Mathematics teachers' professional noticing as an immanent feature of Lesson Study

by

Koketso Clinton Moremi

Submitted in fulfilment of the requirements for the degree

MAGISTER EDUCATIONIS

in the Faculty of Education

at the

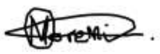
UNIVERSITY OF PRETORIA

Supervisor: Dr RD Sekao

September 2024

Declaration

I, Koketso Clinton Moremi (St number: 18229256), declare that the dissertation titled **Mathematics teachers’ professional noticing as an immanent feature of Lesson Study** which I hereby submit for the degree in Magister Educationis at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.



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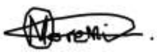
Signature: 

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Month Year: September 2024

Dedication

I dedicate this research to my late sister, Mashilo Pretty Mothemane, who dedicated her life to being a good sister, daughter, friend, and person. I will forever appreciate her presence in my life and will continue to honour her good soul and spirit by trying to be a good human being at all times.

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- My friends and colleagues who supported me throughout this journey and kept on checking up on me. Your support and presence in my life is much appreciated.

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Abstract

Teachers' professional noticing is an essential pedagogical skill for the effective teaching and learning of mathematics. Professional noticing is an inherent attribute of Lesson Study (LS). Therefore, LS is the perfect context to explore how mathematics teachers used professional noticing when offering lessons within the LS setting; thereby making instant instructional decisions while the lesson unfolds. My qualitative case study research is situated within the interpretivist paradigm, wherein two LS groups were studied to gain insights into how mathematics teachers' professional noticing informs their instructional decisions. I used two theoretical lenses, namely, Situated Learning Theory (Lave & Wenger, 1991), to provide a theoretical basis for LS and FOCUS Framework for Productive Noticing (FFPN) (Choy, 2015) to provide a theoretical basis for professional noticing. The two LS groups were purposively selected because of their familiarity with and implementation of LS. Data were collected through observation, document analysis and unstructured interviews to answer the primary research question: How do mathematics teachers use professional noticing to facilitate lessons within LS?

The findings of this research have practical implications for mathematics teaching. The study revealed that while teachers noticed learners struggling with specific mathematics ideas, their noticing was only superficial. This led to instructional decisions that were not optimal for enhancing or developing learners' mathematical thinking. The dominant instructional decision was to, in the same way, re-explain a mathematics idea in the same way when learners were struggling with it. The study also revealed that teachers could reflect on practice cosmetically and suggest alternatives for the future. The recommendations from this research can guide teachers to use purposeful activities when teaching in the LS context to enable them to notice specific issues regarding learners' mathematical thinking and then make suitable instructional decisions. It also suggests that teachers consult curriculum policy when engaging in post-lesson reflections to enrich their reflection-on-practice.

Keywords: Professional noticing, Lesson Study, instructional decisions, mathematical thinking, purposeful activities.

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List of Abbreviations

CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
FET	Further Education and Training
FFPN	FOCUS Framework for Productive Noticing
GET	General Education and Training
HCF	Highest Common Factor
JICA	Japan International Cooperation Agency
LCM	Lowest Common Multiple
LS	Lesson Study
MSSI	Mpumalanga Secondary School Initiative
NCTM	National Council of Teachers of Mathematics
PD	Professional Development
PTM	Pivotal Teaching Moment
SLT	Situated Learning Theory
SRQ	Secondary Research Questions

Description of Key Terms

Learner: The term *learner* used in South Africa is synonymous with *student* or *pupil,* generally used elsewhere globally. The South African Schools Act (1996) defines a *learner* as someone who receives and is obliged to receive education.

Primary school: In South Africa, primary school starts from Grade 1 until Grade 7. Learners who attend primary school are between the ages of 6 to 12.

High school: In South Africa, high school starts in Grade 8 and ends in Grade 12. This type of schooling follows after primary school, and learners who attend high school are typically between the ages of 13 and 19.

Senior Phase: The schooling system in South Africa is divided into phases: the Foundation Phase, which covers Grades R-3; the Intermediate Phase, covering Grades 4 to 6; the Senior Phase, covering Grades 7 to 9; and the Further Education and Training Phase, covering Grades 10 to 12. Evidently, the term Senior Phase refers to Grades 7 to 9.

Noticing: Any articulation of the word 'noticing' about teachers must be understood through the definition of professional noticing. Professional noticing involves the ability of a teacher to identify specific incidents and aspects of learners' mathematical thinking, interpret the incidents, and make instructional decisions concerning learners' current mathematical understanding as the lesson(s) progress (Haverly et al., 2020).

Observation: refers to the observation that members of the LS team do during the lesson *presentation and observation stage* of the LS cycle. Observation in this context should not be confused with professional noticing. Instead, members of the LS team practise professional noticing as they observe lessons being presented.

Pivotal teaching moment: Stockero and Van Zoest (2013) stated that a pivotal teaching moment (PTM) is an instance that disrupts the flow of the lesson but provides teachers with an opportunity to modify their instructional practices to change or extend how learners think about and understand mathematics.

TABLE OF CONTENTS

List of Figures

List of Tables

CHAPTER 1: GENERAL ORIENTATION OF THE STUDY

1.1. INTRODUCTION

In this chapter, I provide a holistic introduction and the general orientation of the entire study. In doing so, I begin my discussion with the concept of mathematical thinking. Human beings, across their lifespan, try to solve problems, be it at school or in life, and to do this, they need mathematical thinking (Çelik & Özdemir, 2020).

Mathematical thinking, according to Çelik and Özdemir (2020), is a dynamic and multifaceted process that involves the utilisation of mathematical ideas, techniques, and concepts to solve problems directly or indirectly. Mathematical thinking can also be understood as a way of thinking related to mathematical processes to solve mathematical tasks (Sukmadewi, 2014).

One of the most fundamental yet elusive goals of teaching mathematics is for learners to be able to use mathematical thinking to solve problems (Stacey, 2006). As such, Stacey (2006) suggested that mathematical thinking is essential in three ways: Mathematical thinking is a significant goal for schooling, it is crucial for learning mathematics, and it is vital for teaching mathematics. Therefore, it is clear that mathematical thinking and methods are essential in various fields beyond mathematical disciplines, such as linguistics, history, and archaeology (Özdemir & Yalçın, 2018).

Teachers must have specific competencies to enhance and promote learners' mathematical thinking. Some of these competencies include the teacher's ability to ask thought-provoking questions and design or select tasks to encourage learners to think mathematically and provide justifications for their solutions (Mukuka et al., 2023; Sukmadewi, 2014; Tohir et al., 2020). Additionally, teachers should plan lessons and design a challenging learning environment for learners that requires them to think mathematically (Mastuti et al., 2022; Sukmadewi, 2014).

One of the contemporary ways of enhancing learners' mathematical thinking is through practising professional noticing (Lee, 2018). Professional noticing moves away from practices of traditional teaching methods that are teacher-centred and do not promote or enhance learners' mathematical thinking (Lessani et al., 2017). Traditional teaching methods are based on the behaviourist learning theory, which places the teacher as the transmitter of knowledge (Ulum & Fauzi, 2023). The teacher reviews previous material and homework, followed by demonstrations of solving low-level problems that the learners must imitate (Lessani et al., 2017). Yue (2024) argues that traditional teaching methods put learners in the position of passive receivers, receiving and accepting information from their teachers to memorise and meet assessment requirements. Such practices overlook the development of learners' mathematical thinking, as they are not cognitively challenging the learners.

On the other hand, professional noticing is directly linked to the effective teaching and learning of mathematics. Amid different conceptualisations, there is consensus among researchers that professional noticing refers to identifying and attending to classroom situations that elicit learners' mathematical thinking and understanding, reasoning about and interpreting these situations and making certain instructional decisions to respond to the learners to ensure that they learn in a way that best suits their current mathematical understanding (Biccard, 2020; Haverly et al., 2020; Kgosi, 2020; Selmer et al., 2021).

Professional noticing is an essential aspect of teaching because it influences the teaching and learning activities the teacher uses, the quality of mathematical instruction and learners' mathematical achievement (Yang et al., 2019). What teachers notice inevitably impacts their classroom decisions and practices (Biccard, 2020). Therefore, the ability of a teacher to notice significant features of learners' mathematical thinking is central to the effective teaching and learning of mathematics (Guner & Akyuz, 2020; Lindstrom & Selmer, 2022). Professional noticing relates to how teachers exploit learners' contributions to elicit learners' mathematical thinking and understanding and how they interpret learner understanding and decide on classroom practices to ensure that learners learn in alignment with their contemporary mathematical knowledge.

Different people refer to professional noticing as teacher noticing, or simply noticing (Khoza, 2023; Sekao, 2023). I chose to align my study with the usage of the term "professional noticing", however, due to grammatical reasons, any encounter of the words 'noticing' or 'notice' in the context of the teachers must be understood through the definition of professional noticing.

Biccard (2020) claims that professional noticing extends beyond providing descriptions or evaluations of an event. Noticing learners' mathematical thinking allows teachers to make instructional decisions based on what they noticed, so as to implement lessons that use what learners know and need to know (Amador, 2016). However, for teachers to practise meaningful noticing, they must engage in interpretive discourses. Notwithstanding, it seldom happens that a single teacher can engage in such discourses alone. It is, therefore, necessary that teachers work together to achieve meaningful professional noticing. Hence, professional noticing was, in this study, explored within the Lesson Study (LS) context.

LS is a teacher development model that emanates from Japan. LS is a cyclic process of identifying the problem and planning the lesson collaboratively, with one member of the LS team teaching the lesson while other members of the LS team observe, reflecting on the lesson, based on the findings and refining practice (Sekao, 2023).

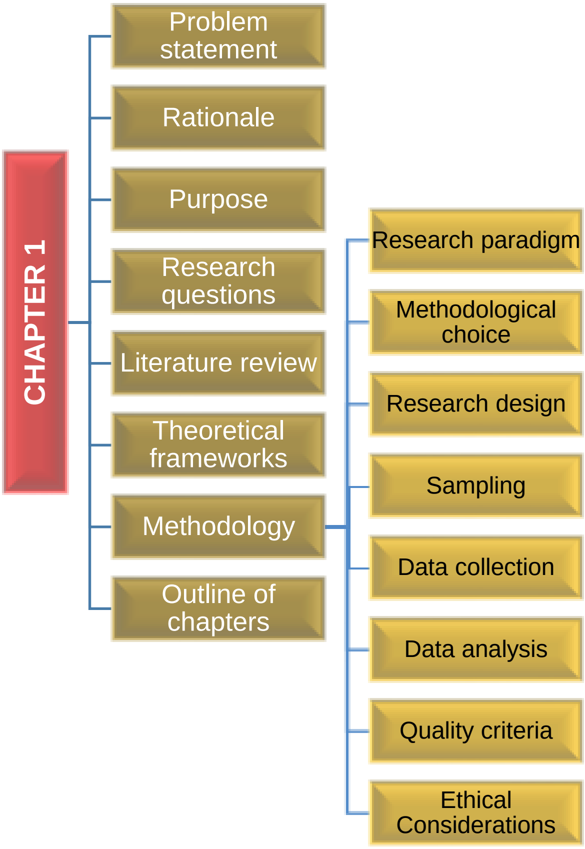
According to Amador and Carter (2018), the reflective discussion in LS allows teachers who form part of the LS team to notice learners' [mathematical] thinking professionally. Çelebi (2023) argued that LS can provide a framework for teachers to collaboratively attend to significant instances during a lesson, thereby allowing them to identify, make sense of, and reason about these incidents.

Guner and Akyuz (2020) contended that LS provides a fitting context for teachers to understand learners' learning, thereby allowing them to modify their instruction accordingly. Arguments by Çelebi (2023) and Guner and Akyuz (2020) resemble key features of professional noticing, such as attending to learners mathematical thinking to support their learning and development. Evidently, professional noticing is entrenched in LS even though it is not necessarily documented in the literature.

This chapter provides an overarching outline of the entire study. For ease of reference, the flow of this chapter is illustrated in Figure 1.1.

Figure 1.1.

Outline and flow of chapter 1



As illustrated in Figure 1.1, I begin the overview of the entire study by providing the problem statement, thereby outlining the problem that led to this study. I then discuss the rationale and purpose of my study. That is followed by the research questions I aimed to answer with my study. I then provide a preliminary literature review and describe the theoretical frameworks I used to underpin my study. I then discuss my study's methodological issues and give an outline of the chapters.

1.2. PROBLEM STATEMENT

Although professional noticing is concerned with identifying features that elicit learners' current mathematical thinking, reasoning about the features and making in-practice instructional decisions, this practice rarely happens in South African mathematics classrooms because teachers often dismiss or disregard learner responses that could have been opportunities to notice their mathematical thinking and understanding. In my experience, teachers tend not to entertain learners' classroom contributions, especially in cases where there is a preconceived response they expect from the learners. Moodliar and Abdulhamid (2021) revealed that dismissing learners' classroom contributions denies the learners an opportunity to access mathematical concepts in the moment. Therefore, dismissing learners' contributions can discourage learners from actively engaging during lessons (Sekao, 2023) and thus eliminating affordances to notice their mathematical thinking and understanding. While Khoza (2023) argued that teachers must be able to notice a germane attribute in learners' contributions that will allow them to unpack the learners' understanding, Biccard (2020) asserted that teachers tend to miss these conspicuous features.

Teachers need to engage in "interpretive discourses" to achieve effective professional noticing. However, it seldom happens that one teacher can engage in such discourses alone (Biccard, 2020, p. 94). It is, therefore, imperative that teachers work together in this regard. Hence, this study will follow the LS approach as a teacher development model, whose collaborative nature has proven to be effective in granting teachers the opportunity to professionally notice learners' mathematical thinking (Guner & Akyuz, 2020), thereby promoting teacher learning, improving instruction (Sekao & Engelbrecht, 2022) and the enhancement of learners' conceptual understanding (Joubert et al., 2020).

Dindyal et al. (2021) stated that professional noticing must be researched in different contexts. However, very few studies in South Africa explored mathematics teachers' professional noticing, especially in the context of LS. Most of the research on teacher noticing has been on pre-service and primary school teachers. Hence, LaRochelle et al. (2020) suggested that more research should be done on teacher noticing among in-service teachers and teachers in secondary schools.

1.3. RATIONALE

Regularly, teachers encounter unanticipated situations when offering lessons and are expected to respond on the spot and make immediate instructional decisions (Chan et al., 2021; Karatsioli et al., 2022). How they respond directly results from their ability to professionally notice learners' mathematical thinking and understanding. Various researchers argued that teachers' professional noticing is essential to teaching and learning (Haverly et al., 2020; Yang et al., 2019).

Professional noticing is a multifaceted process in teaching because it is inextricably linked to a teacher's beliefs, resources, and the knowledge they possess (Biccard, 2020). Guner and Akyuz (2020) argued that noticing is an expertise that pre-service and in-service teachers must develop and maintain. However, noticing is not an easy skill to master. This was evident in a study by Khoza (2023), which revealed that even experienced teachers may not be fully aware of noticing opportunities in their classrooms when they become apparent. The same was revealed by Gibson and Ross (2016) in an earlier study. Teachers have a finite classroom capacity to make sense of incidents they notice in the classroom (Jazby et al., 2023). It is therefore important that they are able to be able to identify noticing moments as they arise (Biccard, 2020; Khoza, 2023).

This study will provide teachers with ideas they can use to comprehend classroom incidents and respond accordingly. Since professional noticing is an immanent feature of LS, the findings from my current study may be helpful to teacher education institutions by preparing and developing pre-service teachers on noticing, primarily when it is facilitated through LS. In-service teachers can also learn to use LS to develop their ability to notice learners' mathematical thinking.

The study may indicate to teachers that every learner's response or contribution provides an opportunity to notice their mathematical thinking and understanding and that they must always be ready to exploit such affordances instead of dismissing them. Teachers may also learn how to modify their teaching practices to help learners and address learner misconceptions through LS. Professional noticing can be practised in any mathematics topic or concept, and this study is not focused on a specific mathematics topic. Therefore, its findings can be applied to any mathematics topic or concept.

1.4. PURPOSE

The purpose of this study is to explore how mathematics teachers employ professional noticing when offering lessons in the context of LS. This purpose will be pursued by:

- identifying pertinent features teachers notice about learners' mathematical thinking during the presentation of their lessons.

- exploring how mathematics teachers use professional noticing to make instructional decisions as they teach; and
- exploring how mathematics teachers use professional noticing to reflect on their teaching collaboratively.

The aforementioned elements will in turn inform the research questions.

1.5. RESEARCH QUESTIONS

The primary research question that underpins my study is: How do mathematics teachers use professional noticing to facilitate lessons within LS?

The following secondary research questions (SRQs), which are informed by the focal points of the FOCUS framework (see Chapter 2, sections 2.3), will be addressed to respond to the primary research question:

SRQ1: What do teachers notice about learners' mathematical thinking during teaching within LS?
SRQ2: How do mathematics teachers use noticing to make instructional decisions in-practice within LS?
SRQ3: How do mathematics teachers use noticing to steer their reflection-on-practice within LS?

I provided a brief explanation of the aim of each research question below to help the reader understand the context in which these questions were explored.

- SRQ1: The National Council of Teachers of Mathematics [NCTM] (2000) proposed five processes of mathematical thinking, namely, (1) problem-solving, (2) reasoning and proof, (3) communication, (4) connections, and (5) representation (these are explored in detail in paragraph 2.2.1.3). Consequently, SRQ1 focuses on exploring what teachers notice regarding learners' mathematical thinking and concentrates exclusively on the mathematical misconceptions, confusions or realisations that learners present.
- SRQ2: Principles of professional noticing dictate that teachers respond or act upon noticing certain features in learners' mathematical thinking and understanding (Guner & Akyuz, 2020). Therefore, SRQ2 focused on the

teacher's instructional decisions upon noticing learners' mathematical misconceptions, confusions, or realisations.

- SRQ3: A teacher must be a reflective practitioner by trade, to make sense of certain decisions they made in-the-moment (during teaching) and to also aid in refining practice (Dewey, 1910; Schön, 1992). Therefore, through SRQ3, I was able to understand why the teacher offering the lesson made certain instructional decisions and why the teachers who were observing also got to indicate what they noticed and what they did upon noticing.

1.6. LITERATURE REVIEW

Literature reviews are intended to reveal a researcher's understanding of a topic or phenomenon under investigation and how existing literature supports the researcher's understanding (Grant & Osanloo, 2014). Consequently, I have conducted a literature review (see more information in Chapter 2) and organised it under the following main headings:

a) *Professional noticing and its worth for effective teaching and learning*. In this section, I intend to explain what professional noticing is, provide the background behind the idea of professional noticing, and describe what and how teachers notice learners' mathematical thinking to make instructional decisions. I also discussed how reflection is important in teachers' professional noticing.

b) *Lesson Study (LS)*. In this section, I intend to explain what LS is. I provide the background behind LS, the global perspective, how different nations are altering the basic LS cycle to fit their context and the types of LS. Lastly, I explain the stages of the LS model used in South Africa. Simultaneously, I also discuss how professional noticing fits into the various stages.

c) *Professional noticing as an inherent feature of Lesson Study*. In this section, I cited different studies to show how LS helps to facilitate professional noticing. Consequently, we learnt that professional noticing is inherent in LS. In other words, teachers are bound to practice professional noticing when they use LS in their practice.

1.7. THEORETICAL FRAMEWORKS

A research study must have some form of framework that guides it. Essentially, a theoretical framework is the blueprint of a research study that assists the researcher in defining the problem, presenting the literature review, determining the type of data to be collected, and presenting, analysing, and discussing the collected data (Adom et al., 2018; Grant & Osanloo, 2014). I guided my study using the Situated Learning Theory (SLT) by Lave and Wenger (1991) and the FOCUS Framework for Productive Noticing (FFPN), coined by Choy (2015). Chapter 2, particularly section 2.3, provides more information regarding the two theoretical frameworks.

1.8. METHODOLOGY

1.8.1. Research paradigm

A paradigm is a researcher's worldview; it inherently mirrors the researcher's views about the environment they live in and comprises overarching assumptions such as ontology, epistemology, and methodology (Alharahsheh & Pius, 2020) that assist the researcher in collecting and analysing data (Pervin & Mokhtar, 2022). There are various paradigms that a researcher can choose from, including positivism, interpretivism, and pragmatism. My study was guided by the interpretivist paradigm because I intended to understand how mathematics teachers employ professional noticing when offering lessons in the LS context (Ugwu et al., 2021). I discuss my paradigmatic approach in detail in Chapter 3, section 3.2.

1.8.2. Methodological choice

In attempt to understand teachers' experiences, I employed the qualitative research approach, which is a direct implication of having adopted the interpretivist paradigm (Upadhyay, 2012). By definition, qualitative research is an approach aimed at exploring and understanding the meaning of a phenomenon for those who are involved in it (Kamal, 2019). In my study, my aim was to understand how mathematics teachers employed professional noticing in the LS context in their organic environment, which is the classroom. I elaborate further on my methodological choice in Chapter 3, section 3.3.

1.8.3. Research design

I followed the multiple case study research design because my study comprised more than one single case study (Gustafsson, 2017). The intention was not to compare. Instead, the second case aimed to address the apparent shortfalls of the first case in terms of the depth of the findings (Yin, 2018) and therefore enhance the findings from it (Zainal, 2007). For more information, see section 3.4.

1.8.4. Sampling

The selection of participants for my study was based on certain predetermined characteristics which are relevant to its purpose. Consequently, I primarily employed purposive sampling (Andrade, 2021). The sampling techniques I used are discussed in more detail in section 3.5.

1.8.5. Data collection

In line with the ontological assumptions of the interpretivist paradigm, I used multiple data collection methods (Ikram & Kenayathulla, 2022). Firstly, I used observation to observe teachers during the lesson presentation and observation stage of the LS cycle. Secondly, I used document analysis to collect data noted by observing teachers during the lessons' presentation. Lastly, I used unstructured interviews for corroboration purposes. See section 3.6 for a detailed account of my data collection methods.

1.8.6. Data analysis

I used content analysis to analyse data collected from the observation sheet used by the teachers who were observing the lesson (Lindgren et al., 2020) and thematic analysis to analyse data collected through observation and unstructured interviews (Braun & Clarke, 2006). More is discussed on Content Analysis and Thematic Analysis are discussed further in Chapter 3, sections 3.7.1 and 3.7.2, respectively.

1.8.7. Quality criteria

In an attempt to enhance the quality of my study, I took measures to respond to the five concepts used to evaluate the quality of a qualitative research study, namely, (1) credibility, (2) transferability, (3) dependability, (4) confirmability, and (5) authenticity

(Treharne & Riggs, 2015). There is more information about the measures I took to enhance the quality of my study in Chapter 3, section 3.8.

1.8.8. Ethical considerations

Studies taking place in higher education institutions need to seek permission to conduct research in designated settings to abide by the required ethical standards (Wa-Mbaleka, 2019). A study must be ethical and protect its human subjects (Roshaidai, 2018). Therefore, I followed the set procedure to ensure that my study was ethical. A discussion of my measures can be found in Chapter 3, section 3.9, and all the relevant documents can be located in the Appendices section.

1.9. OUTLINE OF CHAPTERS

My study is organised into five chapters as follows:

- Chapter 1 - General orientation of the study. The main focus of this chapter is to provide an overview of the entire study.
- Chapter 2 - Literature review and theoretical frameworks. This chapter reviews the literature relevant to my research questions and outlines the two theoretical frameworks I used.
- Chapter 3 - Research methodology. This chapter focuses on the methodological aspects of my study, including issues such as the paradigm, the methodological choice, and the research design.
- Chapter 4 - Presentation of findings. The main focus of this chapter is to present an analysis of the collected data and the findings and,
- Chapter 5 - Discussions and recommendations. This chapter focuses on discussing the findings presented in Chapter 4. I also answered my research questions and discussed the utility of the theoretical frameworks, the limitations of the study, the recommendations, and reflexivity.

CHAPTER 2: LITERATURE REVIEW AND THEORETICAL FRAMEWORKS

2.1. INTRODUCTION

As reflected in the current chapter's name, the literature review comprises two components: a literature review and the theoretical frameworks. Literature reviews intend to provide the researcher's understanding and knowledge concerning the topic under investigation and to show how the research literature supports this understanding and knowledge (Grant & Osanloo, 2014).

The literature review assists the researcher with scrutinising current and relevant studies that are linked to the study a researcher is conducting (Luft et al., 2022).

A theoretical framework is the blueprint of a research study whose primary purpose is to assist the researcher in defining the problem, presenting the literature review, determining the type of data to be collected, the presentation, and analysis, and discussing the collected data (Adom et al., 2018; Grant & Osanloo, 2014).

Figure 2.1 depicts the diagrammatic illustration of this chapter. My discussion in this chapter begins with reviewing relevant literature. I first discussed what professional noticing entails, its significance for the effective teaching and learning of mathematics, and the constructs described in professional noticing, such as what and how teachers notice, instructional decisions teachers make, and the reflective process thereof.

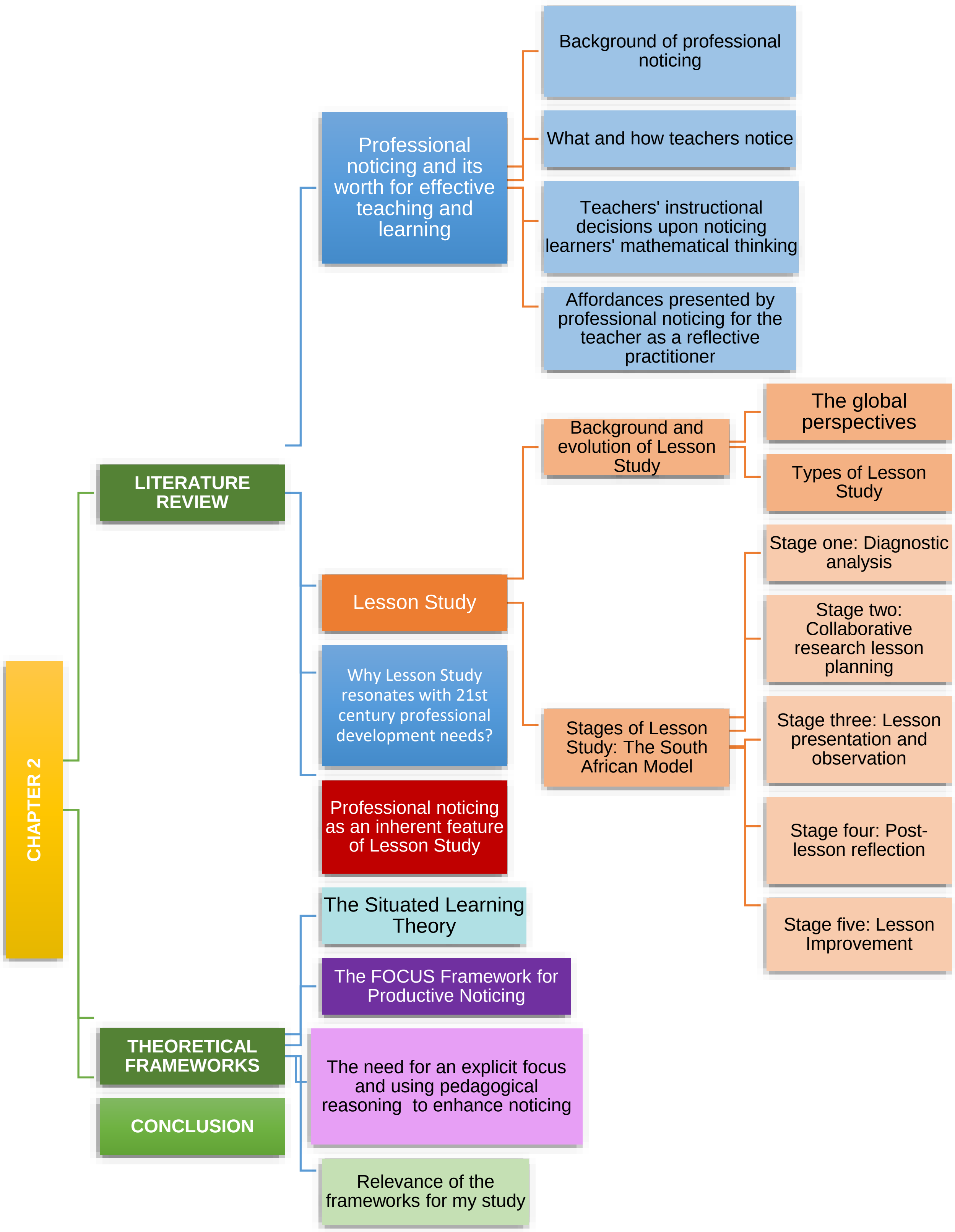
Secondly, I discuss the theory behind LS. I give a brief background of LS, the international perspective, the types of LS, and the stages of the LS cycle. I also indicate how professional noticing fits in with the discussion of the LS stages.

Lastly, I discussed several studies related to my current study, to demonstrate what other researchers did and where the knowledge gap lies.

The second part of Chapter 2 discusses the theoretical frameworks, the SLT and the FFPN. I discuss the background, outline, and conceptions behind the development and application of the frameworks. I also indicate how the frameworks are particularly relevant to my study. I close off the chapter with a brief conclusion of the main issues I discussed in the chapter.

Figure 2.1.

Outline and flow of chapter 2



2.2. LITERATURE REVIEW

2.2.1 Professional noticing and its worth for effective teaching and learning

2.2.1.1. Background of professional noticing

While the origin and roots of professional noticing remain unclear, the literature shows that research on professional noticing dates back to the 1980s. The claim above was cemented by Erickson (2011, p. 17), who outlined findings from an exploratory study he "conducted in the early 1980s in early grades classrooms" on teachers' professional noticing. Notwithstanding, Thomas et al. (2020) asserted that the contemporary ways that researchers understand professional noticing are anchored through the conceptualisation of professional vision by Goodwin (1994) and through Mason's (2002) discussion about the interaction between what teachers notice and how they teach. According to Goodwin (1994), members of every profession develop their ways of seeing and comprehending events. Similarly, teachers as members of the teaching profession develop their own ways of seeing, interpreting, and understanding classroom activity (Jarry-shore & Richardson, 2024), i.e., practising professional noticing.

Konig et al. (2022) claim that Mason's (2002) work was seminal in developing the discipline-specific perspective of teacher noticing, particularly in mathematics education. Mason (2002) advocated for intentional noticing, indicating that this type of noticing is a trait of the mathematics teaching profession, unlike the everyday noticing that everyone does. For Mason, teachers' professional noticing is concerned with their sensitivity regarding their teaching, thus allowing them to be more creative in their teaching and move away from routine teaching. Consequently, Mason's conceptualisation concerns teachers' ability to notice and respond to whatever they notice at particular moments.

It is, therefore, unequivocal that how we understand professional noticing today was shaped mainly by Mason's work. Today, we know that professional noticing is at the heart of the teaching expertise (van Es & Sherin, 2021). As Mason (2002, p. 10) put it, "noticing is at the heart of all practice." Through professional noticing, teachers can positively and directly enhance mathematics teaching and learning (Choy, 2016; Choy & Dindyal, 2021). Professional noticing inevitably impacts a teacher's classroom instructional decisions (Biccard, 2020).

Although researchers have differing views and understanding about teachers' professional noticing, their (researchers') general consensus is that professional noticing comprises three constructs: (a) identifying incidents that portray learners' mathematical understanding and attending to learners' [problem-solving] strategies, (b) interpreting and reasoning about these incidents and learners' thinking and understanding, and (c) making instructional decisions to respond, based on learners' current mathematical understanding (Guner & Akyuz, 2020; Khoza, 2023; Sekao 2023; Tamba & Cendana, 2022). Consequently, professional noticing is central to a teacher's ability to respond to learners' exigent mathematical needs (Yenmez, 2021).

2.2.1.2. What and how teachers notice?

Teaching is a multifaceted activity. A critical trait of professional noticing is discerning what is peculiar about a situation. During all that is happening in a classroom, teachers cannot attend to everything. Therefore, they must identify and select what they will attend and respond to (van Es & Sherin, 2002). Notwithstanding, Amador and Weiland (2015) argued that knowing what to specifically notice can be an arduous task for teachers, hence Biccard (2020) suggested that future research must emphasise the importance of training teachers to develop their noticing skill. Noticing stems from and begins with deciding what is important regarding learners' learning in a lesson (Roller, 2015). Hence, van Es and Sherin (2002) suggested three ways in which teachers can notice learners' mathematical thinking. (see section 2.2.1.3 for a detailed account of mathematical thinking).

The three ways are as follows:

a) Teachers must identify crucial issues during a lesson. In other words, teachers must be able to select what they will give attention to in a lesson because not everything that happens in a lesson is worth paying attention to.
b) When teachers engage in the noticing process (e.g., analysing a video of a lesson), they must not merely give a literal description of what they observe. Instead, they must form connections between specific events they notice and the broader conceptions they represent, including certain concepts and principles they know about teaching and learning, such as how learners learn and;

c) Teachers must use what they know about the context to reason about certain situations; for example, a mathematics teacher should be able to interpret and explain how their own learners may be better able to learn certain mathematical concepts much better than they would with another group of mathematics learners who were taught by a different teacher.

Often, teachers miss opportunities to notice how learners think mathematically, by dismissing them when they offer their ideas, especially in classroom interactions. Consequently, Khoza (2023) argued that every contribution from a learner is significant, however, it is the teacher's responsibility to notice, interpret and respond in a way that can spark further classroom interactions with other learners thereby adding what they think to the overall interaction. Moodliar and Abdulhamid (2021) noted that although it is difficult to respond to an unexpected learner's classroom offering, it is however important that teachers listen attentively and respond accordingly instead of disregarding the learners.

To effectively notice learners' mathematical thinking and understanding, teachers must design lessons following the Three-point template Yang and Ricks (2012) suggested, including the Key Point, Difficult Point, and Critical Point. A template that resonates more with this study is the three focal points (Concept, Confusion and Course of Action) suggested by Choy (2015), based on Yang and Ricks' Three-point template. For instance, a teacher can plan a Grade 8 lesson about adding common fractions (Concept) and find that the confusion learners have is when they add fractions without making the denominators the same (finding the Lowest Common Multiple (LCM) of the denominators and thereby finding the Lowest Common Denominator (LCD)). The latter becomes what the teacher notices about learners' mathematical thinking and understanding. The teacher can then decide on the appropriate Course of Action, such as bringing manipulatives to show the learners that you cannot add parts of unequal wholes.

2.2.1.3. Teachers' instructional decisions upon noticing learners' mathematical thinking

Professional noticing has been found to be effective in focusing on and understanding learners' mathematical thinking (Lee, 2018). Çelik and Özdemir (2020) stated that mathematical thinking can be described as the use of mathematical ideas, techniques, processes, and logical inferences to solve problems. Isa and Ibrahim (2023) added that engaging in mathematical thinking helps one to develop one's ability to think critically, make informed decisions, build solid connections between various mathematical ideas, hence leading to a better understanding of the subject.

Rooney and Boud (2019) argued that noticing is essential in various professional spaces because failure to see specific aspects leads to poor outcomes. In the mathematics education field, the problem can be understood as follows: failure to notice salient elements of learners' mathematical thinking leads to poor instructional decisions by teachers.

Although the concept of mathematical thinking has been in existence since arguably the beginning of mathematics, research on mathematical thinking matured around the 1990s (Goos & Kaya, 2020). Notwithstanding, Mason et al. (1982) identified and described four processes organised in two pairs whereby mathematical thinking takes place. The first pair is specialising (attempting exceptional cases following examples) and generalisation (identifying patterns and relationships). The second pair is conjecturing (pre-empting findings and relationships) and convincing (justifying and explaining why something is true). When faced with a problem, one can engage in mathematical thinking by utilising the four processes mentioned above (Burton, 1984). For instance, if a teacher gives learners a sophisticated activity to do, learners are required to use what they know (mathematically) to try and find an answer or a solution to the problem the teacher has posed. Notably, following the processes described above (Burton 1984; Isa & Ibrahim, 2023; Mason et al., 1982; Stacey, 2006), learners must:

a) explore the meaning of the problem at hand by looking at examples,
b) formulate conjectures about the seemingly apparent relationships between the examples,
c) generalise the relationships and findings they have observed from the examples and
d) provide convincing reasons why those relationships or findings are authentic.

For the teacher, the trick lies in how they interpret the learners' mathematical thinking about the problem at hand, to make the correct instructional decision (respond). The aforementioned process forms the basis of the three constructs of professional noticing (Simpson & Haltiwanger, 2017; Tohir et al., 2020) – attending to learners' mathematical thinking, interpreting, and reasoning about the strategies that learners use, and responding in a way that supports or develops learners' mathematical thinking. However, in making instructional decisions, teachers must be able to attend to the mathematically significant details of the learners' mathematical thinking (Jacobs et al., 2010), which relates quite well to the discussion I presented later in this chapter on having an explicit focus (see paragraph 2.3.2.1).

It may, however, be challenging to identify what is mathematically significant. Hence, there must be some trait a teacher focuses explicitly on. About the latter, the National Council of Teachers of Mathematics [NCTM] (2002) proposed five process standards, which Scusa (2008) described as critical areas that must be addressed so that learners can become effective mathematical thinkers. The five process standards proposed by the NCTM are (1) problem-solving, (2) reasoning and proof, (3) communication, (4) connections, and (5) representation. A brief description of the characteristics of each process is shown in Table 2.1, as explained by Scusa (2008, p. 22).

Table 2.1.

Characteristics of the five process standards

	Characteristics of the Five Processes of a Mathematical Thinker
Process 1	Connections – a student who is successful at making mathematical connections… • likes to see how mathematical ideas are related. • connects new problems to old by asking, "Where have I seen a problem like this before?" • likes to see how mathematical ideas or concepts are connected to other subjects and the real world. • can easily connect familiar ideas to new concepts or skills. • likes to know when others think of a solution strategy in a different way.
Process 2	Representations - a student who is successful at representation… • has an unofficial list of ways to represent a problem and its solution. • uses a range of representation in expressing my thinking, his/her (words, drawings or pictures, charts, or other graphs) • uses representation(s) to help others know exactly what he/she was thinking, how he/she figured it out, and how the problem was solved. • can move easily from one kind of representation to another and knows the right or appropriate representation to use and when to use it.
Process 3	Communication - a student who is successful at communicating mathematically. • is able to explain his/her thinking clearly and concisely. • seeks clarification. • realizes that it is okay to struggle in math and make mistakes. • when others come up with new ideas, asks them to explain or tries to figure and why they make sense
Process 4	Reasoning and Proof - a student who is successful at reasoning and proof. • Can use data to make, test, or argue a conjecture. • Can adequately explain the reasons behind his/her mathematical thinking and can do more than simply just explain the procedure or summarize the answer. • Uses a variety of reasoning methods and proof. • Listens to others mathematical thinking.
Process 5	Problem Solving - a student who is a successful problem solver. • shows confidence is solving problems. • demonstrates persistence when encountering a difficult problem and does not give up. • when given an unfamiliar problem, knows what to do and can switch strategies if one is not working. • has an unofficial list of problem-solving strategies to call upon when solving problems.

Note. *The table is presented here as it is presented in Scusa (2008, p. 22).*

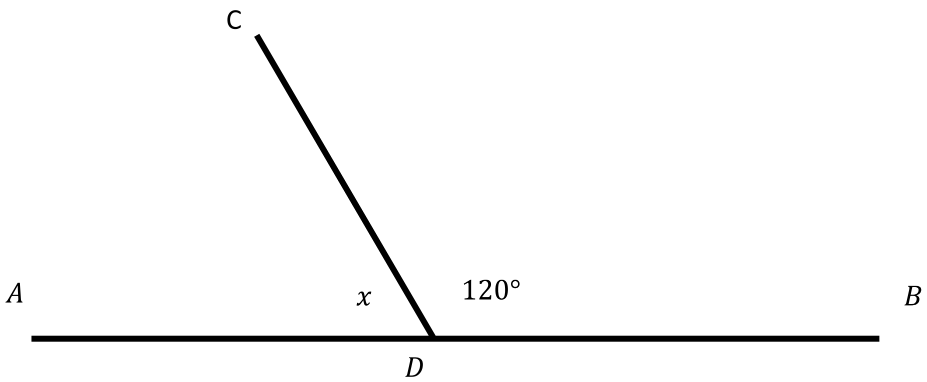
Ideally, all learners must embody all five processes of mathematical thinking as described in Table 2.1. According to the NCTM (2002), successful mathematical thinkers must have certain traits. The four traits are discussed below. Notably, all five traits are interconnected (Sekao, 2023), and dealing with them in isolation is impractical.

Connections

Learners must be able to make connections between and across different mathematical ideas and concepts. Making mathematical connections is important because it helps learners understand how mathematical concepts or ideas work in other contexts and explain why they work (Hatisaru, 2024). According to Hatisaru (2024), mathematical connections are prevalent when learners solve mathematical problems. In fact, mathematics concepts are so connected that mastering one concept depends on mastering another first (Sekao, 2023). For instance, in Euclidean Geometry, learners are required to solve for unknown angles and provide reasons. After they make a statement, they realise that to solve for an unknown angle, they need to use ideas and techniques they learnt in Algebra, thereby creating a mathematical connection. The aforementioned example is described in Figure 2.2 below.

Figure 2.2.

Demonstration of mathematical connections



In Figure 2.2. above, learners can be asked to calculate the value of x. To do so, they will have to they will have to do the following:

Step 1: $C\hat{D}A + C\hat{D}B = 180°$ $[\angle's\ on\ a\ str.line]$

Step 2: $x + 120° = 180°$

In Step 1, learners need to realise that in order to solve for x, they need knowledge about additive inverses to isolate and solve for x. The knowledge and skill involving additive inverses and solving equations are normally taught in Algebra. Through connections, the same knowledge and skill are used when solving equations involving geometry thus:

Step 3: $x + 120° - 120° = 180° - 120°$

Step 4: $\therefore x = 60° = C\hat{D}A$

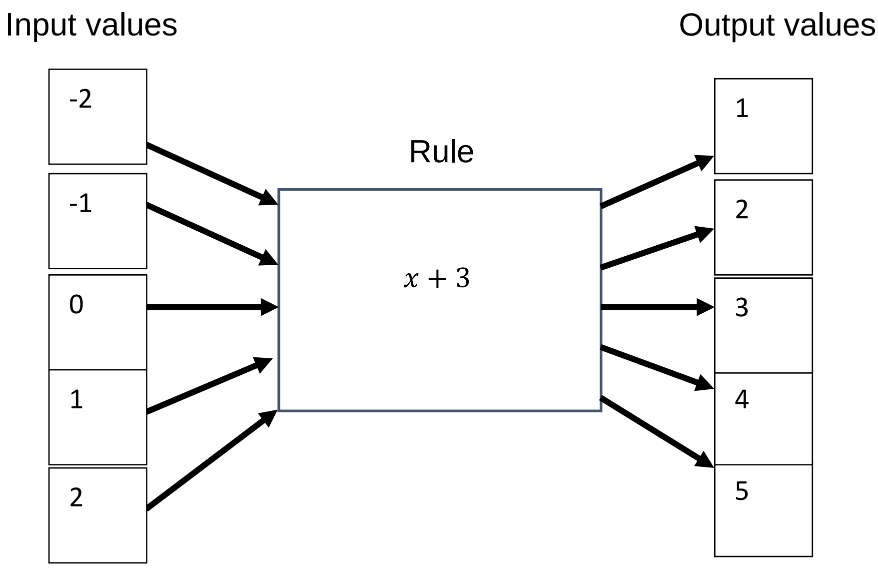
In another scenario, Sekao (2023) makes the example that for learners to be able to add two fractions, $\frac{1}{4} + \frac{5}{12}$, they need to make the denominators the same. They need to understand the concepts of equivalence as well as one (1) as an identity element for multiplication. Learners need to express $\frac{1}{4}$ as an equivalent fraction that has a denominator of 12. To do so, they need to multiply $\frac{1}{4}$ by 1, in the form $\frac{3}{3}$ to obtain $\frac{3}{12}$ subsequently $\frac{3}{12} + \frac{5}{12}$. Having the same denominator allows them to add the numerators to obtain the answer, $\frac{8}{12}$ which can be further simplified to $\frac{2}{3}$. To add the two fractions, learners need to invoke knowledge from equivalent fractions and use one as an identity element for multiplication to make the denominators the same.

Representations

For Scusa (2008), it is often difficult to understand what the learner is thinking unless they provide a fairly good explanation and representation of their solution. Therefore, learners must be able to present their solutions in different forms so that teachers and their classmates can understand what they are thinking. However, the process of obtaining the knowledge of mathematical representation may take a long time; therefore, mathematical representation must be developed per grade, as specified in the CAPS (Sekao, 2023). For instance, learners can be given an equation in the form of $y = x + 3$, whereby they are required to represent the relationship between x and y, given the x values: -2; -1; 0; 1; 2. Different learners can choose to do so differently. While some can choose to use a flow diagram (Figure 2.3), others can use a table to generate ordered pairs (Table 2.2). These are the different ways that learners can represent their mathematical solutions.

Figure 2.3.

Flow diagram representing the relationship between x and y



Learners can choose to use the above flow diagram to represent the input (values on the left representing the given x values), the rule (in the middle), and the output (values on the right representing the y values obtained).

Table 2.2.

Table representing the relationship between x and y

$y = x + 3$					
x	-2	-1	0	1	2
y	1	2	3	4	5

The table above represents the y value obtained when substituting the given x values.

Communication

According to Kaya and Aydin (2016), communication in mathematics provides a platform for learners to express, share, and reflect on their ideas. Therefore, learners must be able to communicate their mathematical thinking clearly and succinctly. Kaya and Aydin (2016) also recognise mathematical communication as a way of enhancing learners' conceptual comprehension, thinking, problem-solving skills, and reasoning. Lomibao et al. (2016) argued that when learners are able to communicate their mathematical procedures orally and in writing, they demonstrate their comprehension of the concepts learnt. We can vividly see the traits of mathematical representation in communication. The connectedness of mathematical representation and connection coincides with my earlier argument that mathematical thinking processes are interlinked. Learners must be allowed and encouraged to communicate their mathematical thinking, regardless of whether they are correct or incorrect (Sekao, 2023). This will enable teachers to identify learners' mathematical ability and understanding so they can give the learners immediate feedback and, if necessary, the teacher can immediately intervene to guide them in the right direction (Rohid et al., 2019).

Returning to the example I made under connections ($x + 120° = 180°$), the teacher can ask the learners to explain their next step. As the learners explain, they must explain why they do certain things. For instance, the learner must explain why the next step must be $x + 120° - 120° = 180° - 120°$. This will help the teacher see if the learner understands the mathematical concepts. Through this example, the learner must be able to make mathematical connections while giving mathematical reasons. This vividly reveals the connection between the mathematical thinking processes involved in this single example i.e., connections, communication, and reasoning.

Reasoning and proof

Agustyaningrum et al. (2019) argued that learners with reasoning and proof abilities are likely to understand mathematical concepts better. Sari et al. (2019) believed that reasoning and proof abilities cannot be inseparable. For Sari et al. (2019), a learners' reasoning can be deduced from their proofs. Therefore, learners must be encouraged to provide reasons for their thinking through their logical mathematical arguments. Sari et al. (2019) argued that learners' reasoning ability can be developed by attempting and solving proof activities, requiring them to use justifications and argumentation. Sekao (2023) argued that learners will gain a deeper and better comprehension of mathematics by doing the latter. Setialesmana et al. (2021) argued that learners' problem-solving abilities are directly linked to their ability to provide mathematical reasons. Through mathematical reasoning, learners can propound propositions, gather evidence, manipulate mathematical problems and come up with correct and precise conclusions (Lestari, 2019). This again shows the connectedness of mathematical thinking processes.

Problem-solving

Anam et al. (2020) contended that mathematical problem-solving reveals a learner's efforts to solve mathematical problems using methods, strategies, and procedures they can account for mathematically. Ersoy and Guner (2015) stated that problem-solving in mathematics allows learners to devise various ways and alternatives to solve a problem and select the best fit the situation. Therefore, learners must be able to come up with mathematical solutions to problems and have a variety of strategies to use when faced with an unfamiliar situation. In fact, Nasution et al. (2018) argued that good mathematical problem-solving skills enable learners to solve problems they encounter in their everyday lives. Çelik and Özdemir (2020) claim that everyone engages in problem-solving throughout their daily lives, at work, and in school, and that for this, they need mathematical thinking. Mason et al. (2010) argued that thinking mathematically helps us to understand the world and attack problems that are not just mathematical or scientific but also general. Therefore, apart from adding to one's success in mathematics, one can argue that mathematical thinking has benefits that extend beyond the mathematical space. Hence, it is imperative that learners are successful mathematical thinkers.

Stacey (2006, p. 39) insists that mathematical thinking is:

a) an important goal for schooling,
b) important as a way of learning mathematics and
c) important for teaching mathematics.

Ultimately, learners who are successful mathematical thinkers can solve mathematical problems with ease and tend to develop a healthy attitude towards the subject (Çelik & Özdemir, 2020). What and how the teachers notice regarding learners' mathematical thinking and understanding has an inevitable impact on the instructional decisions they make. In the teaching of mathematics, teachers are required to make certain instructional decisions during the lesson (Yenmez, 2021). This comes as no surprise because one of the constructs of professional noticing entails responding to learners' mathematical thinking (strategies) once they have been identified through classroom interactions, or lack thereof (Amador et al., 2016). Hence, It has been presented as a valid argument that, without fail, teachers are required to make in-the-moment decisions since classroom occurrences and interactions cannot be precisely pre-empted (Chan et al., 2021).

Jacobs et al. (2010) suggested that if a lesson is built around learners' thinking, then the teachers must be able to attend to and discern learners' strategies from their verbal or written responses, interpret their understanding and most importantly, respond in-the-moment; thus, making instructional decisions. Therefore, teachers must be able to think on their feet and calibrate their instruction according to current circumstances (Gibson & Ross, 2016). The challenge, however, is that all this must happen quickly (Konig et al., 2022) to avoid disrupting the lesson. Therefore, the way teachers make these instructional decisions needs to be investigated to show and even find ways of how teachers can learn and master the skill of professional noticing.

2.2.1.4. Affordances presented by professional noticing for the teacher as a reflective practitioner

Work on reflection in education can be traced back to Dewey (1910), who is known to be a big proponent of reflective practice (Letseka & Zivera, 2013; Norton & Campbell, 2007; Reynolds, 2011; Rodgers, 2002). According to Dewey, reflection is not merely a string of ideas, but a consecutive sequence of ideas arranged in a way that prompts the next as the appropriate outcome, while feeding off its predecessor. In other words, reflection must be intentional and directed. Everyone undergoes involuntary reflection during or after a certain experience (Rodgers, 2002). However, many people would not entertain their thoughts, and according to Dewey (1933, as cited in Rodgers, 2002), that is irresponsible. Hence, Dewey (1910) talks of reflective practice, which he defines as the tenacious and thorough consideration of any beliefs, bearing in mind the basis on which the alleged form of knowledge rests and what deductions it is inclined towards. Be that as it may, I chose to confine my focus on reflection-on-practice, which I elaborate on below. Reflective practice can help teachers move from impulsive and unconscious decisions to active and conscious ones (Arefian & Meihami, 2023). As an attempt to explain how professional noticing works, I draw inspiration from Schön (1983; 1992), who coined *reflection-in-action* and *reflection-on-action*. Although professional noticing can take different forms, Bakker et al. (2022) also supported that professional noticing in-the-moment (during the lesson) is like

Schön's reflection-in-action, while noticing after the moment (the lesson) is similar to reflection-on-action.

I later explain how these two terms coined by Schön fit within the space of professional noticing (see sections 2.2.2.2.3 and 2.2.2.2.4). While I do so, I also indicate the relevant stages of the LS cycle in which these reflective practices happen to show that professional noticing is immersed in LS. However, reflection-in-action takes place during the action, and reflection-on-action takes place after the action; the action, in this case, is the lesson. Reflection is particularly important because if teachers are allowed to reflect on what they observed, processes of metacognition are evoked, thus leading to enhanced professional noticing (Biccard, 2020). For reflection to be effective, it ought to occur in a community through interaction among people (Rodgers, 2002). Hence, this study was situated within the LS context, allowing teachers to collaborate. Wei et al. (2023) pointed out that, amongst other things, research on professional noticing is concerned with what teachers will do upon noticing (instructional decisions they will make), and this then leads to them being reflective practitioners, thus developing them professionally.

2.2.2. Lesson Study

2.2.2.1. Background and evolution of Lesson Study

Lesson Study (LS), or *jugyou kenkyuu* in Japanese, is a teacher-led model of professional development whereby a group of teachers collaboratively plan a series of lessons, teach the lessons, reflect on the lessons, and refine practice (Sekao & Engelbrecht, 2022). The term *jugyou*, when translated to English, means "live lesson", while *kenkyuu* translates to "research or study" (Lewis, 2016, p. 571). Therefore, *jugyou kenkyuu* can be understood as *"the research/study of a live lesson"*. According to Pjanic (2014), the term Lesson Study was coined in 1999 by Makoto Yoshida in his doctoral thesis titled: *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development*.

According to Hervas and Medina (2020), the roots of LS date back to the last quarter of the 19th century. In 1872, the Japanese Meiji government introduced a new school system (Makinae, 2010). According to Makinae (2010), the new school system was called *Gaku-sei*. This system comprised of *Dai-gaku* (which means university), *Tyu-*

gaku (which means secondary school), and *Syo-gaku* (which means elementary school). The Meiji government was determined to ensure that *Syo-gaku* was accessible to the entire nation in attempt to introduce Western civilisation and technology. As a result, an American teacher named Marion Scott and other foreign teachers were invited to Japan to introduce the Western model whereby a single teacher teaches the entire class (Isoda & Baldin, 2023; Ishii, 2017). Japanese teachers were, at the time, only familiar with the individualised teaching model whereby individuals were taught subjects individually and based on their academic abilities (Isoda & Baldin, 2023). In order for other teachers to master the skill of whole-class teaching, teacher training had to take place (Makinae, 2010; Ishii, 2017). Consequently, the Meiji government introduced a normal school in Tokyo to ensure that pre-service teachers learnt the new teaching methods.

The teacher training programmes included the creation of lesson plans, presentation and observation of the lessons, and lesson evaluation meetings (Ishii, 2017). The normal school the Meiji government introduced was designated to train and produce teachers (Makinae, 2010). It is in the normal school where the *object and criticism lessons* were introduced. The *object lesson* was influenced by the Pestalozzian theory, which emphasised that learning must begin with the learners observing an object in order to recognise the concepts intuitively (Makinae, 2010).

The *criticism lesson* involved having a student present a lesson to their class while their classmates observed and later discussed it in terms of what they believed the teacher succeeded or failed in (Makinae, 2010). The paradigm of the *criticism lesson* had a greater influence on the origin of LS when it started to become prominent in the development of pre-service teachers (Sekao, 2023). Pre-service teachers were poised to practice the *object lesson* through the *criticism lesson*. The above discussion describes the origin of LS in Japan (Makinae, 2010).

LS has been practised in Japan for more than a century (Lewis, 2016). Although it predates the 21st century, its relevance makes LS one of the most rapidly advancing and revolutionary teacher development models (Sekao, 2023). The main purpose of LS is to offer teachers more information regarding the challenges they face in their classrooms (Elliot, 2019). Sekao and Engelbrecht (2022) agreed with Elliot and further argued that teachers use LS to target a specific and identified subject area that needs development in their learners' learning.

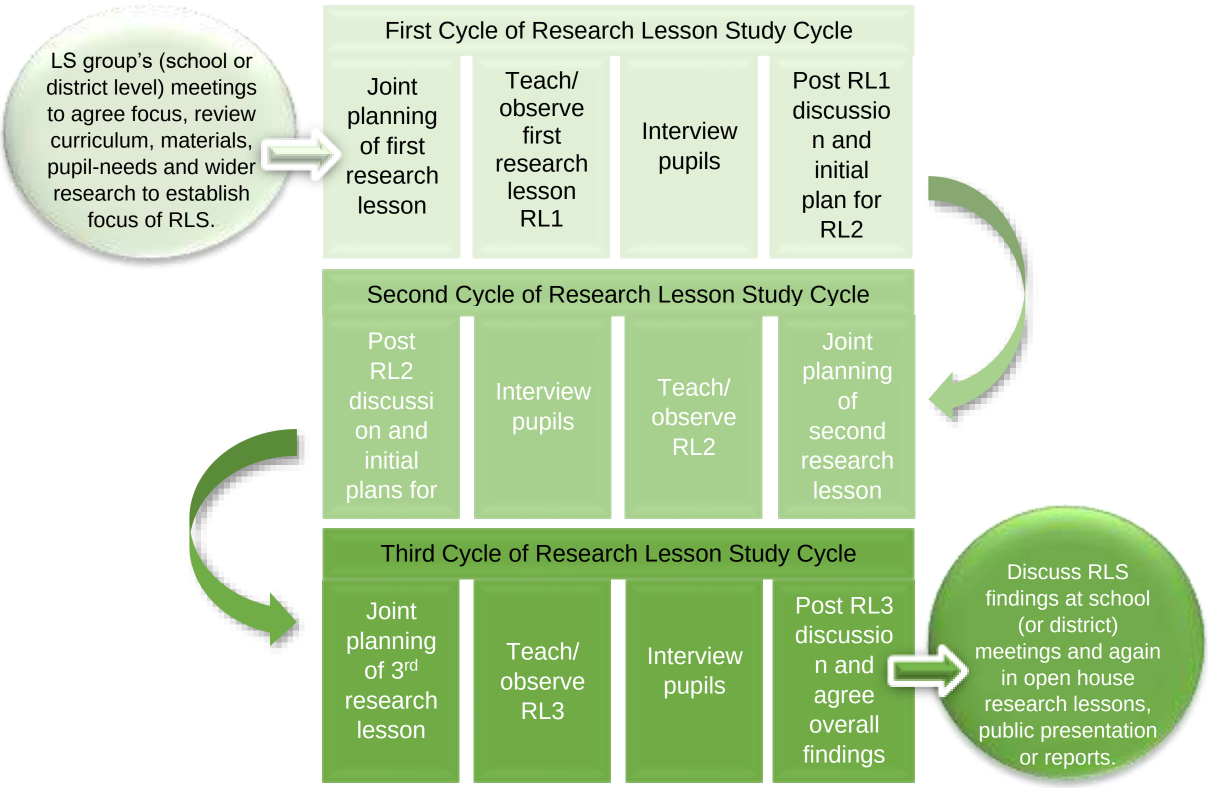
Consequently, LS must be practised only when there is a need to gain insights into a mathematics topic or concept that is difficult for teachers to teach and/or for learners to learn efficiently (Sekao, 2023). Sekao's argument seemingly agrees with an earlier point made by Lewis et al. (2012), claiming that LS can also improve teaching a particular mathematics topic or concept beyond simply investigating it. An LS team typically comprises four to six teachers from the same grade and/or subject who collaborate on planning, teaching, and reflecting on a lesson (Chong & Kong, 2012). When one member of the LS team teaches the lesson, the others take the place of observers and record any information regarding teaching behaviours, learners' classroom behaviours and interactions, and the work done during the lesson. From the above argument, we can vividly see the characteristics of professional noticing implied holistically throughout the LS cycle. Another immanent aspect here is that of the FFPN, which specifies certain actions teachers must take when planning, teaching, and reflecting on the lesson. Consequently, it is evident that professional noticing is inherent in LS and has the ability to allow teachers to notice learners' mathematical thinking and understanding productively.

2.2.2.1.1. The global perspectives

Global interest in LS was triggered when Stigler and Hiebert's (1999) book, *The Teaching Gap*, reported on the TIMSS Video Study (Stigler et al., 1999) and credited LS with Japan's intriguing problem-based learning approaches in primary school mathematics classes. Since then, worldwide interest has grown rapidly (Lewis, 2016; Sekao & Engelbrecht, 2022). Inevitably, different countries have used and adapted LS to suit their contexts better. In their book, Sekao (2023) explored and compared LS models used in the United Kingdom (Figure 2.4), the Netherlands (Figure 2.5), Zambia (Figure 2.6), and South Africa (Figure 2.8). In this section, I am only trying to show that LS is being used in many countries to enhance the teaching and learning of mathematics. Hence, I did not go into detail about what each model comprises. However, I later describe the South African model in detail (see section 2.2.2.2) because it is the country in which this study took place.

Figure 2.4.

Research lesson study cycle in the United Kingdom

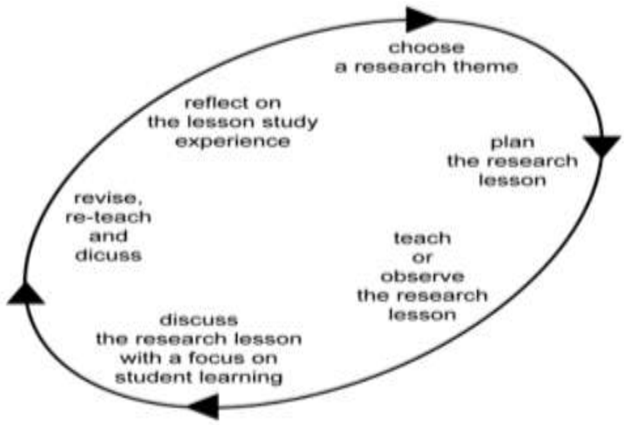


Note. From Dudley (2019). Used with permission from the author.

The research Lesson Study cycle used in the United Kingdom comprises three cycles, each with four steps. Each cycle addresses its own research lesson. The unique feature of the UK model is that it interviews the learners in order to give them a voice (Dudley, 2019).

Figure 2.5.

Research lesson study cycle in the Netherlands

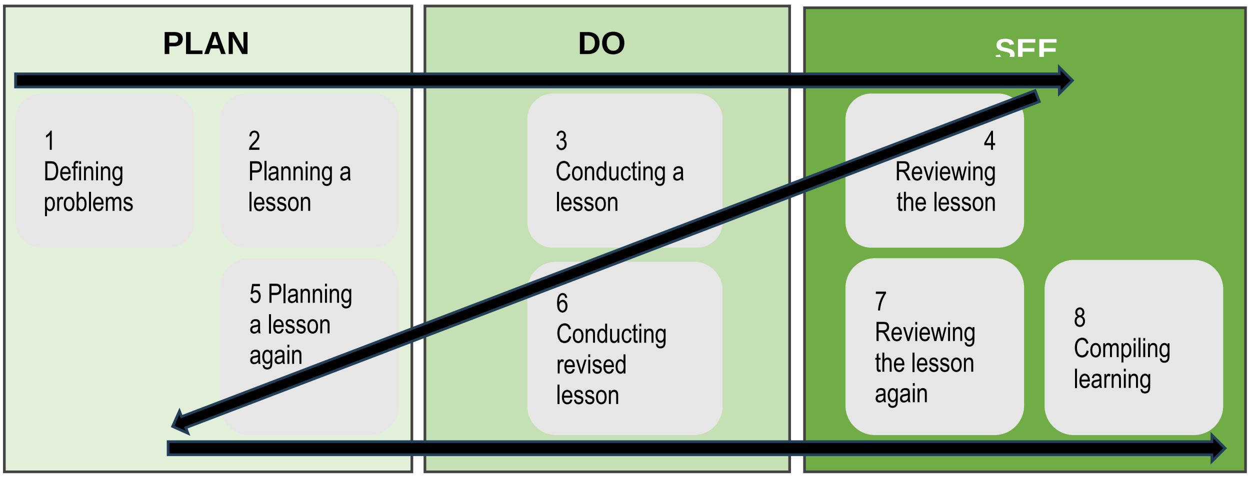


Note. *From Verhoef and Goei (2015).*

The research Lesson Study cycle used in the Netherlands comprises 6 stages. It is a cyclic process whose unique feature is reteaching the research lesson that was the focus of the LS cycle.

Figure 2.6.

Research lesson study cycle in Zambia



Note. *From Ministry of General Education, Republic of Zambia and Japan International Cooperation Agency (JICA) (2016).*

Zambia's research Lesson Study cycle comprises eight steps organised in groups of plan, do, and see. The model is divided into two processes, with steps 1 to 4 forming part of the first process and steps 5 to 8 forming part of the second process. The unique feature of this model is that it features the Deming cycle of continuous improvement, meaning that continuous improvement is implemented throughout the cycle. The three aforementioned models share some characteristics in how they are organised and implemented. However, each one is tailored to fit its local context. Table 2.3 displays some key features of the four (including the South African model) LS models. The South African LS model is described in the next section.

Table 2.3.

Key features relating to the four LS models

Country	*Number of Stages*	*Implementation of LS Cycle*	*Uniqueness of Model*
The United Kingdom	3	3 cycles consisting of 4 steps each	Learners are interviewed to give them a "voice"
The Netherlands	6	A cyclic process	The model involves reteaching the research lesson
Zambia	8	The first process comprises steps 1 to 4. The second process comprises of steps 5 to 8	Features the aspect of the Deming cycle of continuous improvement
South Africa	5	Interlinked cycle	Involves diagnostic assessment/analysis

As Table 2.3 shows, the LS models have various stages. Each one is tailored to meet the country's specific needs and has a unique feature. Although all the models follow different implementation processes, they all share the three stages of planning, teaching, and reflecting on a research lesson.

2.2.2.1.2 Types of Lesson Study

Lewis and Takahashi (2013) indicated that there are four types of LS namely, School-wide Lesson Study, District-level Lesson Study, National school-based Lesson Study, and Association-sponsored Lesson Study. Similarly, Sekao (2023) argued that there are five types of LS, namely, the School-based Lesson Study, the Circuit-based Lesson Study, the District-based Lesson Study, the Provincial-/national-based Lesson Study, and the Association-sponsored Lesson Study, all of which are briefly described in Figure 2.7. My study consisted of two LS groups. One was from a combination of schools, thereby being a Circuit-based Lesson Study, while the other was a School-based Lesson Study. Both LS groups consisted of mathematics teachers working collaboratively to address a mathematics concept/topic that presents difficulties for learners to learn and/or for teachers to teach effectively.

Figure 2.7.

Types of lesson study

School-based Lesson study	• Done at and within a school to improve the teaching and learning of specific subject related concepts/topics.
Circuit-based Lesson Study	• Practised across schools to improve the teaching and learning of subject related concepts and topics.
District-based Lesson Study	• Takes the form of a seminar to showcase teaching expertise on how to best teach a concept or topic chosen from any subject.
Provincial-/national-based Lesson Study	• Takes the form of a conference attended by delegates from various districts (if the conference is provincial) or various provinces (if the conference is national).
Association-sponsored Lesson Study	• Takes place through an association whose interest lies in professionally developing teachers, e.g. The Association of Mathematics Education South Africa (AMESA).

Note. *From Sekao (2023, p. 5).*

Figure 2.7 displays the different types of LS as practised in the South African context. It must be noted that the types of LS presented by Lewis and Takahashi (2013) are in line with the Japanese context.

2.2.2.2. Stages of Lesson Study: The South African model

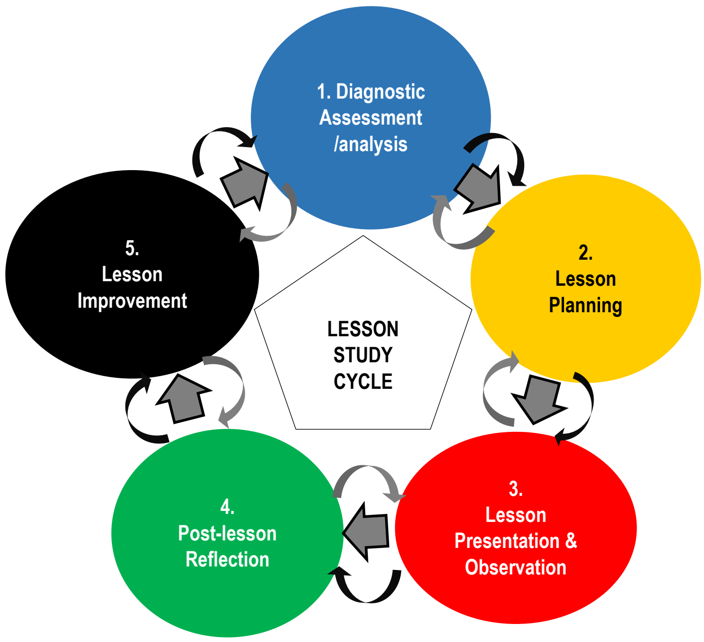
LS, in South Africa, was first implemented in the Mpumalanga province in 1999, the programme through which LS was to be implemented was called the Mpumalanga Secondary School Initiative (MSSI) (Ono & Ferreira, 2010; Ozawa, 2013). When the programme started, the Mpumalanga Department of Education, the Japan International Cooperation Agency (JICA), and the University of Pretoria agreed to collate their resources to collaboratively work on the project to enhance the teaching and learning of mathematics and science by improving teachers' skills and knowledge of these subjects (Ono & Ferreira, 2010). The MSSI was implemented in two phases (Ozawa, 2013). Phase 1 took place between 1999 and 2003, and the target group was mathematics and science teachers who taught Grades 8 and 9 in secondary schools. Phase 2 occurred between 2003 and 2006, and the target group was mathematics and science teachers who taught Grades 10 to 12 (Ozawa, 2013).

According to Ono and Ferreira (2010), the MSSI's plan to institutionalise LS was unsuccessful, so the MSSI was terminated in 2006. However, JICA's efforts to professionally develop South African teachers did not stop there as they continued until 2008 (Ozawa, 2013). South African teachers also did not stop taking an interest in LS, as delegates from South Africa formed part of the 800 participants who attended the World Association of Lesson Studies annual conference in 2014 (Huang & Shimizu, 2016). Research on LS in South Africa is rapidly increasing, and more researchers are using the LS context to explore various phenomena, e.g. Adler and Alshwaikh (2019), Biccard (2020), Vetter (2022), and Thobela et al. (2023).

As I already mentioned, various models are tailored for specific countries; South Africa has a variation of the LS cycle practised in South African (mathematics) classrooms. Although other LS models are used in South Africa, such as the one used by Adler and Alshwaikh (2019), the model I present is considered the South African model because it is used by many schools in South Africa (Sekao & Engelbrecht, 2022). Below I outlined the interlinked five stages (see Figure 2.8) of the LS cycle, as primarily explained by Sekao (2019; 2023) and Sekao and Engelbrecht (2022).

Figure 2.8.

Research lesson study cycle in South Africa



Note. *From Sekao (2023).*

While I outline aspects of the different LS stages, where appropriate, I also indicate, how professional noticing is inherent in LS. It is also notable that although the LS stages comprise an integrated cyclic process (depicted by the curved arrows in Figure

2.8), each stage offers teachers unique experiences to learn from (Sekao & Engelbrecht, 2022). The curved arrows indicate that an LS team can go back to processes entrenched in the previous stage should they feel the need to do so – meaning that they can go back and forth throughout the LS cycle.

2.2.2.2.1. Stage one: Diagnostic assessment/analysis

According to the DBE (2020), diagnostic assessment tests serve three purposes: to unearth learners' misconceptions built from their prior knowledge, gauge the conceptual benefits of the entire class, and identify concepts that need development in the learners' or the whole class's understanding. The diagnostic assessment/ analysis stage of the LS cycle is unique to South Africa. This is because LS in South Africa is only done when there is a need, and therefore, for the need to be addressed, there must be evidence that necessitates the LS cycle. Sekao (2023) stated that the diagnostic assessment/analysis stage of the LS cycle allows the teachers in the LS team to identify learners' difficulties, potential causes of difficulties, and appropriate fitting strategies to address them.

At this stage, teachers use evidence from previous assessments to identify and discern learners' misconceptions that are evident in learners' responses while selecting difficult or problematic topics for their learners. The aforementioned approach aligns very well with the concept of professional noticing because the teachers identify learners' strategies from their responses (learners' written material), interpret them (determine the difficulties learners have, decide on the topic to be taught, and then decide how to respond in alignment with learners' current mathematical thinking and understanding (planning a research lesson).

Alternatively, teachers who form part of the LS team can identify a topic that challenges some or all of them to teach effectively and use it to conduct an LS session. Coe et al. (2010, p. 211) called this first stage "Goal setting" and argued that teachers must decide on the goal of the LS cycle, which is focused on the learners.

2.2.2.2.2. Stage two: Collaborative research lesson planning

During the collaborative lesson-planning stage, teachers come together to devise various methods and strategies that can be used to teach the identified topic (Joubert et al., 2020). Before the brainstorming session, the teachers can individually gather information on how to teach the topic of interest effectively, in order to meaningfully participate in *kyozai-kenkyuu*, which is the study of mathematics content and curriculum materials (Lewis et al., 2012; Lewis & Perry, 2014).

The study of teaching materials is used to study what is currently known about the teaching and learning of the topic, thereby forming the basis of the LS (Lewis et al., 2012). Upon generating ideas, teachers then determine the most fitting methods to integrate the generated ideas so as to plan the lessons effectively. The LS team needs to decide on instructional and assessment activities while planning the lesson. However, the activities they decide on must be purposeful (Sekao, 2023). Purposeful activities allow teachers to determine whether their lesson outcomes have been met (Fujii, 2015). Sekao (2023, p. 50) cited an appropriate example regarding the purposefulness of an activity by saying:

> Suppose we introduce square numbers using $2^2 = 2 \times 2 = 4$. Although this is correct, some learners may think that the second 2 in 2×2 is the same 2 that denotes an exponent in 2^2. Therefore, they are likely to conceive 3^2 as $3 \times 2 = 6$.

Fujii (2018) suggested two key issues teachers must consider when selecting purposeful activities that fit a particular task. Fujii suggested that the activities must (1) be grade-specific and (2) address lesson objectives. Purposeful activities must be designed so that learners can construct meanings of mathematical concepts (Ainley, 2006; 2012). Ainley et al. (2005) argued that purposeful activities provide learners with an opportunity to learn mathematical ideas in ways that make them appreciate the utility of these mathematical ideas. For Sullivan et al. (2021), learning happens as a product of teachers purposefully selecting activities that foster interaction between learners, the teacher and their peers. Therefore, it is important that when teachers decide on activities, they know what the activity intends to reveal about learners' mathematical thinking – thus making it purposeful.

Notwithstanding, the LS team needs to pre-empt learners' responses to the questions they will pose or to the activities they will be given to do. Learners' questions are also anticipated in this stage of the LS. This ties in very well with professional noticing because teachers can then plan how they can respond in alignment with learners' current mathematical thinking (Guner & Akyuz, 2020).

2.2.2.2.3. Stage three: Lesson presentation and observations

For the third stage of the LS cycle, the LS team can agree on a date and time to teach the lesson, and one member of the LS team can then volunteer to teach the lesson at their school while the other members of the LS team observe the lesson and collect data on learners' learning (Huang & Shimizu, 2016). An observation sheet must be handed out to the observing teachers before the lesson presentation to document their feedback. A key aspect of LS is that it must happen organically without any alterations. Therefore, if the LS team decides on a day to teach the lesson, and that particular period is period number eight of the day, they must wait until it is time to teach the lesson. In other words, LS must not disrupt the day-to-day activities of the school.

In an attempt to receive what we can call "the outsider's view," the LS study team can invite a knowledgeable other (Amador & Weiland, 2015), such as a subject advisor, to be part of the lesson observation and presentation, to offer fresh and neutral insights during the post-lesson reflection stage of the LS. In South Africa, subject advisors support and lead teaching and learning in schools and in the implementation of the curriculum (Mdabe, 2019).

An aspect I think is key to this stage of the LS cycle, is the ability of the teacher offering the lesson and those observing to professionally notice learners' mathematical thinking and understanding. The teacher offering the lesson must be able to think on their feet upon professionally noticing certain aspects regarding learners' mathematical thinking and understanding, thus applying the principle of reflection-in-action (Schön, 1992). Although not necessarily documented, reflection-in-action, and therefore professional noticing, is inherent in LS, particularly in the lesson presentation and observation stage. This makes LS a fitting context to investigate professional noticing. The idea of reflection-in-action can be best described following Schön's (1992, p. 125) analogy using a basketball player and a jazz pianist:

> Think of a basketball player's instant maneuvering in response to an opponent's move, or a jazz pianist's on-line improvisation on the melody she has just heard the trumpet play.

If a learner asks an unanticipated question, or if the lesson takes a horrible turn, the teacher offering the lesson must make on-the-spot decisions. In most cases, learners' classroom contributions provide the teacher with what Stockero and Van Zoest (2013) called a pivotal teaching moment (PTM). According to Stockero and Van Zoest, a PTM is an occurrence that interrupts the flow of the lesson and consequently affords the teacher an opportunity to alter the way they are teaching, to change or extend the way learners understand mathematics. Synonymous with professional noticing, teachers need to identify a PTM as and when it occurs and then decide on how to respond.

During the lesson, learners need to be engaged in various instructional and assessment activities. During assessment activities, learners are supposed to work either individually or in collaboration; depending on what the LS team decided on. The teacher offering the lesson can walk around the desks, assisting, monitoring, or guiding the learners as they do the activity; with or without interacting with the learners. The aforementioned practice is known as *kikan-shido*, which is a Japanese term that translates to "an instruction at a student's (learner's) desk" (Pjanic, 2014, p. 89). According to Kriewaldt et al. (2021), *kikan-shido* occurs when a teacher intentionally scans learners' problem-solving strategies when attempting a mathematical activity in order to guide the learners or to select samples to use later on when engaging in a whole-class discussion.

When learners are doing an activity, they can arrive at different answers, whether correct or not. For instance, the teacher can select two answers, one correct and one incorrect to facilitate a classroom discussion, whereby learners exchange ideas on how to solve the mathematical problem.

The latter is a phase called *neriage*. *Neriage* in Japanese, is when a teacher purposefully identifies and selects learners to share, compare, and discuss their strategies; so as to refine and consolidate their understanding (Hourigan & Leavy, 2023; Sekao, 2023; Takahashi, 2008). The main aim of *neriage* is to deepen learners' mathematical thinking and understanding (Fujii, 2016). Datt and Hiroshi (2023) claimed that learners learn more when they teach one another, which can be achieved through *neriage*. Consequently, *kikan-shido* informs *neriage*. Going beyond the *neriage* phase, the teacher must appraise the ideas learners were sharing by implementing a class discussion and summarise what they have learnt. This process is known as *matome* in Japanese (Asami-Johansson, 2021; Pjanic, 2014; Shimizu, 1999).

2.2.2.2.4. Stage four: Post-lesson reflection

In this stage, one of the observers facilitates the post-lesson reflection and the teacher who was offering the lesson is the first one to reflect on the lesson. It is important not to make the presenter of the lesson the centre of the reflection session. Instead, the session must be informed by the lesson objectives by assessing contributors to their achievement or lack thereof. It is important to note that the lesson belongs to everyone who was involved in the planning of the lesson. The presenter was just a mouthpiece for the entire group. However, an important aspect of the post-lesson reflection is the seating arrangement. Sekao (2023) suggested that the seating arrangement during the post-lesson reflection should resemble collaboration.

The seating arrangement must not be organised in a way that the teacher who was offering the lesson becomes the centre of the reflection. A suggested seating arrangement is in a circle, perhaps around a large table or desk, facilitating a seamless flow of ideas and discussions. The process in this stage is linked to Schön's (1983) work on reflection-on-action, which is done after one has completed an action or activity. The teachers who were observing can use their documented findings from the observation sheet to inform their feedback.

Wei et al. (2023) argued that in addition to what and how they notice, observing teachers can also suggest what they could do in their instruction when faced with similar observed situations. Although reflection-on-action happens after the lesson, it is also inextricably linked to professional noticing because all three constructs of professional noticing are inherent in the reflective process. This then means that professional noticing can happen during the lesson as well as after the lesson has been taught. In addition, this stage affords the LS team to discuss any noticing opportune moments which might have been missed by the teacher offering the lesson. It also allows for sharing different views and ideas about what transpired in the lesson. After the data collected from the lesson has been discussed, a report is written (Huang & Shimizu, 2016), which can then be shared with all relevant and interested stakeholders (Coe et al., 2010). Reflecting on action provides the LS team with an opportunity to pre-empt occurrences that might need immediate action as well as occurrences that should be avoided in the next lesson (Bakker, 2022).

2.2.2.2.5. Stage five: Lesson improvement

The LS team uses ideas presented in the post-lesson reflection stage to consolidate and improve the lesson. To determine whether the lesson was effective, the LS team can decide to reteach (Chong & Kong, 2012) the lesson applying various variations, such as having the same teacher teach the same lesson to the same learners or, alternatively, having a different teacher teaching the lesson to a different group of learners. However, reteaching the lesson is optional and should be at the discretion of the LS team (Huang & Shimizu, 2016).

Although these stages follow each other logically, it must be noted that they should not be treated as a linear process as one stage informs the other, and you can go back to a previous stage when you discover certain issues as they arise. Despite the many benefits of LS, it also has shortcomings, such as being time-consuming, and many teachers are afraid of being criticised as they are used to making decisions alone in their classrooms (Guner, 2017; Sekao, 2023). Therefore, they may be reluctant to participate in LS.

2.2.3. Why Lesson Study resonates with 21st century professional development needs?

Notwithstanding the fact that LS is fast approaching its second century since its initial inception in 1872 (Makinae, 2010), its attributes and effectiveness are still relevant to the contemporary classroom (Sekao, 2023). The 21st century skills such as collaboration, critical thinking, communication, and creativity are required to thrive in the world of work (Ghafar, 2020). The classroom presents a fertile ground for the acquisition of 21st century skills. Therefore, the role of teachers in this regard cannot be over-emphasised. However, teachers need to have a firm grasp of these skills so as to instil them into the learners. To this effect, teachers' professional development (PD) must occur (Sancar et al., 2021). Postholm (2012) asserted that teachers' PD includes investigating how teachers learn and how they apply their learning in their practice to support and promote learners' learning. Furthermore, Postholm (2012) argued that teachers' learning can happen through various modes and strategies. For instance, teachers' learning can happen through them reflecting on their practice, observing their colleagues and then reflecting on what they have observed, and even through informal and unplanned conversations with colleagues. Consequently, Darling-Hammond et al. (2017) advocated for contemporary teachers' PD initiatives that consider the context within which teachers teach, i.e., the classroom. According to Darling-Hammond et al. (2017), there are seven traits of effective teachers' PD. The seven characteristics of effective teachers' PD are briefly described below. As I explain the seven characteristics advocated by Darling-Hammond et al. (2017), I also weaved in the traits of LS to demonstrate the resonance between them and subsequently to demonstrate how LS resonates with 21st-century teachers' PD needs.

Content focus

Effective PD focuses on teaching strategies regarding curriculum content and supports teachers in their classroom contexts. Content-focused PD treats curricula specific to a discipline, it is also job-embedded, implying that it takes place in teachers' classrooms. Content-focused PD allows teachers to study their learners' work and to try out new curricula with their learners (Darling-Hammond et al., 2017). During LS, teachers engage in the study of curriculum material, referred to as *kyozai-kenkyuu* in Japanese (Lewis et al., 2012), when planning lessons. Lewis (2016) argued that LS provides educators with the opportunity to engage with other teachers about content and curriculum materials to gather ideas regarding teaching, learning, and content. Furthermore, Lewis (2016) argued that among issues such as improving teachers' instruction through improving teachers' beliefs and their interest in learners' thinking, LS also improves teachers' curriculum knowledge, i.e. the content.

Active learning

Bates and Morgan (2018) argued that teachers must engage in interactive experiences that involve analysing learners' work and doing the activities they will give to them. This will help in deepening the teachers' subject knowledge. Darling-Hammond et al. (2017) argued that active learning provides teachers with the opportunity to actively engage in designing and practising new teaching strategies. Cheung and Wong (2013) indicated that in LS, teachers become active researchers who explore lessons and refine them; to enhance teaching and learning. Lewis et al. (2013) asserted that teachers who participate in LS become more knowledgeable and gain skills to teach mathematics and argue that this type of teacher learning will, in turn, improve learners' learning. According to Lewis et al. (2013), teacher learning means that teachers increase their content knowledge, confidence in teaching mathematics, and gain skills that they could use to help learners to learn mathematics effectively.

Collaboration

Sims and Fletcher-Wood (2021) contended that collaboration allows teachers to challenge one another and clarify any prevalent misconceptions. Furthermore, Bates and Morgan (2018) argued that teachers' collaboration in PD facilitates the addressing of instructional predicaments or issues. Consequently, collaboration in teachers' PD provides the platform for teachers to work together and share ideas (Darling-Hammond et al., 2017). 'Collaboration' is a trait that makes LS what it is and sets it apart from contemporary teachers' PD models. The whole idea behind LS is to have teachers working together. Fredrick (2019) argued that working collaboratively during LS affords teachers the opportunity to learn from one another and allows teachers to develop the spirit of collaboration and teamwork. The aforementioned point is cemented by findings from the study conducted by Sekao and Engelbrecht (2022) where all the participants agreed that working in collaboration with others is better than working alone.

Collaboration in this context included, inter alia: collaborative diagnosis of learners' difficulties in mathematics concepts, collaborative lesson planning to address learners' difficulties; observing the collaboratively planned lesson when one LS team member teaches the lesson as they apply professional noticing, and collaborative post-reflection to improve teaching practice.

The use of models and modelling of best practices

Effective PD is one that provides teachers with the opportunity to use, practise, and apply what they have learnt (Sims & Fletcher-Wood, 2021). Teachers can sharpen their instructional skills as they see others demonstrating their instructional practices (Bates & Morgan, 2018). Teachers can also significantly benefit from the modelling of best practices as they can then apply what they have learnt in real classroom situations.

Therefore, effective PD must provide the platform for teachers to share best practices and view models such as lesson plans (Darling-Hammond et al., 2017). Sharing best practices is a common practice in LS. Huang and Shimizu (2016) indicated that it is common practice to document and share instructional products, which can then be used as resources for other teachers. In fact, the LS model in South Africa also suggests that after the completion of an LS cycle, a report should be written and shared with interested parties (Sekao 2019; 2023)

Coaching and support from experts

Effective PD involves the sharing of expertise about the content whereby experts can share their expertise on a one-on-one basis or within a group (Darling-Hammond et al., 2017). Additionally, Sims and Fletcher-Wood (2021) argue that effective PD happens when teachers from different schools share ideas instead of recycling ideas within the same school. Coaching and expert support is not a concept that is foreign to LS. In fact, teachers taking part in an LS cycle can invite a 'knowledgeable other' who was not part of the lesson planning to offer their input and comments during the post-lesson reflection (Amador & Weiland, 2015).

Takahashi (2014) adds that the knowledgeable other in LS serves three purposes: (1) to provide comments on the work that the LS team has done, (2) to offer and give expert subject matter, and (3) to share what other LS teams have done elsewhere. Kadroon (2023) revealed in a study that experienced teachers coached pre-service teachers and assisted them with developing their teaching skills, and reflective practices. Suh et al. (2020) also revealed that peer coaching in LS helped promote better instructional practices among colleagues. In the South African context, teachers who are members of the LS team can serve as peer coaches, while more experienced teachers and subject advisors can serve as expert coaches.

Affordances for feedback and reflection

Effective PD provides teachers with an opportunity to reflect on their practices and to receive feedback on issues such as the lessons taught and even the teacher's classroom instruction (Darling-Hammond et al., 2017). Darling-Hammond et al. argued that feedback and reflection are two different aspects, however, they cannot be divorced. Teachers must see feedback as constructive rather than critical. Beyond that, they must take time to reflect on the feedback they received (Bates & Morgan, 2018). LS offers a platform for teachers to reflection-on-practice, which is made effective by professional noticing. In the reflection phase of LS, teachers reflect on the lesson by discerning the strong and weak points of the lesson (Guner & Akyuz, 2020). Suggestions are also made to refine the lesson, and if necessary, the lesson is retaught. By virtue of reteaching the lesson, teachers will learn from their practice and further refine their practice. Kager et al. (2024) argue that teacher discourses during the post-lesson reflection can prompt teachers to re-evaluate their beliefs and attitudes regarding teaching and learning.

Duration sustenance

Darling-Hammond et al. (2017) argue that although there is no specific time frame, effective PD takes time; it is not a once-off thing. Sims and Fletcher-Wood (2021) averred that teachers often take a long time to assimilate new knowledge. Therefore, it makes sense that effective PD should be sustained over time. Bates and Morgan (2018) argued that no matter how dynamic a PD session is, if it is once-off, it is likely to be ineffective. The issue of PD taking place over a duration is also inherent in LS. LS's complex and demanding nature does not fit to be done in a single day. This is what Sekao (2023, p. 122) calls an “un-Lesson Study practice”. In fact, Takahashi and McDougal (2016) argued that the typical LS cycle lasts about five weeks. Stokes et al. (2020) pointed out that an LS cycle can even last for an entire semester (6 months) or even a year.

2.2.4. Professional noticing as an inherent feature of Lesson Study

Among other models of PD, LS is one model that explicitly provides teachers with an opportunity to professionally notice learners’ thinking when they engage in reflective discussions, which is, according to Amador and Carter (2018), an essential aspect of growing professionally in LS. Instead of noticing mathematics content-specific issues, teachers might notice more generic stuff (Çelebi, 2023). Hence, Biccard (2020) and Khoza (2023) concurred that it is easy to miss opportunities that elicit learners’ mathematical thinking and understanding, especially when a teacher is individually delivering a lesson due to the many things happening in the classroom at the same time. However, LS provides a fitting context for teachers to effectively notice and make sense of learners' thinking and adapt their instructions accordingly (Guner & Akyuz, 2020; Çelebi, 2023). Unsurprisingly, research showed that LS can impact teachers' professional noticing positively, resulting in improved mathematical instruction (Cater & Amador, 2015; Cater et al., 2016; Baki & Işık, 2018; Çelebi, 2023; Guner & Akyuz, 2020).

As mentioned earlier, teaching a class individually can make one susceptible to missing important episodes one could have used to understand learners' current mathematical thinking and understanding. In light of that, Sekao (2023, p. 74) points out that in LS, the observers of the lesson are at a vantage point of noticing more than the teacher offering the lesson. Sekao asserts that the observers could potentially notice moments that could have been "opportune moments" but were instead "missed opportunities" where the teacher delivering the lesson could have asked probing questions to further understand learners' mathematical thinking and understanding. In fact, Khoza's (2023) study, although done in science education, indicated that teachers' professional noticing can be used as a tool to drive interaction in the classroom; thereby allowing teachers to practise deeper noticing.

Several studies have been conducted on the professional noticing of mathematics teachers in the LS context. An example of such is a study by Guner and Akyuz (2020) who found that the process of LS was able to support mathematics (pre-service) teachers' professional noticing of learners' mathematical thinking. Cater et al. (2016) revealed that when teachers engage in post-lesson discussions, deeper noticing was evident where more regular interpretations of learner learning were made. The finding aforementioned was averred by Karlsen and Helgevold (2019) who revealed that the deep analyses made in the post-lesson reflections practiced in LS led to higher levels of noticing whereby teachers did not just describe what they saw but took an analytic stance involving collective elucidations, enlightenments, and reasoning. Cater and Amador (2015) revealed through their study that when pre-service teachers were afforded the opportunity to mention and explain evidence of learners' learning, more noticing was prevalent. Amador and Cater (2018) also indicated through their study that when pre-service teachers engaged in LS, they were able to engage and discuss how learners were thinking mathematically and, hence, were able to notice at more advanced levels. Through their study, Yilmaz and Ozdemir (2023) revealed that the level of pre-service teachers' professional noticing was very low initially, and increased as they engaged in LS. Yilmaz and Odzermir's findings are supported by Druken (2023) who revealed that LS provided the teachers with an opportunity to understand and engage on leaners' mathematical thinking to a greater extent. Such affordances were not available outside the LS context. The shortcoming here, however, is that most research on professional noticing in the LS context was performed on pre-service teachers. This creates a research gap to investigate how in-service teachers practise and employ professional noticing when offering lessons in the LS context.

2.3. THEORETICAL FRAMEWORKS

A study can be guided by either a conceptual or a theoretical framework. A conceptual framework is used to demonstrate how a researcher understands the key concepts under investigation (Luft et al., 2022). The researcher builds a conceptual framework, and it does not exist elsewhere (Tamene, 2016). A theoretical framework is a set of concepts logically connected and developed from other theories used by researchers to support a study (Varpio et al., 2020). A theoretical framework helps a researcher to develop a lens through which they are going to collect and analyse data, discuss, and interpret findings and make recommendations (Kivunja, 2018). Every study should be guided by a framework (Lederman & Lederman, 2015), whether it be a conceptual or a theoretical framework. Due to the two contemporaneous aspects of my study, i.e., LS and professional noticing, I decided to employ two theoretical frameworks. I used the SLT developed by Lave and Wenger (1991) to provide a theoretical basis for the occurrence of the LS. Similarly, the FFPN by Choy (2015) provided a theoretical basis for mathematics teachers' professional noticing.

2.3.1. The Situated Learning Theory

Outline and background of the framework

The SLT was developed by Lave and Wenger (1991) and was presented in their book titled *Situated Learning: Legitimate Peripheral Participation*. The SLT argues that knowledge must be rendered in realistic contexts (Besar, 2018). In other words, effective learning cannot be achieved outside the context in which the learning takes place (Bell et al., 2013). Therefore, effective learning cannot be divorced from the context in which it occurs. Lave and Wenger (1991) stated that the SLT comprises two fundamental concepts: Legitimate peripheral participation and the community of practice. Legitimate peripheral participation concedes that new members learn to become practitioners when they engage in a community of practice. Prolonged and continued peripheral participation allows newcomers in a community of practice to internalise the culture of practice. Lave and Wenger (1991) defined a community of practice as a system that accommodates relationships between people, activities, and the world. The authors contended that these relationships grow over time. According to Jugdev and Mathur (2013), newcomers in a community of practice must start by engaging in less significant tasks and as they become more knowledgeable and skilled, they can engage in more sophisticated tasks.

The classroom is the context within which the LS takes place. According to the prescripts of the SLT, it only makes sense for teachers to engage in LS in the setting where teaching takes places, i.e., the classroom. In fact, Loose (2014) affirmed that the classroom setting can facilitate sustained PD when used as a context for teachers' PD. The SLT argues that learning does not only happen in human minds but also in the context of where the learning is taking place. In fact, learning happens effectively when there is collaboration among people (Loose, 2014). It is clear that the characteristics of effective teachers' PD, as suggested by Darling-Hammond et al. (2017) (see section 2.2.3), link with the major attributes of the SLT (i.e., working collaboratively within the context where learning takes place) and LS. The LS model is tailored for teachers' PD in collaboration with other teachers. Therefore, the SLT provides a theoretical basis for the implementation and occurrence of LS through the facilitation of a community of practice.

2.3.2. The FOCUS Framework for Productive Noticing

Outline and background of the framework

In his doctoral study, Choy (2015) explored how teachers noticed when they made pedagogically productive instructional decisions. The aim of his study was to unearth and comprehend how teachers think when they teach for mathematical reasoning. To achieve this, he proposed a robust theoretical framework to analyse teachers' noticing to gain insights into how they think in the planning, teaching, and reflection phases of a lesson. The theoretical framework Choy (2015) proposed is called the FFPN. This framework was developed by Choy (2015) as part of a doctoral study at the University of Auckland, New Zealand. The FFPN was developed within the LS setting, allowing Choy to examine the professional noticing of groups of mathematics teachers and to zoom into the professional noticing of a single teacher. The FFPN follows a photographic metaphor, elucidating and portraying teachers' noticing from two perspectives: the wide-angle view (zoom out) and the close-up view (zoom in). The wide-angle view was used to zoom out and view the general development of teachers' professional noticing throughout the lesson cycle. The close-up view was used to zoom into teachers' professional noticing in each phase of the lesson, i.e., in the planning, teaching, and the reviewing phases. The FFNP is based on and developed from Yang and Ricks' (2012) Three-Point Framework: Key Point, Difficult Point, and Critical Point (Choy, 2015; Dindyal et al., 2021).

The Key Point is associated with the main mathematical concept, which is the focus of the lesson; the Difficult Point refers to the cognitive hindrances that learners encounter when attempting to learn the Key Point, and the Critical Point refers to the methods teachers apply to help learners vanquish the Difficult Point to learn the Key Point (Choy, 2015; 2016; Dindyal et al., 2021). Choy (2015) then introduced the three focal points of the FFPN based on the Three-Point Framework, which are Concept (Key Point), Confusion (Difficult Point), and Course of Action (Critical Point) and suggested that the alignment among the three focal points can allow mathematics teachers to practise productive noticing. The alignment in question asks whether the teacher's Course of Action (Critical Point) appropriately addresses learners' Confusions (Difficult Point) to effectively learn the Concept (Key Point). Not only is FFPN useful to teachers to enhance their skills on how they apply professional noticing, but it can also help researchers to investigate teachers' professional noticing (Choy, 2015; Guner, 2017). As already mentioned, the FFPN was developed by Choy (2015) as part of a doctoral thesis at the University of Auckland in New Zealand. The FFPN is concerned with what and how teachers notice regarding learners' mathematical thinking and understanding when learning from the three stages of practice namely, planning a lesson, teaching the lesson, and reviewing the lesson. The FFPN underlines what teachers attend to (focus) and how they notice (focusing) when practising productive noticing.

To promote effective noticing, the FFPN highlights two key dimensions: the need for an *explicit focus* (following the three focal points: Concept, Confusion, and Course of Action) and how *pedagogical reasoning* plays a leading role in ensuring the alignment of the three focal points of the explicit focus. To contextualise the three focal points, I use typical examples of daily occurrences in South African mathematics classrooms, taking lessons from my personal experience as a mathematics teacher. As such, I will refer to the example I gave earlier about adding fractions. The problem of adding fractions is not unique to South African mathematics learners; it has been found to be an issue elsewhere. Research conducted by Mohyuddin and Khalil (2016) in Pakistan also revealed that learners added numerators and denominators separately, thus treating the addition of fractions similar to the addition of whole numbers.

Concept

The focal point, *Concept*, refers to the mathematical ideas or concepts of interest for the lesson. In Figure 2.9, this focal point is illustrated by the green colour. A Grade 8 lesson can focus on the addition of common fractions. This then becomes the lesson's *Concept*. As depicted in Figure 2.9, the *Concept* features across all three phases of the lesson.

Confusion

The focal point, *Confusion*, refers to the mathematical obscurities, errors, cognitive hurdles, ambiguities, and misconceptions learners' display. This focal point is illustrated in red in Figure 2.9. Many learners struggle with adding fractions as they treat them like one does when one adds whole numbers. Many of them add the numerators and denominators separately. The *Confusion* then becomes the idea behind the addition of fractions. Similar to the *Concept* focal point, the *Confusion* focal point also features across all three phases of the lesson cycle. It can also happen that the teacher notices an unanticipated *Confusion* in the learners' learning. The teacher needs to be able to think on their feet and come up with a fitting *Course of Action*.

Course of Action

The *Course of Action* focal point refers to the instructional decisions or responses made by the teacher during the planning, teaching, and reviewing of the lesson. This focal point is illustrated in blue in Figure 2.9. Once the teacher notices that learners are confused about a mathematics concept, they need to develop an appropriate *Course of Action* to address the *Confusion* and ultimately help the learners learn the *Concept*. For instance, a teacher can decide to bring a manipulative where they divide fractions into unequal parts and visually show the learners that it is not possible to add unequal parts because the whole idea behind adding fractions is to determine how many parts we need to make up a whole.

The FFPN also divides the process of noticing into three categories: *attending to*, *making sense of*, and *deciding to respond*. These three categories form the basis of teachers' professional noticing. In professional noticing, teachers *attend to* learners' problem-solving strategies, *make sense of* these strategies, and *decide to respond* in alignment with learners' current understanding (Guner & Akyuz, 2020). According to Choy (2015), the three categories of professional noticing take place throughout the three phases of a lesson (planning, teaching, and reviewing) and are depicted in Figure 2.9. For Choy (2015), a teacher who notices productively follows the following steps when planning a lesson:

1. Classifies what is precise about the concept(s) of the lesson.
2. Distinguishes what learners find difficult or confusing about the concept.
3. Analyses the reasons why learners find the concept challenging or confusing.
4. Analyses potential ways to address learners' confusion(s) when learning the concept, and
5. Develop and implement a cognitively demanding activity to address the learners' confusion.

Steps 1 and 2 fall under the *attending to category*. Steps 3 and 4 fall under the *making sense of category*, and step 5 falls under the *deciding to respond category*. When teaching the lesson, the teacher must be able to do the following:

1. Use learners' responses to identify their understanding of the concept.
2. Recognise potential confusion emanating from the learners' responses regarding the concept.
3. Analyse learners' responses to scrutinise what they are thinking regarding the concept.
4. Think of questions to ask the learners to expose their understanding of the concept and take it further.

5. Ask questions with the aim to unearth learners' thinking of the concept, and
6. Listen and get ready to respond to learners' thinking and/or understanding.

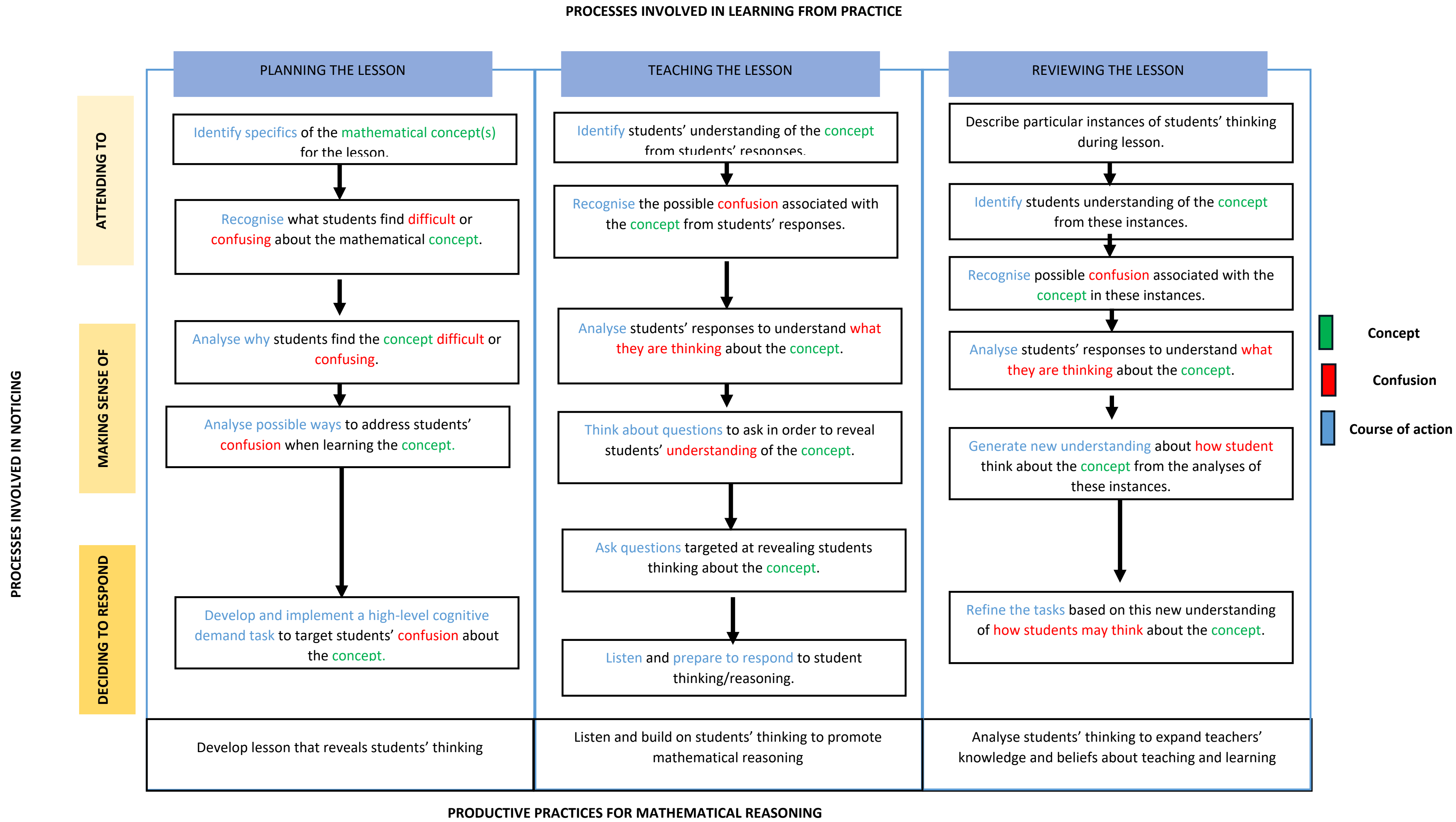
Steps 1 and 2 fall under the *attending to* category, steps 3 and 4 fall under the *making sense of* category, and steps 5 and 6 fall under the *deciding to respond* category. When reviewing the lesson, teachers must be able to do the following:

1. Illustrate peculiar instances of learners' thinking during the lesson.
2. Use these instances to identify learners' understanding of the concept.
3. Use these instances to identify possible confusion about the concept.
4. Analyse learners' responses to comprehend their thoughts regarding the concept.
5. Formulate new understanding regarding how learners think about the concept, taking insights from the identified instances, and
6. Refine the activity based on the newly generated understanding of how learners think about the concept.

Steps 1 to 3 fall under the *attending to* category, steps 4 and 5 fall under the *making sense of* category, and the sixth step falls under the *deciding to respond* category. The blocks below each phase are not colour-coded because they holistically summarise what happens in each phase (i.e., the planning, teaching, and reviewing phases), as per the FFPN.

Figure 2.9.

The FOCUS framework for productive noticing



Source: Choy (2015).

2.3.3. The need for an explicit focus and using pedagogical reasoning to enhance noticing

According to Choy and Dindyal (2021), teachers may be able to notice classroom occurrences; however, at times, what they notice is not of any use. In other words, teachers can attend to various issues and miss key opportune moments to address some mathematical issues requiring development in their learners' learning. Hence, there must be an explicit focus, i.e., the area of weakness in the learners' mathematical thinking and understanding, looking to be developed. An explicit focus allows teachers to notice pertinent instructional specifics (Choy, 2015; Choy & Dindyal, 2017) and allows them to capitalise on crucial opportunities to develop certain aspects of their learners' mathematical thinking (Choy & Dindyal, 2021). Dindyal et al. (2021) also added that most researchers usually investigate teachers' professional noticing without an explicit focus. The explicit focus comprises two key aspects: first, the three focal points (Concept, Confusion, and Course of Action) and second, the alignment of these three points through pedagogical reasoning (Choy & Dindyal, 2017).

In their study, Choy (2015) revealed that if teachers merely apply professional noticing without an explicit focus that guides them, they can notice issues that may be both relevant and irrelevant to the tasks at hand. However, teachers could reason pedagogically and make relevant and required instructional decisions when they were explicit and intentional about what they wanted to notice. Choy (2015) also noted that an explicit focus alone is not enough, but an alignment between what is noticed, and the Course of Action taken is also important. Choy's view proves that if teachers have an explicit focus, such as the three focal points of the FFPN, they can zoom in on events having mathematical importance in their learners' learning.

For Choy and Dindyal (2017), teachers can achieve the alignment of the three focal points through pedagogical reasoning, whereby they decide on the appropriate Course of Action that will help the learners overcome the Confusion they have when trying to learn the Concept. However, Dindyal et al. (2021) argued that aligning these focal points can be challenging because teachers need to form a connection between their understanding of classroom situations and how they respond. In fact, Choy

(2016) stipulated that this alignment is not innate, instead, it is contemplated by the teachers' pedagogical reasoning.

2.3.4. Relevance of the two theoretical lenses for my study

The focus of this study is on the professional noticing of mathematics teachers in the LS context. Through LS, teachers work collaboratively to address a mathematics topic or concept posing difficulties for learners to learn or for teachers to teach effectively (Sekao, 2023). The FFPN, which proposes activities teachers must do to notice learners' mathematical thinking and understanding productively, was specifically designed for the LS context (Choy, 2016).

The basic LS process can be imagined as a three-step process of Plan (a lesson), Do (teach), and See (observe and reflect on a lesson) (Joubert et al., 2020). On the same wavelength, the FFPN is based on three steps of learning from practice – planning a lesson, teaching it, and reviewing it (Choy, 2015). It should be noted that the interests of this study did not include the planning step of learning from practice. Hence, nothing is reported on how the lessons were planned.

The LS model aims to develop teachers professionally; this is achievable through the adaptation of the SLT, which strongly contends that learning must happen in collaboration with other people (Jugdev & Mathur, 2013; Lave & Wenger, 1991) within a specific context, particularly, in the setting where the learning takes place. The SLT suggests that newcomers (teachers, in this case) can practice legitimate peripheral participation, meaning that they can gradually learn the culture of practice. For the purpose of my study, greater emphasis was placed on the aspect of a community of practice where collaboration among teachers and teachers' PD can take place seamlessly.

Evidently, both the FFPN and the SLT are concerned with teachers' PD. While the SLT can develop teachers professionally and facilitate professional noticing through the LS cycle (Biccard, 2020), the FFPN explains the steps that teachers must follow to practise productive professional noticing in the three phases of a lesson, i.e., planning, teaching, and reviewing the lesson. The above argument clearly shows the relationship between the FFPN and LS (situated in the SLT), in attempting to achieve a common goal, which is mathematics teachers' PD and, in this case, mathematics teachers' professional noticing. Therefore, the SLT and the FFPN are two frameworks that are relevant to my study.

2.4. CONCLUSION

In this chapter, I reviewed the literature on professional noticing in the LS context and outlined the theoretical frameworks that guided my study. My discussion began with an outline of how professional noticing is significant for the effective teaching and learning of mathematics, followed by an extensive breakdown of LS as a teacher development model and the stages involved. After that, I referred to studies that explored professional noticing in the LS context, to build a foundation on which to base my study and identify the research gap. I then explained a theoretical framework and the frameworks I used for my study, namely the SLT and the FFPN. Lastly, I made an argument that explains why the two frameworks are suitable for my study.

CHAPTER 3: RESEARCH METHODOLOGY

3.1. INTRODUCTION

Research methodology is referred to as the overarching plan a researcher follows to achieve the aims and objectives of their study (Kothari, 2004), and thus systematically solve the research problem (Sutrisna, 2009). Just as a teacher requires chalk and a blackboard to effectively deliver a lesson, various research methods exist within research methodology, serving as tools that are used to attain a study's aims and objectives (Sutrisna, 2009). The methodology chapter outlines how a researcher used research methodology and research methods to collect, analyse and make sense of data. In this chapter, I discuss the measures I took to answer my research questions. I have presented the synopsis of the flow of Chapter 3 in Figure 3.1.

Figure 3.1.

Outline and flow of chapter 3

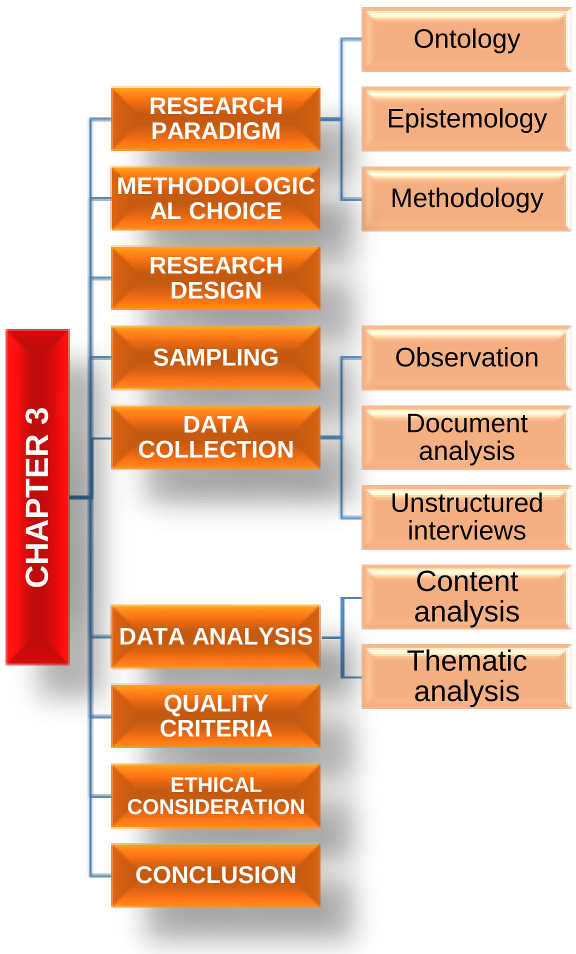


Figure 3.1 shows that the first aspect of the methodology is the research paradigm and its associated paradigmatic assumptions, whose primary role is to assist a researcher in studying and understanding the world (Rehman & Alharthi, 2016). Assumptions of a paradigm include (1) the ontology, which is the nature of reality (Rehman & Alharthi, 2016); (2) the epistemology, which is concerned with how one knows what one knows (Ikram & Kenayathulla, 2022); and lastly; (3) the methodology, which is the research strategy adopted by the researcher (Alharahsheh & Pius, 2020). The second aspect I discussed is the qualitative research approach, which is rooted in the interpretivist paradigm. Typically, a qualitative research approach is used to understand qualitative phenomena such as human behaviour and subjective experiences (Kumar et al., 2023). I then discussed the multiple case study research design as the design I adopted for my study. Multiple case studies are often used when more than one case study is involved (Yin, 2003).

My discussion is followed by an outline of the sampling techniques I used. I used purposive sampling primarily because I wanted to utilise participants who met specific requirements (Andrade, 2021). I found several participants who met my criteria, and I then employed convenience sampling to select conveniently available participants (Andrade, 2021). The fifth aspect of my discussion is the data collection methods I employed. I used observation, document analysis, and unstructured interviews for corroboration purposes. Next, I discussed my data analysis methods: document and thematic analysis. I then discussed the five concepts of the qualitative research quality criteria and the measures I took to enhance the quality of my study (Treharne & Riggs, 2015). Next, I discussed the ethical considerations I followed to ensure my study was ethical and protected its human subjects (Roshaidai, 2018). Lastly, I have provided a brief conclusion of the chapter.

3.2. THE RESEARCH PARADIGM

A paradigm is a researcher's worldview. Sefotho (2015) describes a paradigm as a lens through which researchers agree to be the standard outline of how one conducts research and is rigorously influenced by how one makes sense of the world. There are four types of research philosophy based on how a researcher views and understands the nature of the research process namely, positivism, interpretivism, realism, and pragmatism (Al-Ababneh, 2020). Positivism is the philosophical stance of natural scientists aiming to produce generalisations by observing social reality without the influence of human interpretation and bias (Alharahsheh & Pius, 2020). Contrary to positivism, interpretivism is a philosophical stance prominently used in qualitative studies (Muzari et al., 2022), which is concerned with gaining an in-depth understanding of the variables and factors that are related to the context within which the study takes place (Alharahsheh & Pius, 2020).

Ikram and Kenayathulla (2022) argued that the primary focus of interpretivism is to derive meaning through the understanding of the world and human behaviour. Realism is a philosophical stance based on the idea that reality exists and is independent of human thoughts and beliefs (Al-Ababneh, 2020). Lastly, pragmatism claims that reality exists in the world and assumes that the uniqueness of people may impact how they see the world (Al-Ababneh, 2020). Hence, pragmatists argue that there is no one correct way of conducting research; instead, what is relevant must be used.

In this study, I aligned myself with the interpretivist paradigm. In interpretivism, researchers are required to interpret the constituents of their discoveries. Therefore, the only way that interpretivists can access reality is through social constructions such as language and shared meanings (Ikram & Kenayathulla, 2022). According to Creswell and Poth (2016), a paradigm reveals a researcher's view on what the essence of reality (**ontology**) is about, how they know reality (**epistemology**), and the assumptions they make while conducting research (**methodology**). Therefore, each paradigmatic tradition is underpinned by certain ontological, epistemological, and methodological assumptions. Since I chose the interpretivist paradigm for my study, I confined my articulations of the three assumptions above to interpretivism to demonstrate how they apply to my study.

Ontological assumptions of the interpretivist paradigm: Ontology is concerned with the nature of reality and establishing a phenomenon's existence and nature (Alharahsheh & Pius, 2020; Junjie & Yingxin, 2022). Essentially, ontology asks the question, what is reality? (Ikram & Kenayathulla, 2022). The ontological assumption of interpretivism declares that reality is multiple (Havea et al., 2020) and that multiple interpretations exist for every situation (Ikram & Kenayathulla, 2022).

In fact, Reimer and Johnston (2019) reminds us that reality depends on how one perceives and interprets the world, therefore, knowledge is socially constructed, contextually driven, and culturally situated. The direct implication of the above discussion on my study was that I had to employ multiple ways of collecting data to satisfy the principle of multiple perspectives.

Epistemological assumptions of the interpretivist paradigm: Epistemology can be defined as notions that constitute acceptable knowledge and ways that establish who can be the knower, how they know what they know and how they construct knowledge (Berryman, 2019; Ikram & Kenayathulla, 2022). According to Alharahsheh and Pius (2020), epistemology is an internal constituent within the researcher concerned with how the researcher perceives the world around them.

The *epistemological assumption of interpretivism*, therefore, is that knowledge is subjective and constructed by the researcher and those involved in the setting where a phenomenon is taking place (Havea et al., 2020; Ikram & Kenayathulla, 2022). Therefore, it is important that researchers immerse themselves into the research setting to gain context as well as a specific understanding of the phenomenon under investigation (Tuli, 2010). Therefore, I had to immerse myself in the research setting, where professional noticing was taking place – that is, the classroom, to get a first-hand and context specific experience of how mathematics teachers employ professional noticing in LS.

Methodological assumptions of the interpretivist paradigm: Methodology can be described as an overarching plan or strategy that needs to be followed when conducting research (Alharahsheh & Pius, 2020; Berryman, 2019). In line with the interpretivist paradigm, the methodological choice relevant for my study is the qualitative research approach, whose goal is to provide a specific comprehension of a phenomenon based on those who experience it (Alharahsheh & Pius, 2020). As such, my study was qualitative in nature because I wanted to understand how mathematics teachers practise professional noticing in their classrooms in the LS setting. Given the assumptions articulated in the previous paragraphs, I chose interpretivism because it allowed me to focus on and interpret the professional noticing of mathematics teachers in the context of LS, thereby causing my study to be focused on a particular topic (Alharahsheh & Pius, 2020). Interpretivism also allowed me to collect data in multiple ways (Pervin & Mokhtar, 2022). I do, however, realise that interpretivism is subjective (Muzari et al., 2022) and that the findings from a study facilitated within the interpretivist paradigm cannot be generalised (Junjie & Yingxin, 2022). In fact, another researcher could interpret the data set I collected differently (Alharahsheh & Pius, 2020).

3.3. METHODOLOGICAL CHOICE

I followed the qualitative research approach in my study because it allowed me to produce multiple realities or perspectives of knowledge gained (Rahman, 2017) and, therefore, it aligned with the interpretivist paradigm that guided my study. Ezer and Aksut (2021) defined qualitative research as studies that aim to probe, comprehend and construct meaning from situations, events, and phenomena in their organic environment. The organic environment in this context refers to a natural environment where a particular situation, event, or phenomenon occurred, which was experienced by individuals in real-time.

More specifically, the organic environment in my study is the classroom where teaching and learning are taking place. Various researchers have stipulated that findings from qualitative studies are generally in the form of words and that the data reflects the primary insights and feelings of the people in the environment where the study is taking place (Busetto et al., 2020; Muzari et al., 2022). The advantages of the qualitative research approach include that qualitative studies yield descriptions of participants' feelings, opinions, and experiences, thereby making it possible to interpret the meaning behind their actions (Rahman, 2017). Deriving meaning from participants' actions enabled me to collect data drawn from profound engagements with the participants (Muzari et al., 2022). Due to its interpretive nature, the main source of data collection and analysis in qualitative research is the researcher himself (Kamal, 2019), allowing the researcher to understand the situation and interpret the findings (Haven & van Grootel, 2019). The qualitative research approach is suitable for this study due to its alignment with the interpretivist paradigm (Kivunja & Kuyini, 2017). This alignment has allowed me to understand and interpret how mathematics teachers employ professional noticing when offering lessons in the LS setting to understand learners' mathematical thinking and understanding. In other words, I was able to gain an insider's perspective of the teachers under investigation, or as Kivunja and Kuyini (2017, p. 33) put it, I was able to “get into the heads of the subjects being studied”.

3.4. RESEARCH DESIGN

I followed a multiple case study research design. Multiple case study designs are used when a study involves more than one case study (Gustafsson, 2017). According to Yin (2003), multiple case studies can be used for two purposes: to foretell similar results (known as literal replication) or to produce different results based on predictable reasons (known as theoretical replication). Yin (2003) argued that multiple case studies afford the researcher the opportunity to analyse data within and across cases; and, and this point is further reinforced by Baxter and Jack (2008). Various researchers argue that multiple case studies are used to understand the similarities and differences between cases (Baxter & Jack, 2008; Gustafsson, 2017).

Notwithstanding, Zainal (2007) argued that multiple case studies can enhance and support findings from previous results. Yin (2009) averred that a second case study can be used to uncover a fundamental result in a first case study, in order to replicate the findings. Yin (2018) argued that a researcher can use a “two-case” case study, where the second case study is used to fill a gap left by the first case or to respond better to some explicit shortcomings of the first case study. Yin further argued that one must consider having at least two case studies to make a strong case and diminish the amount of criticism that case single-case studies face, such as scepticism about whether the findings are empirical, especially when a case is very unique.

For my study, I used the “two-case” multiple case study, where I collected data from one school and analysed it before going to the second school, also to collect and analyse data. It should be noted that the purpose of this approach was not to compare the results found in the two cases but to provide supplementary information and data that might have been lacking in the first case and to address the deficiencies better that may or may not have been present in the first case (Yin, 2018; Zainal, 2007).

The small-scale nature of case studies allowed me to gain an in-depth understanding of the phenomenon under investigation (Changyong, 2021; Takahashi & Araujo, 2019). Through my study, I gained an extensive understanding of mathematics teachers’ professional noticing, as practised in LS. Yin (2009), a renowned proponent of case studies, argued that case studies are employed in research that seeks to explore a contemporary phenomenon in a real-life context.

Notwithstanding, case studies are case-specific. Therefore, the results obtained from case studies cannot be generalised elsewhere (Changyong, 2021). However, Bassey (2001) posed a special situation called *fuzzy generalisation*, which argued that research findings can be confirmed elsewhere, provided that the research is conducted under the same conditions and in the same context. Other critics of multiple case studies suggested multiple case studies undermine the single-case study's significance and are expensive and time-consuming (Gustafsson, 2017). However, findings from multiple case studies are generally considered cogent (Yin, 2018), and Zainal (2007) argued that the ability to replicate findings in multiple case studies increases the level of trust in the vigour of the method.

Other researchers have used the case study research design to understand specific cases that were of interest to them. For example, Genc and Erbas (2020) used the case study design to investigate the impressions of secondary mathematics teachers regarding hindrances to the development of mathematical literacy. Selmer et al. (2022) used the case study design to study and describe the professional noticing of pre-service teachers when they examined learners' written material.

Lastly, Khoza (2023) used a case study research design to explore how science teachers can use professional noticing to respond to learners and forge classroom interaction. The three examples explained above used the case study research design to gain an in-depth understanding of a phenomenon that interested them and led them to information-rich conclusions.

Yin (2018) suggested that researchers must define and bound their cases when using a case study research design. For Yin, bounding implies that one clarifies what is and is not part of the case. For instance, if the case examines mathematics teachers, then teachers teaching other subjects cannot be part of the case. As such, my multiple case study is based on understanding the concept of mathematics teachers' professional noticing in the context of LS. Therefore, other teachers who teach subjects other than mathematics cannot form part of the participants of my study.

Similarly, the lessons taught ought to be based on mathematical concepts and must be taught to mathematics learners. Notwithstanding, the notion of professional noticing does not depend on a certain mathematics concept, or a certain number of participants. Therefore, professional noticing can be practised on any mathematics topic. For my study, the topic of interest in School A was *numeric and geometric patterns*, while the topic at School B was *factorising a trinomial*. Basic information about my two cases is shown in Table 3.2.

3.5. SAMPLING

I used purposive sampling to select the participants for my study. Purposive sampling, also known as deliberate sampling (Bhardwaj, 2019), is a cost and time-efficient (Thomas, 2022) type of non-probability sampling (Amir et al., 2020) where participants are selected for a study if they meet predetermined characteristics that are relevant to the purpose of the study (Andrade, 2021). Purposive sampling can help yield real-life results provided that the selected participants have the required knowledge or are familiar with the subject under investigation (Amir et al., 2020).

The participants of my study were mathematics teachers and Grades 7 and 9 mathematics learners. It should be noted that in the LS setting, teachers who teach mathematics in a particular school can constitute a LS team; therefore, participating teachers were not confined to Grades 7 or 9 in my study. The choice for Grade 7 was motivated by the fact that Grade 7 is an entry-level into the Senior Phase, which falls under the General Education and Training (GET) band (DBE, 2011). The Senior Phase is identified as a phase where learners struggle mathematically (Mabena et al., 2021). Furthermore, the choice of Grade 9 learners was motivated by two factors: (1) continuous unsatisfactory performance in mathematics locally and internationally (Mabena et al., 2021; Mokgwathi et al., 2019); and (2) Grade 9 in the South African schooling system is an exit grade of compulsory primary education for learners to proceed to Further Education and Training (FET) (Motshekga, 2019).

Therefore, using Grades 7 and 9 helped me to understand mathematics teachers' professional noticing and to gain insights into learners' mathematical thinking. I used the following criteria to select the participants for my study: mathematics teachers who are familiar with LS and its implementation, i.e., they practice LS in their teaching. The learners had to be in Grades 7 or 9. The group of learners to be taught in the research lesson had to have been taught by their own mathematics teacher who was assigned to teach them – this is in line with the trait of LS as an organic process. Although purposive sampling is beneficial to the researcher in that it allows one to select participants that are relevant to the study (Gill, 2020), it may, however, be challenging to access the sampled participants for a study (Bakkalbasioglu, 2020), especially information-profuse participants (Gill, 2020).

Additionally, I found participants with the characteristics I was looking for in various and scattered locations. Therefore, I used convenience sampling because participants were easily accessible and willing to be part of my study (Andrade, 2021). Convenience sampling helped me cut commuting costs to reach the participants, as I was travelling to places near where I live and my place of employment.

3.6. DATA COLLECTION

In line with the ontological assumption of interpretivism, I used multiple methods to collect data for corroboration and to gain different perspectives on mathematics teachers' professional noticing. The data collection methods I deemed relevant to my study are observation, document analysis, and unstructured interviews. I was guided by the research questions and the theoretical framework when collecting data. I collected data from two schools (School A and School B). At School A, Grade 7 learners were taught *numeric and geometric patterns*, while those in School B were taught how to *factorise a trinomial*. Professional noticing can apply to any mathematics topic; hence, it did not matter what topic was taught when collecting data for this study. In order to obtain enough information and reach a stage of data saturation, I used one LS cycle per school.

Table 3.1 outlines the link between the research questions, the aspects of the theoretical framework (FFPN), the data collection instrument(s), and the appropriate LS stage(s). It must be noted that SLT is not indicated in the table because I mentioned in Chapter 2 that its purpose was to facilitate LS in the classroom. Table 3.2 depicts key information about the two lessons as they took place in the different schools.

Table 3.1.

The link between the key aspects of my research

Research question	Theoretical framework	Data collection instrument	Relevant LS stage
a) What do teachers notice about learners' mathematical thinking during teaching within LS?	• Concept • Confusion (Misconceptions)	• Observation and video recording	• Lesson presentation and observation • Post-lesson reflection
b) How do mathematics teachers use noticing to make instructional decisions in-practice within LS?	• Confusion (Misconceptions) • Course of Action (in action)	• Observation and video recording	• Lesson presentation and observation • Post-lesson reflection
c) How does mathematics teachers' noticing steer their reflection-on-practice within LS?	Course of Action (after action)	• Observation and video recording • Document analysis • Unstructured interview	• Lesson presentation and observation • Post-lesson reflection

Table 3.2.

Details about the lessons offered during data collection

School	*Topic of interest*	*Grade*	*Number of teachers*	*Number of learners*	*Date*	*Time and duration*
A	Numeric and Geometric Patterns	7	9	28	02 – May 2024	13:30 – 14:33 (1 hour and 3 minutes)
B	Factorising a trinomial	9	2	20	15 – May 2024	07:45 – 08:35. 50 Minutes

3.6.1. Observation

Observation is a data collection method where the researcher observes participants' behaviour in the research setting to gain insight into their experiences in their natural environment where the research occurs, i.e., the classroom (Croker, 2009). Evidently, the trait of the *natural setting* as it pertains to observations as a data collection strategy is also an inextricable characteristic of the interpretivist paradigm as well as the LS approach. Data is generated by creating field notes (Cowie, 2009) including taking note of detailed accounts of what the researcher observed and their reflections regarding what they observed (Croker, 2009).

The observation allowed me to study mathematics teachers and learners in their natural setting, i.e., classroom setting. Observation enabled me to identify what the teacher presenting the lesson noticed about learners' mathematical thinking (see Appendix A) as they made certain instructional decisions when teaching. In addition to taking notes of what I observed, I used video recording to capture the moments I might have missed in real-time. The latter gave me something to refer to later when I was attempting to understand learners' thinking during the lesson (Sekao, 2023) and how the teacher offering the lesson employed professional noticing when opportune moments arose. Data collected through observation enabled me to respond to all three SRQs (see Table 3.1).

3.6.2. Document analysis

Document analysis is the review of written materials such as diaries, policies, and reports by the researcher (Busetto et al., 2020). In fact, any document that has text, has the potential to be qualitatively analysed (Morgan, 2022). After planning the lesson but before teaching it, the LS team at School A, independent of me as the researcher, created an observation sheet explicitly tailored for the lesson, as Sekao (2023) suggests.

The latter point coincides directly with an earlier suggestion by Karlsen and Helgevold (2019) that teachers need to design observation (forms) sheets to capture learner learning as an instrument for teachers' professional noticing. However, the LS team in School B decided to remain flexible by using a blank sheet to note anything noteworthy. I, however, did not receive the observation sheets from the teachers at School B since we never met for the post-lesson reflection due to the teachers being heavily involved with mid-year examinations and other school-related and personal commitments.

To avoid confusion, every encounter with the term "observation sheet" must be understood as the document that was used by the observing teachers (teachers who are part of the LS cycle but do not interfere with the teaching of the lesson in the classroom) to note down what they noticed during the lesson. Similarly, every encounter with the term "observation tool" refers to the tool I used as the researcher for my study. I then used *document analysis* to analyse the information and notes written by the teachers on their observation sheet as they relate to professional noticing attributes such as the Confusion learners displayed about mathematics Concepts, the Course of Action and instructional decisions they would make to respond to the learners' confusions during lesson presentation.

At School A, the LS team produced a well-detailed observation sheet in which they could comment on the lesson introduction, presentation, and conclusion. Some key features of the observation sheet were issues such as the coverage of the Annual Teaching Plan provided by the Department of Education for teachers to follow.

3.6.3. Unstructured interviews

Unstructured interviews are questions that are not based on a predetermined written schedule (Thomas, 2021). Unstructured interviews occur in a conversation like manner between the researcher and respondents allowing the respondents to feel free to say what they think without thinking if it is wrong or right to say (Chauhan, 2022). I used *unstructured interviews* as a supplementary data collection method to ask the teacher offering the lesson why they made (or did not make) certain decisions upon certain incidents. I used video recording through an online platform to capture the engagements to avoid making the teachers uncomfortable during the interviews by taking notes as they responded to my questions. Using video recording also allowed me to go back to what the teachers said when analysing the data.

3.7. DATA ANALYSIS

3.7.1 Content analysis

Content analysis is an organised method to analyse written, spoken, and visible transmission messages and create categories and concepts to summarise a phenomenon and describe its behaviour (Stragier et al., 2018). I used qualitative content analysis (Lindgren et al., 2020) to analyse data gleaned from the teachers' observation sheet (document analysis), and thematic analysis to analyse the data collected through observation and unstructured interviews (Braun & Clarke, 2006).

I categorised the data according to the *confusion about mathematical concepts displayed by the learners and the teachers and the Course of Action taken to respond to the confusion*. I employed directed content analysis to categorise data according to the three focal points of the FFPN, namely Concept, Confusion, and Course of Action (Choy, 2015), because they (the three focal points) explain the process of noticing learners' mathematical thinking (Humble & Mozelius, 2022), hence, I used deductive analysis (Azungah, 2018). However, the sub-themes underpinning each focal point emerged from the participants' experiences. Hence, inductive analysis was utilised in this regard (Bingham, 2023). Therefore, my overall analysis was inductive-deductive because although I was open-minded and guided by the participants' experiences, I was also influenced by my knowledge of the LS generally, and the FFPN in particular.

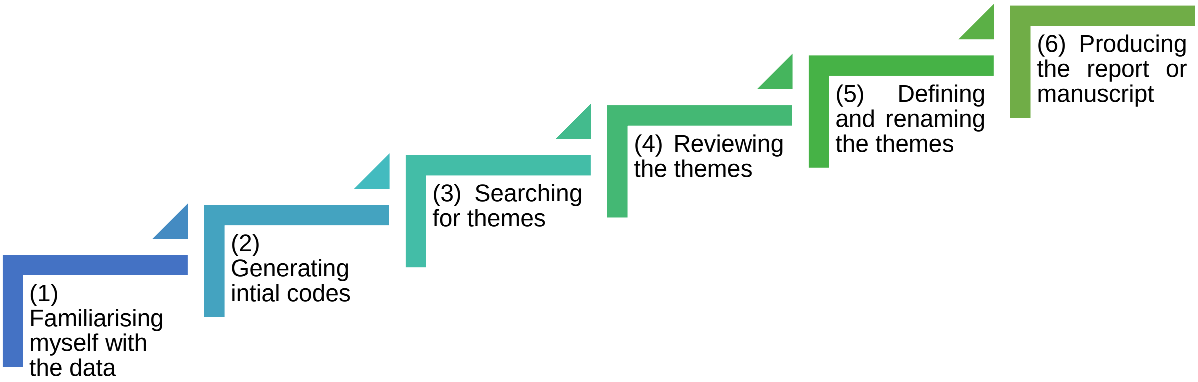
3.7.2. Thematic analysis

Thematic analysis is a method that allows a researcher to identify, analyse and report patterns or themes inherent in a set of data (Braun & Clarke, 2006; Fredriksen, 2021). Scharp and Sanders (2019) contended that researchers engaged in thematic analysis ask whether a data set meaningfully answers the research questions. Therefore, it seemed useful to use thematic analysis in my study because I had a set of research questions that required answers in the form of meaningful data. In fact, Castleberry and Nolen (2018) argued that using thematic analysis can address various research questions. Using thematic analysis is vital because themes capture essential facets of data, regardless of whether they are major or minor experiences (Scharp & Sanders, 2019). Thematic analysis is suitable for researchers who want to use a framework to code emerging themes (Fredriksen, 2021), which is the FFPN in my case. I deductively analysed the data using the main themes of my study as they were informed by the FFPN. However, sub-themes emerged as the teacher presented the lesson and I analysed those inductively. I also transcribed the exchanges I deemed key during the post-lesson reflection and categorised them according to the themes from the FFPN.

Challenges of thematic analysis include viewing the data simplistically and providing a mere summary and description of what the data entails (Kiger & Varpio, 2020). I ensured that I looked beyond the data and sought to unearth the meaning behind it. Researchers practising thematic analysis may be susceptible to bias and may report the data based on personal presumptions (Javadi & Zarea, 2016). However, I used thick descriptions to describe the participants, the context, and the methods I used in collecting the data to report, interpret and discuss the findings of my study (Korstjens & Moser, 2018). To apply thematic analysis correctly, I followed the relevant steps out of the six steps proposed by Braun and Clarke (2006), which are depicted in the diagram below and are briefly elucidated below the diagram.

Figure 3.2.

Six steps of thematic analysis



Note. *From Braun and Clarke (2006).*

Step 1: *Familiarising myself with the data*—In this stage, I immersed myself in the data through reiterated reading and active reading, where I searched for patterns and meanings. I also transcribed data I deemed key from the post-lesson reflection and unstructured interviews, took notes, and marked ideas before starting with the formal coding process.

Step 2: *Generating initial codes*—This stage involves working through the data and giving each aspect equal attention and importance. It also involves identifying interesting aspects that may form the foundation of repeated patterns. Since my analysis is theory driven, I went into the field with predetermined themes from the focal points of the FFPN, namely *Concept*, *Confusion*, and *Course of Action*. Therefore, there was no need for me to generate any codes.

Step 3: *Searching for themes*—This stage involves sorting all the identified and collated codes into the relevant candidate themes.

Step 4: *Reviewing the themes*—This stage involves reviewing the candidate themes and phasing out those that are not themes.

Step 5: *Defining and renaming themes*—This stage involves defining and renaming the themes and establishing their quintessence. My themes were predetermined from the FFPN, although some emerged from the data I collected.

Step 6: *Producing the report or manuscript*—This last stage involves writing the final report, in which the researcher tells the story of the data collected and convinces the reader of the merits of the analysis. This stage also includes extracts from the data that simplistically capture the essence of what the researcher is trying to convey. I did so using thick descriptions in Chapter 4.

3.8. QUALITY CRITERIA

According to Treharne and Riggs (2015), five significant concepts can be used to evaluate the quality of qualitative research namely, credibility, transferability, dependability, confirmability, and authenticity. The next paragraphs outline the steps I took to adhere to the five qualitative research criteria aforementioned, thereby enhancing the quality of my study.

The ***credibility*** of research data is when the researcher provides evidence that the results accurately represent the study's interest (Johnson et al., 2020). I used triangulation to ensure that the findings from my study were credible. Triangulation is using various methodological resources (Natow, 2020), theories, and data sources to comprehensively capture social reality (Farquhar et al., 2020). Firstly, I observed and took field notes as the teacher taught the lesson in their natural environment.

Secondly, I analysed the observation sheet used by other members of the LS team who were observing the lesson as it reflected what they had professionally noticed during the teaching of the lesson. Lastly, I used unstructured interviews for corroboration purposes and to allow the participants to make their observations clear to me as the researcher.

Transferability refers to when a researcher provides enough details about their research for other researchers to determine whether the results could apply to their situation, context, or setting (Johnson et al., 2020; Korstjens & Moser, 2018). I used thick descriptions to enhance the transferability of my study. Thick descriptions are information-profuse descriptions that provide the reader with enough information for them to judge the themes and categories of a study, to determine whether it will be appropriate to apply the findings of that study to other contexts or settings (Byrne, 2001). Using thick descriptions provides meaning to the reader because the behaviour, experiences, and context are well explained (Korstjens & Moser, 2018). I described my participants' setting, context, behaviour, and experiences as the lesson was presented and during the post-lesson reflection.

Confirmability refers to how neutral and accurate the data is (Houghton et al., 2013). Confirmability can be achieved when the researcher informs the reader that the study results depict the information collected from the participants and are not based on the researcher's biased interpretation (Johnson et al., 2020). A study's ***dependability*** is enhanced when the researcher provides extensive details of the research process so that other researchers can repeat their work (Johnson et al., 2020). Houghton et al. (2013) stated that the methods to institute confirmability and dependability are alike Therefore, I used the audit trail strategy, describing the research procedure in detail to reinforce my study's confirmability and dependability (Korstjens & Moser, 2018).

The fundamental principle of ***authenticity*** is that the researcher must ensure that the participants' different constructions and views are heard since people have dissimilar value systems that affect how they see and understand things (Johnson & Rasulova, 2016). Messner et al. (2017) suggested that the authenticity of a study is enhanced when a researcher can provide an account that shows he was in the research setting. They further argued that interviews and observations are some of the data collection methods that can add to the authenticity of a study. The latter aligns very well with my research because I immersed myself in the research setting, observed the lessons as they were being taught, and afterwards used unstructured interviews for corroboration. Using thick descriptions of my observations also added to the authenticity of my study due to the detailed account of my time in the research field I provided.

3.9. ETHICAL CONSIDERATIONS

Wa-Mbaleka (2019) pointed out that it is standard procedure in higher education institutions for researchers to seek permission to conduct research and abide by appropriate ethical standards. Connelly (2014) emphasised the point above and stated that peculiar care was required when dealing with vulnerable people such as children. It is also important that permission to conduct research in a research setting is granted, along with consent from potential participants (Roshaidai, 2018).

To ensure that my research was ethical, I applied for ethics approval from the Research Ethics Committee at the University of Pretoria **(see Appendix C).** Next, I identified two schools within Gauteng I intended to approach for my study. I asked for permission from the following stakeholders: the Gauteng Department of Education **(Appendix D)**, the district within which the identified schools belong **(Appendix E)**, the schools' principals **(Appendix F)**, the teachers to be studied **(Appendix G)**, the parents of the learners who would be taught **(Appendix H)**, and lastly, from the learners **(Appendix I).** After completing that process, I went into the field and collected data following the conditions stipulated in the ethics approval letter.

According to Roshaidai (2018), any research study must use ethical principles to protect its human participants. Therefore, I used the ethical principles discussed below to ensure that my study protected the human subjects and was ethical.

Informed consent – O'Sullivan et al. (2021) view informed consent as a process whereby the researcher approaches the potential participants of their study and provides them with information about their study so that they can make an informed decision to participate in the study voluntarily. They also argued that rapport and trust between the researcher and the participants need to be built during the process. In this case, I provided the potential participants of my study with information about my study and consent forms they needed to sign. These consent forms stipulated that they took part in the study voluntarily. In the process, I also answered any questions they had about my study and provided clarity where needed.

Autonomy or voluntary participation – I made it clear to the participants that their participation in my study was entirely voluntary, that it would take place without any coercion from me or anyone else and that they were free to withdraw from the study at any point should they wish to do so (Cohen et al., 2007; Connelly, 2014).

Privacy—In this case, privacy refers to how a person decides to control access to oneself, whether physical or informational (Resnik, 2010). In my study, participants were not coerced to share private information about themselves. Their privacy was ensured as the LS process, and the lesson's actual teaching occurred in their classrooms' privacy. If a participant wished to ask or answer questions in private, I allowed them to do so.

Confidentiality – since every participant has the right to privacy, I fostered the confidentiality of my participants by ensuring that their personal data was not made available to anyone else but myself as the researcher and the institution where I am registered (Ethicist, 2015). Therefore, any information that may reveal their identity, such as the video recordings or transcripts from the lesson presentation and observation, post-lesson reflection and unstructured interviews, would not be made available to anyone and would be kept in a safe place at the University of Pretoria.

Anonymity—I used pseudonymisation, in which participants' personal data is processed in a way that can no longer be attributed to specific individuals (Class et al., 2021). Therefore, the participants of my study remained anonymous because pseudonymisation allowed me to remove identifying information such as their names and places of work of the participants of the study. Instead, I assigned them codes such as “Teacher A1, Learner A1, Teacher B1, Leaner B1” and so on.

Safety in participation – I employed the principle of beneficence, which is understood as a way of doing good for others and preventing them from any potential harm (Orb et al., 2000). I ensured the participants would be free from social harm, such as embarrassment, especially when they wanted to share their views. Everything they said was an important contribution. My study had no potential for physical harm because we were not doing any physical activity that could harm the participants. The same applies to psychological harm. Participants were not asked or coerced to answer questions that could evoke negative feelings and emotions. The whole research procedure did not interrupt their normal day-to-day activities, thereby reducing the risk of potential harm.

3.10. CONCLUSION

In this chapter, I outlined the methodological aspects of my research, such as the research paradigm I followed and the adopted ontological, epistemological, and methodological stances. I proceeded to explain the methodological paradigm of my study, which included the research design, sampling methods, data collection, and analysis methods; how I aimed to enhance the quality of my study; the ethical considerations I took note of; and the principles I applied in the study.

CHAPTER 4: PRESENTATION OF FINDINGS

4.1. INTRODUCTION

In this chapter, I present the data I collected through observation, document analysis and unstructured interviews. The focal point of my study was how mathematics teachers employ professional noticing in the LS context. Therefore, the FFPN by Choy (2015) was used to guide data collection and analysis.

Although the main themes of the data collected stemmed from the research questions, the three focal points of the FFPN, namely, Concept, Confusion, and Course of Action, were used to inform the themes relevant to the research questions. I looked at what the teacher did [**Course of Action**] when they noticed that learners had misconceptions, expressed confusions or realisations [**Confusion**] when they were trying to learn the mathematical **Concept**(s).

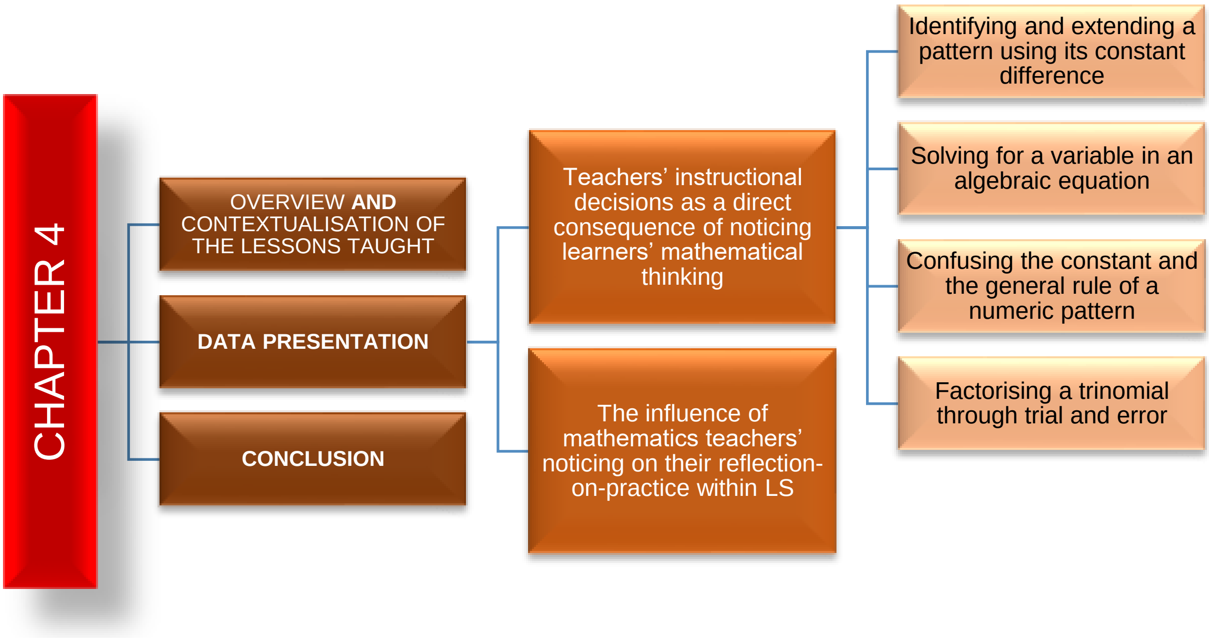
Notwithstanding, I was flexible in my data analysis and identified new sub-themes as and when they emerged. By so doing, my analysis was rendered inductive-deductive in nature. The steps of thematic analysis by Braun and Clarke (2006) helped me develop codes emanating from the field as I collected data. I familiarised myself with the data by reviewing the videos and the observation sheets multiple times and taking notes as and when I recognised patterns and key moments. I grouped data with the same characteristics and then came up with headings as I saw fit. I then began my discussion by outlining the processes I followed when I collected the data.

The two lessons I observed took place in an LS context, as evident. I used three methods of data collection. Firstly, I used observation to immerse myself in the research setting where the phenomenon of professional noticing took place. The primary purpose was to have first-hand experience of how the teacher offering the lesson employed professional noticing regarding learners' mathematical thinking.

Secondly, I used document analysis to gain insights into what the observing teachers noticed regarding learners' mathematical thinking through the notes they took during the lesson presentation. Lastly, I used unstructured interviews during the post-lesson reflection for corroboration purposes. The flow of the current chapter is illustrated in Figure 4.1.

Figure 4.1.

Outline and flow of chapter 4



As outlined in Figure 4.1, I begin the chapter by providing an overview and contextualisation of the two lessons taught. I then present the data that is relevant to the lessons I observed and conclude the chapter.

4.2. OVERVIEW AND CONTEXTUALISATION OF THE LESSONS TAUGHT

I observed two lessons from two schools. In School A, Grade 7 learners were taught *numeric and geometric patterns*. However, geometric patterns were not clearly presented because learners were not exposed to any shapes. Instead, they were given a theoretical scenario presented in a table with a geometric feature (Hexagon) and the number of matchsticks required to make one. Therefore, the lesson was in a way, purely focused on numeric patterns. The objectives of the lesson were as follows: By the end of the lesson, learners should:

(a) know how to investigate and extend numeric and geometric patterns, and
(b) describe and justify the general rule in their own words and in a mathematical representation.

These lesson outcomes were read to the learners by the teacher and were not written on the board or given to them on a document. According to the Department of Basic

Education (DBE) (2011), Grade 7 learners must be exposed to numeric and geometric patterns through patterns that have a constant ratio and/or a constant difference, and patterns with neither. Learners are also prompted to create their own patterns following a set of instructions and are exposed to representing patterns in tables.

In School B, Grade 9 learners were taught to *factorise a trinomial*. Factorisation is first taught in Grade 9, where learners are taught how to factorise algebraic expressions that involve taking out the highest common factor, a difference of two squares, and trinomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$, where a is a common factor (DBE, 2011). However, the lesson presented in School B focused on trinomials of the form $x^2 + bx + c$. The lesson outcome was that by the end of the lesson, learners must be able to factorise a trinomial using the trial-and-error method. As already indicated, this study did not focus on lesson planning, so I did not have the lesson plans used to plan the two lessons.

4.3. DATA PRESENTATION

In this section, I presented data collected from schools A and B. To preserve the identity of the participants, pseudonyms were used. Teachers from School A were classified as Teacher A1 (TA1), Teacher A2 (TA2) and so on. The A refers to them belonging to School A, and the number next to the letter identified them as unique. The same applied to learners; they were termed Learner A1 (LA1), Learner A2 (LA2) and so on. Teachers and learners from School B were identified through the codes Teacher B1 (TB1), Teacher B2 (TB2), Learner B1 (LB1), Learner B2 (LB2), and so on. The letter B refers to School B and the number in front of the letter uniquely identifies them. It should be noted that verbatim language was used as and where appropriate, to provide substance and present the instances as they occurred during the lessons.

4.3.1. Teachers' instructional decisions as a direct consequence of noticing learners' mathematical thinking

The heading of this section (4.3.1) emanates from SRQ1 and SRQ2. The two questions are interlinked and cannot be divorced. Hence, the findings are presented together. SRQ1 focuses on what mathematics teachers notice about learners' mathematical thinking, and SRQ2 focuses on what they do (instructional decisions) upon noticing certain features of learners' mathematical thinking. Consequently, SRQs

1 and 2 are linked by the misconceptions, confusions, or realisations learners had, i.e., what instructional decisions does the teacher make upon professionally noticing certain features about learners' mathematical thinking? Therefore, SRQ1 directly feeds SRQ2. It must be noted that although there are five processes of mathematical thinking, not all of them featured in the lessons, and more often than not, one process was apparent while others were only featured in bits and pieces. I also indicated the relevant mathematical thinking processes when presenting the findings.

4.3.1.1. Identifying and extending a pattern using its constant difference

The teacher interestingly introduced the lesson by asking learners to identify any patterns they saw in the classroom. LA1 recognised a pattern in the learners' seating arrangement, as there were five tables per group. LA2 added that there was a pattern at each table as a boy sat next to a girl. LA3 said, "The tables," TA1 asked LA3 if her answer was not the same as LA1's, and she said "No." TA1 decided to proceed with the lesson and read the lesson outcomes to the learners without asking LA3 to clarify how her answer was different to LA1's answer. It should be noted that TA1 missed an opportunity to understand how the learner was thinking, by simply disregarding the learner.

In continuing the lesson, TA1, asked the learners to complete patterns she wrote on the board. The activity included three questions. Question 1 had the pattern $\mathbf{3; 6; 9; \ldots}$. Question 2 was the pattern $\mathbf{1; 3; 5; \ldots}$ Learners had no difficulties in answering the first two questions. Question 3 was the pattern $\mathbf{1; 3; 7; \ldots}$ TA1 asked that one learner come to the front to write the answers on the board and explain their actions to obtain the next three terms. It must be noted that this practice allows learners to develop mathematical thinking as they have to present their solutions, communicate, and give mathematical reasons why they took the steps they did. LA1 went to the board and wrote the following: $\mathbf{1; 3; 7; 12; 18; 26}$. Directly below the terms in the pattern, the learner continued to write the difference between the terms as follows: $\mathbf{4; 5; 6; 7; 8}$. LA1 said, "To get the answers, I added 4,5,6,7,8 to all these (sic) previous terms each to get the next one." LA1 was trying to say that she had added 4 and 3 to obtain 7, then 7 and 5 to obtain 12, and so on. The mathematical process involved here was communication. In fact, the learner displayed characteristics of being successful at communicating mathematically because they could explain their thinking clearly.

Looking at the differences, the learner wrote below the terms; the learner was incorrect because the terms and the difference between them did not correspond. TA1 noticed that the learner did not get this question correct. Her instructional decision was to call upon another learner to assist LA1. LA2 came to the board and wrote the following: $\mathbf{1; 3; 7; 13; 22; 33,}$ and wrote the differences between the terms as follows: $\mathbf{2; 5; 8; 9; 11.}$ Again, LA2 was incorrect; however, he immediately realised that he was not correct, and he asked TA1 if he could rectify it, to which the teacher responded positively. LA2 erased his differences between the terms, i.e. $\mathbf{2; 5; 8; 9; 11,}$ and wrote the following: $\mathbf{2; 4; 6; 8; 10}$. This second attempt was also incorrect. TA1 noticed that LA2 was struggling; however, she made no response and opted to continue with the lesson. The aforementioned incident raises questions about the purposefulness of the specific activity because it did not align with the DBE (2011) specifications regarding the types of patterns learners in Grade 7 were supposed to know. Therefore, the question was not grade specific.

4.3.1.2. Solving for a variable in an algebraic equation

The second incident in the lesson came about when TA1 drew the following table on the board:

Table 4.1

Class activity 1

Input	1	2	3	4		n
Output	5	8	11			35

TA1 then said to the learners “the first question, they can say to you, explain the rule for the above pattern in your own words”. It became apparent that she was looking for an answer from the learners, who did not respond. She then proceeded to give them the answer and showed them how they could use their rule to find the general term (T_n), to which she wrote the following on the board (It must be noted that all answers that were written on the board by the learners were so small that I could not attach them in this dissertation. Instead, I retyped all occurrences of such):

$$T_n = dn + c$$

$$5 = 3(1) + c$$

$$5 = 3 + c$$

At this point, TA1 asked the learners what the variable was, and one learner in the class said, “3,” and TA1 responded by saying, “Ahh, you are just guessing”. LA1 raised their hand and said, “Ma’am, the variable is c”. While TA1 noticed that the learner who said 3 was the variable did not understand what a variable is, she, however, missed a PTM. TA1 then asked the learners what the next step was from $5 = 3 + c$. The following dialogue ensued.

LA3: $5 + 3$

TA1: Plus?

LA3: Minus

TA1: Why are you saying minus? (Learners just mumbled).

TA1: Why are you saying minus? (Repeated TA1*)*

LA4: Because the 5 is big and the 3 is small.

TA1: LA4 is saying because the 5 is big and the 3 is small, is correct? (sic).

Learners (collectively): No!

TA1: okay, why?

LA1: We are saying negative because 5 is on the negative side.

TA1: She’s, correct?

LA2: She’s almost there.

TA1: Oh, you have the answer; okay, tell us.

LA2: 3 was on the positive side, 5 was on the negative side, so we can’t add the numbers where the symbols are not the same, we have to *multiply the negative plus … negative multiplied by positive equals to negative, so that is why we said 5 minus 3.*

TA1: No! Guys, when a number is on this side of an equal sign (indicating the right side of the equation on the board: $5 = 3 + c$)*, you want to take it this side (indicating the left), it's going to change sign* (sic*). That's why it's no longer a positive 3 it's a negative 3 because it was at that side now it is …* (using hands to indicate the movement).

This interaction between TA1 and the learners clearly shows that learners had misconceptions and were confused about what to do when solving for a particular variable. They did not even know or understand what a variable was. TA1 did notice that they did not understand how to approach the equation and explained it in a way she thought best fitted.

Although TA1 made an instructional decision to explain how to solve an algebraic expression, she missed a PTM where she could have resorted to the curriculum's prescripts regarding solving algebraic equations. The CAPS suggests that Grade 7 learners must be able to solve number sentences by inspection and through trial and improvement (DBE, 2011). The scenario above was a typical case of solving an equation through inspection. Nevertheless, learners were then able to solve another example provided by the teacher, although she guided them through it step by step. For me, this was merely procedural rather than conceptual.

Although the activity was grade specific since it was an algebraic equation that could be solved through inspection and was in line with the learning outcomes, its pedagogical aspect failed to meet the requirements of a purposeful activity, i.e., the approach used to solve the equation was of a higher cognitive level than what the learners could currently manage.

4.3.1.3. Confusing the constant and the general rule of a numeric pattern

The third incident followed when TA1 gave the learners two activities to do in groups. Activity 1 had the following questions:

1. Complete the table below:

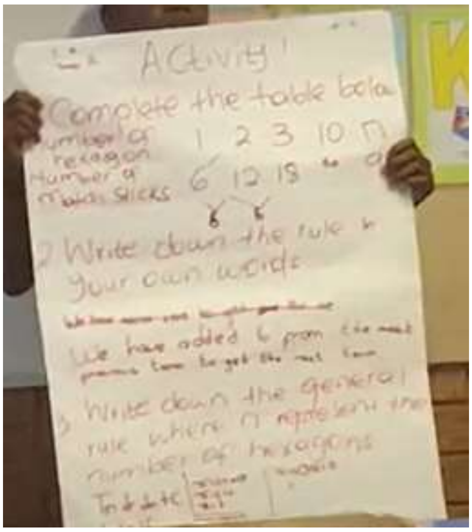
Number of hexagons	1	2	3	10	n
Number of match sticks					

2. Write down the rule in your own words.
3. Write down the general rule where n represents the number of hexagons.

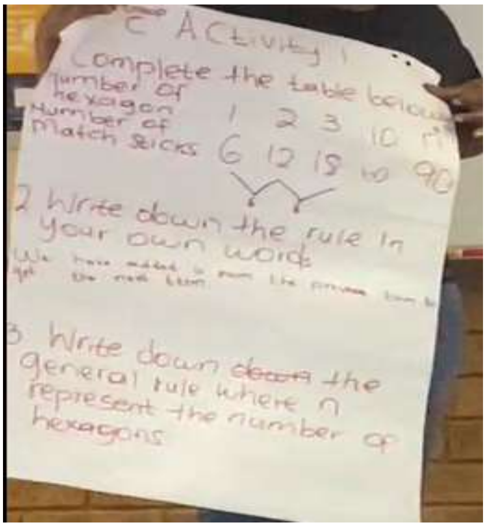
The solutions of groups A and C are shown in Figure 4.2.

Figure 4.2.

Solutions for activity 1



Answers from Group A



Answers from Group C

Learners were required to work in their groups for 10 minutes. After that, TA1 called upon the learners to come and present their work. Group A was called to the front to present their answers, but they went upfront and said nothing (making them unsuccessful communicators as far as mathematical thinking processes are involved). TA1 then called group C to come and present their answers. However, both groups had the same answers. Question 1 required the learners to extend the pattern by writing down the number of matchsticks in the 10^{th} hexagon, to which both correctly answered “60”. Question 2 required the learners to write down the rule in their own words. Group A answered, “We have added 6 from the next previous term to get the next term”. Group C answered, “We have added 6 from the previous term to get the next term”. Although the language used was not completely correct, learners understood that they needed to add 6 each time to obtain the next term in the pattern.

The problem arose with question 3, which read, “Write down the general rule where n represents the number of hexagons.” Group A wrote the following:

$$T_n = dn + c$$
$$6 = 6(1) + c$$
$$6 = 6 + c$$
$$6 - 6 = c$$
$$0 = c$$

Learners in group A thought that by finding c they had automatically found the general rule. This misconception was cemented by group C (who wrote their answer on the board), who wrote the same answer and ended there. TA1 did not attend to this issue but did attend to it in activity 2. TA1 was focused on how the learners had found n (input) when given 90 (output). Group C could not answer, and TA1 asked that another group help them. Different learners from group A came to the front to present. LA5 wrote the following on the board:

$$6 = 6(1) + c$$
$$6 = 6 + c$$
$$6 - 6 = c$$
$$0 = c$$

$$T_n = 60(10) + 0$$
$$T_n = 60 + 0$$
$$T_n = 60$$

$$T_n = 60(2) + 0$$
$$= 120 + 0$$
$$= 120$$

Then LA6 read out what they wrote on their poster, and then TA1 said the following:

> *“Thank you LA6, I think LA6 he have* (sic) *completed our activity 1 for us, and then he got everything right. I don’t know what groups they got* (sic)*. Go nale group e e leng gore [Is there a group that has] they have different answers there? Okay, let’s go to activity 2.”*

At this moment, TA1 missed an opportunity to address learners’ misconceptions regarding finding the general rule. It should also be noted that the question where learners were required to determine the value of n (the input) for 90 as the output was not addressed.

Group activity 2 was more of the same as Activity 1. The questions in activity 2 were as follows:

1. Given the sequence below:

$$8; 15; 22;\ \_;\ \_\_;\ \_\_\_$$

   1.1. Extend the pattern by writing down the next three terms.

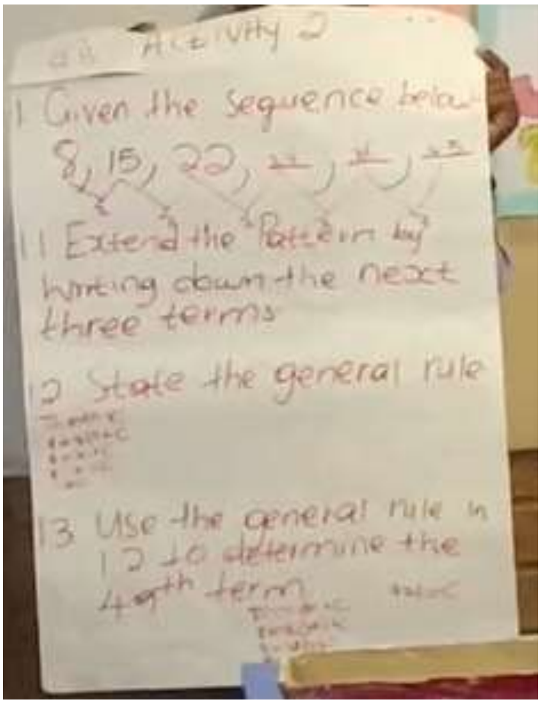
   1.2. State the general rule.

   1.3. Use the general rule in 1.2 to determine the 49th term.

Group B's solutions for activity 2 are shown in Figure 4.3.

Figure 4.3.

Solutions for activity 2



Question 1.1 required the learners to extend the pattern by writing down the next three terms of the pattern $\mathbf{8; 15; 22; 29; 36; 43}$ where the constant difference was 7. Question 1.2. required the learners to state the general rule of the pattern, to which the learners from group B answered:

$$T_n = dn + c$$

$$8 = 7(1) + c$$

$$8 = 7 + c$$

$$8 - 7 = c$$

$$1 = c$$

Again, the learners ended at c when asked to find the general rule. This corroborates my earlier finding that learners seemed not to understand what a general rule is. TA1 then intervened and said the following:

> *TA1: so, 1 is equal to c, is the answer? Guys, I said to you when you apply this formula ($T_n = dn + c$), you are going to generate the general rule, akere? [right?].* (Learners collectively said "yes"). *You find the constant, after finding the constant, it's where you have to write your general rule. This one is not a rule (*pointing at $T_n = dn + c$ *), it's a formula to generate your rule. You didn't find the rule now, you find* (sic) *only the constant, please after finding the constant…* (teacher points at where she was saying learners must find the rule).

TA1 noticed that learners were confused between what was a general rule and what $c\ was$, and how to go about finding the general rule using the standard formula, $T_n = dn + c$. She then intervened and responded by explaining as indicated above.

4.3.1.4. Factorising a trinomial through trial and error

The following narration is about occurrences that took place in School B. The teacher (TB1) indicated at the beginning of the lesson that he would be teaching learners how to factorise a trinomial using the trial and error method. TB1 first wrote the following on the board: $x^2 + 7x + 10$, which was the example used in this lesson to demonstrate how to factorise a trinomial. TB1 asked the learners if there was a highest common factor (HCF) in the given expression, to which learners collectively replied, "Yes,". TB1 asked the learners what the HCF is; LB1 said the HCF is x^2 and LB2 said the common factor is x. TB1 noticed that learners were confused about what the HCF is and used this as a PTM. He saw an opportunity to remind the learners what the HCF is by factorising through taking out an HCF, and the following exchange ensued.

> *TB1: Okay, do you still remember what I said when we are taking about the common factor?*

Learners: Yes.

TB1: If we say something is common, it means it must be available on term one (pointing at x^2), *term two* (pointing at x) *and term three* (pointing at 10)*. Now, if you look at x^2… on these two* ($\mathrm{x}^2 + 7\mathrm{x}$) *we can say we have a common factor of x, but remember, when we take out a common factor, we must make sure that whatever we say is common must be available in all these three terms* (sic)*. Now, do you see something x here* (pointing at 10) *or do you see just a number?*

Learners: Just a number.

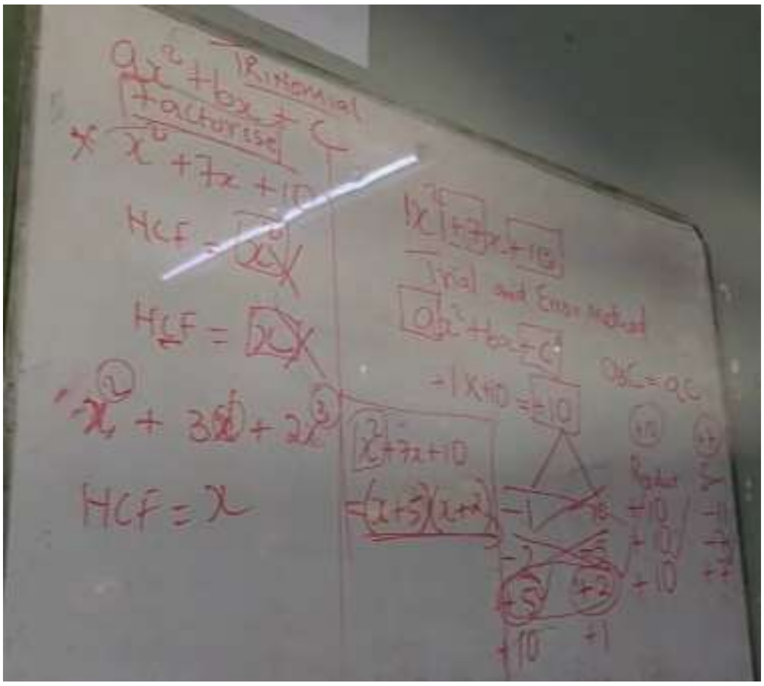
TB1: Now can we say x is a common factor?

Learners: No.

After clarifying what the HCF is and how it was identified, TB1 then proceeded to explain the procedure involved in factorising a trinomial. TB1 taught this through the route of multiplying the coefficient of x^2 and the constant, thus multiplying a and c when following the standard trinomial; $ax^2 + bx + c$. In this case, TB1 multiplied 1 by 10 to obtain 10 (see Figure 4.3). TB1 then asked learners what the factors of 10 were, which learners listed as 1,2,5 and 10. TB1 then asked the learners what he needed to multiply these factors by in order to get 10, and the learners listed 10,5,2 and 1. TB1 then attached signs next to the terms that will produce a product of a positive 10. As the method suggested, TB1 did a trial and error of the factors to see if they would get a positive 10 when multiplied and a positive 7 when added. The first pair of factors did not work out because while their product was $+10$, their sum was -11.

Figure 4.4.

The board where factorisation was demonstrated by the teacher



A PTM rose for TB1 when it came to the second pair. TB1 multiplied -2 by -5, and got $+10$. TB1 then asked the learners what is $-2+(-5)$ and the following conversation took place.

LB3: 7.

TB1: is the 7 *positive or negative?*

LB3: (Unsure) *Negative.*

TB1: It seems as if you are not sure. Why do you say is (sic) *negative?*

LB3: (Mumbles).

TB1: Okay, can you come in and assist? Yes? (Pointing at LB4).

LB4: If the signs are the same, you take the sign of both (sic) *and add the numbers.*

LB4 indicated characteristics of a learner who is successful in connecting mathematical ideas because even though the topic was factorisation, the learner could connect the idea to what they did in integers. TB1 then used this PTM by giving the following explanation:

> *TB1: That is what you did in Term 1. If you have two integers with the same signs, when we add them, if the signs are the same remember we do what? We add and the answer will take the sign of both* (sic).

TB1 then proceeded to show learners how to factorise the trinomial after discovering that the third pair of factors was the correct one. He showed them that once they have their factors, they must write the answer as $(x + 5)(x + 2)$.

After the whole process, LB5 raised his hand and asked a very interesting question, and the teacher and LB5 had the following conversation.

> *LB5: Sir, so when it is x^3 am I going to open three brackets?*
>
> *TB1: At your level, they cannot give you an expression with the power of x^3 whereby you're requested to open three sets of brackets. So, if they give you such a trinomial, it means somewhere, somehow, you have to factorise it first by taking out a common factor. What made you to ask this question maybe? Because here I have opened two sets of brackets, what informs you to say I'll have three sets of brackets?*
>
> *LB5: Akere [right], Sir, when it is x^2, I can see you opened two sets of brackets, yes, sir.*
>
> *TB1: Perfect! Le a mo kwa gore o reng?* [Can you hear what he is saying?]. *O re* [he says] *because the highest exponent of this trinomial it is (sic) two; hence, that's why we are opening two sets of brackets and you're correct.*

LB5 showed traits of connection and communication through his question and his answer when asked what informed his question. He realised that TB1 opened two brackets because the highest power of the expression was 2 and made a connection that if the power of the expression was 3, he would need three brackets for that.

LB6 raised his hand and said: But meneer [sir], in the textbook they say factorise $a^4 - b^4$. The teacher noticed a PTM since the learner's contribution highlighted confusion between factorising a difference of two squares and a trinomial. TB1 then replied as follows.

> *TB1: Okay perfect, it means now I'm going back. Remember now we are factorising a trinomial. a^4 has a square root. What is the square root of a^4? (Learners collectively said a^2). What is the square root of b^4? (Learners collectively said b^2). Now, this one is a square (pointing at a^4), this one is a square (pointing at b^4), in between there is a negative sign, remember I said the difference simply means between the two squares there must be a negative sign.*

It caught my attention that TB1 used "negative" and "minus" synonymously. Conceptually, this can cause confusion for the learners because a negative indicates that a number is less than zero, while a minus indicates subtraction. The same is true for plus and positive, where plus indicates addition and positive indicates that a number is greater than zero.

TB1 then proceeded to factorise the difference of two squares as follows:

$$(a^4 - b^4)$$

$$= (a^2 - b^2)(a^2 + b^2)$$

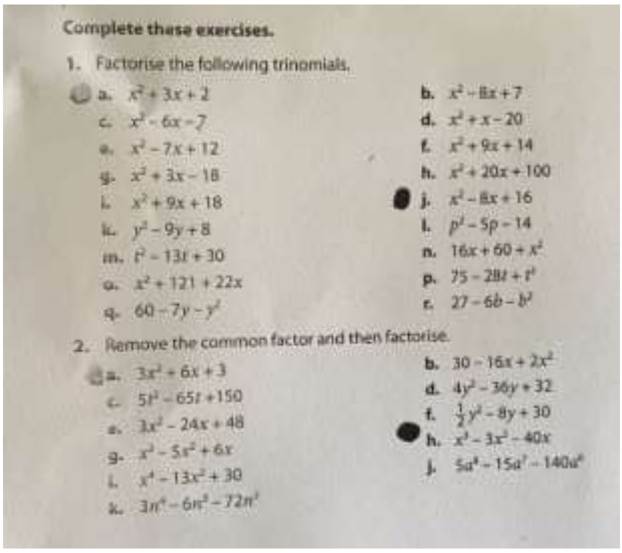
$$= (a - b)(a + b)(a^2 + b^2).$$

TB1 then asked the learners why factorising the last bracket was not possible, and LB6 answered, "Because there is a positive sign." This indicates that the last bracket does not present a difference of two squares because of the plus sign in between. LB6 also used plus and positive synonymously, which could cause conceptual misunderstandings as I have already indicated.

After this presentation, TB1 gave the learners two activities to do (See Figure 4.5). Activity 1 was 1(a) and Activity 2 was 1(j). TB1 also gave the learners 2(a) and 2(h) as homework. TB1 walked around and identified two learners to go and write their answers for the first activity on the board (see Figure 4.6). The stage within the broader teaching and learning process, where the teacher walks around the classroom scanning through learners' problem-solving methods is known as *kikan-shido*. LB7's solution is on the left-hand side and LB8's is on the right.

Figure 4.5.

Learners' classroom activities and homework



Complete these exercises.

1. Factorise the following trinomials.

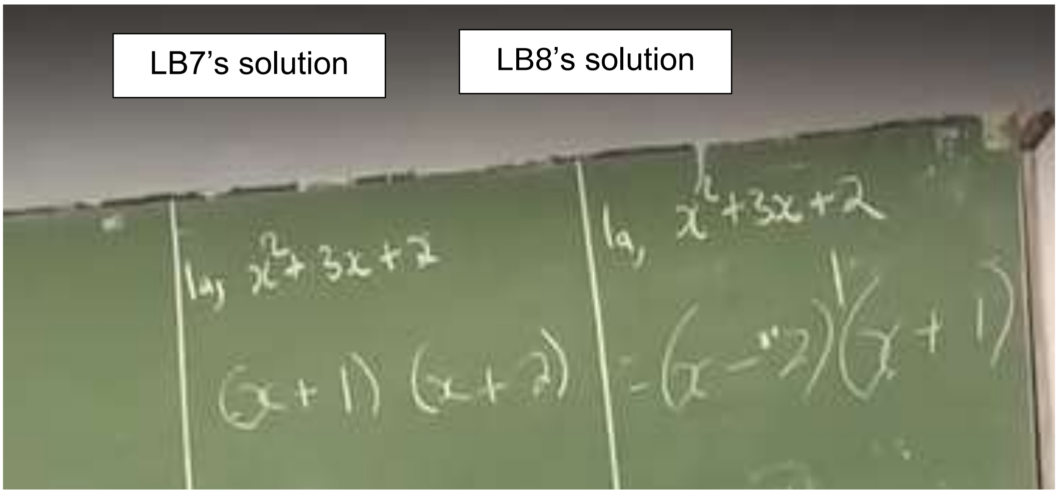
a. $x^2 + 3x + 2$
b. $x^2 - 8x + 7$
c. $x^2 - 6x - 7$
d. $x^2 + x - 20$
e. $x^2 - 7x + 12$
f. $x^2 + 9x + 14$
g. $x^2 + 3x - 18$
h. $x^2 + 20x + 100$
i. $x^2 + 9x + 18$
j. $x^2 - 8x + 16$
k. $y^2 - 9y + 8$
l. $p^2 - 5p - 14$
m. $t^2 - 13t + 30$
n. $16x + 60 + x^2$
o. $x^2 + 121 + 22x$
p. $75 - 28t + t^2$
q. $60 - 7y - y^2$
r. $27 - 6b - b^2$

2. Remove the common factor and then factorise.

a. $3x^2 + 6x + 3$
b. $30 - 16x + 2x^2$
c. $5t^2 - 65t + 150$
d. $4y^2 - 36y + 32$
e. $3x^2 - 24x + 48$
f. $\frac{1}{2}y^2 - 8y + 30$
g. $x^3 - 5x^2 + 6x$
h. $x^3 - 3x^2 - 40x$
i. $x^4 - 13x^2 + 30$
j. $5a^8 - 15a^7 - 140a^6$
k. $3n^4 - 6n^3 - 72n^2$

Figure 4.6.

Learners' solutions to activity 1.



TB1 opted to start with LB8's solution (which was not correct) and asked the learners if they agreed with LB8's solution, to which the learners replied, "No". TB1 then asked the learners if they agreed with LB7's solution, and they replied, "Yes". He then asked if someone could explain why LB7's solution was correct. The teacher's actions symbolise the stimulation of *neriage.* Within the context of the LS, *neriage* refers to a whole class discussion facilitated by the teacher to deepen learners' mathematical understanding (Fujii, 2016). LB9 raised her hand but only mentioned that LB7's solution has plus signs, whereas LB8's solution has a plus and a minus; the teacher acknowledged that but wanted to know why LB7's solution was correct. LB10 raised her hand and said:

> *LB10: As LB5 said that when x is to the power of 2 we must open two brackets, and LB7 wrote x as a constant of the formula, and the other side of the brackets wrote x and then the signs in the …* (inaudible) … *and when you add 2 and 1 it gives us the middle term which is 3 and when we multiply 1 times 2 it gives us positive 2.*

LB10's contribution indicated the communication process of mathematical thinking.

TB1 turned to LB7 and asked if LB10 had explained the process the way she did it and LB7 said, "Yes." TB1 then focused on LB8's solution and asked the learners what was wrong with LB8's solution. LB11 raised her hand and said:

> *LB11: Coz (sic), like sir, we make 2 as a negative. Negatives 2 times 1 it is equals to negative [2], because negative times positive is equals to negative.*

LB11's contribution relates to the connection and communication processes of mathematical thinking.

> *TB1: So, it means, what you are saying, if we take this negative 2 (pointing at -2 in LB8's solution) and multiply by positive 1 is gonna (sic) give us negative 2. Is it giving us the constant? (Learners: No). Now, that's the first step where we see that the two factors are wrong.*

The above utterances by TB1 resemble characteristics of *matome*, whereby the teacher summarises what the learners have learnt after the *neriage* phase, which involves a whole class discussion. TB1 then called LB12 and LB13 to present their answers for activity 2. LB12's and LB13's solutions are displayed in Figures 4.7 and 4.8, respectively.

Figure 4.7.

LB12's solution to activity 2

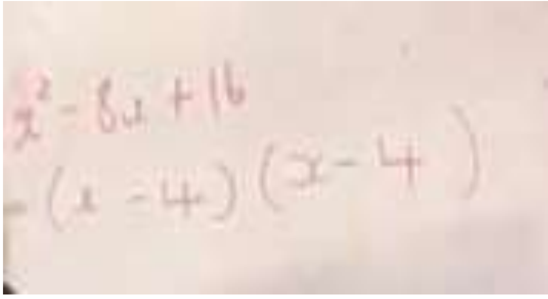
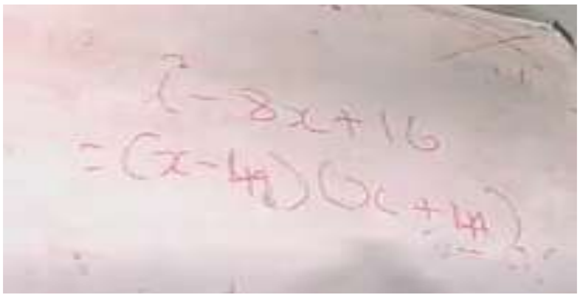


Figure 4.8.

LB13's Solution to activity 2



After LB12 and LB13 wrote their solution, TB1 asked the learners in the classroom if LB13's solution was correct, and the learners said "No" TB1 asked why they were saying "No." LB6 raised his hand, and the following ensued:

> *LB6: When we add the 4's … when we add negative 4. plus 4 it will give us negative zero … it will give us zero, then when we multiply them they will give us negative 8.* [Communication process of mathematical thinking].
>
> *TB1: Negative…?*
>
> *LB6: Oh, negative 16.*
>
> *TB1: Now this one* (LB12's solution) *is it correct or wrong?*
>
> *LB7: When we add 4 plus 4 is equal* (sic) *to 8 and when we times it is equal to 16* (sic). [Communication process of mathematical thinking].
>
> *TB1: But I don't see 4 plus 4, nna* [I] *I see minus 4, minus 4.*
>
> *LB9: Which is negative 8.*
>
> *TB1: Is it the same as the middle term? (Learners: Yes). And then when we multiply minus 4 times minus 4 what is the answer?*
>
> *LB9: Positive 16.*
>
> *TB1: Is it the constant?*
>
> *Learners: Yes.*

Generally, the connection and communication processes of mathematical thinking were dominant as learners were given an opportunity to explain their thinking, which they did very well through the connection of mathematical ideas.

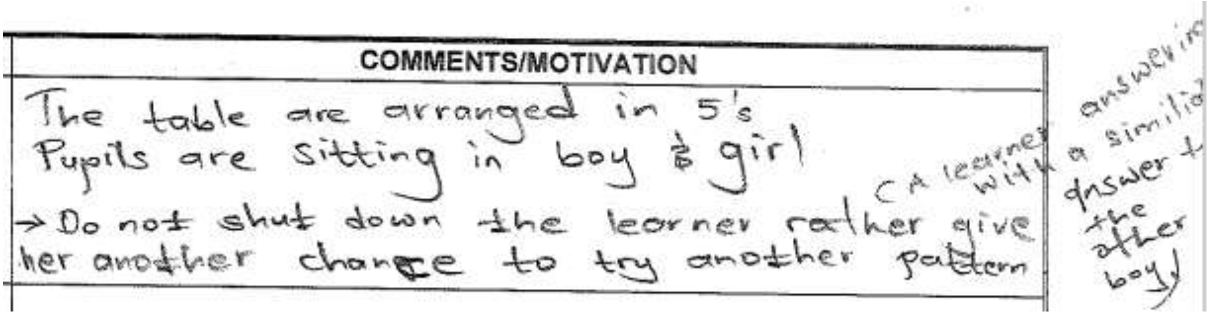
4.3.2. The influence of mathematics teachers' noticing on their reflection-on-practice within LS

The following findings stem from data collected and analysed from School A during their reflection after the lesson presentation and observation. A total of only nine teachers took part in the post-lesson reflection (TA1 to TA9). However, I only received six of the nine observation sheets to analyse what had come from the teachers observing the lesson. Therefore, the presentation here is based on the six observation sheets and the post-lesson reflection. The post-lesson reflection happened in the classroom where the lesson took place immediately after the completion of the lesson. The teachers were seated in a circle. The Mathematics Senior Phase subject advisor for the district in which the school is located and where LS was conducted led the reflection session. The subject advisor was, however, part of the LS cycle from start to finish. The teacher who was offering the lesson, TA1, was given the platform to be the first one to reflect. This practice by the LS team was in line with the prescripts of LS regarding the post-lesson reflection (Sekao, 2023). There were incidents where the teachers were tempted to make the teacher offering the lesson the centre of the reflection, however, the chairperson of the reflection session kept on reminding them that the lesson belongs to everyone who was involved in it. This practice was aligned with the prescripts of the LS post-lesson reflection.

As mentioned, TA1 was the first to reflect on the lesson. However, she merely expressed her disappointment with her learners, who seemed unable to grasp the content well and attributed that to time constraints. TA2 asked TA1 what positives she took from the lesson, and TA1 did not have anything to say. TA8 indicated to the LS team that "there was a learner who gave an answer (regarding identifying a pattern in the classroom) that was similar to another learners' answer and was ignored as TA1 decided to proceed with the lesson". TA8, therefore, suggested that "in such cases, teachers must give the learner another opportunity to try again or should ask the learner to look around and identify another pattern". The suggestion by TA8 was also captured in his lesson observation sheet (see Figure 4.9). TA4 added to TA8's point and mentioned that if a learner incorrectly answers a question posed by a teacher, the teacher must have a way of encouraging and motivating the learner and not just dismiss them.

Figure 4.9.

TA8's suggestion against learner dismissal


COMMENTS/MOTIVATION

The table are arranged in 5's
Pupils are sitting in boy & girl
→ Do not shut down the learner rather give her another change to try another pattern

CA learner with answering a similiar answer to the other boy)

I asked TA1 why she made an instructional decision to proceed with the lesson upon noticing that LA3's answer was the same as LA1's, and I asked her what was going through her mind during this incident. TA1 indicated that some learners were too eager to be involved in classroom engagements even when what they said was not part of the day's lesson. TA1 went on to say:

> *"I think she raised her hand and then realised gore [that] when they are saying tables, then they realised after gore [that] ohh mang mang o boletse ka the tables* [so and so spoke about tables], *so that's why she didn't go further explaining the way they are arranged."*

According to what TA1 said above, it was clear that she thought asking LA3 if her answer was not the same as LA1's answer made LA3 realise that her answer was the same as LA1's. However, it must be noted that the teacher made an instructional decision not to ask LA3 to explain her thinking, thus taking away an opportunity to notice the LA3's mathematical thinking.

Furthermore, TA8 also took note of the fact that learners struggled with finding the general rule of a numeric pattern since they kept getting stuck after finding the constant, c, in the formula $T_n = dn + c$. TA8 went on to suggest that perhaps another strategy must be used (see Figure 4.10). TA2 alluded to this point and indicated that instead of using a formula which seeks to produce procedural fluency rather than a conceptual understanding of the mathematics concept, the learners must be taught to “multiply the constant difference by the position of the term and see what you need to add or subtract to get the next term” (see Figure 4.11). According to TA2, that would make learners understand the concept better. TA3 suggested that revision should be done regarding how to find the general rule of a numeric pattern (see Figure 4.12).

Figure 4.10

TA8's suggestion for finding the general rule of a numeric pattern

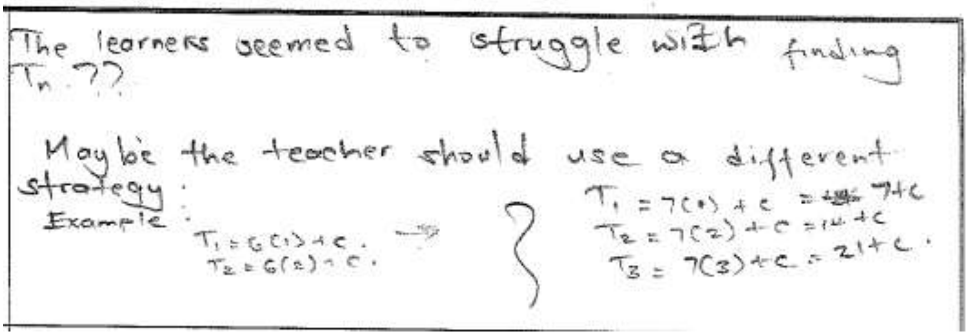


Figure 4.11

TA2's suggestion regarding how to find the general rule of a numeric pattern

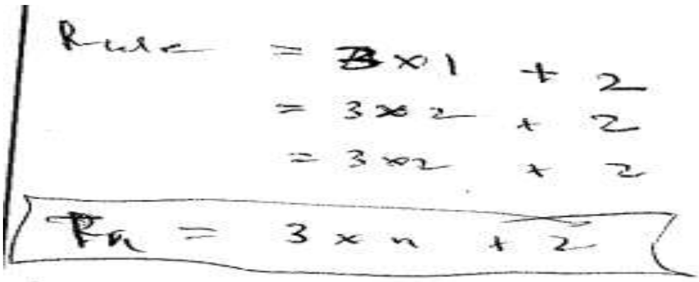
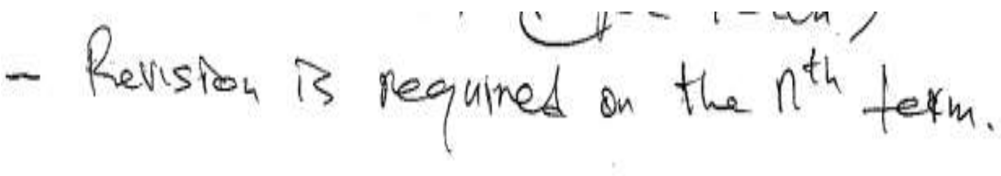


Figure 4.12

TA3's suggestion regarding finding the general rule of a numeric pattern



I then asked TA1 why she thought learners were struggling with finding the general rule and stopped at c. TA1 indicated that numeric and geometric patterns is a new topic in Grade 7. She even added that perhaps the learners could not grasp the content since she had only started the topic with them a day before the LS lesson presentation and observation.

TA9 indicated that learners were unsure what to do when they encountered the equation $5 = 3 + c$. TA9 mentioned that "learners were struggling, saying that when a number jumps an equal sign, automatically, it changes the sign." TA9 further mentioned that perhaps additive inverses could be introduced to teach the learners that what happens on the left-hand side of an equation must also happen on the right-hand side. TA2 did not agree with the idea that a number "jumps" an equal sign. TA2 reasoned that the correct mathematical term for the process described by TA9 is "transposing"; however, he mentioned that transposing is not encouraged at the Senior Phase level. TA2 did, however, agreed with the idea of introducing additive and multiplicative inverse to solve algebraic equations (see Figures 4.13 and 4.14).

Figure 4.13

TA9's noticing of learners' struggle with the equation

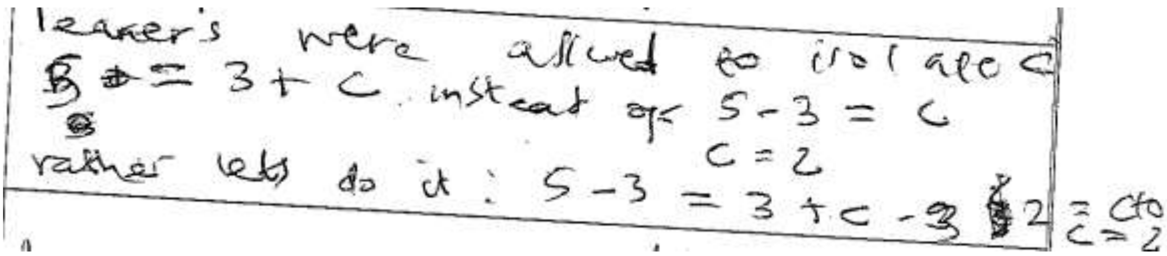
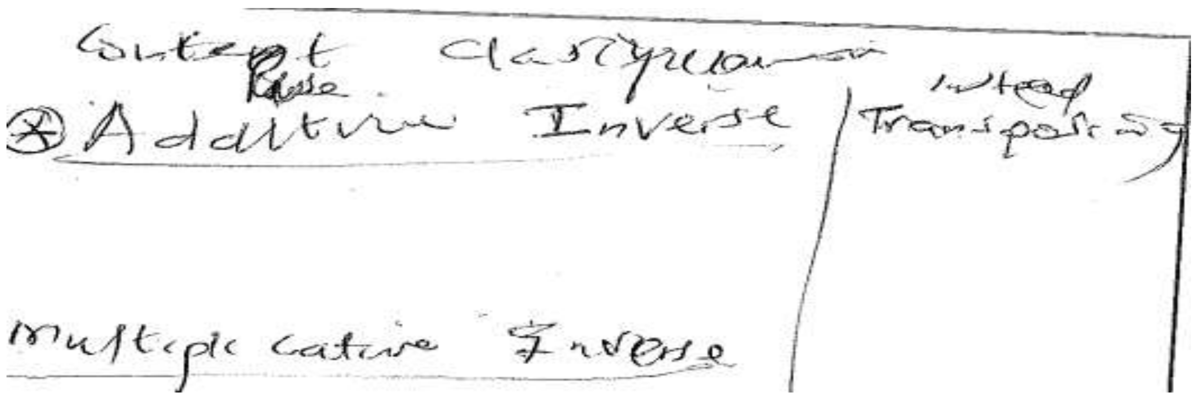


Figure 4.14

TA9's suggestion regarding the equation



I asked TA1 why she made an instructional decision to explain that when a number "jumps" an equal sign, it must change its sign. TA1 said:

"When we are talking about a negative number and a positive number, remember it's where the integers comes (sic) in. Akere [right] remember the integers is also a new topic (sic) to them in Grade 7. These learners neh [right] mo ba hlagang ko teng [where they are coming from – meaning lower grades], if you subtracting a big number to a small number, is possible (sic). But if you're subtracting a small number to a big number is not possible (sic), for them! But after doing the integers it's where they can say 'no Ma'am it's possible now because the answer can be in a negative number.'"

TA1's comments are true because it is common practice in South African mathematics classrooms, particularly in lower grades, that young children are taught that you cannot subtract a "big number from a small number". However, this idea is done away with when learners are introduced to calculations with integers and properties of whole numbers. I then asked TA1 what the curriculum says regarding solving algebraic equations, and she said:

TA1: Okay, the rule says we are doing… ba re ke eng? [what is it again?] *Inverse?*

Interviewer: Additive inverse?

TA1: Additive inverse, whatever that you are doing on the left-hand side you must also do on the right-hand side. That one was a general rule in Grade 7 because when they say you take this number, you take it that side, it's where they get confused."

The suggestions made by the LS team and TA1 that learners must be taught additive inverses directly contradict the Curriculum and Assessment Policy Statement (CAPS) because the curriculum dictates that Grade 7 learners must solve number sentences through inspection and trial and improvement.

TA5 indicated that there was a question where learners had to extend a pattern by three terms, however, learners were not able to do so. TA5 suggested that the question was above the learners' cognitive level and that perhaps the LS team should have discussed the solutions of the instructional activities after planning the lesson.

TA3 also added to TA5's suggestion that a "lesson plan review" must be done after planning a lesson to avoid cases where learners are given activities they cannot solve, which will, in turn, confuse them. Figures 4.15 and 4.16 both cement the comments made by TA5 and TA3, respectively.

Figure 4.15

TA5's noticing of learners' inability to solve a pattern

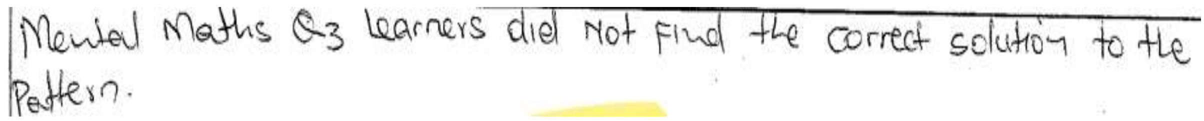
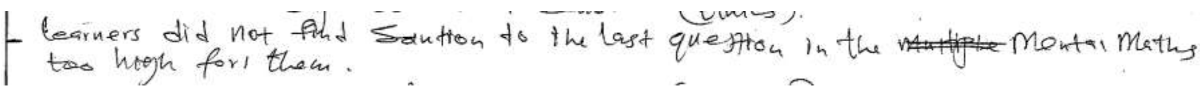

Mental Maths Q3 learners did not find the correct solution to the Pattern.

Figure 4.16

TA3's noticing of learners' inability to solve a pattern


learners did not find Solution to the last question in the ~~Multiple~~ Mental Maths too high for them.

The pattern that the learners were supposed to extend by three more terms was $\mathbf{1; 3; 7; \ldots}$. I asked the TA1 what was the purpose of that activity since it did not align with the specific grade requirements. TA1 insisted that there was a constant difference, and upon explaining to her that the pattern had no constant difference, she then realised that the pattern had a constant difference on the second level, making it a quadratic pattern – which is beyond the scope of Grade 7. I then explained to TA1 that the general term of the pattern is $T_n = n^2 - n + 1$, hence, it is quadratic. TA1 replied by saying:

> *"You know what, in Grade 7, neh? [right?]. We are not allowed to use more than the n^{th}, if you are saying n, you must have one n on that one, you cannot have two n's* (sic). *$n^2 - n\ldots$ in Grade 7, you cannot have double n. I can see gore [that] that one is n^2 but you cannot have $n^2 - n + 1$ in Grade 7."*

TA1 realised that this activity was not suitable for Grade 7 learners, thereby removing its purposefulness because it is not grade specific. TA1 reiterated that she thought it was because numeric and geometric Patterns are a new topic in Grade 7, and therefore, learners need to be drilled on the topic.

Throughout the lesson, TA1 kept asking learners “Why?” whenever they said or did something. I asked her why she continuously made this instructional decision. TA1 indicated that she did not want the learners to use “shortcuts,” especially when they were doing their homework at home using calculators that “skip steps.” TA1 further mentioned that she asks the question “Why?” so that she can specifically identify where the learners got confused and assist them when it is relevant.

4.4. CONCLUSION

In this chapter I presented the analysis and findings of the data I collected through observation, document analysis and unstructured interviews. I began by explaining how I came up with the headings in this chapter, finding guidance from Braun and Clarke (2006). I then described the process I followed to collect the data. The collected data was analysed through thematic and content analysis. The findings presented emerge from the lessons I observed, particularly observing how the teacher offering the lesson during LS used professional noticing to make instructional decisions upon noticing certain features regarding learners’ mathematical thinking. I then observed the post-lesson reflection to see how the LS team used professional noticing to reflect -on-practice.

In the process, I asked the teacher offering the lesson questions regarding moments I deemed key for professional noticing in the lesson. The findings revealed that although noticing happened differently at the two schools, the common thing was that the noticing was superficial. At School A, the important aspect in LS regarding using purposeful activities was not adhered to, while at School B, the teacher offering the lesson was fairly accurate in his practice of professional noticing during the lesson. Post-lesson reflection had only happened in School A; however, the reflection was also merely procedural and did not adequately address issues of practice. An extensive and detailed discussion of the findings follows in the next chapter.

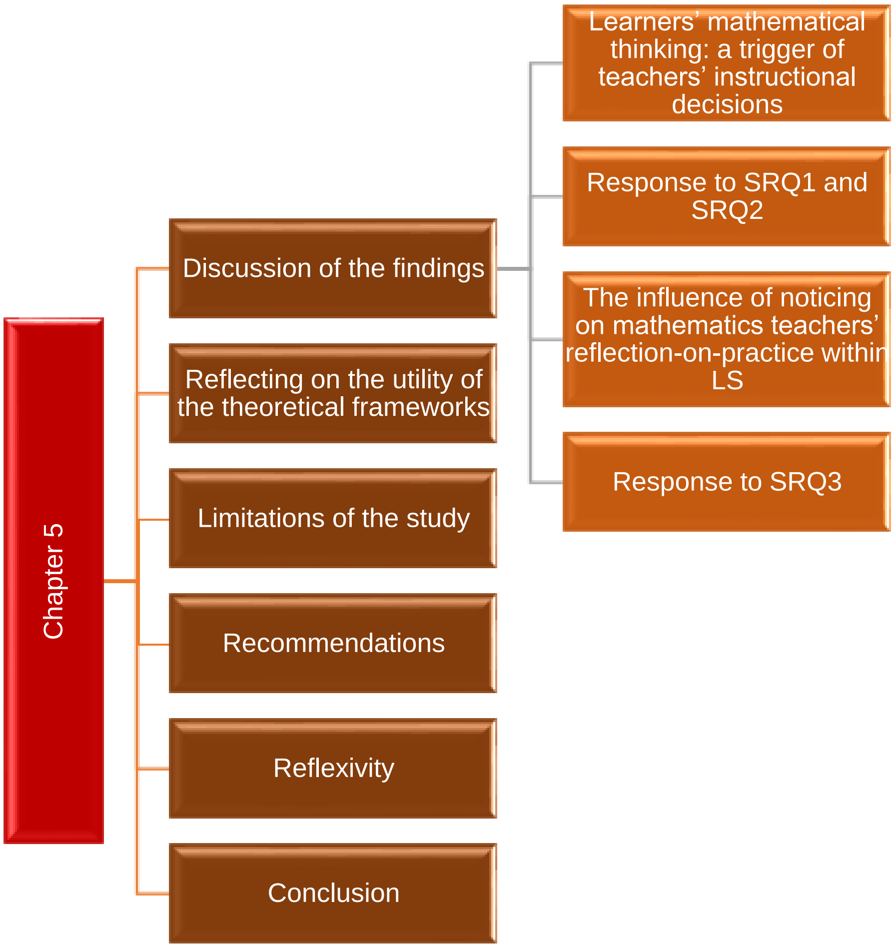
CHAPTER 5: DISCUSSIONS AND RECOMMENDATIONS

5.1. INTRODUCTION

In Chapter 5, I discuss the findings and recommendations for future research. I organised the current chapter according to the following headings: Discussion of findings relating to the research questions, reflecting on the utility of the theoretical frameworks, the limitations of the study, recommendations, reflexivity; and lastly, I provided the overall conclusion to my study. The discussion of findings in this chapter is based on the findings I presented in Chapter 4. In order to ground my discussion, I extensively used cross-referencing to link findings to the discussions I made in Chapter 2, i.e., the literature review. For ease of reference, the flow of the current chapter is illustrated in Figure 5.1.

Figure 5.1.

Outline and flow of chapter 5



Through this study, I intended to answer the primary research question: **How do mathematics teachers use professional noticing to facilitate lessons within LS?**

I then explored three SRQs to respond to the primary research question. The three SRQs are:

> SRQ1: What do teachers notice about learners' mathematical thinking during teaching within LS?
> SRQ2: How do mathematics teachers use noticing to arrive at instructional decisions in-practice within LS?
> SRQ3: How does mathematics teachers' noticing steer their reflection-on-practice within LS?

5.2. DISCUSSION OF THE FINDINGS

5.2.1. Learners' mathematical thinking: a trigger of teachers' instructional decisions

Disregarding learners' classroom contributions

As mentioned in Chapter 4, a lesson on *numeric and geometric patterns* was taught to learners at School A. At the beginning of the lesson, the teacher asked learners to identify any patterns in the classroom. One learner gave an answer that was somewhat similar to another learner's answer. The teacher did not try to investigate what the learner's thoughts were. Instead, the learner was disregarded. The teacher then made an instructional decision to proceed with the lesson even though she had noticed that there was a possible confusion. Although van Es and Sherin (2002) argued that teachers must be able to select what they would attend and respond to, the issue of disregarding learners' contributions or attempts to answer questions posed by the teacher could have far-reaching consequences in mathematics learning.

For instance, the consequences could include the learner being embarrassed in front of his/her classmates. Consequently, the learner might no longer contribute to classroom discussions, thus taking away potential opportunities for the teacher to holistically notice and advance learners' mathematical thinking. The point is in line with Moodliar and Abdulhamid's (2021) argument that dismissing a learner would mean turning down an opportunity for a learner to access mathematical concepts while they are being taught. Sekao (2023) further affirmed that learners tend to cease active participation in mathematics classroom discourses when their contributions are disregarded.

It is fair to say that the teacher missed an opportunity to unpack the learners' mathematical thoughts. This issue was also noted by Biccard (2020), even though Khoza (2023) argued that teachers must be ready to notice features that allow them to unpack learners' understanding, as and when they arise. When I asked the teacher why she had disregarded the learner's contribution, she indicated that it seemed to her that the learner had realised that her answer was the same as the other learner's answer.

The teacher's response showed that the teacher did not practise effective professional noticing. She did not make an instructional decision to ask the learner to clarify what she meant, and, therefore, she denied the learner an opportunity to access mathematics-in-the moment (Moodliar & Abdulhamid, 2021). It is common practice that teachers will disregard a learner's contribution; especially if they have a predetermined answer – which is an issue which particularly led to me undertaking this study.

Teachers need to realise that every classroom contribution made by the learners is important (Khoza, 2023) because it holds the potential to allow them to notice learners' mathematical thinking and understanding. A fitting instructional decision would be to probe the learners further in order to unpack what they are thinking and, therefore, teachers should rather respond in alignment with learners' thinking (Guner & Akyuz, 2020; Tamba & Cendana, 2022).

Teachers' instructional decisions upon noticing certain features regarding learners' mathematical thinking could be better understood through Schön's (1992) analogy of improvisation by a basketball player or a pianist upon encountering an unanticipated occurrence. It cannot be disputed that it is difficult to accurately pre-empt classroom interactions (Chan et al., 2021), and that noticing can be a difficult task for teachers to execute adequately (Amador & Weiland, 2015). However, teachers should make appropriate instructional decisions upon noticing certain issues in learners' mathematical thinking (Moodliar & Abdulhamid, 2021; Yenmez, 2021).

The utility and purposefulness of activities

The purposefulness of the activity that was displayed on the board during teaching, where learners had to find the next three terms of the pattern $\mathbf{1}; \mathbf{3}; \mathbf{7}; \dots$ can be questioned. The first issue with this question is that it was not grade-specific, nor did it target the lesson objectives. Consequently, it fails to meet the principles of what a purposeful mathematics activity should encompass (Fujii, 2018). Fujii (2018) suggested that for an activity to be purposeful, it must (1) be grade-specific and (2) it must address the lessons' learning outcomes. The activity given to the learners was not grade-specific because learners in Grade 7 had not yet been introduced to quadratic patterns. According to the DBE (2011), Grade 7 learners must be exposed to numeric and geometric patterns that have either a constant ratio or a constant difference, and patterns with neither of the two. Nonetheless, the activity did not address the lesson outcomes, therefore, it will be impossible to determine if the lesson outcomes were met or not as suggested by Fujii (2015).

If an activity is not purposeful, it denies the learners the opportunity to make connections between mathematical concepts and construct the meaning of mathematical ideas (Ainley, 2006; 2012). The fact that no clarity was given to the learner regarding this question, which had no constant ratio nor difference, could lead to potential misconceptions and confusion. The outcome was that learning did not happen (Sullivan et al., 2021). Grade 7 learners will come across patterns with neither a constant difference nor a constant ratio, beyond Grade 7. Leaving them hanging might lead to them being confused about what to do when they are faced with similar patterns in the future. In fact, that specific moment has the potential to ruin an entire lesson if the teacher does not to think on his/her feet.

Solving for the variable c in the equation $5 = 3 + c$ also presented difficulties for the learners. In Grade 7, learners have not yet been introduced to issues of solving algebraic equations using additive or multiplicative inverses, i.e. algebraically. In fact, the DBE (2011, p. 25) stated that Grade 7 learners must be able to 'solve and complete number sentences by inspection and trial and improvement.' Through inspection, learners could ask the question, what must c be such that the answer is 5 when added to 3? Other learners could use trial and improvement whereby they pick different values of c and add them to 3 until they obtain 5. Notwithstanding, learners seemed to struggle with the procedural part of solving for c. The teacher noticed what learners were struggling with and made an instructional decision to explain it by indicating that when a number 'jumps' an equal sign, it changes its sign. This is what I would call 'aimless pedagogy' because in as much as the equation was solved even with a conceptually incorrect approach, the underlying ideas used were of a higher level than what the learners were required to know. The underlying ideas were about using an additive inverse, although that was not clearly articulated. The implication could be that learners are finding it difficult because they are being exposed to approaches that are not at their level; thus, creating cognitive overload on the part of the learners. Viewed from afar, the activity is purposeful as it is grade-specific and aligns with the learning outcomes. However, the approach followed to solve it is pedagogically flawed.

The other issue that emerged from School A was that learners did not seem to be able to understand or discern the difference between the general rule and a constant in an equation. This finding was cemented when two different groups, working independently of each other, arrived at the same answer and stopped there, thinking that they had answered the question.

Finding the general rule of a numeric pattern can take various forms. For instance, one can find the general rule by using the formula $T_n = a + (n-1)d$, where a is the first term in the pattern and d is the constant difference of the pattern. Alternatively, which is the method that was used in the lesson, one could use the formula $T_n = dn + c$, where d is the constant difference and c is the constant added (or subtracted) to find a specific term. However, this method requires learners to find the value of c first, which I have already indicated that the approach through which c was found was not grade specific because it involved using an additive inverse. Therefore, this activity follows directly from the previous activity, and together, they make up a combination of activities that are not purposeful. However, the teacher did notice that learners did not seem to know or understand what the general rule is or must be. She made an instructional decision to explain the procedure they needed to follow. However, the representation of the general rule in algebraic language is only clearly articulated by the DBE (2011) for Grade 8 learners.

Factorising by taking out an HCF, difference of two squares, and factorising a trinomial

At School B, learners were taught how to *factorise a trinomial*. The teacher started the lesson by tapping into learners' prior knowledge. He asked them if they could identify a common factor in the trinomial $x^2 + 7x + 10$. The learners indicated that they could see a common factor. One learner said the HCF was x, while the other said that the HCF was x^2. The teacher noticed that learners were confused about the issue of what an HCF is and what it meant. The teacher made an instructional decision to deal with the confusion before continuing with the day's business. The teacher clarified that something must be present across all terms to be deemed a common factor. His intervention was well received by the learners because after that explanation, the learners could tell that x was not there in the last term of the trinomial. The teacher identified a mathematically significant issue (Jacobs et al., 2010; van Es & Sherin, 2002) and saw an opportunity to change how learners thought about an HFC (Stockero & Van Zoest, 2013). As the lesson progressed and the teacher demonstrated how to factorise a trinomial, one learner asked a question regarding a difference of two squares; to which the teacher paused the lesson to clarify the difference between a difference of two squares and a trinomial.

The teacher noticed that the learner was confused and could not distinguish between a difference of two squares and a trinomial. He, therefore, made an instructional decision to explain the difference by pointing out features that learners could look at to identify a difference of two squares. This is good practice from the teacher because the next time learners see a difference of two squares, they will be able to identify it as such and will not confuse it with a trinomial. This was also a clear application of identifying a PTM, as was suggested by Stockero and Van Zoest (2013).

The teacher's intervention is characteristic of a teacher who can reflect-in-the moment (Bakker et al., 2022; Schön, 1983), aligning with the idea that a teacher must be able to "think on their feet". From my experience, learners tend to be confused regarding what to do when faced with factorisation. Attending to learners' confusions and misconceptions as and when they arise is important (Khoza, 2023) in the teaching and learning of mathematics. Therefore, upon noticing learners' confusion, the teacher must make an instructional decision (Course of Action) to help the learners learn the concept (Choy, 2015). However, teachers need to be selective about what they attend to. I believe that attending to the issues that arose in the classroom was a decision of best practice because of the confusion learners tend to have. The teacher's response to the learners is a typical application of Jacobs et al.'s (2010) contention that teachers must be able to attend to the mathematically significant details of the learners' mathematical thinking. Even though, through experience, a teacher might be able to pre-empt what learners' confusion might be regarding a topic, they cannot precisely pre-empt what the confusion will be in a classroom as each case is unique. Yenmez (2021) avers that teachers' instructional decisions directly result from what they notice-in-the moment. Therefore, the teacher responded accurately to the learners' mathematical thinking after identifying how they think mathematically through classroom interactions. Other researchers, such as Amador et al. (2016), Gibson and Ross (2016), and Jacobs et al. (2010), have also argued that teachers must be able to respond-in-the moment to enhance learners' mathematical thinking. The direct implication of the teacher's noticing and appropriate instructional decisions could include that the learner can identify which method of factorisation to use in different situations, thus leading to an improved understanding of the entire topic of factorisation.

Learner engagement – classroom discourses

Throughout the lesson, the teacher sparked classroom discourses, allowing learners to contribute their mathematical thoughts. When the teacher noticed a learner struggling to gather his/her thoughts, contrary to what many teachers would do by being tempted to assist the learner, the teacher brought other learners into the conversation so that they could help each other process the mathematical concepts. Initiating classroom interactions has been advocated for by Khoza (2023), who argued that whatever a learner says in the classroom in the form of contributing towards classroom discourses is important. However, Khoza further pointed out that it is the teacher's responsibility to notice pertinent features in the learners' (mathematical) thinking amid these classroom discourses.

The teacher gave the learners two activities to do; in both cases, he picked two learners, one with the correct answer and one whose answer was not correct, to come and write their answers on the board. This practice resembles features of *kikan-shido* (Kriewaldt et al., 2021), a Japanese term entrenched in LS implying that the teacher offers instruction at learners' desks. Through *kikan-shido*, the teacher can select samples of work to use during the whole class discussion that comes later during the *neriage* stage. Selecting samples of work to be used during classroom discussion can assist in clarifying any misconceptions that might be present among the learners. The teacher then unconventionally asked learners sitting down to explain what the learners (who wrote their answers on the board) did by looking at their solutions. This brought everyone into the discussion, and where one was not clear with their thoughts, other learners would assist. After the explanation was done, the teacher asked the learners who wrote on the board if they thought the learners sitting down explained what went through their minds when they were solving the questions. Through such discourses, the teacher heightened learners' mathematical thinking processes. Engaging in such classroom discourses has been proven to be an effective approach to deepening the learners' mathematical thinking and understanding (Hourigan & Leavy, 2023; Sekao, 2023). These classroom discourses can be characterised as a phase called *neriage* during the lesson in LS. In this whole class discussion, a teacher facilitates and orchestrates the sharing of learners' solutions and helps them to polish their solutions so they can learn the mathematical content (Takahashi, 2008). Through *neriage*, the following three processes of a mathematical thinker as explained by Scusa (2008) were prevalent in the lesson:

a) Connections—Then learners saw the power of 2 in the trinomial $x^2 + 7x + 10$ they thought of a difference of two squares, a mathematical idea that they had previously dealt with. The learners were trying to see how the mathematical ideas were connected.
b) Communication—The learners were able to explain their thinking effectively, clearly, and concisely. When they were not clear about something, they sought clarification.
c) Problem-solving—Although the teacher gave the learners a list of steps to follow when factorising a trinomial, the learners followed the list meticulously, and this

was evident when the learners explained the procedure they followed to factorise the trinomials.

Various researchers have argued that when learners become better mathematical thinkers, they tend to do well in mathematics and in their lives beyond mathematics (Çelik & Özdemir, 2020; Mason et al., 2010; Stacey, 2006). The implication of the aforementioned occurrences suggests positive outcomes in terms of the learners' mathematical thinking capacity. However, even more well-thought (purposeful) activities could be selected, designed, or used to address all five mathematical processes in order to fully develop the learners' mathematical abilities.

In another peculiar instance, one learner asked the teacher if he would have to open three sets of brackets if he happened to have an expression with a power of 3. The learner displayed characteristics of a learner who is successful at connections as far as the processes of mathematical thinking are concerned. He further displayed traits of a good communicator when the teacher asked him to explain what informed his question. The learner indicated that when the trinomial has a power of 2, two sets of brackets were used. The teacher could have made an instructional decision to extend the learners' mathematical knowledge, even though it was beyond the scope of the lesson (Stockero and Van Zoest, 2013). According to the DBE (2011), factorising algebraic expressions that have a power of 3 are introduced in Grade 10. Therefore, in terms of mathematical abilities, one can assume that when the concept of factorising a sum or a difference of two cubes emerges; the learner will have little to no difficulties in terms of understanding the concept. Such thinking must be encouraged to all mathematics learners.

5.2.2. Response to SRQ1 and SRQ2

In this section, I respond to SRQs 1 and 2. SRQ1 states: *What do teachers notice about learners' mathematical thinking during teaching within LS?* and SRQ2 states: *How do mathematics teachers use noticing to make instructional decisions in-practice within LS?* As previously mentioned, SRQ1 and SRQ2 are inextricably linked, therefore, it only makes sense to respond to them simultaneously. In responding to SRQ1, I can conclude that teachers only superficially noticed that learners are either struggling, confused, or even advanced regarding a mathematics concept. When learners were struggling, they were unable to answer a question and even gave incorrect answers. Learners showed confusion when they thought one thing meant the other. For instance, learners thought that determining the value c in the formula $T_n = dn + c$, meant that they had found the general rule of a pattern. Another instance was when learners confused a difference of two squares with a trinomial. One learner showed attributes of a learner who is advanced regarding a mathematics concept when he realised that he would need three brackets when factorising an algebraic expression with a power of 3. Biccard (2020) advised us that the instructional decisions a teacher makes are directly affected by what they notice. If noticing is only cosmetic, then it will be evident in the instructional decisions a teacher makes. A possible reason could be that teachers cannot think on their feet, meaning that they cannot reflect-in-action as suggested by Schön (1983), and therefore the principles of professional noticing are compromised. However, there were instances in School B where the teacher managed to effectively think-on-their feet and make appropriate and relevant instructional decisions. The teacher was able to alter their instruction and match it to the circumstances in the classroom (Gibson & Ross, 2016). Consequently, in responding to SRQ2, I conclude that teachers noticed that learners were struggling with certain mathematics ideas. However, their noticing was only superficial, which led to them making instructional decisions that were not optimal for enhancing or developing learners' mathematical thinking. The dominant instructional decision that teachers made was to re-explain a mathematics idea in the same way, when they noticed that learners were struggling with it. This means that the three constructs of professional noticing were not adequately addressed in the teachers' thought process.

It is no secret that deeper and intentional noticing, results in directed and positive instructional decisions (Choy, 2016; Choy & Dindyal, 2021). In consolidating my response to the two research questions, I can conclude that teachers did not adequately practice the principle of professional noticing. This issue could be due to a number of factors. The lack of curriculum knowledge is one issue. If the teachers' grasp of the curriculum is sound, then they will know not to use certain approaches as they are not encouraged by the South African Curriculum and Assessment Policy Statement (CAPS) for their particular grade. Looking at the example $5 = 3 + c$, when the teacher noticed that learners were struggling, it is clear that deeper and intentional noticing would have resulted in them knowing that CAPS suggests that learners are taught how to solve equations through inspection and trial and improvement. They would not have explained the process as "when a number jumps an equal sign, it changes its sign."

The second issue could be the compartmentalising of mathematical concepts by teachers. If teachers make the links between mathematics concepts clear to the learners, then learners will not have to struggle when faced with problems that involve mathematics concepts they have dealt with in the past. The third issue could be that the activities used were not purposeful in the sense that they were not grade-specific, nor did they align with the lesson outcomes. The fourth and last issue relates to having no clear focus for the lesson in terms of what is needed for the lesson outcomes to be met. I, therefore, suggest that teachers have a list of things to do, something I call 'sub-outcomes' so that the major lesson outcomes are achieved through achieving the sub-outcomes.

5.2.3. The influence of noticing on mathematics teachers' reflection-on-practice within LS

In the post-lesson reflection session at School A, the teacher who was offering the lesson was given the platform to be the first one to reflect – as suggested by the prescriptions of LS (Sekao, 2023). However, as indicated in Chapter 4, the teacher only reflected on her disappointment with the learners, stating that the learners seemed to show a limited conceptual understanding of the content of the day and attributing this shortcoming to the fact that she had a lot to cover in a short space of time. When asked what some of the positives were, she had nothing to say.

This occurrence stamps the fact that noticing was superficial, hence the instructional decisions she made during the lesson. This finding aligns with Rooney and Boud's (2019) argument that failure to notice specific aspects of learners' mathematical thinking will result in poor or uninformed instructional decisions. The teacher is in fact, an experienced professional with over ten years in service. Her inability to notice attributes of learners' mathematical thinking is in line with findings by Khoza (2023) and Gibson and Ross (2016), who argued that the number of years as a teacher is generally not associated with how effective a teacher can notice.

Therefore, the implication can be that since productive noticing is not taking place, the effective teaching and learning of mathematics will be compromised. Thus, teachers need to engage in PD initiatives, such as LS, to sharpen their noticing skills. Guner and Akyuz (2020) argued that all teachers need to develop and maintain the expertise of noticing, and in cases where it is lacking, they must be taught how to notice (Biccard, 2020).

Along with the prescriptions of LS, Rodger (2002) suggests that reflection must occur in a community. Consequently, the LS team engaged in a post-lesson reflection where teachers who observed the lesson shared their insights. The goal was to reflect-on and refine practice (Schön, 1992). As such, members of the LS team offered their perspectives. One teacher noticed the issue of disregarding learners and indicated that teachers must have ‘moves’ they make to avoid shutting learners down. Another teacher in the LS team supported the teacher’s submission. Both teachers noticed that one learner gave an answer that was similar to an answer provided by another learner. However, both did not notice any specifics regarding the learners’ mathematical thinking. When I asked the teacher why she disregarded the answer given by the learner, she revealed that it seemed to her that the learner realised that her answer was the same as that given by the previous learner. The teacher also expressed that she feels that some learners just wish to be involved in classroom engagements even though they do not know the answers to the posed questions. The two points aforementioned reveal that teachers tend to miss conspicuous features of learners’ mathematical thinking, as Khoza (2023) alluded to. Across all the issues the LS team noticed; the noticing was superficial as it did not point out anything specific regarding learners’ mathematical thinking. Essentially, the teachers simply noticed that learners were “struggling” with something.

Notwithstanding, the LS team still managed to reflect on their practice. Offering insights into what they could do better in the next lesson, and in the future. Planning for future lessons and pre-empting what learners could struggle with is in line with Bakker’s (2022) revelation that reflecting-on-action affords the LS team with opportunities to anticipate future occurrences that may be relevant and need immediate attention from those that can be deemed irrelevant and can therefore be discarded. As they reflect on what they have observed, it appeared that teachers have gained insight into what could have been done differently, which ultimately reflects what they would have done in the same situation(s).

The latter corresponds with an observation made by Wei et al. (2023) that the observing teachers can also offer what they would do in their instruction when faced with similar situations. I can confidently say that the observing teachers were able to notice more than the teacher offering the lesson, which is in line with Sekao's (2023) argument that the observing teachers are at a vantage point to notice opportune moments more than the teacher teaching the lesson. It was also evident that as the teachers were engaging and exchanging ideas, they were also able to gain from their counterparts and therefore more noticing happened (Biccard, 2020). This revelation coincides with findings by other researchers such as Guner and Akyuz (2020), Cater et al. (2016), Karlsen and Helgevold (2019), Amador (2015), Amador and Cater (2018) and Druken (2023) who argued that meaningful happens in collaboration with other teachers.

The responses given by the teacher who offered the lesson raised questions regarding the teachers' use of professional noticing. Instead of explaining why she made certain instructional decisions, she described the concepts learners struggling with. Professional noticing dictates that a teacher should respond to learners' contributions in a way that suits how they currently understand mathematics (Guner & Akyuz, 2020; Tamba & Cendana, 2022). Therefore, instead of the teacher shifting blame to earlier grades where learners had been taught not to subtract a bigger number from a smaller one, she could have used something learners already knew, such as a number line to indicate how a bigger number (such as 5) could be subtracted from (3). The teacher also noted that matters such as integers and numeric and geometric patterns are new topics for learners.

This submission by the teacher is not true because the learners are introduced to numeric and geometric patterns as early as Grade 4. However, they are only exposed to those with a constant ratio or a constant difference (DBE, 2011). Learners are also taught about addition and subtraction as inverse operations in Grade 4, a Grade that also includes using a number line (DBE, 2011). This means the teacher is not informed about what the curriculum stipulates, which consequently means she will not be able to gauge the level of learners' current mathematical thinking and understanding.

Consequently, poor instructional decisions are made and therefore learners' urgent mathematical needs will not be addressed (Yenmez, 2021).

5.2.4. Response to SRQ3

In this section, I respond to SRQ3: *How does mathematics teachers' noticing steer their reflection-on-practice within LS?* Members of the LS team offered insights on what they thought must happen when faced with situations like those that emerged in the lesson (Wei et al., 2023). This is a typical example of reflecting-on-action (Schön, 1983). The LS team could also discern things they would not do in the next lesson such as disregarding learners' classroom contributions.

The latter is an affordance of the post-lesson reflection (Bakker, 2022). Therefore, I can conclude that the LS team effectively reflected-on-practice by giving ideas about what they would do or avoid in a lesson. It is important to note that some ideas suggested by the LS team related to the prescripts of the curriculum; however, they were not accurate as they suggested alternatives that are encouraged at higher grades than grade 7. This could be due to a lack of knowledge of the holistic mathematics curriculum from Grade 1 to Grade 12. I think that if the LS team knew the prescripts of the CAPS thoroughly, their reflection-on-practice would have been richer in terms of instructional decisions that they would have made or would make upon noticing certain features of learners' mathematical thinking and understanding.

5.3. REFLECTING ON THE UTILITY OF THE THEORETICAL FRAMEWORKS

The theoretical frameworks I used for this study are the SLT developed by Lave and Wenger (1991) and the FOCUS FFPN developed by Choy (2015) as a product of his doctoral studies. The SLT argues that learning is not a process solely in the human brain and that effective learning must happen in the context in which it occurs. The SLT contends that learning must happen in a community of practice. I can deduce that the SLT was very effective in situating the setting where the LS occurs, the classroom. The classroom has, in fact, been deemed a fitting context to sustain teachers' PD (Loose, 2014).

The FFPN is concerned with what and how mathematics teachers professionally notice learners' mathematical thinking and understanding, particularly in the LS context. The FFPN comprises three focal points: **Concept**, **Confusion** and **Course** of Action. The FFPN asks the question: what does the teacher do (**Course** of Action), when learners have a **Confusion**(s), regarding a specific mathematical **Concept**(s)?

The FFPN was extremely helpful to me in more ways than one, from coining my research questions to the data collection and analysis processes I engaged in and, subsequently, research findings and discussions. The problem that led to this research was dismissing or disregarding learners' classroom contributions, an issue that the FFPN clearly discourages and even adds to what the teachers must do when a learner contributes in a way they did not expect or gives an incorrect answer in the classroom.

The FFPN states that a teacher must be able to identify the learners' mathematical understanding from their classroom contribution and recognise the confusion(s) that the learner might be having. Therefore, the limitation of FFPN is that it provides a broad overview of the action teachers must take when planning, teaching, and reviewing the lesson in order to productively notice learners' mathematical thinking. However, it does not provide the finer details of what the teachers must notice regarding learners' mathematical thinking and how to interpret the noticed incidents. I therefore think that I could have used another framework to complement or make up for the limitation of FFPN to assist me, as the researcher, to identify those pertinent moments teachers took advantage of or missed as opportunities to understand further how learners think mathematically. To this effect, I believe that the framework proposed by Stockero and Van Zoest (2013) to characterise PTMs would have assisted me in having a better understanding of the moments that had underlying characteristics of a PTM.

Stockero and Van Zoest define a PTM as occurrences that disrupt the lesson flow but allow the teacher to alter their pedagogy to change or extend the way their learners think about and understand mathematics. PTMs can be categorised into five types:

a) Extending— a PTM can occur when a learner comments on or asks a mathematically correct question beyond the mathematics a teacher intended to cover in a lesson. For instance, in my study, one learner asked the teacher if he would have to open three brackets if he had an algebraic expression with a power of 3. His thinking was correct; however, the idea was beyond his grade level.

b) Incorrect mathematics—a PTM can occur when learners make their incorrect mathematics or solutions public to the class. Instead of dismissing the answer, the teacher can choose to clarify issues, and thus a PTM has been capitalised on.
c) Sense-making—a PTM can occur when learners try to understand a mathematical concept during the lesson.
d) Contradiction—a PTM can occur if learners have solutions or answers that are mathematically contradictory. The teacher can use this PTM to change the learners' thinking or extend their mathematical thinking and understanding.
e) Confusion—a PTM can also occur when learners can mathematically express their confusion(s).

I think characterising PTMs would have assisted me in the following ways: (1) I would have been able to determine specific features teachers notice regarding learners' mathematical thinking, moving away from superficial noticing, and (2) the instructional decisions teachers would have made would have been a reassurance and a direct implication of what they noticed. I, however, do not take away the affordances provided to me by the FFPN. Therefore, I conclude that synergising the FFPN and PTM would have assisted me in understanding better what teachers noticed about learners' mathematical thinking, as would have been evident in their instructional decisions.

In addition, since I was using the LS model used in South Africa, whose unique feature is the stage of diagnosing learners' difficulties in mathematics concepts (Stage 1—diagnostic assessment/analysis), the FFPN requires alignment with the LS in South Africa. In the FFPN, Choy (2015) describes steps teachers must take when they plan a lesson. This first step coincidentally aligns with the first stage of the South African LS model, the diagnostic assessment/analysis stage. In diagnostic analysis of learners' assessment responses, teachers can follow the following steps Sekao (2023) suggests:

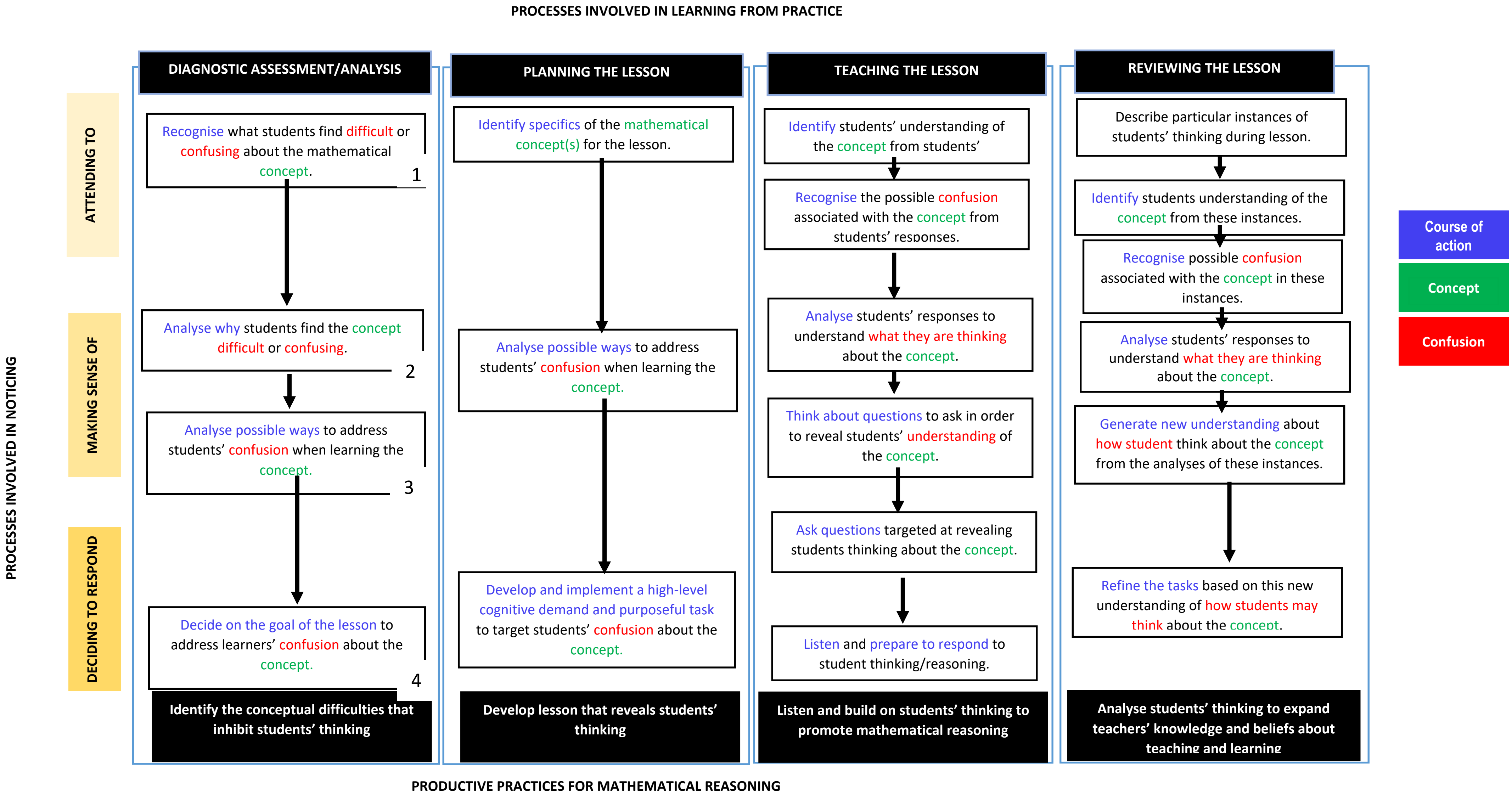
1. Problem identification – an overview of the problem identified is given.
2. Possible causes – assumptions on what could have led to the problem are made.

3. Evidence – Errors and misconceptions in learners’ responses are used as evidence to justify the problem identified.
4. Propose intervention – A solution is devised to address the problem.

I suggest an alignment between the steps suggested by Sekao (2023) and the steps in the planning phase of the lesson as proposed by Choy (2015). The alignment is illustrated in Figure 5.2. I have indicated the steps under the diagnostic assessment/analysis phase in Figure 5.2. The blocks at the bottom of each column represent different phases, i.e., diagnostic assessment/analysis, planning, teaching, and reviewing. They are not colour-coded because they only briefly summarise the occurrences in each phase. For instance, at the bottom of the column named diagnostic assessment/analysis phase, the summary is presented as follows: Identify the conceptual difficulties that inhibit students’ thinking.

Figure 5.2.

Alignment of the Diagnostic Analysis Stage of LS and the Planning Phase of the FFPN



I have named steps 1 to 4 under the diagnostic assessment/analysis phase. These steps coincide with the steps suggested by Sekao (2023), which I explained before the diagram. I believe that teachers who conduct the LS cycle can synergise the propositions from the FFPN and the diagnostic analysis stage of the LS to inform their research lessons further and enhance learners' mathematical thinking.

5.4. LIMITATIONS OF THE STUDY

This study was conducted at two schools, one a primary and the other a secondary school. Therefore, the design of this study was a two-case study. The sample was relatively small, and the findings cannot be generalised to the entire community of mathematics teachers as each case presented its unique characteristics. The second case consisted of only two mathematics teachers. I am of the assumption that more teachers could have brought different perspectives and added to the richness of the data collected.

The other limitation is that the reflection of the lesson offered at School B did not happen, thereby taking away the opportunity to unearth more findings related to the lesson that was taught. It must be noted that pestering the teachers to conduct the reflection would have been tantamount to coercing them, which would have contravened the ethical practices within the research context. Notwithstanding, the reflection conducted in School A provided sufficient insights to enable me to respond to SRQ3.

5.5. RECOMMENDATIONS

The field of mathematics education continues to grow. However, research focusing on professional noticing in the LS context is significantly paltry. My study focused holistically on how mathematics teachers employed professional noticing in the LS context. Several issues emerged that caught my eye but were not necessarily the focus of my study such as teachers' use of purposeful activities to enable deeper professional noticing. I believe further research must be conducted to investigate how teachers can be intentional regarding professional noticing in the LS context.

Further research can also focus on the purposefulness of teachers' instructional and assessment activities when offering lessons particularly in the LS context. This study revealed that some of the activities used did not assist the teachers in any way. Therefore, future research can develop a framework that teachers can follow to identify, select, or even design purposeful activities to use when planning a lesson to enable deeper professional noticing within LS.

Although the primary focus of my study was not to explore the FFPN with the intention of improving it, implementing it as a theoretical framework has revealed some limitations that I had to express. To this effect, I recommend empirically testing the proposed adapted FFPN model (Figure 5.2) to determine its effectiveness and, if necessary, improve its alignment with the LS model practised in South Africa.

Apart from research, I recommend that LS be institutionalised for all mathematics teachers in South Africa as a form of teachers' PD. Teacher education institutions that prepare mathematics teachers must include LS in their programmes, whereby student teachers engage in LS during their Work-Integrated Learning. Professional noticing must also be added as an integral part of the teacher education curricula, whereby student teachers are specifically taught how to notice learners' mathematical thinking.

5.6. REFLEXIVITY

In this section, I reflected on the study's impact on me as a professional. I am a mathematics teacher at a high school, teaching Grades 8 and 10 (Technical mathematics). One of the things that motivated me to embark on this journey was my love and passion for the effective teaching and learning of mathematics. I then realised that it could not be that most learners are struggling with mathematics; somehow, it occurred to me that perhaps the problem might be with us as mathematics teachers. Consequently, I decided to embark on this journey. Throughout my journey in this study, I also developed as a mathematics teacher. The readings I read and my observations in the field made me take my actions as a teacher into cognisance. More peculiar, I began practising professional noticing as much as possible in my classrooms. I even tried to rope in my colleagues into practising LS after I saw how effective it was in terms of self-development as a teacher and how a group of mathematics teachers can work together to benefit the learners.

As a mathematics teacher, I have my personal views on how particular mathematics concepts must be taught. Therefore, my knowledge of mathematics and the CAPS could have also affected how I analysed the lessons I observed. I have also picked up new ways of explaining the concepts taught in my observed lessons and continue honing my mathematics teaching skills.

5.7. CONCLUSION

This chapter concludes this study. In this study, I explored how mathematics teachers employed professional noticing of learners' mathematical thinking in the context of LS. To achieve the purpose of this study, I aimed to (1) identify pertinent features teachers notice regarding learners' mathematical thinking when they offer lessons within the LS context, (2) explore how mathematics teachers use professional noticing to make instructional decisions when they teach, and (3) explore how mathematics teachers use professional noticing to reflect-on-practice within LS collaboratively. This study took place at two schools, employing two types of LS: Circuit-based and School-based.

The findings revealed that teachers practice professional noticing superficially. Hence, poor instructional decisions are often made. The main contributor to this was using activities that were not purposeful for the lessons.

The findings further indicated that teachers forming part of the LS team were able to reflect-on-practice effectively. However, their reflection would have been richer if they had a good grasp of the prescripts of the CAPS for mathematics.

In Chapter 1, I presented an overview and general orientation of the study to make the reader aware of its main components, such as the background relating to the problem statement, the rationale of the study, the purpose of the study, and the research questions the study intended to address.

In Chapter 2, I reviewed the literature relevant to this study's key components and outlined the two theoretical frameworks that served as its theoretical lenses.

In Chapter 3, I presented methodological issues relevant to the study. My discussion began with the paradigmatic approach followed in this study, the methodological choice, and the research design. I then outlined how the samples for this study were selected, how data was collected and analysed, the quality criteria followed and the ethical considerations I made in this study.

In Chapter 4, I presented the data collected to reveal this study's key findings. The data was organised into themes, which the FFPN informed as they related to the research questions.

In this chapter, Chapter 5, I discussed the findings reported in Chapter 4 to respond to the three SRQ and consequently answer the primary research question that underpinned this study. My SRQs were aligned with the FFPN. Additionally, I reflected on the utility of the two frameworks I employed in this study. I discussed the study's limitations and made recommendations for future research and policymakers. Lastly, I reflected on my own beliefs during the journey of this study and how they might have influenced this study.

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APPENDICES

Appendix A: Observation instrument for lesson presentation

<table>
<tr><th>Research question</th><th>Criteria</th><th>Comments</th></tr>
<tr><td rowspan="6">a) What do teachers notice about learners’ mathematical thinking during teaching within LS?</td><td rowspan="3">Concept</td><td></td></tr>
<tr><td></td></tr>
<tr><td></td></tr>
<tr><td rowspan="3">Confusion (Misconceptions)</td><td></td></tr>
<tr><td></td></tr>
<tr><td></td></tr>
<tr><td rowspan="6">b) How do mathematics teachers use noticing to make instructional decisions in-practice within LS?</td><td rowspan="3">Confusion (Misconceptions)</td><td></td></tr>
<tr><td></td></tr>
<tr><td></td></tr>
<tr><td rowspan="3">Course of Action (in action)</td><td></td></tr>
<tr><td></td></tr>
<tr><td></td></tr>
</table>

Appendix B: Observation instrument for post-lesson reflection

Research question	Criteria	Comments
c) What do teachers notice about learners' mathematical thinking during teaching within LS?	Concept	
	Confusion (Misconceptions)	
d) How do mathematics teachers use noticing to make instructional decisions in-practice within LS?	Confusion (Misconceptions)	
	Course of Action (in action)	

Appendix C: Ethics approval letter from the University of Pretoria

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FACULTY OF EDUCATION
Ethics Committee

Amendment

18 August 2023

Dear Mr KC Moremi

The application for ethical clearance for the research project described below served before this committee on 19 July 2023 :

Ethics Protocol No:	**UP 19/03/01 SEKAO23-04**
Principal investigator:	**Mr KC Moremi**
Student/Staff No:	**18229256**
Degree:	**Masters**
Supervisor/Promoter:	**Dr RD Sekao**
Department:	**Science Mathematics and Technology Education**

The decision by the committee is reflected below:

Decision:	**Approved**
Comments:	
Period of approval:	**Two years**

The approval by the Ethics Committee is subject to the following conditions being met:

1. The research will be conducted as stipulated on the application form submitted to the Ethics Committee with the supporting documents.
2. Proof of how you adhered to the Department of Basic Education (DBE) policy for research must be submitted where relevant.
3. In the event that the research protocol changed for whatever reason the Ethics Committee must be notified thereof by submitting an amendment to the application, together with all the supporting documentation that will be used for data collection namely; questionnaires, interview schedules and observation schedules, for further approval before data can be collected. The changes may include the following but are not limited to:
   - Change of investigator,
   - Research methods any other aspect therefore and,
   - Participants.

The Ethics Committee of the Faculty of Education does not accept any liability for research misconduct, of whatsoever nature, committed by the researcher(s) in the implementation of the approved protocol.

Best wishes

____

Prof Funke Omidire
Chair: Ethics Committee
Faculty of Education

Room 3-53, Level 3, Aldoel Building
University of Pretoria, Private Bag X20
Hatfield 0028, South Africa
Tel +27 (0)12 420 5656
Email edu.ethicsadmin@up.ac.za
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Faculty of Education
Fakulteit Opvoedkunde
Lefapha la Thuto

Appendix D: Gauteng Department of Education research permission form

GAUTENG PROVINCE

Department: Education

REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	**27 October 2023**
Validity of Research Approval:	**08 February 2024– 30 September 2024** **2023/476**
Name of Researcher:	**Moremi KC**
Address of Researcher:	**2214 Ntwanano Street , Kanana Ext 4** **Midrand**
Telephone Number:	**072 972 0887/064 896 1743**
Email address:	**U18229256@tuks.co.za**
Research Topic:	**Mathematics Teachers' Professional Noticing as an Immanent Feature of Lesson Study.**
Name of University:	**UP**
Type of qualification	**Masters**
Number and type of schools:	**3 Secondary Schools**
District/s/HO	**Tshwane South**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

Making education a societal priority

Office of the Director: Education Research and Knowledge Management

7th Floor, 17 Simmonds Street, Johannesburg, 2001

Tel: (011) 355 0488

Email: Faith.Tshabalala@gauteng.gov.za

Website: www.education.gpg.gov.za

Appendix E: District director’s consent letter (District Approval)

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

TO:	**The Principal** **District Tshwane South Schools**
FROM:	**Mr. Andries Nkadimeng** **Acting District Director: Tshwane South**
DATE:	**4th March 2024**
SUBJECT :	**PERMISSION TO CONDUCT RESEARCH AT AN EDUCATION INSTITUTION**

Dear Sir/ Madam

Permission is hereby granted to **K.C. Moremi** to conduct academic research at your institution.

The researcher shall make arrangements for research with the school management. The school staff, learners and SGB are requested to co-operate with and give support to the researcher. Research findings and recommendations are critical for policy review in public education sector.

The researcher may however not disrupt the normal school programme in the course of research. The research may only take place between the months of February and September 2024. Attached are other conditions to be observed by the researcher.

The school may request for the research outcome presentation directly from the researcher or obtain research document from Research & Knowledge Management Directorate at GDE Head Office.

Regards

Mr A.M. Nkadimeng
Acting District Director: Tshwane South
Date: 05/03/2024

CONFIDENCE The Game Changer!

Office of the District Director: Tshwane South
President Towers building, 265 Pretorius Street, Pretoria, 0002
Private Bag X198, Pretoria, 0001 Tel: (012) 401 5317 Fax: (012) 401 6318
Website: www.education.gpg.gov.za

Appendix F: Principal's consent letter

Faculty of Education

Dear Mr KC Moremi

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I,………………………………………., principal of ………………………., voluntarily and willingly permit Mr KC Moremi to conduct a research study titled: ***Mathematics Teachers' Professional Noticing as an Immanent Feature of Lesson Study.*** I understand that the participation of both learners and the GET Phase teachers in the afore-mentioned study to which I am consenting, will involve:

a) teachers and learners being observed during the lesson presentation;
b) teachers availing their collaboratively prepared observation tool for analysis;
c) teachers being part of the interview that will be audio/video recorded; and
d) teachers being observed and audio/video recorded during the lesson planning, lesson presentation and reflection stages.

I declare that I understand the purpose of the study and that you (the researcher) subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I permit you to conduct your study in our school.

______________________	______________	____________
(Name and **surname)**	Signature	Date

Appendix G: Teachers' consent letter

Faculty of Education

Dear Mr K.C Moremi

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I,……………………………………., a teacher at ……………………………….., voluntarily and willingly permit Mr K.C. Moremi to conduct a research study titled: ***Mathematics Teachers' Professional Noticing as an Immanent Feature of Lesson Study.*** I understand that my participation in the afore-mentioned study to which I am consenting, will involve:

a) being observed and audio/video recorded during the lesson presentation, and reflection stages of the Lesson Study cycle;
b) being part of the interview that will be audio/video recorded; and
c) availing the collaboratively prepared observation tools for analysis.

I declare that I understand the purpose of the study and that you (the researcher) will subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I consent to participate in your study.

______________________	______________	____________
(**Name** and **surname**)	Signature	Date

Appendix H: Parents' consent letter

Faculty of Education

Dear Mr K.C Moremi

LETTER OF CONSENT FOR MY CHILD TO PARTICIPATE IN THE RESEARCH STUDY

I, ……………….………………………………, parent of…………….………..…………………………, voluntarily and willingly permit my child to participate in the research study titled **Mathematics Teachers' Professional Noticing as an Immanent Feature of Lesson Study.** I understand that the participation of my child in the afore-mentioned study to which I am granting permission, will involve being observed during the lessons taught by their teacher(s). I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including *informed consent, safety, privacy* and *trust*.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I give permission for my child's participation in the study.

______________________	______________	____________
(Name and surname)	Signature	Date

Appendix I: Learners' assent letter

Faculty of Education

Dear Mr K.C. Moremi

ACCEPTANCE TO **PARTICIPATE IN YOUR RESEARCH STUDY**

I,.................................. a GET Phase learner at .... voluntarily and **willingly** agree to participate in the study titled: **Mathematics Teachers' Professional Noticing as an Immanent Feature of Lesson Study.** I understand that as part of the study to which I agree to participate, I will be observed during the lessons when my mathematics teacher teaches me and my classmates.

I declare that I understand the purpose of the study and that you (the researcher) promise to obey the ethical research principles, including safety, privacy (not revealing my name and identity) and trust as you explained to me.

In addition, I grant the University of Pretoria permission to use the data provided for this study, without revealing my name and identity, for further research purposes.

Given the above information, I agree to voluntarily participate in your study.

______________________	______________	____________
(Name and surname)	Signature	Date