} \mu_i u'\left(c_{is'}^*\right) \pi_i\left(\omega_{s'}\right)} \text{ for all } \omega_s \in \Omega \tag{13}$$

whereby the agent weights $\mu_i > 0$, $i \in \{1, ..., n\}$, are arbitrary.

Remark 4. To see the economic intuition why we can freely choose the agent weights μ_i, think of them as agent i's personal currency denominator: while every agent pays price p_s^* in terms of the same common currency, every i translates this price into his own personal currency through the denominator μ_i. This personal denomination does not change the equilibrium price ratios as agent i's λ_i-normalized marginal utility

$$\frac{1}{\lambda_i} u'\left(c_{is}^*\right) \pi_i\left(\omega_s\right)$$

'bought' at price p_s^* is also denominated by the same factor μ_i to the effect that personal price ratios coincide with the equilibrium price ratios. In other words, any personal currency denominations in the form of the agent weights μ_i do not matter for the equilibrium price ratios of the representative agent model.

3.2.1 Discussion: Pareto agent weights and welfare-maximization

By Theorem 2, we have the degree of freedom to fix arbitrary agent weights in (13) before we proceed and transform the belief aggregation formula (13) into analytically more convenient expressions. Traditional representative agent analysis has worked with specifically constructed agent weights, denoted μ_i^*, which I henceforth refer to as Pareto agent weights. The main reason[5] is that this standard analysis has interpreted the representative agent in terms of a benevolent dictator who replicates the Pareto-efficient equilibrium allocation of the underlying complete markets economy through the maximization of the utilitarian welfare function

$$W_{\mu^*}(c_1, ..., c_n) = \sum_{i=1}^{n} \mu_i^* E_{\pi_i} u(c_i). \tag{14}$$

In order to achieve this equivalence between an expected utility and a welfare-maximizing representative agent, the Pareto agent weights μ_i^*, $i \in \{1, ..., n\}$, in (14) must be constructed in such a way that the utilitarian welfare function is maximized at the equilibrium consumption $(c_1^*, ..., c_n^*)$.[6] For our complete markets economy–where competitive equilibria are Pareto-efficient–we can easily describe the Pareto agent weights that must enter the welfare function (14) in order to recast the equilibrium consumption of the underlying economy as a welfare maximizing allocation.

The purpose of the following two propositions is to make our analysis comparable to existing aggregation results in the literature that are formulated for CRRA Bernoulli utility functions in terms of Pareto agent weights.

Proposition 1. *For any given equilibrium* $(p^*, \theta_1^*, ..., \theta_n^*) \in \mathbb{R}^S_{>0} \times \mathbb{R}^{nS}$ *of the underlying economy there exist weights* $\mu^* = (\mu_1^*, ..., \mu_n^*)$ *with* $\mu_i^* > 0$ *and* $\sum_{i=1}^{n} \mu_i^* = 1$ *such that*

[5] Another reason is that utilitarian agent weights result in an analytically convenient expression of aggregate beliefs for CRRA Bernoulli utility functions (cf. Proposition 5).

[6] The idea to equivalently describe equilibrium prices of Pareto-efficient competitive equilibria through the welfare-maximization problem of a benevolent dictator with suitably specified agent weights goes back to Negishi (1960). For details, see, e.g., Chapter 1E in Duffie (2001) who derives a representative agent model from the maximization problem of a utilitarian welfare function whereby he writes: "Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing." (p.8)

the utilitarian welfare function (14) is maximized at $(\theta_1^*, ..., \theta_n^*)$*, i.e.,*

$$\begin{aligned}
(\theta_1^*, ..., \theta_n^*) &\in \arg \max_{(\theta_1,...,\theta_n)\in\mathbb{R}^{nS}} \sum_{i=1}^{n} \mu_i^* \sum_{s=1}^{S} u(\theta_{is} + e_{is}) \pi_i(\omega_s) \\
\text{subject to} \sum_{i=1}^{n} \theta_{is} &= 0 \text{ and } \theta_{is} + e_{is} \geq 0 \text{ for all } s \in \{1, ..., S\}.
\end{aligned}$$

Moreover, these Pareto agent weights are uniquely pinned down as

$$\mu_i^* = \frac{\prod_{k\neq i} u'(c_{ks^*}^*) \pi_k(\omega_{s^*})}{\sum_{j=1}^{n} \prod_{k\neq j} u'(c_{ks^*}^*) \pi_k(\omega_{s^*})} \tag{15}$$

for an arbitrarily chosen $s^* \in \{1, ..., n\}$*.*

Proposition 2. *The Pareto agent weights (15) are equivalently given as follows.*

(i) *In terms of the economic agents' Lagrange multipliers:*

$$\mu_i^* = \frac{\prod_{k\neq i} \lambda_k}{\sum_{j=1}^{n} \prod_{k\neq j} \lambda_k} = \frac{(\lambda_i)^{-1}}{\sum_{j=1}^{n} (\lambda_j)^{-1}}. \tag{16}$$

(ii) *In terms of equilibrium consumption and price levels:*

$$\mu_i^* = \frac{\sum_{s=1}^{S} \frac{p_s^*}{u'\left(c_{is}^*\right)}}{\sum_{j=1}^{n} \left(\sum_{s=1}^{S} \frac{p_s^*}{u'\left(c_{js}^*\right)} \right)}. \tag{17}$$

We will later use the Lagrange multipliers expression (16) of the Pareto agent weights to show the equivalence between our CRRA representative agent and the "consensus characteristic" approach of Jouini and Napp (2007). The expression (17) for agent weights in terms of equilibrium consumption and prices will recover Rubinstein's (1976) classic belief aggregation result for logarithmic Bernoulli utility in terms of a wealth-weighted average of the economic agents' beliefs.

Remark 5. There exist recent models which use the Pareto agent weight approach to derive aggregation results for a continuum of agents $i \in (a,b) \subseteq \mathbb{R}$ from the maximization of a utilitarian welfare function given as

$$W_{\mu^*}(c) = \int_{i \in (a,b)} E_{\pi_i} u(c_i)\, d\mu^* \tag{18}$$

for appropriately constructed measure μ^*. The interval (a,b) thereby corresponds to the possible deviations (measured by some real-number) from any agent's subjective belief to an objective probability measure. The Pareto agent weight measure μ^* in (18) must thus also incorporate the density of agents of a given subjective belief corresponding to a point in (a,b). For example, for a continuum of economic agents on the real line with CRRA Bernoulli utility function, Atmaz and Basak (2018) use a belief aggregation result from Basak (2005) under the assumption that different agents hold heterogeneous beliefs about the mean of the dividend growth rate in a continuous time diffusion process. For a continuum of economic agents on the open unit interval with log Bernoulli utility function, Martin and Papadimitriou (2021) derive a representative agent result for a infinite horizon binomial tree process.[7]

4 Belief aggregation formulas for specific Bernoulli utility functions

This section applies the general belief aggregation formulas of Theorems 1 and 2 to CARA, CRRA, and logarithmic Bernoulli utility functions, respectively.

4.1 CARA Bernoulli utility

Consider a CARA Bernoulli utility function such that for $c > 0$

$$u(c) = -\exp(-\alpha c)$$

where $\alpha > 0$ denotes the agents' constant absolute risk-aversion coefficient. As we have the degree of freedom to choose arbitrary agent coefficients $\nu_i > 0$ in (8), let us simply

[7] These existing aggregation results for a continuum of agents seem to depend on highly specific assumptions about the economic agents' Bernoulli utility functions and their heterogeneous belief distributions on (a,b). While it would be interesting to come up with a more general analysis of aggregation results for a continuum of agents, such technically difficult task is beyond the scope of the present paper.

set $\nu_i = 1$ for all i. Observe that

$$\prod_{i=1}^{n} u'\left(c_{is}^{*}\right) = \alpha^{n}\exp\left(-\alpha\sum_{i=1}^{n} c_{is}^{*}\right) = \alpha^{n}\exp\left(-\alpha e_{s}\right)$$

so that substitution in (8) yields

$$\pi_{\rho}\left(\omega_{s}\right) = \frac{\frac{1}{\exp(-\alpha e_{\rho s})}\left(\exp\left(-\alpha e_{s}\right)\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s}\right)\right)^{\frac{1}{n}}}{\sum_{s'=1}^{S}\frac{1}{\exp\left(-\alpha e_{\rho s'}\right)}\left(\exp\left(-\alpha e_{s'}\right)\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s'}\right)\right)^{\frac{1}{n}}}.$$

Substituting the respective expressions (2) for $e_{\rho s}$, $\rho \in \{r, R\}$, gives us immediately the following characterizations for the aggregate beliefs π_r and π_R, respectively.

Proposition 3.

(i) *There exists an average endowment representative agent model for the underlying CARA economy such that the representative agent's aggregate belief is given as*

$$\pi_{r}\left(\omega_{s}\right) = \frac{\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s}\right)\right)^{\frac{1}{n}}}{\sum_{s'=1}^{S}\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s'}\right)\right)^{\frac{1}{n}}} \text{ for all } \omega_{s} \in \Omega. \tag{19}$$

(ii) *There exists an aggregate endowment representative agent model for the underlying CARA economy such that the representative agent's aggregate belief is given as*

$$\pi_{R}\left(\omega_{s}\right) = \frac{\exp\left(\left(1-\frac{1}{n}\right)\alpha e_{s}\right)\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s}\right)\right)^{\frac{1}{n}}}{\sum_{s'=1}^{S}\exp\left(\left(1-\frac{1}{n}\right)\alpha e_{s'}\right)\left(\prod_{i=1}^{n}\pi_{i}\left(\omega_{s'}\right)\right)^{\frac{1}{n}}} \text{ for all } \omega_{s} \in \Omega.$$

Remark 6. The non-normalized version of (19) for the average endowment model appears already in Rubinstein (1974, Theorem (Aggregation)) who refers to

$$\pi_s = \left(\prod_{i=1}^{n} \pi_i (\omega_s)\right)^{\frac{1}{n}} \text{ for all } \omega_s \in \Omega \tag{20}$$

as 'composite beliefs'. Rubinstein (1974, p.232) writes: "[...] one caveat applies. The composite beliefs [...], a geometric average of individual beliefs, does not fulfill all the properties of a probability measure." The same holds true for Jouini and Napp (2007, Example 2.1) who refer to the non-normalized beliefs (20) as a 'consensus characteristic'. Our normalization in Proposition 3 transforms the 'composite beliefs' or/and 'consensus characteristic' for the CARA economy into additive probability measures without affecting the preferences of the representative agent.

Remark 7. To be precise, Rubinstein (1974) even considers the more general case where different economic agents with CARA Bernoulli utility can have different absolute risk aversion coefficients α_i. Based on Remark 3, we can easily do the same. Suppose that the representative agent has CARA Bernoulli utility with risk coefficient α_ρ. Specify the agent coefficients as

$$\nu_i = \frac{\alpha_\rho}{\alpha_i}$$

for all i and observe that (10) becomes[8]

$$\pi_\rho (\omega_s) = \frac{\frac{1}{\exp(-\alpha_\rho e_{\rho s})} \exp\left(-\alpha_\rho e_s \frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}\right) \left(\prod_{i=1}^{n} [\pi_i (\omega_s)]^{\frac{\alpha_\rho}{\alpha_i}}\right)^{\frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}}}{\sum_{s'=1}^{S} \frac{1}{\exp\left(-\alpha_\rho e_{\rho s'}\right)} \exp\left(-\alpha_\rho e_{s'} \frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}\right) \left(\prod_{i=1}^{n} [\pi_i (\omega_{s'})]^{\frac{\alpha_\rho}{\alpha_i}}\right)^{\frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}}}.$$

Consequently, whenever we have the weighted endowment model $e_{\rho s} = \frac{1}{\sum_{i=1}^{n} \frac{\alpha_\rho}{\alpha_i}} e_s$ for all s, the belief of the representative agent is given as

$$\pi_\rho (\omega_s) = \frac{\left(\prod_{i=1}^{n} [\pi_i (\omega_s)]^{\frac{\alpha_\rho}{\alpha_i}}\right)^{\frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}}}{\sum_{s'=1}^{S} \left(\prod_{i=1}^{n} [\pi_i (\omega_{s'})]^{\frac{\alpha_\rho}{\alpha_i}}\right)^{\frac{1}{\sum_{i=1} \frac{\alpha_\rho}{\alpha_i}}}}$$

which yields (19) as our special case of the average endowment model with a homogeneous CARA Bernoulli utility function, i.e., $\alpha_i = \alpha_\rho = \alpha$ for all i.

[8] For a formal proof, see Zimper (2022).

4.1.1 Discussion: The "adjustment coefficient" approach by Calvet, Grandmont, and Lemaire (2018)

As in our aggregation analysis, Calvet, Grandmont, and Lemaire (2018) also consider aggregate beliefs defined as additive probability measures. Instead of using our ex-post normalization procedure, however, these authors embed–from the very outset–normalization into their aggregation analysis through the concept of an 'adjustment coefficient'.

Belief aggregation result of Calvet et al. (2018, Proposition 4) for CARA Bernoulli utility. *Suppose that all agents share the same CARA Bernoulli utility function with absolute risk-aversion coefficient α. Then there exists a unique adjustment coefficient r^o characterized through the equation*

$$\sum_{s'=1}^{S} \exp\left(\left(r^o - 1\right)\alpha\frac{e_{s'}}{n}\right)\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_{s'}\right)\right) = 1 \tag{21}$$

such that the additive probability measure π^o given by

$$\pi_s^o = \exp\left(\left(r^o - 1\right)\alpha\frac{e_s}{n}\right)\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_s\right)\right) \text{ for all } s \in \{1, ..., S\}$$

forms the aggregate belief of the representative agent for the average endowment model.

The adjustment coefficient r^o ensures, by its very characterization (21), that π^o is an additive probability measure which satisfies $\sum_{s'=1}^{S}\pi_{s'}^o = 1$. Although this normalization through an adjustment coefficient is a clever way to obtain aggregate beliefs that are probability measures, it comes with the drawback that an explicit expression for r^o (rather than just its implicit characterization through (21)) is, in general, not available. What we can do, however, is the next best thing and introduce S different state-dependent adjustment coefficients instead of a single adjustment coefficient for all states. For these state-dependent adjustment coefficients we can easily obtain an explicit expression. To this purpose, introduce state-dependent adjustment coefficients r_s^o, $s \in \{1, .., S\}$, satisfying

$$\sum_{s'=1}^{S} \exp\left(\left(r_{s'}^o - 1\right)\alpha\frac{e_{s'}}{n}\right)\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_{s'}\right)\right) = 1 \tag{22}$$

and define the additive probability measure $\hat{\pi}^o$ such that

$$\hat{\pi}_s^o = \exp\left(\left(r_s^o - 1\right)\alpha\frac{e_s}{n}\right)\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_s\right)\right) \text{ for all } \omega_s \in \Omega \tag{23}$$

Next suppose that this probability measure $\hat{\pi}^o$ coincides with our aggregate belief π_r, given by (19). Straightforward transformation then results into the following expression for the state-dependent adjustment coefficient for state s:

$$\begin{aligned}
\hat{\pi}_s^o &= \pi_r\left(\omega_s\right) && (24)\\
&\Leftrightarrow \\
\exp\left(\left(r_s^o - 1\right)\alpha\frac{e_s}{n}\right)\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_s\right)\right) &= \frac{\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_s\right)\right)}{\sum_{s'=1}^{S}\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_{s'}\right)\right)} \\
&\Leftrightarrow \\
r_s^o &= 1 - \frac{1}{a}\frac{n}{e_s}\ln\sum_{s'=1}^{S}\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_{s'}\right)\right)
\end{aligned}$$

As the $\pi_r\left(\omega_s\right)$ in (24) are additive probabilities, equation (22) must hold. Collecting the above arguments shows that the aggregate beliefs of our Proposition 3 can be equivalently described as beliefs with state-dependent adjustment coefficients whereby state-dependence is exclusively driven by the aggregate endowments in different states.

Proposition 4. *Consider the CARA representative agent model with average endowment.*

(i) *We have that $\hat{\pi}^o = \pi_r$ such that π_r is given by (19) and $\hat{\pi}^o$ is given by (23) with state-dependent adjustment coefficients defined as*

$$r_s^o = 1 - \frac{1}{a}\frac{n}{e_s}\ln\sum_{s'=1}^{S}\exp\left(\sum_{i=1}^{n}\frac{1}{n}\ln\pi_i\left(\omega_{s'}\right)\right) \tag{25}$$

(ii) *For the special case that the aggregate endowments are identical across states, i.e., $e_s = e_{s'}$ for all s, s', Calvet et al.'s (2018) adjustment coefficient r^o, implicitly characterized through (21), coincides with (25).*

4.2 CRRA Bernoulli utility

Consider now a CRRA Bernoulli utility function such that for $c > 0$

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{cases}$$

where $\gamma > 0$ denotes the relative risk aversion coefficient. An application of belief aggregation formula (13) gives us for CRRA Bernoulli utility

$$\pi_\rho(\omega_s) = \frac{(e_s)^\gamma \sum_{i=1}^{n} \mu_i (c_{is}^*)^{-\gamma} \pi_i(\omega_s)}{\sum_{s'=1}^{S} (e_{s'})^\gamma \sum_{i=1}^{n} \mu_i (c_{is'}^*)^{-\gamma} \pi_i(\omega_{s'})} \tag{26}$$

for $\rho \in \{r, R\}$ whereby the agent weights μ_i are arbitrary positive numbers. When we fix the agent weights as Pareto agent weights (15), we obtain the following analytically convenient expression of aggregate beliefs.

Proposition 5. *There exists a CRRA representative agent model–which is the same for the average and the aggregate endowment economy $\rho \in \{r, R\}$–such that the representative agent's belief is given as*

$$\pi_\rho(\omega_s) = \frac{\left(\sum_{i=1}^{n} (\mu_i^* \pi_i(\omega_s))^{\frac{1}{\gamma}}\right)^\gamma}{\sum_{s'=1}^{S} \left(\sum_{i=1}^{n} (\mu_i^* \pi_i(\omega_{s'}))^{\frac{1}{\gamma}}\right)^\gamma} \text{ for all } \omega_s \in \Omega \tag{27}$$

with Pareto agent weights

$$\mu_i^* = \frac{\prod_{k \neq i} (c_{ks^*}^*)^{-\gamma} \pi_k(\omega_{s^*})}{\sum_{j=1}^{n} \prod_{k \neq j} (c_{ks^*}^*)^{-\gamma} \pi_k(\omega_{s^*})}$$

for an arbitrarily chosen $s^* \in \{1, ..., S\}$.

If the economic agents have identical beliefs, i.e., $\pi = \pi_i$ for all i, the specification of the agent weights becomes irrelevant because the aggregate belief (27) reduces to the shared belief

$$\pi_\rho(\omega_s) = \frac{\pi(\omega_s)\left(\sum_{i=1}^{n}(\mu_i^*)^{\frac{1}{\gamma}}\right)^\gamma}{\sum_{s'=1}^{S}\pi(\omega_{s'})\left(\sum_{i=1}^{n}(\mu_i^*)^{\frac{1}{\gamma}}\right)^\gamma} = \pi(\omega_s).$$

If the economic agents have heterogeneous beliefs, however, the agent weights matter for the determination of the aggregate belief. Whenever agent i has a greater weight than agent j, i.e., $\mu_i^* > \mu_j^*$, agent i's belief π_i makes a greater impact on the aggregate belief than agent j's belief π_j. In particular, we have the following limit relationship for any fixed $\gamma > 0$

$$\begin{aligned} \lim_{\mu_i \to 1} \frac{\left(\sum_{i=1}^{n}(\mu_i^*\pi_i(\omega_s))^{\frac{1}{\gamma}}\right)^\gamma}{\sum_{s=1}^{S}\left(\sum_{i=1}^{n}(\mu_i^*\pi_i(\omega_s))^{\frac{1}{\gamma}}\right)^\gamma} &= \pi_i(\omega_s) \\ &\Leftrightarrow \\ \lim_{\mu_{is} \to 1} \pi_r(\omega_s) &= \pi_i(\omega_s). \end{aligned}$$

This is not surprising. By the FOCs for CRRA Bernoulli utility we have the general relationship

$$\frac{\mu_i^*}{\mu_j^*} = \left(\frac{c_{is}^*}{c_{js}^*}\right)^\gamma \frac{\pi_j(\omega_s)}{\pi_i(\omega_s)}$$

for any s. Compared to the belief π_j of agent j the belief of agent i will make a greater impact on the aggregate belief if agent i consumes more in a state s than agent j to which she attaches a smaller probability than agent j. Of course, if $\mu_i^* \to 1$, agent i tends to consume infinitely more than any other agent j so that the underlying economy converges to a single agent economy where only the belief π_i of agent i matters.[9] Moreover, such convergence happens faster for greater values of the relative risk-aversion coefficient γ. That is, the belief of a wealthy economic agent will the stronger dominate the belief of a poor agent, the more risk-averse the CRRA economy is.

[9] This argument comes with the caveat that the agent weights depend on the equilibrium consumption allocation of all economic agents, which, in turn, depends on the risk-aversion coefficient γ, the agents' beliefs as well as their endowments. In other words, the agent weights are not exogenously given but completely determined by the economy's fundamentals, including the beliefs of all economic agents. That is, when we keep the agents' beliefs as well as γ fixed, $\mu_i^* \to 1$ means actually that agent i's initial endowment in Arrow-Debreu securities becomes infinitely more than any other agent's endowment.

4.2.1 Discussion: The "consensus characteristic" approach by Jouini and Napp (2007)

We show in detail that Jouini and Napp's (2007) aggregation result for CRRA preferences in terms of a "consensus characteristic" becomes, after appropriate normalization, identical to our CRRA belief aggregation result in Proposition 5. These authors define subjective and aggregate beliefs with respect to some reference measure P, with full support on Ω, interpreted as the 'true' probability measure.

Aggregation result of Jouini and Napp (2007) for CRRA Bernoulli utility. *The representative agent for the CRRA economy can be described by the following utility function*

$$U(c_1, ..., c_S) = \sum_{s=1}^{S} u(c_s)\, m(\omega_s)\, P(\omega_s) \tag{28}$$

such that the "consensus characteristic" m satisfies

$$m(\omega_s) = \left(\sum_{i=1}^{n} \nu_i \left(\pi_i(\omega_s)\right)^{\frac{1}{\gamma}} \right)^{\gamma} \text{ for } s \in \{1, ..., S\}$$

with agent weights

$$\nu_i = \frac{(\lambda_i)^{-\frac{1}{\gamma}}}{\sum_{j=1}^{n} (\lambda_j)^{-\frac{1}{\gamma}}}$$

where λ_i denotes agent i's Lagrange multiplier.

This aggregation result in terms of a reference measure P combined with a "consensus characteristic" m falls short of a representative agent model in our sense because the aggregate utility function (28) is–with the notable exception of logarithmic Bernoulli utility–not of the expected utility form. More precisely, mP in (28) is an additive probability measure if and only if $\gamma = 1$. To see this, note that

$$\begin{aligned} \sum_{s=1}^{S} m(\omega_s)\, P(\omega_s) &= 1 \\ &\Leftrightarrow \\ \sum_{s=1}^{S} P(\omega_s) \left(\sum_{i=1}^{n} \frac{(\lambda_i)^{-\frac{1}{\gamma}}}{\sum_{j=1}^{n} (\lambda_j)^{-\frac{1}{\gamma}}} \left(\pi_i(\omega_s)\right)^{\frac{1}{\gamma}} \right)^{\gamma} &= 1 \end{aligned}$$

if and only if $\gamma = 1$ because of

$$\sum_{s=1}^{S} m(\omega_s) \lesseqgtr 1 \Leftrightarrow 1 \lesseqgtr \gamma$$

Although the "consensus characteristic" approach by Jouini and Napp (2007) is thus, in general, not of the expected utility form, it must represent the same expected utility preferences as the aggregate belief of the CRRA representative agent of Proposition 5. Based on a straightforward normalization argument, we formally prove the following equivalence result in the Appendix.

Proposition 6. *Introduce the normalized Jouini and Napp (2007)"consensus characteristic"*

$$m_r(\omega_s) = \frac{m(\omega_s)}{\sum_{s'=1}^{S} m(\omega_{s'})} \text{ for } s \in \{1, ..., S\}$$

The aggregate belief (27) of the CRRA representative agent of Proposition 5 is identical to this normalized consensus characteristic, i.e.,

$$m_r(\omega_s) = \pi_r(\omega_s) \text{ for } s \in \{1, ..., S\}$$

The straightforward normalization procedure of Proposition 6 transforms the CRRA representative agent of Jouini and Napp (2007) into an expected utility maximizer with additive beliefs. The only thing that is here special about the logarithmic Bernoulli utility function is that we do not need any normalization to turn the consensus characteristic into a probability measure because $\gamma = 1$ implies $\sum_{s'=1}^{S} m(\omega_{s'}) = 1$ so that we already have $m_r = m$ for logarithmic utility. Jouini and Napp (2007) refer to this specific feature of logarithmic Bernoulli utility in terms of martingales and density processes:

> "The process m represents a consensus characteristic; however, as seen above, except in the logarithmic case, case, it fails to be a martingale. Consequently, it cannot be interpreted as a proper belief, that is, the density process of a given probability measure. It is easy to see that it is not possible in general to recover the consensus characteristic as a martingale [...]." (p.1157)

In what follows we show how the normalized consensus characteristic, or rather its density process with respect to Jouini and Napp (2007)'s reference measure P, can be recovered as a martingale.[10] To this purpose, introduce an information-filtration process $\{\mathcal{F}_t\}_{t\in\{0,...,T\}}$ on the probability space $(\Omega, \mathcal{F}_0, P)$ where Ω coincides with the finite state space from the static set-up, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = 2^{\Omega}$, and the reference measure P has full support on Ω. For any probability measure $\hat{\pi}$ defined on $(\Omega, \mathcal{F}_0)$ there exists the $\mathcal{F}_t$-adapted *density process* $\{Z_t\}_{t\in\{0,...,T\}}$ such that

$$Z_t = \left.\frac{d\hat{\pi}}{dP}\right|_{\mathcal{F}_t} \text{ for } t \in \{0, ..., T\} \tag{29}$$

where the $\mathcal{F}_t$-measurable function $\frac{d\hat{\pi}}{dP}$ is the Radon-Nikodym derivative of $\hat{\pi}$ with respect to P defined as

$$\hat{\pi}(A) = \int_{\omega\in A} \frac{d\hat{\pi}}{dP}(\omega)\, dP \text{ for all } A \in \mathcal{F}_t$$

The density process $\{Z_t\}_{t\in\{0,...,T\}}$ is a P-martingale satisfying

$$\left.\frac{d\hat{\pi}}{dP}\right|_{\mathcal{F}_t} = E_P\left(\left.\frac{d\hat{\pi}}{dP}\right|_{\mathcal{F}_{t+1}}\right) \text{ for } t \in \{0, ..., T\} \tag{30}$$

To see this, let the $\mathcal{F}_t$, $t \in \{0, ..., T\}$, be generated by partitions Π_t, $t \in \{0, ..., T\}$, with information cell $I_t \in \Pi_t$ as generic element. For any $A \in \mathcal{F}_t$ we have that

$$\int_{\omega\in A} \left.\frac{d\hat{\pi}}{dP}\right|_{\mathcal{F}_t}(\omega)\, dP = \sum_{I_t\subseteq A}\sum_{\omega\in I_t} \frac{d\hat{\pi}}{dP}(\omega)\, P(I_t) \tag{31}$$

Since $A \in \mathcal{F}_t$ implies $A \in \mathcal{F}_{t+1}$, we also have

$$\begin{aligned}
\int_{\omega\in A} E_P\left(\left.\frac{d\hat{\pi}}{dP}\right|_{\mathcal{F}_{t+1}}(\omega)\right) dP &= \sum_{I_{t+1}\subseteq A}\sum_{\omega\in I_{t+1}} \frac{d\hat{\pi}}{dP}(\omega)\, P(I_{t+1}) \\
&= \sum_{I_t\subseteq A}\sum_{I_{t+1}\subseteq I_t}\left(\sum_{\omega\in I_{t+1}} \frac{d\hat{\pi}}{dP}(\omega)\, P(I_{t+1} \mid I_t)\right) P(I_t) \\
&= \sum_{I_t\subseteq A}\sum_{\omega\in I_t} \frac{d\hat{\pi}}{dP}(\omega)\, P(I_t) \qquad (32)
\end{aligned}$$

[10]We use a more precise terminology than Jouini and Napp (2007) who interchangeably speak of the consensus characteristic m for logarithmic utility as a 'belief'/'prior probability measure' or 'density process'/'martingale'. In contrast, we refer to this consensus characteristic–or any normalized consensus characteristic–as 'belief'/'prior probability measure' whereas we use the notions 'density process'/'martingale' only for the adapted Radon-Nikodym derivatives of (normalized) consensus characteristics with respect to the reference measure P.

whereby the last step follows from

$$\sum_{I_{t+1}\subseteq I_t}\left(\sum_{\omega\in\Omega_{t+1}}\frac{d\hat{\pi}}{dP}(\omega)\,P(I_{t+1}\mid I_t)\right)=\sum_{\omega\in I_t}\frac{d\hat{\pi}}{dP}(\omega)\,\frac{P(I_t)}{P(I_t)}$$

as $\frac{d\hat{\pi}}{dP}$ must be constant on any I_t due to $A\in\mathcal{F}_t$. Finally, note that the integrals (31) and (32) are identical which gives us the P-martingale property (30) for the density process (29).

Since the normalized consensus characteristic m_r of Proposition 6 is a probability measure on $(\Omega,\mathcal{F}_0)$ (coinciding with the aggregate belief (27)), we can set $\hat{\pi}=m_r$ to obtain the density process $\{Z_t\}_{t\in\{0,...,T\}}$ such that

$$Z_t=\left.\frac{dm_r}{dP}\right|_{\mathcal{F}_t}\quad\text{for } t\in\{0,...,T\}\tag{33}$$

where the Radon-Nikodym derivatives of m_r with respect to P form a P-martingale. Using the normalized consensus characteristic of Proposition 6, such density process can thus be defined for general CRRA Bernoulli utility functions with $\gamma>0$ and not only for the special case of logarithmic Bernoulli utility function with $\gamma=1$. In other words, there is nothing special about the Bernoulli utility function in terms of martingales or density processes when we use the normalized consensus characteristic of Proposition 5 rather than the non-normalized consensus characteristic originally used by Jouini and Napp (2007).

4.3 Logarithmic Bernoulli utility

For the special case of logarithmic CRRA Bernoulli utility, i.e., $\gamma=1$, the aggregate belief (27) becomes the μ_i^*-weighted convex combination of agents' beliefs

$$\pi_\rho(\omega_s)=\sum_{i=1}^{n}\mu_i^*\pi_i(\omega_s)\ \text{ for all } s\in\{1,...,S\}$$

as the denominator in (27) reduces to one, i.e.

$$\sum_{s=1}^{S}\sum_{i=1}^{n}\mu_i^*\pi_i(\omega_s)=\sum_{i=1}^{n}\mu_i^*\sum_{s=1}^{S}\pi_i(\omega_s)=1.$$

By (17), the Pareto agent weights for CRRA Bernoulli utility functions are equivalently given as

$$\mu_i^*=\frac{\sum_{s=1}^{S}p_s^*\left(c_{is}^*\right)^\gamma}{\sum_{i=1}^{n}\left(\sum_{s=1}^{S}p_s^*\left(c_{is}^*\right)^\gamma\right)}$$

which becomes for logarithmic Bernoulli utility

$$\mu_i^* = \frac{\sum_{s=1}^{S} p_s^* c_{is}^*}{\sum_{i=1}^{n} \left(\sum_{s=1}^{S} p_s^* c_{is}^* \right)}$$

Using the budget condition

$$\sum_{s=1}^{S} p_s^* c_{is}^* = \sum_{s=1}^{S} p_s^* e_{is}$$

gives us a classic belief-aggregation result that is already mentioned in Rubinstein (1976).

Proposition 7. (Rubinstein 1976). *For logarithmic Bernoulli utility the representative agent's aggregate belief is the wealth-weighted average of all economic agents' beliefs, i.e., for $\rho \in \{r, R\}$*

$$\pi_\rho (\omega_s) = \sum_{i=1}^{n} \mu_i^* \pi_i (\omega_s) \text{ for all } \omega_s \in \Omega$$

such that the Pareto agent weights are given as

$$\mu_i^* = \frac{w_i^*}{\sum_{j=1}^{n} w_j^*} \tag{34}$$

where

$$w_i^* = \sum_{s=1}^{S} p_s^* e_{is}$$

denotes agent i's wealth evaluated at equilibrium prices.

Since the endogenously determined equilibrium prices are central to wealth-weighted average aggregation, Rubinstein's (1976) belief aggregation result does, in general, not show how the agent weights are determined through the economic fundamentals. However, there are two notable exceptions in terms of endowment distributions for which we do not need to know the equilibrium prices (which cancel out) in order to characterize agent weights through economic fundamentals.

Corollary 1. *Consider the Pareto agent weights (34).*

(i) *Suppose that all economic agents have identical endowments, i.e.,* $e_{is} = \frac{1}{n} e_s$ *for all* $i \in \{1, ..., n\}$ *and all* $s \in \{1, ..., S\}$*. Then all Pareto agent weights are also identical, i.e.,*

$$\mu_i^* = \frac{1}{n} \text{ for all } i \in \{1, ..., n\}$$

(ii) *Suppose that there are two economic agents* i *and* j *such that, for all* $s \in \{1, ..., S\}$, $e_{is} = a e_{js}$ *for some factor* $a > 0$*. Then we have for the relationship of both Pareto agent weights that*

$$\mu_i^* = a \mu_j^*$$

Assume now that the heterogeneous beliefs of the agents are *unbiased* in the sense that "the heterogeneous beliefs have an average nonzero bias" (Levy et al. 2006, p.1319). For the generic case in which not all agents have the same equilibrium wealth levels the beliefs of the representative agent will be biased whenever the average belief over all economic agents is unbiased. That is, even if we follow the argument of Levy et al. (2006, p.1319) that "[...] it seems reasonable to assume that expectations are not systematically dependent on wealth and that investors do not have a systematic bias in their expectations", Corollary 1 implies that the aggregate beliefs of the representative agent cannot be described as rational expectations in an economy with different wealth levels.

Next we go beyond Rubinstein's (1976) characterization of aggregate beliefs in terms of endogenous equilibrium wealth levels and provide a characterization of the representative agent's aggregate belief exclusively in terms of the exogenous economic fundamentals, i.e., in terms of the beliefs and endowments in the underlying economy.

Proposition 8. *Suppose that all economic agents have a logarithmic Bernoulli utility function, i.e.,* $\gamma = 1$*. Then the aggregate beliefs (27) become the* μ_i^**-weighted convex combination of the economic agents' beliefs*

$$\pi_\rho(\omega_s) = \sum_{i=1}^{n} \mu_i^* \pi_i(\omega_s) \text{ for all } \omega_s \in \Omega$$

such that the Pareto agent weights are given as

$$\mu_i^* = \frac{\prod_{k \neq i} \left(\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} e_{ks'} \right) + e_{ks^*} \right)^{-1}}{\sum_{h=1}^{n} \prod_{k \neq h} \left(\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} e_{ks'} \right) + e_{ks^*} \right)^{-1}} \tag{35}$$

for an arbitrarily chosen $s^* \in \{1, ..., S\}$.

Observe that the characterization of the Pareto agent weights (35) is mathematically equivalent to the following expression

$$\mu_i^* = \frac{\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} \right) e_{is'} + e_{is^*}}{\sum_{h=1}^n \left(\sum_{s' \neq s^*} \left(\frac{\sum_{j=1}^n e_{js^*} \pi_j(\omega_{s'})}{\sum_{j=1}^n e_{js'} \pi_j(\omega_{s^*})} \right) e_{hs'} + e_{hs^*} \right)}. \tag{36}$$

By comparing the mathematically equivalent expression (17) of agent weights from Proposition 2, i.e.,

$$\mu_i^* = \frac{\sum_{s=1}^S p_s^* e_{is}}{\sum_{j=1}^n \left(\sum_{s=1}^S p_s^* e_{js} \right)},$$

with the expression (36), we can immediately recover the equilibrium price ratios

$$\frac{p_s^*}{p_{s^*}^*} = \frac{\sum_{j=1}^n e_{js^*} \pi_j (\omega_s)}{\sum_{j=1}^n e_{js} \pi_j (\omega_{s^*})} \text{ for all } s \in \{1, ..., S\}$$

with Arrow-Debreu security s^* being the numeraire. The next result reformulates Rubinstein's (1976) belief aggregation formula exclusively in terms of economic fundamentals.

Corollary 2. *The Pareto agent weights (35) are equivalently given as*

$$\mu_i^* = \frac{\sum_{s' \neq s^*} \left(\frac{p_{s'}^*}{p_{s^*}^*} \right) e_{is'} + e_{is^*}}{\sum_{h=1}^n \left(\sum_{s' \neq s^*} \left(\frac{p_{s'}^*}{p_{s^*}^*} \right) e_{hs'} + e_{hs^*} \right)} \text{ for } i \in \{1, ..., n\}$$

such that the equilibrium price ratios satisfy

$$\frac{p_{s'}^*}{p_{s^*}^*} = \frac{\sum_{j=1}^n e_{js^*} \pi_j (\omega_{s'})}{\sum_{j=1}^n \pi_j (\omega_{s^*}) e_{js'}} \text{ for all } s' \in \{1, ..., S\}$$

for an arbitrarily chosen numeraire Arrow-Debreu security $s^* \in \{1, ..., S\}$.

5 Concluding remarks: Related literature and an outlook on multiperiod complete markets economies

There exists a large literature which convincingly argues that heterogeneous beliefs are the rule rather than the exception for real-life decision makers whereby heterogeneity of beliefs might help to explain empirically observed asset price patterns.[11] While many authors simply treat different beliefs as 'a fact of life', other authors offer deeper decision-theoretic, behavioral or/and psychological explanations why beliefs might differ (such explanations range from different degrees of *overconfidence* as, e.g., in Scheinkman and Xiong (2003) to different *belief-optimization* problems as in Brunnermeier, Gollier, and Parker (2009)).

Most closely related to our own analysis is the belief-aggregation literature that considers a complete markets framework with finitely many agents and states of the world, namely, Rubinstein (1974, 1976), Jouini and Napp (2007), Calvet, Grandmont, and Lemaire (2018). Within this framework–which is also the framework of this paper–the arbitrage-free equilibrium prices of a static economy are characterized in terms of a finite, strictly positive state-price vector, i.e., a finite vector of equilibrium prices for Arrow-Debreu securities.[12]

To see the novelty of our belief aggregation formulas, note that Rubinstein (1974, 1976) as well as Jouini and Napp (2007) describe aggregate beliefs as additive probability measures only for the special case of logarithmic but not for CARA or for general CRRA Bernoulli utility functions. For non-logarithmic CRRA Bernoulli utility functions, Jouini and Napp (2007) introduce a 'consensus characteristic' instead of an additive probability measure. Although Calvet et al. (2018) consider aggregate beliefs given as additive probability measures, they only provide an implicit characterization of these aggregate additive beliefs in terms of an 'adjustment coefficient', which exists but whose value is hard to pin down in a closed form representation of aggregate beliefs. These existing aggregation results for CARA and CRRA Bernoulli utility functions and our CARA and CRRA representative agent models provide mathematically equivalent characterizations of equilibrium prices. Given this mathematical equivalence, what exactly is the advantage of our representative agent models–in terms of expected utility maximizing representative

[11] For early contributions see, e.g., Rubinstein (1974, 1976), Williams (1977), Detemple and Murthy (1994), Zapatero (1998), Chiarella and He (2002), Basak (2005). For more recent articles see, e.g., Jouini and Napp (2007), Chiarella, Dieci, and He (2011), Carlin, Longstaff, and Matoba (2014), Bhamra and Uppal (2014), Chabakauri (2015), Baker, Hollifield, and Osambela (2016), and Borovička (2018). Campbell (2003; Chapter 5.3), Xiong (2013), Calvet, Grandmont, and Lemaire (2018), Zhang and Zhang (2020) as well as the references in Martin and Papadimitriou (2021).

[12] Other authors like Bhamra and Uppal (2014), Atmaz and Basak (2018) and Martin and Papadimitriou (2021) also consider a complete markets set-up but with non-finite state or/and agent spaces.

agents–over these existing aggregation models whose single agent is not an expected utility maximizer?

The major advantage of working with a representative agent who is an expected utility maximizer becomes clear when we move from a static to a multiperiod Arrow-Debreu economy. In the companion paper Zimper (2022), I consider the notion of an equilibrium for a multiperiod Arrow-Debreu economy in which the expected utility maximizing agents are not necessarily Bayesian decision makers.[13] For a fixed number of time periods $T \geq 1$ introduce the following family of information partitions of the state space Ω

$$\Pi_t = \left\{ I_t^1, ..., I_t^{m_t} \right\} \text{ for } t \in \{0, ..., T\}$$

such that Π_{t+1} is strictly finer than Π_t. At any given information cell $I_t \in \Pi_t$, $t \in \{0, ..., T\}$, the economic agents have subjective beliefs $\pi_i [I_t]$ defined as additive probability measures on $\left(I_t, 2^{I_t}\right)$. Moreover, at any given information cell the economic agents can trade Arrow-Debreu securities in order to maximize their expected utility over final consumption in period T. According to a standard argument from multiperiod equilibrium analysis, aggregation results for the static complete markets economy immediately carry over to aggregation results for this multiperiod complete markets economy in the specific sense that all price-ratios of Arrow-Debreu securities in the multiperiod economy are already pinned down as the price-ratios of the static economy in which trade only happens in the initial period (cf. Arrow 1974; Chapter 19 in Mas-Colell, Whinston, and Green 2006). This standard argument is based on the notion that equilibrium prices in a multiperiod economy must be arbitrage-free, which in turn is only satisfied under the implicit assumption that all economic agents are Bayesian decision makers. That is, equilibrium prices in a multiperiod economy are arbitrage-free if and only if, for all $i \in \{1, ..., n\}$,

$$\pi_i [I_t] (\omega) = \pi_{i0} (\omega \mid I_t) \text{ for all } \omega \in I_t, \, t \in \{0, ..., T\}$$

where π_{i0} denotes agent i's prior defined on $\left(\Omega, 2^{\Omega}\right)$. For underlying economies which allow at every information cell for a representative agent model, I show the following result for multiperiod Arrow-Debreu economies:

The equilibrium prices are arbitrage-free if and only if the expected utility maximizing representative agent is a Bayesian decision maker, i.e., if and only if

$$\pi_\rho [I_t] (\omega) = \pi_{\rho 0} (\omega \mid I_t) \text{ for all } \omega \in I_t, \, t \in \{0, ..., T\}$$

[13]To be precise, I consider in Zimper (2022) the notion of a *naive* equilibrium where economic agents are boundedly rational in the specific sense that the agents are not aware that they might be non-Bayesian decision makers. *Naivety* has here the same meaning as in the multiperiod decision theoretic literature where naive agents are not aware that their preferences might be dynamically inconsistent. Of course, equilibria in multiperiod economies with fully rational, i.e., *sophisticated*, economic agents must be arbitrage-free.

where $\pi_{\rho 0}$ *denotes representative agent's prior defined on* $\left(\Omega, 2^{\Omega}\right)$ *(cf. Theorem 2 in Zimper 2022).*[14]

The question whether equilibrium prices are arbitrage-free or not is a fundamental one. However, in order to judge whether an expected utility maximizing representative agent is a Bayesian decision maker or not, the beliefs of this representative agent must be given at any given information cell $I_t \in \Pi_t$, $t \in \{0, ..., T\}$, as additive probability measures $\pi_i [I_t]$. At this point, the 'consensus characteristic' approach of Jouini and Napp (2007) might cause some confusion. This possible confusion is best illustrated by the following statement in Atmaz and Basak (2018):

> "The main idea, as discussed in detail by Jouini and Napp (2007), is to summarize the heterogeneous beliefs in the economy by a single consensus belief so that when the consensus investor has that consensus belief and is endowed with the aggregate consumption in the economy, the resulting equilibrium is the same as in the heterogeneous-investors economy. In a model with intertemporal consumption and a finite number of agents that have CRRA preferences, Jouini and Napp (2007) show that when investors' preferences are not logarithmic, the consensus belief is not necessarily well defined since the process that aggregates investors' beliefs is not a martingale and hence not a proper belief process. Unlike their analysis, as we demonstrate in the proof of Proposition 1 in Appendix A, it turns out that this issue does not arise in our setting and we obtain a well-defined consensus belief process for all values of risk aversion due to investors' preferences being over horizon wealth." (Footnote 8, p.1234)

Firstly, notice that Atmaz and Basak (2018) appear to be surprised that they have obtained an expected utility maximizing representative agent for general CRRA Bernoulli utility functions which is seemingly different from the findings in Jouini and Napp (2007). Moreover, these authors speculate that the absence of intertemporal consumption in their model might be responsible for this difference. Our analysis shows, however, that there is

[14] To say that the representative agent is a Bayesian decision maker is equivalent to saying that his belief process is a martingale with respect to the prior $\pi_{\rho 0}$ in the following sense: for all $\omega \in \Omega$,

$$\begin{aligned} \pi_\rho [I_t] (\omega) &= \sum_{I_{t+1} \subseteq I_t} \pi_\rho [I_{t+1}] (\omega) \, \pi_\rho [I_t] (I_{t+1}) \\ &\Leftrightarrow \\ \pi_{\rho 0} (\omega \mid I_t) &= \sum_{I_{t+1} \subseteq I_t} \pi_{\rho 0} (\omega \mid I_{t+1}) \, \pi_{\rho 0} (I_{t+1} \mid I_t) . \end{aligned}$$

only an interpretational difference because one can mathematically equivalently transform the non-expected utility maximizing 'consensus investor' of Jouini and Napp (2007) into an expected utility maximizing representative agent. The issue of a not-well defined belief process does not arise in the setting of Atmaz and Basak (2018) because these authors derive for their diffusion process a CRRA representative agent model which is the analogue of the CRRA representative agent model for our static Arrow-Debreu economy. That is, contrary to the speculation by Atmaz and Basak (2018), the question whether intertemporal consumption enters into the model or not has nothing to do with their seemingly different finding from Jouini and Napp (2007).

Secondly, to observe about Jouini and Napp's 'consensus characteristic' approach that "the process that aggregates investors' beliefs is not a martingale and hence not a proper belief process" could mean two different things. On the one hand, it could mean that the consensus investor is not an expected utility maximizer because his beliefs are not given as additive probability measures $\pi_\rho [I_t]$ for all $I_t \in \Pi_t$, $t \in \{0, ..., T\}$. On the other hand, it could mean that this consensus investor can be equivalently described as an expected utility maximizer at any information cell I_t but that he is not a Bayesian decision maker, that is, the $\pi_\rho [I_t]$ are well-defined probability measures on $\left(I_t, 2^{I_t}\right)$ but there is no prior $\pi_{\rho 0}$ such that $\pi_\rho [I_t] = \pi_{\rho 0} (\cdot \mid I_t)$ for all I_t. If this latter meaning was the correct one, then the equilibrium prices pinned down by the Jouini and Napp (2007) consensus investor would give rise to unrealized arbitrage opportunities (cf. Theorem 1 in Zimper 2022). Again, our representative agent analysis clarifies this ambiguity: Since we can mathematically equivalently transform the Jouini and Napp (2007) consensus investor into an expected utility maximizing representative agent, the modeller has to decide after this transformation whether the representative agent is either described as a Bayesian or a non-Bayesian decision maker.

To summarize the major advantage of the representative agent models derived in this paper: In a multiperiod extension of the complete markets static economy, our expected utility maximizing representative agent can be modelled either as a Bayesian or a non-Bayesian decision maker. This distinction is crucial for the question of whether equilibrium prices are arbitrage-free or not. In contrast, one cannot meaningfully speak of Bayesian versus non-Bayesian decision making under the 'consensus characteristic' approach of Jouini and Napp (2007) because this 'consensus characteristic' is not a probability measure for non-logarithmic Bernoulli utility.

Appendix: Mathematical proofs

Proof of Proposition 1. The corresponding FOCs for the constraint maximization of the welfare function $W_{\mu^*}(c_i, ..., c_I)$ results in

$$\frac{u'(c_{is}^*)\,\pi_i(\omega_s)}{u'\left(c_{js}^*\right)\pi_j(\omega_s)} = \frac{\mu_j^*}{\mu_i^*} = \frac{1 - \sum_{k \neq j} \mu_k^*}{\mu_i^*} \tag{37}$$

where $\sum_{k=1}^{n} \mu_k^* = 1$. We claim that the equation system (37) is solved through the agent weights

$$\mu_i^* = \frac{\prod_{k \neq i} u'(c_{ks}^*)\,\pi_k(\omega_s)}{\sum_{j=1} \prod_{k \neq j} u'(c_{ks}^*)\,\pi_k(\omega_s)} \tag{38}$$

for all $i \in \{1, .., n\}$. To verify this claim, rewrite (37) as

$$\mu_i^* = \frac{u'\left(c_{js}^*\right)\pi_j(\omega_s)}{\left(u'(c_{is}^*)\,\pi_i(\omega_s) + u'\left(c_{js}^*\right)\pi_j(\omega_s)\right)}\left(1 - \sum_{k \neq j,i} \mu_k^*\right)$$

and substitute (38) for the LHS to obtain

$$\begin{aligned} &\frac{\prod_{k \neq i} u'(c_{ks}^*)\,\pi_k(\omega_s)}{\sum_{i=1} \prod_{k \neq i} u'(c_{ks}^*)\,\pi_k(\omega_s)} \\ = \;& \frac{u'\left(c_{js}^*\right)\pi_j(\omega_s)}{\left(u'(c_{is}^*)\,\pi_i(\omega_s) + u'\left(c_{js}^*\right)\pi_j(\omega_s)\right)}\left(1 - \sum_{k \neq j,i} \mu_k^*\right) \\ &\Leftrightarrow \\ &\left(u'(c_{is}^*)\,\pi_i(\omega_s) + u'\left(c_{js}^*\right)\pi_j(\omega_s)\right) \prod_{k \neq j,i} u'(c_{ks}^*)\,\pi_k(\omega_s) \\ = \;& \sum_{i=1} \prod_{k \neq i} u'(c_{ks}^*)\,\pi_k(\omega_s) - \sum_{i=1} \prod_{k \neq i} u'(c_{ks}^*)\,\pi_k(\omega_s) \sum_{k \neq j,i} \mu_k^* \end{aligned} \tag{39}$$

Next substitute

$$\mu_k^* = \frac{\prod_{m \neq k} u'(c_{ms}^*)\,\pi_m(\omega_s)}{\sum_{i=1} \prod_{m \neq i} u'(c_{ms}^*)\,\pi_m(\omega_s)}$$

in the RHS of (37) to obtain, after some transformations,

$$
\begin{aligned}
& \left(u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right) + u'\left(c_{js}^*\right)\pi_j\left(\omega_s\right)\right)\prod_{k\neq j,i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) \\
= & \sum_{i=1}\prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) - \sum_{k\neq j,i}\prod_{m\neq k} u'\left(c_{ms}^*\right)\pi_m\left(\omega_s\right)
\end{aligned}
$$

$$\Leftrightarrow$$

$$
\begin{aligned}
& \prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) + \prod_{k\neq j} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) \\
= & \sum_{i=1}\prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) - \sum_{k\neq j,i}\prod_{m\neq k} u'\left(c_{ms}^*\right)\pi_m\left(\omega_s\right)
\end{aligned}
$$

$$\Leftrightarrow$$

$$\sum_{i=1}\prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right) = \sum_{i=1}\prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right)$$

This proves that (38) for all $i \in \{1, .., n\}$ indeed solves the equation system (37).□□

Proof of Proposition 2. Part (i). Because of

$$u'\left(c_{is}^*\right)\pi_i\left(\omega_s\right) = \lambda_i p_s^*, \tag{40}$$

the Pareto agent weights are equivalently given in terms of the agents' Lagrange multipliers as

$$\mu_i^* = \frac{\prod_{k\neq i} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right)}{\sum_{j=1}^{n}\prod_{k\neq j} u'\left(c_{ks}^*\right)\pi_k\left(\omega_s\right)} = \frac{\prod_{k\neq i}\lambda_k}{\sum_{j=1}^{n}\prod_{k\neq j}\lambda_k} \tag{41}$$

Next introduce the notational convention

$$\eta_k = \left(\lambda_k\right)^{-1}$$

and rewrite (41) as

$$\mu_i^* = \frac{\prod_{k\neq i}\left(\eta_k\right)^{-1}}{\sum_{j=1}^{n}\prod_{k\neq j}\left(\eta_k\right)^{-1}}$$

Without loss of generality let $i = 1$ and observe that

$$\begin{aligned}
\frac{\prod_{k\neq i}(\eta_k)^{-1}}{\sum_{j=1}^{n}\prod_{k\neq j}(\eta_k)^{-1}} &= \frac{1}{\sum_{j=1}^{n}\frac{\prod_{k\neq i}(\eta_k)}{\left(\prod_{k\neq j}(\eta_k)\right)^{-1}}} \\
&= \frac{1}{1+\frac{\eta_2\prod_{k=2}^{n}(\eta_k)}{\eta_2\prod_{k\neq 2}^{n}(\eta_k)}+\frac{\eta_3\prod_{k=2}^{n}(\eta_k)}{\eta_3\prod_{k\neq 3}^{n}(\eta_k)}+\cdots+\frac{\eta_n\prod_{k=2}^{n}(\eta_k)}{\eta_n\prod_{k\neq n}^{n}(\eta_k)}} \\
&= \frac{\prod_{k=1}^{n}(\eta_k)}{\prod_{k=1}^{n}(\eta_k)+\eta_2\prod_{k=2}^{n}(\eta_k)+\eta_3\prod_{k=2}^{n}(\eta_k)+\cdots+\eta_n\prod_{k=2}^{n}(\eta_k)} \\
&= \frac{\eta_1}{\eta_1+\eta_2+\eta_3+\cdots+\eta_n}
\end{aligned}$$

which gives us the desired expression for μ_1.

Part (ii). Rewrite the FOCs (40) as

$$\frac{p_s^*}{u'(c_{ks}^*)} = \eta_k \pi_k(\omega_s)$$

Summing up over all states gives

$$\begin{aligned}
\sum_{s=1}^{S}\frac{p_s^*}{u'(c_{ks}^*)} &= \sum_{s=1}^{S}\eta_k\pi_{ik}(\omega_s) \\
&\Leftrightarrow \\
\sum_{s=1}^{S}\frac{p_s^*}{u'(c_{ks}^*)} &= \eta_k
\end{aligned}$$

Substitution gives (17).□□

Proof of Proposition 5. Using expression (16) for Pareto agent weights in (12) implies

$$\mu_i^* u'(c_{is}^*)\,\pi_i(\omega_s) = \frac{1}{\sum_{j=1}^{n}(\lambda_j)^{-1}}p_s^*$$

for all i which yields for CRRA Bernoulli utility

$$\begin{aligned} \frac{\mu_i^*}{\mu_j^*} &= \left(\frac{c_{is}^*}{c_{js}^*}\right)^\gamma \frac{\pi_j(\omega_s)}{\pi_i(\omega_s)} \\ &\Leftrightarrow \\ \left(\frac{\mu_i^*\pi_i(\omega_s)}{\mu_j^*\pi_j(\omega_s)}\right)^{\frac{1}{\gamma}} &= \frac{c_{is}^*}{c_{js}^*} \end{aligned}$$

at the equilibrium consumption $(c_1^*, ..., c_n^*)$. Recall from Remark 2 that we can restrict attention to $\rho = R$ as the two representative agent models r and R must coincide for CRRA Bernoulli utility. Since $e_{Rs} = e_s = \sum_{i=1}^{n} c_{is}^*$, this system of equations over all $i \in \{1, ..., n\}$ has the solution

$$c_{is}^* = \frac{(\mu_i^*\pi_i(\omega_s))^{\frac{1}{\gamma}}}{\sum_{j=1}^{n}\left(\mu_j^*\pi_j(\omega_s)\right)^{\frac{1}{\gamma}}} e_{Rs}.$$

Consequently,

$$\begin{aligned} & (e_{Rs})^\gamma \sum_{i=1}^{n} \mu_i^* (c_{is}^*)^{-\gamma} \pi_i(\omega_s) \\ = \; & (e_{Rs})^\gamma \sum_{i=1}^{n} \mu_i^* \left(\frac{(\mu_i^*\pi_i(\omega_s))^{\frac{1}{\gamma}}}{\sum_{j=1}^{n}\left(\mu_j^*\pi_j(\omega_s)\right)^{\frac{1}{\gamma}}} e_{Rs}\right)^{-\gamma} \pi_i(\omega_s) \\ = \; & (e_{Rs})^\gamma \sum_{i=1}^{n} (e_{Rs})^{-\gamma} \left(\frac{\mu_i^*(\mu_i^*\pi_i(\omega_s))^{-1}}{\left(\sum_{j=1}^{n}\left(\mu_j^*\pi_j(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{-\gamma}}\right) \pi_i(\omega_s) \\ = \; & n\left(\sum_{j=1}^{n}\left(\mu_j^*\pi_j(\omega_s)\right)^{\frac{1}{\gamma}}\right)^\gamma. \end{aligned}$$

Substituting back into (26) for s and s', respectively, and using expression (15) for the agent weights yields the desired belief aggregation result.□□

Proof of Proposition 6. Step 1. Using the Lagrange multipliers expression (16)

$$\mu_i^* = \frac{(\lambda_i)^{-1}}{\sum_{j=1}^{n} (\lambda_j)^{-1}}$$

for the Pareto agent weights in (27) of Proposition 5 gives us

$$\begin{aligned}
\left(\sum_{i=1}^{n} \left(\mu_i^* \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma} &= \left(\sum_{i=1}^{n} \left(\frac{(\lambda_i)^{-1}}{\sum_{j=1}^{n} (\lambda_j)^{-1}} \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma} \\
&= \frac{\left(\sum_{i=1}^{n} \left((\lambda_i)^{-1} \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(\sum_{j=1}^{n} (\lambda_j)^{-1}\right)}
\end{aligned}$$

and therefore

$$\sum_{s\prime=1}^{S} \left(\sum_{i=1}^{n} \left(\mu_i^* \pi_i(\omega_{s\prime})\right)^{\frac{1}{\gamma}}\right)^{\gamma} = \sum_{s\prime=1}^{S} \frac{\left(\sum_{i=1}^{n} \left((\lambda_i)^{-1} \pi_i(\omega_{s'})\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(\sum_{j=1}^{n} (\lambda_j)^{-1}\right)}$$

implying, by (27),

$$\pi_r(\omega_s) = \frac{\left(\sum_{i=1}^{n} \left((\lambda_i)^{-1} \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\sum_{s\prime=1}^{S} \left(\sum_{i=1}^{n} \left((\lambda_i)^{-1} \pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma}}.$$

Step 2. Next use the Jouini and Napp (2007) characterization of the consensus characteristic

$$m(\omega_s) = \left(\sum_{i=1}^{n} \frac{(\lambda_i)^{-\frac{1}{\gamma}}}{\sum_{j=1}^{n} (\lambda_j)^{-\frac{1}{\gamma}}} \left(\pi_i(\omega_s)\right)^{\frac{1}{\gamma}}\right)^{\gamma}$$

to see that

$$\frac{m(\omega_s)}{\sum_{s\prime=1}^{S} m(\omega_{s'})} = \frac{\left(\sum_{i=1}^{n} (\lambda_i)^{-\frac{1}{\gamma}} (\pi_i(\omega_s))^{\frac{1}{\gamma}}\right)^{\gamma}}{\sum_{s\prime=1}^{S}\left(\sum_{i=1}^{n} (\lambda_i)^{-\frac{1}{\gamma}} (\pi_i(\omega_{s'}))^{\frac{1}{\gamma}}\right)^{\gamma}}$$

which gives us the desired result

$$\pi_r(\omega_s) = \frac{m(\omega_s)}{\sum_{s\prime=1}^{S} m(\omega_{s'})}.$$

□□

Proof of Proposition 8. Step 1. Observe that the optimal net-trade function of agent i in Arrow-Debreu security s, which satisfies for all s' the MRS

$$\frac{\pi_i(\omega_s)}{\pi_i(\omega_{s'})} \frac{(\theta_{is} + e_{is})^{-1}}{(\theta_{is'} + e_{is'})^{-1}} = \frac{p_s}{p_{s'}}$$

as well as the budget condition

$$\sum_{s=1}^{S} p_s \theta_{is} = 0$$

is given as

$$\theta_{is}(p, e_i, \pi_i) = \pi_i(\omega_s) \sum_{s' \neq s} \frac{p_{s'}}{p_s} e_{is'} - e_{is} \sum_{s' \neq s} \pi_i(\omega_{s'}) \text{ for all } s \in \{1, ..., S\} \tag{42}$$

Step 2. From the market-clearing condition we obtain

$$\begin{aligned} \sum_{j=1}^{n} \theta_{js}(p^*, e_j, \pi_j) &= 0 \\ &\Leftrightarrow \\ \sum_{j=1}^{n}\left(\pi_j(\omega_s) \sum_{s' \neq s} \frac{p^*_{s'}}{p^*_s} e_{js'} - e_{js} \sum_{s' \neq s} \pi_j(\omega_{s'})\right) &= 0 \\ &\Leftrightarrow \\ \sum_{s' \neq s} \frac{p^*_{s'}}{p^*_s} \sum_{j=1}^{n} \pi_j(\omega_s) e_{js'} &= \sum_{s' \neq s} \sum_{j=1}^{n} e_{js} \pi_j(\omega_{s'}) \\ &\Leftrightarrow \\ \frac{p^*_{s'}}{p^*_s} &= \frac{\sum_{j=1}^{n} e_{js} \pi_j(\omega_{s'})}{\sum_{j=1}^{n} \pi_j(\omega_s) e_{js'}} \end{aligned} \tag{43}$$

Substituting (43) in the optimal net-trade function (42) gives, after straightforward transformation, the equilibrium consumption of agent i in Arrow-Debreu security s:

$$c_{is}^{*}=\theta_{is}\left(p^{*},e_{i},\pi_{i}\right)+e_{is}=\pi_{i}\left(\omega_{s}\right)\left(\sum_{s'\neq s}\left(\frac{\sum_{j=1}^{n}e_{js}\pi_{j}\left(\omega_{s'}\right)}{\sum_{j=1}^{n}\pi_{j}\left(\omega_{s}\right)e_{js'}}e_{is'}\right)+e_{is}\right)$$

Finally, substituting the equilibrium consumption into the Pareto agent weights for logarithmic utility, i.e.,

$$\mu_{i}^{*}=\frac{\prod_{k\neq i}\left(c_{ks}^{*}\right)^{-1}\pi_{k}\left(\omega_{s}\right)}{\sum_{h=1}^{n}\prod_{k\neq h}\left(c_{ks}^{*}\right)^{-1}\pi_{k}\left(\omega_{s}\right)},$$

gives the desired result.□□

References

Arrow, K.J. (1974) *Essays in the theory of risk-bearing.* Amsterdam: North-Holland Publishing Company.

Atmaz, A., and S. Basak (2018) "Belief dispersion in the stock market" *Journal of Finance* **73**, 1225-1279.

Baker, S.D., Hollifield, B., and E. Osambela (2016) "Disagreement, speculation, and aggregate investment" *Journal of Financial Economics* **119**, 210-225.

Basak, S. (2005) "Asset pricing with heterogeneous beliefs" *Journal of Banking and Finance* **29**, 2849-2881.

Bhamra, H.S., and R. Uppal (2014) "Asset prices with heterogeneity in preferences and beliefs" *Review of Financial Studies* **27**, 519-580.

Borovička, J. (2020) "Survival and long-run dynamics with heterogeneous beliefs under recursive preferences" *Journal of Political Economy* **128**, 206-251.

Brunnermeier, M.K., C. Gollier, and J.A. Parker (2007) "Optimal beliefs, asset prices, and the preference for skewed returns" *American Economic Review. Papers and Proceedings* **97**, 159-165.

Calvet, L.-E., Grandmont, J.-M., and I. Lemaire (2018) "Aggregation of heterogenous beliefs, asset pricing, and risk sharing in complete financial markets" *Research in Economics* **72**, 117-146.

Campbell, J.Y. (2003) "Consumption-based asset pricing" in: G.M. Constantinides, M. Harris and R.M. Stulz, eds., *Handbook of the Economics and Finance.* Amsterdam: Elsevier, 803-887.

Carlin, B.I., Longstaff, F.A, and K. Matoba (2014) "Disagreement and asset prices" *Journal of Financial Economics* **114**, 226-238.

Chabakauri, G. (2015) "Asset pricing with heterogeneous preferences, beliefs, and portfolio constraints" *Journal of Monetary Economics* **75**, 21-34.

Chiarella, C., and X-Z. He (2002) "Heterogeneous beliefs, risk and learning in a simple asset pricing model" *Computational Economics* **19**, 95-132.

Chiarella, C., Dieci, R., and X-Z. He (2011) "Do heterogeneous beliefs diversify market risk?" *European Journal of Finance* **17**, 241-258.

Detemple, J., and S. Murthy (1994) "Intertemporal asset pricing with heterogeneous beliefs" *Journal of Economic Theory* **62**, 294-320.

Duffie, D. (2001) *Dynamic Asset Pricing Theory.* Princeton: Princeton University Press.

Gorman, W. (1953) "Community preference fields" *Econometrica* **21**, 63-80.

Harrison, J.M., and D.M. Kreps (1978) "Speculative investor behavior in a stock market with heterogeneous expectations" *Quarterly Journal of Economics* **92**, 323-336.

Jouini E., and C. Napp (2007) "Consensus consumer and intertemporal asset pricing with heterogeneous beliefs" *Review of Economic Studies* **74**, 1149-1174.

Krueger, D., and H. Lustig (2010) "When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?" *Journal of Economic Theory* **145**, 1-45.

Levy, H., Levy, M., and G. Benita (2006) "Capital asset prices with heterogeneous beliefs" *The Journal of Business* **79**, 1317-1353.

Lucas, R.E. Jr. (1978) "Asset prices in an exchange economy" *Econometrica* **46**, 1429-1445.

Martin, I.W.R., and D. Papadimitriou (2021) "Sentiment and speculation in a market with heterogeneous beliefs" Working paper.

Mas-Colell, A., M.D. Whinston, and J.R. Green (1995) *Microeconomic theory.* Oxford: Oxford University Press.

Negishi, T. (1960) "Welfare economics and existence of an equilibrium for a competitive economy" *Metroeconomica* **12**, 92-97.

Rubinstein, M. (1974) "An aggregation theorem for securities markets" *Journal of Financial Economics* **1**, 225-244.

Rubinstein, M. (1976) "The strong case for the generalized logarithmic utility model as the premier model of financial markets" *Journal of Finance* **31**, 551-571.

Scheinkman, J.A., and W. Xiong (2003) "Overconfidence and speculative bubbles" *Journal of Political Economy* **111**, 1183-1219.

Williams, J.T. (1977) "Capital asset prices with heterogeneous beliefs" *Journal of Financial Economics* **5**, 219-239.

Xiong, W. (2013) "Bubbles, crises, and heterogeneous beliefs" in: J-P. Fouque and J.A. Langsam, eds., *Handbook of Systemic Risk*. New York: Cambridge University Press, 663-713.

Zapatero, F. (1998) "Effects of financial innovation on market volatility when beliefs are heterogeneous" *Journal of Economic Dynamics and Control* **22**, 597-626.

Zhang R., and T. Zhang (2020) "AH Premium: A Natural Experiment" Working paper.

Zimper, A. (2022) "Unrealized arbitrage opportunities in naive equilibria with non-Bayesian belief processes" Mimeo.