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FREDHOLM CLASSES IN OPERATOR ALGEBRAS

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Fredholm classes in operator algebras

By

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CONTENTS

NOTATION

NOTATION

We give a (not necessarily complete) list of symbols used in this thesis. A page reference is given where it might be useful.

Symbol	Meaning	Page
$\mathcal{A}$	a von Neumann algebra	8
$\mathcal{A}^p$	complete lattice of projections in $\mathcal{A}$	8
$\mathcal{A}_\tau$	trace class of τ in $\mathcal{A}$	35
$\mathcal{A}'$	commutant of $\mathcal{A}$	
$\tilde{\mathcal{A}}$	*-algebra of τ-measurable operators affiliated to $\mathcal{A}$	65
$\tilde{\mathcal{A}}(\varepsilon, t)$	basic neighbourhood of 0 in the τ_{cm}-topology on $\tilde{\mathcal{A}}$	65
$\mathcal{B}(\mathcal{H})$	von Neumann algebra of all bounded linear operators on $\mathcal{H}$	
$\mathcal{B}_\mathcal{H}$	unit ball of $\mathcal{H}$	41
$\mathbf{C}$	set of complex numbers	
$\mathcal{C}$, $\mathcal{D}$	C^*-algebras	
$\mathcal{C}^+$	positive elements in $\mathcal{C}$	
$dim(.)$	dimension function	10
$\mathcal{D}(S)$	domain of the operator S	
$dist(T, B)$	distance from the operator T to the set B	
E, F	projections	
$E_{(.)}$, $F_{(.)}$	spectral measures	

E_λ	spectral projection $E_{[0,\lambda]}$	
$\mathcal{G}(\mathcal{A})$	group of invertible operators in $\mathcal{A}$	
$\gamma(T)$	minimum modulus of T	60
$\mathcal{H}$	a Hilbert space	8
$\mathcal{I}$	closed two-sided ideal	
$ind(T)$	index of bounded operator T	11
$index(S)$	index of not necessarily bounded operator S	71
$\mathcal{K}(\mathcal{A})$	closed ideal generated by the finite projections in $\mathcal{A}$	17
$\mathcal{K}(\mathcal{A},\tau)$	closed ideal generated by the finite rank projections in $\mathcal{A}$	18
$\mathcal{K}(\mathcal{H})$	ideal of compact elements of $\mathcal{B}(\mathcal{H})$	
$\mathcal{K}(\tilde{\mathcal{A}},\tau)$	τ_{cm}-compact ideal in $\tilde{\mathcal{A}}$	66
$m_{\mathcal{I}}(T)$	lower bound of T relative to $\mathcal{I}$	13
$m(T)$	$\inf\sigma(\|T\|)$	53
$\mu_t(S)$	't th' generalised singular number of $S \in \tilde{\mathcal{A}}$	68
$\mu_\infty(S)$	$\lim_{t\to\infty}\mu_t(S)$	68
$\mathbf{N}$	set of natural numbers	
$\mathcal{N}_l(\mathcal{A},\mathcal{I})$	the T in $\mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is a left algebraic divisor of zero in $\mathcal{A}/\mathcal{I}$	23
$\mathcal{N}_r(\mathcal{A},\mathcal{I})$	the T in $\mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is a right algebraic divisor of zero in $\mathcal{A}/\mathcal{I}$	23
$\mathcal{N}(\mathcal{A},\mathcal{I})$	the T in $\mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is an algebraic divisor of zero in $\mathcal{A}/\mathcal{I}$	25

$\mathcal{N}_u(\mathcal{A}, \mathcal{I})$	union of $\mathcal{N}_l(\mathcal{A}, \mathcal{I})$ and $\mathcal{N}_r(\mathcal{A}, \mathcal{I})$	25
N_T	null projection of T	
P, Q	projections	
$P \leq Q$	order relation in $\mathcal{A}^p$	8
$P \sim Q$	equivalent projections	8
$\mathcal{P}(B)$	perturbation class associated with B	34
$\pi_{\mathcal{I}}(.)$	quotient map	9
$\Phi_l(\mathcal{A}, \mathcal{I})$	left Fredholm operators in $\mathcal{A}$ relative to $\mathcal{I}$	9
$\Phi_r(\mathcal{A}, \mathcal{I})$	right Fredholm operators in $\mathcal{A}$ relative to $\mathcal{I}$	9
$\Phi(\mathcal{A}, \mathcal{I})$	Fredholm operators in $\mathcal{A}$ relative to $\mathcal{I}$	9
$\Phi_s(\mathcal{A}, \mathcal{I})$	semi-Fredholm operators in $\mathcal{A}$ relative to $\mathcal{I}$	9
$\Phi_u(\mathcal{A}, \mathcal{I})$	union of $\Phi_l(\mathcal{A}, \mathcal{I})$ and $\Phi_r(\mathcal{A}, \mathcal{I})$	9
$P\sigma(a)$	set of all eigenvalues of $a \in \mathcal{A}/\mathcal{K}(\mathcal{A}, \tau)$	40
$Q\mathcal{A}$	the von Neumann algebra $\mathcal{A}$ reduced by the projection Q	20
$q(\Omega)$	quantity measuring relative precompactness of Ω	41
$\mathbf{R}$	set of real numbers	
$\mathbf{R}^*$	extended real number system	27
R_T	range projection of T	
$Reg(\mathcal{C})$	regular elements in $\mathcal{C}$	60
$\sigma_e(T)$	essential spectrum of T	36
$supp(T)$	support projection of T	

T, S, U, V	operators	
$\tau(.)$	trace function on $\mathcal{A}^+$	
T^*	adjoint operator of T	
$\mathbf{T}$	unit circle in $\mathbf{C}$	47
τ_{cm}	topology of convergence in measure	65
$\mathcal{Z}$	center of $\mathcal{A}$	
$\mathcal{Z}_l(\mathcal{A}, \mathcal{I})$	the T in $\mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is a left topological divisor of zero in $\mathcal{A}/\mathcal{I}$	23
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INTRODUCTION

The theory of Fredholm operators in von Neumann algebras was developed by Breuer [Bre 68, Bre 69]. The main reason for introducing his theory became clear in [Bre 73], where Fredholm theory was used in the study of vector bundles relative to a von Neumann algebra. Another important application of Breuer's index theory is the generalisation of the well-known Gohberg–Krein index theorem for Toeplitz operators with continuous symbol on the unit circle. In this case Toeplitz operators were studied with symbols continuous functions on more general groups, and hence the index theory of Breuer was necessary [CDSS71, Mur 95]. Since then the theory of Fredholm-type operators in operator algebras has been studied by several authors, for instance Kaftal [Kaf 77, Kaf 78], Olsen [Ols 84], Peligrad and Zsido [PeZ 73, Zsi 75], Phillips and Raeburn [PhR 94] and Ströh and Swart [Str 89, StS 89, StS 91, StZ 95]. In [Bre 68, Bre 69] most of the fundamental results from the classical index theory were generalised to a von Neumann algebra relative to the ideal of compact elements. An index map is defined with all the desired properties. In [Kaf 77, Kaf 78] the theory of left and right Fredholm elements relative to the compact ideal was introduced and results like the Weyl–von Neumann theorem and the Berg–Sikonia–Halmos theorem were obtained.

In [Ols 84] Olsen defined a natural analytic index function on an arbitrary von Neumann algebra $\mathcal{A}$ relative to an arbitrary closed ideal $\mathcal{I}$ using a relative dimension function defined on the projections of $\mathcal{A}$ due to Tomiyama [Tom 58]. This index map enabled her to develop a complete Fredholm theory in this setting which is parallel to classical Fredholm theory. If the ideal $\mathcal{I}$ is contained in the ideal of compact elements, the index map enjoys all the desired properties, e.g. it is stable under perturbations of elements of $\mathcal{I}$, locally constant on components of the Fredholm elements, etc. If $\mathcal{I}$ is not contained in the ideal of compact elements, then the properties fail to hold and Olsen modified this index map in such a way that these properties were recovered.

In the first part of the thesis we use the divisors of zero in the "Calkin" algebra to define classes of Fredholm-type operators. The geometrical, algebraic and topological properties of these classes are studied completely and shown to be similar to properties possessed by the Fredholm classes. We also investigate precisely in which cases these classes coincide with the Fredholm classes, hence to obtain useful characterisations of the semi-Fredholm operators in the von Neumann algebra setting. We refer the reader to the well-known paper [FSW 72] where Fillmore, Stampfli and Williams obtained characterisations for the classical semi-Fredholm operators on a Hilbert space in terms of the set of divisors of zero in the Calkin algebra. For operators on Banach spaces such characterisations also exist [Pfa 70].

Closely related to Fredholm theory is the lifting of properties of elements from the "Calkin" algebra to the algebra. In the second part of the thesis we prove lifting theorems for a number of such properties and state numerous problems still unsolved.

We conclude our study on a Fredholm theory for closed densely defined operators affiliated to a von Neumann algebra.

Although unanswered questions (which we describe later) remain, our results are reasonably complete, especially with respect to certain norm closed ideals which are of principal interest in the theory of operator algebras. The following is a more detailed description of the thesis:

Chapter 1 contains a summary of the notation used throughout the thesis as well as some preliminary results which are needed in the other chapters.

It was shown by Pfaffenberger [Pfa 70] that under certain conditions on a Banach space the left and right Fredholm operators can be characterised respectively in terms of the left and right algebraic divisors of zero in the Calkin algebra. Again under additional conditions on the Banach space Lebow and Schechter [LeS 71] showed that the left and right topological divisors of zero coincide with the left and right algebraic divisors of zero, respectively. In Chapter 2 we characterise the left and right Fredholm operators relative to any closed ideal in any von Neumann algebra in terms of the left and right topological divisors of zero in the appropriate quotient algebra.

Chapter 2 also describes a perturbation theory for certain classes of operators related to the left and right algebraic divisors of zero in the quotient algebra. In this chapter we show that these classes are semigroups that are stable under perturbations of elements in the ideal under consideration. The purpose of these results are to show that these classes have properties very similar to those of the Fredholm classes and therefore equality of the respective classes are expected in some cases.

In the classical theory of operators on a Hilbert space numerous characterisations of the left and right Fredholm operators exist. Some of these characterisations are used in the study of the structure of the essential spectrum. An important purpose of Chapter 2 is to show that several of these results still hold when a Fredholm theory is studied relative to any von Neumann algebra.

In a well-known paper [FSW 72] Fillmore, Stampfli and Williams made use of an elementary but very useful sequential characterisation on semi-Fredholm operators, due to Wolf [Wol 59], to study the structure of the essential spectrum of a Hilbert space operator. Moreover, this characterisation is used to show that the left Fredholm operators on a Hilbert space can be characterised in terms of the set of divisors of zero in the Calkin algebra. Ströh and Swart [StS 89] proved that if a Fredholm theory is considered in a semifinite von Neumann algebra with non-large center relative to the ideal of compact operators, i.e. the norm closure of the ideal generated by the finite projections (see [Bre 68, Bre 69] for a Fredholm theory relative to this ideal), then the left and right Fredholm operators can be characterised in terms of the left and right algebraic divisors of zero in the Calkin algebra and the left (right, resp.) algebraic divisors of zero coincide with the left (right, resp.) topological divisors of zero in the Calkin algebra.

In this chapter we show that similar results hold when a Fredholm theory is studied in any semifinite von Neumann algebra relative to a specific closed ideal. The ideal under consideration will be the norm closed ideal generated by the projections with finite trace. It is shown in [Fac 83, Son 71, StW 93] that this ideal has properties similar to the ideal of compact operators.

To motivate the study of this ideal, we refer the reader to [PhR 94]. In this paper Phillips and Raeburn proved an index theorem for Toeplitz operators with noncommutative symbol space. They showed that in their study the existance of traces is essential, hence the study of Fredholm theory relative to the closed ideal generated by the projections with finite trace. Of course in the case of factors this ideal coincides with Breuer's ideal.

We continue by proving a characterisation of the semi-Fredholm operators, from which a description of the essential spectrum in terms of the eigenvalues in the Calkin algebra follows. This characterisation is then used to obtain some more equivalent descriptions of the semi-Fredholm operators. The first one in terms of the left and right algebraic divisors of zero in the Calkin algebra and the second one generalising a geometrical type of characterisation of Yood [Yoo 51].

Some of the results in Chapter 2 appear in [StV 94] and the other results are submitted for publication [StV 2].

The lifting results are contained in Chapter 3. It is easy to see that if $a \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is self-adjoint then there exists a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(T) = a$. In this case we say that T is a self-adjoint *lifting* for a. To ask whether a normal element in the Calkin algebra can be lifted is a much harder question. A well-known theorem of Brown, Douglas and Fillmore (see [BDF 73]) gives a complete answer to this question. In Chapter 3 we prove a lifting theorem for a certain type of normal element, viz. unitary elements, in the quotient algebra. These lifting theorems are important, but in general hard to prove. There is for instance the well-known West decomposition theorem, which states that any quasinilpotent element in the Calkin algebra has a quasinilpotent lifting. See [BMSW82] for a proof of this decomposition in a C^*-algebra relative to any inessential ideal. It is still an open question whether this theorem holds, generally, in any von Neumann algebra. Partial answers to this question can be found in [Rog 90, StS 91].

In [BDF 73, Theorem 6.2] a lifting theorem is proved on the isometries in the Calkin algebra. In Chapter 3 we show a similar result in the case when $\mathcal{A}$ is a factor. We also show when the lifting of invertibility is possible and we even show that the lifting of divisors of zero and regularity are possible in any von Neumann algebra relative to any closed ideal. The results of Chapter 3 are submitted for publication [StV 3].

Many of the properties and characterisations of left and right Fredholm operators relative to a closed ideal in $\mathcal{A}$ carry over to the case of unbounded operators, more specific, the left and right Fredholm operators affiliated to a semifinite von Neumann algebra. In Chapter 4 we show that if the algebra possesses a faithful semifinite normal trace and we consider the τ-measurable Fredholm operators relative to the closure of the trace class in the topology of convergence in measure, then some of the results in Chapter 2 extend to this setting.

Chapter 1

PRELIMINARIES

Throughout the thesis $\mathcal{A}$ will denote a concrete von Neumann algebra of operators on an infinite dimensional Hilbert space $\mathcal{H}$. We denote by $\mathcal{A}^p$ the complete lattice of projections in $\mathcal{A}$ with the usual order relation $P \leq Q$ if and only if $PQ = P$ where P, $Q \in \mathcal{A}^p$. In this case P is called a *subprojection* of Q. The well known Murray - von Neumann equivalence on the projection lattice $\mathcal{A}^p$ will be essential throughout the thesis:

Call $P, Q \in \mathcal{A}^p$ *equivalent* ($P \sim Q$) if there exists a $U \in \mathcal{A}$ such that $P = U^*U$ and $Q = UU^*$. We also refer to equivalent projections as projections with the same dimension. A projection P is of dimension less than Q if P is equivalent to a subprojection of Q. In this case we write $P \precsim Q$. See [KaR 83, Section 2.5], [KaR 86, Chapter 6] and [Tak 79, Chapter V] for more properties of the projection lattice.

Presently there exists a substantial Fredholm theory in the framework of von Neumann algebras. This theory was at first introduced in [Bre 68, Bre 69] where the Fredholm elements were studied with respect to a specific norm closed two-sided ideal in the algebra. However, since most of the results hold with respect to any norm closed ideal we state the necessary results in this general framework.

For any norm closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ let $\pi_{\mathcal{I}} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denote the canonical quotient map. An operator $T \in \mathcal{A}$ is called *left Fredholm relative to* $\mathcal{I}$ if $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{A}/\mathcal{I}$. We denote this semigroup of operators by $\Phi_l(\mathcal{A}, \mathcal{I})$. The class $\Phi_r(\mathcal{A}, \mathcal{I})$ of *right Fredholm operators* is defined in the obvious similar way. The *Fredholm operators relative to* $\mathcal{I}$ are those operators contained in $\Phi(\mathcal{A}, \mathcal{I}) := \Phi_l(\mathcal{A}, \mathcal{I}) \cap \Phi_r(\mathcal{A}, \mathcal{I})$. As in [Ols 84] we define an operator $T \in \mathcal{A}$ to be *semi-Fredholm relative to* $\mathcal{I}$ if there is a central projection $P \in \mathcal{A}$ such that $PT \in \Phi_l(P\mathcal{A}, P\mathcal{I})$ and $(I-P)T \in \Phi_r((I-P)\mathcal{A}, (I-P)\mathcal{I})$. We denote this class by $\Phi_s(\mathcal{A}, \mathcal{I})$. In the case where $\mathcal{A}$ is a factor, i.e. the center $\mathcal{Z} = \mathbf{C}I$, $\Phi_s(\mathcal{A}, \mathcal{I}) = \Phi_l(\mathcal{A}, \mathcal{I}) \cup \Phi_r(\mathcal{A}, \mathcal{I})$. In general we shall denote the class $\Phi_l(\mathcal{A}, \mathcal{I}) \cup \Phi_r(\mathcal{A}, \mathcal{I})$ by $\Phi_u(\mathcal{A}, \mathcal{I})$. Although it can be shown that most of our results hold for $\Phi_s(\mathcal{A}, \mathcal{I})$ we shall concentrate only on $\Phi_u(\mathcal{A}, \mathcal{I})$.

In [Ols 84] Olsen used the well known Tomiyama's dimension function to define an index for Fredholm elements and show that this index map has properties similar to the classical index. A representation of the index group for the algebra and the given ideal as a group of continuous functions on the spectrum of the center is obtained. We include a brief discussion of this theory.

In order to do this generalisation one needs to investigate the dimension function *dim* due to Tomiyama [Tom 58]:
Let $\mathcal{Z}$ be the center of $\mathcal{A}$, with spectrum Ω, and identify $\mathcal{Z}$ with $C(\Omega)$. The compact space Ω is hyperstonean, i.e. the closure of every open set in Ω is open and Ω admits the structure of a perfect Borel measure [Dix 51]. Then every projection P in $\mathcal{Z}$ corresponds to the characteristic function of a unique clopen subset G of the compact set Ω. Since any von Neumann algebra can be decomposed into three different types we can partition Ω into three clopen subsets Ω_j, such that $\mathcal{A}_{\chi_{\Omega_j}}$ is of type j where $j = I,\ II,\ III$. Let

$$\begin{aligned} V_I &:= \{0\} \cup \mathbf{N} \cup \{\mathcal{N} : \mathcal{N} \text{ an infinite cardinal}, \mathcal{N} \leq \text{ dimension of } \mathcal{H}\}; \\ V_{II} &:= [0,\infty) \cup \{\mathcal{N} : \mathcal{N} \text{ an infinite cardinal}, \mathcal{N} \leq \text{ dimension of } \mathcal{H}\}; \\ V_{III} &:= \{0\} \cup \{\mathcal{N} : \mathcal{N} \text{ an infinite cardinal}, \mathcal{N} \leq \text{ dimension of } \mathcal{H}\}. \end{aligned}$$

Each V_j is compact when considered with the order topology. Let

$$\mathcal{D} := \{f : \Omega \to V : f \text{ continuous and } f(\Omega_j) \subset V_j \times \{j\}\},$$

where V denotes the disjoint union $\bigcup_{j\in\{I,II,III\}} V_j \times \{j\}$ of the V_j.

The following theorem is due to Tomiyama [Tom 58].

Theorem 1.1 *For any von Neumann algebra* $\mathcal{A}$ *on a Hilbert space* $\mathcal{H}$ *there exists a function* $dim : \mathcal{A}^p \to \mathcal{D}$ *with the following properties:*

(a) $0 \leq dim\, P \leq$ *dimension of* $\mathcal{H}$, *for each* $P \in \mathcal{A}^p$, *and* $dim\, P = 0$ *iff* $P = 0$;

(b) $dim\, P \leq dim\, Q$ *iff* $P \precsim Q$;

(c) *for mutually orthogonal projections* P *and* Q $dim(P + Q) = dim\, P + dim\, Q$;

(d) *for a central projection* Z $dim(ZP) = Z\, dim\, P$ *for each* $P \in \mathcal{A}^p$. ■

Olsen defined the index function in terms of this dimension function. Recall that in the classical index theory every Fredholm operator has index some positive or negative integer. In order to include 'negative' values in the general theory we need to extend $\mathcal{D}$ to a larger class of continuous functions.

For V_j, let $-V_j = \{-a : a \in V_j\}$ with the natural ordering $-a \leq -b$ if $a \geq b$. Identify -0 and 0 and with the order topology on $-V_j \cup V_j$ we let $C_c(\Omega)$ be the set of all continuous functions f such that $f(\Omega_j) \subset -V_j \cup V_j$. We define addition on $C_c(\Omega)$ as follows:

For f and g in $C_c(\Omega)$ let $X = \{t \in \Omega : f(t) \neq -g(t)\}$. Then $(f+g)(t) = f(t) + g(t)$ is well defined and continuous on X, hence continuously extends to $\beta X = \overline{X}$. Since $\overline{X}$ is clopen we may define $f + g$ on $\Omega \backslash \overline{X}$ as zero.

Define the map $ind : \mathcal{A} \to C_c(\Omega)$ by

$$ind(T) := dim\, N_T - dim\, N_{T^*}, \text{ for each } T \in \mathcal{A}.$$

Then, clearly, $ind(T^*) = -ind(T)$ and from Theorem 1.1 $ind(T) \geq 0$ if and only if $N_{T^*} \precsim N_T$.

To illustrate we give two examples:

Examples 1.2

1. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and suppose that $\mathcal{H}$ has dimension $\mathcal{N}_1$. Then the center $\mathcal{Z}$ of $\mathcal{A}$ is $\mathbf{C}I$. Since $\mathcal{A}$ is of type I

   $$V = -V_1 \cup V_1 = \{-\mathcal{N}_1, -\mathcal{N}_0\} \cup \mathbf{Z} \cup \{\mathcal{N}_0, \mathcal{N}_1\}$$

   and $C_c(\Omega) = V$. In this case $ind(\Phi(\mathcal{A}, \mathcal{K}(\mathcal{H}))) = \mathbf{Z}$ which implies the classical index theory.

2. Let $\mathcal{A} = l^\infty$ and $\mathcal{I} = c_0$. It is well known that $\mathcal{A}$ can be embedded as a w^*-subalgebra of $\mathcal{B}(l^2)$. Then $\mathcal{Z} = l^\infty = C(\beta \mathbf{N})$. For this type I algebra

   $$V = \{-\mathcal{N}_0\} \cup \mathbf{Z} \cup \{\mathcal{N}_0\}$$

   and $C_c(\beta\mathbf{N}) = C(\beta\mathbf{N}, V) = V^{\mathbf{N}}$. ■

For a detailed exposition of a Fredholm theory relative to any closed ideal in $\mathcal{A}$ the reader is referred to [Ols 84].

In Chapter 2 we shall prove several characterisations of operators in $\Phi_u(\mathcal{A}, \mathcal{I})$. In order to obtain these results a considerable amount of important techniques and results from [Ols 84] will be used. For this reason we include the following:

The following quantity gives a very useful characterisation of left and right Fredholm operators (see Theorem 1.7). For $T \in \mathcal{A}$ we define

$$m_{\mathcal{I}}(T) \;:=\; \inf\; \sigma(\pi_{\mathcal{I}}(|T|)).$$

Clearly $m_{\mathcal{I}}(T) \leq \|T\|$.

By an easy application of the Spectral Theorem one obtains (see [Ols 84])

Lemma 1.3 *Let $T \in \mathcal{A}$ and let $E_{(.)}$ denote the spectral measure of $|T|$.*

Then

$$m_{\mathcal{I}}(T) = \inf\{\beta \geq 0 : E_{[0,\beta+\varepsilon]} \notin \mathcal{I} \textit{ for each } \varepsilon > 0\}.$$

■

The first equality in the following proposition can be found in [Str 89]. The second and third equalities are new.

Proposition 1.4 *Let $T \in \mathcal{A}$, then*

$$\begin{aligned} m_{\mathcal{I}}(T) &= \inf\{\|\pi_{\mathcal{I}}(TP)\| : P \in \mathcal{A}^p \textit{ and } P \notin \mathcal{I}\} \\ &= \inf\{\|\pi_{\mathcal{I}}(TS)\| : S \in \mathcal{A} \textit{ and } \|\pi_{\mathcal{I}}(S)\| = 1\} \\ &= \inf\{\frac{\|\pi_{\mathcal{I}}(TS)\|}{\|\pi_{\mathcal{I}}(S)\|} : S \in \mathcal{A} \textit{ and } S \notin \mathcal{I}\}. \end{aligned}$$

PROOF. Let $E_{(.)}$ denote the spectral measure for $|T|$ and let $F = \{\|\pi_{\mathcal{I}}(TP)\| : P \in \mathcal{A}^p \text{ and } P \notin \mathcal{I}\}$ and let $G = \{\|\pi_{\mathcal{I}}(TS)\| : S \in \mathcal{A} \text{ and } \|\pi_{\mathcal{I}}(S)\| = 1\}$.

Let $\varepsilon > 0$ be given and let $Q_\varepsilon = E_{[0,\, m_{\mathcal{I}}(T)+\varepsilon]}$. Then by using Lemma 1.3, we have for any $\varepsilon > 0$ that $Q_\varepsilon \notin \mathcal{I}$. Moreover, $\|\pi_{\mathcal{I}}(TQ_\varepsilon)\| \leq \|TQ_\varepsilon\| \leq m_{\mathcal{I}}(T) + \varepsilon$ and therefore $\inf F \leq m_{\mathcal{I}}(T)$.

Now consider any $\varepsilon > 0$ and let $S_\varepsilon = |T|\, E_{[m_{\mathcal{I}}(T)-\varepsilon,\, \infty)}$ and $R \in \mathcal{A}$ arbitrary. Then, since $E_{[0,\, m_{\mathcal{I}}(T)-\varepsilon)} \in \mathcal{I}$ from Lemma 1.3, it follows directly that $\pi_{\mathcal{I}}(R^*T^*TR) = \pi_{\mathcal{I}}(R^*S_\varepsilon^2R)$. Compute

$$\begin{aligned} \|\pi_{\mathcal{I}}(TR)\|^2 &= \|\pi_{\mathcal{I}}(R^*T^*TR)\| \\ &= \|\pi_{\mathcal{I}}(R^*S_\varepsilon^2R)\| \\ &\geq (m_{\mathcal{I}}(T)-\varepsilon)^2\|\pi_{\mathcal{I}}(R^*E_{[m_{\mathcal{I}}(T)-\varepsilon,\infty)}R)\| \\ &= (m_{\mathcal{I}}(T)-\varepsilon)^2\|\pi_{\mathcal{I}}(R^*R)\| \\ &= (m_{\mathcal{I}}(T)-\varepsilon)^2\|\pi_{\mathcal{I}}(R)\|^2, \end{aligned}$$

so that $m_{\mathcal{I}}(T) \leq \inf G$.

All that is still needed to be shown in order to obtain the first two equalities is that $\inf G \leq \inf F$ and this is clear since $F \subset G$.

The third equality is clear by similar arguments. ∎

The proof of Proposition 1.4 says a bit more:

Proposition 1.5 *Let $T \in \mathcal{A}$, then*

$$\begin{aligned} m_{\mathcal{I}}(T) &= \inf\{\|TP\| : P \in \mathcal{A}^p \textit{ and } P \notin \mathcal{I}\} \\ &= \inf\{\|TS\| : S \in \mathcal{A} \textit{ and } \|\pi_{\mathcal{I}}(S)\| = 1\}. \end{aligned}$$

PROOF. Clearly

$$\begin{aligned}\inf\{\|\pi_{\mathcal{I}}(TS)\| : S \in \mathcal{A} \text{ and } \|\pi_{\mathcal{I}}(S)\| = 1\} &\leq \inf\{\|TS\| : S \in \mathcal{A} \text{ and } \|\pi_{\mathcal{I}}(S)\| = 1\} \\ &\leq \inf\{\|TP\| : P \in \mathcal{A}^p \text{ and } P \notin \mathcal{I}\}.\end{aligned}$$

The remaining part follows from the proof of Proposition 1.4. ■

That $m_{\mathcal{I}}(.)$ is a continuous function on $\mathcal{A}$ follows by the inequalities in the following result.

Proposition 1.6 *Let T, $S \in \mathcal{A}$, then*

$$|m_{\mathcal{I}}(T) - m_{\mathcal{I}}(S)| \leq \|T - S\|$$

and

$$m_{\mathcal{I}}(T)m_{\mathcal{I}}(S) \leq m_{\mathcal{I}}(TS) \leq \|T\|m_{\mathcal{I}}(S).$$

PROOF. Using Proposition 1.5 the first statement follows easily from the inequalities:

$$m_{\mathcal{I}}(T) \leq \|TP\| = \|(T-S)P + SP\| \leq \|T - S\| + \|SP\|$$

for any projection $P \notin \mathcal{I}$.

Using Proposition 1.4 the second statement follows from the following inequalities:

$$m_{\mathcal{I}}(T)m_{\mathcal{I}}(S) = \inf_{R \notin \mathcal{I}} \frac{\|\pi_{\mathcal{I}}(TR)\|}{\|\pi_{\mathcal{I}}(R)\|} . \inf_{R \notin \mathcal{I}} \frac{\|\pi_{\mathcal{I}}(SR)\|}{\|\pi_{\mathcal{I}}(R)\|}$$

$$\begin{aligned}
&\le \inf_{R\notin\mathcal{I}} \frac{\|\pi_{\mathcal{I}}(TSR)\|}{\|\pi_{\mathcal{I}}(SR)\|}\cdot \inf_{R\notin\mathcal{I}} \frac{\|\pi_{\mathcal{I}}(SR)\|}{\|\pi_{\mathcal{I}}(R)\|} \\
&\le \inf_{R\notin\mathcal{I}} \left(\frac{\|\pi_{\mathcal{I}}(TSR)\|}{\|\pi_{\mathcal{I}}(SR)\|}\cdot \frac{\|\pi_{\mathcal{I}}(SR)\|}{\|\pi_{\mathcal{I}}(R)\|}\right) \\
&= m_{\mathcal{I}}(TS) \\
&\le \|T\| m_{\mathcal{I}}(S).
\end{aligned}$$

■

Theorem 1.7 [Ols 84] *If* $T \in \mathcal{A}$, *then* $T \in \Phi_l(\mathcal{A}, \mathcal{I})$ *iff* $m_{\mathcal{I}}(T) > 0$.

PROOF. $T \in \Phi_l(\mathcal{A}, \mathcal{I})$ iff $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{A}/\mathcal{I}$

iff $\pi_{\mathcal{I}}(|T|)$ is invertible in $\mathcal{A}/\mathcal{I}$, since

$\pi_{\mathcal{I}}(|T|) = |\pi_{\mathcal{I}}(T)|$ is positive in $\mathcal{A}/\mathcal{I}$

iff $m_{\mathcal{I}}(T) > 0$. ■

Remark 1.8

It is clear by taking adjoints in Theorem 1.7 that we can characterise right Fredholmness of $T \in \mathcal{A}$ by $m_{\mathcal{I}}(T^*) > 0$. It is direct from Lemma 1.3 and Theorem 1.7 that if $T \in \Phi_l(\mathcal{A}, \mathcal{I})$ then $N_T = E_{[0]}(|T|) \in \mathcal{I}$. Moreover in [Ols 84] an operator $T \in \mathcal{A}$ is characterised as left Fredholm if and only if the range of T^* contains the range of $I - P$ for some projection $P \in \mathcal{I}$. This implies $N_T \in \mathcal{I}$. ■

Examples

Before presenting the properties of the index map we introduce examples of important ideals of which some were already studied considerably by various authors.

We start by defining the ideal $\mathcal{K}(\mathcal{A})$ of compact elements relative to $\mathcal{A}$. This ideal plays a central role in the whole index theory. The index group with respect to any smaller ideal is a subgroup of the index group with respect to $\mathcal{K}(\mathcal{A})$, and the index group with respect to a "completely noncompact" ideal is a quotient of a subgroup of the index group with respect to $\mathcal{K}(\mathcal{A})$.

Call $P \in \mathcal{A}^p$ *finite* if for $Q \in \mathcal{A}^p$ the relation $P \sim Q \leq P$ implies $P = Q$. Let $\mathcal{K}(\mathcal{A})$ be the norm closed two-sided ideal in $\mathcal{A}$ generated by the finite projections. In the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ we have $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{H})$. It is for this reason that the elements in $\mathcal{K}(\mathcal{A})$ are called compact relative to $\mathcal{A}$. In [Bre 68, Bre 69] Breuer has developed an abstract Fredholm theory relative to the ideal $\mathcal{K}(\mathcal{A})$. In [Ols 84] it is shown that a projection P in $\mathcal{A}$ is finite if and only if $dimP$ is finite on a dense open subset of Ω. From this it follows that the index group $ind(\Phi(\mathcal{A}, \mathcal{K}(\mathcal{A})))$ is the abelian group $\mathcal{G} := \{f \in C_c(\Omega) : f \text{ is finite a.e.}\}$.

We call a closed two-sided ideal $\mathcal{I}$ *compact* if $\mathcal{I}$ is contained in $\mathcal{K}(\mathcal{A})$. We define a specific example of a compact ideal which will be important throughout the thesis. This ideal is important in the theory of von Neumann algebras for many different reasons. See for instance [DDd 89, DDd 90, HiN 89, PhR 94] where non-commutative L^p spaces were studied. Our main purpose is to study Fredholm properties with respect to this ideal:

Let $\mathcal{A}$ be a semifinite von Neumann algebra. Then $\mathcal{A}$ possesses a faithful semifinite normal trace τ. For more information on the theory of traces we refer the reader to

[Tak 79]. We call $P \in \mathcal{A}^p$ *finite rank* if $\tau(P) < \infty$. Let $\mathcal{K}(\mathcal{A}, \tau)$ be the norm closed two-sided ideal generated by the finite rank projections in $\mathcal{A}$. An easy exercise, using the trace properties, is to show that a finite rank projection is finite. Hence $\mathcal{K}(\mathcal{A}, \tau)$ is a compact ideal.

In the case where $\mathcal{A}$ is a factor the notions of finite rank and finite coincide, hence $\mathcal{K}(\mathcal{A}, \tau) = \mathcal{K}(\mathcal{A})$. If $\mathcal{A} = l^\infty$, then $\mathcal{K}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{K}(\mathcal{A}, \tau) = c_0$. We shall refer to this ideal as the *τ-compact ideal.* Recently Dodds, Dodds and de Pagter [DDd 89, DDd 90, DDd 91] have made some use of this class in their study of symmetric non-commutative Banach function spaces. Furthermore Hiai and Nakamura [HiN 89, HiN 91] have used the τ-compact operators in their study of unitary orbits in von Neumann algebras. ∎

For the compact ideals the index map has properties exactly the same as the classical index. We include those properties that will be used. It is shown in [Ols 84] that the property

$$ind(TS) = ind(T) + ind(S)$$

holds under very general conditions. In fact T, $S \in \mathcal{A}$ can satisfy any one of the following conditions:

(i) T, $S \in \Phi_l(\mathcal{A}, \mathcal{I})$;

(ii) $T \in \Phi(\mathcal{A}, \mathcal{I})$ and $ind(T) = 0$;

(iii) $TS \in \Phi_l(\mathcal{A}, \mathcal{I})$ and $ind(TS) = 0$.

Clearly by taking adjoints we can replace the conditions on left Fredholmness with conditions on right Fredholmness. We include proofs of some of the following crucial lemmas (which appear in [Ols 84]) in which $\mathcal{I}$ will be a compact ideal in $\mathcal{A}$.

Lemma 1.9 *Let* $T \in \mathcal{A}$ *and* $S \in \mathcal{A}$ *such that* $R_S \in \mathcal{I}$. *Then* $ind(T+S) = ind(T)$.

PROOF. Since $R_{S^*} \sim R_S$, it follows that R_{S^*} is in $\mathcal{I}$. Hence $N_S = I - R_{S^*}$ is Fredholm of index zero. Then applying property (ii) above twice we get

$$\begin{aligned} ind(T+S) + ind(N_S) &= ind((T+S)N_S) \\ &= ind(TN_S) \\ &= ind(T) + ind(N_S). \end{aligned}$$

Therefore $ind(T+S) = ind(T)$. ■

Let $\mathcal{I}_{finite} = \{S \in \mathcal{A} : R_S \in \mathcal{I}\}$. It is an easy application of the Spectral Theorem to show that the norm closure of $\mathcal{I}_{finite}$ equals $\mathcal{I}$. In fact if $T \in \mathcal{I}$ then $|T| = U^*T \in \mathcal{I}$. Let $|T| = \int_0^\infty \lambda \, dE_\lambda$ and let $\varepsilon > 0$ be given. Then since $E_{(\varepsilon,\infty)} \leq \frac{1}{\varepsilon}|T|E_{(\varepsilon,\infty)}$, it is clear that $E_{(\varepsilon,\infty)} \in \mathcal{I}$. Hence $E_{(\varepsilon,\infty)} \in \mathcal{I}_{finite}$. Now since $\mathcal{I}_{finite}$ is a two-sided ideal it follows from the inequality $\|T - TE_{(\varepsilon,\infty)}\| \leq \varepsilon$ that $T \in \overline{\mathcal{I}_{finite}}$.

To extend Lemma 1.9 from elements in $\mathcal{I}_{finite}$ to arbitrary elements in $\mathcal{I}$ we need the following perturbation result concerning $m_{\mathcal{I}}(T)$.

Lemma 1.10 *Let* $T, S \in \mathcal{A}$. *If* $\|T - S\| < m_{\mathcal{I}}(T)$, *then* $ind(T) = ind(S)$ *and* $T, S \in \Phi_l(\mathcal{A}, \mathcal{I})$. ■

Lemma 1.11 *Let* $T \in \mathcal{A}$. *If* $T \in \Phi_u(\mathcal{A},\mathcal{I})$ *and* $K \in \mathcal{I}$, *then* $T+K \in \Phi_u(\mathcal{A},\mathcal{I})$ *and* $ind(T + K) = ind(T)$.

PROOF. It follows easily from the definition of Φ_u-operators that $T + K \in \Phi_u(\mathcal{A},\mathcal{I})$. Assume T is left Fredholm relative to $\mathcal{I}$. Then $m_{\mathcal{I}}(T) > 0$, so that there exists a $C \in \mathcal{I}_{finite}$ with $\|K - C\| < m_{\mathcal{I}}(T)$. Therefore $\|(T + K) - (T + C)\| < m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T + K)$, so that we may apply Lemmas 1.9 and 1.10 to conclude that $ind(T + K) = ind(T + C) = ind(T)$. ■

We call a two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ completely noncompact if there is no nonzero central projection Q such that the ideal $Q\mathcal{I}$ is a compact ideal in the reduced algebra $Q\mathcal{A}$. In case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ let $\mathcal{I} = \{T \in \mathcal{B}(\mathcal{H}) : T(\mathcal{H}) \text{ is separable}\}$. Then clearly $\mathcal{I}$ is completely noncompact. In fact any norm closed two-sided ideal in $\mathcal{B}(\mathcal{H})$ which is different from $\mathcal{K}(\mathcal{H})$ is completely noncompact.

It is shown in [Ols 84] that for any norm closed ideal $\mathcal{I}$ in $\mathcal{A}$ there exists a unique central projection P such that $P\mathcal{I}$ is relatively compact in $P\mathcal{A}$, and $(I - P)\mathcal{I}$ is completely noncompact in $(I-P)\mathcal{A}$. For the compact summand (i.e. with index map ind_P) the index theory above applies; for the remaining completely noncompact part of $\mathcal{I}$ the index map on $(I-P)\mathcal{A}$ has to be modified to obtain an index ind_{I-P} with all the desired properties. It is shown in [Ols 84] that ind_{I-P} satisfies, among others, Lemmas 1.9, 1.10 and 1.11. For the general case we let $ind(T) := ind_P(PT) + ind_{I-P}((I-P)T)$. Since the major part of the thesis deals with compact ideals we shall not go into further details and refer the reader to [Ols 84].

Other definitions and notation will be introduced as needed, primarily at the beginning of each chapter.

Chapter 2

CLASSES OF FREDHOLM-TYPE OPERATORS

In the classical theory of operators on a Hilbert space numerous characterisations of the semi-Fredholm operators exist. Some of these characterisations are used in the study of the structure of the essential spectrum. In this chapter the Fredholm theory in a von Neumann algebra relative to any closed two-sided ideal, as introduced by Olsen [Ols 84], is used to show that several of these results actually hold in any von Neumann algebra.

It was shown by Pfaffenberger [Pfa 70] and by Lebow and Schechter [LeS 71] that under certain conditions on a Banach space X, the semi-Fredholm operators can be characterised in terms of the algebraic and topological divisors of zero of the Calkin algebra. In [Str 89] Ströh and Swart proved that similar results hold in a semifinite von Neumann algebra with non-large center where the Fredholm theory is studied relative to Breuer's compact ideal.

For any closed ideal $\mathcal{I}$ in $\mathcal{A}$ let $\mathcal{Z}_l(\mathcal{A}, \mathcal{I})$ denote the class of operators $T \in \mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is a left topological divisor of zero in $\mathcal{A}/\mathcal{I}$; i.e., $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$ if and only if there exists a normalised sequence $(\pi_{\mathcal{I}}(S_n))$ in $\mathcal{A}/\mathcal{I}$ such that $\lim_{n\to\infty} \|\pi_{\mathcal{I}}(T)\pi_{\mathcal{I}}(S_n)\| = 0$. The class $\mathcal{Z}_r(\mathcal{A}, \mathcal{I})$ is defined in an obvious similar way. Let $\mathcal{N}_l(\mathcal{A}, \mathcal{I})$ denote the class of operators $T \in \mathcal{A}$ for which $\pi_{\mathcal{I}}(T)$ is a left algebraic divisor of zero in $\mathcal{A}/\mathcal{I}$; i.e., $T \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$ if and only if there exists an $S \notin \mathcal{I}$ with $TS \in \mathcal{I}$. A similar definition holds for $\mathcal{N}_r(\mathcal{A}, \mathcal{I})$. These classes possess a number of algebraic and topological properties.

Lemma 2.1 *Let* $T \in \mathcal{A}$.

(a) $T \in \Phi_r(\mathcal{A}, \mathcal{I})$ *if and only if* $T^* \in \Phi_l(\mathcal{A}, \mathcal{I})$

(b) $T \in \mathcal{Z}_r(\mathcal{A}, \mathcal{I})$ *if and only if* $T^* \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$

(c) $T \in \mathcal{N}_r(\mathcal{A}, \mathcal{I})$ *if and only if* $T^* \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$

where T^* *denotes the adjoint operator of* T.

PROOF. (a) $T \in \Phi_r(\mathcal{A}, \mathcal{I})$ implies that there exists an $S \in \mathcal{A}$ such that $ST - I \in \mathcal{I}$, so that $T^*S^* - I \in \mathcal{I}$. Hence $T^* \in \Phi_l(\mathcal{A}, \mathcal{I})$.

Similar for the converse.

(b) Suppose $T \in \mathcal{Z}_r(\mathcal{A}, \mathcal{I})$ and let $(\pi_{\mathcal{I}}(S_n))$ be a normalised sequence in $\mathcal{A}/\mathcal{I}$ such that $\lim_{n\to\infty} \|\pi_{\mathcal{I}}(S_n)\pi_{\mathcal{I}}(T)\| = 0$. Then $(\pi_{\mathcal{I}}(S_n^*))$ is also a normalised sequence in $\mathcal{A}/\mathcal{I}$ and $\lim_{n\to\infty} \|\pi_{\mathcal{I}}(T^*)\pi_{\mathcal{I}}(S_n^*)\| = 0$. Hence $T^* \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$.

Similar for the converse.

(c) Suppose $T \in \mathcal{N}_r(\mathcal{A}, \mathcal{I})$ and let $S \notin \mathcal{I}$ with $ST \in \mathcal{I}$. Then $S^* \notin \mathcal{I}$ and $T^*S^* \in \mathcal{I}$, so that $T^* \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$.

Similar for the converse. ■

In Theorem 1.7 we proved that the left Fredholm operators can be characterised in terms of the quantity $m_{\mathcal{I}}(T)$, i.e. $T \in \Phi_l(\mathcal{A}, \mathcal{I})$ if and only if $m_{\mathcal{I}}(T) > 0$. In the following proposition we use the bounded function $m_{\mathcal{I}}(.)$ to characterise $\mathcal{Z}_l(\mathcal{A}, \mathcal{I})$.

Proposition 2.2 *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{I}$ any norm closed two-sided ideal in $\mathcal{A}$. Then $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$ if and only if $m_{\mathcal{I}}(T) = 0$.*

PROOF. Suppose $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$, then by definition there exists a sequence (S_n) not in $\mathcal{I}$ with $(\pi_{\mathcal{I}}(S_n))$ normalised such that $\lim_{n\to\infty} \|\pi_{\mathcal{I}}(TS_n)\| = 0$. Then it follows, using Proposition 1.4, that $m_{\mathcal{I}}(T) = 0$.

For the converse inclusion suppose $m_{\mathcal{I}}(T) = 0$, i.e. $\inf \{\|\pi_{\mathcal{I}}(TP)\| : P \in \mathcal{A}^p$ and $P \notin \mathcal{I}\} = 0$ (using Proposition 1.4). So we can find a sequence (P_n) of projections not in $\mathcal{I}$ such that $\|\pi_{\mathcal{I}}(TP_n)\| < 1/n$ for every n. It is easy to see that $\|\pi_{\mathcal{I}}(P_n)\| = 1$ for every n. For if $\|\pi_{\mathcal{I}}(P_n)\| < 1$, then $\|\pi_{\mathcal{I}}(P_n)\| = \|\pi_{\mathcal{I}}(P_n^k)\| = \|\pi_{\mathcal{I}}(P_n)^k\| \leq \|\pi_{\mathcal{I}}(P_n)\|^k$ which tends to 0 as $k \to \infty$, so that $\|\pi_{\mathcal{I}}(P_n)\| = 0$ for every n, contradicting the fact that $P_n \notin \mathcal{I}$. Therefore $\|\pi_{\mathcal{I}}(P_n)\| = 1$ for every n

and $\lim_{n\to\infty}\|\pi_{\mathcal{I}}(TP_n)\| = 0$ so that $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$. ■

It follows directly from Lemma 2.1 and Proposition 2.2 that:

Corollary 2.3 *$T \in \mathcal{Z}_r(\mathcal{A}, \mathcal{I})$ if and only if $m_{\mathcal{I}}(T^*) = 0$.* ■

Due to Theorem 1.7, Proposition 2.2 and Corollary 2.3 we obtain the following topological characterisation of the semi-Fredholm operators:

Theorem 2.4 *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{I}$ any norm closed two-sided ideal in $\mathcal{A}$. Then $\Phi_l(\mathcal{A}, \mathcal{I}) = (\mathcal{Z}_l(\mathcal{A}, \mathcal{I}))^c$ and $\Phi_r(\mathcal{A}, \mathcal{I}) = (\mathcal{Z}_r(\mathcal{A}, \mathcal{I}))^c$.* ■

Remark 2.5

According to Theorem 2.4 and the fact that any left (right, resp.) algebraic divisor of zero is also a left (right, resp.) topological divisor of zero, the following relations hold in any von Neumann algebra $\mathcal{A}$ relative to any closed ideal $\mathcal{I}$, where $\mathcal{N}_u(\mathcal{A}, \mathcal{I}) := \mathcal{N}_l(\mathcal{A}, \mathcal{I}) \cup \mathcal{N}_r(\mathcal{A}, \mathcal{I})$ and $\mathcal{N}(\mathcal{A}, \mathcal{I}) := \mathcal{N}_l(\mathcal{A}, \mathcal{I}) \cap \mathcal{N}_r(\mathcal{A}, \mathcal{I})$ and similarly for $\mathcal{Z}_u(\mathcal{A}, \mathcal{I})$ and $\mathcal{Z}(\mathcal{A}, \mathcal{I})$.

$$\begin{array}{ccccc}
\Phi_u(\mathcal{A}, \mathcal{I}) & = & (\mathcal{Z}(\mathcal{A}, \mathcal{I}))^c & \subset & (\mathcal{N}(\mathcal{A}, \mathcal{I}))^c \\
\cup & & \cup & & \cup \\
\Phi_l(\mathcal{A}, \mathcal{I}) & = & (\mathcal{Z}_l(\mathcal{A}, \mathcal{I}))^c & \subset & (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c \\
\cup & & \cup & & \cup \\
\Phi(\mathcal{A}, \mathcal{I}) & = & (\mathcal{Z}_u(\mathcal{A}, \mathcal{I}))^c & \subset & (\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c
\end{array}$$

A similar diagram is valid for the classes $\Phi_r(\mathcal{A},\mathcal{I})$, $(\mathcal{Z}_r(\mathcal{A},\mathcal{I}))^c$ and $(\mathcal{N}_r(\mathcal{A},\mathcal{I}))^c$. ■

It follows from Proposition 2.2 and Corollary 2.3 that

$$\mathcal{Z}(\mathcal{A},\mathcal{I}) = \{T \in \mathcal{A} : m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) = 0\}.$$

In [Gal 94] Galaz-Fontes used the class

$$\{T \in \mathcal{B}(\mathcal{H}) : m_{\mathcal{K}(\mathcal{H})}(T) = m_{\mathcal{K}(\mathcal{H})}(T^*) = 0\}$$

to calculate distances from an operator to sets related to semi-Fredholm operators in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable. It is shown there that the class $\{T \in \mathcal{B}(\mathcal{H}) : m_{\mathcal{K}(\mathcal{H})}(T) = m_{\mathcal{K}(\mathcal{H})}(T^*) = 0\}$ is a closed semigroup of $\mathcal{B}(\mathcal{H})$. For a general von Neumann algebra $\mathcal{A}$ and a closed ideal $\mathcal{I} \subset \mathcal{A}$, knowing that $\{T \in \mathcal{A} : m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) = 0\}$ equals $\mathcal{Z}(\mathcal{A},\mathcal{I})$, it follows directly from $\mathcal{Z}(\mathcal{A},\mathcal{I}) = (\Phi_u(\mathcal{A},\mathcal{I}))^c$ that $\{T \in \mathcal{A} : m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) = 0\}$ is closed. Furthermore, $\{T \in \mathcal{A} : m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T^*) = 0\}$ is trivially a semigroup.

Suppose now $\mathcal{H}$ is a separable Hilbert space. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the type I_∞ factor, then by a well-known result of Calkin $\mathcal{K}(\mathcal{H})$ is the unique closed two-sided ideal in $\mathcal{B}(\mathcal{H})$. It was shown in [Bou 81] that if $T \in \mathcal{Z}(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}))$, then T is in the closure of the group of invertible operators.

Suppose now that $\mathcal{A}$ is a type II_∞ factor on a separable Hilbert space. It is not hard to show [Son 71, Theorem 2] that, similar to Calkin's result, the ideal $\mathcal{K}(\mathcal{A})$ generated by the finite projections is the unique closed two-sided ideal in $\mathcal{A}$. In this case we can prove a similar theorem as the one in [Bou 81]. Let $\mathcal{G}(\mathcal{A})$ denote the group of invertible elements in $\mathcal{A}$.

Theorem 2.6 *Let $\mathcal{A}$ be a type II_∞ factor on a separable Hilbert space $\mathcal{H}$. Then*

$$\mathcal{Z}(\mathcal{A}, \mathcal{K}(\mathcal{A})) \subset \overline{\mathcal{G}(\mathcal{A})}.$$

PROOF. Let $T \in \mathcal{Z}(\mathcal{A}, \mathcal{K}(\mathcal{A}))$. Then $m_{\mathcal{K}(\mathcal{A})}(T) = m_{\mathcal{K}(\mathcal{A})}(T^*) = 0$. Let $E_{(.)}$ ($F_{(.)}$, resp.) denote the spectral measure for $|T|$ ($|T^*|$, resp.). From Lemma 1.3 $E_{[0,\varepsilon]} \notin \mathcal{K}(\mathcal{A})$ and $F_{[0,\varepsilon]} \notin \mathcal{K}(\mathcal{A})$ for all $\varepsilon > 0$. Since all infinite projections in a factor on a separable Hilbert space are equivalent [Nai 59], $E_{[0,\varepsilon]} \sim F_{[0,\varepsilon]}$ for all $\varepsilon > 0$. By [Ols 89, Theorem 2.2]

$$dist(T, \mathcal{G}(\mathcal{A})) = \inf\{\lambda : E_{[0,\lambda]} \sim F_{[0,\lambda]}\} = 0.$$

Hence $T \in \overline{\mathcal{G}(\mathcal{A})}$. ■

Remark 2.7

In a type II_∞ factor $\mathcal{A}$ on a separable Hilbert space the index group is the additive group $\mathbf{R}$ of real numbers. Let $\mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}$ and let $\Phi_t(\mathcal{A}, \mathcal{K}(\mathcal{A}))$ be the set of semi-Fredholm operators in $\mathcal{A}$ with index $t \in \mathbf{R}^*$. Then we can repeat the proof of Theorem 2.2 in [Gal 94] to show that $\mathcal{Z}(\mathcal{A}, \mathcal{K}(\mathcal{A}))$ equals the boundary of $\Phi_t(\mathcal{A}, \mathcal{K}(\mathcal{A}))$ for any $t \in \mathbf{R}^*$.

Moreover, in [Ols 84, Section 12] Olsen studied the closure of the semi-Fredholm components for general von Neumann algebras and closed ideals. ■

By definition the elements in $\mathcal{N}_l(\mathcal{A}, \mathcal{I})$ are precisely those operators T whose images $\pi_{\mathcal{I}}(T)$ in $\mathcal{A}/\mathcal{I}$ are annihilated when multiplied from the right by some non-zero element in $\mathcal{A}/\mathcal{I}$, i.e. there exists an $S \notin \mathcal{I}$ such that $TS \in \mathcal{I}$. Moreover, we have control over $\|TS\|$ in the sense that we can guarantee $\|TS\|$ to be arbitrarily small for some choice of $S \notin \mathcal{I}$. To show this we need the following lemma. See [Str 89, Proposition 2.4] from which the proof of this lemma can be deduced.

Lemma 2.8 *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{I}$ any closed two-sided ideal in $\mathcal{A}$. Then $T \in \mathcal{I}$ if and only if for every $\varepsilon > 0$, $E_{(\varepsilon,\infty)} \in \mathcal{I}$ where $E_{(.)}$ is the spectral measure of $|T|$.*

Proof. Let $T \in \mathcal{I}$, then since $|T| = U^*T$ it is clear that $|T| \in \mathcal{I}$. Since for $\varepsilon > 0$ $E_{(\varepsilon,\infty)} = |T| \int_\varepsilon^\infty \frac{1}{\lambda}\, dE_\lambda$ and $\mathcal{I}$ is an ideal, we have that $E_{(\varepsilon,\infty)} \in \mathcal{I}$ for every $\varepsilon > 0$.

The converse is clear since $\mathcal{I}$ is norm closed and for every $\varepsilon > 0$ $\|T - TE_{(\varepsilon,\infty)}\| \leq \varepsilon$. ∎

Theorem 2.9 *Let $\mathcal{I}$ be a closed two-sided ideal in $\mathcal{A}$. Then $T \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$ if and only if for every $\varepsilon > 0$ there exists an $S_\varepsilon \notin \mathcal{I}$ such that $TS_\varepsilon \in \mathcal{I}$ and $\|TS_\varepsilon\| \leq \varepsilon$.*

Proof. Suppose $T \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$. Then there exists an $S \notin \mathcal{I}$ such that $TS \in \mathcal{I}$. Let E be the spectral measure for $|TS|$ and let $\varepsilon > 0$ be given. If we let $S_\varepsilon = SE_{[0,\varepsilon]}$ then clearly $\|TS_\varepsilon\| \leq \varepsilon$ and $TS_\varepsilon \in \mathcal{I}$. Moreover, it follows from Lemma 2.8 that $S_\varepsilon \notin \mathcal{I}$. For if $S_\varepsilon \in \mathcal{I}$, Lemma 2.8 would imply that $S \in \mathcal{I}$.

The converse is direct. ∎

Remark 2.10

By studying the proof of Theorem 2.9 carefully one can show that the $S_\varepsilon \notin \mathcal{I}$ can be chosen to be a projection, for every $\varepsilon > 0$. From Propositions 1.5 and 2.2 $T \in (\Phi_l(\mathcal{A}, \mathcal{I}))^c$ $(= \mathcal{Z}_l(\mathcal{A}, \mathcal{I}))$ if and only if for every $\varepsilon > 0$ there exists a projection $S_\varepsilon \notin \mathcal{I}$ such that $\|TS_\varepsilon\| \leq \varepsilon$. This comparison suggests that for general ideals the class $(\Phi_l(\mathcal{A}, \mathcal{I}))^c$ contains $\mathcal{N}_l(\mathcal{A}, \mathcal{I})$ strictly. ■

The following examples illustrate that the inclusions can be strict.

Examples 2.11

1. Let $\mathcal{A}$ be a purely infinite von Neumann algebra and let $\mathcal{I}$ be the trivial ideal consisting only of the zero element. Since $\mathcal{A}$ is purely infinite there exists a sequence (P_n) of mutually orthogonal non-zero projections such that

$$I = \sum_{n=1}^{\infty} P_n$$

(confer [Dix 91]).

Let $T = \sum_{n=1}^{\infty} \frac{1}{n} P_n$, then $T \geq 0$ and $E[0, \epsilon) := \sum_{1/n<\epsilon} P_n$ is non-zero for every $\epsilon > 0$. Hence from Lemma 1.3 and Theorem 1.7 it follows that $T \notin \Phi_l(\mathcal{A}, \mathcal{I})$. If there exists an $S \in \mathcal{A}$ such that $TS = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} P_n S = TS = 0.$$

Multiplying on both sides from the left by P_k we get that $P_k S = 0$ for every $k \in \mathbf{N}$ and it follows that $S = 0$. Hence $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$.

2. We illustrate with the following example that even in a type I von Neumann algebra the inclusion can be strict:

   Let $\mathcal{A}$ be the direct sum $\sum_{n=1}^{\infty} \oplus \mathcal{B}(\mathcal{H}_n)$, where $\mathcal{H}_n = \mathcal{H}$ for every $n \in N$. Denote by $\mathcal{K}$ the closed two-sided ideal in $\mathcal{A}$ generated by the finite projections in $\mathcal{A}$. It is easy to see that $\mathcal{K} = \sum_{n=1}^{\infty} \oplus \mathcal{K}(\mathcal{H}_n)$, where $\mathcal{K}(\mathcal{H})$ is the compact operators on $\mathcal{H}$. Consider the direct sum $T = \sum_{n=1}^{\infty} \frac{1}{n} I_n$, where $I_n = I$ for every $n \in N$. Then $T \geq 0$ and $E_{[0,\varepsilon)} = \sum\{I_n : \frac{1}{n} < \varepsilon\}$ is infinite for every $\varepsilon > 0$. Hence by using Lemma 1.3 and Theorem 1.7 T is not left Fredholm relative to $\mathcal{K}$. However, if $Q \in \mathcal{A}^p$ such that $TQ \in \mathcal{K}$, then $I_n Q = nTQI_n \in \mathcal{K}$ for every $n \in N$. Hence $Q = \sum_{n=1}^{\infty} I_n Q \in \sum_{n=1}^{\infty} \oplus \mathcal{K}(\mathcal{H}_n) = \mathcal{K}$. ■

Since the classes $(\mathcal{Z}_l(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{Z}_r(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{Z}(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{Z}_u(\mathcal{A}, \mathcal{I}))^c$ coincide with the respective Fredholm classes in $\mathcal{A}$ (see Remark 2.5), these classes have the same properties than the Fredholm classes. Therefore $(\mathcal{Z}_l(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{Z}_r(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{Z}_u(\mathcal{A}, \mathcal{I}))^c$ are open semigroups and $(\mathcal{Z}(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{Z}_u(\mathcal{A}, \mathcal{I}))^c$ are involutive. The classes $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{N}(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$ satisfy various properties of the Fredholm classes. Before we illustrate some of these properties we notice that in general $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$ are not open. In fact we have the following:

Proposition 2.12 $\overline{\mathcal{N}_l(\mathcal{A}, \mathcal{I})} = \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$.

PROOF. Since $\mathcal{N}_l(\mathcal{A}, \mathcal{I}) \subset \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$ and $\mathcal{Z}_l(\mathcal{A}, \mathcal{I})$ is closed it is obvious that

$\overline{\mathcal{N}_l(\mathcal{A}, \mathcal{I})} \subset \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$.

Suppose that $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$. Then by Proposition 2.2 $m_{\mathcal{I}}(T) = 0$. Hence by Proposition 1.5 there exists a sequence of projections $P_n \notin \mathcal{I}$ such that $\|TP_n\| \to 0$, i.e. $\|T - T(I - P_n)\| \to 0$. But $T(I - P_n) \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$ since $T(I - P_n)P_n = 0$ and $P_n \notin \mathcal{I}$. Hence $T \in \overline{\mathcal{N}_l(\mathcal{A}, \mathcal{I})}$. ■

We have seen from Examples 2.11 that in general $\Phi_l(\mathcal{A}, \mathcal{I})$ is strictly contained in $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$. Therefore we conclude from the proposition above that in general $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ is not an open set in $\mathcal{A}$.

Corollary 2.13 *Let $T \in \mathcal{A}$ and $\mathcal{I}$ any closed ideal in $\mathcal{A}$. Then*

$$m_{\mathcal{I}}(T) = dist(T, \mathcal{N}_l(\mathcal{A}, \mathcal{I})).$$

PROOF. By [LSS 95, Theorem 2.6] $m_{\mathcal{I}}(T) = dist(T, (\Phi_l(\mathcal{A}, \mathcal{I}))^c)$. Since $(\Phi_l(\mathcal{A}, \mathcal{I}))^c = \mathcal{Z}_l(\mathcal{A}, \mathcal{I}) = \overline{\mathcal{N}_l(\mathcal{A}, \mathcal{I})}$ we have $m_{\mathcal{I}}(T) = dist(T, \mathcal{N}_l(\mathcal{A}, \mathcal{I}))$. ■

Remark 2.14

By taking adjoints where necessary in the proofs of Proposition 2.12 and Corollary 2.13 we obtain similar results for $\mathcal{N}_r(\mathcal{A}, \mathcal{I})$ and $m_{\mathcal{I}}(T^*)$. ■

Theorem 2.15 (a) *$(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$ are semigroups.*

(b) *$(\mathcal{N}(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$ are involutive.*

(c) *If* $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ $(T \in (\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$, *resp.), then* $N_T \in \mathcal{I}$ $(N_{T^*} \in \mathcal{I}$, *resp.).*

(d) $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ *if and only if* $|T| \in (\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$.

PROOF. (a) We consider $\mathcal{N}_l(\mathcal{A}, \mathcal{I})$, the other cases follow similarly. Suppose $T, S \notin \mathcal{N}_l(\mathcal{A}, \mathcal{I})$. Then we need to show that $TS \notin \mathcal{N}_l(\mathcal{A}, \mathcal{I})$. If $R \in \mathcal{A}$ such that $(TS)R \in \mathcal{I}$, then $SR \in \mathcal{I}$ (since $T \notin \mathcal{N}_l(\mathcal{A}, \mathcal{I})$). Therefore $R \in \mathcal{I}$ (since $S \notin \mathcal{N}_l(\mathcal{A}, \mathcal{I})$).

(b) This is clear from Lemma 2.1.

(c) Let $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$, then since $TN_T = 0 \in \mathcal{I}$ it follows directly that $N_T \in \mathcal{I}$.

(d) Suppose $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and let $|T|S \in \mathcal{I}$. Then $U|T|S \in \mathcal{I}$, so that $TS \in \mathcal{I}$ which implies $S \in \mathcal{I}$. Hence $|T| \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and using Lemma 2.1 $|T| \in (\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$.

Conversely, suppose $|T| \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c \cap (\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$ and let $TS \in \mathcal{I}$. Then $U^*TS \in \mathcal{I}$ so that $|T|S \in \mathcal{I}$ which implies $S \in \mathcal{I}$. Hence $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$. ∎

The index map $ind(.)$ has similar properties for elements in $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$ than for elements in $\Phi(\mathcal{A}, \mathcal{I})$:

Theorem 2.16 *Let* $T, S \in (\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$. *Then* $ind(TS) = ind(T) + ind(S)$ *and* $ind(T^*) = -ind(T)$.

PROOF. Let $T, S \in (\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$. From [Ols 84, Proposition 9.3] there exists a central projection $P \in \mathcal{A}^p$ such that $P\mathcal{I}$ is a compact ideal in $P\mathcal{A}$ and $(I - P)\mathcal{I}$

is completely noncompact in $(I-P)\mathcal{A}$. By direct computation it can be shown that $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c = (\mathcal{N}_u(P\mathcal{A}, P\mathcal{I}))^c \oplus (\mathcal{N}_u((I-P)\mathcal{A}, (I-P)\mathcal{I}))^c$. Hence PT, $PS \in (\mathcal{N}_u(P\mathcal{A}, P\mathcal{I}))^c$ and $(I-P)T$, $(I-P)S \in (\mathcal{N}_u((I-P)\mathcal{A}, (I-P)\mathcal{I}))^c$. From Theorem 2.15(c) N_{PT}, $N_{PS} \in P\mathcal{I}$ and $N_{(I-P)T}$, $N_{(I-P)S} \in (I-P)\mathcal{I}$. By [Ols 84, Theorems 6.1 and 10.2] $ind_P(PTS) = ind_P(PT) + ind_P(PS)$ and $ind_{I-P}((I-P)TS) = ind_{I-P}((I-P)T) + ind_{I-P}((I-P)S)$. Hence $ind(TS) = ind(T) + ind(S)$.

It was noted in the preliminaries that the second identity holds for any $T \in \mathcal{A}$. ■

Remark 2.17

Note that $\Phi(\mathcal{A}, \mathcal{I})$ contains the invertible operators of $\mathcal{A}$; for if $S \in \mathcal{A}$ is invertible, then $\pi_{\mathcal{I}}(S) \in \mathcal{A}/\mathcal{I}$ is invertible. Therefore S is Fredholm. Remark 2.5 implies that each of the classes $(\mathcal{N}(\mathcal{A}, \mathcal{I}))^c$, $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c$ contains $(\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c$ and therefore all four of these classes contain the invertible operators. ■

It is a well-known and important fact in the classical Fredholm theory that the semi-Fredholm operators are stable when perturbed by compact operators. Lemma 1.11 and further remarks in the preliminaries show that relative to any closed ideal $\mathcal{I}$, $\Phi_u(\mathcal{A}, \mathcal{I})$ is stable under perturbations of elements in $\mathcal{I}$ with constant index. With respect to $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ we have the following result.

For a subset $B \subset \mathcal{A}$ we define the perturbation class associated with B (see [LeS 71]) as

$$\mathcal{P}(B) := \{K \in \mathcal{A} : T + K \in B \text{ for each } T \in B\}.$$

Proposition 2.18 $\mathcal{P}((\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c) = \mathcal{I}$.

PROOF. Suppose $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and $K \in \mathcal{I}$. Let $S \in \mathcal{A}$ such that $(T+K)S \in \mathcal{I}$. Then $TS \in \mathcal{I}$ and hence $S \in \mathcal{I}$. This proves $T + K \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and therefore $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ is stable under perturbations of elements of $\mathcal{I}$.

For the converse inclusion, note that if $\mathcal{G}(\mathcal{A})$ denotes the group of invertible operators in $\mathcal{A}$, then $\mathcal{G}(\mathcal{A})(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c \subset (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ and $(\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c\mathcal{G}(\mathcal{A}) \subset (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ (see Remark 2.17 and Theorem 2.15). It follows from [CPY 74, Lemma 5.5.5] that $\mathcal{P}(\mathcal{N}_l(\mathcal{A}, \mathcal{I})^c)$ is a two-sided ideal. Therefore, to show that $\mathcal{P}((\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c) \subset \mathcal{I}$ it suffices to show that the projections in $\mathcal{P}((\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c)$ are contained in $\mathcal{I}$. Let $Q \in \mathcal{P}((\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c)$ be a projection. Then since $I \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$, $I - Q \in (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$. But $(I - Q)Q = 0 \in \mathcal{I}$ and therefore $Q \in \mathcal{I}$. ■

Similar results for $\mathcal{P}((\mathcal{N}_r(\mathcal{A}, \mathcal{I}))^c)$, $\mathcal{P}((\mathcal{N}(\mathcal{A}, \mathcal{I}))^c)$ and $\mathcal{P}((\mathcal{N}_u(\mathcal{A}, \mathcal{I}))^c)$ follow in the same way. It was shown in [LeS 71] that if a Banach space possesses properties like subprojectivity and the compact approximation property, then the semi-Fredholm operators can be characterised in terms of the divisors of zero in the Calkin algebra. In the case of Hilbert spaces it was shown in [FSW 72] that an operator $T \in \mathcal{B}(\mathcal{H})$ is not left Fredholm if and only if there exists an infinite-dimensional projection P such that TP is compact (i.e. $T \in \mathcal{N}_l(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}))$). This useful characterisation is used to study the essential spectrum of an operator. It is thus important to ask for which cases such characterisations do exist in the general framework of von Neumann algebras.

Let $\mathcal{A}$ be a semifinite von Neumann algebra. Hence $\mathcal{A}$ possesses a faithful semifinite normal trace τ. We have seen in the preliminaries that elements in the compact ideal $\mathcal{K}(\mathcal{A},\tau)$ possess properties very similar to the classical compact operators on a Hilbert space. For this important ideal we shall obtain the expected characterisations. For the remaining part of this chapter, whenever $\mathcal{K}(\mathcal{A},\tau)$ is involved we shall assume that $\mathcal{A}$ is semifinite.

Note that the ideal $\mathcal{K}(\mathcal{A},\tau)$ is the norm closure of the two-sided ideal $\mathcal{A}_\tau :=$ $\{T \in \mathcal{A} : \tau(|T|) < \infty\}$. To see this suppose $T \in \mathcal{A}_\tau$, then from the spectral theorem $|T| = \int_0^\infty \lambda \, dE_\lambda$ is a norm limit of linear combinations of projections. We have that $\lambda\tau(I - E_\lambda) \leq \tau(|T|)$ for all $\lambda \geq 0$ and hence $\tau(I - E_\lambda) < \infty$ for all $\lambda > 0$. It now follows from $\|T - T(I - E_\lambda)\| \leq \lambda$ that T is in $\mathcal{K}(\mathcal{A},\tau)$. Hence $\overline{\{T \in \mathcal{A} : \tau(|T|) < \infty\}}^{\|\cdot\|}$ is contained in $\mathcal{K}(\mathcal{A},\tau)$. The converse inclusion is clear.

The C^*-algebra $\mathcal{A}/\mathcal{K}(\mathcal{A},\tau)$ is called the τ-*Calkin algebra.* For properties of these ideals see [Fac 83, Son 71, StW 93]. It is clear that any projection P in $\mathcal{A}_\tau$ is finite rank (i.e. $\tau(P) < \infty$).

We start by proving the following lemmas using spectral theory.

The *essential spectrum* $\sigma_e(T)$ of $T \in \mathcal{A}$ is defined to be the spectrum of $\pi_{\mathcal{I}}(T)$ in $\mathcal{A}/\mathcal{I}$. When $\mathcal{I}$ is replaced by a specific ideal we shall still use the notation $\sigma_e(T)$ since the related quotient algebra should be clear from the context.

Lemma 2.19 *Let $T \in \mathcal{A}$ be a positive operator and let $E_{(.)}$ denote the spectral measure of T. Then $\sigma_e(T) = \{\beta \geq 0 : E_{(\beta-\varepsilon,\beta+\varepsilon)} \notin \mathcal{A}_\tau$ for all $\varepsilon > 0\}$.*

PROOF. Suppose there exists an $\varepsilon > 0$ such that $E_{(\beta-\varepsilon,\beta+\varepsilon)} \in \mathcal{A}_\tau$. If we let $E = I - E_{(\beta-\varepsilon,\beta+\varepsilon)}$ it follows that $\beta \notin \sigma_{\mathcal{A}_E}(T_E)$, where $\mathcal{A}_E$ is the reduced algebra. Thus there exists an operator $S \in \mathcal{A}$ such that $E(T-\beta)ESE = ESE(T-\beta)E = E$. Since $\pi_{\mathcal{K}(\mathcal{A},\tau)}(E) = \pi_{\mathcal{K}(\mathcal{A},\tau)}(I)$, it follows that $T - \beta$ is invertible modulo $\mathcal{K}(\mathcal{A},\tau)$. Hence $\beta \notin \sigma_e(T)$.

Conversely, suppose $\beta \notin \sigma_e(T)$. Denote $T - \beta$ by T_β and let S be the inverse of T_β modulo $\mathcal{K}(\mathcal{A},\tau)$. Let $\varepsilon < \|S\|_e^{-1}$. Then

$$\|E_{(\beta-\varepsilon,\beta+\varepsilon)}\|_e = \|E_{(\beta-\varepsilon,\beta+\varepsilon)}T_\beta S\|_e \leq \varepsilon\|S\|_e < 1$$

From this and the fact that $E_{(\beta-\varepsilon,\beta+\varepsilon)}$ is a projection it follows that $\|E_{(\beta-\varepsilon,\beta+\varepsilon)}\|_e = 0$, i.e. $\tau(E_{(\beta-\varepsilon,\beta+\varepsilon)}) < \infty$. ∎

Remark 2.20

The above lemma characterises the essential spectrum of a positive operator in terms of its spectral projections. It can however be shown, in a similar way, that such a characterisation also holds for normal operators. ∎

Lemma 2.21 *Suppose $S \in \mathcal{A}_\tau$, $P \in \mathcal{A}^p$ with $0 < S < P$. Then there exists a projection $Q_S \in \mathcal{A}_\tau$ such that $Q_S \leq P$ and $\tau(Q_S) > \tau(S)$.*

PROOF. Let $E_{(.)}$ be the spectral measure for S. Since $0 < S < I$, it follows directly that $S \leq supp(S)$. Moreover, since $S < P$ it follows from $0 = (I-P)0(I-P) \leq (I-P)S(I-P) \leq (I-P)P(I-P) = 0$ that $(I-P)S = 0$. Hence $SP = PS = S$ which implies that $supp(S) \leq P$. If $\tau(supp(S)) < \infty$ we choose $Q_S = supp(S)$ and the result follows. Suppose $\tau(supp(S)) = \infty$. Then since $E_{(0,1]} = supp(S)$ and $E_{(t,1]} \uparrow E_{(0,1]}$ as $t \downarrow 0$ it follows from the normality of τ that there exists a $t_0 > 0$ such that $\tau(E_{(t_0,1]}) > \tau(S)$. In this case let $Q_S = E_{(t_0,1]}$. Since $S = \int_0^1 \lambda \, dE_\lambda \geq \int_{t_0}^1 \lambda \, dE_\lambda \geq t_0 Q_S$ it is clear that $Q_S \in \mathcal{A}_\tau$ and the result follows. ■

We are now ready to prove a characterisation of Φ_u-operators which will be very useful in characterising these operators in terms of the algebraic divisors of zero in the related quotient algebra. For the case of closed densely defined operators on a separable infinite dimensional Hilbert space a proof of the following theorem can be found in [FSW 72].

Theorem 2.22 *Let $T \in \mathcal{A}$. Then $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$ if and only if for every $P \in \mathcal{A}^p$ with $TP \in \mathcal{K}(\mathcal{A},\tau)$ it follows that P is finite rank.*

PROOF. Suppose $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$, then there exists an $S \in \mathcal{A}$ such that $I - ST \in \mathcal{K}(\mathcal{A},\tau)$. Consider any $P \in \mathcal{A}^p$ for which $TP \in \mathcal{K}(\mathcal{A},\tau)$. Then it follows that $P = (I - ST)P + STP \in \mathcal{K}(\mathcal{A}, \tau)$, so $\tau(P) < \infty$.

Conversely, suppose $T \notin \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$ and let $E_{(.)}$ be the spectral measure of $|T|$. Then $|T| \notin \Phi(\mathcal{A},)$, which implies that $0 \in \sigma_e(|T|)$. Hence it follows from Lemma 2.19 that for each $n \in \mathbf{N}$, $F_n := E_{[0,\frac{1}{n})} \notin \mathcal{A}_\tau$. Let $G_n = F_{n-1} - F_n$ and $G_0 = N_T = N_{|T|} = E_{[0]}$. If $\tau(G_0) = \infty$, it follows from $TG_0 = 0 \in \mathcal{K}(\mathcal{A}, \tau)$ that $T \notin \{S \in \mathcal{A} : P \in \mathcal{A}^p$ and $SP \in \mathcal{K}(\mathcal{A}, \tau)$ then $P \in \mathcal{K}(\mathcal{A}, \tau)\}$ and the theorem is proved. If $\tau(G_0) < \infty$, it follows from $F_{n-1} = G_0 + \sum_{j=n}^{\infty} G_j$ that $\sum_{j=n}^{\infty} \tau(G_j) = \tau(\sum_{j=n}^{\infty} G_j) = \infty$ for every $n \in \mathbf{N}$. Choose a subsequence (F_{n_k}) of the sequence (F_n) such that $\tau(F_{n_k} - F_{n_{k+1}}) > 1$ for all $k \in \mathbf{N}$. Let $P_k = F_{n_k} - F_{n_{k+1}}$. Since τ is semifinite, there exists for each $k \in \mathbf{N}$ an $S_k \in \mathcal{A}_\tau$ such that $0 < S_k < P_k$ with $\tau(S_k) > 1$. By Lemma 2.21 there exist projections $Q_k \in \mathcal{A}_\tau$ such that $Q_k \leq P_k$ and $\tau(Q_k) > 1$. Clearly the projections Q_k are disjoint. If we define $Q := \sum_{k=1}^{\infty} Q_k$, then it follows from the normality of τ that $\tau(Q) = \infty$. We conclude the theorem by showing that $TQ \in \mathcal{K}(\mathcal{A}, \tau)$. Let $\varepsilon > 0$ be given. Choose $k \in \mathbf{N}$ so large that $\frac{1}{k} < \varepsilon$. Since $\sum_{j=k}^{\infty} Q_j \leq \sum_{j=k}^{\infty} P_j \leq F_k$, it follows that

$$\|TQ - T\sum_{j=1}^{k-1} Q_j\| = \|T\sum_{j=k}^{\infty} Q_j\| \leq \|TF_k\| < \varepsilon$$

Then, since $T\sum_{j=1}^{k-1} Q_j \in \mathcal{A}_\tau$ for all $k \in \mathbf{N}$ it follows that $TQ \in \mathcal{K}(\mathcal{A}, \tau)$. Hence there exists a $Q \notin \mathcal{K}(\mathcal{A}, \tau)$ such that $TQ \in \mathcal{K}(\mathcal{A}, \tau)$ and the theorem is proved. ■

By taking adjoints and applying the above result, we obtain

Corollary 2.23 *Let $T \in \mathcal{A}$. Then $T \in \Phi_r(\mathcal{A}, \mathcal{I})$ if and only if for every $P \in \mathcal{A}^p$*

and $PT \in \mathcal{K}(\mathcal{A},\tau)$ *it follows that* P *is finite rank.* ∎

Now for the long awaited characterisation.

Theorem 2.24 $\Phi_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)) = (\mathcal{N}_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)))^c$ *and*

$$\Phi_r(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)) = (\mathcal{N}_r(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)))^c.$$

PROOF. It is clear from Remarks 2.5(1) that $\Phi_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)) \subset (\mathcal{N}_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)))^c$ and it is clear from Theorem 2.22 that if $T \notin \Phi_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau))$, then $\pi_{\mathcal{K}(\mathcal{A},\tau)}(T)$ is a left algebraic divisor of zero in $\mathcal{A}/\mathcal{K}(\mathcal{A},\tau)$, in other words $T \in \mathcal{N}_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau))$.

The characterisation of the right Fredholm operators follows by taking adjoints and applying the first result. ∎

From Theorems 2.4 and 2.24 we conclude that the classes $\Phi_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau))$, $(\mathcal{N}_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)))^c$ and $(\mathcal{Z}_l(\mathcal{A},\mathcal{K}(\mathcal{A},\tau)))^c$ all coincide. However by Examples 2.11 we notice that even for a type I algebra $\mathcal{N}_l(\mathcal{A},\mathcal{K}(\mathcal{A}))$ and $\mathcal{Z}_l(\mathcal{A},\mathcal{K}(\mathcal{A}))$ can differ. *It is an open question whether* $\mathcal{N}_l(\mathcal{A},\mathcal{I})$ *and* $\mathcal{Z}_l(\mathcal{A},\mathcal{I})$ *will coincide for any closed ideal* $\mathcal{I}$ *contained in* $\mathcal{K}(\mathcal{A},\tau)$. In the case where $\mathcal{I} = \{0\}$ it is clear that the elements in $\mathcal{N}_l(\mathcal{A},\mathcal{I})$ ($\mathcal{Z}_l(\mathcal{A},\mathcal{I})$, resp.) are exactly the left algebraic (left topological, resp.) divisors of zero in the algebra $\mathcal{A}$. In the following result we show that in a von Neumann algebra the algebraic and topological divisors of zero coincide. Hence this solves our open question for the case where $\mathcal{I} = \{0\}$.

Theorem 2.25 *Let $\mathcal{A}$ be a von Neumann algebra and $T \in \mathcal{A}$. The following are equivalent:*

(a) *T is a left algebraic divisor of zero.*

(b) *T is a left topological divisor of zero.*

(c) *T is not left invertible in $\mathcal{A}$.*

PROOF. (a) implies (b) follows directly from the definitions. That (b) implies (c) follows from Theorem 2.4 by choosing $\mathcal{I} = \{0\}$. We show that (c) implies (a). Suppose that T is not a left algebraic divisor of zero, i.e. if $TS = 0$ for $S \in \mathcal{A}$ then $S = 0$. By an easy application of the polar decomposition of T it is clear that $|T|$ is not a left algebraic divisor of zero. Now from $|T|N_{|T|} = 0$ it follows that $N_{|T|} = 0$. Hence $R_{|T|} = I - N_{|T|} = I$. Hence $|T|$ is a bijection and by the Open Mapping Theorem $|T|$ is invertible in $\mathcal{B}(\mathcal{H})$ and hence in $\mathcal{A}$. Again by using the polar decomposition of T we conclude that T is left invertible in $\mathcal{A}$. ■

By restating Theorem 2.22, we get a useful description of the essential spectrum of an operator in terms of the eigenvalues in the τ-Calkin algebra. Let $a \in \mathcal{A}/\mathcal{K}(\mathcal{A}, \tau)$. Call $\lambda \in \mathbf{C}$ an *eigenvalue* of a if there exists a nonzero projection $p \in \mathcal{A}/\mathcal{K}(\mathcal{A}, \tau)$ such that $ap = \lambda p$. We denote the set of all eigenvalues of $a \in \mathcal{A}/\mathcal{K}(\mathcal{A}, \tau)$ by $P\sigma(a)$.

Theorem 2.26 *Let $T \in \mathcal{A}$ and let $a = \pi(T)$. Then*

$$\sigma_e(T) = \sigma(a) = P\sigma(a) \cup \overline{P\sigma(a^*)}$$

PROOF. Suppose $\lambda \in \sigma_e(T)$, then $\lambda I - T \notin \Phi(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$. If $\lambda I - T \notin \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$, then Theorem 2.22 implies that there exists a projection $P \notin \mathcal{K}(\mathcal{A}, \tau)$ such that $(\lambda I - T)P \in \mathcal{K}(\mathcal{A}, \tau)$. Let $p = \pi(P)$, then $ap = \lambda p$ and so $\lambda \in P\sigma(a)$. Similarly, if $\lambda I - T \notin \Phi_r(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$, it follows that $\overline{\lambda} \in P\sigma(a^*)$. Hence $\sigma_e(T) \subset P\sigma(a) \cup \overline{P\sigma(a^*)}$.

The converse is obvious. ■

In [Yoo 51] Yood characterised the Φ_l-operators on a Banach space X in terms of sets which are precompact in X. The result states that an operator T is a Φ_l-operator on X if and only if for any bounded set $\Omega \subset X$ the property that $T(\Omega)$ is precompact implies that Ω is precompact. To prove a similar result in our case we need to generalise the notion of a finite ε-net. This generalisation is actually suggested in [Son 71].

Let $\mathcal{B}_{\mathcal{H}}$ be the unit ball of $\mathcal{H}$ and let $\Omega \subset H$ be a bounded set. We say Ω possesses a *finite ε-net* if there exists a projection P_ε which is finite rank such that $\Omega \subset P_\varepsilon(\Omega) + \varepsilon\mathcal{B}_{\mathcal{H}}$. For a set $\Omega \subset \mathcal{H}$, define

$$q(\Omega) := \inf\{\varepsilon > 0 : \text{there exists a finite } \varepsilon\text{-net for } \Omega\}.$$

Then clearly $q(\Omega) < \infty$ if and only if Ω is a bounded set. We call Ω *precompact* *relative to* $\mathcal{A}$ if $q(\Omega) = 0$.

It is clear that if we choose $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and τ the canonical trace, then precompactness of a bounded set relative to $\mathcal{A}$ coincides with the well-known notion of total-boundedness of the set.

The following properties of the function q will be needed.

Lemma 2.27 *For bounded sets* Ω *and* Ω_1 *in* $\mathcal{H}$ *and* $T \in \mathcal{A}$ *we have*

(a) $q(\Omega + \Omega_1) \leq q(\Omega) + q(\Omega_1)$,

(b) $q(\alpha\Omega) = |\alpha|\, q(\Omega)$ *for* $\alpha \in \mathbf{C}$,

(c) $q(T(\Omega)) \leq \|T\|\, q(\Omega)$.

PROOF. (a) Let $\varepsilon > 0$ be given. Using the definition of q it follows that there exists a μ-net for Ω such that $\mu < q(\Omega) + \varepsilon/2$. Similarly one can choose a μ_1-net for Ω_1 with $\mu_1 < q(\Omega_1) + \varepsilon/2$. Now put $\alpha = \mu + \mu_1$ and $P_\alpha = sup(P_\mu,\, P_{\mu_1})$, then P_α is finite rank and $\Omega + \Omega_1 \subset P_\alpha(\Omega + \Omega_1) + \alpha\, \mathcal{B}_{\mathcal{H}}$. Hence $q(\Omega + \Omega_1) \leq \alpha < q(\Omega) + q(\Omega_1) + \varepsilon$ and the result follows.

(b) Let $\varepsilon > 0$ be given. From the definition of q we have the existance of a μ-net such that $\mu < q(\Omega) + \varepsilon$. Now let $\beta = |\alpha|\,\mu$ and $P_\beta = P_\mu$, then P_β is finite rank and $\alpha\,\Omega \subset P_\beta(\alpha\,\Omega) + \beta\,\mathcal{B}_{\mathcal{H}}$. Hence $q(\alpha\,\Omega) \leq \beta < |\alpha|(q(\Omega) + \varepsilon)$ and therefore $q(\alpha\,\Omega) \leq |\alpha|\, q(\Omega)$. From this result we also have $q(\Omega) \leq \frac{1}{|\alpha|}\, q(\alpha\,\Omega)$ and the result follows.

(c) Let $\varepsilon > 0$ be given. By application of the definition of q there exists a μ-net such that $\mu < q(\Omega) + \varepsilon$. Now let $\alpha = \|T\|\,\mu$ and $P_\alpha = R_{TP_\mu}$, then P_α is finite rank and $T(\Omega) \subset P_\alpha(T(\Omega)) + \alpha\,\mathcal{B}_{\mathcal{H}}$. Hence $q(T(\Omega)) \leq \alpha < \|T\|(q(\Omega) + \varepsilon)$ and the result follows. ■

It was proven in [Son 71] that $T \in \mathcal{K}(\mathcal{A}, \tau)$ if and only if $T(\Omega)$ is relatively precompact for any bounded set $\Omega \subset \mathcal{H}$. With the use of Lemma 2.19 we give a more elegant proof.

Proposition 2.28 *$T \in \mathcal{K}(\mathcal{A}, \tau)$ if and only if $q(T(\Omega)) = 0$ for any bounded set $\Omega \subset \mathcal{H}$.*

PROOF. Suppose $T \in \mathcal{K}(\mathcal{A}, \tau)$ and $\varepsilon > 0$ is given. Then from $|T| = U^*T$ it is clear that $|T| \in \mathcal{K}(\mathcal{A}, \tau)$. From Lemma 2.19, since $\sigma_e(|T|) = \{0\}$, $\tau(E_{(\delta,\infty)}) < \infty$ for any $\delta > 0$, where $E_{(.)}$ is the spectral measure for $|T|$. Now let $\Omega \subset \mathcal{H}$ be any bounded set and $c := \sup_{x\in\Omega} \|x\|$. If we let $P_\varepsilon = E_{(\varepsilon/c,\infty)}$, then $\tau(P_\varepsilon) < \infty$ and $|T|(\Omega) \subset P_\varepsilon|T|(\Omega) + \varepsilon\mathcal{B}_{\mathcal{H}}$, for if $x \in \Omega$ then $\||T|x - P_\varepsilon|T|x\| \leq \||T|E_{[0,\varepsilon/c]}\|\|x\| \leq \frac{\varepsilon}{c}.c = \varepsilon$. This shows that $q(|T|(\Omega)) = 0$. By Lemma 2.27 (c) $q(T(\Omega)) \leq \|U\|q(|T|(\Omega)) = 0$.

Conversely, suppose for any bounded set $\Omega \subset \mathcal{H}$ $q(T(\Omega)) = 0$. Then for any $\varepsilon > 0$ there exists a finite ε-net for $T(\mathcal{B}_{\mathcal{H}})$. Hence there exists a projection P_ε such that $\tau(P_\varepsilon) < \infty$ and $T(\mathcal{B}_{\mathcal{H}}) \subset P_\varepsilon T(\mathcal{B}_{\mathcal{H}}) + \varepsilon\mathcal{B}_{\mathcal{H}}$. From this it follows that $\|T - P_\varepsilon T\| \leq \varepsilon$. Since $P_\varepsilon T \in \mathcal{K}(\mathcal{A}, \tau)$ it follows that $T \in \mathcal{K}(\mathcal{A}, \tau)$. ■

Theorem 2.29 *Let* $T \in \mathcal{A}$. *Then* $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$ *if and only if the only bounded sets that* T *maps onto relatively precompact sets are those which are themselves relatively precompact.*

Proof. Let $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$ and let $\Omega \subset \mathcal{H}$ be any bounded set such that $T(\Omega)$ is relatively precompact. We show that Ω is relatively precompact.
Since $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$ there exist operators $S \in \mathcal{A}$, $K \in \mathcal{K}(\mathcal{A},\tau)$ such that $ST - I = K$. Hence $\Omega = I(\Omega) \subset ST(\Omega) - K(\Omega)$. By a straightforward argument it follows that $q(S(T(\Omega))) \leq \|S\| q(T(\Omega))$ which implies that $S(T(\Omega))$ is precompact. Since $K \in \mathcal{K}(\mathcal{A},\tau)$, it follows that $q(K(\Omega)) = 0$. Hence $K(\Omega)$ is relatively precompact. By a direct application of our definition of relative precompactness it can be shown that sums of precompact sets relative to $\mathcal{A}$ are relatively precompact. Hence Ω is relatively precompact.

Conversely, suppose $T \notin \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$. Then it follows from Theorem 2.22 that there exists a $P \in \mathcal{A}^p$ such that $\tau(P) = \infty$ and $TP \in \mathcal{K}(\mathcal{A},\tau)$. Let $\Omega = P(\mathcal{B}_{\mathcal{H}})$. Since $TP \in \mathcal{K}(\mathcal{A},\tau)$, it follows that $T(\Omega) = TP(\mathcal{B}_{\mathcal{H}})$ is relatively precompact. It is clear, however, from the definition of precompactness that precompactness of Ω relative to $\mathcal{A}$ would imply that $\tau(P) < \infty$. Hence there exists a bounded set $\Omega \subset \mathcal{H}$ for which $T(\Omega)$ is relatively precompact but Ω itself is not relatively precompact. ■

Chapter 3

LIFTING RESULTS

To explain the basic ideas around lifting theorems we consider the following short exact sequence:

$$0 \longrightarrow \mathcal{I} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi_{\mathcal{I}}} \mathcal{A}/\mathcal{I} \longrightarrow 0$$

where $\mathcal{I}$ is a closed two-sided ideal in a von Neumann algebra $\mathcal{A}$, i the inclusion map and $\pi_{\mathcal{I}}$ the canonical quotient map.

Let $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$, and suppose a has a certain property P (algebraic, analytic, spectral, etc.). If there exists an element $S \in \mathcal{A}$ with property P such that $T - S \in \mathcal{I}$, then we call S a P-lifting of a. To lift self-adjointness of an element from the quotient algebra to $\mathcal{A}$ is trivial: for if $a = \pi_{\mathcal{I}}(T)$ is self-adjoint, let $S = 1/2\ (T + T^*)$, then clearly S is self-adjoint and $T - S = 1/2\ (T - T^*) \in \mathcal{I}$.

It can be seen from [Cal 41, Theorem 2.4] that the lifting of projections from $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ is less trivial. By using the spectral theorem for bounded operators we show by a more streamlined proof that projections in $\mathcal{A}/\mathcal{I}$ can be lifted to $\mathcal{A}$. A special case of this result (i.e. where $\mathcal{I} = \mathcal{K}$) can be found in [Kft 82].

Theorem 3.1 *Let $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ be a projection. Then a has a projection lifting in $\mathcal{A}$.*

PROOF. Let S be a self-adjoint lifting of $a = \pi_{\mathcal{I}}(T)$. Also let $E_{(.)}$ denote the spectral measure of S. Then $S = \int_{-\|S\|}^{\|S\|} \lambda \, dE_\lambda$. Since $\sigma_e(S) = \sigma_e(T) = \{0, 1\}$, it is clear by using functional calculus in $\mathcal{A}/\mathcal{I}$ that $\sigma_e(SE_{[-\|S\|, 1/2]}) = \{0\}$, $\sigma_e(SE_{(1/2, \|S\|]}) = \{1\}$ and hence $\sigma_e(SE_{(1/2, \|S\|]} - E_{(1/2, \|S\|]}) = \{0\}$. Since $\pi_{\mathcal{I}}(SE_{[-\|S\|, 1/2]})$ and $\pi_{\mathcal{I}}((S - I)E_{(1/2, \|S\|]})$ are self-adjoint elements in $\mathcal{A}/\mathcal{I}$ with zero spectrum, both these elements are zero in $\mathcal{A}/\mathcal{I}$. Hence $SE_{[-\|S\|, 1/2]}$, $(S - I)E_{(1/2, \|S\|]} \in \mathcal{I}$. From this it is clear that $S - E_{(1/2, \|S\|]} \in \mathcal{I}$. Then $E_{(1/2, \|S\|]}$ is a projection lifting for $a = \pi_{\mathcal{I}}(T)$. ■

Remark 3.2

Theorem 3.1 does not hold for a general C^*-algebra $\mathcal{C}$ and a general closed ideal $\mathcal{I}$ in $\mathcal{C}$. For example if $\mathcal{C} = C[0, 1]$ and $\mathcal{I} = \{f \in C[0, 1] : f(0) = f(1) = 0\}$, then $\mathcal{I}$ is a closed ideal in $\mathcal{C}$. If we let $h(t) = t$, then $h + \mathcal{I}$ is a non-trivial projection in $\mathcal{C}/\mathcal{I}$. However since $\mathcal{C}$ contains only the trivial projections 0 and 1, there exists no projection lifting for $h + \mathcal{I}$.

In [Had 95] Hadwin studied the problem of lifting algebraic elements from $\mathcal{C}/\mathcal{I}$ to $\mathcal{C}$, where $\mathcal{I}$ is any norm closed ideal in a C^*-algebra $\mathcal{C}$. He showed that the lifting of algebraic elements is equivalent to the problem of lifting finite orthogonal families of projections. It is therefore important to ask for which C^*-algebras and ideals lifting of projections is possible. ∎

The natural question arising is whether normality of elements can be lifted. This is however a very hard problem, for instance if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{I} = \mathcal{K}(\mathcal{H})$ the well-known theorem of Brown, Douglas and Fillmore [BDF 73] provides us with precise conditions under which normality of elements in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ can be lifted. Before stating this theorem we indicate with an example the importance of lifting theorems: Consider $L^2(\mathbf{T}, m)$ and let $H^2(\mathbf{T}) = \overline{span}\{z^n : n \geq 0\}$, where $\mathbf{T}$ denotes the unit circle. Let $P : L^2 \to H^2$ be the orthogonal projection and for any $f \in C(\mathbf{T})$ let $T_f \in \mathcal{B}(H^2(\mathbf{T}))$ be defined by $T_f g = P(fg)$. Let $\mathcal{E} = C^*\{T_z\}$ be the C^*-algebra generated by T_z. Then by a result of Coburn $\mathcal{E} = \{T_f + K : f \in C(\mathbf{T}), K \in \mathcal{K}(H^2)\}$. Hence $\mathcal{E}$ contains $\mathcal{K}(H^2)$ and if we define a *-homomorphism $\phi : \mathcal{E} \to C(\mathbf{T})$ by $\phi(T_f + K) = f$, then the sequence

$$0 \longrightarrow \mathcal{K}(H^2) \xrightarrow{i} \mathcal{E} \xrightarrow{\phi} C(\mathbf{T}) \longrightarrow 0$$

is exact. We call $(\mathcal{E}, \phi)$ an extension of $\mathcal{K}(H^2)$ by $C(\mathbf{T})$. All elements in $\mathcal{E}$ are *essentially normal*, which means $\pi_{\mathcal{I}}(e) \in \mathcal{B}(H^2)/\mathcal{K}(H^2)$ is normal.

Two extensions $(\mathcal{E}_1,\ \phi_1)$ and $(\mathcal{E}_2,\ \phi_2)$ of $\mathcal{K}(\mathcal{H})$ are called equivalent if there exists an isomorphism $\rho : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ such that $\phi_2 o \rho = \phi_1$. It turns out that if say $\mathcal{E}_1 = C^*(\mathcal{K}(\mathcal{H}),\ T_1,\ I)$ and $\mathcal{E}_2 = C^*(\mathcal{K}(\mathcal{H}),\ T_2,\ I)$, then $\mathcal{E}_1 \sim \mathcal{E}_2$ if and only if T_1 and T_2 are unitarily equivalent mod $\mathcal{K}(\mathcal{H})$. Hence classifying extensions is equivalent to the problem of classifying essentially normal operators.

The important Brown-Douglas-Fillmore theorem states that *an essentially normal operator T is the sum of a normal and a compact operator if and only if ind$(\lambda - T) = 0$ for all $\lambda \notin \sigma_e(T)$*. The main result in classifying extensions is the following:
Two essentially normal operators T_1 and T_2 are unitarily equivalent modulo $\mathcal{K}(\mathcal{H})$ if and only if $\sigma_e(T_1) = \sigma_e(T_2)$ and *ind*$(\lambda - T_1) =$ *ind*$(\lambda - T_2)$ for all $\lambda \notin \sigma_e(T_1) = \sigma_e(T_2)$. The Brown-Douglas-Fillmore theory motivates the important link between problems in K-theory and operator theory. There exists to our knowledge no proof of the Brown-Douglas-Fillmore theorem that doesn't rely on ideas from homological algebra. For more results on lifting properties we refer the reader to [Wes 66, BMSW82, Str 94, Had 95].

Our aim in this chapter is to prove a number of lifting theorems for elements in a von Neumann algebra relative to any closed two-sided ideal in the algebra.

An essential result in the Brown-Douglas-Fillmore theory is the Weyl-von Neumann theorem which states that *two self-adjoint operators T_1 and T_2 are unitarily equiva-*

lent if and only if $\sigma_e(T_1) = \sigma_e(T_2)$. This result was generalised to semifinite factors by Zsido [Zsi 75]. For a semifinite factor $\mathcal{A}$ there exists up to multiplication by a positive constant a unique trace and the ideals $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A}, \tau)$ coincide. In [Fil 76] a substantial part of the Brown-Douglas-Fillmore extention theory was generalised to this setting.

Our first result illustrates the lifting of isometries. Recall that an operator $U \in \mathcal{A}$ is called an *isometry* if $U^*U = I$, a *co-isometry* if $UU^* = I$, and *unitary* if it is both an isometry and a co-isometry.

Theorem 3.3 *If $\mathcal{A}$ is a factor and $T \in \mathcal{A}$, then $I - T^*T$ is compact if and only if $T = W + K$ where K is compact and W is either an isometry or a co-isometry with finite null projection.*

PROOF. If $T = W + K$, then $I - T^*T = (I - W^*W) + (W^* + K^*)K + K^*W = I - W^*W + C$ where C is a compact element.

Now, if W is an isometry, then $I - T^*T = C$ which is compact. On the other hand if W is a co-isometry with finite null projection, then it follows from $N_W = I - W^*W$ that $I - T^*T$ is compact.

Conversely, suppose $I - T^*T$ is compact. Then T is left invertible modulo $\mathcal{K}(\mathcal{A})$ and hence left Fredholm. It follows easily from the polar decomposition of T that $|T|$ is Fredholm. Since $I - T^*T = (I - |T|)(I + |T|)$ and $I + |T|$ is invertible it follows that $I - |T|$ is compact. Hence, if $T = U|T|$ is the polar decomposition of T, then

$T = U + K$ for some compact K. N_T is finite, since $TN_T = 0$ and $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A}))$ (cf. Theorem 2.22). Since $\mathcal{A}$ is a factor either $N_T \lesssim I - R_T$ or $I - R_T \lesssim N_T$.
Suppose $N_T \lesssim I - R_T$. We may assume that T is one-to-one for if not, we can add a compact operator to T to obtain a one-to-one operator. To illustrate this:
Let E be a subprojection of $I - R_T$ such that $N_T \sim E \leq I - R_T$ and let V be the partial isometry such that $V^*V = N_T$ and $VV^* = E$. Then $T + VN_T$ is one-to-one. Hence we may assume that T is one-to-one. Then, since $U^*U = I - N_T = I$, U is an isometry. In this case let $W = U$.
On the other hand if $I - R_T \lesssim N_T$, then since $I - R_T = N_{T^*}$ one has that $N_{T^*} \lesssim I - R_{T^*}$. The above argument implies that $T^* = U^* + K^*$ where U^* is an isometry and K is compact. By taking adjoints it follows that $T = U + K$, in which case we choose $W = U$. It follows from

$$\begin{aligned} N_W = I - W^*W &= I - U^*U \\ &= I - R_{T^*} \\ &= N_T \end{aligned}$$

that N_W is finite. ■

In the case when $\mathcal{A}$ is not necessarily a factor we can make use of the index theory discussed in Chapter 1 and arguments similar to those used in Theorem 3.3 to obtain the following result for $T \in \mathcal{A}$. We prove the theorem only for the case where $\mathcal{I}$ is a compact ideal.

Theorem 3.4 (a) *If T is an isometry mod $\mathcal{I}$, then $\pi_{\mathcal{I}}(T)$ has an isometry lifting if and only if ind(T) ≤ 0.*

(b) *If T is a co-isometry mod $\mathcal{I}$, then $\pi_{\mathcal{I}}(T)$ has a co-isometry lifting if and only if ind(T) ≥ 0.*

(c) *If T is a unitary mod $\mathcal{I}$, then $\pi_{\mathcal{I}}(T)$ has a unitary lifting if and only if ind(T) = 0.*

PROOF. (a) Suppose $I - T^*T \in \mathcal{I}$.

Suppose there exists an isometry $S \in \mathcal{A}$ such that $\pi_{\mathcal{I}}(S) = \pi_{\mathcal{I}}(T)$, i.e. $T - S = K \in \mathcal{I}$. Then $ind(T) = ind(S + K) = ind(S)$ (see Lemma 1.11). Any isometry is one-to-one and therefore $N_S = 0$, so that $N_S \lesssim N_{S^*}$, and hence $ind(S) \leq 0$.

Conversely, suppose $ind(T) \leq 0$. Then $N_T \lesssim N_{T^*}$. As in the proof of Theorem 3.3 we can show that $I - |T| \in \mathcal{I}$. We can proceed exactly as in the proof of Theorem 3.3, assuming without loss of generality that T is one-to-one and using the polar decomposition of T to obtain an isometry lifting for T.

(b) follows by applying part (a) to T^* and using the fact that $ind(T^*) = -ind(T)$.

(c) The important thing to note in the proofs of (a) and (b) is that the partial isometry U in the polar decomposition of T is both the isometry lifting in (a) and the co-isometry lifting in (b). Then (c) follows immediately. ■

Remark 3.5

By inspection it can be seen that most of these lifting theorems still hold relative to any closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$. In fact one can even show that an invertible element mod $\mathcal{I}$ has an invertible lifting if and only if $ind(T) = 0$. To prove this result relative to any ideal one makes extensive use of the index map introduced in [Ols 84]. We only prove the result for any compact ideal $\mathcal{I}$ in a semifinite von Neumann algebra $\mathcal{A}$. ■

Theorem 3.6 *Let $\mathcal{I}$ be a compact ideal in $\mathcal{A}$. If $\pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ is invertible, then it has an invertible lifting if and only if $ind(T) = 0$.*

PROOF. $T \in \Phi(\mathcal{A}, \mathcal{I})$ since $\pi_{\mathcal{I}}(T)$ is invertible.

Suppose there exists an invertible operator $S \in \mathcal{A}$ such that $\pi_{\mathcal{I}}(S) = \pi_{\mathcal{I}}(T)$, i.e. $T - S = K \in \mathcal{I}$. Then $ind(T) = ind(S + K) = ind(S)$ (see Lemma 1.11), so that $ind(T) = 0$ follows from the fact that S invertible implies S^* invertible.

Conversely, let $ind(T) = 0$, i.e. $N_T \sim N_{T^*}$ (see Lemma 1.11). Let V be a partial isometry such that $V^*V = N_T$ and $VV^* = N_{T^*}$.

First assume $T(\mathcal{H})$ is closed. Let $x \in \mathcal{H}$, say $x = y + z$ with $y \in N_T(\mathcal{H})$ and $z \in (I - N_T)(\mathcal{H})$. Then $(T + VN_T)(x) = Tz + Vy$. So $\|(T + VN_T)(x)\|^2 = \|Tz + Vy\|^2 = \|Tz\|^2 + \|Vy\|^2$, since it follows from $V(\mathcal{H}) = N_{T^*}(\mathcal{H}) = (I - R_T)(\mathcal{H})$ that $T(\mathcal{H}) \perp V(\mathcal{H})$. Now $\|Vy\|^2 = \|y\|^2$, from the definition of V, and $\|Tz\|^2 \geq c\|z\|^2$: For note that $T^{-1} : T((I - N_T)(\mathcal{H})) \to (I - N_T)(\mathcal{H})$ exists and is bounded from the Open Mapping Theorem. Hence we have that $\|(T+VN_T)(x)\|^2 \geq min\{c,1\}(\|z\|^2 + \|y\|^2) = min\{c,1\}\|x\|^2$. Therefore $T + VN_T$ is bounded from below on $\mathcal{H}$ and $(T + VN_T)^{-1}$ exists. Now since $N_T \in \mathcal{I}$ it follows that $T = (T + VN_T) - VN_T$ is of the required form.

Suppose $T(\mathcal{H})$ is not closed. Since T is Fredholm [Ols 84, Theorem 4.7] implies that there exists a $Q \in \mathcal{A}^p$ such that $I - Q \in \mathcal{I}$ and $Q(\mathcal{H}) \subset T(\mathcal{H})$. From Lemma 1.11 $QT = T-(I-Q)T$ is Fredholm with index $ind(QT) = 0$. Now QT has closed range: Let $y \in \overline{QT(\mathcal{H})}$, then there exists a sequence (x_n) in $\mathcal{H}$ such that $QTx_n \rightarrow y$. Since $Q(\mathcal{H})$ is closed it follows that $y \in Q(\mathcal{H}) \subset T(\mathcal{H})$. Hence there exists an $x \in \mathcal{H}$ such that $y = Tx$ and thus $QTx_n \rightarrow QTx$. Hence $y = QTx \in QT(\mathcal{H})$.

Applying the above argument to QT there exists an invertible operator $S \in \mathcal{A}$ and a $K \in \mathcal{I}$ such that $QT = S + K$. Hence $T = S + K + (I - Q)T$ and since $I - Q \in \mathcal{I}$ the result follows. ■

From the proof of Theorem 3.6 it is clear that we can also formulate the result as follows: *If $T \in \mathcal{A}$ is Fredholm relative to $\mathcal{I}$, then $T = S + K$, where S is invertible and $K \in \mathcal{I}$, if and only if $ind(T) = 0$.* This decomposition of Fredholm operators with zero index is quite an important result in the classical theory where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{I} = \mathcal{K}(\mathcal{H})$ (see [Ber 70, Corollary 2.8]).

Consider $m_{\mathcal{I}}(T) = \inf \sigma(\pi_{\mathcal{I}}(|T|))$ and let $m(T) := \inf \sigma(|T|)$.

Theorem 3.7 *For $T \in \mathcal{A}$ and $\mathcal{I}$ any closed two-sided ideal in $\mathcal{A}$*

$$m_{\mathcal{I}}(T) = \sup_{K \in \mathcal{I}} m(T - K).$$

PROOF. $m_{\mathcal{I}}(T) = m_{\mathcal{I}}(T + K) \geq m(T + K)$ for each $K \in \mathcal{I}$.

Hence $m_{\mathcal{I}}(T) \geq \sup_{K \in \mathcal{I}} m(T + K)$.

If $m_{\mathcal{I}}(T) = 0$ we are done. Therefore we may assume that $m_{\mathcal{I}}(T) > 0$, i.e. $\pi_{\mathcal{I}}(T)$ is left invertible in $\mathcal{A}/\mathcal{I}$. Let $E_{(.)}$ denote the spectral measure of $|T|$. Since $m_{\mathcal{I}}(T) > 0$, it follows from Lemma 1.3 that $E_{[0,\beta)} \in \mathcal{I}$ for any $\beta < m_{\mathcal{I}}(T)$. Hence $TE_{[0,\beta)} \in \mathcal{I}$ and $m(T - TE_{[0,\beta)}) \geq \beta$ for every $\beta < m_{\mathcal{I}}(T)$. Hence $\sup_{K \in \mathcal{I}} m(T - K) \geq m_{\mathcal{I}}(T)$ and the result follows. ■

It is not clear whether the supremum in Theorem 3.7 is attained. However, if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{I} = \mathcal{K}(\mathcal{H})$, then we have the following lifting result.

Theorem 3.8 *If $T \in \mathcal{B}(\mathcal{H})$, then there exists a $K \in \mathcal{K}(\mathcal{H})$ such that $m_{\mathcal{K}(\mathcal{H})}(T) = m(T - K)$.*

PROOF. Let $T = U|T|$ be the polar decomposition of T. The Stampfli decomposition theorem ensures the existance of a compact operator C such that $\sigma(|T| - C) = \sigma(\pi_{\mathcal{K}(\mathcal{H})}(|T|))$. Studying the proofs in [Sta 74, Lemma 5 and Theorem 4] carefully we notice that since $|T|$ is a positive operator, the operator C is self-adjoint. In fact C is of the form $\sum_{n=1}^{\infty} \lambda_n P_n$ where $\lambda_n \in \mathbf{R}$ for each n, $\lambda_n \to 0$

and the P_n's are finite rank spectral projections of $|T|$ corresponding to the isolated eigenvalues with finite multiplicity. Let $K = UC$ and compute

$$\begin{aligned} |T-K|^2 &= (T-K)^*(T-K) \\ &= |T|^2 - T^*UC - CU^*T + CU^*UC \\ &= |T|^2 - |T|C - C|T| + C^2 \\ &= (|T| - C)^2 \end{aligned}$$

Note that since the P_n's are spectral projections of $|T|$, $U^*UC = C$ and $|T|C = C|T|$. Further, since $|T| - C$ is positive, it follows from the uniqueness of the square root of positive elements that $|T-K| = |T| - C$. Therefore $\sigma(|T-K|) = \sigma(\pi_{\mathcal{K}(\mathcal{H})}(|T|))$. Hence $m_{\mathcal{K}(\mathcal{H})}(T) = m(T-K)$. ■

Theorem 3.9 *Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{A}$. If $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ is left invertible, then there exists a $K \in \mathcal{I}$ such that $T - K$ is left invertible in $\mathcal{A}$.*

PROOF. The result follows directly from Theorem 1.7, Theorem 3.7 and the fact that $m(T) = m_{\{0\}}(T)$ where $\{0\}$ is the zero ideal in $\mathcal{A}$. ■

Theorem 3.10 *Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{A}$. If $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ is a left topological divisor of zero (i.e. $T \in \mathcal{Z}_l(\mathcal{A}, \mathcal{I})$), then T is a left topological divisor of zero in $\mathcal{A}$.*

PROOF. This follows directly by Proposition 2.2 and the fact that $m(T) \leq m_{\mathcal{I}}(T) = \sup_{K \in \mathcal{I}} m(T + K)$. ∎

We obtain a similar result for the left algebraic divisors of zero.

Theorem 3.11 *Let $\mathcal{I}$ be any closed two-sided ideal in $\mathcal{A}$. If $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ is a left algebraic divisor of zero (i.e. $T \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$), then T is a left algebraic divisor of zero in $\mathcal{A}$.*

PROOF. We claim that $T \in \mathcal{N}_l(\mathcal{A}, \mathcal{I})$ implies that $N_T \neq 0$. For suppose $N_T = 0$ and let $S \in \mathcal{A}$ be such that $TS = 0$. Then $S(\mathcal{H}) \subset N_T(\mathcal{H}) = \{0\}$. Hence $S = 0$. This shows that T is not a left algebraic divisor of zero in $\mathcal{A}$. By Theorem 2.25 T is left invertible in $\mathcal{A}$. Hence $T \in \Phi_l(\mathcal{A}, \mathcal{I}) \subset (\mathcal{N}_l(\mathcal{A}, \mathcal{I}))^c$ which is a contradiction. Now since $TN_T = 0$, T is a left algebraic divisor of zero. ∎

Remarks 3.12

1. By replacing T by T^* in Theorems 3.7, 3.9, 3.10 and 3.11 similar lifting results for right invertible elements, right topological divisors of zero and right algebraic divisors of zero can be obtained.

2. From Theorem 3.11 and (1) above we obtain the following result: *If $T \in \mathcal{N}(\mathcal{A}, \mathcal{I}) = \mathcal{N}_l(\mathcal{A}, \mathcal{I}) \cap \mathcal{N}_r(\mathcal{A}, \mathcal{I})$, then T is an algebraic divisor of zero in $\mathcal{A}$, i.e. T is a left and a right algebraic divisor of zero.* ∎

There exist many examples in the literature of an invertible element $a = \pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ which has no invertible lifting. From Theorem 3.9 and Remarks 3.12(1) we can easily find elements K_1 and K_2 in $\mathcal{I}$ such that $T - K_1$ is left invertible and $T - K_2$ is right invertible in $\mathcal{A}$. However, to ensure that we can choose a single element $K \in \mathcal{I}$ such that $T - K$ is left and right invertible we have to assume that $ind(T) = 0$; cf. the proof of Theorem 3.6.

It is also interesting to ask whether lifting results for the algebraic divisors of zero in a general C^*-algebra hold. The next lemma is probably well known but for completeness we illustrate it by a proof.

Lemma 3.13 *Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras.*

(a) *If $a, b \in \mathcal{C}$ are positive elements with $ab = ba = 0$, then $|a - b| = a + b$.*

(b) *If $\phi : \mathcal{C} \longrightarrow \mathcal{D}$ is a *-homomorphism and $a \in \mathcal{C}$ is positive, then $(\phi(a))^{1/2} = \phi(a^{1/2})$.*

PROOF. (a) follows directly from the following computation and the fact that the square root of a positive element is unique:

$$\begin{aligned} |a-b|^2 = (a-b)^2 &= a^2 - ab - ba + b^2 \\ &= a^2 + ab + ba + b^2 \\ &= (a+b)^2. \end{aligned}$$

(b) Let $\mathcal{C}^+$ denote the positive elements in $\mathcal{C}$. Now (b) follows since $\phi(\mathcal{C}^+) \subset \mathcal{D}^+$, $(\phi(a^{1/2}))^2 = \phi((a^{1/2})^2) = \phi(a)$ and by the uniqueness of the square root of a positive element. ∎

We noticed after some investigation was done that the following theorem can be deduced from [AkP 77, Proposition 2.3]. We however indicate how the proof of their result can be modified to obtain the lifting theorem.

Theorem 3.14 *Let $\mathcal{I}$ be any closed two-sided ideal in a C^*-algebra $\mathcal{C}$. If $a = \pi_{\mathcal{I}}(T) \in \mathcal{C}/\mathcal{I}$ is a left algebraic divisor of zero, then there exists a $K \in \mathcal{I}$ such that $T - K$ is a left algebraic divisor of zero in $\mathcal{C}$.*

PROOF. Let $S \notin \mathcal{I}$ be such that $TS \in \mathcal{I}$.

We first assume that both T and S are positive elements in $\mathcal{C}$. Clearly $\pi_{\mathcal{I}}(T)$ and $\pi_{\mathcal{I}}(S)$ commute in $\mathcal{C}/\mathcal{I}$ and $\pi_{\mathcal{I}}(T)\pi_{\mathcal{I}}(S) = 0$. Let

$$A = \frac{T-S}{2} + \frac{|T-S|}{2}$$

and

$$B = \frac{T-S}{2} - \frac{|T-S|}{2}.$$

Then $AB = BA = 0$ and by applying Lemma 3.13 we obtain $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(T)$ and $\pi_{\mathcal{I}}(B) = \pi_{\mathcal{I}}(S)$. It is important to note that both A and B are non-zero, in fact A and B are not in $\mathcal{I}$, since $A \in \mathcal{I}$ would imply $T \in \mathcal{I}$ and $B \in \mathcal{I}$ would imply $S \in \mathcal{I}$.

For the general case let $T = U|T|$ and $S^* = V|S^*|$ denote the polar decompositions of T and S^* in the enveloping W^*-algebra $\mathcal{C}^{**}$. By assumption $TS \in \mathcal{I}$, so by Gelfand theory and since $\mathcal{I}$ is closed,

$$|T|^{1/2}|S^*|^{1/2} = \lim_{\varepsilon \to 0}(\varepsilon + |T|)^{-3/2}(T^*T)(SS^*)(\varepsilon + |S^*|)^{-3/2} \in \mathcal{I}.$$

Clearly $|S^*|^{1/2} \notin \mathcal{I}$ so by the first part of the proof there exist A_1, B_1 non-zero, such that $A_1B_1 = B_1A_1 = 0$ and $\pi_\mathcal{I}(A_1) = \pi_\mathcal{I}(|T|^{1/2})$ and $\pi_\mathcal{I}(B_1) = \pi_\mathcal{I}(|S^*|^{1/2})$. Now let $A = U|T|^{1/2}A_1$ and $B = B_1|S^*|^{1/2}V^*$. From [KaR 86, Exercise 10.5.11] $U|T|^{1/2}$ and $|S^*|^{1/2}V^*$ are elements of $\mathcal{C}$. Therefore $T - A = U|T|^{1/2}(|T|^{1/2} - A_1) \in \mathcal{I}$ and $S - B = (|S^*|^{1/2} - B_1)|S^*|^{1/2}V^* \in \mathcal{I}$, so that $\pi_\mathcal{I}(A) = \pi_\mathcal{I}(T)$ and $\pi_\mathcal{I}(B) = \pi_\mathcal{I}(S)$. Moreover, $AB = U|T|^{1/2}A_1B_1|S^*|^{1/2}V^* = 0$ and A and B are non-zero. ∎

Remark 3.15

By taking adjoints it is clear that a similar lifting theorem holds for right algebraic divisors of zero. It is however not clear from the proof of Theorem 3.14 that if $a = \pi_\mathcal{I}(T) \in \mathcal{C}/\mathcal{I}$ is an algebraic divisor of zero, then there exists a $K \in \mathcal{I}$ such that $T - K$ is an algebraic divisor of zero in $\mathcal{C}$, i.e. both a left and a right algebraic divisor of zero.

We therefore state the following question in general C^*-algebra theory:

Is it possible in general to lift algebraic divisors of zero in $\mathcal{C}/\mathcal{I}$ to $\mathcal{C}$, and if not under which conditions will such a lifting theorem hold?

Recall from Theorem 3.6 that although left and right invertibility in von Neumann algebra theory can be lifted without any conditions, the lifting of invertibility only holds for those elements $\pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ with $ind(T) = 0$. ■

We conclude this chapter with a few remarks on regular elements in von Neumann algebras.

In [HaM 92, HaM 93] Harte and Mbekhta introduced the notion of generalised inverses in C^*-algebras. Let $\mathcal{C}$ denote a C^*-algebra. An element $a \in \mathcal{C}$ is called *regular* if there exists an element $b \in \mathcal{C}$ such that $a = aba$. We denote the class of regular elements in $\mathcal{C}$ by $Reg(\mathcal{C})$. The element b is called a *generalised inverse* of a. It is clear from the definition that $ab = (ab)^2$ and $ba = (ba)^2$. In [HaM 92] it is shown that there exists a unique $b \in \mathcal{C}$ such that ab and ba are projections (i.e. self-adjoint idempotents). In this case b is called the *Moore-Penrose inverse* of a.

Moreover, if $a \in \mathcal{C}$ and $L_a : \mathcal{C} \longrightarrow \mathcal{C}$ is defined by $L_a(b) = ab$, then it is shown in [HaM 92] that a is regular if and only if L_a has a closed range, generalising a well-known result of Atkinson for operators on a Hilbert space, viz. that an operator T is regular if and only if $T(\mathcal{H})$ is closed. For $a \in \mathcal{C}$ let $\gamma(a) := \inf \sigma(|a|)\backslash\{0\}$. It is shown in [HaM 93] that $a \in \mathcal{C}$ is regular if and only if $\gamma(a) > 0$ (i.e. 0 is not a point of accumulation of $\sigma(|a|)$).

In [Str 94, MbP 95] it was shown independently with different proofs that

$$\gamma(\pi_{\mathcal{K}(\mathcal{H})}(T)) = \sup_{K\in\mathcal{K}(\mathcal{H})} \gamma(T+K).$$

Hence any regular element in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ has a regular lifting in $\mathcal{B}(\mathcal{H})$. In fact, it was shown in [Str 94] that the supremum above is attained, solving an open problem stated in [MbP 95].

It was shown in [LSS 95] that if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{I}$ any closed ideal in $\mathcal{A}$, then $\gamma(\pi_{\mathcal{I}}(T)) = \sup_{K\in\mathcal{I}} \gamma(T+K)$. Hence the following lifting theorem:

Theorem 3.16 *Let $T \in \mathcal{A}$ such that $\pi_{\mathcal{I}}(T) \in \mathcal{A}/\mathcal{I}$ is regular, then there exists a $K \in \mathcal{I}$ such that $T - K$ is regular in $\mathcal{A}$.*

PROOF. Suppose $\pi_{\mathcal{I}}(T)$ is regular. Then $\gamma(\pi_{\mathcal{I}}(T)) > 0$. Hence there exists a $K \in \mathcal{I}$ such that $\gamma(T-K) > 0$. So $T-K$ is regular in $\mathcal{A}$. ■

Remarks 3.17

1. Theorem 3.16 is not valid for general C^*-algebras and general closed ideals: Let $\mathcal{C} = C[0,1]$ and $\mathcal{I} = \{f \in C[0,1] : f(0) = f(1) = 0\}$. If we choose $h(t) = t$, as in Remark 3.2, then $h + \mathcal{I}$ is a non-trivial projection, hence a regular element of $\mathcal{C}/\mathcal{I}$. However, since for any $f \in \mathcal{I}$ $(h+f)(0) = 0$, $h + f$ is not invertible in $\mathcal{C}$. Since for this algebra the nonzero regular elements are exactly the invertible functions, $h + f$ is not regular for every $f \in \mathcal{I}$. Hence $h + \mathcal{I}$ has no regular lifting.

2. If $\mathcal{C} = \mathcal{B}(\mathcal{H})$ and $\mathcal{I} = \mathcal{K}(\mathcal{H})$, it follows from the well-known Atkinson theorem that $\Phi_u(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})) \subset Reg(\mathcal{C})$. Hence $Reg(\mathcal{B}(\mathcal{H}))$ is dense in $\mathcal{B}(\mathcal{H})$. In general von Neumann algebras it is not in general true that $\Phi_u(\mathcal{A}, \mathcal{I})$ is contained in $Reg(\mathcal{A})$. If $T \in \Phi_u(\mathcal{A}, \mathcal{I})$, say T is right Fredholm, then by [Ols 84, Theorem 4.5] there exists a projection $E \in \mathcal{I}$ such that $(I - E)(\mathcal{H}) \subset T(\mathcal{H})$. Hence from $T = (I - E)T + ET$ it follows that T is a sum of a regular element and an element in $\mathcal{I}$, i.e. $\Phi_u(\mathcal{A}, \mathcal{I}) \subset Reg(\mathcal{A}) + \mathcal{I}$. ■

We complete the investigation in this chapter with more open questions.

Open questions.

1. *From Remarks 3.17(1) we have seen that regular liftings do not hold relative to any closed ideal in C^*-algebras. In [Str 94] it was shown that if the ideal $\mathcal{I}$ is the closure of the socle in the algebra, then regular elements can be lifted. Hence our question is to characterise the closed two-sided ideals in a C^*-algebra $\mathcal{C}$ for which a regular lifting theorem exists.*

2. *Is Theorem 3.3 still valid in the case where $\mathcal{A}$ is not necessarily a factor?*

3. *Is the supremum attained in Theorem 3.7 for general von Neumann algebras and general ideals?*

4. *For a general C^*-algebra $\mathcal{C}$ and closed ideal $\mathcal{I}$ in $\mathcal{C}$, can any topological divisor of zero in $\mathcal{C}/\mathcal{I}$ be lifted to $\mathcal{C}$?*

5. *We call* $a \in \mathcal{A}/\mathcal{I}$ quasinilpotent *if* $\sigma(a) = \{0\}$, *i.e.* $r(a) = 0$, *where* r *denotes the spectral radius. The question whether a quasinilpotent element of* $\mathcal{A}/\mathcal{I}$ *has a quasinilpotent lifting is still unresolved in general. For the case where* $\mathcal{A} = \mathcal{B}(\mathcal{H})$ *and* $\mathcal{I} = \mathcal{K}(\mathcal{H})$ *we have the well-known West decomposition theorem [Wes 66], which states that any quasinilpotent element in* $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ *has a quasinilpotent lifting. The method of proof of this result is analogous to the process of super-diagonalizing a matrix and then splitting it into the sum of diagonal and nilpotent matrices. Super-diagonalization of compact operators depends essentially on the existence of proper closed invariant subspaces.*

   The best result in this direction can be found in Rogers [Rog 90]. In this paper Rogers proved that if $\mathcal{I}$ *is any closed two-sided ideal in a* C^**-algebra* $\mathcal{C}$, *then a quasinilpotent element* $\pi_{\mathcal{I}}(T)$ *of* $\mathcal{C}/\mathcal{I}$ *has a quasinilpotent lifting if* $\sigma(T)$ *is totally disconnected.*

   The open question mentioned would be answered in the affirmative if one could 'lift' the spectral radius for arbitrary elements of $\mathcal{A}/\mathcal{I}$. *In other words we need to prove that for any* $T \in \mathcal{A}$ *there exists a* $K \in \mathcal{I}$ *such that* $r(T - K) = r(\pi_{\mathcal{I}}(T))$, *where* $r(T)$ *denotes the spectral radius of* T. *In [AkP 77] Akemann and Pedersen proved this result for any* T *with* $r(\pi_{\mathcal{I}}(T)) > 0$. *Further partial answers to the question under consideration can be found in [StS 91].*

Chapter 4

UNBOUNDED OPERATORS

The Fredholm theory for unbounded operators on Banach spaces is well established and plays an intimate role in the theory of differential equations [Gol 66]. We show that relative to a von Neumann algebra $\mathcal{A}$ we can obtain a Fredholm theory for a certain class of closed densely defined operators "affiliated" to the algebra $\mathcal{A}$. To obtain similar results on the algebraic and topological properties of the Fredholm classes involved, as in Chapter 2, we shall restrict the classes of unbounded operators to a certain extent. We start by introducing the relevant concepts. Most of the preliminary results in this chapter are known. We however need to include the proofs to illustrate the techniques which will be used in our results.

Let $\mathcal{A}$ be a semifinite von Neumann algebra with a distinguished faithful semifinite normal trace $\tau : \mathcal{A}^+ \to [0, \infty]$, where $\mathcal{A}^+$ denotes the positive cone of $\mathcal{A}$. A closed operator S with domain $\mathcal{D}(S)$ dense in $\mathcal{H}$ is said to be *affiliated to* $\mathcal{A}$ if $RS \subset SR$ for all R in the commutant $\mathcal{A}'$ of $\mathcal{A}$. Any closed densely defined operator S on $\mathcal{H}$ has a polar decomposition $S = U|S|$, and if S is affiliated to $\mathcal{A}$, then $U \in \mathcal{A}$, $|S|$ is affiliated to $\mathcal{A}$, and the spectral projections $E_{(.)}$ of $|S| = \int_0^\infty \lambda \, dE_\lambda(|S|)$ are in $\mathcal{A}$.

We call an affiliated operator S *τ-measurable* if for every $t > 0$ there exists a $P \in \mathcal{A}^p$ such that $P(\mathcal{H}) \subset \mathcal{D}(S)$ and $\tau(I - P) \leq t$. Denote the class of τ-measurable operators by $\widetilde{\mathcal{A}}$. For T, $S \in \widetilde{\mathcal{A}}$ it is shown in [Nel 74] that $T + S$ and TS are closable and densely defined. By defining the sum and product of T, $S \in \widetilde{\mathcal{A}}$ to be the closure of $T + S$ and TS respectively, and the *-operation the usual adjoint of T, it was shown in [Nel 74] that $\widetilde{\mathcal{A}}$ is a *-algebra. We shall denote this sum and product also by $T + S$ and TS. The sets

$$\widetilde{\mathcal{A}}(\varepsilon, t) = \{S \in \widetilde{\mathcal{A}}\text{: there exists a } P \in \mathcal{A}^p \text{ such that } \|SP\| \leq \varepsilon \text{ and } \tau(I - P) \leq t\} \cdot$$

form a neighbourhood base at 0 (where $\varepsilon, t > 0$ are allowed to vary) for a metrisable vector topology τ_{cm} on $\widetilde{\mathcal{A}}$, called the topology of convergence in measure. Equipped with this topology, $\widetilde{\mathcal{A}}$ is a complete topological *-algebra containing $\mathcal{A}$. For proofs of these facts the reader may consult [Nel 74] or [Ter 81]. In fact, $\mathcal{A}$ is dense in $\widetilde{\mathcal{A}}$, for if $S = U|S| \in \widetilde{\mathcal{A}}$ and $|S| = \int_0^\infty \lambda \, dE_\lambda(|S|)$, then the sequence $\{U \int_0^n \lambda \, dE_\lambda(|S|)\}_{n=1,2,\ldots}$ in $\mathcal{A}$ tends to S in the topology of convergence in measure as $n \to \infty$. S is τ-measurable if and only if $\tau(E_{(\lambda,\infty)}) < \infty$ for λ large enough, or equivalently, $\lim_{\lambda\to\infty} \tau(E_{(\lambda,\infty)}) = 0$, since τ is normal [Ter 81, Proposition I.21].

This leads to the following lemma which provides a useful characterisation of τ-measurable operators in terms of projections.

Lemma 4.1 *An affiliated operator S is τ-measurable if and only if there exists a non-decreasing sequence (E_n) in $\mathcal{A}^p$ satisfying*

(a) $E_n(\mathcal{H}) \subset \mathcal{D}(S)$ *for all n,*

(b) $\sup_n E_n = I$, *and*

(c) $\tau(I - E_m) < \infty$ *for some m*

Proof. Since $S \in \tilde{\mathcal{A}}(\varepsilon, t)$ if and only if $|S| \in \tilde{\mathcal{A}}(\varepsilon, t)$ it suffices to prove the result for positive operators. Suppose $S = \int_0^\infty \lambda \, dE_\lambda$ is τ-measurable and let $E_n = E_{[0,n]}$. Then (E_n) is non-decreasing and $E_n(\mathcal{H}) \subset \mathcal{D}(S)$ for all n. Since S is τ-measurable there exists an $m \in \mathbf{N}$ such that $\tau(I - E_m) = \tau(E_{(m,\infty)}) < \infty$. Furthermore, it is clear that $E_n \uparrow E_{[0,\infty)} = I$.

Conversely, suppose (a), (b) and (c) hold. From (c) there exists an $m \in \mathbf{N}$ such that $\tau(I - E_m) < \infty$. It follows from $E_n \uparrow I$ that $\tau(I - E_n) \leq \tau(I - E_m) < \infty$ for all $n \geq m$. By normality of τ one has $\lim_{n\to\infty} \tau(I - E_n) = 0$ and hence for any $\varepsilon > 0$ there exists a projection E_{n_0} such that $E_{n_0}(\mathcal{H}) \subset \mathcal{D}(S)$ and $\tau(I - E_{n_0}) < \varepsilon$. So S is τ-measurable. ■

Let $\mathcal{K}(\tilde{\mathcal{A}}, \tau)$ denote the closure of $\mathcal{A}_\tau$ (defined on page 35) in the topology of convergence in measure and recall that $\mathcal{K}(\mathcal{A}, \tau)$ is the norm closure of $\mathcal{A}_\tau$. Then $\mathcal{K}(\tilde{\mathcal{A}}, \tau)$

is a τ_{cm}-closed two-sided *-ideal in $\tilde{\mathcal{A}}$ that behaves somewhat like the ideal of compact operators on $\mathcal{H}$. For this reason we call it the *τ_{cm}-compact ideal* in $\tilde{\mathcal{A}}$ and its elements are called the *τ_{cm}-compact operators* of $\tilde{\mathcal{A}}$.

Examples 4.2

1. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with the canonical trace, then $\tilde{\mathcal{A}} = \mathcal{A}$ and $\mathcal{K}(\tilde{\mathcal{A}}, \tau) = \mathcal{K}(\mathcal{H})$.
2. Let $\mathcal{A} = l^\infty$ with the canonical trace, then $\tilde{\mathcal{A}} = \mathcal{A}$ and $\mathcal{K}(\tilde{\mathcal{A}}, \tau) = c_0$.
3. Let $\mathcal{A} = \mathcal{L}^\infty(X, \Sigma, \mu)$ and $\tau = \int . \, d\mu$, then $\tilde{\mathcal{A}} = \{f : X \to \mathbf{R}^*$ measurable function : f bounded except on a set of finite measure$\}$, where $\mathbf{R}^*$ denotes the extended real number system. $\mathcal{A}_\tau = \mathcal{L}^\infty(X, \Sigma, \mu) \cap \mathcal{L}^1(X, \Sigma, \mu)$ and $\mathcal{K}(\tilde{\mathcal{A}}, \tau) = \{f \in \tilde{\mathcal{A}} : \mu\{x \in X : |f(x)| > t\} < \infty$ for all $t > 0\}$. ■

We shall see that a *-algebra norm is canonically induced on the quotient algebra $\tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}}, \tau)$. We call this algebra with its norm the *τ_{cm}-Calkin algebra.* It was shown in [StW 93] that $\tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}}, \tau)$ is a C^*-algebra. Moreover, it is *-isomorphic to $\mathcal{A}/\mathcal{K}(\mathcal{A}, \tau)$. Since this result will play an important role in our work, we shall include a proof.

Fack [Fac 82] developed the notion of generalised singular numbers for bounded operators. Kosaki [Kos 84] showed some applications of its generalisation to τ-measurable operators. For $t > 0$ we define the 't th' *generalised singular number* of

$S \in \tilde{\mathcal{A}}$ by

$$\mu_t(S) := \inf\{\|SP\| : P \in \mathcal{A}^p \text{ with } \tau(I-P) \leq t\} = \inf\{s \geq 0 : \tau(E_{(s,\infty)}) \leq t\}$$

where $E_{(.)}$ denotes the spectral measure of $|S|$. This function characterises the topology of convergence in measure as we shall see in the following lemma due to Fack and Kosaki [FaK 86, Lemma 3.1]:

Lemma 4.3 *A sequence (S_n) in $\tilde{\mathcal{A}}$ converges to $S \in \tilde{\mathcal{A}}$ in the measure topology if and only if*

$$\lim_{n\to\infty} \mu_t(S_n - S) = 0 \quad \textit{for each } t > 0.$$

∎

According to [FaK 86, Lemma 2.5] the map $t \in (0,\infty) \mapsto \mu_t(S)$ is non-increasing and continuous from the right with $\lim_{t\downarrow 0} \mu_t(S) = \|S\| \in [0,\infty]$. Using the notation of [StW 93] we define

$$\mu_\infty(S) := \lim_{t\to\infty} \mu_t(S) = \inf_{t>0} \mu_t(S) = \inf\{\|SP\| : P \in \mathcal{A}^p \text{ with } \tau(I-P) < \infty\}.$$

It can be shown that μ_∞ is a continuous *-algebra semi-norm on $\tilde{\mathcal{A}}$ with kernel $\mathcal{K}(\tilde{\mathcal{A}},\tau)$.

Hence a *-algebra norm, which we shall also denote by μ_∞, is canonically induced on the τ_{cm}-Calkin algebra $\tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}},\tau)$.

The results contained in the following lemma will be used a few times and we include proofs for some of them.

Lemma 4.4 (a) *For any* $P \in \mathcal{A}^p$ $P \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$ *iff* $\tau(P) < \infty$.

(b) $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{K}(\tilde{\mathcal{A}}, \tau)$, *i.e. any* $S \in \tilde{\mathcal{A}}$ *admits a decomposition* $S = T + K$ *with* $T \in \mathcal{A}$ *and* $K \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$.

Proof. (a) For any $P \in \mathcal{A}^p$, and $t \in (0, \infty)$, $\mu_t(P) = \chi_{(0,\tau(P))}(t)$, so that $\mu_\infty(P)$ is either 0 or 1.

If $P \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$, then $\mu_\infty(P) = 0$ and it follows that $\tau(P) < \infty$.

Conversely, if $\tau(P) < \infty$, then $\mu_\infty(P) = \lim_{t\to\infty} \mu_t(P) = \lim_{t\to\infty} \chi_{(0,\tau(P))}(t) = 0$. Hence $P \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$.

(b) It follows directly from the definition of τ-measurability that any $S \in \tilde{\mathcal{A}}$ admits a decomposition $SP + S(I - P)$, where $P \in \mathcal{A}^p$, $\tau(I - P) < \infty$, and $SP \in \mathcal{A}$. So, since $I - P \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$, $S(I - P) \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. It follows that $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. ∎

From Lemma 4.4(a) it follows that if $\tau(I) = \infty$ then $\tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}}, \tau)$ is non-trivial and has unit $I + \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. We henceforth assume this to be the case.

In [StW 93] Ströh and West introduced Fredholm operators in $\tilde{\mathcal{A}}$ relative to $\mathcal{K}(\tilde{\mathcal{A}}, \tau)$. Call $S \in \tilde{\mathcal{A}}$ *Fredholm relative to* $\mathcal{K}(\tilde{\mathcal{A}}, \tau)$ if S is invertible modulo $\mathcal{K}(\tilde{\mathcal{A}}, \tau)$. In accordance with the notation in Chapter 1 we denote the class of Fredholm operators by $\Phi(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$. The classes $\Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$, $\Phi_r(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$, $\mathcal{N}_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$,

$\mathcal{N}_r(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau))$, $\mathcal{Z}_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau))$, and $\mathcal{Z}_r(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau))$ are defined similarly as in Chapter 1 and Chapter 2. The first statement of the following proposition can be found in [StW 93].

Proposition 4.5 (a) $\mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A} \; = \; \mathcal{K}(\mathcal{A},\tau)$

(b) $\Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau)) \cap \mathcal{A} \; = \; \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$

(c) $\mathcal{N}_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau)) \cap \mathcal{A} \; = \; \mathcal{N}_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$.

PROOF. (a) Since convergence in norm implies convergence in measure it is clear that $\mathcal{K}(\mathcal{A},\tau) \; \subset \; \mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A}$.

Conversely, let $K = U|K| \in \mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A}$ and define $K_n := U \int_{\frac{1}{n}}^{\|K\|} \lambda \; dE_\lambda(|K|)$. By the Spectral Theorem (K_n) converges to K in the norm topology. Furthermore $supp(K_n) = E_{(\frac{1}{n},\infty)}(|K|)$, and so $\tau(supp(K_n)) = \tau(E_{(\frac{1}{n},\infty)}(|K|)) < \infty$ (since $K \in \mathcal{K}(\tilde{\mathcal{A}},\tau)$). Hence $K_n \in \mathcal{A}_\tau$ for each n, so that $K \in \overline{\mathcal{A}_\tau}^{\|\cdot\|} = \mathcal{K}(\mathcal{A},\tau)$.

(b) Obviously $\Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau)) \; \subset \; \Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau)) \cap \mathcal{A}$. Let $T \in \Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau)) \cap \mathcal{A}$. Then there exists an $S \in \tilde{\mathcal{A}}$ such that $ST - I \in \mathcal{K}(\tilde{\mathcal{A}},\tau)$. From Lemma 4.4(b) $S = R + K$ where $R \in \mathcal{A}$ and $K \in \mathcal{K}(\tilde{\mathcal{A}},\tau)$. Hence $RT - I \in \mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A} \; = \; \mathcal{K}(\mathcal{A},\tau)$ (by part (a)) which implies that $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A},\tau))$.

(c) follows similarly. ∎

The following theorem which is due to [StW 93] provides us with a useful tool to deduce almost all the results of the Fredholm theory of bounded operators for the unbounded case.

Theorem 4.6 *$\tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}},\tau)$ is isometrically *-isomorphic to $\mathcal{A}/\mathcal{K}(\mathcal{A},\tau)$.*

PROOF. For any $S \in \tilde{\mathcal{A}}$ we can decompose $S = T + K$ where $T \in \mathcal{A}$, and $K \in \mathcal{K}(\tilde{\mathcal{A}},\tau)$. Define

$$\phi : \tilde{\mathcal{A}} \to \mathcal{A}/\mathcal{K}(\mathcal{A},\tau) : S \mapsto T + \mathcal{K}(\mathcal{A},\tau).$$

Then ϕ is well defined, for if $S = T_1 + K_1 = T_2 + K_2$ then

$$T_1 - T_2 = K_2 - K_1 \in \mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A} = \mathcal{K}(\mathcal{A},\tau).$$

It is not hard to show that ϕ is a *-algebra homomorphism onto $\mathcal{A}/\mathcal{K}(\mathcal{A},\tau)$, with kernel $\mathcal{K}(\tilde{\mathcal{A}},\tau)$. Therefore a *-algebra isomorphism $\tilde{\phi} : \tilde{\mathcal{A}}/\mathcal{K}(\tilde{\mathcal{A}},\tau) \to \mathcal{A}/\mathcal{K}(\mathcal{A},\tau)$ is induced and $\tilde{\phi}$ is also isometric:

$$\begin{aligned}
\mu_\infty(S + \mathcal{K}(\tilde{\mathcal{A}},\tau)) &= \mu_\infty(S) \\
&= \mu_\infty(T) \\
&= \inf\{\|TP\| : P \in \mathcal{A}^p \text{ and } \tau(I-P) < \infty\} \\
&= dist(T, \mathcal{K}(\mathcal{A},\tau)) \\
&= \|\pi_{\mathcal{K}(\mathcal{A},\tau)}(T)\|
\end{aligned}$$

■

If $S \in \Phi(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau))$ we define the index of S to be $index(S) := dim(N_S) - dim(N_{S^*})$. The index theory from [Ols 84] extends completely to an index theory for elements in $\Phi(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}},\tau))$.

Corollary 4.7 *Let $S \in \tilde{\mathcal{A}}$ and let $T \in \mathcal{A}$ be any operator for which $S-T \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. Then*

(a) $S \in \Phi_l(\tilde{\mathcal{A}},\, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ *if and only if* $T \in \Phi_l(\mathcal{A},\, \mathcal{K}(\mathcal{A}, \tau))$

(b) $S \in \Phi(\tilde{\mathcal{A}},\, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ *if and only if* $T \in \Phi(\mathcal{A},\, \mathcal{K}(\mathcal{A}, \tau))$, *and* $index(S) = ind(T)$

(c) $S \in \mathcal{N}_l(\tilde{\mathcal{A}},\, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ *if and only if* $T \in \mathcal{N}_l(\mathcal{A},\, \mathcal{K}(\mathcal{A}, \tau))$.

PROOF. From Theorem 4.6 we notice that we only need to prove the second statement of (b).

Let $S = U|S|$ be the polar decomposition of S and let $R = |S|E_{[0,\mu_\infty(|S|)]} + \mu_\infty(|S|)E_{(\mu_\infty(|S|),\infty)}$, where $E_{(.)}$ denotes the spectral measure of $|S|$. Then $R \in \mathcal{A}$ and $|S| - R \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$ and hence $S - UR \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. It is clear that $N_{|S|} = N_R$ and therefore $R_{|S|} = R_R$. Now $U^*U = R_R$ so that the operator UR is indeed in polar decomposition form, by the uniqueness clause in the polar decomposition theorem [KaR 86, Theorem 6.1.11]. Hence

$$N_S = N_{|S|} = N_R = N_{UR}$$

and

$$N_{S^*} = I - R_S = I - UU^* = I - R_{UR} = N_{(UR)^*}.$$

It follows that $index(S) = ind(UR)$. Let $T \in \mathcal{A}$ be any operator for which $S - T \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. Then $UR - T = UR - S + S - T \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$ so that, since $UR - T \in \mathcal{A}$, $UR - T = K \in \mathcal{K}(\mathcal{A}, \tau)$. From Lemma 1.11 $ind(UR) = ind(T)$ and hence $index(S) = ind(T)$. ∎

We use Corollary 4.7 and Theorem 2.22 to prove the following characterisation of left and right Fredholm operators. Fillmore, Stampfli and Williams [FSW 72] proved this result for the class of closed operators which are densely defined on a separable infinite dimensional Hilbert space.

Theorem 4.8 *Let* $S \in \tilde{\mathcal{A}}$, *then* $S \in \Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ *if and only if for every* $P \in \mathcal{A}^p$ *with* $SP \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$ *it follows that* P *is finite rank.*

PROOF. Let $S \in \tilde{\mathcal{A}}$ and let $T \in \mathcal{A}$ be any operator for which $S - T \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$.

$S \in \Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ iff $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$

iff for $P \in \mathcal{A}^p$ with $TP \in \mathcal{K}(\mathcal{A}, \tau)$ it follows that $\tau(P) < \infty$

iff for $P \in \mathcal{A}^p$ with $SP \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$ it follows that $\tau(P) < \infty$. ■

Proposition 4.9 $\Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau)) = (\mathcal{N}_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau)))^c$ *and*

$$\Phi_r(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau)) = (\mathcal{N}_r(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau)))^c.$$

PROOF. Let $S \in \tilde{\mathcal{A}}$ and let $T \in \mathcal{A}$ be any operator for which $S - T \in \mathcal{K}(\tilde{\mathcal{A}}, \tau)$. From Corollary 4.7 and Theorem 2.24 $S \in \Phi_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau))$ if and only if $T \in \Phi_l(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau))$ if and only if $T \in (\mathcal{N}_l(\mathcal{A}, \mathcal{K}(\mathcal{A}, \tau)))^c$ if and only if $S \in (\mathcal{N}_l(\tilde{\mathcal{A}}, \mathcal{K}(\tilde{\mathcal{A}}, \tau)))^c$.

The characterisation of the τ-measurable right Fredholm operators follows in a similar way. ■

Remark 4.10

Although very straight forward we can continue to show that almost all the results from Chapter 2 can be extended to this setting. The quantity $m_{\mathcal{K}(\tilde{\mathcal{A}},\tau)}(.)$ will also make sense and moreover, it can be shown that $\Phi_l(\tilde{\mathcal{A}},\ \mathcal{K}(\tilde{\mathcal{A}},\tau)) = (\mathcal{Z}_l(\tilde{\mathcal{A}},\ \mathcal{K}(\tilde{\mathcal{A}},\tau)))^c$.

■

In [Son 71] it was shown that if $\mathcal{A}$ is a factor on a separable Hilbert space, then $\mathcal{K}(\mathcal{A},\tau)$ is the unique closed two-sided ideal in $\mathcal{A}$. On separable Hilbert spaces we have the following generalisation of Calkin's theorem.

Theorem 4.11 *If $\mathcal{A}$ is a factor on a separable Hilbert space, then $\mathcal{K}(\tilde{\mathcal{A}},\tau)$ is the unique two-sided ideal in $\tilde{\mathcal{A}}$ which is closed with respect to the measure topology.*

PROOF. Suppose $\mathcal{J}$ is any closed (in measure) two-sided ideal in $\tilde{\mathcal{A}}$. Then $\mathcal{J} \cap \mathcal{A}$ is clearly a two-sided ideal in $\mathcal{A}$.

We show that $\overline{\mathcal{J} \cap \mathcal{A}}^{\tau cm} = \mathcal{J}$: It is clear that $\overline{\mathcal{J} \cap \mathcal{A}}^{\tau cm} \subset \mathcal{J}$. Using the ideal properties we derive that we may prove the converse inclusion by using positive elements only. Let $S = \int_0^\infty \lambda \, dE_\lambda$ be any positive element in $\mathcal{J}$. Define $S_n = \int_{1/n}^n \lambda \, dE_\lambda$ for $n \in \mathbf{N}$, then $S_n \in \mathcal{J}$, $S_n \in \mathcal{A}$ and $S_n \xrightarrow{\tau cm} S$. Hence $S \in \overline{\mathcal{J} \cap \mathcal{A}}^{\tau cm}$. Therefore we have equality.

If we assume that $\mathcal{J} \cap \mathcal{A}$ is closed with respect to the norm topology, then from [Son 71, Theorem 2.2] $\mathcal{J} \cap \mathcal{A} = \mathcal{K}(\mathcal{A},\tau) = \mathcal{K}(\tilde{\mathcal{A}},\tau) \cap \mathcal{A}$. Hence $\mathcal{J} = \overline{\mathcal{J} \cap \mathcal{A}}^{\tau cm} =$

$\overline{\mathcal{K}(\tilde{\mathcal{A}},\tau)\cap\mathcal{A}}^{\tau cm} = \mathcal{K}(\tilde{\mathcal{A}},\tau)$.

It remains to prove that $\mathcal{J}\cap\mathcal{A}$ is norm closed: Let (T_n) be a sequence in $\mathcal{J}\cap\mathcal{A}$ which converges to T in norm. Clearly $T\in\mathcal{A}$. We need to show that $T\in\mathcal{J}$. From the definition of the 't th' generalised singular number of $T_n - T$ (see page 67) we obviously have $\mu_t(T_n-T)\leq \|T_n-T\|\to 0$ for all $t>0$. Hence $\lim_{n\to\infty}\mu_t(T_n-T)=0$ for all $t>0$. From Lemma 4.3 it follows that $T_n\to T$ in measure. Therefore $T\in\mathcal{J}$. ■

Throughout our study of Fredholm theory for unbounded operators affiliated to a von Neumann algebra we assumed that all operators involved are τ-measurable. It remains an open area to develop a Fredholm theory for closed densely defined operators affiliated to a von Neumann algebra which are not necessarily τ-measurable.

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Fredholm classes in operator algebras

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SUMMARY

In the first part of the thesis the divisors of zero in the "Calkin" algebra of a von Neumann algebra relative to an arbitrary closed ideal are used to define classes of Fredholm-type operators. The geometrical, algebraic and topological properties of these classes are studied completely and shown to be similar to properties possessed by the Fredholm classes. It is also investigated precisely in which cases these classes coincide with the Fredholm classes, and hence useful characterisations of the semi-Fredholm operators in the von Neumann algebra setting are obtained. In the second part of the thesis lifting theorems for a number of properties of elements in the "Calkin" algebra are proved and numerous unsolved problems are stated. The study is concluded with a Fredholm theory for closed densely defined operators affiliated to a von Neumann algebra.

Although unanswered questions (which are described in the thesis) remain, the results are reasonably complete, especially with respect to certain norm closed ideals which are of principal interest in the theory of operator algebras.

Chapter 1 contains a summary of the notation used throughout the thesis as well as some preliminary results. In Chapter 2 the left and right Fredholm operators relative to any closed ideal $\mathcal{I}$ in any von Neumann algebra $\mathcal{A}$ are characterised in terms of the left and right topological divisors of zero in the quotient algebra $\mathcal{A}/\mathcal{I}$. A characterisation of the semi-Fredholm operators in a semifinite von Neumann algebra, relative to the closed ideal generated by the projections with finite trace, is proved and then used to give a description of the essential spectrum in terms of the eigenvalues in the Calkin algebra. Chapter 3 contains a few lifting results on properties of elements in the "Calkin" algebra. In Chapter 4 some of the results contained in Chapter 2 are extended to similar results for τ-measurable Fredholm operators relative to the closure of the trace class in the topology of convergence in measure.

Fredholm klasse in operator algebras

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OPSOMMING

In die eerste deel van die proefskrif word die nuldelers in die "Calkin" algebra van 'n von Neumann algebra, met betrekking tot 'n willekeurige geslote ideaal, gebruik om Fredholm-tipe operatore te definieer. Die meetkundige, algebraïse en topologiese eienskappe van hierdie klasse word volledig bestudeer en daar word aangetoon dat hierdie eienskappe soortgelyk aan dié van die Fredholm-klasse is. Die gevalle waarin hierdie klasse met die Fredholm-klasse saamval, word volledig ondersoek en gevolglik word bruikbare karakteriserings van die semi-Fredholm operatore in die von Neumann algebra opset verkry. In die tweede deel van die proefskrif word hefstellings vir 'n aantal eienskappe van elemente in die "Calkin" algebra bewys en baie

onopgeloste probleme word gestel. Die studie word afgerond met 'n Fredholm-teorie vir geslote diggedefinieerde operatore wat tot 'n von Neumann algebra geaffilieer is.

Alhoewel onopgeloste probleme (wat in die proefskrif beskryf word) steeds bestaan, is die resultate redelik volledig, veral met betrekking tot sekere normgeslote ideale wat van besondere belang is in die teorie van operatoralgebras.

Hoofstuk 1 bevat 'n opsomming van die notasie wat regdeur die proefskrif gebruik word, sowel as sommige voorafgaande resultate. In Hoofstuk 2 word die links- en regs-Fredholm operatore met betrekking tot enige geslote ideaal $\mathcal{I}$ in 'n von Neumann algebra $\mathcal{A}$ gekarakteriseer in terme van die linker en regter topologiese nuldelers in die kwosiëntalgebra $\mathcal{A}/\mathcal{I}$. 'n Karakterisering van die semi-Fredholm operatore in 'n semi-eindige von Neumann algebra, met betrekking tot die geslote ideaal voortgebring deur die projeksies met eindige spoor, word bewys en dan gebruik om 'n beskrywing van die essensiële spektrum in terme van die eiewaardes in die "Calkin" algebra te gee. Hoofstuk 3 bevat 'n paar hefresultate ten opsigte van eienskappe van elemente in die "Calkin" algebra. In Hoofstuk 4 word sommige resultate van Hoofstuk 2 uitgebrei na soortgelyke resultate vir τ-meetbare Fredholm operatore met betrekking tot die afsluiting van die spoorklas in die topologie van konvergensie in maat.