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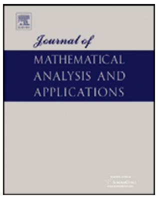

Free joinings of C^*-dynamical systems

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ABSTRACT

Joinings of C^*-dynamical systems are defined in terms of free products of C^*-algebras, as an analogue of joinings of classical dynamical systems. We then consider disjointness in this context, in particular for ergodic versus identity systems. Lastly we show how multi-time correlation functions appearing in quantum statistical mechanics naturally fit into this joining framework.

1. Introduction

Joinings, and more specifically disjointness, of measure theoretic dynamical systems were introduced in [16] and has since become an important tool in classical ergodic theory (see for example [10] and [17] for overviews). In noncommutative dynamical systems, in an operator algebraic framework, a generalization of joinings in terms of tensor products of operator algebras has been applied in the study of dynamical entropy [22], and a more systematic study of such generalized joinings was initiated in [12,13]. An early exploration of disjointness in the noncommutative setting, but from the point of view of topological dynamics, can be found in [6, Section 5].

However, although the tensor product approach gives a direct generalization of classical joinings, it does have its limitations, for example if one wants to consider noncommutative versions of so-called graph joinings of more than two copies of the same system. In [12, Construction 3.4] and [13, Section 5] this problem could be handled for the case of two copies of the same system by considering the commutant of the one copy, since this alleviates commutation problems sufficiently, but for more than two copies this simple approach does not help. This is unfortunate, as "self-joinings" of this type have proven very useful in classical ergodic theory; see for example [20,21,11] as sample of papers that have appeared over the years applying this idea.

In this paper we replace the tensor product with a free product of operator algebras as an alternative way to approach this problem. In this case we get an analogy rather than a generalization of classical joinings. We could refer to joinings based on free products as "free joinings", but since we consider only the free product setting in this paper, no confusion will arise if we simply use the term "joinings". We will however consider two types of free products, namely the unital C^*-algebra free product, and the reduced free product of unital C^*-algebras with specified states, and will correspondingly use the terms "joining" and "r-joining", respectively.

Free products of operator algebras were initially studied in [9,5,23], and useful sources for the ideas and tools from this area that we will use are [25] and [24]. We can add that free products of operator algebras have already appeared in work on noncommutative ergodic theory. See for example the recent papers [1,14,15]. In particular we will use a result from [1] in Section 3.

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We discuss the setting and basic definitions and constructions regarding joinings in Section 2. In Section 3 we consider disjointness, and in particular how it relates to ergodicity. In the tensor product case ergodicity can be characterized in terms of disjointness from identity systems [13, Theorem 2.1], and here we explore a similar connection in the free product case for certain classes of systems. In Section 3 we only consider joinings of two systems at a time, and the goal is to give an idea of how free joinings compare with tensor joinings in one of the standard applications of joinings (characterizing ergodicity). Lastly in Section 4 we briefly study multi-time correlation functions. This concept has it origins in quantum statistical mechanics [7,4], but here we also motivate it from a mathematical point of view in terms of higher order mixing of strongly mixing systems based on reduced group C^*-algebras. We show that multi-time correlation functions and their so-called asymptotic states (if they exist), are examples of joinings. This illustrates how joinings of more than two copies of a system occur naturally in applications.

2. Joinings

We fix an arbitrary group G and define a *C^*-dynamical system*, or *system* for short, as a $\mathbf{B} = (B, \nu, \beta)$ where B is a unital C^*-algebra with a state ν, and $\beta : G \to \mathrm{Aut}(B) : g \mapsto \beta^g$ is an automorphism group, i.e. a representation of G as $*$-automorphisms of B, such that $\nu \circ \beta^g = \nu$ for all $g \in G$. We write β^g rather than β_g, since we will shortly add other indices and this will then be a convenient notation; β^g is simply the value of the function β at g. The identity of G will be denoted by 1, so for example $\beta^1 = \mathrm{id}_B$. We will call $\mathbf{B}$ an *identity system* if $\beta^g = \mathrm{id}_B$ for all g, and we will call it *trivial* if $B = \mathbb{C}1_B$.

Throughout the rest of this section we consider a family $(\mathbf{A}_\iota)_{\iota \in I}$ of systems (but keep in mind they all use the same group G), where we use the notation $\mathbf{A}_\iota = (A_\iota, \mu_\iota, \alpha_\iota)$, and let $A := *_{\iota \in I} A_\iota$ be the unital C^*-algebra free product [25, Definition 1.4.1], and let $\mu := *_{\iota \in I} \mu_\iota$ be the free product state on A [25, Definition 1.5.4]. Using the universal property of A we can define a free product α of the automorphism groups as follows in terms of a commutative diagram:

$$\begin{tikzcd}
A_\iota \arrow[r, "\psi_\iota"] \arrow[d, "\alpha_\iota^g"'] & A \arrow[d, "\alpha^g"] \\
A_\iota \arrow[r, "\psi_\iota"'] & A
\end{tikzcd}$$

Here $\psi_\iota : A_\iota \to A$ is the injective unital $*$-homomorphism that appears in the universal property of A. Then one can verify that $\mu \circ \alpha^g = \mu$ for all $g \in G$, i.e. (A, μ, α) is a system. In terms of this we give:

Definition 2.1. A *joining* of $(\mathbf{A}_\iota)_{\iota \in I}$ is any state ω on A such that $\omega \circ \psi_\iota = \mu_\iota$ for all $\iota \in I$, and such that $\omega \circ \alpha^g = \omega$ for all $g \in G$. Let $J((\mathbf{A}_\iota)_{\iota \in I})$ be the set of all such joinings.

Note that $\mu \in J((\mathbf{A}_\iota)_{\iota \in I})$ and we call μ the *trivial* joining of $(\mathbf{A}_\iota)_{\iota \in I}$.

We will also consider a second type of joining in terms of the reduced free product $(R, \varphi) := *_{\iota \in I}(A_\iota, \mu_\iota)$, the definition of which is discussed for example in [24]. Here one considers the GNS representation $(H_\iota, \pi_\iota, \Omega_\iota)$ of (A_ι, μ_ι). Denoting $H_\iota \ominus (\mathbb{C}\Omega_\iota)$ by H_ι° and setting

$$H := \mathbb{C}\Omega \oplus \bigoplus_{n \geqslant 1} \left(\bigoplus_{\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n} H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ \right)$$

(where we can view Ω as the number 1 in $\mathbb{C}$ if we want to be concrete) one can obtain a representation Λ_ι of A_ι on H (see [25, Definition 1.5.1]), and R is the C^*-algebra in $B(H)$ generated by $\bigcup_{\iota \in I} \Lambda_\iota(A_\iota)$ while $\varphi(a) := \langle \Omega, a\Omega \rangle$ for all $a \in R$. Defining a unitary representation U_ι of α_ι on H_ι by

$$U_\iota^g \pi_\iota(a)\Omega_\iota := \pi_\iota\big(\alpha_\iota^g(a)\big)\Omega_\iota$$

for all $a \in A_\iota$ and all $g \in G$, we obtain a unitary representation U of G on H by setting $U^g\Omega := \Omega$ and $U^g(x_1 \otimes \cdots \otimes x_n) := (U_{\iota_1}^g x_1) \otimes \cdots \otimes (U_{\iota_n}^g x_n)$ for all elementary tensors $x_1 \otimes \cdots \otimes x_n \in H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ$ and all $g \in G$. Setting $\pi := *_{\iota \in I} \Lambda_\iota : A \to B(H)$ it can be shown that

$$U^g \pi(a)\Omega = \pi\big(\alpha^g(a)\big)\Omega$$

for all $a \in A$ and $g \in G$. It can furthermore be shown that

$$\rho^g(a) := U^g a \big(U^g\big)^*$$

gives a well-defined automorphism group $\rho : G \to \mathrm{Aut}(R)$ which satisfies $\varphi \circ \rho^g = \varphi$ for all $g \in G$. In other words we have again obtained a system (R, φ, ρ). In terms of this we give:

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Definition 2.2. A *reduced free product joining* (or *r-joining* for short) of $(\mathbf{A}_\iota)_{\iota\in I}$ is any state ω on R such that $\omega \circ \Lambda_\iota = \mu_\iota$ for all $\iota \in I$, and such that $\omega \circ \rho^g = \omega$ for all $g \in G$. Let $J_r((\mathbf{A}_\iota)_{\iota\in I})$ be the set of all such joinings.

Note that $\varphi \in J_r((\mathbf{A}_\iota)_{\iota\in I})$ and we call φ the *trivial* r-joining of $(\mathbf{A}_\iota)_{\iota\in I}$.

In the case of joinings we now show how to construct an analogue of (a class of) graph joinings that appear in classical ergodic theory. We will use this construction in the subsequent sections.

Construction 2.3. Assume that the systems $(\mathbf{A}_\iota)_{\iota\in I}$ are *factors* of some system $\mathbf{B} = (B, \nu, \beta)$, i.e. there are unital $*$-homomorphisms $h_\iota : A_\iota \to B$ such that $\nu \circ h_\iota = \mu_\iota$ and $\beta^g \circ h_\iota = h_\iota \circ \alpha_\iota^g$. (A simple instance of this is $\mathbf{A}_\iota = \mathbf{B}$ for all ι, in which case we will end up with a so-called self-joining, since it will be a joining of copies of $\mathbf{B}$.) Now we use the universal property of $\mathbf{A}$ to define a unital $*$-homomorphism $\delta : A \to B$ by $\delta \circ \psi_\iota = h_\iota$ for all $\iota \in I$, i.e. by the following commutative diagram:

$$\begin{tikzcd}
A_\iota \arrow[r, "\psi_\iota"] \arrow[d, "h_\iota"'] & A \arrow[dl, "\delta"] \\
B &
\end{tikzcd}$$

Then we set

$$\Delta := \nu \circ \delta$$

which is a joining of $(\mathbf{A}_\iota)_{\iota\in I}$ as we now show.

Clearly Δ is a state on A, and $\Delta \circ \psi_\iota = \nu \circ h_\iota = \mu_\iota$. One can view Δ as an analogue of a diagonal measure. Combining the diagram for δ with the diagram for α, we obtain the commutative diagram

$$\begin{tikzcd}
A_\iota \arrow[r, "\psi_\iota"] \arrow[d, "h_\iota\circ\alpha_\iota^g"'] & A \arrow[dl, "\delta\circ\alpha^g"] \\
B &
\end{tikzcd}$$

However, combining δ's diagram with

$$\begin{tikzcd}
B & A_\iota \arrow[l, "h_\iota\circ\alpha_\iota^g"'] \arrow[d, "h_\iota"] \\
 & B \arrow[ul, "\beta^g"]
\end{tikzcd}$$

we obtain the commutative diagram

$$\begin{tikzcd}
A_\iota \arrow[r, "\psi_\iota"] \arrow[d, "h_\iota\circ\alpha_\iota^g"'] & A \arrow[dl, "\beta^g\circ\delta"] \\
B &
\end{tikzcd}$$

It follows from the two diagrams that we have just obtained, together with the universal property, that $\beta^g \circ \delta = \delta \circ \alpha^g$, and therefore $\Delta \circ \alpha^g = \nu \circ \beta^g \circ \delta = \nu \circ \delta = \Delta$. Hence Δ is indeed a joining of $(\mathbf{A}_\iota)_{\iota\in I}$.

Assuming that G is abelian, we can take this construction further by considering a family $\bar{g} := (g_\iota)_{\iota\in I}$ of elements of G, and defining a unital $*$-homomorphism $\delta_{\bar{g}} : A \to B$ by $\delta_{\bar{g}} \circ \psi_\iota = h_\iota \circ \alpha_\iota^{g_\iota}$ using the universal property of A. We then set

$$\Delta_{\bar{g}} := \nu \circ \delta_{\bar{g}}$$

which is again a joining by similar arguments as for Δ: We obtain $(\delta_{\bar{g}} \circ \alpha^g) \circ \psi_\iota = h_\iota \circ \alpha_\iota^{g_\iota g}$ from α and $\delta_{\bar{g}}$'s diagrams, and $(\beta^g \circ \delta_{\bar{g}}) \circ \psi_\iota = h_\iota \circ \alpha^{gg_\iota}$ from $\delta_{\bar{g}}$'s diagram combined with $\beta^g \circ (h_\iota \circ \alpha_\iota^{g_\iota}) = h_\iota \circ \alpha_\iota^{gg_\iota}$. Since G is abelian, it follows that $\beta^g \circ \delta_{\bar{g}} = \delta_{\bar{g}} \circ \alpha^g$.

As we will see in the next section (in Theorem 3.3's proof), the joinings in Construction 2.3 are generally not trivial. Simple nontrivial r-joinings will also be discussed in the next section.

Another construction that will be used in the next section is the following:

Construction 2.4. Let ω be any state on A such that $\omega \circ \psi_\iota = \mu_\iota$ and let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of (A, ω). Set $\gamma_\omega := \pi_\omega(\cdot)\Omega_\omega$ and $\gamma_\iota := \gamma_\omega \circ \psi_\iota$, and let H_ι be the closure of $\gamma_\iota(A_\iota)$ in H_ω, so we in effect obtain $\gamma_\iota : A_\iota \to H_\iota$.

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It is easily seen that we can take the GNS representations of (A_ι, μ_ι)'s discussed earlier to be given by $\Omega_\iota := \Omega_\omega$ and $\pi_\iota(a)\gamma_\iota(b) = \gamma_\iota(ab)$ on the H_ι that we have just obtained.

Denote the projection of H_ω onto H_ι by P_ι and set $P_\iota^\kappa := P_\iota|_{H_\kappa}$ for all $\iota, \kappa \in I$. It is easy to verify that P_ι^κ is the unique function $H_\kappa \to H_\iota$ satisfying $\langle P_\iota^\kappa x, y\rangle = \langle x, y\rangle$ for all $x \in H_\kappa$ and $y \in H_\iota$, and we can call it a *conditional expectation operator*.

Assume furthermore that ω is a joining of $(\mathbf{A}_\iota)_{\iota\in I}$, define a unitary representation U_ω of α on H_ω by

$$U_\omega^g \pi_\omega(a)\Omega_\omega := \pi_\omega\big(\alpha_g(a)\big)\Omega_\omega$$

and let U_ι be defined as before. (We can note that if $\omega = \mu$, then $(H_\omega, \pi_\omega, \Omega_\omega)$ will be (unitarily equivalent to) (H, π, Ω) which we defined earlier.) It is straightforward to show that $U_\omega^g|_{H_\iota} = U_\iota^g$ for all g and ι.

Combining all this we find that

$$P_\iota^\kappa U_\kappa^g = U_\iota^g P_\iota^\kappa$$

for all g, ι and κ, since $\langle (U_\iota^g)^* P_\iota^\kappa U_\kappa^g x, y\rangle = \langle U_\kappa^g x, U_\iota^g y\rangle = \langle U^g x, U^g y\rangle = \langle x, y\rangle = \langle P_\iota^\kappa x, y\rangle$ for all $x \in H_\kappa$ and $y \in H_\iota$.

3. Disjointness and ergodicity

In classical ergodic theory disjointness of two systems refers to them having only one joining, for example any ergodic system is disjoint from all identity systems and this in fact characterizes ergodicity. The same situation holds in the non-commutative case in terms of tensor product joinings. In this section we study analogous results in the free product case. We begin with the relevant definitions in terms of the notation in Section 2, with $I = \{1, 2\}$. We will study these two definitions in turn for two different classes of dynamical systems.

Definition 3.1. Two systems $\mathbf{A}_1$ and $\mathbf{A}_2$ are called *tensorially disjoint* if for every $\omega \in J(\mathbf{A}_1, \mathbf{A}_2)$ one has $\omega(a_1 a_2) = \mu_1(a_1)\mu_2(a_2)$, or equivalently $\omega(a_2 a_1) = \mu_2(a_2)\mu_1(a_1)$, for all $a_1 \in A_1$ and $a_2 \in A_2$.

Definition 3.2. Two systems $\mathbf{A}_1$ and $\mathbf{A}_2$ are called *r-disjoint* if $J_r(\mathbf{A}_1, \mathbf{A}_2) = \{\varphi\}$.

We call the system $\mathbf{A}_\iota$ *ergodic* if

$$\big\{x \in H_\iota\colon\ U_\iota^g x = x \text{ for all } g \in G\big\} = \mathbb{C}\Omega_\iota$$

In Theorem 3.3 below we have to assume additional structure, namely that the systems involved are actually W^*-dynamical systems. The system $\mathbf{A}_\iota$ is called a *W^*-dynamical system* if A_ι is a σ-finite von Neumann algebra and μ_ι is a faithful normal state. For such a system ergodicity is equivalent to the *fixed point algebra*

$$A_\iota^{\alpha_\iota} := \big\{a \in A_\iota\colon\ \alpha_\iota^g(a) = a \text{ for all } g \in G\big\}$$

being trivial, i.e. $A_\iota^{\alpha_\iota} = \mathbb{C}1_{A_\iota}$, according to [8, Theorem 4.3.20].

Theorem 3.3. *A W^*-dynamical system is ergodic if and only if it is tensorially disjoint from all identity W^*-dynamical systems.*

Proof. Let $\mathbf{A}_1$ be an ergodic W^*-dynamical system, and $\mathbf{A}_2$ an identity W^*-dynamical system, i.e. $\alpha_2^g = \mathrm{id}_{A_2}$ for all $g \in G$. Consider any $\omega \in J(\mathbf{A}_1, \mathbf{A}_2)$ and apply Construction 2.4 to see that for any $a_2 \in A_2$,

$$U_1^g P_1^2 \gamma_2(a_2) = P_1^2 U_2^g \gamma_2(a_2) = P_1^2 \gamma_2(a_2)$$

since $\mathbf{A}_2$ is an identity system. However, since $\mathbf{A}_1$ is ergodic, the fixed point space of the unitary group U_1 is $\mathbb{C}\Omega_\omega$, and therefore $P_1^2\gamma_2(a_2) = \mu_2(a_2)\Omega_\omega$. Hence

$$\omega(a_1 a_2) = \big\langle \gamma_1\big(a_1^*\big), \gamma_2(a_2)\big\rangle = \big\langle \gamma_1\big(a_1^*\big), P_1^2\gamma_2(a_2)\big\rangle = \mu_1(a_1)\mu_2(a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$ as required.

Conversely, suppose the W^*-dynamical system $\mathbf{A}_1$ is not ergodic, and set $A_2 := A_1^{\alpha_1}$ and $\mathbf{A}_2 := (A_2, \mu_1|_{A_2}, \mathrm{id}_{A_2})$. Then $\mathbf{A}_2$ is a nontrivial identity W^*-dynamical system and also a factor of $\mathbf{A}_1$ via the embedding $h_2 : A_2 \to A_1$. Apply Construction 2.3 to obtain the joining $\Delta \in J(\mathbf{A}_1, \mathbf{A}_2)$, where we have set $\mathbf{B} := \mathbf{A}_1$ and $h_1 := \mathrm{id}_{A_1}$. Now note that we do not have $\Delta(a_1 a_2) = \mu_1(a_1)\mu_2(a_2)$ for all $a_1 \in A_1$ and $a_2 \in A_2$, for if we did it would follow that

$$\big\langle \pi_1(a_1)\Omega_1, \pi_1\big(h_2(a_2)\big)\Omega_1\big\rangle = \Delta\big(a_1^* a_2\big) = \mu_1\big(a_1^*\big)\mu_2(a_2) = \big\langle \pi_1(a_1)\Omega_1, \mu_2(a_2)\Omega_1\big\rangle$$

hence $\pi_1(h_2(a_2))\Omega_1 = \mu_2(a_2)\Omega_1$, but Ω_1 is separating for $\pi_1(A_1)$ (since μ_1 is faithful) therefore $\pi_1(h_2(a_2)) = \mu_2(a_2)1_{B(H_1)}$. Since π_1 and h_2 are injective, we conclude that $a_2 = \mu_2(a_2)1_{A_2}$ which contradicts the fact that $\mathbf{A}_2$ is not trivial. This proves that $\mathbf{A}_1$ and $\mathbf{A}_2$ are not tensorially disjoint. □

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Note that the second part proof of Theorem 3.3 provides an instance of a joining Δ which is not trivial (when $\mathbf{A}_1$ is not ergodic).

The proof of Theorem 3.3 is very similar to the tensor product case in [12,13], but somewhat simpler, since in the tensor product analogue of Construction 2.3, namely [12, Construction 3.4], we had to resort to Tomita–Takesaki theory. The result itself is of course not quite the same as the tensor product case, since we have not been able to prove that an ergodic W^*-dynamical system only has the trivial joining with any identity W^*-dynamical system.

In the remainder of this section we look at this last problem from the perspective of r-joinings, but only for a very special class of systems. We find that for this class of systems ergodicity implies r-disjointness from identity systems (i.e. only the trivial r-joining occurs), which is a better analogy with the tensor product case (including the classical case).

In the rest of this section (and the next) we only consider $G = \mathbb{Z}$. We now consider systems $\mathbf{A}_1$ and $\mathbf{A}_2$ of the following sort: Let $\varGamma_\iota$ be the free group on the alphabet of symbols S_ι and let T_ι be an automorphism of $\varGamma_\iota$ obtained from a bijection of S_ι. We set $H_\iota := l^2(\varGamma_\iota)$. Let A_ι be the reduced group C^*-algebra $C_r^*(\varGamma_\iota)$ and define $\mu_\iota(a) := \langle \varOmega_\iota, a\varOmega_\iota \rangle$ where $\varOmega_\iota \in H_\iota$ is defined by $\varOmega_\iota(1) = 1$ and $\varOmega_\iota(g) = 0$ for $g \neq 1$. Using the unitary operator $U_\iota : H_\iota \to H_\iota : f \mapsto f \circ T_\iota^{-1}$ we obtain a well-defined $*$-automorphism $\alpha_\iota : A_\iota \to A_\iota : a \mapsto U_\iota a U_\iota^*$ which of course gives an automorphism group $\mathbb{Z} \ni n \mapsto \alpha_\iota^n$ which we also simply denote as α_ι. This gives a system $\mathbf{A}_\iota = (A_\iota, \mu_\iota, \alpha_\iota)$ which we will call a *group system*. Note that for the generators $\lambda_\iota(g)$ of A_ι given by the left regular representation λ_ι of $\varGamma_\iota$ (defined as $[\lambda_\iota(g)f](h) := f(g^{-1}h)$ for $f \in H_\iota$ and $g, h \in \varGamma_\iota$) one has the simple relation $\alpha_\iota(\lambda_\iota(g)) = \lambda_\iota(T_\iota g)$. We will consider these systems in the next section as well. The group system $\mathbf{A}_\iota$ is ergodic (as defined earlier) if and only if the orbits $(T_\iota^n g)_{n\in\mathbb{Z}}$ are infinite for all $g \in \varGamma_\iota \backslash \{1\}$.

Theorem 3.4. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be group systems as above, with $\mathbf{A}_1$ ergodic and S_1 countably infinite, while $\mathbf{A}_2$ is an identity system and S_2 is finite or countably infinite. Then $\mathbf{A}_1$ and $\mathbf{A}_2$ are r-disjoint.*

Proof. The key point of this proof is a recent result by Abadie and Dykema [1, Proposition 3.5] regarding unique ergodicity relative to fixed point algebras. In the notation of Section 2, namely $(R, \varphi) := (A_1, \mu_1) * (A_2, \mu_2)$, we have $R = C_r^*(\varGamma_1 * \varGamma_2)$ (see for example [24, Example 1.9]) and we can view $\varGamma_2$ as the subgroup $\{g \in \varGamma_1 * \varGamma_2\colon (T_1 * T_2)g = g\}$ of $\varGamma_1 * \varGamma_2$, and $A_2 = C_r^*(\varGamma_2)$ as the fixed point algebra of ρ. Also keep in mind that $\varGamma_1 * \varGamma_2$ is a free group on a countably infinite alphabet. Since φ is invariant under ρ, and can be viewed as an extension of μ_2 to R, it follows from [1, Proposition 3.5] that φ is the unique ρ invariant state on R which restricts to μ_2. In particular φ is the only r-joining of $\mathbf{A}_1$ and $\mathbf{A}_2$. □

Note that in this proof we did not use the property $\omega \circ \varLambda_1 = \mu_1$ of a joining ω of $\mathbf{A}_1$ and $\mathbf{A}_2$, but only $\omega \circ \varLambda_2 = \mu_2$ and $\omega \circ \rho = \omega$. So it would seem that unique ergodicity relative to the fixed point algebra is in this situation a stronger property than r-disjointness from identity group systems. We can also mention that Abadie and Dykema's result actually applies more generally than the way that we have used it in Theorem 3.4, but the setting of Theorem 3.4 is a very concrete situation which illustrates r-disjointness very clearly.

It is not clear if the converse of Theorem 3.4 holds, since the proof technique in Theorem 3.3 relies on Construction 2.3, which doesn't apply in the case of r-joinings. Theorem 3.4 would therefore not be very interesting if there were not at least complementary cases of non-ergodic group systems $\mathbf{A}_1$ which are not r-disjoint from identity group systems. So let us provide as a simple example the other extreme:

Example 3.5. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be identity group systems (and in fact, in this example the relevant groups $\varGamma_1$ and $\varGamma_2$ could even be arbitrary, they need not be free, as long as they are not trivial, i.e. $\varGamma_1 \neq \{1\}$ and $\varGamma_2 \neq \{1\}$). Remember that as in Section 2 the trivial r-joining is ϕ given by $\phi(a) = \langle \varOmega, a\varOmega \rangle$. Our goal is to exhibit a very simple nontrivial r-joining of $\mathbf{A}_1$ and $\mathbf{A}_2$. Since we are working with identity systems, any state ω on R is automatically ρ invariant. We only need to check whether ω restricts correctly to A_1 and A_2. We consider the following variation on ϕ: For $h \in \varGamma_1 \backslash \{1\}$ and $k \in \varGamma_2 \backslash \{1\}$, we consider $z := \delta_h \otimes \delta_k \in H_1^\circ \otimes H_2^\circ \subset H$ in terms of the notation in Section 2, but in this example δ_h is the function such that $\delta_h(h) = 1$ while $\delta_h(g) = 0$ for $g \in \varGamma_1 \backslash \{h\}$, and similarly for δ_k. Then set

$$\eta := \frac{\varOmega + z}{\|\varOmega + z\|}$$

and define the state ω by

$$\omega(a) := \langle \eta, a\eta \rangle$$

for all $a \in R$. It can be checked, using the definition of $\varLambda_\iota$ in Section 2, that ω is indeed an r-joining by first considering it on the generators of A_1 and A_2 given by the left regular representations λ_1 and λ_2 of $\varGamma_1$ and $\varGamma_2$. It can similarly be verified that $\phi(\varLambda_1(\lambda_1(h))\varLambda_2(\lambda_2(k))) = 0$ while $\omega(\varLambda_1(\lambda_1(h))\varLambda_2(\lambda_2(k))) = 1/2$. So $\omega \neq \phi$ is indeed nontrivial, and therefore $\mathbf{A}_1$ is not r-disjoint from $\mathbf{A}_2$.

There has recently been some activity around topics related to [1] and to various ergodicity and mixing conditions more generally, for example [2,14,15,18]. It might also be interesting to explore where exactly various forms of free product disjointness from identity systems (or from other classes of systems) fit into the hierarchy of ergodicity and mixing conditions. We will not study this issue further in this paper, and instead now turn to another aspect of joinings.

4. Mixing and multi-time correlations functions

A variety of mixing (or clustering) conditions for quantum systems have appeared in the physics literature (see for example [19]), and this is related to so-called multi-time correlations which have been studied in [7,4,3] using free products of operator algebras. In this section we first study higher order mixing of strongly mixing group systems (as defined in the previous section) as a motivating example for multi-time correlation functions, and then we show more generally how multi-time correlation functions fit into a joining framework, although we work in a slightly simplified setting compared to above mentioned physics literature in order to make the connection with joinings very clear. Throughout this section all systems have $G = \mathbb{Z}$. We will often use the notation $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

Recall that a system $\mathbf{B} = (B, \nu, \beta)$ is called *strongly mixing* if

$$\lim_{n\to\infty} \nu\bigl(a\beta^n(b)\bigr) = \nu(a)\nu(b)$$

for all $a, b \in B$. It turns out that a group system is strongly mixing if and only if it is ergodic (see for example [13, Theorem 3.4]). We now show that a strongly mixing group system is k-mixing for all $k \in \mathbb{N}$, i.e. "mixing of all orders", and we formulate it in terms of joinings (essentially the same result is quoted in [7, Section 3.1] in a slightly different context and not in the language of joinings). For $\bar{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ we use the notation $\bar{n} \to \infty$ to mean $n_1 \to \infty, n_2 - n_1 \to \infty, \dots, n_k - n_{k-1} \to \infty$.

Theorem 4.1. *Let* $\mathbf{B}$ *be a strongly mixing group system and consider any* $k \in \mathbb{N}$. *For* $\bar{n} \in \mathbb{Z}^k$, *let* $\Delta_{\bar{n}}$ *be the joining given by Construction* 2.3 *in terms of* $I = [k]$ *and* $\mathbf{A}_\iota = \mathbf{B}$. *Then*

$$\lim_{\bar{n}\to\infty} \Delta_{\bar{n}}(a) = (*_{\iota\in I}\nu)(a)$$

for all $a \in *_{\iota\in I} B$.

Proof. Let the dynamics β be given by the automorphism T of the free group Γ obtained from a bijection of the alphabet S of symbols of the group, as explained in Section 3. It is convenient to work explicitly in terms of the index ι, e.g. the left regular representation $\lambda_\iota = \lambda$ of Γ will be indexed by ι when we view $\lambda(g)$ as an element of A_ι. Consider any $g_1, \dots, g_m \in \Gamma\backslash\{1\}$ and $\iota_1, \dots, \iota_m \in I$ with $\iota_j \neq \iota_{j+1}$. Since $\mathbf{B}$ is strongly mixing, all the orbits of T must be infinite, except on the identity of Γ. Hence for $\bar{n}$ "large enough" (in the sense of $\bar{n} \to \infty$) the group elements $T^{n_{\iota_p}} g_p$ and $T^{n_{\iota_q}} g_q$ will have no symbols in common for any $\iota_p \neq \iota_q$ and therefore

$$\Delta_{\bar{n}}\bigl(\lambda_{\iota_1}(g_1)\dots\lambda_{\iota_m}(g_m)\bigr) = \nu\bigl(\lambda\bigl(T^{n_{\iota_1}} g_1 \dots T^{n_{\iota_m}} g_m\bigr)\bigr) = 0$$

but $(*_{\iota\in I}\mu_\iota)(\lambda_{\iota_1}(g_1)\dots\lambda_{\iota_m}(g_m)) = 0$, since $\mu_{\iota_j}(\lambda_{\iota_j}(g_j)) = 0$. We conclude that for any a in a dense subset of $*_{\iota\in I} A_\iota$ we have $\Delta_{\bar{n}}(a) = (*_{\iota\in I}\mu_\iota)(a)$ for $\bar{n}$ large enough, and the result follows. □

This theorem implies for example that

$$\lim_{\bar{n}\to\infty} \nu\bigl(\beta^{n_1}(a_1)\dots\beta^{n_k}(a_k)\bigr) = \nu(a_1)\dots\nu(a_k)$$

for all $a_1, \dots, a_k \in B$, which is why we view it as expressing k-mixing.

More generally consider an arbitrary system $\mathbf{B} = (B, \nu, \beta)$ and a fixed $k \in \mathbb{N}$. For any $a_1, \dots, a_m \in B$ and $\iota_1, \dots, \iota_m \in [k]$ with $\iota_j \neq \iota_{j+1}$ for all j we call

$$\mathbb{Z}^k \ni \bar{n} \mapsto \nu\bigl(\beta^{n_{\iota_1}}(a_1)\dots\beta^{n_{\iota_m}}(a_m)\bigr)$$

a *multi-time correlation function* of $\mathbf{B}$. All of these multi-time correlation functions are subsumed in the single function $\mathbb{Z}^k \ni \bar{n} \mapsto \Delta_{\bar{n}}$ where $\Delta_{\bar{n}}$ is again the joining obtained in Construction 2.3 in terms of $I = [k]$ and $\mathbf{A}_\iota = \mathbf{B}$. So our first conclusion is that multi-time correlation functions are in fact given by joinings. Furthermore, in terms of this notation we have the following simple theorem regarding an average of the $\Delta_{\bar{n}}$'s:

Theorem 4.2. *Let* $(\Phi_N)_{N\in\mathbb{N}}$ *be a Følner sequence in the group* $\mathbb{Z}^k$. *Then*

$$\bar{\Delta}_N := \frac{1}{|\Phi_N|}\sum_{\bar{n}\in\Phi_N} \Delta_{\bar{n}}$$

is a joining of $(\mathbf{A}_\iota)_{\iota\in I}$ *for every* $N \in \mathbb{N}$. *If* $\omega(a) := \lim_{N\to\infty} \bar{\Delta}_N(a)$ *exists for every* $a \in *_{\iota\in I} B$ *then* ω *is a joining of* $(\mathbf{A}'_\iota)_{\iota\in I}$ *where* $\mathbf{A}'_\iota := (B, \nu, \beta^{m_\iota})$ *for any* $m_\iota \in \mathbb{Z}$.

Proof. The first part is clear. So assume $\omega(a)$ exists. Then it is clear that ω is a state on $*_{\iota\in I}B$, and since $\bar{\Delta}_N$ is a joining, we see that $\omega\circ\psi_\iota=\nu$. Let τ denote dynamics on $*_{\iota\in I}A$ obtained from the β^{m_ι}'s. Setting $\bar{m}:=(m_1,\dots,m_k)$ and using the universal property of the free product one easily finds that

$$\bar{\Delta}_N\circ\tau=\frac{1}{|\Phi_n|}\sum_{\bar{n}\in\Phi_N+\bar{m}}\Delta_{\bar{n}}$$

and since $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence, which means that

$$\left|\Phi_N\mathbin{\triangle}(\Phi_N+\bar{m})\right|/|\Phi_N|\to 0$$

as $N\to\infty$, it follows that $\omega\circ\tau=\omega$ so ω is indeed a joining of $(\mathbf{A}'_\iota)_{\iota\in I}$. □

The joining ω in Theorem 4.2 is a simplified version of the "asymptotic state" considered in [4], and the fact that it is a joining as described, corresponds to part of [4, Proposition 3.1]. In that paper they however use a countably infinite free product instead of $*_{\iota\in[k]}A_\iota$ as we did, to allow for variable k, and they use a more abstract averaging procedure. The essential point remains the same though.

Note that Theorem 4.1 provides an illustration of Theorem 4.2: It is easy to see that $([N]+sN)_{N\in\mathbb{N}}$ is a Følner sequence in $\mathbb{Z}$ for any $s\in\mathbb{Z}$, and since the cartesian product of the terms of Følner sequences leads to a Følner sequence in the cartesian product of the involved groups, we see that

$$\Phi_N:=\bigl([N]+2N\bigr)\times\bigl([N]+4N\bigr)\times\cdots\times\bigl([N]+2kN\bigr)$$

provides us with a Følner sequence in $\mathbb{Z}^k$ for which $\omega(a):=\lim_{N\to\infty}\bar{\Delta}_n(a)=(*_{\iota\in I}\nu)(a)$ is simple to verify for the situation in Theorem 4.1. In this case of course $\omega=*_{\iota\in I}\nu$ is trivially a joining of $(\mathbf{A}'_\iota)_{\iota\in I}$.

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