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Appendix A. Proof of Theorem 3.1

Proof. We want to prove that for non-negative initial condition, at all time $t \geq 0$, the system (11) has a unique non-negative solution which is contained in Ω. The proof is done in three steps: Firstly we show the non-negativity of the solutions for any non-negative initial data; secondly we establish the boundedness of the solution and finally we establish uniqueness of the solution.

To prove the non-negativity, we use the method of contradiction as it is in.[46,47] Without loss of generality, we may assume that the trajectory of R will pass to the negative region before others, i.e., we consider the trajectory of R crosses to the region $R < 0$ at some positive time t_1, such that

$$\begin{aligned} &R(t_1) = 0,\ R'(t_1) < 0,\ \text{and} \\ &A(t_1) > 0,\ I(t_1) > 0, A_1(t_1) > 0, I_1(t_1) > 0,\ \ Q(t_1) > 0. \end{aligned} \tag{A.1}$$

Then, from equation (11h) we have,

$$R'(t_1) = \gamma_1 A(t_1) + \gamma_2 I(t_1) + \gamma_3 Q(t_1) + \gamma_4 A_1(t_1) + \gamma_5 I_1(t_1). \tag{A.2}$$

Observe that, due to assumption (A.1), the left hand side of (A.2) is negative while the right hand side is positive, which is a contradiction. Hence, $R(t)$ remains non-negative for all $t \geq 0$. From equation (11a) we have

$$S'(t) = \Lambda - [\lambda + \sigma + \mu]S + \varphi R \geq -[\lambda + (\sigma + \mu)]S. \tag{A.3}$$

Applying simple calculus techniques to (A.3), we obtain

$$S(t) \geq S(0)\exp\left(-\int_0^t (\lambda(u) + \sigma + \mu)\, du\right) \geq 0.$$

Thus, $S(t)$ remains non-negative for all $t \geq 0$. Similarly, from equation (11b) we have

$$V'(t) = \sigma S - [(1-\rho)\lambda + \mu]V + (\omega - \varphi)R \geq -[(1-\rho)\lambda + \mu]V, \tag{A.4}$$

which yields

$$V(t) \geq V(0)\exp\left(-\int_0^t ((1-\rho)\lambda(u) + \mu)\, du\right) \geq 0.$$

Hence, $V(t)$ also remains non-negative for any $t \geq 0$.

To show the non-negativity of the variables A, A_1, I, I_1 and Q one can follow a procedure similar to the one used to show the non-negativity of R.

To proof the boundedness of the system, we use principle of conservation. From (1) and (11), we obtain

$$N'(t) = \Lambda - \mu N - \delta(I + Q) - \delta_1 I_1 \leq \Lambda - \mu N. \tag{A.5}$$

For an initial population N_0, implementing Gronwall's inequality on (A.5) gives

$$N(t) \leq \Lambda/\mu + (N_0 - \Lambda/\mu)\exp(-\mu t) < \infty. \tag{A.6}$$

Hence, the solution of the model is bounded for every time $t \geq 0$.

Finally, the uniqueness follows from Steps 1 and 2 and Theorem 2.1.5 of.[48] Thus we are guaranteed that any solution of (11) is non-negative and bounded for $t \geq 0$. Thus, the model equation (11) is a dynamical system on Ω. This completes the proof of Theorem 3.1.

□

Remark Appendix A.1. *Equation*(A.6) *tells us that the total population at any given time* $N(t)$ *remains bounded. In fact*

$$\lim_{t\to\infty} (\Lambda/\mu + (N_0 - \Lambda/\mu)\exp(-\mu t)) = \frac{\Lambda}{\mu}.$$

Thus, the set

$$\tilde{\Omega} = \left\{(S, V, A, I, A_1, I_1, Q, R) \in \mathbb{R}^8_+ : 0 \leq S + V + A + I + A_1 + I_1 + Q + R = N \leq \frac{\Lambda}{\mu}\right\} \subset \Omega$$

is an attractor set of the system (11).

Appendix B. Calculation of the Basic Reproduction Number

The basic reproduction number refers the average number of secondary cases produced in a completely susceptible population by an infectious individual during his/her entire infectious period.[20] We compute $\mathcal{R}_0$ by using the method of Next Generation Matrix, see.[18,19,20,21]

For the model under consideration, we denote infected classes by $\mathcal{A}$ and define vector valued functions $\mathcal{F} : \mathcal{A} \to \mathbb{R}^5$ and $\mathcal{U} : \mathcal{A} \to \mathbb{R}^5$ by

$$\mathcal{F}(X) = \begin{pmatrix} \eta\lambda S \\ (1-\eta)\lambda S \\ \phi(1-\rho)\lambda V \\ (1-\phi)(1-\rho)\lambda V \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{U}(X) = \begin{pmatrix} k_1 A \\ k_2 I \\ k_3 A_1 \\ k_4 I_1 \\ -\theta A - \theta_1 A_1 - \epsilon I - \epsilon_1 I_1 + k_5 Q \end{pmatrix}$$

where

$$\mathcal{A} = \{(A, I, A_1, I_1, Q) : (S, V, A, I, A_1, I_1, Q, R) \in \Omega\}.$$

The function $\mathcal{F}$ represents the rate of appearance of new infection and $\mathcal{U}$ denotes the rate of transfer of individuals among the infective classes, respectively, where

$$k_1 = \theta + \gamma_1 + \mu,\ k_2 = \epsilon + \gamma_2 + \delta + \mu,\ k_3 = \theta_1 + \gamma_4 + \mu,\ k_4 = \epsilon_1 + \gamma_5 + \delta_1 + \mu$$

and

$$k_5 = \gamma_3 + \delta + \mu.$$

The next generation matrix is given by

$$\mathcal{K} = J_{\mathcal{F}} J_{\mathcal{U}}^{-1}, \tag{B.1}$$

where

$$J_{\mathcal{F}} = \begin{pmatrix} B_1\nu & B_1 & B_1\nu_1 & B_1\kappa & 0 \\ B_2\nu & B_2 & B_2\nu_1 & B_2\kappa & 0 \\ B_3\nu & B_3 & B_3\nu_1 & B_3\kappa & 0 \\ B_4\nu & B_4 & B_4\nu_1 & B_4\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_{\mathcal{U}} = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ -\theta & -\theta_1 & -\epsilon & -\epsilon_1 & k_5 \end{pmatrix} \tag{B.2}$$

are the Jacobian matrices of $\mathcal{F}$ and $\mathcal{U}$ at E_0, respectively with

$$B_1 = \frac{\eta\beta\mu}{\sigma + \mu},\ B_2 = \frac{(1-\eta)\beta\mu}{\sigma + \mu},\ B_3 = \frac{\phi(1-\rho)\beta\sigma}{\sigma + \mu} \text{ and } B_4 = \frac{(1-\phi)(1-\rho)\beta\sigma}{\sigma + \mu}.$$

Notice that

$$J_{\mathcal{U}}^{-1} = \begin{pmatrix} \frac{1}{k_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} & 0 \\ \frac{\theta}{k_1 k_5} & \frac{\theta_1}{k_2 k_5} & \frac{\epsilon}{k_3 k_5} & \frac{\epsilon_1}{k_4 k_5} & \frac{1}{k_5} \end{pmatrix}. \tag{B.3}$$

We now combine equations (B.2) and (B.3), to get

$$\mathcal{K} = \begin{pmatrix} \frac{B_1\nu}{k_1} & \frac{B_1}{k_2} & \frac{B_1\nu_1}{k_3} & \frac{B_1\kappa}{k_4} & 0 \\ \frac{B_2\nu}{k_1} & \frac{B_2}{k_2} & \frac{B_2\nu_1}{k_3} & \frac{B_2\kappa}{k_4} & 0 \\ \frac{B_3\nu}{k_1} & \frac{B_3}{k_2} & \frac{B_3\nu_1}{k_3} & \frac{B_3\kappa}{k_4} & 0 \\ \frac{B_4\nu}{k_1} & \frac{B_4}{k_2} & \frac{B_4\nu_1}{k_3} & \frac{B_4\kappa}{k_4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.4}$$

Thus, we have

$$\mathcal{R}_0 = \mathcal{R}_A + \mathcal{R}_I + \mathcal{R}_{A_1} + \mathcal{R}_{I_1} \tag{B.5}$$

where

$$\mathcal{R}_A = \frac{\nu B_1}{k_1},\ \mathcal{R}_I = \frac{B_2}{k_2},\ \mathcal{R}_{A_1} = \frac{\nu_1 B_3}{k_3},\ \mathcal{R}_{I_1} = \frac{\kappa B_4}{k_4}.$$

Appendix C. Proof of Theorem 3.7

The proof is based on the Center manifold theory and Theorem 4.1 from.[22] For this we introduce a change of variables by setting , $x_1 = S,\ x_2 = V,\ x_3 = A,\ x_4 = I,\ x_5 = A_1,\ x_6 = I_1,\ x_7 = Q$ and $x_8 = R$ and use the new variables to write model (11) as

$$X'(t) = f(X) = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T,$$

with $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T$, and

$$\begin{aligned} f_1 &:= x_1'(t) = \Lambda - [\lambda + k_0]x_1 + \varphi x_8, \\ f_2 &:= x_2'(t) = \sigma x_1 - [(1-\rho)\lambda + \mu]x_2 + (\omega - \varphi)x_8, \\ f_3 &:= x_3'(t) = \eta\lambda x_1 - k_1 x_3, \\ f_4 &:= x_4'(t) = (1-\eta)\lambda x_1 - k_2 x_4, \\ f_5 &:= x_5'(t) = \phi(1-\rho)\lambda x_2 - k_3 x_5, \\ f_6 &:= x_6'(t) = (1-\phi)(1-\rho)\lambda x_2 - k_4 x_6, \\ f_7 &:= x_7'(t) = \theta x_3 + \theta_1 x_5 + \epsilon x_4 + \epsilon_1 x_5 - k_5 x_7, \\ f_8 &:= x_8'(t) = \gamma_1 x_3 + \gamma_2 x_4 + \gamma_3 x_7 + \gamma_4 x_5 + \gamma_5 x_6 - k_6 x_8, \end{aligned} \tag{C.1}$$

where $k_0 = \sigma + \mu,\ k_1 = \theta + \gamma_1 + \mu,\ k_2 = \epsilon + \gamma_2 + \delta + \mu,\ k_3 = \theta_1 + \gamma_4 + \mu,\ k_4 = \epsilon_1 + \gamma_5 + \delta_1 + \mu,\ k_5 = \gamma_3 + \delta + \mu,$ and $k_6 = \omega + \mu$. Then, the force of infection λ in (2) and the disease-free equilibrium E_0 in (14) in terms of the new variables are given by

$$\lambda = \beta \frac{x_4 + \nu x_3 + \nu_1 x_5 + \kappa x_6}{N}, \tag{C.2}$$

and

$$E_0 = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \left(\frac{\Lambda}{\sigma + \mu}, \frac{\sigma\Lambda}{\mu(\sigma + \mu)}, 0, 0, 0, 0, 0, 0 \right), \tag{C.3}$$

respectively.

Next, we calculate the Jacobean matrix of the system (C.1) at (C.3) and we get

$$J(E_0) = \begin{pmatrix} -k_0 & 0 & -\nu m_1 & -m_1 & -\nu_1 m_1 & -\kappa m_1 & 0 & \varphi \\ \sigma & -\mu & -(1-\rho)\nu m_2 & -(1-\rho)V_0 m_2 & -(1-\rho)\nu_1 m_2 & -(1-\rho)\kappa m_2 & 0 & \omega - \varphi \\ 0 & 0 & \eta\nu m_1 - k_1 & \eta m_1 & \eta\nu_1 m_1 & \eta\kappa m_1 & 0 & 0 \\ 0 & 0 & (1-\eta)\nu m_1 & (1-\eta)m_1 - k_2 & (1-\eta)\nu_1 m_1 & (1-\eta)\kappa m_1 & 0 & 0 \\ 0 & 0 & \phi(1-\rho)\nu m_2 & \phi(1-\rho)m_2 & \phi(1-\rho)\nu_1 m_2 - k_3 & \phi(1-\rho)\kappa m_2 & 0 & 0 \\ 0 & 0 & (1-\phi)(1-\rho)\nu m_2 & (1-\phi)(1-\rho)m_2 & (1-\phi)(1-\rho)\nu_1 m_2 & (1-\phi)(1-\rho)\kappa m_2 - k_4 & 0 & 0 \\ 0 & 0 & \theta & \epsilon & \theta_1 & \epsilon_1 & -k_5 & 0 \\ 0 & 0 & \gamma_1 & \gamma_2 & \gamma_4 & \gamma_5 & \gamma_3 & -k_6 \end{pmatrix}, \tag{C.4}$$

where $m_1 = \frac{\mu\beta}{\sigma+\mu}$ and $m_2 = \frac{\sigma\beta}{\sigma+\mu}$. We now consider $\beta^* := \beta$ as a bifurcation parameter at $\mathcal{R}_0 = 1$. Solving for β from (B.5) at $\mathcal{R}_0 = 1$ gives

$$\beta = \frac{k_0 k_1 k_2 k_3 k_4}{\nu\eta\mu k_2 k_3 k_4 + (1-\eta)\mu k_1 k_3 k_4 + \nu\phi(1-\rho)\sigma k_1 k_2 k_4 + \kappa(1-\phi)(1-\rho)\sigma k_1 k_2 k_3}. \tag{C.5}$$

At $\mathcal{R}_0 = 1$ the matrix $J(E_0)$ has a simple zero eigenvalue and all other eigenvalues have negative real parts. This can be verified by substituting (C.5) into (C.4) and follow the usual way of calculating the eigenvalue of a matrix. Borrowing the notation of,[22] let $w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)$ and $v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^T$ respectively be the left and right eigenvector associated with the zero eigenvalue of $J(E_0)$ such that $v \cdot w = 1$. Therefore, w and v satisfy the system

$$wJ(E_0) = 0 \quad \text{and} \quad J(E_0)v = 0. \tag{C.6}$$

Solving for v and w from (C.6) yields

$$v_1 = \alpha_1 v_3,\ \ v_2 = \alpha_2 v_3,\ \ v_3 = v_3 > 0,\ \ v_4 = \alpha_4 v_3,\ \ v_5 = \alpha_5 v_3,\ \ v_6 = \alpha_6 v_3,\ \ v_7 = \alpha_7 v_3,\ \ v_8 = \alpha_8 v_3,$$

where

$$\begin{aligned}
\alpha_1 &= \frac{-\nu m_1 - m_1\alpha_4 - \nu_1 m_1\alpha_5 - \kappa m_1\alpha_6 + \varphi\alpha_8}{k_0},\\
\alpha_2 &= \frac{\sigma\alpha_1 - (1-\rho)\nu m_2 - (1-\rho)V_0 m_2\alpha_4 - (1-\rho)\nu_1 m_2\alpha_5 - (1-\rho)\kappa m_2\alpha_6 + (\omega-\varphi)\alpha_8}{\mu},\\
\alpha_4 &= \frac{-(\eta\nu m_1 - k_1) - \eta\nu_1 m_1\alpha_5 - \eta\kappa m_1\alpha_6}{\eta m_1},\\
\alpha_5 &= \frac{(1-\rho)\phi m_2\left(\frac{\eta\nu m_1 - k_1}{\eta m_1} - \nu\right)}{\phi(1-\rho)\nu_1 m_2\left(1-\frac{m_2}{m_1}\right) - k_3},\\
\alpha_6 &= \frac{(1-\rho)(1-\phi)m_2\left(\nu - \frac{\eta\nu m_1 - k_1}{\eta m_1}\right)}{(1-\phi)(1-\rho)\kappa m_2 - k_4}.\\
\alpha_7 &= \frac{\theta + \epsilon\alpha_4 + \theta_1\alpha_5 + \epsilon_1\alpha_6}{k_5},\\
\alpha_8 &= \frac{\gamma_1 + \gamma_2\alpha_4 + \gamma_4\alpha_5 + \gamma_5\alpha_6 + \gamma_3\alpha_7}{k_6}.
\end{aligned}$$

and

$$\begin{aligned}
& w_1 = w_2 = 0, \quad w_3 = w_3 > 0,\\
& w_4 = \left[\frac{k_1 - \eta\nu m_1}{(1-\eta)\nu m_1}\right] w_3 - \left[\frac{\phi(1-\rho)\nu m_2}{(1-\eta)\nu m_1}\right] w_5 - \left[\frac{(1-\phi)(1-\rho)\nu m_2}{(1-\eta)\nu m_1}\right] w_6\\
& w_5 = \left[\frac{\eta\nu_1 m_1\left(k_4 - (1-\phi)(1-\rho)\kappa m_2\right) - (1-\eta)\nu_1 m_1 \mathcal{C}}{\mathcal{D}}\right] w_3\\
& w_6 = \left[\frac{\eta\kappa m_1\left(k_3 - \phi(1-\rho)\nu_1 m_2\right) + (1-\eta)\kappa m_1 \mathcal{C}}{\mathcal{D}}\right] w_3\\
& w_7 = w_8 = 0\\
& \mathcal{C} = \frac{(\eta\nu m_1 - k_1)(1-\rho)\kappa\nu_1 m_2}{(1-\eta)\nu m_1}\\
& \mathcal{D} = k_3 k_4 - (1-\rho)m_2\left(1 - \frac{m_2}{\nu m_1}\right)\left[\phi\nu_1 k_4 + (1-\phi)\kappa k_3\right].
\end{aligned}$$

Therefore, the bifurcation coefficients a and b as defined in[22] at the disease-free equilibrium (C.3) are given by

$$\begin{aligned}
a &= \sum_{k,i,j=1}^{8} w_k v_i v_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}, \qquad \text{(C.7)}\\
&= w_3 \sum_{i,j=3}^{6} v_i v_j \frac{\partial^2 f_3}{\partial x_i \partial x_j} + w_4 \sum_{i,j=3}^{6} v_i v_j \frac{\partial^2 f_4}{\partial x_i \partial x_j}\\
&\quad + w_5 \sum_{i,j=3}^{6} v_i v_j \frac{\partial^2 f_5}{\partial x_i \partial x_j} + w_6 \sum_{i,j=3}^{6} v_i v_j \frac{\partial^2 f_6}{\partial x_i \partial x_j},
\end{aligned}$$

$$
\begin{aligned}
&= 2m_1\Big[\eta\,(w_3v_3v_4\nu + w_3v_3v_5\nu\nu_1 + w_3v_3v_6\nu\kappa)\\
&\quad +(1-\eta)\,(w_4v_3v_4\nu + w_4v_3v_5\nu\nu_1 + w_4v_3v_6\nu\kappa)\,\Big]\\
&\quad +2m_2\Big[\phi(1-\rho)\,(w_5v_3v_4\nu + w_5v_3v_5\nu\nu_1 + w_5v_3v_6\nu\kappa)\\
&\quad +(1-\phi)(1-\rho)\,(w_6v_3v_4\nu + w_6v_3v_5\nu\nu_1 + w_6v_3v_6\nu\kappa)\,\Big],\\
&= 2\nu\Big[m_1\Big(\eta w_3 + (1-\eta)w_4\Big) + m_2\Big(\phi w_5 + (1-\phi)w_6\Big)\Big]v_3\Big(v_4 + \nu_1 v_5 + \kappa v_6\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
b &= \sum_{k,i=1}^{8} w_k v_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(E_0,\beta^*), \qquad (C.8)\\
&= w_3\sum_{i=1}^{8} v_i\frac{\partial^2 f_3}{\partial x_i\partial\beta} + w_4\sum_{i=1}^{8} v_i\frac{\partial^2 f_4}{\partial x_i\partial\beta} + w_5\sum_{i=1}^{8} v_i\frac{\partial^2 f_5}{\partial x_i\partial\beta} + w_6\sum_{i=1}^{8} v_i\frac{\partial^2 f_6}{\partial x_i\partial\beta},\\
&= \frac{\eta\mu}{\sigma+\mu}\Big(v_3w_3\nu + v_4w_3 + v_5w_3\nu_1 + v_6w_3\kappa\Big) + \frac{(1-\eta)\mu}{\sigma+\mu}\Big(v_3w_4\nu + v_4w_4 + v_5w_4\nu_1 + v_6w_4\kappa\Big),\\
&+ \frac{\phi(1-\rho)\sigma}{\sigma+\mu}\Big(v_3w_5\nu + v_4w_5 + v_5w_5\nu_1 + v_6w_5\kappa\Big) + \frac{(1-\phi)(1-\rho)\sigma}{\sigma+\mu}\Big(v_3w_6\nu + v_4w_6 + v_5w_6\nu_1 + v_6w_6\kappa\Big),\\
&= \Big[\frac{\mu}{\sigma+\mu}\Big(\eta w_3 + (1-\eta)w_4\Big) + \frac{(1-\rho)\sigma}{\sigma+\mu}\Big(\phi w_5 + (1-\phi)w_6\Big)\Big]\Big(\nu v_3 + v_4 + \nu_1 v_5 + \kappa v_6\Big),\\
&= \frac{\mu(\eta + (1-\eta)\alpha_4) + (1-\rho)\sigma(\phi\alpha_5 + (1-\phi)\alpha_6)}{\sigma+\mu}\cdot\Big(\nu + \alpha_4 + \nu_1\alpha_5 + \kappa\alpha_6\Big)v_3w_3, \qquad (C.9)
\end{aligned}
$$

where $v_3 > 0$ and $w_3 > 0$.

Since all parameters are positive and the components v_3, and w_3 are positive, it is not difficult to verify that $b > 0$ with $m_1 = \frac{\mu\beta^*}{\sigma+\mu}$ and $m_2 = \frac{\sigma\beta^*}{\sigma+\mu}$. Thus, it follows from Theorem 4.1 in[22] the model (11) exhibits a backward bifurcation at $\mathcal{R}_0 = 1$ whenever $a > 0$.

Appendix D. Proof of Theorem 3.8

To prove the global stability of the disease-free equilibrium at $\omega = 0$ and $\rho = 1$, we use LaSalle Invariance Principle.[49] For this we first define a Lyapunove function $L : \mathcal{G} \to \mathbb{R}$, by

$$
L(E) = \frac{\mu}{3(\sigma+\mu)}A + \frac{\mu}{6(\sigma+\mu)}I,
$$

where

$$
\mathcal{G} = \{(S, V, A, I, A_1, I_1, Q, R) \in \Omega : A_1 = 0,\ I_1 = 0\} \subset \Omega, \quad \text{and} \quad E \in \mathcal{G}.
$$

Now observe that

$$
L(E_0) = 0, \qquad L(E) > 0 \quad \text{for all } E \in \mathcal{G}\backslash\{E_0\}.
$$

Hence the function L is positive definite.

We now rewrite (11) in a vector form by

$$
X'(t) = f(X),
$$

where

$$X = (S, V, A, I, 0, 0, Q, R)^T$$
$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T$$

with

$$\begin{aligned}
f_1 &= \Lambda - [\lambda + \sigma + \mu]S, \\
f_2 &= \sigma S - \mu V, \\
f_3 &= \eta\lambda S - (\theta + \gamma_1 + \mu)A, \\
f_4 &= (1-\eta)\lambda S - (\epsilon + \delta + \gamma_2 + \mu)I, \\
f_5 &= 0, \\
f_6 &= 0, \\
f_7 &= \theta A + \epsilon I - (\gamma_3 + \delta + \mu)Q, \\
f_8 &= \gamma_1 A + \gamma_2 I + \gamma_3 Q - \mu R.
\end{aligned}$$

Let $\dot{L}$ represent the directional derivative of L in the direction of f. Then we have

$$\begin{aligned}
\dot{L} &= \nabla L \cdot f, \\
&= \left(0, 0, \frac{\mu}{3(\sigma+\mu)}, \frac{\mu}{6(\sigma+\mu)}, 0, 0, 0\right) \cdot f, \\
&= \frac{\mu}{3(\sigma+\mu)}\left(\eta\lambda S - k_1 A\right) + \frac{\mu}{6(\sigma+\mu)}\left((1-\eta)\lambda S - k_2 I\right), \\
&\leq \frac{\mu}{3(\sigma+\mu)}\left(\eta\beta(\nu A + I) - k_1 A\right) + \frac{\mu}{6(\sigma+\mu)}\left((1-\eta)\beta(\nu A + I) - k_2 I\right), \\
&\leq \frac{\mu}{2(\sigma+\mu)}(\beta\eta\nu - k_1)A + \frac{\mu}{2(\sigma+\mu)}(\beta(1-\eta) - k_2)I, \\
&\leq \left(\frac{\nu B_1}{k_1} - 1\right)\frac{\mu}{\sigma+\mu}k_1 A + \left(\frac{B_2}{k_2} - 1\right)\frac{\mu}{\sigma+\mu}k_2 I, \\
&\leq \left(\frac{\nu B_1}{k_1} + \frac{B_2}{k_2} - 1\right)\frac{\mu}{\sigma+\mu}(k_1 A + k_2 I), \\
&= (\mathcal{R}_0 - 1)\frac{\mu}{\sigma+\mu}\left(k_1 A + k_2 I\right).
\end{aligned}$$

Here we used the fact that $\frac{S}{N} \leq 1$.

Thus, $\dot{L} \leq 0$ on $\mathcal{G}$ whenever $\mathcal{R}_0 \leq 1$. Hence, L is a Lyapunov function for E_0 on $\mathcal{G}$. Furthermore, at E_0 we have $\lambda = 0$. Which implies that

$$\dot{L} = 0 \Longleftrightarrow E = E_0.$$

Hence, the largest invariant set contained in $\mathcal{M} = \left\{E \in \mathcal{G} : \dot{L}(E) = 0\right\}$ is $\{E_0\}$, i. e.,

$$\lim_{t\to\infty} E(t) = E_0.$$

Therefore, we conclude by LaSalle Invariance Principle[49] that the disease-free equilibrium E_0 of the model with $\omega = 0$ and $\rho = 1$ is globally asymptotically stable on $\mathcal{G}$ for $\mathcal{R}_0 \leq 1$. This completes the proof of the theorem. □