TESTING THE GENERAL RELATIVITY THEORY THROUGH THE ESTIMATION OF PPN PARAMETERS γ AND β USING SATELLITE LASER RANGING DATA

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ABSTRACT

Parameterised post Newtonian (PPN) parameters γ and β (which should equal unity if general relativity is valid) are estimated through the analyses of four years of LAGEOS 1 and LAGEOS 2 data. To reduce the chance of possible imprinting of general relativistic effects which could be contained in the Earth gravity model used, several low order spherical harmonic coefficients are solved in the orbit estimation and propagation process. After convergence is reached, the solved PPN parametersand γ and β are kept fixed at their solved values and scaling coefficients are introduced to explore the possibility of open parameter space in the accelerations due to the Schwarzschild field, frame dragging and de Sitter (geodesic) precession. It is found that some parameter space is available in the radial acceleration component, which could be taken up by including post-post-Newtonian components of general relativity, or it could be due to some other radial acceleration component that is not adequately modelled. The inclusion of scale coefficients in the least-squares process proves to be a valuable tool in evaluating the solutions for γ and β. A comparison with results obtained for the PPN parameters by other authors utilising different techniques such as VLBI and radar measurements to planets and probes are made.

Introduction

Although Space Geodesy data are used in a variety of research topics such as geophysics, plate tectonics, gravity, meteorology and space science, such data have unique application in fundamental physics. In particular, Satellite Laser Ranging (SLR), which allows very accurate range determination to a satellite, can be used to evaluate General Relativity Theory (GRT). In the SLR technique, a very short laser pulse (typically 200 pico-seconds long) is transmitted to satellites containing an array with corner cube reflectors (CCRs) that reflect the incoming laser light back to the SLR system. Using precise timing, the round-trip time from pulse transmission to reception can be determined as the time-of-flight (tof) observable. The LAGEOS-1 and LAGEOS-2 (LAser GEOdynamics Satellite) satellites are covered completely with CCRs and have small area-to-mass ratios which minimize the effects of non-gravitational forces, such as experienced from atmospheric drag and solar radiation pressure. Accuracies in orbit determination using SLR and advanced orbit determination software are currently at the 2 to 3 cm level in the case of LAGEOS satellites.

Einstein (1920) theorized that the geometrical properties of space-time depend on the distribution of matter in space-time and that as the accuracy of our measurements improves, we will eventually start to detect small deviations from Newton's theory, though they may still escape our observational tests as these excursions from pure Newtonian theory are extremely small. The accuracies achieved today in the four space geodetic techniques, Very Long Baseline Interferometry (VLBI), Global Navigation Satellite Systems (GNSS), Doppler Orbitography and Radiopositioning Integrated by Satellite (DORIS) and SLR, are at such a level that the data from these techniques must be analysed within the framework of a post-Newtonian formalism (Nordtvedt 1968; Will 1971). Correct analyses of the data require that GRT must be considered within the complete context within which the modelling is performed. Therefore solar body ephemerides, reference and time frames ((JPL DE405 and J2000 respectively in our case), signal propagation and observables such as satellite clock frequency and laser pulse travel time, need to account for GRT (Müller et al. 2008). All of these aspects are taken into account in the processing of data in this work. For the purpose of this work, International Earth Rotation Service (IERS) recommendations (McCarthy and Petit, 2003) as updated from time to time are used for the determination of the accelerations and increased range delay due to GRT. These equations (discussed in detail below) are valid in the weak-field and slow motion approximation (linearised and resembles Maxwellian equations of electromagnetism), whereas Einstein's field equations represent a non-linear Lorentz-covariant (locally) theory of gravitation.

SLR analysis strategy

This work utilises HartRAO developed software that processes normal point data obtained by International Laser Ranging Service (ILRS) SLR stations; see for instance Pearlman et al. (2002). Certain specific analysis strategies were applied and developed to improve the

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validity of the approach used. The zonal coefficients of the gravity field could be imprinted (Iorio, 2009) in the gravity field model used (GGM03C for these tests) therefore software was developed to estimate J_2 to J_5. The GRACE gravity model used in this work is GGM03C, which contains terrestrial and satellite data, GGM03S in contrast contains satellite data only.

The GRACE data analyses does not solve for GRT and so could create an imprint effect on the even zonal harmonic coefficients at a level which could influence estimates and evaluations of relativistic accelerations. Iorio (2009) estimates these imprinting effects could be at the 10^{-10} to 10^{-11} level, which will obviously affect the results given that the GRACE model accuracies are (for $l = 4,6$) at the 10^{-12} level. To reduce such a possible imprinting effect, the strategy therefore is to include in the least-squares process, in addition to the other solve-for parameters, the zonal harmonic coefficients J_2 to J_5 as solve-for parameters, and J_6 to J_{20} are included as consider parameters. A-priori error levels for these coefficients are set at the formal published levels. It is important to note here that the software does not estimate a new gravity model; it only allows the parameter space to be filled at the a-priori error level, as constrained by the SLR data. Tests indicated that variations in J_2 can be estimated which is comparable to previously published estimates. In addition, no adjustments for linear trends of the coefficients are made; these linear trends should be treated with caution, as they do not represent the true value of date. The SLR arcs processed are one day arcs; therefore an estimate of the values rather than a trend value is used. The literature contains different trend values, reflecting the temporal variations of earth's oblateness (Chao, 2006), which depend to some extent on the period of evaluation, processing software, models and strategies used. For instance J_2 (linked to Earth's dynamic oblateness) generally linked to post-glacial rebound was decreasing (as indicated by SLR data since 1979 (Yoder, 1983)) until 1997, when it exhibited a reversal in trend and started to increase, due to global-scale mass redistribution. The reversing trend seems to have reversed itself recently, so that one cannot just adopt a trend value, the real trend will depend on the SLR data date and the 'trend' suitable for that specific period. Estimated published trend values (over varying time series period lengths, typically ~20 years) for J_2 vary from $-2.6 \pm 0.6\times10^{-11}$/yr to $-3.3 \pm 0.3\times10^{-11}$/yr (see Cox and Chao, 2002), whereas the IERS recommended value (epoch 2000, IERS 2003), is $\dot{\bar{C}}_{20}=1.162755\times10^{-11}$/yr These values are normalised coefficients. It is clear that due to the trend variability, uncertainties and oscillatory nature of the variations involved, an estimate of the coefficients will be more suitable.

A modified Harris-Priester model for drag was used (Montenbruck and Gill, 2000), which was extended to be useful at heights of 6000 km, the height of the LAGEOS satellites. At this altitude, drag is negligible and is normally not included in SLR data analysis; however, we found small drag accelerations (~2.5 × 10^{-12})m.s^{-2} and a coefficient of drag of ~4.9 for this analysis. The basic equation for drag which can be used to estimate atmospheric drag is given by:

$$\vec{f}_{Drag} = -\frac{1}{2} C_D \frac{A}{m} \rho v_r^2 \vec{e}_v \qquad 1.1$$

where the drag coefficient is described by a dimensionless quantity, C_D, a solve-for parameter in the least-squares adjustment of the orbit, which reflects the interaction of the satellite with the atmosphere. In Equation (1.1) the velocity v is the sum of the vector of the satellite velocity and the vector of the velocity of the atmosphere (Xu, 2008). Drag coefficient values for lower orbit satellites are normally in the range 1.5 to 3.0; the value of 4.9 found during this work is higher and is related to the change in the constituents of the atmosphere at high altitude and interaction between air molecules and the satellite surface. In Equation (1.1) the mass and effective area of the satellite are given by m and A respectively. Atmospheric density is denoted by ρ. The acceleration experienced by the satellite is parallel but opposite in direction to the satellite velocity vector which is given by $\vec{e}_v = \vec{v}_r/\vec{v}_r$. The drag coefficient is estimated every 24 hours.

Once-per-cycle-per-revolution parameters, which are of empirical nature, are typically used to describe and account for unmodelled forces (to some extent) due to model limitations, and have the form:

$$\ddot{\vec{r}} = \vec{a}_0 + \vec{a}_1 \sin v + \vec{a}_2 \cos v \qquad 1.2$$

This mismodelling occurs mainly at a frequency of one-cycle-per-revolution (1CPR). In Equation (1.2) $\vec{a}_0$ is a constant acceleration bias, with 1CPR coefficients $\vec{a}_1$ and $\vec{a}_2$; v is the true anomaly. In total there are nine parameters accounted for here, which were set to a very low level so that they would not interfere with the GRT PPN estimates. A slightly different strategy was employed by Lucchesi and Peron (2010), in a very stringent and robust determination of the advancement of the perigee of the LAGEOS satellites (and specifically LAGEOS II). The perigee advancement resulting from general-relativistic precession was determined with the complete disablement of estimations of empirical accelerations to avoid possible absorption of physical effects. However in this work, so as to minimise or avoid possible interaction with other parameter estimates, the a-priori error levels of the empirical accelerations are set at 1×10^{-12}m.s^{-2} and not completely disabled. Typical 1CPR accelerations found during a one day arc are at the 10^{-15} to 10^{-14}m.s^{-2} level for radial and normal components and 10^{-14} to 10^{-13}m.s^{-2} for the tangential component of acceleration, well below the level of the GRT accelerations and essentially within the noise floor of detectability. It can therefore be assumed that in this case 1CPR modelling does not overshadow modelling of other small accelerations. In this work, a total of 48 parameters are estimated, although only some are discussed here.

Basic Methodology

The basic method used is the classical approach, utilising numerical integration of Newton's second law of motion, including other perturbing forces, and then we add post-Newtonian corrections which allow a slow-motion and weak field approximation to GRT. Acceleration of the satellite can be described by:

$$\ddot{\vec{r}} = -\frac{GM_{\oplus}}{r^3}\vec{r} + \vec{f}. \qquad 1.4$$

In Equation (1.4) $\ddot{\vec{r}}$ represents acceleration in a geocentric inertial reference frame, where $-\vec{r}/r$ is the unit vector from satellite to geocentre (position vector of the satellite is $\vec{r}$), $GM_{\oplus}$ is the product of the gravitational constant and of Earth's mass, and r is the geocentric range given by $r = \sqrt{(\vec{r}\cdot\vec{r})}$. The inclusion of the second term in Equation (1.4) (extending the notation of Tapley et al. 2004) includes the perturbing force $\vec{f}$, which consists of additional forces acting on the satellite:

$$\vec{f} = \vec{f}_{NS} + \vec{f}_{TC} + \vec{f}_{3B} + \vec{f}_{g} + \vec{f}_{Drag} + \vec{f}_{SRP} + \vec{f}_{ERP} + \vec{f}_{Other} + \vec{f}_{Emp} \qquad 1.5$$

In Equation (1.5) $\vec{f}_{NS}$ is the force resulting from the uneven mass-distribution in Earth and is found from the gradient of the gravitational potential U. Added to this static gravity field, the contribution of the variations in time of the static gravity field which includes the variation of Earth's mass-distribution due to ocean and earth tides, is included in $\vec{f}_{TC}$. Perturbations caused by the gravitational forces of the Moon, Sun and planets are given by $\vec{f}_{3B}$, the effects of General Relativity by $\vec{f}_{g}$, $\vec{f}_{Drag}$, is atmospheric drag, $\vec{f}_{SRP}$ the solar radiation force contribution, $\vec{f}_{ERP}$ is the earth radiation pressure and $\vec{f}_{Other}$ contains additional forces such as thermal, satellite rotation dependent effects. Empirical corrections (1CPR), expressed in a local reference frame, and normally divided into radial, tangential and normal (RTN) components are denoted as $\vec{f}_{Emp}$. All of these mentioned perturbations are modelled, except the thermal, satellite rotation dependent effects. Four direct/indirect solar radiation parameters and 9 RTN parameters are estimated.

Station positions are not estimated; they are fixed but adjusted per observational epoch for International Terrestrial Reference Frame (ITRF) velocity. Further adjustments to SLR station positions include those due to ocean loading, earth tide, pole tide and atmospheric loading. Specific centre-of–mass corrections for each SLR station were added.

Estimation of General Relativistic contributions

Space does not allow discussion of these different perturbing components of a force model; instead focus is on just the calculation of acceleration due to general relativity, obtained from the force ($\vec{f}_{g}$) of Equation (1.5) as well as estimating the gravity field coefficients (section 3.2).

Shapiro delay

The Shapiro delay results from the fact that the length of the path of light is increased due to bending in the gravity field. Raw data gathered by the SLR stations are converted to a normal point (NP) which consists of many (at 5 Hz, hundreds) of single shot ranges. Using the speed of light and some additional corrections, one can calculate the Normal Point Range (NPR), as given by the range equation:

$$NPR_i = \left(\frac{NPtof_i}{1\times10^{12}}\times c\right)\Big/2 - \Delta a_i + \Delta CoM_i - \Delta R_{b_i} - \Delta GR_i - \Delta\varepsilon_i, \qquad 1.6$$

In Equation (1.6), $NPtof_i$ is the normal point time-of-flight (in picoseconds, i.e. 10^{-12}) at a certain instance of time and c is (Kaplan 2005) the velocity of light (299,792,458.0 m/s). Corrections include the effects of the atmosphere Δa_i, a satellite dependent (0.251 m for LAGEOS) centre-of-mass correction ΔCoM_i, SLR station range-bias ΔR_{b_i}, as well as the relativistic correction (Shapiro delay) ΔGR_i and other ($\Delta\epsilon_i$) errors. Considering LAGEOS, the Shapiro delay correction is about 7 mm. According to IERS (2003):

$$t_2 - t_1 = \frac{\left|\vec{x}_2(t_2) - \vec{x}_1(t_1)\right|}{c} + \sum_J \frac{(1+\gamma)GM_J}{c^3}\ln\left(\frac{r_{J1}+r_{J2}+\rho}{r_{J1}+r_{J2}-\rho}\right) \qquad 1.7$$

In Equation (1.7), γ is the PPN parameter which should equal unity if GRT is valid, $t_2 - t_1$ denotes the total time delay considering a laser pulse emitted from coordinate x_1 (SLR station) at time t_1 and the return pulse is received at coordinate x_2 (SLR station) at time t_2. In Equation (1.7) the range defined by $\rho = \left|\vec{x}_2 - \vec{x}_1\right|$ is the uncorrected (for GRT) range, in addition $r_{J1} = \left|\vec{x}_1 - \vec{x}_J\right|$ and $r_{J2} = \left|\vec{x}_2 - \vec{x}_J\right|$.

Similar to the numerator of the first term in Equation (1.7) the relativistically uncorrected range ρ is not simply the subtraction of two vectors, but is determined through iterative solutions of two light-time equations for the uplink and downlink path. This procedure is described in Montenbruck and Gill (2001) and Combrinck (2010). For the uplink path (SLR to satellite) a fixed-point iteration with:

$$\tau_u^{(i+1)} = 1/c \cdot \left|\vec{r}\left(t-\tau_d\right) - \vec{R}\left(t-\tau_d-\tau_u^{(i)}\right)\right| \qquad 1.8$$

is executed in a loop until τ_u achieves an accuracy threshold that has been defined previously. For the downlink, the algorithm starts from an initial value of $\tau_u = 0$, then consecutive solutions are done using the fixed-point iteration:

$$\tau_d^{(i+1)} = 1/c \cdot \left|\vec{r}\left(t-\tau^{i}\right) - \vec{R}\left(t\right)\right|. \qquad 1.9$$

The range (determined from two-way ranging) $\rho = \left|\overrightarrow{x_2 - x_1}\right|$ in Equation (1.7) is then determined from the average of uplink range ρ_u and downlink range ρ_d, so that:

$$\rho = 0.5\left(\rho_u + \rho_d\right). \qquad 1.10$$

In this (IERS, 2003) formulation (Equation (1.7)), the sum is carried over all bodies J with mass M_J centred at x_J. According to Ries (1988), only the Earth needs to be considered as J, for near-Earth satellites (including LAGEOS), due to the fact that analysis is done in the geocentric frame of reference.

GRT accelerations

The IERS 2003 (McCarthy and Petit 2003) recommendations for GRT accelerations to be included in precise orbit determination models as applied to a satellite in Earth orbit are:

$$\begin{aligned}\Delta\ddot{\vec{r}} = {} & \frac{GM_{\oplus}}{c^2r^3}\left\{\left[2(\beta+\gamma)\frac{GM_{\oplus}}{r} - \gamma\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right)\right]\vec{r} + 2(1+\gamma)\left(\vec{r}\cdot\dot{\vec{r}}\right)\dot{\vec{r}}\right\} + \\ & (1+\gamma)\frac{GM_{\oplus}}{c^2r^3}\left[\frac{3}{r^2}\left(\vec{r}\times\dot{\vec{r}}\right)\left(\vec{r}\cdot\vec{J}\right) + \left(\dot{\vec{r}}\times\vec{J}\right)\right] + \\ & \left\{(1+2\gamma)\left[\dot{\vec{R}}\times\left(\frac{-GM_s\vec{R}}{c^2R^3}\right)\right]\times\dot{\vec{r}}\right\}.\end{aligned} \qquad 1.11$$

The terms in Equation (1.11) are, with representative acceleration values (Combrinck, 2010),

1. the nonlinear Schwarzschild field of the Earth ($\approx 10^{-9}\text{ms}^{-2}$),
2. Lense-Thirring precession (frame dragging) ($\approx 10^{-11}\text{ms}^{-2}$); and
3. de Sitter (geodesic) precession ($\approx 10^{-11}\text{ms}^{-2}$).

In Equation (1.11), the velocity of light is c the PPN parameters to be evaluated in this work are β, γ. Satellite position relative to the Earth is given by $\vec{r}$ whereas $\vec{R}$ is Earth's position relative to the Sun. Earth's angular momentum per unit mass is $|\vec{J}| \cong 9.8 \times 10^8\text{m}^2\text{s}^{-1}$ and $GM_{\oplus}$ is the gravitational coefficient of Earth. The Sun's gravitational coefficient is denoted by GM_S. Parameterised post-Newtonian parameter β is only present in the Schwarzschild term, which makes the estimates of β less sensitive comparative to γ which is present in all three terms of Equation (1.11) as well as in Equation (1.7).

Evaluation of PPN parameters

Similar to the way (see Section 3.2) the gravity field coefficients are estimated, the partial derivatives of the GRT acceleration components are included in the sensitivity matrix as part of the linearization of the orbit, and are estimated together with various parameters that describe the other perturbing forces which affect the satellite orbit. Inclusion of the PPN parameters in the estimation process occurs only after many other parameters have been estimated (e.g. coefficients for solar pressure and reflected sunlight from Earth), so as to ensure stable solutions (i.e. after n iterations, a selectable parameter). The estimates of PPN parameter γ is also passed back into the Shapiro delay, Equation (1.7), so that it is included in all the relativistic equations.

Estimation of Earth gravity field coefficients

As it has been suggested by Iorio (2009) that possible imprinting of GRT in the gravity field models could adversely affect tests of GRT, certain gravity field spherical harmonic coefficients, ($J_2 - J_5$), C_{21} and S_{21} are estimated. It is important to stress that a new gravity field model is not developed, the a-priori gravity field is basically 'tuned' to fit the observations, at the a-priori error level. Therefore, the gravity field is allowed to be slightly reconstructed by the SLR data, and is constrained by the ranging data.

Earth gravity field coefficients

As an introduction and acknowledgement of the fact that readers of this journal are mainly geologists, a short background and notation description on the Earth's gravity field is given to enable easy reading of the section where the spherical harmonic coefficients are estimated. Following Tapley et al. (2004), and Combrinck (2010), the gravitational potential between two point masses can be described by:

$$U = \frac{GM_1M_2}{r}\ , \qquad 1.12$$

Here r is the distance between the two masses. The gradient of U allows finding the gravitational force on M_2, the LAGEOS satellite in our case:

$$\vec{F} = \nabla U = -\frac{GM_1M_2\vec{r}}{r^3}, \qquad 1.13$$

where $\vec{r} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$ describes the position vector of M_2 relative to M_1, and

$$\nabla = \frac{\partial}{\partial x}\vec{u}_x + \frac{\partial}{\partial y}\vec{u}_y + \frac{\partial}{\partial z}\vec{u}_z, \qquad 1.13$$

with $\vec{u}_x$, $\vec{u}_y$ and $\vec{u}_z$ being unit vectors. The gravitational potential U (Tapley et al., 2004) affects a point mass, m' at a location external to a body M which has an arbitrary mass distribution and if modelled as a collection of point masses we have:

$$U = m' \iiint \frac{G\ \sigma\ dx\ dy\ dz}{\rho}. \qquad 1.14$$

If σ represents the mass density of mass element dm, then the differential volume is $dx\ dy\ dz$. Here ρ is the distance between dm and m'. The potential given by Equation (1.14) can be compacted if the external mass is taken as unity so that $m' = 1$ and the integral notation is taken to include the total mass of the body so that:

$$U = \int_M \frac{G\ dm}{\rho}. \qquad 1.15$$

Equation (1.15) as an infinite series is then:

$$U = \frac{G}{r} \int_M \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l P_l(\cos S)\, dm \qquad 1.16$$

The distance R is between the origin of the total mass M and the differential mass dm. The Legendre polynomial of degree l is P_l whereas the argument of P_l is $\cos S$. Here S denotes the angle between the vectors $\vec{R}$ and $\vec{r}$. To evaluate Equation (1.16) the Legendre polynomial is expanded into spherical harmonics:

$$\begin{aligned} U &= \frac{\mu}{r} + U' \\ U' &= -\frac{\mu^*}{r}\sum_{l=1}^{\infty}\left(\frac{a_e}{r}\right)^l P_l(\sin\phi) J_l \\ &\quad + \frac{\mu^*}{r}\sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\frac{a_e}{r}\right)^l P_{lm}(\sin\phi)\left[C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\right]. \end{aligned} \qquad 1.17$$

In Equation (1.17) m', (in our case LAGEOS) has spherical coordinates (r,ϕ,λ), where:

$$\begin{aligned} x &= r\cos\phi\cos\lambda \\ y &= r\cos\phi\sin\lambda \\ z &= r\sin\phi \end{aligned} \qquad 1.18$$

with the (x,y,z) system fixed in the body's origin O. A reference distance a_e and reference mass ($\mu^* = GM^*$) are included in Equation (1.17) as scale factors to non-dimensionalise the mass coefficients C_{lm} and S_{lm}. Legendre's associated functions are given by P_{lm} of degree l and order m.

Mass properties of the body are represented by spherical harmonic coefficients J_l, C_{lm}, and S_{lm}. Zonal coefficients describe the part of the potential that are not dependent on longitude, and are related to $C_{l,m}$ through the relation $J_l = -C_{l,0}$. The degree 2, order 0, term (zonal) models the contribution due to Earth's oblateness and is the second largest contributor to the overall potential in addition to that of the central body. The degree 1 term is zero if one assumes that the centre of the Earth fixed coordinate system coincides with the centre of mass of the Earth. Tesseral harmonics are represented by C_{lm} and S_{lm} with $l \neq m$ and sectorial harmonics are considered if $l = m$. As the values of geopotential coefficients vary over a range of ten or more orders of magnitude, they can be normalised, so that the magnitudes are more similar.

Gravity models used for SLR are usually published in normalised format. These coefficients are defined as (Montenbruck and Gill, 2000):

$$\begin{Bmatrix} \bar{C}_{lm} \\ \bar{S}_{lm} \end{Bmatrix} = \sqrt{\frac{(l+m)!}{(2-\delta_{0m})(2l+1)(l-m)!}} \begin{Bmatrix} C_{lm} \\ S_{lm} \end{Bmatrix}. \qquad 1.19$$

Currently the HartRAO software can utilise 55 different gravity models, dating from 1990 (soon to be expanded and modified to include models from the early 1960s), these are all published in normalised format. As there is a direct relationship between the degree one terms (J_1, $C_{1,1}$ and $S_{1,1}$) the offset from the origin O to the centre of mass of the body, degree one terms are zero, as SLR uses geocentric coordinate systems. If m' represents a satellite one can write (Tapley et al. 2004), ignoring the other forces for the moment:

$$\ddot{\vec{r}} = \nabla U = -\frac{GM_\oplus}{r^3}\vec{r} + \vec{f}_{NS} \qquad 1.20$$

as $m'/M_\oplus$ is very small. The force contribution in Equation (1.20) resulting from non-spherical terms is presented by, f_{NS}, i.e. $\nabla U'$ When the acceleration term is represented in body-fixed coordinates and the gravitational potential in spherical coordinates:

$$\nabla U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\vec{u}_\phi + \frac{1}{r\cos\phi}\frac{\partial U}{\partial \lambda}\vec{u}_\lambda \qquad 1.21$$

where the gradient gives force components in spherical coordinates which can be rotated via a coordinate transformation into (x,y,z) components using:

$$T_{r\phi\lambda}^{xyz} = \begin{bmatrix} \cos\phi\cos\lambda & -\sin\phi\cos\lambda & -\sin\lambda \\ \cos\phi\sin\lambda & -\sin\phi\sin\lambda & \cos\lambda \\ \sin\phi & \cos\phi & 0 \end{bmatrix} \tag*{1.22}$$

To provide $\ddot{\vec{r}}$ in a non-rotating system a further transformation is required, e.g. if axes Z and z coincide and need to be rotated through an angle α, then:

$$T_{xyz}^{XYZ} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag*{1.23}$$

The portion of the perturbing force resulting from the mass distribution of Earth is written as (Tapley et al. 2004):

$$\vec{f}_{NS} = T_{xyz}^{XYZ} T_{r\phi\lambda}^{xyz} \nabla U' \tag*{1.24}$$

when $\vec{r}$ is in the non-rotating system (X,Y,Z). In the software of HartRAO, precise orbit estimation is done in an inertial reference system, and complex transformations need to be made to transform from the Earth-fixed geocentric system to the J2000 Earth-Centred-Inertial (ECI) system, utilising International Earth Rotations Service (IERS) products and recommendations (McCarthy and Petit, 2003), and in particular Earth orientation file eopc04_62.now, available from the IERS at http://hpiers.obspm.fr/iers/eop/eopc04_05/. This file contains several parameters required for precise orbit determination, including pole offsets, corrections and errors commencing 1 January 1962 to the present.

If T_{XYZ}^{xyz} is the transformation matrix from J2000 to Earth-fixed, then:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECF} = T_{XYZ}^{xyz} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{ECI} \tag*{1.25}$$

with the transformation matrix expanded to:

$$T_{XYZ}^{xyz} = WS'NP \tag*{1.26}$$

where the matrix NP introduces precession-nutation from some epoch to current time, S' applies rotation to make provision for sidereal time and W applies polar motion to enable alignment of the true pole (z axis) with the pole of the Earth Centred Fixed (ECF) system. Further details can be had from Tapley et al. (2004), Montenbruck and Gill (2000) and McCarthy and Petit (2003).

Solving Earth gravity field coefficients

In solving for the spherical harmonic coefficients, I adapt Montenbruck and Gill (2000), based on derivations by Cunningham (1970). Firstly harmonic relations are evaluated by setting:

$$\begin{aligned} V_{lm} &= \left(\frac{R_\oplus}{r}\right)^{l+1} \cdot P_{lm}\left(\sin\phi\right) \cdot \cos m\lambda \\ W_{lm} &= \left(\frac{R_\oplus}{r}\right)^{l+1} \cdot P_{lm}\left(\sin\phi\right) \cdot \sin m\lambda \end{aligned} \tag*{1.27}$$

The gravity potential is then:

$$U = \frac{GM_\oplus}{R_\oplus} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \left(C_{lm} V_{lm} + S_{lm} W_{lm}\right), \tag*{1.28}$$

where V_{lm} and W_{lm} satisfy recurrence relations:

$$\begin{aligned} V_{mm} &= \left(2m-1\right)\left\{\frac{xR_\oplus}{r^2} V_{m-1,m-1} - \frac{yR_\oplus}{r^2} W_{m-1,m-1}\right\} \\ W_{mm} &= \left(2m-1\right)\left\{\frac{xR_\oplus}{r^2} W_{m-1,m-1} - \frac{yR_\oplus}{r^2} V_{m-1,m-1}\right\} \end{aligned} \tag*{1.29}$$

as well as:

$$\begin{aligned} V_{lm} &= \left(\frac{2l-1}{l-m}\right) \cdot \frac{zR_\oplus}{r^2} \cdot V_{l-1,m} - \left(\frac{l+m-1}{l-m}\right) \cdot \frac{R_\oplus^2}{r^2} \cdot V_{l-2,m} \\ W_{lm} &= \left(\frac{2l-1}{l-m}\right) \cdot \frac{zR_\oplus}{r^2} \cdot W_{l-1,m} - \left(\frac{l+m-1}{l-m}\right) \cdot \frac{R_\oplus^2}{r^2} \cdot W_{l-2,m} \end{aligned} \tag*{1.30}$$

The acceleration which is equal to, (see Equation (1.20)), can be found directly (Montenbruck and Gill, 2000) from V_{lm} and W_{lm}, so that:

$$\ddot{x} = \sum_{l,m} \ddot{x}_{lm}, \quad \ddot{\mathrm{y}} = \sum_{l,m} \ddot{y}_{lm}, \quad \ddot{\mathrm{z}} = \sum_{l,m} \ddot{z}_{lm}, \tag*{1.31}$$

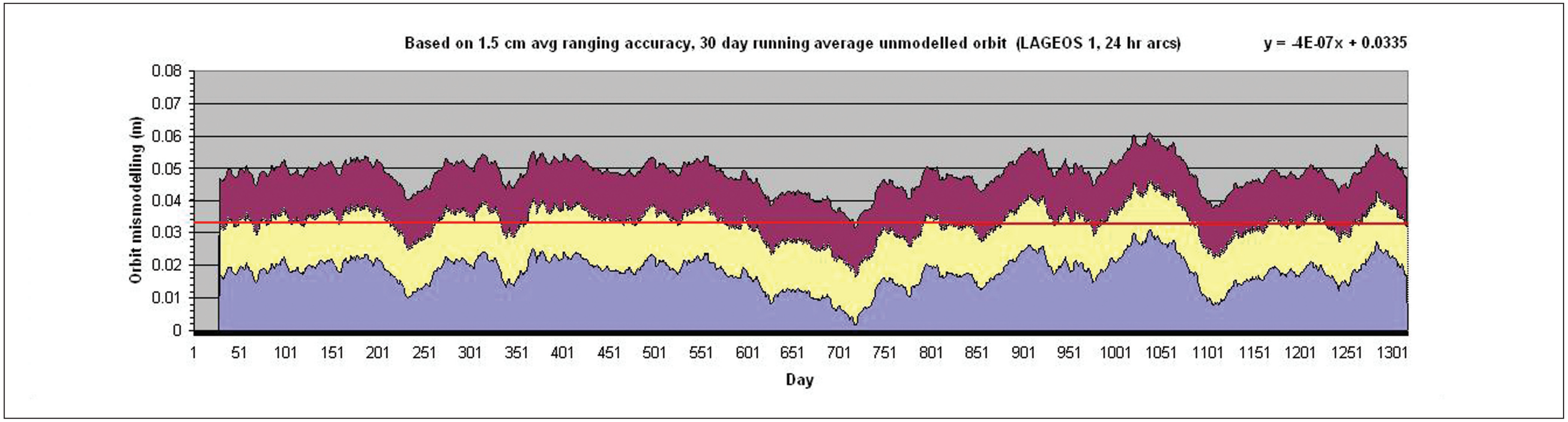

Figure 1. *Thirty day running average of the Observed minus Computed range (O-C) for LAGEOS 1, indicating orbit mismodelling. The linear fit equation indicates a very small slope and average O-C of 3.35 cm. Assuming average SLR ranging accuracy of 1.5 cm, the bottom and top lines indicate worst and best possible orbital fits.*

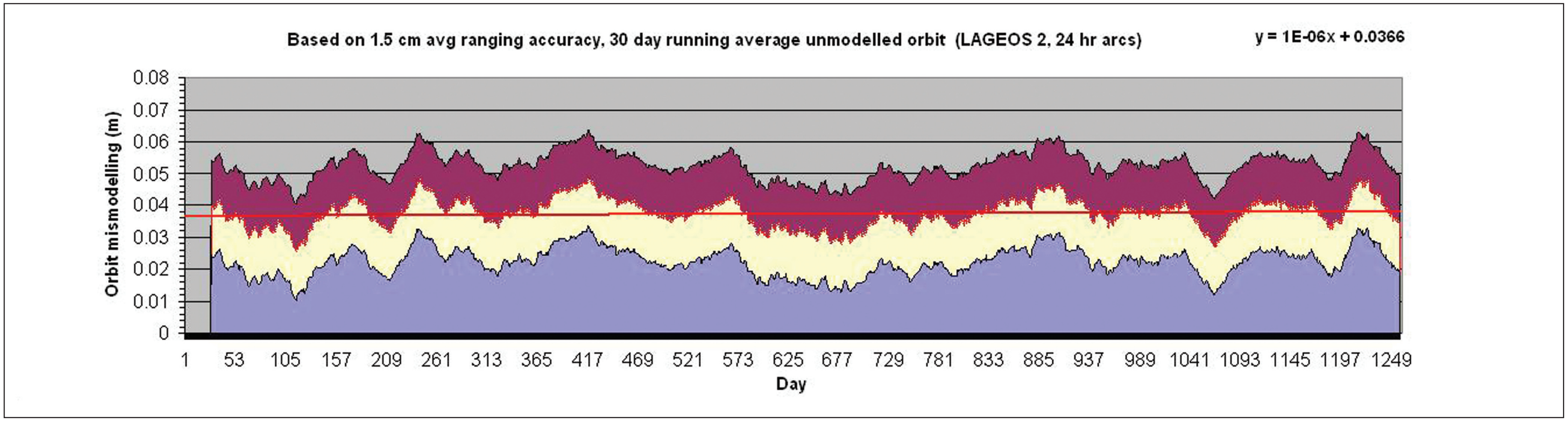

Figure 2. *O-C for LAGEOS 2. Similar to Figure 1, the linear fit equation indicates a very small slope and average O-C of 3.66 cm.*

where the partial accelerations are given by:

$$\ddot{x}_{lm} \overset{m=0}{=} \frac{GM}{R_\oplus^2}\cdot\left\{-C_{l0}V_{l+1,1}\right\}$$

$$\overset{m>0}{=} \frac{GM}{R_\oplus^2}\cdot\frac{1}{2}\cdot\left\{\begin{matrix}\left(-C_{lm}V_{l+1,m+1}-S_{lm}W_{l+1,m+1}\right)\\ +\dfrac{(l-m+2)!}{(l-m)!}\cdot\left(+C_{lm}V_{l+1,m-1}+S_{lm}W_{l+1,m-1}\right)\end{matrix}\right\}$$

$$\ddot{y}_{lm} \overset{m=0}{=} \frac{GM}{R_\oplus^2}\cdot\left\{-C_{l0}W_{l+1,1}\right\}$$

$$\overset{m>0}{=} \frac{GM}{R_\oplus^2}\cdot\frac{1}{2}\cdot\left\{\begin{matrix}\left(-C_{lm}\cdot W_{l+1,m+1}+S_{lm}\cdot V_{l+1,m+1}\right)\\ +\dfrac{(l-m+2)!}{(l-m)!}\cdot\left(-C_{lm}W_{l+1,m-1}+S_{lm}V_{l+1,m-1}\right)\end{matrix}\right\}$$

$$\ddot{z}_{lm} = \frac{GM}{R_\oplus^2}\cdot\left\{(l-m+1)\cdot\left(-C_{nm}V_{l+1,m}-S_{lm}W_{l+1,m}\right)\right\}. \qquad 1.32$$

In order to estimate the spherical harmonic coefficients for selected $-C_{lm} = J_{lm}$ the accelerations as determined in Equation (1.32) are subtracted in the code during the process of finding $\nabla U = \ddot{\vec{r}}$, currently for selected J_{lm} where m = 0, l = 2, 3, 4, 20, and for C_{21} and S_{21}. Coefficients C_{21} and S_{21} together with J_2 are used to estimate pole tide. These selected accelerations are determined in their own subroutines according to Equation (1.32). Their partial derivatives are passed to the sensitivity matrix as part of the rigorous linearization of the orbit trajectory, together with the different parameters that determine the various forces (e.g. gravitational attraction of the moon, sun and planets) affecting the satellite orbit.

The accelerations in Equation (1.32) are in an Earth-fixed coordinate system and need to be transformed into an inertial reference system (J2000) utilising current IERS recommendations and standards for precession-nutation, Earth rotation and polar motion. Currently solving for the spherical harmonic coefficients is done for each satellite being processed in an independent way, so that each satellite has its own state transition matrix and sensitivity matrix. Combined solutions thus require statistical addition of solved parameters and their errors.

Results and discussion

Figures 1 and 2 contain Observed-Computed (O-C) residuals for LAGEOS 1 and LAGEOS 2 respectively. The SLR data covers the period from 1 December 2005 to 30 April 2010. For illustration of the best and worst case, based on 1.5 cm average ranging accuracy, the averages as well as worst and best cases are plotted. An average O-C value of 3.4 cm for LAGEOS 1 and an average value of 3.7 cm for LAGEOS 2 resulted. Calculated slopes of a linear regression indicate no noticeable long term linear bias, neither are seasonal periodicities apparent.

The estimated PPN parameters γ and β (as function of time) are plotted in Figures 3 and 4. After the final iteration a 5 sigma outlier rejection was applied in the final analysis to exclude weak solutions for some arcs. Statistically, the solutions and their precisions have nearly the same magnitude. Normally, when the results

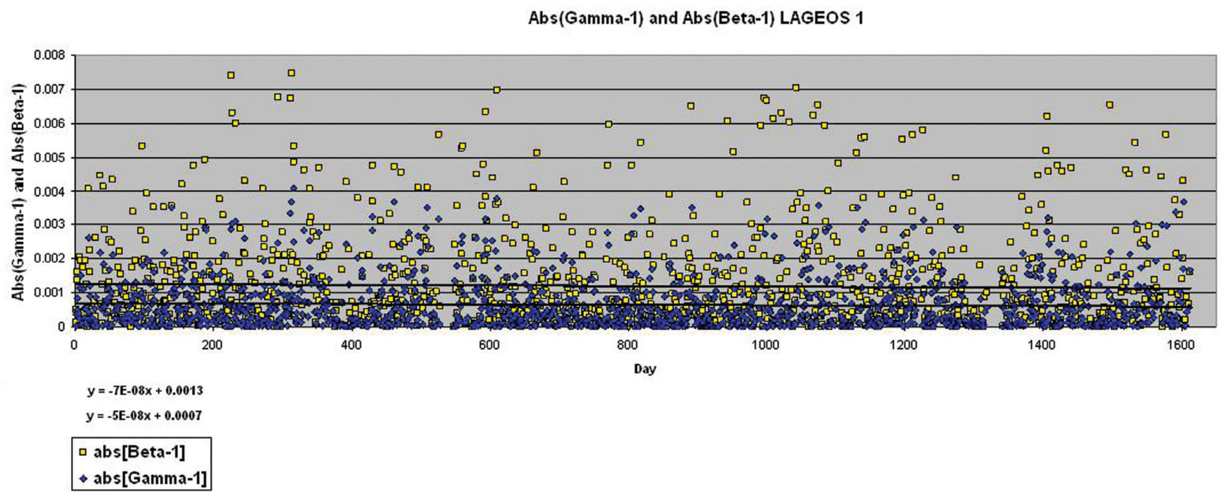

Figure 3. *Absolute values of Gamma-1 and Beta-1 for LAGEOS 1. The two linear fits are for Beta-1 (0.0013) and Gamma-1 (0.0007) respectively, the slopes are small indicating no significant long term effects. Beta-1 values exhibit larger deviations.*

Table 1. Comparison of recent estimates of PPN parameters Gamma and Beta using different techniques.

Parameter	**Effect or Experiment**	**Value**	**Bound (1 Sigma)**	**Remarks**
$\gamma-1$ How much space curvature per unit mass?	Time delay	-1.3 X 10-5	5.2 X 10-5	Cassini-Earth-Sun conjunction microwave tracking (Anderson et al. 2004)
	Light deflection (bending of signal passing the Sun)	-6 X 10-5	3.1 X 10-4	Astrometric VLBI (Eubanks et al. 1997)
	Light deflection (bending of signal passing the Sun)	2 X 10-4	3 X 10-4	VLBI (Fomalont et al. 2009) - Standard error not sigma?
	Radar observations of planets and spacecraft	-1 X 10-4	2 X 10-4	Pitjeva (2005)
	GRT Satellite acceleration			GRT components of satellite acceleration
	LAGEOS 1	6.5 X 10-4	7.4 X 10-4	+ Shapiro delay (this study
	LAGEOS 2	9.0 X 10-4	9.6 X 10-4	
$\beta-1$ How "non-linear" is gravity	Radar observations of planets and spacecraft	0.0000	1.0 X 10-4	Pitjeva (2005)
	Light deflection	-1.9 X 10-4	2.6 X 10-4	Astrometric VLBI (Eubanks et al. 1997)
	GRT satellite acceleration			
	LAGEOS 1	1.2 X 10-3	1.4 X 10-3	GRT components of satellite acceleration
	LAGEOS 2	1.4 X 10-3	1.5 X 10-3	(this study)

of the solutions are different (more or less) than 3 sigmas, the solutions are considered insignificant. Applying a 5 sigma outlier rejection will allow outliers beyond 3 sigmas to be included; however, this creates a very conservative approach and reduces subjectivity. Comparisons between alternative solutions are not always as straightforward as one may think, as the complete analysis and statistical strategy employed in obtaining final solutions are not always available. Values for LAGEOS 1 are slightly better than for LAGEOS 2, this could be due to several factors, including unequal number of normal points (1317 for LAGEOS 1 and 1265 for LAGEOS 2) processed due to uneven tracking coverage of the two satellites, different geometry of coverage, and independent errors in modelling. Correlation between the PPN parameters and the O-C values are high. For LAGEOS 1 (0.407 and 0.418, γ and β respectively) the correlation is slightly lower than for LAGEOS 2, (0.429 and 0.431, γ and β respectively) indicating a direct and strong relationship between model accuracies (as reflected by the O-C residuals) and accuracy of the PPN parameter

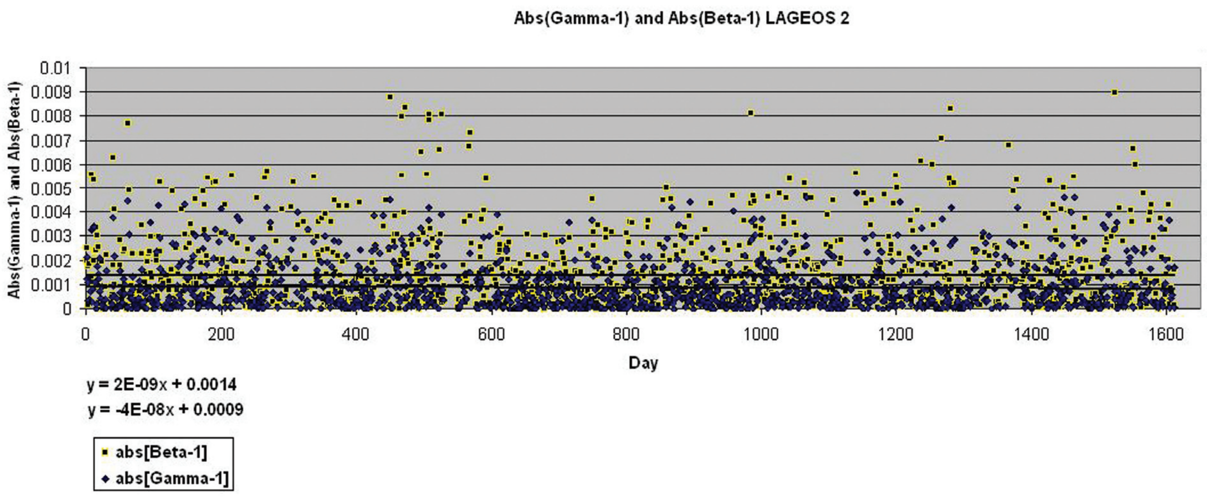

Figure 4. *Absolute values of Gamma-1 and Beta-1 for LAGEOS 2. Similar to Figure 3, the linear fits are for Beta-1 (0.0014) and Gamma-1 (0.0009) respectively, the slopes indicate no significant long term effects. Results are slightly inferior to that of LAGEOS 1; this could be due to different data coverage or modelling imperfections.*

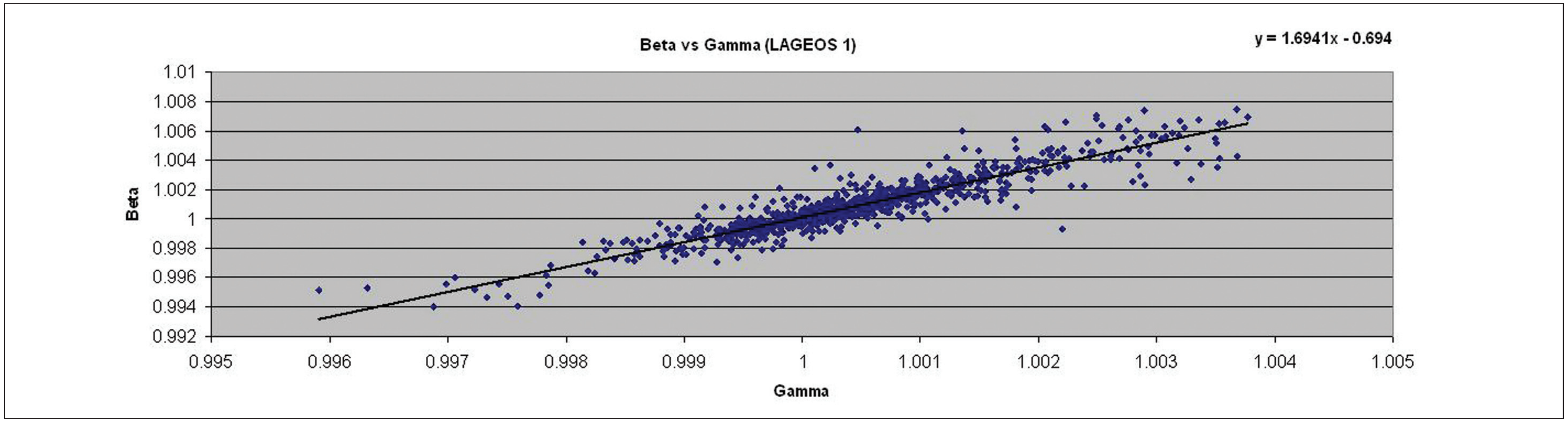

Figure 5. *PPN Beta as a function of Gamma for LAGEOS 1, indicating a high level of correlation as indicated by the coefficients of the linear fit equation. There is some clustering in the region Gamma > 1, indicating a bias towards finding Gamma and Beta larger than unity.*

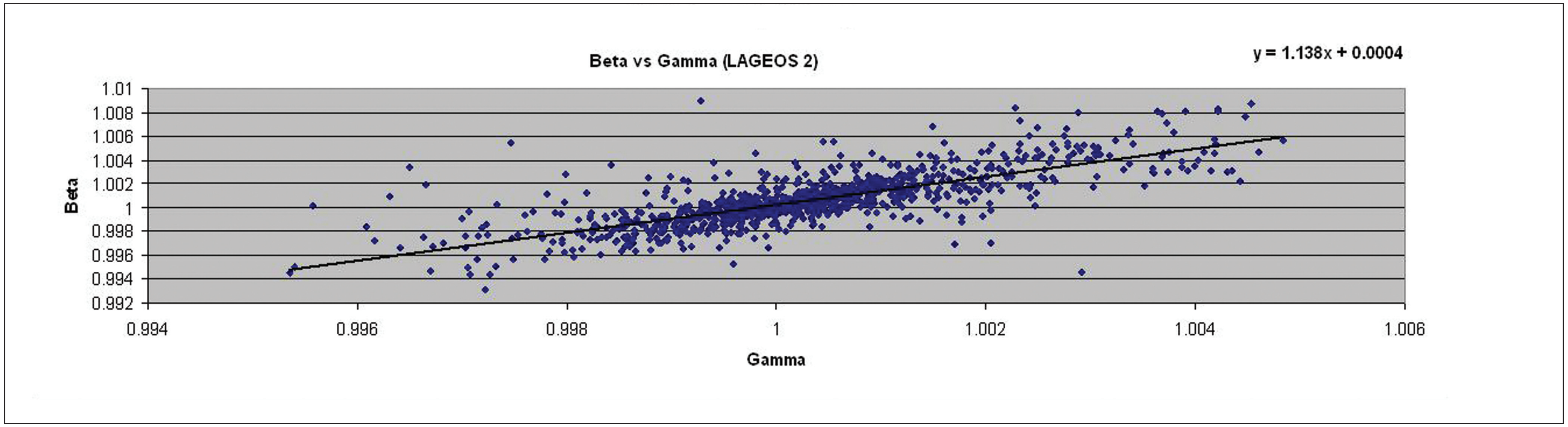

Figure 6. *PPN Beta as a function of Gamma for LAGEOS 2, also indicating a high level of correlation as indicated by the coefficients of the linear fit equation. There is a larger spread in comparison to Figure 5, indicating a weaker solution for LAGEOS 2. Some outliers, which have not been removed by the conservative 5 sigma filter are evident.*

estimates. Figures 5 and 6 contain plots of the correlations; the solution for LAGEOS 2 is slightly weaker, values are slightly more spread. To achieve improved values of the PPN parameters, it is clear that other errors in modelling (atmospheric drag, solar radiation pressure, unmodelled effects etc.) will have to be reduced. In comparison to a selection of other published estimates of PPN parameters γ and β, the estimate for β is weaker using this technique, as it is only evaluated in the Schwarzschild term of the GRT acceleration, whereas γ is present in all the terms of Equation (1.11) and in the second term of Equation (1.7).

A comparison of the results of this study and other published results using different techniques is made in Table 1. Current best estimates are indicated in bold, the Cassini-Earth-Sun conjunction microwave tracking (Anderson et al. 2004) technique providing best value for $\gamma - 1$ ($-1.3\times10^{-5} \pm 5.2\times10^{-5}$). Radar observations of planets and spacecraft (Pitjeva, 2005) currently provide the best value of $\beta - 1$ ($0.0000 \pm 1\times10^{-4}$). The most recent test of the combination of γ and β was made by Lucchesi and Peron (2010). In the PPN framework, their results for the general relativistic advance of the perigee of LAGEOS II satellite can be considered as a 0.03% measurement of the combination of the γ and β parameters. Their result, essentially from the Schwarzschild signal, puts a constraint only on the combination of γ and β. Other, separate constraints are needed to disentangle them (personal communication, R Peron), so it is not possible to make a direct comparison with the other published estimates at this stage.

The values for $\gamma - 1$ ($6.5\times10^{-4} \pm 7.4\times10^{-4}$ and ($9.0\times10^{-4} \pm 9.6\times10^{-4}$ for LAGEOS 1 and 2 respectively) and $\beta - 1$ ($1.2\times10^{-3} \pm 1.4\times10^{-3}$ and $1.4\times10^{-3} \pm 1.5\times10^{-3}$ for LAGEOS 1 and 2, respectively) found in this study based on the GRT acceleration are an order of magnitude less certain than the best published values (cf. Table 1).

Standard deviation is the important criterion here as it represents the uncertainty bounds set by the test (not standard error). Considering that further improvements in modelling can be made, it does have potential for higher precision estimates.

To estimate possible empty parameter space, scaling parameters were inserted in Equation (1.11) which result in Equation (1.33):

$$\begin{aligned}\Delta\ddot{\vec{r}} = GR_S &\left\{\frac{GM_\oplus}{c^2r^3}\left\{\left[2(\beta+\gamma)\frac{GM_\oplus}{r}-\gamma\left(\dot{\vec{r}}\cdot\dot{\vec{r}}\right)\right]\vec{r}+2(1+\gamma)\left(\vec{r}\cdot\dot{\vec{r}}\right)\dot{\vec{r}}\right\}\right\}+\\ GR_f &\left\{(1+\gamma)\frac{GM_\oplus}{c^2r^3}\left[\frac{3}{r^2}\left(\vec{r}\times\dot{\vec{r}}\right)\left(\vec{r}\cdot\vec{J}\right)+\left(\dot{\vec{r}}\times\vec{J}\right)\right]\right\}+\\ GR_d &\left\{(1+2\gamma)\left[\dot{\vec{R}}\times\left(\frac{-GM_s\vec{R}}{c^2R^3}\right)\right]\times\dot{\vec{r}}\right\}.\end{aligned} \qquad 1.33$$

Here GR_S, GR_f and GR_d are scaling parameters or residual coefficients, (for Schwarzschild, frame dragging and de Sitter precession terms respectively) which should equal unity if no empty parameter space is available. After γ and β are determined, these parameters are set fixed to the solved values (i.e. not estimated again). Then the GR residual coefficients are solved for to determine estimates of the un-modelled (residual) GRT acceleration; constraints are set very loosely, so the coefficients are allowed to be freely adjusted within available parameter space during the least-squares process. These solved for residual coefficients are converted to acceleration using average values of the 3 GR components, allowing evaluation of the modelling. It was found that:

$1\times10^{-11} \geq GR_S \geq 1\times10^{-12}$ m.s$^{-2} \geq GR_f \geq 1\times10^{-14}$ m.s^{-2}
$1\times10^{-13} \geq GR_d \geq 1\times10^{-14}$ m.s^{-2}

It is possible that some parameter space is available in the Schwarzschild term of the acceleration; however the other two terms are filled as accelerations of the order 10^{-13} m.s^{-2} are believed to be below the detectable limit currently. This available parameter space could be due to unmodelled radial acceleration due to mismodelling of for instance Earth radiation pressure relating from reflected sunlight off the earth (determined by Earth's reflection albedo, average ~1.34), or a requirement to include post-post-Newtonian components of GRT. In addition, parameter space could be influenced by over constrained (too tight) parameters. Setting the constraints of the scaling parameters very loosely will ensure filling of empty parameter space and in addition will be an indicator of too tight constraints on the parameters (γ and β) to be solved.

Conclusion

This work describes a technique using Satellite Laser Ranging data to estimate the Parameterised post-Newtonian parameters γ and β which should equal unity if GRT is valid. The results are promising and further development and improvement of this PPN parameter estimation technique will depend on orbital perturbation modelling improvements.

Other parameters such as Earth's elasticity coefficient are also estimated. Spherical harmonic coefficients J_2 to J_5 were estimated to mitigate possible effects of GRT "imprinting". In addition C_{21} and S_{21} were estimated as part of pole-tide calculations. These gravity coefficient parameters were tightly constrained with a-priori error estimates set to gravity model formal errors. Residual coefficients were introduced in the GRT acceleration formulation to estimate possible free parameter space. It was found that some parameter space could be available in the Schwarzschild term of the GRT acceleration, whereas the other terms seem to have no parameter space left within the context of the techniques' sensitivity.

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