Zabrejko's lemma and the fundamental principles of functional analysis in the asymmetric case

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1. Preliminary results

A *quasi-semimetric* on a set X is a mapping $\rho : X \times X \to [0;\infty)$ satisfying the following conditions:

$$\begin{aligned} &(\text{QM1}) \qquad \rho(x,y) \ge 0, \quad \text{and} \quad \rho(x,x) = 0\,;\\ &(\text{QM2}) \qquad \rho(x,z) \le \rho(x,y) + \rho(y,z)\,, \end{aligned}$$

for all $x, y, z \in X$. If, further,

$$(\text{QM3}) \qquad \rho(x,y) = \rho(y,x) = 0 \Rightarrow x = y\,,$$

for all $x, y \in X$, then ρ is called a *quasi-metric*. The pair (X, ρ) is called a *quasi-semimetric space*, respectively a *quasi-metric space*. The conjugate of the quasi-semimetric ρ is the quasi-semimetric $\bar\rho(x,y) = \rho(y,x)$, $x, y \in X$. The mapping $\rho^s(x,y) = \max\{\rho(x,y), \bar\rho(x,y)\}$, $x, y \in X$, is a semimetric on X which is a metric if and only if ρ is a quasi-metric.

An *asymmetric norm* on a real vector space X is a functional $p : X \to [0,\infty)$ satisfying the conditions

$$\begin{aligned} &(\text{AN1})\ p(x) = p(-x) = 0 \Rightarrow x = 0; \quad (\text{AN2})\ p(\alpha x) = \alpha p(x)\,;\\ &(\text{AN3})\ p(x+y) \le p(x) + p(y)\,, \end{aligned}$$

for all $x, y \in X$ and $\alpha \ge 0$.

If p satisfies only the conditions (AN2) and (AN3), then it is called an *asymmetric seminorm*. The pair (X, p) is called an *asymmetric normed* (respectively *seminormed*) *space*.

If (X, p) is an asymmetric normed space, then all topological and metric notions are considered with respect to the quasi-metrics

$$\rho_p(x,y) = p(y-x) \quad \text{and} \quad \bar\rho_p(x,y) = p(x-y)\,.$$

The conjugate asymmetric norm to p is $\bar p(x) = p(-x)$, so that

$$\bar\rho_p = \rho_{\bar p} \quad \text{and} \quad p^s(x) = \max\{p(x), p(-x)\},\ x \in X.$$

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If (X,ρ) is a quasi-semimetric space, then for $x \in X$ and $r > 0$ one defines the balls in X by the formulae

$$\begin{aligned} B_\rho(x,r) =&\{y \in X : \rho(x,y) < r\} - \text{ the open ball, and} \\ B_\rho[x,r] =&\{y \in X : \rho(x,y) \le r\} - \text{ the closed ball}. \end{aligned}$$

In the case of an asymmetric seminormed space (X,p) the balls are given by

$$B_p(x,r) = \{y \in X : p(y-x) < r\}, \text{ respectively } B_p[x,r] = \{y \in X : p(y-x) \le r\}.$$

The closed unit ball of X is $B_p = B_p[0,1]$ and the open unit ball is $B'_p = B_p(0,1)$. In this case the following formulae hold true

$$B_p[x,r] = x + rB_p \quad \text{and} \quad B_p(x,r) = x + rB'_p, \tag{1.1}$$

that is, the unit ball of X completely determines its quasi-metric structure. If necessary, these balls will be denoted by $B_{p,X}$ and $B'_{p,X}$, respectively.

The topology $\tau(\rho)$ (or τ_ρ) of a quasi-semimetric space (X,ρ) can be defined starting from the family $\mathcal{V}_\rho(x)$ of neighborhoods of an arbitrary point $x \in X$:

$$\begin{aligned} V \in \mathcal{V}_\rho(x) \iff& \exists r > 0 \text{ such that } B_\rho(x,r) \subset V \\ \iff& \exists r' > 0 \text{ such that } B_\rho[x,r'] \subset V. \end{aligned}$$

The topology generated by a quasi-metric ρ is only T_0. It is T_1 if and only if $\rho(x,y) > 0$ for any pair of distinct elements $x, y \in X$. A characterization of asymmetric norms inducing a Hausdorff topology was given in [8], see also [5].

The convergence of sequences in a quasi-metric space (X,ρ) is characterized by the conditions

$$\begin{aligned} x_n \xrightarrow{\rho} x &\iff \rho(x,x_n) \to 0; \\ x_n \xrightarrow{\bar\rho} x &\iff \bar\rho(x,x_n) \to 0 \iff \rho(x_n,x) \to 0. \end{aligned}$$

If (X,p) is an asymmetric normed space, then

$$\begin{aligned} &\text{(i)} \quad x_n \xrightarrow{p} x \iff p(x_n - x) \to 0; \\ &\text{(ii)} \quad x_n \xrightarrow{\bar p} x \iff p(x - x_n) \to 0; \\ &\text{(iii)} \quad x_n \xrightarrow{p} x \iff -x_n \xrightarrow{\bar p} -x. \end{aligned} \tag{1.2}$$

From the equivalence (iii) in (1.2) one obtains the following result.

Remark 1.1. Let (X,p) be an asymmetric normed space and Z a linear subspace of X. Then

$$Z \text{ is } p\text{-closed} \iff Z \text{ is } \bar p\text{-closed}.$$

The following topological properties are true for quasi-semimetric spaces. We use the abbreviations lsc for lower semicontinuous and usc for upper semicontinuous.

Proposition 1.2 ([5], P. 1.1.8). *If (X,ρ) is a quasi-semimetric space, then*

1. *Any ball* $B_\rho(x,r)$ *is* $\tau(\rho)$*-open and a ball* $B_\rho[x,r]$ *is* $\tau(\bar\rho)$*-closed. The ball* $B_\rho[x,r]$ *need not be* $\tau(\rho)$*-closed.*
   Also, the following inclusions hold
   $$B_{\rho^s}(x,r) \subset B_\rho(x,r) \ \text{ and } \ B_{\rho^s}(x,r) \subset B_{\bar\rho}(x,r)\,,$$
   with similar inclusions for the closed balls.
2. *For every fixed* $x \in X$*, the mapping* $\rho(x,\cdot): X \to (\mathbb{R}, |\cdot|)$ *is* τ_ρ*-usc and* $\tau_{\bar\rho}$*-lsc.*
   For every fixed $y \in X$*, the mapping* $\rho(\cdot,y): X \to (\mathbb{R}, |\cdot|)$ *is* τ_ρ*-lsc and* $\tau_{\bar\rho}$*-usc.*
3. *The mapping* $\rho(x,\cdot): X \to (\mathbb{R}, |\cdot|)$ *is* τ_ρ*-continuous at* $x \in X$*, if and only if* τ_ρ*-*$\operatorname{cl}(B_\rho(x,r)) \subset B_\rho[x,r]$ *for all* $r > 0$*.*

Similar results hold for an asymmetric norm p*, its conjugate* $\bar p$ *and the associated norm* p^s*.*

Example 1.3. Consider on $\mathbb{R}$ the asymmetric norm $u(t) = t^+$, $t \in \mathbb{R}$, Then $\bar u(t) = (-t)^+ = t^-$ and $u^s(t) = |t|$. The closed unit ball is $B_u = (-\infty; 1]$ and its complement is $\mathbb{R} \setminus B_u = (1;\infty)$, and for any point $t \in (1;\infty)$ and $r > 0, B_u(t,r) = (-\infty; t+r) \not\subseteq (1;\infty)$. Consequently, B_u is not τ_u-closed.

The situation considered above can be extended to normed lattices.

Example 1.4. Let $(X, \|\cdot\|)$ be a normed lattice. Define the functional $p: X \to \mathbb{R}_+$ by $p(x) = \|x^+\|$, $x \in X$, where $x^+ = x \vee 0$. Then (X,p) is an asymmetric normed space, which is biBanach if X is a Banach lattice. The (symmetric) norm p^s associated to p is equivalent to the original norm $\|\cdot\|$ and the topology τ_p is only T_0 (see [2, 3, 6, 7]).

Completeness in quasi-metric spaces

In the case of a quasi-metric space (X,ρ) there are several notions of Cauchy sequence and more notions of completeness, see [11]. We present only the following two notions of Cauchy sequence.

A sequence (x_n) in X is called *left (right) K-Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$\forall n,k, \ \ n_0 \le k \le n \ \Rightarrow \ \rho(x_k,x_n) < \varepsilon \ \ (\text{respectively } \rho(x_n,x_k) < \varepsilon)\,.$$

Sometimes, to emphasize the quasi-metric ρ, we shall say that a sequence is left (right) ρ-K-Cauchy.

The quasi-semimetric space (X,ρ) is called:
- *left (right)* ρ*-K-complete* if every left (right) ρ-K-Cauchy sequence in X is ρ-convergent;
- *left (right) Smyth complete* if every left (right) ρ-K-Cauchy sequence in X is ρ^s-convergent;
- *bicomplete* if the metric space (X,ρ^s) is complete.

A bicomplete asymmetric normed space is called sometimes a *biBanach space.*
Similar notions can be defined for the conjugate quasi-metric $\bar\rho$.

Proposition 1.5 ([5], P. 1.2.6 and P.1.2.9).

1. *A quasi-semimetric space* (X,ρ) *is right K-complete if and only if any decreasing sequence of closed* $\bar\rho$*-balls* $B_{\bar\rho}[x_1,r_1] \supset B_{\bar\rho}[x_1,r_1] \supset \dots$ *with* $\lim_{n\to\infty} r_n = 0$ *has nonempty intersection.*
   If the topology τ_ρ *is Hausdorff, then* $\cap_{n=1}^\infty B_{\bar\rho}[x_n,r_n]$ *contains exactly one element.*

2. *A closed subset of a right (left) K-complete quasi-metric space is right (left) K-complete.*
3. *An asymmetric normed space (X,p) is left K-complete if and only if every absolutely convergent series is convergent.*

As usual, a series $\sum_n x_n$ in an asymmetric normed space (X,p) is called *absolutely convergent* if $\sum_{n=1}^{\infty} p(x_n) < \infty$.

Remark 1.6 ([1]). In an asymmetric normed space (X,p) the following equivalences hold true:

$$\begin{aligned}&\text{(i)} \qquad (x_n) \text{ is left (right) } p\text{-}K\text{-Cauchy} \iff (-x_n) \text{ is left (right) } \bar{p}\text{-}K\text{-Cauchy;}\\ &\text{(ii)} \qquad X \text{ is left (right) } p\text{-}K\text{-complete} \iff X \text{ is left (right) } \bar{p}\text{-}K\text{-complete.}\end{aligned}$$

Continuity of linear operators

Let (X,p) and (Y,q) be asymmetric normed spaces. A linear operator $T : X \to Y$ is (p,q)-continuous if and only if there exists $\beta > 0$ such that

$$\forall x \in X, \quad q(Tx) \le \beta p(x)\,.$$

Based on this property it is easy to check that the (p,q)-continuity of T is equivalent to its $(\bar{p},\bar{q})$-continuity. Indeed

$$\forall x \in X,\ q(Tx) \le \beta p(x) \iff \forall x \in X,\ \bar{q}(Tx) = q(T(-x)) \le \beta p(-x) = \beta \bar{p}(x)\,.$$

If T is linear and (p,q)-continuous then for every p-convergent series $x = \sum_n x_n$ in X, the series $\sum_n Tx_n$ is q-convergent in Y and $Tx = \sum_n Tx_n$. Indeed,

$$q\big(\sum_{k=1}^{n} Tx_k - Tx\big) = q\big(T(\sum_{k=1}^{n} x_k - x)\big) \le \beta p\big(\sum_{k=1}^{n} x_k - x\big) \xrightarrow[(n\to\infty)]{} 0\,.$$

Remark 1.7. All these properties, concerning the equivalence of various kinds of p- and $\bar{p}$-completeness or closedness (see Remarks 1.1 and 1.6), the equivalence of the (p,q) and $(\bar{p},\bar{q})$ continuity of linear operators, etc, cease to be true if instead of linear subspaces or spaces we consider arbitrary subsets, or cones in particular. For examples as well as for completeness results in some concrete asymmetrically normed Banach lattices see the paper [6].

Baire category in quasi-metric spaces

As it is known, many properties in Banach space theory depend, via completeness, on the Baire category. This is true in the asymmetric framework as well, so we need to see how Baire category looks in this case.

Proposition 1.8 ([5], P. 1.2.43). *A second category asymmetric LCS is a Baire space.*

Theorem 1.9 ([5], T. 1.2.44). *Let (X,ρ) be a quasi-semimetric space. If X is right $\bar{\rho}$-K-complete, then (X,τ_ρ) is of second category in itself.*

Baire category in bitopological spaces

A bitopological space, denoted by (T,τ,ν), is simply a set T endowed with two topologies τ,ν. Bitopological spaces were introduced by Kelly [9] and their study involves some notions relating the topologies τ,ν. A quasi-metric space (X,ρ) can be viewed as a bitopological space with respect to the topologies τ_ρ and $\tau_{\bar{\rho}}$.

We shall present now, following the paper [3], some Baire properties of bitopological spaces.

Let (T,τ,ν) be a bitopological space. A subset S of T is called

- (τ,ν)-*nowhere dense* if $\operatorname{int}_\tau(\operatorname{cl}_\nu(S))=\emptyset$;
- of (τ,ν)-*first category* if it is the union of a countable family of (τ,ν)-nowhere dense sets;
- of (τ,ν)-*second category* if it is not of (τ,ν)-first category;
- (τ,ν)-*residual* if $T\setminus S$ is of (τ,ν)-first category.

The space (T,τ,ν) is called

- (τ,ν)-*Baire* if each nonempty τ-open subset of T is of (τ,ν)-second category;
- *pairwise Baire* if it is both (τ,ν)-Baire and (ν,τ)-Baire.

Remark 1.10. We have slightly changed the terminology with respect to [3], in order to be in concordance with the notions of $(p,\bar{p})$-Baire category in an asymmetric normed space (X,p).

The following characterizations of the Baire property was proved in [3]. For the sake of completeness and since the terminology adopted here differs from that in [3], we include the proofs.

Theorem 1.11. *For any bitopological space (T,τ,ν) the following are eequivalent.*

1. *The space T is (τ,ν)-Baire.*
2. *For every family G_n, $n\in\mathbb{N}$, of ν-open τ-dense subsets of T, the intersection $\bigcap_n G_n$ is τ-dense in T.*
3. *For every family F_n, $n\in\mathbb{N}$, of ν-closed sets with $\operatorname{int}_\tau(F_n)=\emptyset$, for all $n\in\mathbb{N}$, it follows $\operatorname{int}_\tau\left(\bigcup_n F_n\right)=\emptyset$.*
4. *If $S\subset T$ is of (τ,ν)-first category, then $T\setminus S$ is τ-dense in T.*

For the proof we need the following lemma.

Lemma 1.12. *For a subset S of T the following equivalence holds*

$$\operatorname{int}_\tau \operatorname{cl}_\nu(S)=\emptyset \iff \operatorname{cl}_\tau \operatorname{int}_\nu(T\setminus S)=T\,. \tag{1.3}$$

Proof. Suppose $\operatorname{int}_\tau \operatorname{cl}_\nu(S)=\emptyset$. If $V\in\tau$ is nonempty, then, by hypothesis, $V\cap(T\setminus\operatorname{cl}_\tau(S))\neq\emptyset$. Since $T\setminus\operatorname{cl}_\tau(S)\subset\operatorname{int}_\nu(T\setminus S)$ implying $V\cap(T\setminus S)\neq\emptyset$. Since the nonempty set $V\in\tau$ was arbitrarily chosen, it follows $\operatorname{cl}_\tau\operatorname{int}_\nu(T\setminus S)=T$.

To prove the converse suppose that $\operatorname{int}_\tau\operatorname{cl}_\nu(S)\neq\emptyset$ and let $t\in\operatorname{int}_\tau\operatorname{cl}_\nu(S)$. Then $\operatorname{cl}_\nu(S)$ is a ν-neighborhood of t. If we show that $\operatorname{cl}_\nu(S)\cap(\operatorname{int}_\nu(T\setminus S)=\emptyset$, then $t\notin\operatorname{cl}_\tau\operatorname{int}_\nu(T\setminus S)$, so that $\operatorname{cl}_\tau\operatorname{int}_\nu(T\setminus S)\neq T$.

Indeed, if it would exist $s\in\operatorname{cl}_\nu(S)\cap\operatorname{int}_\nu(T\setminus S)$, then $\operatorname{int}_\nu(T\setminus S)$ would be a ν-open neighborhood of s, yielding the contradiction

$$\emptyset\neq S\cap\operatorname{int}_\nu(T\setminus S)\subset S\cap(T\setminus S)=\emptyset\,.$$

□

Proof of theorem 1.11. $1 \Rightarrow 2$. If the sets G_n are ν-open and τ-dense, then $\operatorname{cl}_\tau \operatorname{int}_\nu(G_n) = \operatorname{cl}_\tau(G_n) = T$, so that, by (1.3) $\operatorname{int}_\tau \operatorname{cl}_\nu(G_n) = \emptyset$, that is the sets $T \setminus G_n$ are ν-closed and (τ, ν)-nowhere dense. But

$$\bigcup_n (T \setminus G_n) = T \setminus \big(\bigcap_n G_n\big) \supset T \setminus \operatorname{cl}_\tau \big(\bigcap_n G_n\big).$$

If $G := T \setminus \operatorname{cl}_\tau \left(\bigcap_n G_n\right) \neq \emptyset$, then G would be nonempty τ-open and of (τ, ν)-first category, in contradiction to the fact that T is (τ, ν)-Baire. Consequently $G = \emptyset$, that is $\operatorname{cl}_\tau \left(\bigcap_n G_n\right) = T$.

$2 \Rightarrow 3$. Let $F_n \subset T$, $n \in \mathbb{N}$, be ν-closed with $\operatorname{int}_\tau(F_n) = \emptyset$. It follows $\operatorname{int}_\tau \operatorname{cl}_\nu(F_n) = \operatorname{int}_\tau(F_n) = \emptyset$, so that, by (1.3), $\operatorname{cl}_\tau(T \setminus F_n) = \operatorname{cl}_\tau \operatorname{int}_\nu(T \setminus F_n) = T$. Consequently the sets $G_n := T \setminus F_n$ are ν-open and τ-dense in T, so that, by hypothesis, $\operatorname{cl}_\tau \left(\bigcap_n G_n\right) = T$.

If $V := \operatorname{int}_\tau \left(\bigcup_n F_n\right) \neq \emptyset$, then $T \setminus V$ is τ-closed. The inclusion

$$\bigcap_n G_n = T \setminus \bigcup_n F_n \subset T \setminus V,$$

yields the contradiction

$$\operatorname{cl}_\tau \big(\bigcap_n G_n\big) \subset T \setminus V \neq T.$$

$3 \Rightarrow 4$. Let $S = \bigcup_n S_n$ be a set of (τ, ν)-first category, that is $\operatorname{int}_\tau \operatorname{cl}_\nu(S_n) = \emptyset$ for all $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, the set $F_n := \operatorname{cl}_\nu(S_n)$ is ν-closed and satisfies $\operatorname{int}_\tau(F_n) = \emptyset$, so that, by hypothesis, $\operatorname{int}_\tau \left(\bigcup_n F_n\right) = \emptyset$. For an arbitrary nonempty set $V \in \tau$, $V \cap \left(T \setminus \bigcup_n F_n\right) = \emptyset$ would imply $V \subset \bigcup_n F_n$, and so $\operatorname{int}_\tau(F_n) \neq \emptyset$, a contradiction. Consequently, $V \cap \left(T \setminus \bigcup_n F_n\right) \neq \emptyset$ for every nonempty set $V \in \tau$, showing that $T \setminus \bigcup_n F_n$ is τ-dense in T. But then the set $T \setminus S \supset T \setminus \bigcup_n F_n$ will be also τ-dense in T.

$4 \Rightarrow 1$. Suppose that T is not (τ, ν)-Baire. Then there exists a nonempty τ-open subset G of T that is of (τ, ν)-first category. It follows that $\operatorname{cl}_\tau(T \setminus G) = T \setminus G \neq T$, that is the set $T \setminus G$ is not τ-dense in T.

In the case of an asymmetric normed space (X, p) we use the notation $(p, \bar{p})$ instead of $(\tau_p, \tau_{\bar{p}})$. As in the case of Baire category we have the following result.

Proposition 1.13. *If an asymmetric normed space (X, p) is of second $(p, \bar{p})$-category, then it is $(p, \bar{p})$-Baire.*

Proof. Suppose that there exists a nonempty τ-open subset G of X that is of first $(p, \bar{p})$-category. If $x_0 \in G$ then $U := -x_0 + G$ is a τ-open neighborhood of 0 which is still of first $(p, \bar{p})$-category. If we prove that $X = \bigcup_{k=1}^{\infty} kU$, then the space X would be of first $(p, \bar{p})$-category too, in contradiction to the hypothesis.

Let $x \in X$. Then $p(n^{-1}x) = n^{-1}p(x) \longrightarrow 0$ as $n \to \infty$, that is $n^{-1}x \xrightarrow{p} 0$. It follows that there exists $k \in \mathbb{N}$ such that $k^{-1}x \in U \iff x \in kU$, so that $X = \bigcup_{k=1}^{\infty} kU$. □

2. Zabrejko's lemma

The aim of the present Note is to prove asymmetric versions of the open mapping theorem, the closed graph theorem and the uniform boundedness principle, based on the Zabrejko's lemma [12], see also [10, p. 172].

Let (X, p) be an asymmetric normed space. The notation $x = p\text{-}\sum_{n=1}^{\infty} x_n$ means that the series is p-convergent with sum x, that is $\lim_{n\to\infty} p\left(\sum_{k=1}^{n} x_k - x\right) = 0$. The sum of a $\bar{p}$-convergent series is denoted by $x = \bar{p}\text{-}\sum_{n=1}^{\infty} x_n$, which is equivalent to $\lim_{n\to\infty} p\left(x - \sum_{k=1}^{n} x_k\right) = 0$.

Observe that

$$x = \bar{p}\text{-}\sum_{n=1}^{\infty} x_n \implies p(x) \le \sum_{n=1}^{\infty} p(x_n)\,. \tag{2.1}$$

Indeed, the inequality

$$p(x) \le p\big(x - \sum_{k=1}^{n} x_k\big) + \sum_{k=1}^{n} p(x_k)\,,$$

yields for $n \to \infty$,

$$p(x) \le \sum_{k=1}^{\infty} p(x_k)\,.$$

A positive sublinear functional φ defined on an asymmetric normed space (X, p) is called p-σ-subadditive if for every p-convergent series $\sum_n x_n$ in X

$$x = p\text{-}\sum_{n=1}^{\infty} x_n \implies \varphi(x) \le \sum_{n=1}^{\infty} \varphi(x_n)\,. \tag{2.2}$$

The notion of $\bar{p}$-σ-subadditive functional is defined similarly.

Remark 2.1. A functional $\varphi : X \to \mathbb{R}$ is called sublinear if it is positively homogeneous and subbaditive. Note that the p-σ-subadditivity of φ implies its subadditivity, so it is sufficient to suppose that φ is only positive and positively homogeneous.

Proposition 2.2 (Zabrejko's lemma). *Suppose that the asymmetric normed space (X, p) is of second $(p, \bar{p})$-Baire category. If the sublinear functional $\varphi : X \to [0; \infty)$ is $\bar{p}$-σ-subadditive, then there exists $\beta > 0$ such that*

$$\varphi(x) \le \beta p(x)$$

for all $x \in X$.

The proof will be based on the following lemma.

Lemma 2.3. *Let A be a subset of an asymmetric normed space (X, p). If $B_p \subset \operatorname{cl}_{\bar{p}}(A)$, then for every $x \in B_p$ there exists a sequence (x_n) in A such that*

$$x = \bar{p}\text{-}\sum_{n=1}^{\infty} \frac{x_n}{2^{n-1}}\,.$$

Proof. The proof goes by induction. Let x in B_p. By hypothesis there exists $x_1 \in A$ such that

$$p(x - x_1) \le \frac{1}{2}\,.$$

Suppose that we found $x_1, \dots, x_n$ in B_p such that

$$p\Big(x - \sum_{k=1}^{n} \frac{x_k}{2^{k-1}}\Big) \le \frac{1}{2^n}\,.$$

Then $2^n\left(x - \sum_{k=1}^{n} \frac{x_k}{2^{k-1}}\right) \in B_p$, so that there exists $x_{n+1} \in A$ such that

$$p\Big(2^n\big(x - \sum_{k=1}^{n} \frac{x_k}{2^{k-1}}\big) - x_{n+1}\Big) \le \frac{1}{2} \iff p\Big(x - \sum_{k=1}^{n+1} \frac{x_k}{2^{k-1}}\Big) \le \frac{1}{2^{n+1}}\,.$$

□

Proof of Proposition 2.2. Let $E_n := \{x \in X : \varphi(x) \le n\}$, $n \in \mathbb{N}$. Then $X = \bigcup_n E_n$. Since the space X is of second $(p, \bar{p})$-Baire category, there exists $j \in \mathbb{N}$ such that $\operatorname{int}_p(\operatorname{cl}_{\bar{p}}(E_j) \ne \emptyset$. It follows that there exists $x_0 \in X$ and $r > 0$ such that

$$x_0 + rB_p \subset \operatorname{cl}_{\bar{p}}(E_j) \iff B_p \subset \frac{1}{r}\big(-x_0 + \operatorname{cl}_{\bar{p}}(E_j)\big) = \operatorname{cl}_{\bar{p}}\Big(\frac{1}{r}(-x_0 + E_j)\Big)\,.$$

Let $x \in B_p$. By Lemma 2.3 there exists a sequence (x_n) in E_j such that

$$x = \bar{p}\text{-}\sum_{n=1}^{\infty} \frac{-x_0 + x_n}{2^{n-1}r}\,,$$

implying

$$\varphi(x) \le \sum_{n=1}^{\infty} \frac{\varphi(-x_0 + x_n)}{2^{n-1}r}\,.$$

By the definition of the set E_j,

$$\varphi(-x_0 + x_n) \le \varphi(-x_0) + \varphi(x_n) \le \varphi(-x_0) + j\,,$$

implying

$$\varphi(x) \le \frac{2(j + \varphi(-x_0))}{r}\,.$$

Since this inequality holds for every $x \in B_p$, it follows

$$\varphi(x) \le \frac{2(j + \varphi(-x_0))}{r}\, p(x)\,,$$

for all $x \in X$.

3. The open mapping theorem

The following version of the open mapping theorem was proved by Alegre [1].

Theorem 3.1 (see [5], T.2.3.1). *Let (X, p) and (Y, q) be asymmetric normed spaces. Suppose that (X, p) is right p-K-complete and Y is Hausdorff and of second $(q, \bar{q})$-Baire category. If $T : X \to Y$ is linear, surjective and (p, q)-continuous, then for every p-open subset G of X, $T(G)$ is q-open in Y.*

Based on Zabrejko's lemma we can prove the following version of the open mapping theorem.

Theorem 3.2. *Let (X,p), (Y,q) be asymmetric normed spaces. Suppose that X is right p-K-complete and that Y is of second $(q,\bar{q})$-Baire category. If $T:X\to Y$ is a linear, (p,q)-continuous and surjective, then for every p-open subset G of X the set $T(G)$ is q-open in Y.*

Proof. Define $\varphi : Y \to \mathbb{R}_+$ by

$$\varphi(y)=\inf\{p(x): x\in X, Tx=y\},\quad y\in Y,$$

and show that φ is $\bar{q}$-σ-subadditive.

Let $y=\bar{q}\text{-}\sum_{n=1}^{\infty} y_n$ and $\varepsilon>0$. Without restricting the generality we can suppose $\sum_{n=1}^{\infty}\varphi(y_n)$ $<\infty$. By the definition of the function φ, for every $n\in\mathbb{N}$ there exists $x_n\in X$ such that

$$Tx_n=y_n \quad \text{and} \quad p(x_n)\le \varphi(y_n)+\frac{\varepsilon}{2^n}\,.$$

Then

$$\sum_{n=1}^{\infty} p(x_n)\le \varepsilon+\sum_{n=1}^{\infty}\varphi(y_n)<\infty\,. \tag{3.1}$$

Because $p(s_{n+k}-s_n)\le\sum_{i=1}^{k}p(x_{n+i})$, it follows that the partial sum sequence $s_n=\sum_{k=1}^{n}x_k$, $n\in\mathbb{N}$, is right $\bar{p}$-K-Cauchy. By hypothesis and Remark 1.6.(ii), there exists $x\in X$ such that $x=\bar{p}\text{-}\sum_{n=1}^{\infty}x_n$. By the linearity and the $(\bar{p},\bar{q})$-continuity of T,

$$Tx=\bar{q}\text{-}\sum_{n=1}^{\infty}Tx_n=\bar{q}\text{-}\sum_{n=1}^{\infty}y_n=y\,. \tag{3.2}$$

Taking into account (3.1) and (3.2) one obtains

$$\varphi(y)\le p(x)\le\sum_{n=1}^{\infty}p(x_n)\le\varepsilon+\sum_{n=1}^{\infty}\varphi(y_n)\,,$$

that is

$$\varphi(y)\le\varepsilon+\sum_{n=1}^{\infty}\varphi(y_n)\,.$$

Since $\varepsilon>0$ is arbitrary, this implies

$$\varphi(y)\le\sum_{n=1}^{\infty}\varphi(y_n)\,,$$

so that, by Proposition 2.2, there exists $\beta>0$ such that

$$\varphi(y)\le\beta q(y)\,,$$

for all $y\in Y$.

Taking $\gamma:=(1+\beta)^{-1}$ one obtains the implication

$$q(y)<\gamma \implies \varphi(y)<1\,,$$

which in its turn yields

$$\gamma B'_q \subset T(B'_p)\,. \tag{3.3}$$

Indeed, if $q(y) < \gamma$, then $\varphi(y) < 1$. By the definition of φ there exists $x \in X$ with $Tx = y$ such that $p(x) < 1$, that is $y \in T(B'_p)$.

Finally, show that (3.3) implies the openness of the mapping T. Let G be an open subset of X and let $y_0 \in T(G)$. If $x_0 \in G$ is such that $Tx_0 = y_0$, then, taking $r > 0$ such that $x_0 + rB'_p \subset G$, one obtains

$$y_0 + r\gamma B'_q \subset Tx_0 + rT(B'_p) = T(x_0 + rB'_p) \subset T(G)\,.$$

□

A consequence of the open mapping theorem is the inverse mapping theorem which, in essence, is an equivalent form of it.

Corollary 3.3. *Let (X, p) and (Y, q) be two asymmetric normed spaces. If (X, p) is right p-K-complete and (Y, q) is of second $(q, \bar{q})$-Baire category, then the inverse of any bijective (p, q)-continuous linear mapping $T : X \to Y$ is (q, p)-continuous.*

4. The closed graph theorem

As in the case of Banach spaces, the closed graph theorem can easily be derived from the open mapping theorem, but can be also proved directly, based on Zabrejko's lemma.

The graph Γ_f of a mapping $f : X \to Y$ is the subset of $X \times Y$ given by $\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}$. For two asymmetric normed spaces (X, p) and (Y, q) consider $X \times Y$ endowed with the asymmetric norm

$$r(x, y) = p(x) + q(y),\ (x, y) \in X \times Y. \tag{4.1}$$

The proof of the results from the following proposition are similar to those in the symmetric case.

Proposition 4.1. *Let (X, p), (Y, q) be asymmetric normed spaces.*

1. *The asymmetric norm r defined by (4.1) generates the product topology $\tau_p \times \tau_q$ on $X \times Y$.*
2. *If (X, p) and (Y, q) are right (left) K-complete, then $(X \times Y, r)$ is right (left) K-complete.*

The same assertions hold with respect to the conjugate norms $\bar{p}$, $\bar{q}$ and $\bar{r}$.

Theorem 4.2 (The closed graph theorem). *Let $(X, p), (Y, q)$ be asymmetric normed spaces. Suppose that (X, p) is right p-K-complete and of second $(p, \bar{p})$-Baire category, and (Y, q) is right q-K-complete. If $T : X \to Y$ is a linear operator with $\tau_p \times \tau_q$-closed graph, then T is (p, q)-continuous.*

Proof. In the proof we shall use again Remark 1.6.(ii).

Show first that the functional $\varphi : X \to \mathbb{R}_+$ defined by

$$\varphi(x) = q(Tx),\ x \in X,$$

is $\bar{p}$-σ-subadditive. Let $x = \bar{p}\text{-}\sum_{n=1}^{\infty} x_n$. Suppose that $\sum_{n=1}^{\infty} q(Tx_n) = \sum_{n=1}^{\infty} \varphi(x_n) < \infty$. Then the sequence $\eta_n = \sum_{k=1}^{n} Tx_k$ is right $\bar{q}$-K-Cauchy, so there exists $y \in Y$ such that $\eta_n \xrightarrow{\bar{q}} y$. Denoting $\xi_n = \sum_{k=1}^{n} x_k$ it follows $\eta_n = T\xi_n$, and we have the following situation

$$\xi_n \xrightarrow{\bar{p}} x, \quad \eta_n \xrightarrow{\bar{q}} y \quad \text{and} \quad \forall n \in \mathbb{N},\ (\xi_n, \eta_n) \in \Gamma_T\,.$$

The $\tau_p \times \tau_q$-closedness of the linear subspace Γ_T of $X \times Y$ implies its $\tau_{\bar{p}} \times \tau_{\bar{q}}$-closedness, so that the above conditions imply

$$(x, y) \in \Gamma_T \iff y = Tx\,.$$

It follows $Tx = \bar{q}\text{-}\sum_{n=1}^{\infty} Tx_n$ so that

$$\varphi(x) = q(Tx) \le \sum_{n=1}^{\infty} q(Tx_n) = \sum_{n=1}^{\infty} \varphi(x_n)\,,$$

proving that φ is $\bar{p}$-σ-subadditive. By Proposition 2.2 there exists $\beta > 0$ such that for all $x \in X$

$$\varphi(x) \le \beta p(x) \iff q(Tx) \le \beta p(x)\,,$$

which is equivalent to the continuity of T.

A proof based on the open mapping theorem.

The projection $P : \Gamma_T \to X$ defined by $P(x, Tx) = x$, $x \in X$, is continuous and bijective. By Corollary 3.3, $P^{-1} : X \to \Gamma_T$ is also continuous, so that there exists $\beta > 0$ such that $r(P^{-1}(x)) \le \beta p(x)$, which is equivalent to $p(x) + q(Tx) \le \beta p(x)$. Consequently,

$$q(Tx) \le (\beta - 1)p(x)\,,$$

for all $x \in X$, proving the continuity of T. □

5. The uniform boundedness principle

Let $(X, p), (Y, q)$ be asymmetric normed space. One denotes by $L_{p,q}(X, Y)$ the cone of all (p, q)-continuous linear operators from X to Y. A family $\mathcal{T} = \{T_i : i \in I\} \subset L_{p,q}(X, Y)$ is called *pointwisely bounded* if

$$\forall x \in X, \quad \sup_{i \in I} q(T_i x) < \infty\,. \tag{5.1}$$

In this case the condition (5.1) is equivalent to

$$\forall x \in X, \quad \sup_{i \in I} \bar{q}(T_i x) < \infty\,. \tag{5.2}$$

The following version of the uniform boundedness principle was proved in [5].

Theorem 5.1. *Let (X, p) be a right p-K-complete asymmetric normed space, (Y, q) an asymmetric normed space and $\mathcal{T} = \{T_i : i \in I\} \subset L_{p,q}(X, Y)$ Suppose that the family $\mathcal{T}$ is pointwisely bounded, that is*

$$M_{\mathcal{T}}(x) := \sup_{i \in I} q(T_i x) < \infty\,, \tag{5.3}$$

for every $x \in X$. Then

$$\sup_{i \in I} \sup_{x \in B_p} \bar{q}(T_i x) < \infty \quad \text{and} \quad \sup_{i \in I} \sup_{x \in B_{\bar{p}}} q(T_i x) < \infty\,. \tag{5.4}$$

Proof. As the proof given in [5] is not correct (see the comments in the next section) we give here a different one. Since X is right p-K-complete, it is of second $\bar{p}$-category (see Theorem 1.9). For $n \in \mathbb{N}$ and $i \in I$ let

$$E_{n,i} = \{x \in X : q(T_i x) \le n\} \quad \text{and} \quad E_n = \{x \in X : \forall i \in I,\, q(T_i x) \le n\}\,.$$

Since the function $q(\cdot)$ is $\bar{q}$-lsc and T_i is also $(\bar{p}, \bar{q})$-continuous, it follows that $q \circ T_i$ is $\bar{p}$-lsc, so that $E_{n,i}$ is $\bar{p}$-closed and $E_n = \bigcap_{i\in I} E_{n,i}$ as well.

The pointwise boundedness of the family $\mathcal{T}$ implies $X = \bigcup_{n\in\mathbb{N}} E_n$. Since X is of second $\bar{p}$-category, there exists $n_0 \in \mathbb{N}$ such that $\mathrm{int}_{\bar{p}}(E_{n_0}) \ne \emptyset$. It follows that there exist $x_0 \in X$ and $r_0 > 0$ such that $B_{\bar{p}}[x_0, r_0] = x_0 + r_0 B_{\bar{p}} \subset E_{n_0}$. Consequently,

$$\forall i \in I,\ \forall x' \in x_0 + r_0 B_{\bar{p}}, \quad q(T_i x') \le n_0\,.$$

For $x \in B_{\bar{p}}$, $x' = x_0 + r_0 x \in x_0 + r_0 B_{\bar{p}}$ and $x = r_0^{-1}(x' - x_0)$. It follows that

$$q(T_i x) \le \frac{q(T_i x') + q(T_i(-x_0))}{r_0} \le \frac{n_0 + M_{\mathcal{T}}(-x_0)}{r_0}\,,$$

for all $x \in B_{\bar{p}}$ and all $i \in I$. Consequently

$$\sup_{i\in I}\sup_{x\in B_{\bar{p}}} q(T_i x) \le \frac{n_0 + M_{\mathcal{T}}(-x_0)}{r_0} < \infty\,.$$

The first inequality in (5.4) can be proved similarly by considering the sets

$$F_{n,i} = \{x \in X : \bar{q}(T_i x) \le n\} \quad \text{and} \quad F_n = \{x \in X : \forall i \in I,\, \bar{q}(T_i x) \le n\}\,,$$

although the conditions from (5.4) are in fact equivalent, see Remark 5.2 below.

□

Remark 5.2. The conditions from (5.4) are equivalent.

Indeed

$$\sup_{i\in I}\sup_{x\in B_p} \bar{q}(T_i x) = \infty \iff \forall n \in \mathbb{N},\ \exists (i_n, x_n) \in I \times B_p \quad \text{s.t.} \quad \bar{q}(T_{i_n} x_n) > n\,.$$

But

$$(i_n, x_n) \in I \times B_p \quad \text{and} \quad \bar{q}(T_{i_n} x_n) > n \quad \overset{y_n = -x_n}{\iff} \quad (i_n, y_n) \in I \times B_{\bar{p}} \quad \text{and} \quad q(T_{i_n} y_n) > n\,,$$

showing that

$$\sup_{i\in I}\sup_{x\in B_p} \bar{q}(T_i x) = \infty \iff \sup_{i\in I}\sup_{y\in B_{\bar{p}}} q(T_i y) = \infty\,.$$

The following version of the Uniform Boundedness Principle was proved in [4].

Theorem 5.3. *Let $(X,p),(Y,q)$ be asymmetric normed spaces such that (X,p) is of second $(p,\bar{p})$-Baire category and let T_i, $i\in I$, be a family of (p,q)-continuous linear operators from X to Y. If the family $\{T_i\}$ is pointwisely q-bounded, that is*

$$\forall x\in X,\quad \sup_{i\in I} q(T_i x)<\infty\,,$$

then it is uniformly bounded on B_p, that is

$$\sup_{i\in I}\sup_{x\in B_p} q(T_i x)<\infty\,.$$

Proof. We give a proof based on Zabrejko's lemma. Let the functional $\varphi: X\to\mathbb{R}_+$ be defined by

$$\varphi(x)=\sup_{i\in I} q(T_i x),\ x\in X.$$

and show first that φ is $\bar{p}$-σ-subadditive. Indeed, if $x=\bar{p}\text{-}\sum_{n=1}^{\infty}x_n$, then, for every $i\in I$, the (p,q)-continuity of T_i implies its $(\bar{p},\bar{q})$-continuity, so that $T_i x=\bar{q}\text{-}\sum_{n=1}^{\infty}T_i x_n$ and

$$q(T_i x)=q\big(\sum_{n=1}^{\infty}T_i x_n\big)\le\sum_{n=1}^{\infty}q(T_i x_n)\le\sum_{n=1}^{\infty}\varphi(x_n)\,.$$

Consequently,

$$\varphi(x)=\sup_{i\in I} q(T_i x)\le\sum_{n=1}^{\infty}\varphi(x_n)\,.$$

By Proposition 2.2, there exists $\beta>0$ such that

$$\varphi(x)\le\beta p(x)\,,$$

for all $x\in X$. It follows

$$\forall x\in B_p,\quad \varphi(x)\le\beta\,,$$

that is

$$\sup_{i\in I}\sup_{x\in B_p} q(T_i x)=\sup_{x\in B_p}\sup_{i\in I} q(T_i x)\le\beta\,.$$

□

Remarks 5.4. 1. In [4] the term half second Baire category is used instead of second $(p,\bar{p})$-Baire category.

2. As it is shown by Example 2.2 in [4], if the asymmetric normed space (X,p) is biBanach, then the uniform boundedness principle could not hold, even for linear functionals.

3. If (X,p) is the asymmetric normed space associated to a Banach lattice $(X,\|\cdot\|)$ (as in Example 1.4), then (X,p) is not right K-complete, but it is of second $(p,\bar{p})$-Baire category (see [4]).

6. Some remarks on the condensation of singularities in the asymmetric case

This is a stronger form of the uniform boundedness principle. Let $T_i : X \to Y$, $i \in I$, be a family of continuous linear operators between two asymmetric normed spaces (X, p) and (Y, q). Put $\mathcal{T} = \{T_i : i \in I\}$. The set

$$S_{\mathcal{T}} = \{x \in X : \sup_{i \in I} q(T_i x) = \infty\} \tag{6.1}$$

is called the set of singularities of the family $\mathcal{T}$.

In the symmetric case the principle of the condensation of singularities asserts that if a family $\mathcal{T} := \{T_i : i \in I\} \subset L(X, Y)$ of continuous linear mappings between a Banach space X and a normed space Y is not uniformly bounded, then the set of singularities is G_δ and dense in X. This means that

$$\sup_{i \in I} \|T_i\| = \infty \implies S_{\mathcal{T}} \text{ is of } G_\delta\text{-type and dense in } X.$$

In [5] one asserts that the proof given to the Uniform Boundedness Principle ([5, Theorem 2.3.7]) yields the principle of the condensation of singularities. In the proof one supposes that the family $\mathcal{T}$ is q-pointwise bounded but not $\bar{q}$-pointwise bounded, that is $\mathcal{T}$ satisfies (5.1) but not (5.2), that is obviously false since, as we have remarked, these two conditions are equivalent.

We want to point out the main difficulty that arises in the tentative to prove a Principle of Condensation of Singularities in the asymmetric case.

Consider a family $\mathcal{T} = \{T_i : i \in I\} \subset L_{p,q}(X, Y)$ of (p, q)-continuous linear mappings between two asymmetric normed spaces (X, p), (Y, q) and suppose that (X, p) is $(p, \bar{p})$-Baire. Suppose that the family $\mathcal{T}$ is not uniformly bounded, that is

$$\sup_{i \in I} \sup_{x \in B_p} q(T_i x) = \infty\,. \tag{6.2}$$

Consider the sets

$$X_{n,i} := \{x \in X : q(T_i x) > n\}, \quad (n, i) \in \mathbb{N} \times I, \tag{6.3}$$

and

$$X_n := \bigcup_{i \in I} X_{n,i} = \{x \in X : \exists i \in I,\ \ q(T_i x) > n\}, \quad n \in \mathbb{N}\,. \tag{6.4}$$

It is obvious that

$$S_{\mathcal{T}} = \bigcap_{n \in \mathbb{N}} X_n\,,$$

so if we show that each set X_n is $\bar{p}$-open and p-dense, then, by Theorem 1.11, the set $S_{\mathcal{T}}$ will be $\bar{p}$-G_δ and p-dense.

Since the asymmetric norm q is $\bar{q}$-lsc (see Proposition 1.2) and the operator T_i is also $(\bar{p}, \bar{q})$-continuous, it follows that $q \circ T_i$ is $\bar{p}$-lsc, implying that the set $X_{n,i}$ is $\bar{p}$-open for every $i \in I$, and X_n as well.

The proof will be done if we show that X_n is p-dense in X. Supposing the contrary, there exist $x_0 \in X$ and $r > 0$ such that $B_p[x_0, r] \cap X_n = \emptyset$, implying

$$q(T_i x') \le n\,,$$

for all $x' \in B_p[x_0, r]$ and all $i \in I$. Since $B_p[x_0, r] = x_0 + rB_p$, it follows that every $x' \in B_p[x_0, r]$ is of the form

$$x' = x_0 + rx \iff x = \frac{1}{r}(x' - x_0)\,,$$

for some $x \in B_p$. But then for every $i \in I$ and $x \in B_p$,

$$q(T_i x) \le \frac{q(T_i(x' - x_0)}{r} \le \frac{q(T_i x') + q(-T_i x_0)}{r} \le \frac{n + q(T_i(-x_0))}{r}\,.$$

To go further and to deduce the uniform boundedness of the family $\mathcal{T}$ and to obtain so a contradiction to (6.2), we need that $\sup_{i\in I} q(T_i(-x_0)) < \infty$. In the symmetric case this bound is n, because $q(T_i(-x_0)) = q(-T_i(x_0)) = q(T_i(x_0)) \le n$, for every $i \in I$, an inequality that could not hold in the asymmetric case.

Consequently, versions of the Principle of Condensation of Singularities in the asymmetric case remain to be proved.

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