

Error analysis for Galerkin finite element approximations of general second order hyperbolic equations

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Title	Error analysis for Galerkin finite element approximations of general second order hyperbolic equations
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Summary

The vibration of elastic bodies and structures consisting of elastic bodies is an active research field in Applied Mathematics and Engineering. A mathematical model for such a vibrating system is a complex system of partial differential equations. However, the variational form is in each case of the same type as the variational form for the well known wave equation. The generalized problem is to solve a second order differential equation for a function with values in a Hilbert space. In this thesis this problem is referred to as the general second order hyperbolic equation.

Obviously numerical approximation of solutions is inevitable. In this thesis we consider the continuous Galerkin finite element approximation of the solution. The aim in the research is to generalize the theory of convergence for hyperbolic problems to accommodate complex systems.

A previous attempt to generalize the theory was made in a thesis in 2000, but the result is only valid for modal damping. This thesis and two articles that had a significant impact on the literature were considered for the present investigation. The methods in the two articles were successfully generalized and the results in the thesis slightly improved to create a comprehensive convergence theory.

In 2002 an article on existence for the general second order hyperbolic equation appeared. The existence results are presented in variational form, convenient for the Galerkin method. Furthermore, different types of damping and their significance for the theory are identified.

In this thesis the variational form of the general second order hyperbolic equation is used to create a general framework for convergence theory. It

transpired that sufficient conditions for existence are also used to prove convergence. As for existence theory, the type of damping needed to be taken into account for convergence theory.

This thesis is not merely about generalization, other contributions to the theory are also made. In all publications considered, error estimates are derived for the semi discrete approximation and then for the fully discrete approximation without using the results already obtained. We derive an error estimate for the semi-discrete approximation and then an estimate for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm. The final estimate follows trivially by the triangle inequality. This approach has two advantages. It is not necessary to assume the existence of a third or fourth order derivative for the exact solution and the convergence analysis for the fully discrete approximation is simplified.

It is well known that the smoothness of a solution is important in convergence analysis. In this thesis great care is taken to formulate regularity assumptions.

Numerous applications are given in the thesis where it is clear that it is not trivial to apply the general results.

This thesis also includes a number of new existence results.

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Chapter 1

Introduction

1.1 The wave equation

To explain the aim of the research and the results obtained, it is useful to have an example. The one-dimensional wave equation is the simplest second order hyperbolic partial differential equation.

1.1.1 Model problems

We consider three types of damping. The damping term may be in the partial differential equation:

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u, \\ \partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u,\end{aligned}$$

with $\alpha > 0$ and $\gamma > 0$. The constant α represent physical properties of the bar (or string) and γ is the damping constant. The damping in the first equation is known as viscous damping which one may expect in the vibrating string model due to air resistance. The damping in the second equation is known as Kelvin-Voigt damping due to internal friction in the material of a vibrating bar.

Problem 1 Viscous damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u + f \quad \text{in } (0, \ell) \quad \text{for } t \in \mathbb{R}, \\ u(0, t) &= \partial_x u(\ell, t) = 0 \quad \text{for } t \in \mathbb{R}, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

Problem 2 Kelvin-Voigt damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u + f \quad \text{in } (0, \ell) \quad \text{for } t > 0, \\ u(0, t) &= \partial_x u(\ell, t) = 0 \quad \text{for } t > 0, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

Now consider the undamped wave equation but with damping at the boundary $x = \ell$:

$$\partial_x u(\ell, t) = -\gamma \partial_t u(\ell, t).$$

Problem 3 Boundary damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u + f \quad \text{in } (0, \ell) \quad \text{for } t > 0, \\ u(0, t) &= 0 \quad \text{for } t > 0, \\ \alpha \partial_x u(\ell, t) &= -\gamma \partial_t u(\ell, t) \quad \text{for } t > 0, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

The term f in the wave equation represents a force per unit length.

1.1.2 Variational forms

The test functions for Problems 1 to 3 are the same:

$$T[0, \ell] = \{v \in C^1[0, \ell] \mid v(0) = 0\}.$$

Now consider Problem 1. Multiply the partial differential equation by a function $\phi \in T[0, \ell]$ and integrate. Using integration by parts, we have

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t u(\cdot, t) \phi + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$. It is now possible to present the variational form of Problem 1.

Problem 1V Viscous damping

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t u(\cdot, t) \phi + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$, with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Next we consider Problem 2. Repeating the process above and also performing integration by parts on the damping term yields

$$\begin{aligned} \int_0^\ell \partial_t^2 u(\cdot, t) \phi &= -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot, t) \phi' \\ &\quad + (\alpha \partial_x u(\ell, t) + \gamma \partial_t \partial_x u(\ell, t)) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi \end{aligned}$$

if $\phi \in T[0, \ell]$. It is given that $\partial_x u(\ell, t) = 0$ and it follows that $\partial_t \partial_x u(\ell, t) = 0$. We have the variational form of Problem 2.

Problem 2V Kelvin-Voigt damping

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot, t) \phi' + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$, with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Lastly, we consider Problem 3. In this case we have

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' + \alpha \partial_x u(\ell, t) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi$$

if $\phi \in T[0, \ell]$. Using the boundary condition at $x = \ell$ we arrive at the variational form of Problem 3.

Problem 3V Boundary damping

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \partial_t u(\ell, t) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$, with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

General variational form

Using suitable notation, all three variational problems may be written in the same form. First, consider Problem 1V. Let

$$c(u, v) = \int_0^\ell uv, \quad a(u, v) = \gamma \int_0^\ell uv \quad \text{and} \quad b(u, v) = \alpha \int_0^\ell u'v'.$$

The variational equation in Problem 1V may now be written in the form

$$c(\partial_t^2 u(\cdot, t), \phi) + a(\partial_t u(\cdot, t), \phi) + b(u(\cdot, t), \phi) = c(f(t), \phi) \quad (1.1.1)$$

where ϕ is a test function.

The variational equation in Problems 2V and 3V may also be written in the form above if we define the form a for damping appropriately:

$$\begin{aligned} a(u, v) &= \gamma \int_0^\ell u'v' \quad \text{for Problem 2V,} \\ a(u, v) &= \gamma u(\ell)v(\ell) \quad \text{for Problem 3V.} \end{aligned}$$

1.1.3 The weak formulation

It is important to note that a and c are defined for functions in $\mathcal{L}^2(0, \ell)$ and b is defined for functions $u \in H^1(0, \ell)$ (see Appendix A). But the partial derivatives $\partial_t u$ and $\partial_t^2 u$ do not make sense when a function may be changed arbitrarily on a set of measure zero. If we define a function w by $w(t) = u(\cdot, t)$, then $w'(t)$ may be defined with respect to the norm of $\mathcal{L}^2(0, \ell)$ (see Section 2.1):

$$\lim_{h \rightarrow 0} \|h^{-1}[w(t+h) - w(t)] - w'(t)\| = 0,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{L}^2(0, \ell)$. Equation (1.1.1) may now be rewritten:

$$c(w''(t), \phi) + a(w'(t), \phi) + b(w(t), \phi) = c(f(t), \phi), \quad (1.1.2)$$

for each test function ϕ . Recall that the space of test functions $T[0, \ell]$ is defined by

$$T[0, \ell] = \{\phi \in C^1[0, \ell] \mid \phi(0) = 0\}.$$

It is generally accepted that Equation (1.1.2) is a generalization of Equation (1.1.1) but it is not easy to prove. Consider the following quote in [Sho77]: "... the construction of a representative $u(\cdot, t)$ of $w(t)$ is non-trivial."

Theorem 1.1.1 ([Sho77], Theorem 7.A, p104). *Let $I = [a, b]$, a closed interval in $\mathbb{R}$ and G be an open (or measurable) set in $\mathbb{R}^n$.*

(a) *If $w \in C(I, \mathcal{L}^2(G))$, then there is a measurable function $u : G \times I \rightarrow \mathbb{R}$ such that*

$$w(t) = u(\cdot, t), \quad t \in I. \quad (1.1.3)$$

(b) *If $w \in C^1(I, \mathcal{L}^2(G))$, u and v are measurable real-valued functions on $G \times I$ for which (1.1.3) holds for a.e. $t \in I$ and*

$$w'(t) = v(\cdot, t), \quad \text{a.e. } t \in I,$$

then $v = \partial_t u$ in $\mathcal{D}^*(G \times I)$.

Let $V(0, \ell)$ be the closure of $T[0, \ell]$ in $H^1(0, \ell)$. It is easy to show that Equation (1.1.2) holds for all $\phi \in V(0, \ell)$.

It is now possible to formulate the weak variational form of Problem 1V.

Problem 1W

Find a function w with $w(t) \in V(0, \ell)$ such that

$$c(w''(t), \phi) + a(w'(t), \phi) + b(w(t), \phi) = c(f(t), \phi)$$

for each $\phi \in V(0, \ell)$. The initial conditions are $w(0) = u_0$ and $w'(0) = u_1$.

We used Problem 1V to show how the weak variational form is derived. The connection between the variational form and the weak variational form is similar for Problems 2 and 3.

1.1.4 Existence of solutions

It is not realistic to expect classical solutions in all cases. It is important to note that the existence of a solution is not a simple issue. It depends on the

definition of a solution. If a classical solution does not exist, it is possible to find a weak solution using the theory in Section 2.2.

Consider Problem 1W. It is a special case of the general problem, Problem G, in Section 2.1. The following results are used to apply the existence theory and general convergence theory. The proofs are omitted since similar results are proved in Section 5.3.

Proposition 1.1.1. *There exists a constant β such that $\|v\|^2 \leq \beta^2 b(v, v)$ for each $v \in V(0, \ell)$. The bilinear form b is an inner product on $V(0, \ell)$.*

We define the norm associated with the inner product b as

$$\|u\|_V = \sqrt{b(u, u)} \quad \text{for any } u \in V(0, \ell).$$

Proposition 1.1.2. *The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(0, \ell)$ on $V(0, \ell)$. Also, $V(0, \ell)$ is dense in $\mathcal{L}^2(0, \ell)$.*

The theory in Section 2.2 may be applied with spaces $X = W = \mathcal{L}^2(0, \ell)$ and $V = V(0, \ell)$.

Three types of damping

Three types of damping are defined in Chapter 2: Weak damping, strong damping and damping that is neither weak nor strong. Let us consider the types of damping in the wave equation. We have weak damping in Problem 1

$$|a(u, v)| \leq \gamma \|u\|_W \|v\|_W$$

and strong damping in Problem 2

$$|a(u, u)| = \gamma \|u'\|^2 = \frac{\gamma}{\alpha} \|u\|_V^2.$$

The boundary damping in Problem 3 is neither weak nor strong. Let $\phi(x) = x(\ell - x)$, then $\phi \in T[0, \ell]$ and $a(\phi, \phi) = 0$ but $\|\phi\|_V \neq 0$. It is, on the other hand, not difficult to construct a sequence $\{\phi_n\} \subset T[0, \ell]$ such that

$$\frac{a(\phi_n, \phi_n)}{\|\phi_n\|_W^2} \rightarrow \infty.$$

The solution of Problem 1W is a weak solution of the boundary value problem, Problem 1.

1.1.5 The Galerkin finite element method

Consider the application of the finite element method to Problem 1W. A finite dimensional subspace S^h of $V(0, \ell)$ is constructed. Then the problem is to find a function u_h with $u_h(t) \in S^h$ such that

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = c(f(t), v), \quad (1.1.4)$$

for each $v \in S^h$. If the Galerkin approximation, u_h , converges to the solution of Problem 1W, it does not imply that it converges to a classical solution of the partial differential equation. All that can be guaranteed is that the limit is in $V(0, \ell)$.

More complex vibration problems are considered in Chapters 5 to 9. In each case the weak variational form is the same as Problem 1W.

In [VV02] existence is considered for different types of damping. In this thesis the convergence of the Galerkin approximation is considered.

1.2 Overview of the literature

In this section we consider the literature study in [Bas11] where the focus is on the “standard” Galerkin method, also referred to as the continuous Galerkin method. In [Bas11] an in depth study was made of [Dup73] and [Bak76] since their articles had a significant impact on the literature and other authors adapted their methods or were influenced by it. In the articles error estimates are derived for the Galerkin approximation for the multi-dimensional wave equation. These error estimates are for the semi discrete approximation and the fully discrete approximation.

The articles [Dup73] and [Bak76] were followed by numerous articles on the multi-dimensional wave equation (mostly without damping) and other vibration problems e.g. beams and plates. The basic approach stayed the same and attention was given to different aspects e.g. numerical integration.

It appears as if mixed finite elements became more interesting in the nineties, e.g. [CDW90], [Sem94], [CDW96], [FXX99]. Lately discontinuous Galerkin methods received a lot of attention, e.g. [GSS06] and numerous others. However, the continuous Galerkin method is still of interest. Consider for example recent articles on the continuous Galerkin method: [Wu03], [Wu05], [Wu06] and [Kar11].

The absence of a unifying theory is noted in [Bas11]. (Even notation varies to the extent that comparison of different articles is very difficult.) For this purpose a general approach is necessary. The additional advantage is that the general results may be applied to complex systems as is done in Chapters 7, 8 and 9 of this thesis.

In [Bas11] the author followed [Zie00] and [VV02] and considered the abstract variational form. (Problem G in Section 2.1 of this thesis.) In [Zie00] an attempt is made to generalize the theory, but only for modal damping. On the other hand, the damping in [Dup73] is not modal but the general problem (Problem G) is not considered; only the multi-dimensional wave equation.

It is noted that the four assumptions (E1, E2, E3 and E4 in Section 2.2) made to prove existence theorems, are also fundamental for convergence analysis. Furthermore, different types of damping are identified in [VV02] (see Section 1.1 of this thesis). Existence theorems are different for different types of damping: the assumptions are different and so are the properties of solutions.

Considering these facts and the presentation of the convergence theory in [Bas11], one concludes that damping does not receive sufficient attention in the literature.

1.3 Aim

The aim in the research was to generalize the theory of convergence for hyperbolic problems to accommodate complex systems. More precisely, we consider the Galerkin finite element approximation of a solution of a general second order hyperbolic equation. This includes the general second order wave equation but also vibrating beams, plates etc., since the variational forms are the same as for the wave equation. Even systems of linked elastic bodies can be accommodated. We restrict our attention to the continuous Galerkin method.

One objective was to determine how far one can proceed with the convergence theory using only the assumptions E1, E2, E3, E4 in existence theory (see Section 2.2).

Another objective was to determine how the type of damping affects the convergence theory.

It is well known that the smoothness of a solution is important in convergence analysis. In this thesis we attempt to pinpoint exactly where the regularity in space and time is used in the analysis.

1.4 Results obtained

We consider the continuous Galerkin finite element approximation of a solution of a general second order hyperbolic equation. The general second order hyperbolic equation is formulated in Section 2.1. It is in variational form and referred to as Problem G. This includes the general second order wave equation but also vibrating beams, plates etc., since the variational forms are the same as for the wave equation. Even systems of linked elastic bodies can be accommodated, see Chapter 9.

The existence results in Section 2.2 are presented in variational form. This is more convenient for the finite element method as one can compare conditions for existence to conditions for convergence. We see that sufficient conditions for existence are also used to prove convergence. They seem not to be sufficient for convergence however, and more regularity of solutions are assumed.

In the rest of Chapter 2 we lay the foundation for a general theory on convergence. Apart from the assumptions E1 to E4 we found that an additional assumption (labeled Assumption A1), was necessary to proceed for general damping but not for inertia norm estimates for weak damping. Regarding interpolation, the assumption made (labeled Assumption A3), is realistic for many applications but not for all.

We generalize the results of Baker [Bak76] and Dupont [Dup73]. Note that the problems in [Bak76] and [Dup73] are special cases of Problem G in Section 2.1.

The methods in [Bak76] to derive error estimates for the semi-discrete and fully discrete approximations are adapted for the general case with weak damping. The general results are presented in Sections 3.4, 4.2 and 4.3. Applications are given in Sections 5.1, 5.3, 6.7, 9.1, 9.2 and 9.3. The general results also appear in [BV13].

The method in [Dup73] to derive error estimates for the semi-discrete approximation is adapted in Section 3.3 for the general case. Applications are

given in Sections 5.2, 6.7, 7.7, 8.1 and 8.2.

This thesis is not merely about generalization but other contributions to the theory are also made. Consider for example the significance of the semi-discrete approximation. In [Dup73], [Bak76] and other publications error estimates are derived for the semi discrete approximation and then for the fully discrete approximation without using the results already obtained. In some cases the semi-discrete approximation is ignored. We derive an error estimate for the semi-discrete approximation with respect to the inertia norm or energy norm and then an estimate for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm. The final estimate follows trivially by the triangle inequality. This approach has two advantages. It is not necessary to assume the existence of a third or fourth order derivative for the exact solution and the convergence analysis for the fully discrete approximation is simplified.

It is well known that the smoothness of a solution is important in convergence analysis. An issue that deserves attention here is the differentiability of the projection of the exact solution. It is possible that the exact solution does not have a second order derivative with respect to the energy norm but its projection may (see Section 2.3). For the convergence of the semi-discrete approximation (in general) the existence of a second order derivative w.r.t. the energy norm is required - a serious restriction. Only for convergence w.r.t. the inertia norm for weak damping is this not required. To derive order h or h^α error estimates for the semi-discrete approximation, spatial regularity is required. Assumptions need to be made and to verify that they hold in some application requires a deep study of the theory of partial differential equations.

The advantages of deriving error estimates for the error in approximating the semi-discrete approximation by the fully discrete approximation are mentioned above. Convergence of the fully discrete approximation with central differences to the semi-discrete problem in an abstract setting, is done in Section 4.4.

Existence proofs using the results in [VV02] are given in Sections 6.6, 7.6, 8.2, 9.1 and 9.2. In Sections 7.6, 8.2 and 9.2 the results are new and in the other two cases the proofs are new. To apply the theorems in [VV02] requires the proof of auxiliary results which are not trivial.

It is important to realize that the general results in this thesis do not make

convergence analysis for specific model problems redundant. Application of general results to specific model problems is not trivial, as can be seen in Chapters 5 to 9.

From experience we know that all linear vibration problems can be written in the same form but there is no theorem that states this “fact”. It is therefore necessary to show in each application how this is done. It is also necessary to determine the correct bilinear forms. Unfortunately this necessitated the inclusion of a substantial amount of work that is not new. Although it is indicated where relevant in the text, we mention three prominent examples: Subsections 8.1.1 to 8.1.6, 9.1.1 to 9.1.6 and 9.2.1 to 9.2.3 are not new. However, minor modifications were made and in some places errors corrected.

1.5 Conclusion and future research

The existence results in [VV02] turned out to be a convenient starting point. There are three theorems for the existence of a unique solution: for general damping, strong damping and weak damping. In this thesis we show that the type of damping also affects the convergence theory. Also, we see that the conditions for existence are used to prove convergence but they are not sufficient and more regularity of solutions is required.

We consider the general damping case first. Here we generalized the approach of [Dup73]. An additional assumption (labeled Assumption A1), was necessary to make any progress. For the convergence of the semi-discrete approximation the existence of a second order derivative w.r.t. the energy norm is required - a significant restriction.

In [Dup73] error estimates are derived for the semi-discrete approximation and then for the fully discrete approximation without using the results already obtained. We derive an error estimate for the semi-discrete approximation with respect to the energy norm and then an estimate for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm. The final estimate follows trivially by the triangle inequality. This approach has the advantage that it is not necessary to assume the existence of a third or fourth order derivative for the exact solution, a significant contribution that can be used in applications. We did the fully discrete approximation only for the standard central differences algorithm, but many other options are possible and further

research should be done.

Clearly, the theory for general damping applies to strong damping. However, we believe that the theory can be improved for the case of strong damping and this may be a topic for further research.

Next we consider the case of weak damping. The methods in [Bak76] to derive error estimates for the semi-discrete approximations were adapted for the general case and it has been modified to include weak damping. For convergence w.r.t. the inertia norm for weak damping it is not necessary to make the additional assumption (Assumption A1) required for general damping.

Recall that the fact that we split the semi-discrete and fully discrete approximations has the advantage that it is not necessary to assume the existence of a third or fourth order derivative for the exact solution. In this case there is the additional advantage that the convergence analysis for the fully discrete approximation is simplified.

To derive order h or h^α error estimates for the semi-discrete approximation, spatial regularity is required. Assumptions were made in the abstract theory and it is necessary to verify that they hold in applications. This requires a deep study of the theory of partial differential equations. In only one of the articles considered, was the regularity of a solution mentioned. [Wu03] cited [Eva98], but the result in [Eva98] is only for the wave equation and not sufficient for three-dimensional elasticity or plate problems. This “gap” in the literature needs to be investigated.

Finally we emphasize that the general results in this thesis do not make convergence analysis for specific model problems redundant. In fact, to apply these results the necessary estimates (the assumptions in the general theory) need to be verified for a specific application.

Chapter 2

General framework

2.1 General second order hyperbolic equation in variational form

We consider the general second order hyperbolic equation or general linear vibration problem in variational form, which we define in this section.

For completeness we present a summary of the existence theory in [VV02].

Let X , W and V denote real Hilbert spaces such that $V \subset W \subset X$.

X has inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$.

W has inner product c and norm $\|\cdot\|_W$.

V has inner product b and norm $\|\cdot\|_V$.

We will refer to W as the inertia space and to V as the energy space. The norms $\|\cdot\|_W$ and $\|\cdot\|_V$ will be referred to as the inertia norm and energy norm respectively.

Let J be a bounded or unbounded interval, either an open interval containing zero or of the form $[0, \tau)$ or $[0, \infty)$. Let Y be any Hilbert space and consider a function u on J with values in Y .

Notation

We write $u'(t) \in Y$ if the derivative exists with respect to the norm of Y ;

$u \in C(J; Y)$ if u is continuous on J with respect to the norm of Y ;

$u \in C^k(J; Y)$ if $u^{(k)} \in C(J; Y)$.

Remark $u'(t) \in Y$ implies $\|h^{-1}(u(t+h) - u(t)) - u'(t)\|_Y \rightarrow 0$. In some cases $u'(t) \in Y$ will have a different meaning, but it will be stated clearly.

Consider a bilinear form a defined on V and the following general problem.

Problem G

Given a function $f : J \rightarrow X$, find a function $u \in C(J; V) \cap C^1(J, W)$ such that u' is continuous at 0 and for each $t \in J$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W,$$

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each } v \in V, \quad (2.1.1)$$

while $u(0) = u_0$ and $u'(0) = u_1$.

In many applications X is equal to W and the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent. For the wave equation $\|\cdot\|_X$ is just the $\mathcal{L}^2$ -norm.

2.2 Existence

It is clearly desirable to determine whether a given problem is well posed before considering a numerical approximation for the solution. Also, assumptions are made to obtain convergence results and it is natural to compare them to the assumptions made for existence theory. In this section we present a summary of the existence theory in [VV02]. Existence results may also be found in other publications e.g. [LM72], [Sho77], [Kut86], [AKS96] and [Eva98] but the results in [VV02] are convenient for this thesis as we explain below. (The model problems in [LM72] and [Eva98] do not include damping.)

2.2.1 General damping term

Assumptions The following assumptions are used for existence theory as well as convergence theory.

E1 V is dense in W and W is dense in X ,

E2 There exists a constant C_b such that $\|v\|_W \leq C_b \|v\|_V$ for each $v \in V$,

E3 There exists a constant C_c such that $\|v\|_X \leq C_c \|v\|_W$ for each $v \in W$,

E4 The bilinear form a is nonnegative, symmetric and bounded on V , i.e.

$$|a(u, v)| \leq K_a \|u\|_V \|v\|_V.$$

The following theorem is proved in [VV02].

Theorem 2.2.1. *Suppose Assumptions E1, E2, E3 and E4 hold. If for $u_0 \in V$ and $u_1 \in V$ there exists some $y \in W$ such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each } v \in V, \quad (2.2.1)$$

then for each $f \in C^1([0, \tau]; X)$ there exists a unique solution

$$u \in C([0, \tau]; V) \cap C^1([0, \tau]; W) \cap C^1((0, \tau); V) \cap C^2((0, \tau); W),$$

for Problem G. If $f = 0$ then $u \in C^1([0, \infty); V) \cap C^2([0, \infty); W)$.

The theorem above is convenient to use since the assumptions are in terms of the bilinear forms a , b and c and it is not necessary to construct linear operators with suitable properties as in [Sho77], [Kut86] and [AKS96].

The approach in [VV02] is relatively new and therefore we discuss briefly how it is related to semigroup theory. Problem G is equivalent to a first order differential equation in the product space $H = V \times W$. For $x \in H$ we write $x = \langle x_1, x_2 \rangle$ with $x_1 \in V$ and $x_2 \in W$.

To reformulate Problem G, a linear operator A is constructed in [VV02]. The operator is determined by the three bilinear forms c , b and a . The domain $\mathcal{D}(A) \subset V \times W$ and $\langle u_0, u_1 \rangle \in \mathcal{D}(A)$ if and only if $u_0, u_1 \in V$ and there exists a $y \in W$ such that condition (2.2.1) holds. The function f is replaced by a function $\hat{f} \in C^1(J, W)$ where $c(\hat{f}(t), w) = (f(t), w)_X$ for each $w \in W$.

Now, if u is a solution of Problem G, $g(t) = \langle 0, \hat{f}(t) \rangle$ and $w(t) = \langle u(t), u'(t) \rangle$, then

$$w' = Aw + g \text{ with } w(0) = \langle u_0, u_1 \rangle. \quad (2.2.2)$$

Conversely, if w is a solution of the initial value problem (2.2.2), then w_1 is a solution of Problem G.

Due to Assumptions E1, E2 and E4, A is the infinitesimal generator of a C_0 semigroup of contractions. For the existence of a solution of (2.2.2) conditions are imposed on $w(0)$ and g : $w(0) \in \mathcal{D}(A)$ and $g \in C^1(J, H)$. Now $w(0) \in \mathcal{D}(A)$ is equivalent to the condition imposed in the theorem and $g \in C^1(J, H)$ if and only if $\hat{f} \in C^1(J, W)$.

Remark A majority of the articles considered, provide no reference for existence. If they do they cite [LM72], where the model problem has no damping.

2.2.2 Weak damping and strong damping

In [VV02] three types of damping are identified. It is shown that the type of damping determines the assumptions used in proofs and the properties of the solution. In this thesis we show that the type of damping also affects the convergence theory.

More convenient existence results can be obtained if additional assumptions are made regarding the damping term a . Firstly, condition (2.2.1) may be replaced by simpler conditions. Secondly, solutions have more convenient properties for analysis.

We first consider so called **weak damping**.

Assumption E4W The bilinear form a is nonnegative, symmetric and bounded on W , i.e.

$$|a(u, v)| \leq K_a \|u\|_W \|v\|_W.$$

If the bilinear form a satisfies Assumption E4W the damping may be referred to as viscous type since it is valid for viscous damping in the wave equation and other vibration problems. In [VV02] it is referred to as weak damping.

Definition

$$E_b = \{ x \in V \mid \text{there exists a } y \in W \text{ such that } c(y, v) = b(x, v) \text{ for all } v \in V \}.$$

It is shown in [VV02] that Assumptions E1, E2 and E4W imply that $\mathcal{D}(A) = E_b \times V$ and that A is the infinitesimal generator of a C_0 group of contractions. The following theorem is also proved there.

Theorem 2.2.2. *Let J be an open interval containing zero. Suppose Assumptions E1, E2, E3 and E4W hold. Then there exists a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W),$$

for Problem G for each $u_0 \in E_b$, $u_1 \in V$ and each $f \in C^1(J; X)$. If $f = 0$ then $u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W)$.

Remark The theorem above is valid for the case where there is no damping, i.e. $a = 0$. This is the case in most of the finite element method literature considered.

Next we consider **strong damping** or material damping where the bilinear form a satisfies an additional condition. In this case it is also possible to improve the conclusion of Theorem 2.2.1.

Assumption E5S The bilinear form a is positive definite, i.e. there exists a positive constant C_a such that $a(u, u) \geq C_a \|u\|_V^2$ for each $u \in V$.

Note that a is now actually an inner product for V and the operator A is the infinitesimal generator of an analytic semigroup. Details and the following theorem can be found in [VV02].

Theorem 2.2.3. *Suppose Assumptions E1, E2, E3, E4 and E5S hold. If f is locally Hölder continuous on $(0, \tau)$ and $u_0 \in V$, $u_1 \in W$, then Problem G has a unique solution*

$$u \in C([0, \tau), V) \cap C^1([0, \tau), W) \cap C^2((0, \tau), W).$$

If $f = 0$ then $u \in C([0, \infty); V) \cap C^1([0, \infty); W) \cap C^\infty((0, \infty); V)$.

Remarks

1. Locally Hölder continuous may be replaced by $f \in C^1(0, \tau)$ in Theorem 2.2.3.
2. It is important to note that boundary damping is neither weak nor strong, as shown in Section 1.1. See also Sections 5.2 and 6.6.

2.3 Galerkin approximation

Recall that the aim in this thesis is to establish a general theory of convergence for second order hyperbolic equations. Based on [SF73] a general theory was developed in [ZVV99] and [Zie00]. However, the convergence results obtained are valid only for modal damping (see Section 3.2). This is a restriction since boundary damping, for example, is not modal. In this section no assumption (apart from E4) is made regarding the damping term. In Chapter 3 we show how different approaches are necessary for different types of damping.

The Galerkin approximation was introduced in Chapter 1. In this section we consider the general case. Suppose S^h is a finite dimensional subspace of V . (The symbol h has no meaning at this stage, but it is customary to use the notation.)

Problem G^h

Given a function $f : J \rightarrow X$, find a function $u_h \in C^2(J)$ such that u'_h is continuous at 0 and for each $t \in J$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \text{ for each } v \in S^h, \quad (2.3.1)$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

The initial values u_0^h and u_1^h are elements of S^h as close as possible to u_0 and u_1 . Full details are given in Chapter 3.

Since c is an inner product and S^h is finite dimensional, existence is no problem if f is continuous.

The existence result is given below, but the proof depends on an existence result for systems of ODE's and it is therefore postponed to Section 2.5.

Theorem 2.3.1. *Let J be an interval containing zero. If $f \in C(J, X)$, then there exists a unique solution $u_h \in C^2(J)$ for Problem G^h for each u_0^h and u_1^h in S^h . If $f \in C^k(J, X)$, then $u_h \in C^{2+k}(J)$ and if $f = 0$ then $u_h \in C^\infty((-\infty, \infty))$.*

In the rest of this section we consider the error for the semi-discrete approximation u_h . To begin we introduce the projection operator.

Projection

To find an estimate for the discretization error $\|u(t) - u_h(t)\|_V$, a projection is used. This was the case in the first two publications, [SF73] and [Dup73]. The projection operator P_h is defined by

$$b(u - P_h u, v) = 0 \quad \text{for all } v \in S^h.$$

If no confusion is possible we will write P for P_h . The symbol P is also used to denote the “projection” Pu of the solution u , i.e. $(Pu)(t) = Pu(t)$ for each $t \in J$.

The idea of the projection is to split the error. Let $e_h(t) = u(t) - u_h(t)$, $e(t) = Pu(t) - u_h(t)$ and $e_p(t) = u(t) - Pu(t)$. Then

$$e_h(t) = e_p(t) + e(t) \tag{2.3.2}$$

and

$$\|u(t) - u_h(t)\|_V \leq \|e_p(t)\|_V + \|e(t)\|_V. \tag{2.3.3}$$

Estimates for the norm of $e_p(t)$ are obtained from approximation theory in applications (see the next section). The problem is to find an estimate for $e(t)$, the difference between the projection of u and the Galerkin approximation. The estimates that can be obtained depend on the differentiability of Pu and that depends on the properties of the solution and these may vary from problem to problem.

Remark Assumptions E1, E2, E3 and E4 are basic and assumed to hold for the general convergence theory.

The usual way to start is to subtract Equation (2.3.1) from (2.1.1) and then use the fact that $e_p(t)$ is orthogonal to S^h to prove the result in Lemma 2.3.2 below. In the literature study [Bas11] it was found that the result in Lemma 2.3.2 was used but not explicitly formulated as a separate result. However, it is undesirable in the case of weak damping as explained in Section 3.4 where we generalize and improve the theory in [Bak76]. In this new approach we have the following trivial yet important lemma.

Lemma 2.3.1. *If u is the solution of Problem G and u_h is the solution of Problem G^h , then*

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e(t), v) = 0 \quad \text{for all } v \in S^h. \tag{2.3.4}$$

Proof. By subtracting Equation (2.3.1) from (2.1.1) we obtain the following identity. For any $v \in S^h$,

$$c(u''(t) - u_h''(t), v) + a(u'(t) - u_h'(t), v) + b(u(t) - u_h(t), v) = 0.$$

The result follows from the fact that

$$b(u(t) - Pu(t), v) = 0 \text{ for all } v \in S^h.$$

□

We now consider the differentiability of Pu .

Proposition 2.3.1. *If $u \in C^{(k)}(J, V)$, then $Pu \in C^{(k)}(J, V)$ and $(Pu)^{(k)}(t) = Pu^{(k)}(t)$.*

Proof. The projection operator P is a bounded linear operator with norm less than one, i.e. $\|Px\|_V \leq \|x\|_V$. It follows that

$$\|(\delta t)^{-1}(Pu(t + \delta t) - Pu(t)) - Pu'(t)\|_V \leq \|(\delta t)^{-1}(u(t + \delta t) - u(t)) - u'(t)\|_V.$$

Since the term on the right converges to 0 as $\delta t \rightarrow 0$, $Pu \in C^1(J, V)$ and $(Pu)'(t) = Pu'(t)$. Now use induction to complete the proof. □

To prove Lemma 2.3.2 below the existence of $(Pu)''$ is required. If u is a solution of Problem G, then $u \in C^1(J, V)$ from the existence theorem, Theorem 2.2.1, and hence $Pu \in C^1(J)$. However, $(Pu)''$ may not exist.

Assumption A1

The solution u of Problem G has the property that $(Pu) \in C^2(J)$.

Assumption A2

The solution u of Problem G satisfies $u \in C^2(J, V)$.

Remark In [Dup73] Assumption A2 is used and it is then possible to prove that $Pu \in C^2(J)$. (This was done in [Zie00].) Baker [Bak76] derived error estimates for the undamped case using a weaker assumption, but assumed A1 without explicitly mentioning it. The assumption in [Dup73] is sufficient to prove A1 but not necessary. In Chapter 3 this is considered in detail.

Proposition 2.3.2. *If $u \in C^2(J, W)$ and u satisfies Assumption A1, then*

$$e_p \in C^2(J, W).$$

Proof. Recall that $e_p(t) = u(t) - (Pu)(t)$. Since norms in S^h are equivalent, $Pu \in C^2(J, W)$. $\square$

Lemma 2.3.2. *If the solution u of Problem G satisfies Assumption A1, then $c(e_p''(t), v) + a(e_p'(t), v) + c(e''(t), v) + a(e'(t), v) + b(e(t), v) = 0$ for all $v \in S^h$.*

Proof. The result follows from Proposition 2.3.1 since $e_h = e_p + e$. $\square$

As mentioned above, to estimate the error e_h , it is necessary to estimate e . This is done in Chapter 3, starting with either Lemma 2.3.1 or Lemma 2.3.2. Estimates for e are in terms of e_p . Different methods are used for different types of damping.

Remark The notation used in [Dup73], [Bak76] and other articles varies significantly and it is difficult to compare results. We use the notation from [SF73] and [Zie00], with minor modifications.

2.4 Projection error

To prove the convergence results, it is necessary to estimate the projection errors e_p , e_p' and e_p'' . We use interpolation errors.

With few exceptions, partial differential equations for scalar or vector valued functions on a single domain are considered in the literature. For a system of linked elastic bodies more than one domain is encountered.

It is necessary to generalize the approach in the literature for systems. The assumption in [Bas11] generalizes [Zie00]: There exists a subspace $H(V, k)$ of V , a positive constant $\widehat{C}$ and a positive integer α such that for any $u \in H(V, k)$

$$\inf_{v \in S^h} \|u - v\|_V \leq \widehat{C} h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$. This generalizes also the assumptions in [Dup73] and [Bak76].

The assumption in [Bas11] is inconvenient for estimating errors in initial approximations. For the general theory in this thesis it is necessary to generalize the notion of an interpolation operator.

Definition Generalized interpolation operator.

$$\Pi u = \sum_{k=1}^n \phi_k(u) w_k \text{ where } \{w_k\} \text{ is a basis for } S^h \text{ and } \phi_k \text{ are linear functionals.}$$

To formulate an assumption regarding the error when an element of V is approximated by an element of S^h , we suppose h is a parameter related to the dimension n of S^h and $h \rightarrow 0$ as $n \rightarrow \infty$.

Assumption A3

There exists a subspace $H(V, k)$ of V , an interpolation operator Π and positive constants C_Π and α (depending on V and k) such that for $u \in H(V, k)$:

$$\|u - \Pi u\|_V \leq C_\Pi h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

The following trivial consequence will be used in the general theory in Chapter 3.

Proposition 2.4.1. *There exists a subspace $H(V, k)$ of V and positive constants C_Π and α (depending on V and k) such that for $u \in H(V, k)$:*

$$\|u - Pu\|_V \leq C_\Pi h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

2.5 Fully discrete approximation

2.5.1 A system of ordinary differential equations

Consider Problem G^h formulated in Section 2.3.

Problem G^h

Find u_h such that for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(\cdot, t), v)_X \quad (2.5.1)$$

for each $v \in S^h$ while $u_h(0) = d^h$ and $u_h'(0) = v^h$. (In this section it is convenient to use d^h and v^h for initial conditions instead of u_0^h and u_1^h .)

Problem G^h is equivalent to a system of ordinary differential equations as we prove below. This equivalence enables us to prove Theorem 2.3.1 and to derive differentiability properties of the solution.

Suppose that S^h is the span of the set of basis functions $\{\phi_1, \phi_2, \dots, \phi_n\}$. It is convenient to introduce the following transformation. For any $\bar{x} \in R_n$, let

$$T_h \bar{x} = \sum_{i=1}^n x_i \phi_i \in S^h.$$

The mapping T_h is obviously a linear bijection.

Consider u and v in S^h and let $\bar{u} = T_h^{-1}u$ and $\bar{v} = T_h^{-1}v$. Then,

$$c(u, v) = \sum_{i=1}^n \sum_{j=1}^n u_j v_i c(\phi_i, \phi_j).$$

If the matrix M is defined by $M_{ij} = c(\phi_i, \phi_j)$, then $c(u, v) = M\bar{u} \cdot \bar{v}$. Similarly, if the matrices K and C are defined by $K_{ij} = b(\phi_j, \phi_i)$ and $C_{ij} = a(\phi_j, \phi_i)$, then $a(u, v) = C\bar{u} \cdot \bar{v}$ and $b(u, v) = K\bar{u} \cdot \bar{v}$.

If a function w has values in S^h , then we define a function $\bar{w}$ by

$$\bar{w}(t) = T_h^{-1}w(t),$$

with values in R_n . Consequently w is differentiable if and only if w_j is differentiable for each j and

$$\frac{d}{dt} \left(\sum_{j=1}^n w_j(t) \phi_j \right) = \sum_{j=1}^n w'_j(t) \phi_j.$$

The same is true for the second order derivative.

Define a function F with values on $\mathbb{R}_n$ by $F_i(t) = (f(\cdot, t), \phi_i)_X$. We find that u_h satisfies the variational equation in Problem G^h if and only if

$$M\bar{u}'' \cdot \bar{v} + C\bar{u}' \cdot \bar{v} + K\bar{u} \cdot \bar{v} = F \cdot \bar{v} \quad \text{for each } \bar{v} \in \mathbb{R}_n.$$

Problem G-ODE

Find $\bar{u} \in C^2(J)$ such that

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F(t) \quad \text{with } \bar{u}(0) = \bar{d} \text{ and } \bar{u}'(0) = \bar{v},$$

where $\bar{d} = T_h^{-1}d^h$ and $\bar{v} = T_h^{-1}v^h$.

We have proved the following result.

Proposition 2.5.1. *Suppose M , K , C , F , $\bar{d}$ and $\bar{v}$ are defined as above. Then, the function u_h is a solution of Problem G^h if and only if the function $\bar{u}$ is a solution of Problem G -ODE.*

Proposition 2.5.2. *If $F \in C(J)$, then Problem G -ODE has a unique solution for each pair of vectors $\bar{d}$ and $\bar{v}$. If $F \in C^k(J)$, then $\bar{u} \in C^{2+k}(J)$.*

Proof. Since c is an inner product, M is invertible and the differential equation in Problem ODE may be written in the form $\bar{u}'' + M^{-1}C\bar{u}' + M^{-1}K\bar{u} = M^{-1}F(t)$. It follows from the theory of linear differential equations that the initial value problem has a unique solution.

The differentiability of $\bar{u}$ is obvious. □

Proof of Theorem 2.3.1. Since S^h is finite dimensional, $f \in C(J, X)$ implies that $F \in C(J)$. It follows that Problem G -ODE has a unique solution (by Proposition 2.5.2) and hence Problem G^h has a unique solution (by Proposition 2.5.1).

Note that $F \in C^k(J)$ if $f \in C^k(J, X)$. Then, obviously $\bar{u} \in C^{2+k}(J)$ which implies that $u_h \in C^{2+k}(J)$. □

Proposition 2.5.3. *Suppose u_h is the solution of Problem G^h . There exists a constant K such that if $f \in C^2(J, X)$, then*

$$\begin{aligned} \|u_h^{(4)}(t)\|_W &\leq K \left(5 \int_0^t \|f(\cdot)\|_X + 2\|f(t)\|_X + \|f'(t)\|_X + \|f''(t)\|_X \right) \\ &\quad + K\|u_h'(0)\|_W + K\|u_h(0)\|_V. \end{aligned}$$

Proof. Consider first $u_h(0) = u_h'(0) = 0$. Let

$$E(t) = \frac{1}{2}\|u_h'(t)\|_W^2 + \frac{1}{2}\|u_h(t)\|_V^2. \quad (2.5.2)$$

Since $E'(t) = -a(u_h'(t), u_h'(t)) + (f(t), u_h'(t))_X$,

$$E(t) \leq \int_0^t \|u_h'(\cdot)\|_X \|f(\cdot)\|_X \leq K \max \|u_h'(t)\|_W \int_0^t \|f(\cdot)\|_X. \quad (2.5.3)$$

From (2.5.2) we have that, $\|u_h'(t)\|_W^2 \leq 2E(t)$, therefore

$$\|u_h'(t)\|_W^2 \leq K \max \|u_h'(t)\|_W \int_0^t \|f(\cdot)\|_X$$

and hence

$$\max \|u'_h(t)\|_W \leq K \int_0^t \|f(\cdot)\|_X. \quad (2.5.4)$$

Combining (2.5.3) and (2.5.4) we obtain $E(t) \leq K \left[\int_0^t \|f(\cdot)\|_X \right]^2$. Hence

$$\|u_h(t)\|_V^2 \leq K \left[\int_0^t \|f(\cdot)\|_X \right]^2 \quad \text{and} \quad (2.5.5)$$

$$\|u'_h(t)\|_W^2 \leq K \left[\int_0^t \|f(\cdot)\|_X \right]^2. \quad (2.5.6)$$

From Problem G^h ,

$$c(u''_h(t), v) + a(u'_h(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for all } v \in S^h.$$

Let $v = u''_h$ in Problem G^h . Using the Schwartz inequality we obtain

$$\|u''_h(t)\|_W^2 \leq \|u'_h(t)\|_V \|u''_h(t)\|_V + \|u_h(t)\|_V \|u''_h(t)\|_V + \|f(t)\|_X \|u''_h(t)\|_X.$$

Since u_h is in the finite dimensional space S^h we have equivalence of norms, therefore

$$\|u''_h(t)\|_W^2 \leq K \|u''_h(t)\|_W (\|u'_h(t)\|_V + \|u_h(t)\|_V + \|f(t)\|_X).$$

Using (2.5.5) and (2.5.6) we obtain

$$\|u''_h(t)\|_W \leq K \left(2 \int_0^t \|f(\cdot)\|_X + \|f(t)\|_X \right). \quad (2.5.7)$$

We also have that $c(u'''_h(t), v) + a(u''_h(t), v) + b(u'_h(t), v) = (f'(t), v)_X$ for all $v \in S^h$, and therefore

$$\|u'''_h(t)\|_W^2 \leq \|u''_h(t)\|_V \|u'''_h(t)\|_V + \|u'_h(t)\|_V \|u'''_h(t)\|_V + \|f'(t)\|_X \|u'''_h(t)\|_X.$$

Rewriting the above equation and using (2.5.6) and (2.5.7)

$$\|u'''_h(t)\|_W \leq K \left(3 \int_0^t \|f(\cdot)\|_X + \|f(t)\|_X + \|f'(t)\|_X \right).$$

Similarly,

$$\|u_h^{(4)}(t)\|_W \leq K \left(5 \int_0^t \|f(\cdot)\|_X + 2\|f(t)\|_X + \|f'(t)\|_X + \|f''(t)\|_X \right). \quad (2.5.8)$$

Now consider $f = 0$, but $u_h(0) \neq 0$ and $u'_h(0) \neq 0$. It is obvious that

$$2E(t) \leq \|u'_h(0)\|_W^2 + \|u_h(0)\|_V^2 \quad (2.5.9)$$

The result follows from (2.5.8) and (2.5.9). $\square$

Remark Note that the constant K in the previous proposition depends on the dimension of the space S^h .

2.5.2 Fully discrete approximation in variational form

A finite difference method is used to approximate the solution of Problem G-ODE. Suppose an approximation for the solution is required on the time interval $[0, T]$. The interval is divided into N steps resulting in a step length $\tau = T/N$. The approximation of $\bar{u}(t_k)$ is denoted by $\bar{u}_k^h$.

In Sections 2.3 and 2.4 we considered the error $u - u_h$. Estimates for the error $\bar{u}(t_k) - \bar{u}_k^h$ are available, see e.g. [Hug87]. However, there is a problem, $u(t)$ and $u^h(t)$ are elements of V and error estimates are in the norm of V or W and $\bar{u}(t_k)$ and $\bar{u}_k^h$ are in $\mathbb{R}_n$. To circumvent this difficulty a fully discrete approximation of G^h in variational form is considered.

Consider for example the following scheme where we denote the approximation of $u_h(t_k)$ by u_k^h .

Problem G^h -D-Central

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) + b(u_k^h, v) = (f(t_k), v)_X,$$

for each $v \in S^h$, while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Our aim is to prove convergence of the solution of Problem G^h -D-Central to the solution of Problem G and derive error estimates. Consider the error

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h]. \quad (2.5.10)$$

In Chapter 3 we consider the convergence of the solution of the semi-discrete problem (Problem G^h) to the solution of the general second order problem (Problem G). But we need also to consider convergence of the solution of the fully discrete problem to the solution of the semi-discrete problem. What is required, is an estimate for $\|u_h(t_k) - u_k^h\|_W$.

In [Dup73], [Bak76] and other publications error estimates are derived for the semi discrete approximation and then for the fully discrete approximation without using the results already obtained. In some cases the semi-discrete approximation is ignored. This became the standard approach in the literature.

Splitting the error as in (2.5.10) was done in [Zie00], but only modal damping is considered.

In this thesis we split the error to obtain general results. In Chapter 3 we derive estimates for the semi discrete approximation and in Chapter 4 we consider the error $u_h(t_k) - u_k^h$.

Chapter 3

Convergence of the semi-discrete approximation

3.1 General energy method

This section is from [Bas11] based on [Zie00, Section 5.3]. It is presented here for completeness. Recall that an estimate for $e(t) = Pu(t) - u_h(t)$ is required (Section 2.3).

The following “energy” expression for e is convenient:

$$\begin{aligned} E(t) &= \frac{1}{2}c(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) \\ &= \frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2. \end{aligned} \tag{3.1.1}$$

Lemma 3.1.1. *If the solution u of Problem G satisfies Assumption A1, then*

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)).$$

Proof. Since $u^h \in C^2[0, T]$ and $Pu \in C^2[0, T]$, it follows that $e \in C^2[0, T]$. Also, since Pu and u_h are in S^h , $e(t) \in S^h$, it follows that $e'(t) \in S^h$. Choose $v = e'(t)$ in Lemma 2.3.2, then

$$c(e_p''(t), e'(t)) + c(e''(t), e'(t)) + a(e_p'(t), e'(t)) + a(e'(t), e'(t)) + b(e(t), e'(t)) = 0. \tag{3.1.2}$$

From the definition of the “energy” expression, and since b and c are inner products, we have

$$E'(t) = c(e''(t), e'(t)) + b(e(t), e'(t)).$$

Combining this with (3.1.2) gives us

$$E'(t) = -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)) - a(e'(t), e'(t)).$$

Since $a(e'(t), e'(t)) \geq 0$, the result follows. $\square$

From this inequality the aim is to obtain an estimate for $E(t)$ in terms of projection errors. Since $\|e'(t)\|_W^2 + \|e(t)\|_V^2 = 2E(t)$, it will enable us to estimate $e(t)$ and consequently the error $e_h(t)$. However, this has not been done for the general case. Energy norm estimates were obtained for $e(t)$ in [Zie00], but only for modal damping. On the other hand, in [Dup73] estimates for $e(t)$ were obtained for boundary damping but only for the wave equation. In Section 3.2 we consider the results of [Zie00] and generalize the work of [Dup73] in Section 3.3. In Section 3.4 we obtain inertia norm estimates for weak damping.

Remark The energy method is due to [SF73] and [Dup73]. The book as well as the article appeared in 1973.

3.2 Energy norm estimates for weak damping and modal damping

Definition Modal damping

There exist nonnegative constants μ and k such that

$$a(u, v) = \mu b(u, v) + kc(u, v).$$

3.2.1 Fundamental estimate

Recall assumptions E1, E2, E3 and E4. We also assume that the solution of Problem G satisfies Assumption A1, i.e. $(Pu)''$ exists. Note that we do not assume that $u \in C^2([0, T], V)$ or $(Pu)'' = Pu''$. In [Dup73] and [Zie00] the authors assume that $u \in C^2([0, T], V)$.

Modal damping

The fundamental estimate below is proved in [Zie00] and presented with minor modifications in [Bas11]. It is presented here for completeness and to facilitate comparison with the estimate for general damping in the next section. Also, the proof is valid with trivial modification for weak damping.

Lemma 3.2.1. *Suppose the damping is modal. If the solution u of Problem G satisfies Assumption A1, then for any $t \in [0, T]$,*

$$\|e(t)\|_V + \|e'(t)\|_W \leq \|e(0)\|_V + \|e'(0)\|_W + \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) \, d\tau.$$

Proof. Using Lemma 3.1.1 and substituting the bilinear form a we get

$$E'(t) \leq -c(e_p''(t), e'(t)) - \mu b(e_p'(t), e'(t)) - kc(e_p'(t), e'(t)). \quad (3.2.1)$$

Since $b(u(t) - Pu(t), v) = 0$ for all $v \in S^h$, we have that $b(u'(t) - (Pu)'(t), v) = 0$ for all $v \in S^h$. Therefore $\mu b(e_p'(t), e'(t)) = 0$ and (3.2.1) becomes

$$E'(t) \leq -c(e_p''(t), e'(t)) - kc(e_p'(t), e'(t)). \quad (3.2.2)$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |c(e_p''(t), e'(t))| &\leq \|e_p''(t)\|_W \|e'(t)\|_W, \\ |c(e_p'(t), e'(t))| &\leq \|e_p'(t)\|_W \|e'(t)\|_W, \end{aligned}$$

and therefore we obtain that

$$E'(t) \leq (\|e_p''(t)\|_W + k\|e_p'(t)\|_W) \|e'(t)\|_W. \quad (3.2.3)$$

From (3.1.1), $\|e'(t)\|_W \leq \sqrt{2E(t)}$. Thus

$$E'(t) \leq \sqrt{2E(t)} (\|e_p''(t)\|_W + k\|e_p'(t)\|_W).$$

Also,

$$\frac{d}{dt} \sqrt{E(t)} = \frac{1}{2} \frac{1}{\sqrt{E(t)}} E'(t),$$

and consequently

$$\frac{d}{dt} \sqrt{E(t)} \leq \frac{1}{\sqrt{2}} (\|e_p''(t)\|_W + k\|e_p'(t)\|_W).$$

Integrating the above on both sides yields

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \frac{1}{\sqrt{2}} \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) . \quad (3.2.4)$$

Since

$$E(0) = \frac{1}{2}\|e'(0)\|_W^2 + \frac{1}{2}\|e(0)\|_V^2,$$

we have that

$$\sqrt{2E(t)} \leq \sqrt{\|e'(0)\|_W^2 + \|e(0)\|_V^2} + \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) .$$

Finally, $\|e(t)\|_V + \|e'(t)\|_W \leq \sqrt{2E(t)}$, again from (3.1.1), and the result follows. $\square$

Weak damping

Recall that for weak damping, the bilinear form a satisfies Assumption E4W. The fundamental estimate can be obtained without the assumption that damping is modal.

Where the damping parameters are constant for modal damping, it need not be the case here (see applications in Chapter 5).

The result is almost trivial yet important and not observed before.

Lemma 3.2.2. *Suppose the bilinear form a satisfies assumption E4W. If the solution u of Problem G satisfies Assumption A1, then for any $t \in [0, T]$,*

$$\|e(t)\|_V + \|e'(t)\|_W \leq \|e(0)\|_V + \|e'(0)\|_W + \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) .$$

Proof. From Lemma 3.1.1 we have

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \quad (3.2.5)$$

Using Assumption E4W and the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} |c(e_p''(t), e'(t))| &\leq \|e_p''(t)\|_W \|e'(t)\|_W, \\ |a(e_p'(t), e'(t))| &\leq K_a \|e_p'(t)\|_W \|e'(t)\|_W, \end{aligned}$$

and therefore we obtain that

$$E'(t) \leq (\|e_p''(t)\|_W + K_a \|e_p'(t)\|_W) \|e'(t)\|_W.$$

This inequality is the same as (3.2.3) hence the rest of the proof is the same as the proof of Lemma 3.2.1. $\square$

3.2.2 Error estimates

To obtain an estimate for the error e_h , recall that $e_h = e_p + e$. We now use Assumption A2: The solution of Problem G satisfies $u \in C^2([0, T], V)$. Consequently $(Pu)'' = Pu''$ from Proposition 2.3.1.

The next result follows from either Lemma 3.2.1 or Lemma 3.2.2 using standard arguments.

Lemma 3.2.3. *Assume that the damping is weak or modal and that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \|u(t) - Pu(t)\|_V + \|u'(t) - Pu'(t)\|_W + \|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \\ & \quad + \int_0^t (\|u'' - Pu''\|_W + k\|u' - Pu'\|_W). \end{aligned} \quad (3.2.6)$$

Proof. Note that

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \|e_p(t)\|_V + \|e'_p(t)\|_W + \|e(t)\|_V + \|e'(t)\|_W. \end{aligned}$$

Now use either Lemma 3.2.1 or Lemma 3.2.2. □

Projection errors

To prove the convergence results, it is now necessary to consider the projection errors e'_p and e''_p . Recall Proposition 2.4.1: There exists a subspace $H(V, k)$ of V , and positive constants C_Π and α (depending on V and k) such that for $u \in H(V, k)$:

$$\|u - Pu\|_V \leq C_\Pi h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

Notation $u^{(k)} \in \mathcal{L}^2(J; Y)$ if $u^{(k)}(t) \in Y$ for each t and $\int_J \|u^{(k)}\|_Y^2 < \infty$.

Theorem 3.2.1. *Assume Assumption A3 holds for the space V and the damping is either weak or modal. Assume also that the solution u of Problem G*

satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V, k)} + C_b \|u'(t)\|_{H(V, k)}) + \|Pu_0 - u_0^h\|_V \\ & \quad + \|Pu_1 - u_1^h\|_W + C_b C_\Pi h^\alpha \int_0^t (\|u''(\cdot)\|_{H(V, k)} + k \|u'(\cdot)\|_{H(V, k)}) . \end{aligned}$$

Proof. We use Lemma 3.2.3. The result follows from $\|u\|_W \leq C_b \|u\|_V$ (Assumption E2) and Proposition 2.4.1. $\square$

Initial conditions

The error depends on u_0^h and u_1^h , but u_0^h and u_1^h are not given. In applications one must choose u_0^h and u_1^h . If we choose $Pu_0 = u_0^h$, for example, then $\|Pu_0 - u_0^h\|_V = 0$. In practice, it is difficult if not impossible. An obvious choice is to use the interpolants of u_0 and u_1 (see Section 2.4).

Corollary 3.2.1. *Suppose Assumption A3 holds for the space V and u_0 and u_1 are in $H(V, k)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume that the damping is modal or weak, that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V, k)} + C_b \|u'(t)\|_{H(V, k)} + 2\|u_0\|_{H(V, k)} + 2C_b \|u_1\|_{H(V, k)} \\ & \quad + C_b \int_0^t (\|u''(\cdot)\|_{H(V, k)} + k \|u'(\cdot)\|_{H(V, k)})), \end{aligned}$$

for $t \in [0, T]$.

Proof. Use Theorem 3.2.1 and Proposition 2.4.1 and note that

$$\begin{aligned} \|Pu_0 - \Pi u_0\|_V & \leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\ & \leq 2C_\Pi h^\alpha \|u_0\|_{H(V, k)}; \\ \|Pu_1 - \Pi u_1\|_W & \leq C_b \|Pu_1 - \Pi u_1\|_V \\ & \leq C_b \|Pu_1 - u_1\|_V + C_b \|u_1 - \Pi u_1\|_V \\ & \leq 2C_b C_\Pi h^\alpha \|u_1\|_{H(V, k)}. \end{aligned}$$

$\square$

3.2.3 Higher order inertia norm estimate

The Aubin-Nitsche trick (due to [Aub67] and [Nit69]), can be applied to find higher order estimates in the inertia norm for the discretization error for the case of modal damping. The presentation is based on [Zie00]. The results are not new and we omit some of the proofs. (Proofs are in [SF73], [Zie00] and [Bas11].)

The main factor that determines the rate of convergence of the solution u_h of Problem G^h to the solution u of Problem G as h tends to zero, is the regularity of the weak solution u . However, all that can be guaranteed from the existence theory is that $u \in C^1(J, V)$. The regularity of u depends on the regularity of the initial values u_0 and u_1 . Rauch [Rau85] gave an example to show that the regularity of the solution is necessary to obtain optimal order convergence.

Recall the definition of the set E_b in Section 2.2.

Assumption A4

$E_b \subset H(V, k)$ and for any $u \in E_b$ there exists a constant $\widehat{c}_b$ such that $\|u\|_{H(V, k)} \leq \widehat{c}_b \|y\|_W$, where

$$b(u, v) = (y, v) \quad \forall v \in V.$$

Lemma 3.2.4. *Suppose that Assumptions A3 and A4 hold for the space V . Then, for all $u \in E_b$,*

$$\|u - Pu\|_W \leq C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u\|_{H(V, k)},$$

where $\widehat{c}_b$ is the constant from Assumption A4.

Theorem 3.2.2. *Assume that the damping is weak or modal and that Assumptions A3 and A4 hold for the space V . Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t > 0$,*

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V, k)} + C_b \|Pu_0 - u_0^h\|_V + C_b \|Pu_1 - u_1^h\|_W \\ &\quad + C_b C_\Pi^2 h^{2\alpha} \widehat{c}_b \left(\int_0^t \|u''(\cdot)\|_{H(V, k)} + k \int_0^t \|u'(\cdot)\|_{H(V, k)} \right). \end{aligned}$$

Proof. We have that

$$\begin{aligned} \|u(t) - u_h(t)\|_W &= \|u(t) - Pu(t)\|_W + \|e(t)\|_W \\ &\leq \|u(t) - Pu(t)\|_W + C_b \|e(t)\|_V. \end{aligned}$$

Now use Lemma 3.2.4 and either Lemma 3.2.1 or Lemma 3.2.2. $\square$

To obtain higher order inertia norm estimates for the discretization error, the initial values u_0^h and u_1^h must be chosen appropriately. Either $u_0^h = Pu_0$ and $u_1^h = Pu_1$ or a sufficiently accurate approximation must be used. This was pointed out in [Dup73]. (Dupont suggested a numerical approximation for Pu_0 .)

Following [Dup73], we assume $\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$ and obtain the result below.

Corollary 3.2.2. *Assume that*

- a) *the damping is weak or modal,*
- b) *Assumptions A3 and A4 hold for the space V ,*
- c) *the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2),*
- d) *$u'' \in \mathcal{L}^2([0, T], H(V, k))$ and*
- e) *$\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$.*

Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_{\Pi}^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V, k)} + C_b C_0 h^{2\alpha} \\ &\quad + C_b C_{\Pi}^2 h^{2\alpha} \widehat{c}_b \left(\int_0^t \|u''(\cdot)\|_{H(V, k)} + k \int_0^t \|u'(\cdot)\|_{H(V, k)} \right). \end{aligned}$$

Some authors consider the special case where $u_0^h = Pu_0$ and $u_1^h = Pu_1$.

Corollary 3.2.3. *Let $u_0^h = Pu_0$ and $u_1^h = Pu_1$. Assume that the damping is weak or modal and that Assumptions A3 and A4 hold for the space V . Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t > 0$,*

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_{\Pi}^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V, k)} \\ &\quad + C_b C_{\Pi}^2 h^{2\alpha} \widehat{c}_b \left(\int_0^t \|u''(\cdot)\|_{H(V, k)} + k \int_0^t \|u'(\cdot)\|_{H(V, k)} \right). \end{aligned}$$

3.2.4 Convergence

The assumption that $u'' \in \mathcal{L}^2([0, T], H(V, k))$ is necessary to obtain an error estimate involving h^α . Under less strict regularity conditions the solution u_h of Problem G^h converges to the solution u of Problem G in the energy norm as h tends to zero. The result is not new and we omit the proofs (see [Zie00] and [Bas11]).

Assumption A5

$H(V, k)$ is a dense subset of V .

Lemma 3.2.5. *Suppose Assumptions A3 and A5 hold for the space V . For any $\varepsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that*

$$\|u - Pu\|_V < \varepsilon \quad \text{for } h < \delta.$$

Theorem 3.2.3. *Suppose that Assumptions A3 and A5 hold for the space V and that the damping is modal or weak. Assume that the solution $u \in C^2([0, T], V)$, (Assumption A2). If*

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\|_W = 0,$$

then

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_V = 0 \quad \text{for } t \in [0, T].$$

Remark In most cases the conditions of the theorem will be satisfied if we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$.

3.3 General damping

Boundary damping is nonmodal and also it is neither weak nor strong. In this section we generalise the method of [Dup73] to deal with the damping term. This is a new result.

3.3.1 Fundamental estimate

Consider Problem G and recall assumptions E1, E2, E3 and E4.

We assume that the unique solution satisfies Assumption A1. Recall that the challenge is to estimate $e = Pu - u^h$. From Lemma 3.1.1, we have

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \quad (3.3.1)$$

There is a problem with the last term in Equation (3.3.1) since

$$a(e_p'(t), e'(t)) \leq K \|e_p'(t)\|_V \|e'(t)\|_V,$$

and $\|e'(t)\|_V$ is not bounded by $E(t)$. Lemma 3.3.1 below enables us to estimate this term. We need the following auxiliary result.

Proposition 3.3.1. *If $e_p \in C^2([0, T], V)$, then for any $t \in [0, T]$,*

$$\int_0^t a(e_p'(\cdot), e'(\cdot)) = a(e_p'(t), e(t)) - a(e_p'(0), e(0)) - \int_0^t a(e_p''(\cdot), e(\cdot)).$$

Proof. If $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a basis for S^h , then

$$e(t) = \sum_{i=1}^n e_i(t) \phi_i$$

and hence

$$a(e_p', e) = \sum_{i=1}^n e_i a(e_p', \phi_i). \quad (3.3.2)$$

As $e \in C^1[0, T]$, it follows that $e_i \in C^1[0, T]$ for $i = 1, 2, \dots, n$ and

$$a(e_p', e') = \sum_{i=1}^n e_i' a(e_p', \phi_i). \quad (3.3.3)$$

Since $e_p \in C^2([0, T], V)$, for any $t \in [0, T]$,

$$\begin{aligned} & |(\delta t)^{-1} (a(e_p'(t+h), \phi_i) - a(e_p'(t), \phi_i)) - a(e_p''(t), \phi_i)| \\ & \leq K \|(\delta t)^{-1} (e_p'(t+h) - e_p'(t)) - e_p''(t)\|_V \|\phi_i\|_V \end{aligned}$$

and it follows that, for $i = 1, 2, \dots, n$, $a(e_p'(\cdot), \phi_i)$ is differentiable with

$$\frac{d}{dt} a(e_p'(t), \phi_i) = a(e_p''(t), \phi_i). \quad (3.3.4)$$

Combining (3.3.2), (3.3.3) and (3.3.4) yields

$$\frac{d}{dt}a(e'_p(t), e(t)) = a(e''_p(t), e(t)) + a(e'_p(t), e'(t)).$$

Now, integrate both sides and use the fundamental theorem of calculus. The result follows. $\square$

A result similar to the result above is used in [Dup73] as part of the proof with a different explanation, using results without sufficient references.

Remark For Proposition 3.3.1, $e_p \in C^2([0, T], V)$ is required and therefore $u \in C^2([0, T], V)$ (Assumption A2) is needed.

Lemma 3.3.1 below generalize the approach in [Dup73].

Lemma 3.3.1. *If the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2), then for any $t \in [0, T]$*

$$\|e(t)\|_V + \|e'(t)\|_W \leq \sqrt{24e^{3t}}K_T, \quad \text{where}$$

$$K_T = \int_0^T \|e''_p(\cdot)\|_W + 3K_a \max \|e'_p(t)\|_V + 3K_a \int_0^T \|e''_p(\cdot)\|_V + \|e'(0)\|_W \\ + \sqrt{1 + K_a} \|e(0)\|_V + \sqrt{K_a} \|e'_p(0)\|_V.$$

Proof. We have the following estimates from Cauchy's inequality, Young's inequality and Equation (3.1.1)

$$|c(e''_p(t), e'(t))| \leq \|e''_p(t)\|_W \|e'(t)\|_W \leq \frac{1}{2} \|e''_p(t)\|_W^2 + \frac{1}{2} \|e'(t)\|_W^2, \\ \|e'(t)\|_W^2 \leq 2E(t).$$

Using the above estimates in Equation (3.3.1) and integrating on both sides yields

$$E(t) \leq E(0) + \frac{1}{2} \int_0^t \|e''_p(\cdot)\|_W^2 + \int_0^t E(\cdot) - \int_0^t a(e'_p(\cdot), e'(\cdot)). \quad (3.3.5)$$

Using Proposition 3.3.1 we obtain

$$E(t) \leq E(0) + \frac{1}{2} \int_0^t \|e''_p(\cdot)\|_W^2 + \int_0^t E(\cdot) \\ - \left(a(e'_p(t), e(t)) - a(e'_p(0), e(0)) - \int_0^t a(e''_p(\cdot), e(\cdot)) \right).$$

We need estimates for the terms in brackets. We use Young's inequality and the definition of $E(t)$.

$$\begin{aligned}
|a(e'_p(t), e(t))| &\leq K_a \|e'_p(t)\|_V \|e(t)\|_V \\
&\leq K_a \varepsilon^{-2} \|e'_p(t)\|_V^2 + K_a \varepsilon^2 \|e(t)\|_V^2 \\
&\leq K_a \varepsilon^{-2} \|e'_p(t)\|_V^2 + 2K_a \varepsilon^2 E(t),
\end{aligned}$$

with ε arbitrary. Also,

$$\begin{aligned}
\left| \int_0^t a(e''_p(\cdot), e(\cdot)) \right| &\leq \int_0^t |a(e''_p(\cdot), e(\cdot))| \\
&\leq K_a \varepsilon^{-2} \int_0^t \|e''_p(\cdot)\|_V^2 + K_a \varepsilon^2 \int_0^t \|e(\cdot)\|_V^2 \\
&\leq K_a \varepsilon^{-2} \int_0^t \|e''_p(\cdot)\|_V^2 + 2K_a \varepsilon^2 \int_0^t E(\cdot).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
E(t) &\leq E(0) + \frac{1}{2} \int_0^t \|e''_p(\cdot)\|_W^2 + \int_0^t E(\cdot) + 2K_a \varepsilon^2 \int_0^t E(\cdot) + 2K_a \varepsilon^2 E(t) \\
&\quad + K_a \varepsilon^{-2} \|e'_p(t)\|_V^2 + a(e'_p(0), e(0)) + K_a \varepsilon^{-2} \int_0^t \|e''_p(\cdot)\|_V^2 \\
&\leq E(0) + \frac{1}{2} \int_0^t \|e''_p(\cdot)\|_W^2 + \frac{3}{2} \int_0^t E(\cdot) + \frac{1}{2} E(t) + 4K_a^2 \|e'_p(t)\|_V^2 \\
&\quad + a(e'_p(0), e(0)) + 4K_a^2 \int_0^t \|e''_p(\cdot)\|_V^2, \tag{3.3.6}
\end{aligned}$$

by choosing ε such that $2K_a \varepsilon^2 = \frac{1}{2}$.

Note that $|a(e'_p(0), e(0))| \leq \frac{1}{2} K_a \|e'_p(0)\|_V^2 + \frac{1}{2} K_a \|e(0)\|_V^2$. Rewriting (3.3.6) we obtain

$$\begin{aligned}
\frac{1}{2} E(t) &\leq \frac{3}{2} \int_0^t E(\cdot) + \frac{1}{2} \int_0^t \|e''_p(\cdot)\|_W^2 + 4K_a^2 \|e'_p(t)\|_V^2 \\
&\quad + 4K_a^2 \int_0^t \|e''_p(\cdot)\|_V^2 + E(0) + \frac{1}{2} K_a \|e'_p(0)\|_V^2 + \frac{1}{2} K_a \|e(0)\|_V^2,
\end{aligned}$$

which can be written in the form needed for the Gronwall inequality (see

Appendix D):

$$E(t) \leq 3 \int_0^t E(\cdot) + K_T^*, \text{ where the constant}$$

$$K_T^* = \int_0^T \|e_p''(\cdot)\|_W^2 + 8K_a^2 \max \|e_p'(t)\|_V^2 + 8K_a^2 \int_0^T \|e_p''(\cdot)\|_V^2$$

$$+ \|e'(0)\|_W^2 + (1 + K_a)\|e(0)\|_V^2 + K_a\|e_p'(0)\|_V^2.$$

By applying the Gronwall inequality, we have $E(t) \leq e^{3t} K_T^*$, and therefore

$$\frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2 \leq e^{3t} K_T^*.$$

Using the fact that $\|\cdot\|_2^2 \leq n\|\cdot\|_\infty^2 \leq n\|\cdot\|_1^2$, we obtain

$$K_T^* \leq 6 \left[\int_0^T \|e_p''(\cdot)\|_W + \sqrt{8}K_a \max \|e_p'(t)\|_V + \sqrt{8}K_a \int_0^T \|e_p''(\cdot)\|_V \right. \\ \left. + \|e'(0)\|_W + \sqrt{1 + K_a}\|e(0)\|_V + \sqrt{K_a}\|e_p'(0)\|_V \right]^2.$$

The result then follows from the above inequality and the fact that $(a + b)^2 \leq 2a^2 + 2b^2$. $\square$

3.3.2 Convergence and error estimates

Lemma 3.3.2. *Assume that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,*

$$\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W$$

$$\leq \|u(t) - Pu(t)\|_V + \|u'(t) - Pu'(t)\|_W + \sqrt{24e^{3t}} K_T, \quad \text{where}$$

$$K_T = \int_0^T \|u'' - Pu''\|_W + 3K_a \max \|u'(t) - Pu'(t)\|_V + 3K_a \int_0^T \|u'' - Pu''\|_V$$

$$+ \|Pu_1 - u_1^h\|_W + \sqrt{1 + K_a}\|Pu_0 - u_0^h\|_V + \sqrt{K_a}\|u_1 - Pu_1\|_V.$$

Proof. Note that

$$\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W$$

$$\leq \|e(t)\|_V + \|e_p(t)\|_V + \|e'(t)\|_W + \|e_p'(t)\|_W.$$

The result follows from Proposition 2.3.1 and Lemma 3.3.1. $\square$

Next we substitute the estimates for the projection errors.

Theorem 3.3.1. *Suppose Assumption A3 holds for the space V . Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V, k)} + C_b \|u'(t)\|_{H(V, k)}) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b \|u''(\cdot)\|_{H(V, k)} \right. \\ & \quad \left. + 3K_a \max \|u'(t)\|_{H(V, k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V, k)} \right] \\ & \quad + \sqrt{24e^{3t}} \left[\|Pu_1 - u_1^h\|_W + \sqrt{1 + K_a} \|Pu_0 - u_0^h\|_V + \sqrt{K_a} \|u_1 - Pu_1\|_V \right]. \end{aligned}$$

Proof. The result follows from Lemma 3.3.2, $\|u\|_W \leq C_b \|u\|_V$ (Assumption E2) and Proposition 2.4.1. $\square$

Initial conditions

Concerning initial conditions, the same considerations as in Subsection 3.2.3 are relevant and need not be discussed again.

Corollary 3.3.1. *Suppose Assumption A3 holds for the space V and u_0 and u_1 are in $H(V, k)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V, k)} + C_b \|u'(t)\|_{H(V, k)}) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b \|u''(\cdot)\|_{H(V, k)} \right. \\ & \quad \left. + 3K_a \max \|u'(t)\|_{H(V, k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V, k)} \right. \\ & \quad \left. + 2C_b \|u_1\|_{H(V, k)} + 2\sqrt{1 + K_a} \|u_0\|_{H(V, k)} + \sqrt{K_a} \|u_1\|_{H(V, k)} \right]. \end{aligned}$$

for $t \in [0, T]$.

Proof. The result follows from Theorem 3.3.1 and note that

$$\begin{aligned}
\|Pu_0 - \Pi u_0\|_V &\leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\
&\leq 2C_\Pi h^\alpha \|u_0\|_{H(V,k)}; \\
\|Pu_1 - \Pi u_1\|_W &\leq C_b \|Pu_1 - \Pi u_1\|_V \\
&\leq C_b \|Pu_1 - u_1\|_V + C_b \|u_1 - \Pi u_1\|_V \\
&\leq 2C_b C_\Pi h^\alpha \|u_1\|_{H(V,k)}.
\end{aligned}$$

□

Higher order inertia norm estimate

The Aubin-Nitsche trick (due to [Aub67] and [Nit69]), can be applied to find higher order estimates in the inertia norm for the discretization error for general damping. The theory is adapted in exactly the same way as for weak and modal damping.

As mentioned in Section 3.2, the initial values u_0^h and u_1^h must be chosen appropriately. We consider two cases, $u_0^h = Pu_0$ and $u_1^h = Pu_1$ and the assumption used in [Dup73].

Following [Dup73], we assume $\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$ and obtain the following result.

Corollary 3.3.2. *Assume that*

- a) *Assumptions A3 and A4 hold for the space V ,*
- b) *the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2),*
- c) *$u'' \in \mathcal{L}^2([0, T], H(V, k))$ and*
- d) *$\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0 h^{2\alpha}$.*

Then, for any $t > 0$,

$$\begin{aligned}
\|u(t) - u_h(t)\|_W &\leq C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V,k)} + \sqrt{24e^{3t}} C_b C_\Pi^2 h^{2\alpha} \widehat{c}_b \left(\int_0^T C_b \|u''(\cdot)\|_{H(V,k)} \right. \\
&\quad \left. + 3K_a \max \|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)} + \sqrt{K_a} \|u_1\|_{H(V,k)} \right) \\
&\quad + \sqrt{24e^{3t}} C_b K h^{2\alpha},
\end{aligned}$$

where K depends on C_0 and K_a .

As mentioned in Section 3.2, some authors consider the special case where $u_0^h = Pu_0$ and $u_1^h = Pu_1$.

Corollary 3.3.3. *Assume $u_0^h = Pu_0$ and $u_1^h = Pu_1$ and that Assumptions A3 and A4 hold for the space V . Assume also that the solution u of Problem G satisfies $u \in C^2([0, T], V)$ (Assumption A2) and $u'' \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t > 0$,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_W \\ & \leq C_\Pi^2 h^{2\alpha} \widehat{c}_b \|u(t)\|_{H(V, k)} + \sqrt{24e^{3t}} C_b C_\Pi^2 h^{2\alpha} \widehat{c}_b \left(\int_0^T C_b \|u''(\cdot)\|_{H(V, k)} \right. \\ & \quad \left. + 3K_a \max \|u'(t)\|_{H(V, k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V, k)} + \sqrt{K_a} \|u_1\|_{H(V, k)} \right). \end{aligned}$$

The multi-dimensional wave equation with boundary damping is considered in Section 5.2 where we also discuss [Dup73].

Strong damping

Strong damping is a special case and the general result of this section may be applied. However, intuition suggests that we should be able to obtain better results.

Due to assumption E5S, Theorem 2.2.3 is applicable. Recall that $u \in C^2([0, T], V)$ if $f = 0$ and it is not necessary to use Assumption A2.

It should be possible to determine conditions on f that have $u \in \mathcal{L}^2([0, T], V)$ as a consequence, but to prove this is beyond the scope of this thesis.

3.4 Inertia norm estimate for weak damping

The results of this section are new and may well be a significant contribution to the literature.

In 1976 Baker [Bak76] derived error estimates with “minimal smoothness requirements on the solution”, but no damping. This article had a significant impact on the literature and other authors adapted his methods or were influenced by it, see e.g. [CDW90], [Sem94], [FXX99], [Wu03] and [GSS06].

We generalize the results of Baker [Bak76]. The spaces $\mathcal{L}^2(\Omega)$ and $H_0^1(\Omega)$ are replaced by general Hilbert spaces W and V , where V is a subset of W .

In the general problem the $\mathcal{L}^2$ -norm is replaced by the general inertia norm $\|\cdot\|_W$ and the well known energy norm for the wave equation (equivalent to the H^1 norm) is replaced by a general energy norm $\|\cdot\|_V$ (see Section 2.2).

We derive an error estimate for the semi-discrete approximation with respect to the inertia norm in this section and then an estimate for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm in Section 4.2. The final estimate follows trivially by the triangle inequality. This approach has two advantages. It is not necessary to assume the existence of a third or fourth order derivative for the exact solution and the convergence analysis for the fully discrete approximation is simplified.

In [Bak76] error estimates are derived “using $\mathcal{L}^2$ -projections of the initial data as starting values”. The fact that the starting values for the approximation are projections, is used to derive estimates. But it is not always possible to find the projections exactly. It is not clear in such an event how the error estimates can be interpreted. In our approach projections are not used as initial approximations in the fundamental estimate. The errors for the initial approximations are dealt with at a later stage.

3.4.1 Fundamental estimate

The following lemma is a generalization of an estimate for $e = Pu - u_h$ in [Bak76] (Estimate (3.15)). We also include weak damping.

Lemma 3.4.1. *If the damping term a satisfies Assumption E_4W , then for $t \in [0, T]$,*

$$\begin{aligned} \|e(t)\|_W \leq & \sqrt{2} \left(\|e(0)\|_W + 3T\|e'_h(0)\|_W + 3 \int_0^T \|e'_p(t)\|_W \right. \\ & \left. + 3K_a T \|e_h(0)\|_W + 3K_a \int_0^T \|e_p(t)\|_W \right). \end{aligned}$$

Proof. In Lemma 2.3.1 we have

$$c(e''_h(t), v) + a(e'_h(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h.$$

Now, if v is a function on $[0, T]$ with values in S^h , then

$$c(e''_h(t), v(t)) + a(e'_h(t), v(t)) + b(e(t), v(t)) = 0. \quad (3.4.1)$$

Let $v'(t) = e(t)$, then

$$\begin{aligned}
& \frac{d}{dt} \left[\frac{1}{2} c(e, e) - \frac{1}{2} b(v, v) - c(e'_h, v) - a(e_h, v) \right] \\
&= c(e', e) - b(e, v) - c(e''_h, v) - a(e'_h, v) - c(e'_h, e) - a(e_h, e) \\
&= -c(e'_p, e) - a(e_h, e)
\end{aligned} \tag{3.4.2}$$

from (3.4.1).

Choose $v(\tau) = 0$ for some $\tau \in (0, T)$, and integrate (3.4.2) between 0 and τ . We obtain

$$\begin{aligned}
& - \left[\frac{1}{2} c(e(0), e(0)) - \frac{1}{2} b(v(0), v(0)) - c(e'_h(0), v(0)) - a(e_h(0), v(0)) \right] \\
& + \frac{1}{2} c(e(\tau), e(\tau)) = - \int_0^\tau c(e'_p, e) - \int_0^\tau a(e_p, e) - \int_0^\tau a(e, e).
\end{aligned}$$

Since $a(e, e) \geq 0$ and $b(v(0), v(0)) \geq 0$, we have

$$\begin{aligned}
\frac{1}{2} c(e(\tau), e(\tau)) & \leq \frac{1}{2} c(e(0), e(0)) - c(e'_h(0), v(0)) - a(e_h(0), v(0)) \\
& - \int_0^\tau c(e'_p, e) - \int_0^\tau a(e_p, e)
\end{aligned} \tag{3.4.3}$$

and consequently

$$\begin{aligned}
c(e(\tau), e(\tau)) & \leq c(e(0), e(0)) + 2 \int_0^\tau c(e'_h(0), e) - 2 \int_0^\tau c(e'_p, e) \\
& + 2 \int_0^\tau a(e_h(0), e) - 2 \int_0^\tau a(e_p, e).
\end{aligned} \tag{3.4.4}$$

Estimates for the terms on the right are now required. We use Young's inequality (see Appendix D) and denote $\max_{t \in [0, T]}$ by $\max$.

$$\begin{aligned}
\left| 2 \int_0^\tau c(e'_h(0), e) \right| & \leq 2 \int_0^\tau |c(e'_h(0), e)| \\
& \leq 2 \int_0^\tau \|e'_h(0)\|_W \|e\|_W \\
& \leq 2T \max \|e(t)\|_W \|e'_h(0)\|_W \\
& \leq \frac{1}{9} [\max \|e(t)\|_W]^2 + 9T^2 \|e'_h(0)\|_W^2.
\end{aligned}$$

Similarly

$$\begin{aligned} \left| 2 \int_0^\tau a(e_h(0), e) \right| &\leq 2K_a T \max \|e(t)\|_W \|e_h(0)\|_W \\ &\leq \frac{1}{9} [\max \|e(t)\|_W]^2 + 9K_a^2 T^2 \|e_h(0)\|_W^2, \end{aligned}$$

using also E4W. Next

$$\begin{aligned} \left| 2 \int_0^\tau c(e'_p, e) \right| &\leq 2 \max \|e(t)\|_W \int_0^\tau \|e'_p\|_W \\ &\leq \frac{1}{9} [\max \|e(t)\|_W]^2 + \left(3 \int_0^\tau \|e'_p\|_W \right)^2 \end{aligned}$$

and

$$\begin{aligned} \left| 2 \int_0^\tau a(e_p, e) \right| &\leq 2K_a \max \|e(t)\|_W \int_0^\tau \|e_p\|_W \\ &\leq \frac{1}{9} [\max \|e(t)\|_W]^2 + \left(3K_a \int_0^\tau \|e_p\|_W \right)^2. \end{aligned}$$

Using these estimates in (3.4.4), we obtain

$$\begin{aligned} \|e(\tau)\|_W^2 &\leq \frac{4}{9} \max \|e(t)\|_W^2 + \|e(0)\|_W^2 + 9T^2 \|e'_h(0)\|_W^2 \\ &\quad + 9 \int_0^T \|e'_p(t)\|_W^2 + 9K_a^2 T^2 \|e_h(0)\|_W^2 + 9K_a^2 \int_0^T \|e_p(t)\|_W^2. \end{aligned}$$

Since this is true for all $\tau \in [0, T]$, we have that

$$\begin{aligned} \max \|e(t)\|_W^2 &\leq \frac{1}{2} \max \|e(t)\|_W^2 + \|e(0)\|_W^2 + 9T^2 \|e'_h(0)\|_W^2 \\ &\quad + 9 \int_0^T \|e'_p(t)\|_W^2 + 9K_a^2 T^2 \|e_h(0)\|_W^2 + 9K_a^2 \int_0^T \|e_p(t)\|_W^2, \end{aligned}$$

and hence

$$\begin{aligned} \max \|e(t)\|_W^2 &\leq 2 \left(\|e(0)\|_W^2 + 9T^2 \|e'_h(0)\|_W^2 + 9 \int_0^T \|e'_p(t)\|_W^2 \right. \\ &\quad \left. + 9K_a^2 T^2 \|e_h(0)\|_W^2 + 9K_a^2 \int_0^T \|e_p(t)\|_W^2 \right) \\ &\leq 2 \left(\|e(0)\|_W + 3T \|e'_h(0)\|_W + 3 \int_0^T \|e'_p(t)\|_W \right. \\ &\quad \left. + 3K_a T \|e_h(0)\|_W + 3K_a \int_0^T \|e_p(t)\|_W \right)^2. \end{aligned}$$

□

For completeness, we discuss how Baker's results can be obtained.

Suppose that there is no damping, i.e. $a = 0$ in (3.4.4). Then, using the same arguments as in the proof above, we have the following corollary.

Corollary 3.4.1. *If $a = 0$ in Problem G, then*

$$\|e(t)\|_W \leq \sqrt{2}\|e(0)\|_W + 2T\|e'_h(0)\|_W + 4\sqrt{T} \max_{t \in [0, T]} \|e'_p(t)\|_W,$$

for $t \in [0, T]$.

Now, consider the undamped case with the additional assumption (made in [Bak76]) that

$$c(u_1 - u_1^h, v(0)) = c(e'_h(0), v(0)) = 0.$$

The estimate in (3.4.3) reduces to

$$\frac{1}{2} c(e(\tau), e(\tau)) \leq \frac{1}{2} c(e(0), e(0)) - \int_0^\tau c(e'_p, e)$$

and we have the following result.

Corollary 3.4.2. *Consider Problem G with $a = 0$. If $c(u_1 - u_1^h, v) = 0$ for all $v \in S^h$, then*

$$\|e(t)\|_W \leq \sqrt{2}\|e(0)\|_W + 2\sqrt{T} \max_{t \in [0, T]} \|e'_p(t)\|_W,$$

for $t \in [0, T]$.

This is a generalization of the estimate for $e = Pu - u_h$ in (3.13) in [Bak76]. However, it is not necessary to make the assumption at this stage.

3.4.2 Convergence and error estimates for the semi-discrete approximation

The error estimates follow readily from Lemma 3.4.1.

Lemma 3.4.2. *Suppose the damping term satisfies Assumption E4W and assume that $u'(t) \in \mathcal{L}^2([0, T], H(V, k))$. If u is the solution of Problem G*

then

$$\begin{aligned}
\|u(t) - u_h(t)\|_W \leq & \|u(t) - Pu(t)\|_W + \sqrt{2}(\|Pu_0 - u_0\|_W + 3T\|u_1 - u_1^h\|_W \\
& + 3 \int_0^T \|u'(t) - Pu'(t)\|_W + (1 + 3K_a T)\|u_0 - u_0^h\|_W \\
& + 3K_a \int_0^T \|u(t) - Pu(t)\|_W) , \tag{3.4.5}
\end{aligned}$$

for each $t \in [0, T]$.

Proof. From (2.3.3), $\|u(t) - u_h(t)\|_W \leq \|e_p(t)\|_W + \|e(t)\|_W$. The result follows from Lemma 3.4.1 since $\|Pu_0 - u_0^h\|_W \leq \|Pu_0 - u_0\|_W + \|u_0 - u_0^h\|_W$. $\square$

In applications it is necessary to consider the errors on the right hand side of the Estimate (3.4.5).

To formulate an assumption regarding the error when an element of V is approximated by an element of S^h , we suppose h is a parameter related to the dimension n of S^h and $h \rightarrow 0$ as $n \rightarrow \infty$.

Using Assumption A3, estimates for the projection errors in (3.4.5) can be obtained. For example, $\|u'(t) - Pu'(t)\|_W \leq C_b \|u'(t) - Pu'(t)\|_V$ by Assumption E2. Therefore, by Proposition 2.3.1

$$\|u'(t) - Pu'(t)\|_W \leq C_b C_\Pi h^\alpha \|u'(t)\|_{H(V,k)}.$$

Theorem 3.4.1. *Assume Assumption A3 holds for the space V and the damping term satisfies Assumption E4 W. Assume also that $u'(t) \in \mathcal{L}^2([0, T], H(V, k))$. Then, for any $t \in [0, T]$,*

$$\begin{aligned}
\|u(t) - u_h(t)\|_W \leq & C_b C_\Pi h^\alpha \|u(t)\|_{H(V,k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V,k)} \right. \\
& \left. + 3K_a \int_0^T \|u(t)\|_{H(V,k)} + \|u_0\|_{H(V,k)} \right) \\
& + \sqrt{2} (3T\|u_1 - u_1^h\|_W + (1 + 3K_a T)\|u_0 - u_0^h\|_W) .
\end{aligned}$$

Proof. We use Lemma 3.4.2. The result follows from Assumption E2 and Assumption A3. $\square$

Initial conditions

Corollary 3.4.3. *Suppose u_0 and u_1 are in $H(V, k)$, Assumption A3 holds and $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem G satisfies $u'(t) \in \mathcal{L}^2([0, T], H(V, k))$, then*

$$\begin{aligned} & \|u(t) - u_h(t)\|_W \\ & \leq C_b C_\Pi h^\alpha \|u(t)\|_{H(V, k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V, k)} \right. \\ & \quad \left. + 3K_a \int_0^T \|u(t)\|_{H(V, k)} + (2 + 3K_a T) \|u_0\|_{H(V, k)} + 3T \|u_1\|_{H(V, k)} \right), \end{aligned}$$

for each $t \in [0, T]$.

Proof. The result follows from Theorem 3.4.1. □

For a discussion on initial conditions see Section 3.2.

Higher order estimates

The value of Baker's theory lies in the fact that convergence is proved for a weak solution. It is therefore questionable whether one should try to obtain higher order estimates for the error. The Aubin-Nitsche trick requires spatial regularity which is unlikely.

To obtain higher order estimates there are two possibilities. One possibility is to take u_0^h and u_1^h to be projections of u_0 and u_1 with respect to the inner product c , see e.g. [GSS06] and [Kar12]. The other option is to make an assumption on the initial errors, see Subsection 3.2.3.

Chapter 4

Fully discrete approximation

4.1 Introduction

Recall that the interval $[0, T]$ is divided into N steps of length $\tau = T/N$ and the approximation of $u_h(t_k)$ is denoted by u_k^h . Our aim is to estimate the difference between the solution of Problem G and the fully discrete approximation

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h].$$

Recall that we derived error estimates for the semi-discrete approximation in Chapter 3 and then estimates for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm in this chapter. The final estimate follows trivially by the triangle inequality. This approach has the advantage that it is not necessary to assume the existence of a third or fourth order derivative for the exact solution. The convergence analysis for the fully discrete approximation is simplified as well.

We consider two algorithms for the fully discrete approximation. In one algorithm we use central differences, formulated in Problem G^h -D-Central. It is conditionally stable but has the advantage that it is “almost” explicit (see Section 4.4).

The other algorithm is due to Baker [Bak76]. It is unconditionally stable but has the disadvantage that it is implicit.

4.2 An implicit fully discrete scheme for weak damping

In this section we introduce Baker's algorithm. We show that it can also be used for weak damping. Also, we generalize the proofs in [Bak76] and include the damping term.

The results of this section are completely general; we consider spaces and norms as in Section 2.2. One additional assumption is made, i.e. Assumption E4W. Note that there is no damping in [Bak76].

4.2.1 Baker's fully discrete Galerkin scheme

Notation For any sequence $\{x_k\} \subset R_n$:

$$\delta_t x_k = \tau^{-1}[x_{k+1} - x_k],$$

$$x_{k+\frac{1}{2}} = \frac{1}{2}[x_{k+1} + x_k].$$

Problem G^h -D-Baker

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \dots, N-1$,

$$\delta_t u_k^h = v_{k+\frac{1}{2}}, \tag{4.2.1}$$

$$c(\delta_t v_k, \psi) + a(v_{k+\frac{1}{2}}, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) = \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \tag{4.2.2}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Adapting the proof in [Bak76], we prove that the solution of Problem G^h -D is well defined. In the process an algorithm is derived. No confusion can arise if we write u_k for u_k^h .

Proposition 4.2.1. *Problem G^h -D-Baker has a unique solution for any pair of vectors d^h and v^h in S^h .*

Proof. For any $\psi \in S^h$,

$$\begin{aligned} \frac{\tau}{2}b(u_{k+1} + u_k, \psi) &= \frac{\tau^2}{2}b(\tau^{-1}[u_{k+1} - u_k], \psi) + \frac{\tau}{2}b(u_k, \psi) + \frac{\tau}{2}b(u_k, \psi) \\ &= \frac{\tau^2}{2}b(\frac{1}{2}[v_{k+1} + v_k], \psi) + \tau b(u_k, \psi) \\ &= \frac{\tau^2}{4}b(v_{k+1}, \psi) + \frac{\tau^2}{4}b(v_k, \psi) + \tau b(u_k, \psi). \end{aligned}$$

Substitution into (4.2.2) yields

$$\begin{aligned} c(v_{k+1}, \psi) - c(v_k, \psi) + \frac{\tau}{2}a(v_{k+1} + v_k, \psi) + \frac{\tau^2}{4}b(v_{k+1}, \psi) + \frac{\tau^2}{4}b(v_k, \psi) \\ + \tau b(u_k, \psi) = \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X \end{aligned}$$

or

$$\begin{aligned} c(v_{k+1}, \psi) + \frac{\tau}{2}a(v_{k+1}, \psi) + \frac{\tau^2}{4}b(v_{k+1}, \psi) \\ = c(v_k, \psi) - \frac{\tau}{2}a(v_k, \psi) - \frac{\tau^2}{4}b(v_k, \psi) - \tau b(u_k, \psi) + \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X. \end{aligned} \quad (4.2.3)$$

for any $\psi \in S^h$. The bilinear forms b and c are positive definite and as a consequence $c + \frac{\tau}{2}a + \frac{\tau^2}{4}b$ is positive definite. Therefore v_{k+1} is uniquely determined by (4.2.3) and $u_{k+1} = u_k + \tau v_{k+\frac{1}{2}}$ from (4.2.1). $\square$

The following algorithm may be used.

Problem FD

Find a sequence $\{\bar{u}_k\} \subset R_n$ such that for each k ,

$$\bar{u}_{k+1} = \bar{u}_k + \tau \bar{v}_{k+\frac{1}{2}},$$

$$\left(M + \frac{\tau}{2}C + \frac{\tau^2}{4}K\right) \bar{v}_{k+1} = \left(M - \frac{\tau}{2}C - \frac{\tau^2}{4}K\right) \bar{v}_k - \tau K \bar{u}_k + \frac{\tau}{2}[F(t_k) + F(t_{k+1})]$$

with $\bar{u}_0 = \bar{d}$ and $\bar{v}_0 = \bar{v}$.

Truncation

Consider Problem G^h -D-Baker. Substituting $t = t_k$ and $t = t_{k+1}$ respectively into Problem G^h (Equation (2.5.1)), we obtain

$$\begin{aligned} c(\tau^{-1}[v_h(t_{k+1}) - v_h(t_k)], \psi) + \frac{1}{2}a([v_h(t_{k+1}) + v_h(t_k)], \psi) \\ + \frac{1}{2}b([u_h(t_{k+1}) + u_h(t_k)], \psi) = \frac{1}{2}([f(t_{k+1}) + f(t_k)], \psi)_X + c(\rho_k, \psi) \end{aligned} \quad (4.2.4)$$

where $v_h(t) = u'_h(t)$ and

$$\rho_k = \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - \frac{1}{2}[v'_h(t_{k+1}) + v'_h(t_k)].$$

We denote the errors by e_k and q_k :

$$e_k = u_h(t_k) - u_k \quad \text{and} \quad q_k = u'_h(t_k) - v_k.$$

Note that in our approach $e_0 = 0$ and $q_0 = 0$.

By subtracting (4.2.2) from (4.2.4) we obtain

$$c(\tau^{-1}[q_{k+1} - q_k], \psi) + a(\frac{1}{2}[q_{k+1} + q_k], \psi) + b(\frac{1}{2}[e_{k+1} + e_k], \psi) = c(\rho_k, \psi),$$

which can be rewritten as follows

$$c(\delta_t q_k, \psi) + a(q_{k+\frac{1}{2}}, \psi) + b(e_{k+\frac{1}{2}}, \psi) = c(\rho_k, \psi). \quad (4.2.5)$$

Substitution of u_h and v_h into (4.2.1) yields

$$\tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] = \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)] + \sigma_k), \quad (4.2.6)$$

where (obviously)

$$\sigma_k = \tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] - \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)]).$$

It follows from (4.2.1) and (4.2.6) that

$$\delta_t e_k = q_{k+\frac{1}{2}} + \sigma_k. \quad (4.2.7)$$

4.2.2 Stability

To obtain our general result, we adapt the proof in [Bak76] (which is for the wave equation).

Consider any integer $n > 0$. It follows from (4.2.7) that

$$e_n = \tau \sum_{k=0}^{n-1} \delta_t e_k = \tau \sum_{k=0}^{n-1} q_{k+\frac{1}{2}} + \tau \sum_{k=0}^{n-1} \sigma_k.$$

Consequently

$$\begin{aligned} e_{n+\frac{1}{2}} &= \frac{1}{2}(e_{n+1} + e_n) \\ &= \frac{\tau}{2} \left(\sum_{k=0}^n q_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} q_{k+\frac{1}{2}} \right) + \frac{\tau}{2} \left(\sum_{k=0}^n \sigma_k + \sum_{k=0}^{n-1} \sigma_k \right). \end{aligned} \quad (4.2.8)$$

But from (4.2.7) we also have

$$\begin{aligned} \delta_t e_n &= \frac{1}{2}[q_{n+1} + q_n] + \sigma_n \\ &= \frac{\tau}{2} \left[\sum_{k=0}^n \delta_t q_k + \sum_{k=0}^{n-1} \delta_t q_k \right] + \sigma_n. \end{aligned} \quad (4.2.9)$$

Let $s_0 = 0$ and $s_n = \tau \sum_{k=0}^{n-1} e_{k+\frac{1}{2}}$, then

$$s_{n+\frac{1}{2}} = \frac{\tau}{2} \left[\sum_{k=0}^n e_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} e_{k+\frac{1}{2}} \right]. \quad (4.2.10)$$

In the analysis that follows, ψ denotes an arbitrary element of S^h . Using (4.2.8), (4.2.9) and (4.2.10), we have for $n = 1, 2, \dots, N-1$

$$\begin{aligned} &c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) \\ &= \frac{\tau}{2} c \left(\left[\sum_{k=0}^n \delta_t q_k + \sum_{k=0}^{n-1} \delta_t q_k \right], \psi \right) + c(\sigma_n, \psi) \\ &\quad + \frac{\tau}{2} a \left(\left[\sum_{k=0}^n q_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} q_{k+\frac{1}{2}} \right], \psi \right) + \frac{\tau}{2} b \left(\left[\sum_{k=0}^n e_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} e_{k+\frac{1}{2}} \right], \psi \right). \end{aligned}$$

Using (4.2.5), it follows that

$$c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) = \frac{\tau}{2} c \left(\left[\sum_{k=0}^n \rho_k + \sum_{k=0}^{n-1} \rho_k \right], \psi \right) + c(\sigma_n, \psi).$$

Let

$$\epsilon_n = \frac{\tau}{2} \rho_n + \tau \sum_{k=0}^{n-1} \rho_k + \sigma_n \quad \text{for } n = 1, 2, \dots, N-1, \quad (4.2.11)$$

then

$$c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) = c(\epsilon_n, \psi).$$

Choose $\psi = e_{n+\frac{1}{2}}$, then

$$\begin{aligned} \|e_{n+1}\|^2 - \|e_n\|^2 + 2\tau a(e_{n+\frac{1}{2}}, e_{n+\frac{1}{2}}) + \|s_{n+1}\|_V^2 - \|s_n\|_V^2 \\ = 2\tau c(\epsilon_n, e_{n+\frac{1}{2}}). \end{aligned} \quad (4.2.12)$$

Consider any integer ν with $2 \leq \nu \leq N$. Using (4.2.12) and the fact that $a(e_{n+\frac{1}{2}}, e_{n+\frac{1}{2}}) \geq 0$, we have

$$\sum_{n=1}^{\nu-1} [\|e_{n+1}\|^2 - \|e_n\|^2 + \|s_{n+1}\|_V^2 - \|s_n\|_V^2] \leq 2\tau \sum_{n=1}^{\nu-1} c(\epsilon_n, e_{n+\frac{1}{2}}).$$

It follows that

$$\begin{aligned} \|e_\nu\|_W^2 - \|e_1\|_W^2 + \|s_\nu\|_V^2 - \|s_1\|_V^2 \\ \leq 4T\tau \sum_{n=1}^{\nu-1} \|\epsilon_n\|_W^2 + \frac{\tau}{4T} \sum_{n=1}^{\nu-1} \|e_{n+\frac{1}{2}}\|_W^2, \end{aligned} \quad (4.2.13)$$

using the Cauchy-Schwartz inequality and Young's inequality.

Now, consider $n = 0$. Let $k = 0$ in (4.2.5):

$$c(\delta_t q_0, \psi) + a(q_{\frac{1}{2}}, \psi) + b(e_{\frac{1}{2}}, \psi) = c(\rho_0, \psi). \quad (4.2.14)$$

Also, let $k = 0$ in (4.2.7):

$$\delta_t e_0 = q_{\frac{1}{2}} + \sigma_0.$$

Since $q_0 = 0$ and $e_0 = 0$ we have $\delta_t e_0 = \frac{e_1}{\tau}$, $e_{\frac{1}{2}} = \frac{e_1}{2}$, $\delta_t q_0 = \frac{q_1}{\tau}$ and $q_{\frac{1}{2}} = \frac{q_1}{2}$ and hence

$$q_1 = \frac{2}{\tau} e_1 - 2\sigma_0.$$

Making the appropriate substitutions into (4.2.14) we obtain

$$\begin{aligned} \frac{2}{\tau^2} c(e_1, \psi) + \frac{1}{\tau} a(e_1, \psi) + \frac{1}{2} b(e_1, \psi) \\ = c(\rho_0, \psi) + \frac{2}{\tau} c(\sigma_0, \psi) + a(\sigma_0, \psi). \end{aligned} \quad (4.2.15)$$

From the definition of s_0 and s_1 , we have $e_1 = \frac{2}{\tau} s_1$. Substituting this into the bilinear form b in (4.2.15) (and multiplying by $\frac{\tau^2}{2}$) yields

$$c(e_1, \psi) + \frac{\tau}{2} a(e_1, \psi) + \frac{\tau}{2} b(s_1, \psi) = \frac{\tau^2}{2} c(\rho_0, \psi) + \tau c(\sigma_0, \psi) + \frac{\tau^2}{2} a(\sigma_0, \psi).$$

Choose $\psi = e_1$, then

$$\begin{aligned} \|e_1\|_W^2 + \|s_1\|_V^2 + \frac{\tau}{2} a(e_1, e_1) \\ = \frac{\tau^2}{2} c(\rho_0, e_1) + \tau c(\sigma_0, e_1) + \frac{\tau^2}{2} a(\sigma_0, e_1). \end{aligned}$$

Since $a(e_1, e_1) \geq 0$,

$$\begin{aligned} \|e_1\|_W^2 + \|s_1\|_V^2 &\leq \frac{\tau^2}{2} \|\rho_0\|_W \|e_1\|_W + \tau \|\sigma_0\|_W \|e_1\|_W + \frac{\tau^2}{2} K_a \|\sigma_0\|_W \|e_1\|_W \\ &\leq \tau^4 \|\rho_0\|_W^2 + \frac{1}{16} \|e_1\|_W^2 + 4\tau^2 \|\sigma_0\|_W^2 + \frac{1}{16} \|e_1\|_W^2 \\ &\quad + \tau^4 K_a^2 \|\sigma_0\|_W^2 + \frac{1}{16} \|e_1\|_W^2 \\ &= \frac{3}{16} \|e_1\|_W^2 + \tau^4 \|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a) \|\sigma_0\|_W^2, \quad (4.2.16) \end{aligned}$$

using the Cauchy-Schwartz and Young's inequalities.

To proceed, we combine (4.2.13) and (4.2.16):

$$\begin{aligned} \|e_\nu\|_W^2 &\leq 4T\tau \sum_{n=1}^{\nu-1} \|\epsilon_n\|_W^2 + \frac{\tau}{4T} \sum_{n=1}^{\nu-1} \|e_{n+\frac{1}{2}}\|_W^2 \\ &\quad + \frac{3}{16} \|e_1\|_W^2 + \tau^4 \|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a) \|\sigma_0\|_W^2 \\ &\leq 4T\tau \sum_{n=1}^{\nu-1} \|\epsilon_n\|_W^2 + \frac{1}{2} \max \|e_n\|_W^2 + \tau^4 \|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a) \|\sigma_0\|_W^2. \end{aligned}$$

The stability result follows.

Lemma 4.2.1. *Stability*

$$\max \|e_n\|_W^2 \leq 8T\tau \sum_{n=0}^{N-1} \|\epsilon_n\|_W^2 + 2\tau^4 \|\rho_0\|_W^2 + (8\tau^2 + 2\tau^4 K_a) \|\sigma_0\|_W^2.$$

4.2.3 Error estimates

In this subsection we derive an estimate for the error $u_h(t_k) - u_k^h$. Recall that it must be with respect to $\|\cdot\|_W$. Usually the error $u(t_k) - u_k^h$ is estimated directly, see e.g. [Dup73], [Bak76], [OR76], [Sem94], [FXX99] and [Wu03]. To do this, the assumption is made that the exact solution has derivatives $u^{(k)} \in \mathcal{L}^2([0, T], V)$ for $k \leq 4$ or $k \leq 3$. This assumption is very restrictive.

It remains to find estimates for ρ_k and σ_k . To find an estimate for ρ_k , we consider the following two parts separately

$$\begin{aligned}\rho_A &= \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - v'_h(t_k + \frac{\tau}{2}) \quad \text{and} \\ \rho_B &= -\frac{1}{2}[v'_h(t_{k+1}) + v'_h(t_k)] + v'_h(t_k + \frac{\tau}{2}).\end{aligned}$$

Recall that $v_h(t_k) = \sum_{i=1}^n v_i(t_k)\phi_i$. Using Taylor expansions we obtain

$$\begin{aligned}&\tau^{-1}[v_i(t_{k+1}) - v_i(t_k)] - v'_i(t_k + \frac{\tau}{2}) \\ &= \frac{1}{4\tau} \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^2 v_i'''(\theta) d\theta + \frac{1}{4\tau} \int_{t_k}^{t_k + \frac{\tau}{2}} (t_k - \theta)^2 v_i'''(\theta) d\theta.\end{aligned}$$

From the Cauchy-Schwartz inequality

$$\begin{aligned}\left(\int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^2 v_i'''(\theta) d\theta \right)^2 &\leq \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^4 d\theta \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2 d\theta \\ &= \frac{\tau^5}{160} \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2 d\theta \quad \text{and} \\ \left(\int_{t_k}^{t_k + \frac{\tau}{2}} (t_k - \theta)^2 v_i'''(\theta) d\theta \right)^2 &\leq \frac{\tau^5}{160} \int_{t_k}^{t_k + \frac{\tau}{2}} (v_i'''(\theta))^2 d\theta.\end{aligned}$$

Using the estimate above and assuming that $\{\phi_j\}$ is orthonormal, we have

$$\begin{aligned}
\|\rho_A\|_W^2 &= \left\| \tau^{-1} [v_h(t_{k+1}) - v_h(t_k)] - v'_h(t_k + \frac{\tau}{2}) \right\|_W^2 \\
&\leq \sum_{i=1}^n 2 \left(\frac{1}{4\tau} \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^2 v_i'''(\theta) d\theta \right)^2 \\
&\quad + \sum_{i=1}^n 2 \left(\frac{1}{4\tau} \int_{t_k}^{t_k + \frac{\tau}{2}} (t_k - \theta)^2 v_i'''(\theta) d\theta \right)^2 \\
&\leq \frac{\tau^3}{1280} \left[\int_{t_k + \frac{\tau}{2}}^{t_{k+1}} \|v_h'''(\theta)\|_W^2 d\theta + \int_{t_k}^{t_k + \frac{\tau}{2}} \|v_h'''(\theta)\|_W^2 d\theta \right] \\
&\leq \frac{\tau^4}{1280} \max \|v_h'''\|_W^2,
\end{aligned}$$

where $\max_{\theta \in [0, T]}$ is denoted by $\max$.

To find an estimate for ρ_B , the same steps are followed.

From Taylor expansions

$$\begin{aligned}
&\frac{1}{2} [v'_i(t_{k+1}) + v'_i(t_k)] - v'_i(t_k + \frac{\tau}{2}) \\
&= \frac{1}{2} \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta) v_i'''(\theta) d\theta - \frac{1}{2} \int_{t_k}^{t_k + \frac{\tau}{2}} (t_k - \theta) v_i'''(\theta) d\theta.
\end{aligned}$$

Using the Cauchy-Schwartz inequality again, we obtain

$$\begin{aligned}
\left(\int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta) v_i'''(\theta) d\theta \right)^2 &\leq \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (t_{k+1} - \theta)^2 d\theta \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2 d\theta \\
&= \frac{\tau^3}{24} \int_{t_k + \frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2 d\theta \quad \text{and} \\
\left(\int_{t_k}^{t_k + \frac{\tau}{2}} (t_k - \theta)^2 v_i'''(\theta) d\theta \right)^2 &\leq \frac{\tau^3}{24} \int_{t_k}^{t_k + \frac{\tau}{2}} (v_i'''(\theta))^2 d\theta.
\end{aligned}$$

Therefore

$$\|\rho_B\|_W^2 \leq \frac{\tau^4}{48} \max \|v_h'''\|_W^2.$$

Combining the estimates for ρ_A and ρ_B yields

$$\|\rho_k\|_W^2 \leq \tau^4 \max \|v_h'''\|_W^2. \quad (4.2.17)$$

Now consider $\sigma_k = \tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] - \frac{1}{2}[u_h'(t_{k+1}) + u_h'(t_k)]$. Following the same method as for ρ_k , we obtain

$$\|\sigma_k\|_W^2 \leq \tau^4 \max \|u_h'''\|_W^2. \quad (4.2.18)$$

The following estimate for ϵ_n (see (4.2.11)) is obtained from (4.2.17) and (4.2.18).

$$\|\epsilon_n\|_W^2 \leq 5T^2\tau^4 \max \|v_h'''\|_W^2 + 4\tau^4 \max \|u_h'''\|_W^2.$$

Using the result in Lemma 4.2.1 yields

$$\begin{aligned} \max \|e_n\|_W^2 &\leq 40T^4\tau^4 \max \|v_h'''\|_W^2 + 32T^2\tau^4 \max \|u_h'''\|_W^2 \\ &\quad + 2\tau^8 \max \|v_h'''\|_W^2 + 8\tau^6 \max \|u_h'''\|_W^2 + 2\tau^8 K_a \max \|u_h'''\|_W^2. \end{aligned}$$

Note that if $f \in C^2([0, T], X)$, then $u_h \in C^4([0, T], W)$ and consequently the following error estimate is obtained.

Theorem 4.2.1. *Error estimate.*

If $f \in C^2([0, T], X)$, then

$$\begin{aligned} \|u_h(t_k) - u_k^h\|_W &\leq 7T^2\tau^2 \max \|u_h^{(4)}\|_W + 7T\tau^2 \max \|u_h'''\|_W \\ &\quad + \sqrt{2K_a}\tau^4 \max \|u_h'''\|_W \end{aligned} \quad (4.2.19)$$

for each $t_k \in (0, T)$.

4.3 Convergence of the fully discrete approximation for weak damping

Error estimates for the fully discrete approximation of the solution of Problem G is obtained by combining the error estimate for the semi-discrete approximation from Section 3.4 with the error estimate obtained in the previous section.

Recall that we assume Assumptions E1, E2, E3 and E4W hold and we may use the properties of a solution in Theorem 2.2.2.

From Corollary 3.4.3, if $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, we have that for weak damping

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_b C_\Pi h^\alpha \|u(t)\|_{H(V,k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V,k)} \right. \\ &\quad \left. + 3K_a \int_0^T \|u(t)\|_{H(V,k)} + (2 + 3K_a T) \|u_0\|_{H(V,k)} + 3T \|u_1\|_{H(V,k)} \right), \end{aligned}$$

for each $t \in [0, T]$.

From Theorem 4.2.1, if $f \in C^2([0, T], X)$, then

$$\begin{aligned} \|u_h(t_k) - u_k^h\|_W &\leq 7T^2 \tau^2 \max \|u_h^{(4)}\|_W + 7T \tau^2 \max \|u_h'''\|_W \\ &\quad + \sqrt{2K_a} \tau^4 \max \|u_h'''\|_W \end{aligned}$$

for each $t_k \in (0, T)$.

We can combine these results using the triangle inequality:

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h].$$

Therefore we have the following result.

Theorem 4.3.1. *Suppose*

- (a) *the damping term satisfies Assumption E4W (weak damping),*
- (b) *Assumption A3 holds for the space V ,*
- (c) *$u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$,*
- (d) *$u'(t) \in \mathcal{L}^2([0, T], H(V, k))$,*
- (e) *$f \in C^2([0, T], X)$ and*
- (f) *the sequence $\{u_k^h\}$ is a solution of Problem G^h -D-Baker.*

Then

$$\begin{aligned}
& \|u(t_k) - u_k^h\|_W \\
& \leq C_b C_\Pi h^\alpha \|u(t)\|_{H(V,k)} + \sqrt{2} C_b C_\Pi h^\alpha \left(3 \int_0^T \|u'(t)\|_{H(V,k)} \right. \\
& \quad \left. + 3K_a \int_0^T \|u(t)\|_{H(V,k)} + (2 + 3K_a T) \|u_0\|_{H(V,k)} + 3T \|u_1\|_{H(V,k)} \right) \\
& \quad + 7T^2 \tau^2 \max \|u_h^{(4)}\|_W + 7T \tau^2 \max \|u_h'''\|_W + \sqrt{2K_a} \tau^4 \max \|u_h'''\|_W,
\end{aligned}$$

for each $t_k \in (0, T)$.

Remarks

1. Estimates for $\|u_h'''\|_W$ and $\|u_h^{(4)}\|_W$ in terms of the data is available, see Proposition 2.5.3.
2. Other results from Chapter 3 can also be combined with Theorem 4.2.1. For the energy estimates the additional assumption, $u \in C^2([0, T], V)$ (Assumption A2), must be made.

4.4 Approximation with central differences for general damping

4.4.1 Central differences

We generalize the proofs in [OR76] and include the damping term. Note that the results of this section are completely general; we consider spaces and norms as in Section 2.2. Also, in the process of generalizing the proofs, we corrected some errors and added material for completeness.

Since our approach is abstract, one additional assumption is necessary.

Assumption I Inverse property of the subspace S^h

$$\|u\|_V \leq C_I h^{-\lambda} \|u\|_W. \quad (4.4.1)$$

Notation For any sequence $\{x_k\} \subset R_n$:

$$\delta_t x_k = \tau^{-1}[x_{k+1} - x_k],$$

$$x_{k+\frac{1}{2}} = \frac{1}{2}[x_{k+1} + x_k].$$

Problem G^h -D-Central

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) + b(u_k^h, v) \\ = (f(t_k), v)_X \end{aligned} \quad (4.4.2)$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

No confusion can arise if we write u_k for u_k^h .

We denote the error by e_k :

$$e_k = u_h(t_k) - u_k.$$

Truncation

Consider Problem G^h in Section 2.3. Substituting $t = t_k$ into Problem G^h (Equation (2.3.1)), we obtain

$$c(u_h''(t_k), v) + a(u_h'(t_k), v) + b(u_h(t_k), v) = (f(t_k), v)_X. \quad (4.4.3)$$

By subtracting (4.4.2) from (4.4.3), we obtain:

$$\begin{aligned} c(\tau^{-2}[e_{k+1} - 2e_k + e_{k-1}], v) + \frac{1}{2\tau} a(e_{k+1} - e_{k-1}, v) + b(e_k, v) \\ = c(\varepsilon_k, v) + a(\rho_k, v), \end{aligned} \quad (4.4.4)$$

where

$$\begin{aligned} \varepsilon_k &= \tau^{-2}[u_h(t_{k+1}) - 2u_h(t_k) + u_h(t_{k-1})] - u_h''(t_k) \quad \text{and} \\ \rho_k &= (2\tau)^{-1}[u_h(t_{k+1}) - u_h(t_{k-1})] - u_h'(t_k). \end{aligned}$$

Note that in our approach $e_0 = 0$. If $k = 0$, we have

$$c(u_h''(0) - \tau^{-2}[u_1 - 2u_0 + u_{-1}], v) = 0,$$

and this implies that

$$c(e_1 + e_{-1}, v) = \tau^2 c(\varepsilon_0, v).$$

Since $u_1 - u_{-1} = 2\tau u'_h(0)$,

$$c(e_1 - e_{-1}, v) = 2\tau c(\rho_0, v).$$

It follows that

$$c(e_1, v) = \frac{1}{2}\tau^2 c(\varepsilon_0, v) + \tau c(\rho_0, v).$$

Thus

$$\|e_1\|_W \leq \frac{1}{2}\tau^2 \|\varepsilon_0\| + \tau \|\rho_0\|.$$

4.4.2 Stability

In the following lemma it is shown that a sufficient condition for stability is

$$\left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda}\right) > 0. \quad (4.4.5)$$

Lemma 4.4.1. *If $\tau^2 h^{-\lambda} < C_I^{-1}$, then*

$$\begin{aligned} & R\|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2}\|e_N\|_V^2 \\ & \leq \left(1 + \frac{Te^{CT}}{R}\right) \left(2\|\tau^{-1}e_1\|_W^2 + \frac{1}{2}\|e_1\|_V^2 + \tau\|\varepsilon_1\|_W^2 + \tau \sum_{k=2}^{N-2} \|\varepsilon_k\|_W^2\right) \\ & \quad + \frac{\tau^2 e^{CT}}{R} \sum_{k=1}^{N-2} \|\varepsilon_k\|_W^2 + \tau\|\varepsilon_{N-1}\|_W^2, \end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda}\right)$.

Proof. Let $v = \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]$ in (4.4.4). It follows that

$$\begin{aligned} & c\left(\tau^{-2}[e_{k+1} - 2e_k + e_{k-1}], \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right) \\ & \quad + b\left(e_k, \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right) \\ & \quad = c\left(\varepsilon_k, \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]\right), \end{aligned}$$

since $a \geq 0$. But

$$\begin{aligned} b\left(e_k, \tau^{-1}[e_{k+1} - e_k]\right) &= -\frac{\tau}{2}b\left(\tau^{-1}[e_{k+1} - e_k], \tau^{-1}[e_{k+1} - e_k]\right) \\ & \quad + \frac{1}{2\tau}[b(e_{k+1}, e_{k+1}) - b(e_k, e_k)] \\ &= -\frac{\tau}{2}\|\tau^{-1}(e_{k+1} - e_k)\|_V^2 + \frac{1}{2\tau}[\|e_{k+1}\|_V^2 - \|e_k\|_V^2]. \end{aligned}$$

Similarly,

$$\begin{aligned}
b(e_k, \tau^{-1}[e_k - e_{k-1}]) &= \frac{\tau}{2} b(\tau^{-1}[e_k - e_{k-1}], \tau^{-1}[e_k - e_{k-1}]) \\
&\quad + \frac{1}{2\tau} [b(e_k, e_k) - b(e_{k-1}, e_{k-1})] \\
&= \frac{\tau}{2} \|\tau^{-1}(e_k - e_{k-1})\|_V^2 + \frac{1}{2\tau} [\|e_k\|_V^2 - \|e_{k-1}\|_V^2]
\end{aligned}$$

and

$$\begin{aligned}
c(\tau^{-2}[e_{k+1} - 2e_k + e_{k-1}], \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]) \\
= \frac{1}{\tau} [\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 - \|\tau^{-1}[e_k - e_{k-1}]\|_W^2].
\end{aligned}$$

Combining these results, we obtain the following

$$\begin{aligned}
&\frac{1}{\tau} [\|\tau^{-1}[e_{k+1} - e_k]\|_W^2 - \|\tau^{-1}[e_k - e_{k-1}]\|_W^2] - \frac{\tau}{2} \|\tau^{-1}(e_{k+1} - e_k)\|_V^2 \\
&+ \frac{\tau}{2} \|\tau^{-1}(e_k - e_{k-1})\|_V^2 + \frac{1}{2\tau} [\|e_{k+1}\|_V^2 - \|e_k\|_V^2] + \frac{1}{2\tau} [\|e_k\|_V^2 - \|e_{k-1}\|_V^2] \\
&= c(\varepsilon_k, \tau^{-1}[e_{k+1} - e_k] + \tau^{-1}[e_k - e_{k-1}]).
\end{aligned}$$

Summing from 1 to $n - 1 \leq N - 1$, we obtain

$$\begin{aligned}
&\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 - \frac{\tau^2}{2} \|\tau^{-1}(e_n - e_{n-1})\|_V^2 \\
&+ \frac{\tau^2}{2} \|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2} [\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2] \\
&\leq \tau c(\varepsilon_1, \tau^{-1}[e_1 - e_0]) + \tau c(\varepsilon_{n-1}, \tau^{-1}[e_n - e_{n-1}]) \\
&+ \tau \sum_{k=1}^{n-2} c(\varepsilon_k, \tau^{-1}[e_{k+1} - e_k]) + \tau \sum_{k=1}^{n-2} c(\varepsilon_{k+1}, \tau^{-1}[e_{k+1} - e_k]).
\end{aligned}$$

Using the Schwartz inequality and Young's inequality,

$$\begin{aligned}
&\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 - \frac{\tau^2}{2} \|\tau^{-1}(e_n - e_{n-1})\|_V^2 \\
&+ \frac{\tau^2}{2} \|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2} [\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2] \\
&\leq \frac{\tau}{2} (\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2) + \frac{\tau}{2\kappa} \|\varepsilon_{n-1}\|_W^2 + \frac{\tau\kappa}{2} \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 \\
&+ 2 \sum_{k=1}^{n-2} \frac{\tau}{2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2 + 2 \sum_{k=2}^{n-2} \frac{\tau}{2} \|\varepsilon_k\|_W^2 + \frac{\tau}{2} \|\varepsilon_1\|_W^2 + \frac{\tau}{2} \|\varepsilon_{n-1}\|_W^2.
\end{aligned}$$

Now choose κ such that $\frac{\tau\kappa}{2} = \frac{1}{2}$, then

$$\begin{aligned}
& \frac{1}{2} \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 - \frac{\tau^2}{2} \|\tau^{-1}(e_n - e_{n-1})\|_V^2 \\
& + \frac{\tau^2}{2} \|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2} [\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2] \\
& \leq \frac{\tau}{2} (\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2) + \frac{\tau^2}{2} \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=1}^{n-2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\
& + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \frac{\tau}{2} \|\varepsilon_1\|_W^2 + \frac{\tau}{2} \|\varepsilon_{n-1}\|_W^2.
\end{aligned} \tag{4.4.6}$$

From Assumption I

$$\|\tau^{-1}(e_n - e_{n-1})\|_V^2 \leq C_I h^{-\lambda} \|\tau^{-1}(e_n - e_{n-1})\|_W^2. \tag{4.4.7}$$

Substituting (4.4.7) into (4.4.6) we obtain

$$\begin{aligned}
& \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda} \right) \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 - \|\tau^{-1}[e_1 - e_0]\|_W^2 \\
& + \frac{\tau^2}{2} \|\tau^{-1}(e_1 - e_0)\|_V^2 + \frac{1}{2} [\|e_n\|_V^2 + \|e_{n-1}\|_V^2 - \|e_1\|_V^2 - \|e_0\|_V^2] \\
& \leq \frac{\tau}{2} (\|\varepsilon_1\|_W^2 + \|\tau^{-1}[e_1 - e_0]\|_W^2) + \frac{\tau^2}{2} \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=1}^{n-2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\
& + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \frac{\tau}{2} \|\varepsilon_1\|_W^2 + \frac{\tau}{2} \|\varepsilon_{n-1}\|_W^2,
\end{aligned} \tag{4.4.8}$$

which can be written as

$$\begin{aligned}
& R \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 \\
& \leq \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \frac{\tau}{2} (\|\varepsilon_1\|_W^2 + \|\tau^{-1}e_1\|_W^2) + \frac{\tau^2}{2} \|\varepsilon_{n-1}\|_W^2 \\
& + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \frac{\tau}{2} \|\varepsilon_1\|_W^2 + \frac{\tau}{2} \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=1}^{n-2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2,
\end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda} \right) > 0$.

Using the discrete Gronwall inequality (see Appendix D) we obtain

$$\|\tau^{-1}[e_n - e_{n-1}]\|_W^2 \leq R^{-1} K_n e^{CT},$$

where (using $\tau < 1$)

$$\begin{aligned}
K_n &= (1 + \frac{\tau}{2}) \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \tau \|\varepsilon_1\|_W^2 + \frac{\tau^2 + \tau}{2} \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 \\
&\leq 2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \tau \|\varepsilon_1\|_W^2 + \tau \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2. \quad (4.4.9)
\end{aligned}$$

Now, going back to (4.4.8) and rewriting it again we obtain

$$\begin{aligned}
&R \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 + \frac{1}{2} \|e_n\|_V^2 \\
&\leq \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \frac{\tau}{2} (\|\varepsilon_1\|_W^2 + \|\tau^{-1}e_1\|_W^2) + \frac{\tau^2}{2} \|\varepsilon_{n-1}\|_W^2 \\
&\quad + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \frac{\tau}{2} \|\varepsilon_1\|_W^2 + \frac{\tau}{2} \|\varepsilon_{n-1}\|_W^2 + \tau \sum_{k=1}^{n-2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\
&\leq 2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \tau \|\varepsilon_1\|_W^2 + \tau \|\varepsilon_{n-1}\|_W^2 \\
&\quad + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \tau \sum_{k=1}^{n-2} \|\tau^{-1}[e_{k+1} - e_k]\|_W^2 \\
&\leq 2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \tau \|\varepsilon_1\|_W^2 + \tau \|\varepsilon_{n-1}\|_W^2 \\
&\quad + \tau \sum_{k=2}^{n-2} \|\varepsilon_k\|_W^2 + \frac{\tau e^{CT}}{R} \sum_{k=1}^{n-2} K_{k+1}.
\end{aligned}$$

Using the fact that $\tau N = T$ and substituting (4.4.9), we find that

$$\begin{aligned}
&R \|\tau^{-1}[e_n - e_{n-1}]\|_W^2 + \frac{1}{2} \|e_n\|_V^2 \\
&\leq \left(1 + \frac{T e^{CT}}{R}\right) \left(2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_V^2 + \tau \|\varepsilon_1\|_W^2 + \tau \sum_{k=2}^{N-2} \|\varepsilon_k\|_W^2\right) \\
&\quad + \frac{\tau^2 e^{CT}}{R} \sum_{k=1}^{N-2} \|\varepsilon_k\|_W^2 + \tau \|\varepsilon_{n-1}\|_W^2.
\end{aligned}$$

□

Remark In [OR76] a general constant is used. That has the disadvantage of not indicating that the constant depends on time and could be large if T is large.

Corollary 4.4.1. *If $\tau^2 h^{-\lambda} < C_I^{-1}$, then*

$$\begin{aligned} & R \|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2} \|e_N\|_V^2 \\ & \leq \left(1 + \frac{Te^{CT}}{R}\right) \left(2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2\tau^4} \|e_1\|_W^2 + \tau \|\varepsilon_1\|_W^2 + \tau \sum_{k=2}^{N-2} \|\varepsilon_k\|_W^2\right) \\ & \quad + \frac{\tau^2 e^{CT}}{R} \sum_{k=1}^{N-2} \|\varepsilon_k\|_W^2 + \tau \|\varepsilon_{N-1}\|_W^2. \end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda}\right)$.

Proof. From Assumption I we have that $\|e_1\|_V^2 \leq C_I^2 h^{-2\lambda} \|e_1\|_W^2$, and therefore

$$\|e_1\|_V^2 \leq \tau^{-4} \|e_1\|_W^2,$$

since $\tau^2 h^{-\lambda} < C_I^{-1}$. □

Corollary 4.4.2. *If $\tau^2 h^{-\lambda} < C_I^{-1}$, then*

$$\begin{aligned} & R \|\tau^{-1}[e_N - e_{N-1}]\|_W^2 + \frac{1}{2} \|e_N\|_W^2 \\ & \leq \left(1 + \frac{Te^{CT}}{R}\right) \left(2 \|\tau^{-1}e_1\|_W^2 + \frac{1}{2} \|e_1\|_W^2 + \tau \|\varepsilon_1\|_W^2 + \tau \sum_{k=2}^{N-2} \|\varepsilon_k\|_W^2\right) \\ & \quad + \frac{\tau^2 e^{CT}}{R} \sum_{k=1}^{N-2} \|\varepsilon_k\|_W^2 + \tau \|\varepsilon_{N-1}\|_W^2. \end{aligned}$$

where $R = \left(\frac{1}{2} - \frac{\tau^2 C_I}{2h^\lambda}\right)$.

4.4.3 Error estimate

In this subsection t_k is often denoted by t to simplify the notation.

Consider $u_h(t) = \sum_{j=1}^n u_j(t) \phi_j$, where $\{\phi_j\}$ is an orthonormal basis. Using Taylor expansions on the real valued functions u_j we obtain the following.

$$u_j(t + \tau) = u_j(t) + \tau u_j'(t) + \frac{\tau^2}{2!} u_j''(t) + \frac{\tau^3}{3!} u_j'''(t) + \frac{1}{3!} \int_t^{t+\tau} (t + \tau - s)^3 u_j^{(4)}(s) ds \quad (4.4.10)$$

and

$$u_j(t - \tau) = u_j(t) - \tau u_j'(t) + \frac{\tau^2}{2!} u_j''(t) - \frac{\tau^3}{3!} u_j'''(t) + \frac{1}{3!} \int_t^{t-\tau} (t - \tau - s)^3 u_j^{(4)}(s) ds. \quad (4.4.11)$$

Adding (4.4.10) and (4.4.11) yields

$$\begin{aligned} & \tau^{-2} (u_j(t + \tau) - 2u_j(t) + u_j(t - \tau)) - u_j''(t) \\ &= \frac{1}{6\tau^2} \left(\int_t^{t+\tau} (t + \tau - s)^3 u_j^{(4)}(s) ds - \int_{t-\tau}^t (t - \tau - s)^3 u_j^{(4)}(s) ds \right) \\ &= \varepsilon_{k,j}. \end{aligned}$$

Note that $\varepsilon_k = \sum_{j=1}^n \varepsilon_{k,j} \phi_j$, therefore

$$\begin{aligned} \varepsilon_k &= \frac{1}{6\tau^2} \sum_{j=1}^n \left(\int_t^{t+\tau} (t + \tau - s)^3 u_j^{(4)}(s) ds - \int_{t-\tau}^t (t - \tau - s)^3 u_j^{(4)}(s) ds \right) \phi_j \\ &= \frac{1}{6\tau^2} \int_t^{t+\tau} (t + \tau - s)^3 \sum_{j=1}^n u_j^{(4)}(s) \phi_j ds - \frac{1}{6\tau^2} \int_{t-\tau}^t (t - \tau - s)^3 \sum_{j=1}^n u_j^{(4)}(s) \phi_j ds \\ &= \frac{1}{6\tau^2} \left(\int_t^{t+\tau} (t + \tau - s)^3 u_h^{(4)}(s) ds - \int_{t-\tau}^t (t - \tau - s)^3 u_h^{(4)}(s) ds \right). \end{aligned}$$

Using the above, we obtain

$$\begin{aligned}
\|\varepsilon_k\|^2 &\leq \left[\frac{1}{6\tau^2} \left\| \int_t^{t+\tau} (t+\tau-s)^3 u_h^{(4)}(s) ds + \int_{t-\tau}^t (t-\tau-s)^3 u_h^{(4)}(s) ds \right\| \right]^2 \\
&\leq \frac{1}{36\tau^4} \left[\left\| \int_t^{t+\tau} (t+\tau-s)^3 u_h^{(4)}(s) ds \right\| + \left\| \int_{t-\tau}^t (t-\tau-s)^3 u_h^{(4)}(s) ds \right\| \right]^2 \\
&\leq \frac{1}{36\tau^4} \left[\int_t^{t+\tau} \left\| (t+\tau-s)^3 u_h^{(4)}(s) \right\| ds + \int_{t-\tau}^t \left\| (t-\tau-s)^3 u_h^{(4)}(s) \right\| ds \right]^2 \\
&\leq \frac{1}{36\tau^4} \left[2 \left(\int_t^{t+\tau} \left\| (t+\tau-s)^3 u_h^{(4)}(s) \right\| ds \right)^2 \right. \\
&\quad \left. + 2 \left(\int_{t-\tau}^t \left\| (t-\tau-s)^3 u_h^{(4)}(s) \right\| ds \right)^2 \right] \\
&\leq \frac{1}{18\tau^4} \left[\int_t^{t+\tau} (t+\tau-s)^6 ds \int_t^{t+\tau} \left\| u_h^{(4)}(s) \right\|^2 ds \right. \\
&\quad \left. + \int_{t-\tau}^t (t-\tau-s)^6 ds \int_{t-\tau}^t \left\| u_h^{(4)}(s) \right\|^2 ds \right] \\
&= \frac{1}{18\tau^4} \left(\frac{\tau^7}{7} \right) \int_t^{t+\tau} \left\| u_h^{(4)}(s) \right\|^2 ds + \frac{1}{18\tau^4} \left(\frac{\tau^7}{7} \right) \int_{t-\tau}^t \left\| u_h^{(4)}(s) \right\|^2 ds \\
&= \frac{\tau^3}{126} \int_{t-\tau}^{t+\tau} \left\| u_h^{(4)}(s) \right\|^2 ds \tag{4.4.12}
\end{aligned}$$

Using similar arguments we have

$$\begin{aligned}
&2\tau^{-1} (u_j(t+\tau) - u_j(t-\tau)) - u_j'(t) \\
&= \frac{1}{4\tau} \left(\int_t^{t+\tau} (t+\tau-s)^2 u_j'''(s) ds + \int_{t-\tau}^t (t-\tau-s)^2 u_j'''(s) ds \right) \\
&= \rho_{k,j},
\end{aligned}$$

and therefore

$$\begin{aligned}
\|\rho_k\|^2 &\leq \left[\frac{1}{4\tau} \left\| \int_t^{t+\tau} (t+\tau-s)^2 u_h'''(s) ds - \int_{t-\tau}^t (t-\tau-s)^2 u_h'''(s) ds \right\| \right]^2 \\
&\leq \frac{1}{16\tau^2} \left[\left\| \int_t^{t+\tau} (t+\tau-s)^2 u_h'''(s) ds \right\| + \left\| \int_{t-\tau}^t (t-\tau-s)^2 u_h'''(s) ds \right\| \right]^2 \\
&\leq \frac{1}{16\tau^2} \left[\int_t^{t+\tau} \|(t+\tau-s)^2 u_h'''(s)\| ds + \int_{t-\tau}^t \|(t-\tau-s)^2 u_h'''(s)\| ds \right]^2 \\
&\leq \frac{1}{16\tau^2} \left[2 \left(\int_t^{t+\tau} \|(t+\tau-s)^2 u_h'''(s)\| ds \right)^2 \right. \\
&\quad \left. + 2 \left(\int_{t-\tau}^t \|(t-\tau-s)^2 u_h'''(s)\| ds \right)^2 \right] \\
&\leq \frac{1}{8\tau^2} \left[\int_t^{t+\tau} (t+\tau-s)^4 ds \int_t^{t+\tau} \|u_h'''(s)\|^2 ds \right. \\
&\quad \left. + \int_{t-\tau}^t (t-\tau-s)^4 ds \int_{t-\tau}^t \|u_h'''(s)\|^2 ds \right] \\
&= \frac{1}{8\tau^2} \left(\frac{\tau^5}{5} \right) \int_t^{t+\tau} \|u_h'''(s)\|^2 ds + \frac{1}{18\tau^2} \left(\frac{\tau^5}{5} \right) \int_{t-\tau}^t \|u_h'''(s)\|^2 ds \\
&= \frac{\tau^3}{40} \int_{t-\tau}^{t+\tau} \|u_h'''(s)\|^2 ds \tag{4.4.13}
\end{aligned}$$

Substitution of the estimates (4.4.12) and (4.4.13) in Corollary 4.4.1 yields the error estimate.

Theorem 4.4.1. *Error estimate.*

If $f \in C^2([0, T], X)$, then, for each $t_k \in (0, T)$,

$$\|u_h(t_k) - u_k^h\|_V \leq C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\|_W + \tau \max \|u_h'''(s)\|_W \right),$$

where C_T depends on T .

Proof. Recall that $\|e_1\|_W \leq \frac{1}{2} \tau^2 \|\varepsilon_0\| + \tau \|\rho_0\|$. From (4.4.12) and (4.4.13) we have that

$$\|\varepsilon_0\|^2 \leq \frac{\tau^3}{126} \int_{-\tau}^{+\tau} \|u_h^{(4)}(s)\|^2 ds \quad \text{and} \quad \|\rho_0\|^2 \leq \frac{\tau^3}{40} \int_{-\tau}^{+\tau} \|u_h'''(s)\|^2 ds.$$

Therefore

$$\|e_1\|_W^2 \leq \frac{\tau^7}{252} \int_{-\tau}^{\tau} \|u_h^{(4)}(s)\|_W^2 ds + \frac{\tau^5}{20} \int_{-\tau}^{\tau} \|u_h'''(s)\|_W^2 ds. \quad (4.4.14)$$

Substitution of (4.4.12), (4.4.13) and (4.4.14) into Corollary 4.4.1 yields

$$\begin{aligned} \|e_N\|_V^2 &\leq \left(1 + \frac{Te^{CT}}{R}\right) \left[\frac{4\tau^2 + 1}{\tau^4} \left(\frac{\tau^7}{252} \int_{-\tau}^{\tau} \|u_h^{(4)}(s)\|_W^2 ds + \frac{\tau^5}{20} \int_{-\tau}^{\tau} \|u_h'''(s)\|_W^2 ds \right) \right. \\ &\quad \left. + 2\tau \left(\frac{\tau^3}{126} \int_{t_1-\tau}^{t_1+\tau} \|u_h^{(4)}(s)\|_W^2 ds \right) + 2\tau \sum_{k=2}^{N-2} \frac{\tau^3}{126} \int_{t-\tau}^{t+\tau} \|u_h^{(4)}(s)\|_W^2 ds \right] \\ &\quad + \frac{2\tau^2 e^{CT}}{R} \sum_{k=1}^{N-2} \frac{\tau^3}{126} \int_{t-\tau}^{t+\tau} \|u_h^{(4)}(s)\|_W^2 ds + 2\tau \left(\frac{\tau^3}{126} \int_{t_{N-1}-\tau}^{t_{N-1}+\tau} \|u_h^{(4)}(s)\|_W^2 ds \right) \\ &\leq \left(1 + \frac{Te^{CT}}{R}\right) \left[\frac{4\tau^5 + \tau^3}{252} \int_{-\tau}^{\tau} \|u_h^{(4)}(s)\|_W^2 ds + \frac{4\tau^3 + \tau}{20} \int_{-\tau}^{\tau} \|u_h'''(s)\|_W^2 ds \right. \\ &\quad \left. + \frac{\tau^4}{63} \int_{t_1-\tau}^{t_1+\tau} \|u_h^{(4)}(s)\|_W^2 ds + \frac{\tau^4}{63} \sum_{k=2}^{N-2} \int_{t-\tau}^{t+\tau} \|u_h^{(4)}(s)\|_W^2 ds \right] \\ &\quad + \frac{\tau^5 e^{CT}}{63R} \sum_{k=1}^{N-2} \int_{t-\tau}^{t+\tau} \|u_h^{(4)}(s)\|_W^2 ds + \frac{\tau^4}{63} \int_{t_{N-1}-\tau}^{t_{N-1}+\tau} \|u_h^{(4)}(s)\|_W^2 ds \\ &\leq \left(1 + \frac{Te^{CT}}{R}\right) \left[\frac{4\tau^6 + \tau^4}{126} \max \|u_h^{(4)}(s)\|_W^2 + \frac{4\tau^4 + \tau^2}{10} \max \|u_h'''(s)\|_W^2 \right. \\ &\quad \left. + \frac{2\tau^5}{63} \max \|u_h^{(4)}(s)\|_W^2 + \frac{2\tau^5}{63} \sum_{k=2}^{N-2} \max \|u_h^{(4)}(s)\|_W^2 \right] \\ &\quad + \frac{2\tau^6 e^{CT}}{63R} \sum_{k=1}^{N-2} \max \|u_h^{(4)}(s)\|_W^2 + \frac{2\tau^5}{63} \max \|u_h^{(4)}(s)\|_W^2 \\ &\leq \left(1 + \frac{Te^{CT}}{R}\right) \left[\frac{4\tau^6 + \tau^4}{126} \max \|u_h^{(4)}(s)\|_W^2 + \frac{4\tau^4 + \tau^2}{10} \max \|u_h'''(s)\|_W^2 \right. \\ &\quad \left. + \frac{2\tau^5}{63} \max \|u_h^{(4)}(s)\|_W^2 + \frac{2T\tau^4}{63} \max \|u_h^{(4)}(s)\|_W^2 \right] \\ &\quad + \frac{2T\tau^5 e^{CT}}{63R} \max \|u_h^{(4)}(s)\|_W^2 + \frac{2\tau^5}{63} \max \|u_h^{(4)}(s)\|_W^2, \\ &\leq \left[\left(1 + \frac{Te^{CT}}{R}\right) \frac{4\tau^6 + \tau^4 + 4\tau^5 + 4T\tau^4}{126} + \frac{2T\tau^5 e^{CT} + 2\tau^5 R}{63R} \right] \max \|u_h^{(4)}(s)\|_W^2 \\ &\quad + \left(1 + \frac{Te^{CT}}{R}\right) \frac{4\tau^4 + \tau^2}{10} \max \|u_h'''(s)\|_W^2, \end{aligned}$$

where $\max$ denotes the maximum over the entire interval $[0, T]$. $\square$

Remark [OR76] uses the following estimate from [Dup73]

$$\|\varepsilon_k\|^2 \leq C\tau^3 \int_{t_{k-1}}^{t_{k+1}} \|u^{(4)}(\tau)\|^2 d\tau,$$

where $\varepsilon_k = \tau^{-2}[u_{k+1} - 2u_k + u_{k-1}] - u''(k\tau)$.

4.5 Convergence of the fully discrete approximation for general damping

Finally, error estimates for the fully discrete approximation of the solution of Problem G are obtained by combining results from Chapter 3 with Theorem 4.4.1.

Recall that we assume Assumptions E1, E2, E3 and E4 hold and we may use the properties of a solution in Theorem 2.2.1.

From Corollary 3.3.1, we have that for general damping

$$\begin{aligned} & \|u(t) - u_h(t)\|_V \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V,k)} + C_b \|u'(t)\|_{H(V,k)}) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b \|u''(\cdot)\|_{H(V,k)} \right. \\ & \quad + 3K_a \max \|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)} \\ & \quad \left. + 2C_b \|u_1\|_{H(V,k)} + 2\sqrt{1+K_a} \|u_0\|_{H(V,k)} + \sqrt{K_a} \|u_1\|_{H(V,k)} \right]. \end{aligned}$$

for $t \in [0, T]$.

From Theorem 4.4.1, if $f \in C^2([0, T], X)$, then

$$\|u_h(t_k) - u_k^h\|_V \leq C_T R^{-\frac{1}{2}} \left(\tau^2 \max \left\| u_h^{(4)}(s) \right\|_W + \tau \max \|u_h'''(s)\|_W \right),$$

for each $t_k \in (0, T)$.

As mentioned in Section 4.3, we combine these results using the triangle inequality and obtain the following result.

Theorem 4.5.1. *Suppose*

- (a) *the damping term satisfies Assumption E4,*
- (b) *Assumption A3 holds for the space V ,*
- (c) $u_0^h = \Pi u_0$ *and* $u_1^h = \Pi u_1$,
- (d) $u \in C^2([0, T], V)$ *(Assumption A2),*
- (e) $u''(t) \in \mathcal{L}^2([0, T], H(V, k))$,
- (f) $f \in C^2([0, T], X)$ *and*
- (g) *the sequence $\{u_k^h\}$ is a solution of Problem G^h -D-Central.*

Then

$$\begin{aligned}
& \|u(t_k) - u_k^h\|_V \\
& \leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + C_b \|u'(t)\|_{H(V,k)} \right) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T C_b \|u''(\cdot)\|_{H(V,k)} \right. \\
& \quad \left. + 3K_a \max \|u'(t)\|_{H(V,k)} + 3K_a \int_0^T \|u''(\cdot)\|_{H(V,k)} \right. \\
& \quad \left. + 2C_b \|u_1\|_{H(V,k)} + 2\sqrt{1+K_a} \|u_0\|_{H(V,k)} + \sqrt{K_a} \|u_1\|_{H(V,k)} \right] \\
& \quad + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\|_W + \tau \max \|u_h'''(s)\|_W \right).
\end{aligned}$$

Remark Estimates for $\|u_h'''\|_W$ and $\|u_h^{(4)}\|_W$ in terms of the data are available, see Proposition 2.5.3.

In the introduction to [Wu06] the importance of the explicit method is stressed. “The explicit finite element method has been extensively developed for the transient dynamic analysis to meet the increasing demand of engineering application.” A diagonal mass matrix is needed for an explicit method. In Section 3 of the paper, piecewise linear and bilinear basis functions are considered for one-, two- and three-dimensional cases. The equivalent diagonal mass matrices are derived and the errors estimated in Lemma 3.3. Using this result it is proved in Theorem 4.1 that the rate of convergence is the same as for using consistent mass matrices.

Chapter 5

Routine applications

5.1 Wave equation with weak or modal damping

In this section we apply the general results in Sections 3.2, 3.4 and 4.3.

5.1.1 The multi-dimensional wave equation

Consider the wave equation in a n -dimensional bounded domain denoted by Ω . If $n = 3$ it models the propagation of sound waves [Wei65, Section 34], and in the two-dimensional case it models the vibration of a membrane [Inm94].

Problem MW

Let Σ be part of the boundary $\partial\Omega$ with unit outer normal vector n . We have different boundary conditions on Σ and $\partial\Omega - \Sigma$. Given functions f, u_0 and u_1 , find u defined on $\bar{\Omega} \times [0, T]$ such that

$$\begin{aligned}\rho \partial_t^2 u &= \nabla \cdot (A \nabla u) - k \partial_t u + f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega - \Sigma, \\ (A \nabla u) \cdot n &= 0 \quad \text{on } \Sigma,\end{aligned}$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

The given parameters in the problem are the matrix of functions $A = (a_{ij})$ and the functions k and ρ .

Assumptions on the parameters

1. $a_{ij} \in C(\bar{\Omega}) \cap C^1(\Omega)$.
2. The matrix A is uniformly positive definite, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2,$$

for all $x \in \bar{\Omega}$ and all $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

3. There exist positive constants c_1, c_2 and c_3 such that $c_1 \leq \rho \leq c_2$ and $0 \leq k \leq c_3$.

The damping is weak in this problem (see Proposition 5.1.2). The undamped wave equation is a special case where $k = 0$, see e.g. [Bak76], [Dup73] and [OR76].

Problem MW is also a mathematical model for hyperbolic heat conduction, see e.g. [TY05].

5.1.2 Variational form

To write the problem in variational form, we use the well known Green's formula (a consequence of the divergence theorem).

Green's formula

If $u \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$, then

$$\iiint_{\Omega} -(\nabla \cdot (A \nabla u))v \, dV = \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV - \iint_{\partial\Omega} v(A \nabla u) \cdot n \, dA.$$

Remark If $n = 2$, then $\partial\Omega$ is a curve and the surface integral must be replaced by a line integral.

If u is a solution of Problem MW, then for each $v \in C^1(\bar{\Omega})$

$$\begin{aligned} \iiint_{\Omega} \rho(\partial_t^2 u)v \, dV &= \iiint_{\Omega} (\nabla \cdot (A \nabla u))v \, dV + \iiint_{\Omega} f v \, dV \\ &= - \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV + \iint_{\partial\Omega} v(A \nabla u) \cdot n \, dA \\ &\quad + \iiint_{\Omega} f v \, dV. \end{aligned} \tag{5.1.1}$$

Next consider the variational form. The test functions $T(\Omega)$ are functions in $C^1(\bar{\Omega})$ which vanish on $\partial\Omega - \Sigma$.

Test functions

$$T(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega - \Sigma\}.$$

From Equation (5.1.1) we have, if u is a solution of Problem MW, then for each $v \in T(\Omega)$

$$\iiint_{\Omega} \rho(\partial_t^2 u) v \, dV = - \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV + \iiint_{\Omega} f v \, dV.$$

Notation

$$(f, g)_{\Omega} = \iiint_{\Omega} f g \, dV.$$

Bilinear forms for Problem MW

$$\begin{aligned} b(u, v) &= \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV \\ c(u, v) &= \iiint_{\Omega} \rho u v \, dV, \\ a(u, v) &= \iiint_{\Omega} k u v \, dV. \end{aligned}$$

Note that (f, g) is the usual inner product for $\mathcal{L}^2(\Omega)$.

Problem MWV

Find u such that for each $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot, t), v) + a(\partial_t u(\cdot, t), v) + b(u(\cdot, t), v) = (f(\cdot, t), v)_{\Omega}$$

for each $v \in T(\Omega)$.

5.1.3 Weak variational form

Let $V = V(\Omega)$ denote the closure of $T(\Omega)$ in $H^1(\Omega)$. The bilinear form b is defined as in Subsection 5.1.2:

$$b(u, v) = \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV,$$

where $\partial_i u$ and $\partial_i v$ denote weak derivatives.

Problem MWW

Find u such that for each $t > 0$, $u(t) \in V(\Omega)$, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(\cdot, t), v)_\Omega \quad \text{for each } v \in V(\Omega)$$

while $u(0) = u_0$, and $u'(0) = u_1$.

5.1.4 Existence of a solution

Since $c_1 \leq \rho \leq c_2$, the bilinear form c is clearly an inner product for $X = \mathcal{L}^2(\Omega)$ and the corresponding norm is equivalent to the norm of $\mathcal{L}^2(\Omega)$. The space $\mathcal{L}^2(\Omega)$ with inner product c is the space W (with norm $\|\cdot\|_W$).

Poincare-Friedrichs inequality

There exists a constant β such that $\|v\|^2 \leq \beta b(v, v)$ for each $v \in V(\Omega)$. If $\partial\Omega - \Sigma$ has positive area (positive length for $n = 2$), then the assumption holds, see [Bra01, p.30] or [Eva98].

Baker [Bak76] states this without proof or reference. It is not even mentioned in [Dup73].

If the Poincare-Friedrichs inequality is true, then b is positive definite on $V(\Omega)$ with respect to the norm of $H^1(\Omega)$ and the bilinear form b is an inner product on $V(\Omega)$. (The bilinear form b is symmetric by definition.)

Definition The norm corresponding to the inner product b is

$$\|v\|_V = \sqrt{b(v, v)} \quad \text{for any } v \in V(\Omega).$$

The bilinear form b is bounded on $H^1(\Omega)$, hence the norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(\Omega)$ on $V(\Omega)$.

Proposition 5.1.1. $V(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

Proof. From Appendix A we have that $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$. Since $C_0^\infty(\Omega) \subset V(\Omega)$, $V(\Omega)$ is also dense in $\mathcal{L}^2(\Omega)$. $\square$

We have shown that Assumptions E1, E2 and E3 are satisfied. Note that $X = \mathcal{L}^2(\Omega)$. For existence we also need Assumption E4W (weak damping).

Proposition 5.1.2. *The bilinear form a is nonnegative, symmetric and bounded on W .*

$$|a(u, v)| \leq K_a \|u\|_W \|v\|_W.$$

Proof. We have that

$$a(u, v) \leq c_3 \iiint_{\Omega} |uv| \leq c_3 \|u\| \|v\|,$$

and the $\mathcal{L}^2$ norm is equivalent to the norm $\|\cdot\|_W$ □

Existence of a unique solution for the weak variational problem depends on the smoothness of the function $\tilde{f}: t \rightarrow f(\cdot, t)$.

Recall the definition of E_b :

$$E_b = \{x \in V \mid \text{there exists a } y \in W \text{ such that } c(y, v) = b(x, v) \text{ for all } v \in V\}.$$

Theorem 5.1.1. *Suppose $\tilde{f} \in C^1([0, T], \mathcal{L}^2(\Omega))$. Problem MWW has a unique solution*

$$u \in C^1([0, T]; V) \cap C^2([0, T]; W)$$

if $u_0 \in E_b$ and $u_1 \in V$.

Proof. Assumptions E1, E2, E3 and E4W hold. The result follows from Theorem 2.2.2. □

5.1.5 The semi-discrete approximation

Now consider the Galerkin finite element approximation of Problem MWW on a two-dimensional domain. A finite dimensional subspace S^h of $V(\Omega)$ is constructed, e.g. using piecewise linear basis functions on triangle elements or piecewise bilinear basis functions on quadrilateral elements. Note that the basis functions must satisfy the forced boundary condition: zero on $\partial\Omega - \Sigma$.

Consider the special case of Problem G^h .

Problem MWW^h

Find a function $u_h \in C^2(0, T)$ such that u'_h is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u''_h(t), v) + a(u'_h(t), v) + b(u_h(t), v) = (f(\cdot, t), v)_{\Omega},$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u'_h(0) = u_1^h$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{f}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$.

We need to investigate whether the Assumptions used in Sections 3.2 and 3.4 hold for this problem in order to apply the theorems. (The Assumptions are summarized in Appendix C for convenience.)

Interpolation (See Appendix B)

Consider Assumption A3 regarding the existence of an interpolation operator. For this problem we have that

$$H(V, k) = H^k(\Omega) \cap V(\Omega).$$

Instead of Assumption A3 we have a more specific estimate which depends on S^h and the interpolation operator Π . For simplicity we consider the two-dimensional wave equation on a rectangle and suppose we use triangle elements. For piecewise linear basis functions (Corollary B.2.1):

If $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\|_1 \leq \hat{C}_\Delta h |u|_2,$$

where the semi-norm $|u|_j$ is defined in Appendix B.

Since the bilinear form b is bounded on $H^1(\Omega)$ we have the estimate

$$\|\Pi u - u\|_V \leq \hat{C} h |u|_2. \quad (5.1.2)$$

Remark The estimate for piecewise bilinear basis functions on rectangle elements is the same as for piecewise linear basis functions above.

5.1.6 Energy error estimates

We may apply Theorem 4.5.1 to obtain error estimates for the fully discrete problem directly; however, one may use the average acceleration method instead of standard central differences. For this reason, the error estimates for the semi-discrete approximation remain important.

Problem MW is a model with weak damping. In this subsection we apply Section 3.2. The theorems that may be applied depend on the regularity of the solution, which depends on the given data. The smoothness of the

solution will depend on the smoothness of the functions f , u_0 and u_1 . If f , u_0 and u_1 satisfy the conditions for Theorem 2.2.2, then a weak solution exists. However, we do not know if the solution u has a second derivative $u''(t) \in V(\Omega)$ (Assumption A2).

In each of the applications below we assume $u \in C^2([0, T], V)$ and use estimate (5.1.2).

Application of Corollary 3.2.1

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and suppose that $u'' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\| \\ & \leq \hat{C}h \left[|u(t)|_2 + C_b |u'(t)|_2 + 2|u_0|_2 + 2C_b |u_1|_2 \right. \\ & \quad \left. + C_b \int_0^t (|u''(\cdot)|_2 + k|u'(\cdot)|_2) \right], \end{aligned}$$

for $t \in [0, T]$.

Application of Theorem 3.2.3

If we do not make the assumptions on u' and u'' , we may still use Theorem 3.2.3. Since $H^2(\Omega)$ is dense in V , Assumption A5 is satisfied. If we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\| = 0.$$

Consider for example the first limit.

$$\begin{aligned} \|Pu_0 - u_0^h\|_V & \leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\ & \leq 2\|u_0 - \Pi u_0\|_V \rightarrow 0 \text{ as } h \rightarrow 0 \text{ (from (5.1.2))}. \end{aligned}$$

Therefore, by Theorem 3.2.3,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_V = 0 \text{ for } t \in [0, T].$$

It is sometimes possible to use the Aubin-Nitsche trick to obtain **higher order estimates** for the projection error. Note that the following is a $\mathcal{L}^2$ error estimate.

Application of Corollary 3.2.2

In some cases we have that Assumption A4 is satisfied: for a given $f \in \mathcal{L}^2(\Omega)$ the solution u of $b(u, v) = (f, v)$ for all $v \in V$ is in $H^2(\Omega) \cap V$ and $\|u\|_2 \leq C\|f\|$. Regularity theory for partial differential equations is beyond the scope of this thesis.

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\| \leq C_0 h^2$ and $u'' \in \mathcal{L}^2([0, T], H^2(\Omega))$. Then, for any $t > 0$,

$$\|u(t) - u_h(t)\| \leq \widehat{C}^2 h^2 \widehat{c}_b |u(t)|_2 + C_b C_0 h^2 + C_b \widehat{C}^2 h^2 \widehat{c}_b \left(\int_0^t |u''(\cdot)|_2 + k \int_0^t |u'(\cdot)|_2 \right).$$

If we choose the initial values as $u_0^h = Pu_0$ and $u_1^h = Pu_1$ (as some authors do, for example [Bak76] and [Sem94]), we apply Corollary 3.2.3 and obtain the estimate:

$$\|u(t) - u_h(t)\| \leq \widehat{C}^2 h^2 \widehat{c}_b |u(t)|_2 + C_b \widehat{C}^2 h^2 \widehat{c}_b \left(\int_0^t |u''(\cdot)|_2 + k \int_0^t |u'(\cdot)|_2 \right).$$

5.1.7 $\mathcal{L}^2$ error estimates for weak damping

In this subsection we apply the results of Section 3.4, where less stringent assumptions on the smoothness of the solution are made. For the projection error we have the estimate

$$\|u - Pu\| \leq C_b \|u - Pu\|_V \leq C_b \widehat{C} h |u|_2,$$

using (5.1.2).

Application of Corollary 3.4.3

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem MW satisfies $u' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$, then

$$\begin{aligned} \|u(t) - u_h(t)\| \leq & C_b \widehat{C} h |u(t)|_2 + \sqrt{2} C_b \widehat{C} h \left(3 \int_0^T |u'(\cdot)|_2 \right. \\ & \left. + 3K_a \int_0^T |u(\cdot)|_2 + (2 + 3K_a T) |u_0|_2 + 3T |u_1|_2 \right), \end{aligned}$$

for each $t \in [0, T]$.

5.1.8 Fully discrete

Since we have weak damping, we can apply Section 4.3 (Baker) or Section 4.5 (central differences).

Suppose one chooses to use the algorithm in Problem G^h -D-Baker.

Problem MW^h -D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \dots, N-1$,

$$\begin{aligned} \delta_t u_k^h &= v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + a(v_{k+\frac{1}{2}}, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) &= \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \end{aligned}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Application of Theorem 4.3.1

Recall that in Section 4.3 the fully discrete result (Theorem 4.2.1) is combined with the inertia norm estimate from Section 3.4 (Corollary 3.4.3). We apply this result to our problem, using the inequality (5.1.2).

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If $u'(t) \in \mathcal{L}^2([0, T], H^2(0, 1) \cap V)$, $f \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem MW^h -D, then

$$\begin{aligned} &\|u(t_k) - u_k^h\| \\ &\leq C_b \widehat{C} h |u(t)|_2 + \sqrt{2} C_b \widehat{C} h \left(3 \int_0^T |u'(\cdot)|_2 \right. \\ &\quad \left. + 3K_a \int_0^T |u(\cdot)|_2 + (2 + 3K_a T) |u_0|_2 + 3T |u_1|_2 \right) + 7T^2 \tau^2 \max \|u_h^{(4)}\| \\ &\quad + 7T \tau^2 \max \|u_h'''\| + \sqrt{2K_a} \tau^4 \max \|u_h'''\|, \end{aligned}$$

for each $t_k \in (0, T)$.

Proposition 2.5.3 can now be used to obtain an estimate for $\|u_h^{(4)}\|$ in terms of the function f and the initial conditions $u_h(0) = d^h$ and $u_h'(0) = v^h$.

Remarks

1. Recall that the result above was obtained by using the triangle inequality and combining a semi-discrete and fully discrete estimate:

$$\|u(t_k) - u_k^h\| \leq \|u(t_k) - u_h(t_k)\|_W + \|u_h(t_k) - u_k^h\|_W.$$

2. There are other possible combinations of semi-discrete and fully discrete approximations. For example, one of the energy error estimates in the previous subsection can be combined with Theorem 4.2.1 or another algorithm can be used for the fully discrete approximation. Recall that $u \in C^2([0, T], V)$ (Assumption A2) must be assumed if an energy error estimates is used.
3. Note that the energy error estimates are in terms of $\|\cdot\|_V$ and Theorem 4.2.1 gives an estimate in the inertia norm $\|\cdot\|_W$.

5.2 The multi dimensional wave equation with boundary damping

In 1973 Dupont considered “Galerkin methods for second order hyperbolic equations” [Dup73]. The problem was the “generalized” wave equation in an n -dimensional space. He considered two cases: The undamped case with zero Dirichlet boundary conditions and the case with “general” boundary conditions.

5.2.1 Model problem

Problem WBD

Let Σ be part of the boundary $\partial\Omega$ with unit outer normal vector n . We have different boundary conditions on Σ and $\partial\Omega - \Sigma$. Given functions f, g, u_0 and u_1 , find u defined on $\bar{\Omega} \times [0, T]$ such that

$$\begin{aligned} \rho \partial_t^2 u &= \nabla \cdot (A \nabla u) + f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega - \Sigma, \\ \beta_1 \partial_t u &= -(A \nabla u) \cdot n - \beta_2 u + g \quad \text{on } \Sigma, \end{aligned}$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

The given parameters in the problem are the matrix of functions $A = (a_{ij})$ and the functions ρ, β_1 and β_2 .

Assumptions on the parameters

1. $a_{ij} \in C(\bar{\Omega}) \cap C^1(\Omega)$.

2. The matrix A is uniformly positive definite, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2,$$

for all $x \in \bar{\Omega}$ and all $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

3. There exist positive constants c_1 and c_2 such that
 $c_1 \leq \rho \leq c_2$, $c_1 \leq \beta_1$ and $c_1 \leq \beta_2$.
4. $\beta_i \in C(\partial\Omega)$ for $i = 1, 2$.

The problem in [Dup73] is a special case of Problem WBD with $\Sigma = \partial\Omega$ and $A = \alpha I$.

Remarks

1. No explanation is given in [Dup73] for the boundary conditions on Σ . If $\beta_2 = g = 0$, then we have $\beta_1 \partial_t u = -A \nabla u \cdot n$ which represents linear boundary damping.
2. If $n = 2$, then Σ is a curve and the surface integral must be replaced by a line integral.
3. Dupont [Dup73] considered the case where n is any natural number.

5.2.2 Variational form

To derive the variational form is the same as in Section 5.1.

Test functions

$$T(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega - \Sigma\}.$$

From Equation (5.1.1) we have that if u is a solution of Problem WBD, then for each $v \in T(\Omega)$

$$\begin{aligned} \iiint_{\Omega} \rho(\partial_t^2 u) v \, dV &= - \iint_{\Sigma} \beta_1(\partial_t u) v \, dS - \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV \\ &\quad + \iint_{\Sigma} (-\beta_2 u v + g v) \, dS + \iiint_{\Omega} f v \, dV \end{aligned}$$

(since $\beta_1 \partial_t u = -(A \nabla u) \cdot n - \beta_2 u + g$ on Σ).

Bilinear forms for Problem WBD

$$\begin{aligned} b_B(u, v) &= \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV + \iint_{\Sigma} \beta_2 uv \, dS, \\ c(u, v) &= \iiint_{\Omega} \rho uv \, dV, \\ a(u, v) &= \iint_{\Sigma} \beta_1 uv \, dA. \end{aligned}$$

Note that the bilinear form $a(u, v)$ differs from Problem MW in Section 5.1 and that b_B equals b plus the boundary term.

Notation

$$\begin{aligned} (f, g)_{\Omega} &= \iiint_{\Omega} fg \, dV, \\ (f, g)_{\Sigma} &= \iint_{\Sigma} fg \, dS. \end{aligned}$$

Problem WBDV

Find u such that for each $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot, t), v) + a(\partial_t u(\cdot, t), v) + b(u(\cdot, t), v) = (f(\cdot, t), v)_{\Omega} + (g(\cdot, t), v)_{\Sigma}$$

for each $v \in T(\Omega)$.

5.2.3 Weak variational form

Let $V(\Omega)$ denote the closure of $T(\Omega)$ in $H^1(\Omega)$. The trace operator Γ is defined in Appendix A.5 and the bilinear form b_B is defined on $V(\Omega)$ by

$$b_B(u, v) = \iiint_{\Omega} (A \nabla u) \cdot \nabla v \, dV + \iint_{\Sigma} \beta_2 \Gamma u \Gamma v \, dS,$$

where $\partial_i u$ and $\partial_i v$ denote weak derivatives.

The bilinear form a is defined on $V(\Omega)$ by

$$a(u, v) = \iint_{\Sigma} \beta_1 \Gamma u \Gamma v \, dS.$$

Problem WBDW

Find u such that for each $t > 0$, $u'(t) \in V(\Omega)$, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) + a(u'(t), v) + b_B(u(t), v) = (f(\cdot, t), v)_\Omega + (g(\cdot, t), v)_\Sigma \quad \text{for each } v \in V(\Omega)$$

while $u(0) = u_0$ and $u'(0) = u_1$.

5.2.4 Existence of a solution**Inner products and norms**

As in 5.1, the bilinear form c is an inner product for $X = \mathcal{L}^2(\Omega)$ and the corresponding norm is equivalent to the norm of $\mathcal{L}^2(\Omega)$. The space $\mathcal{L}^2(\Omega)$ with inner product c is denoted by W .

If the Poincare-Friedrichs inequality holds for b (see Section 5.1), then it also holds for b_B since $b_B \geq b$.

Definition The norm corresponding to the inner product b_B is

$$\|v\|_V = \sqrt{b_B(v, v)} \quad \text{for any } v \in V.$$

Proposition 5.2.1. *The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(\Omega)$ on V .*

Proof. From the Poincare-Friedrichs inequality, b_B is positive definite on V with respect to the norm of H^1 , i.e. $b_B(u, u) \geq C_1 \|u\|_1^2$. From the definition of b it is bounded on $H^1(\Omega)$ and from the Schwartz inequality and Theorem A.5.3 in Appendix A we have that

$$\left| \iint_{\Sigma} \beta_2 (\Gamma u)^2 dS \right| \leq \max \beta_2 \|\Gamma u\|_{\Sigma}^2 \leq \max \beta_2 K^2 \|u\|_1^2.$$

Therefore b_B is bounded on $H^1(\Omega)$ and the result follows. $\square$

The energy space V is dense in $\mathcal{L}^2(\Omega)$ from Proposition 5.1.1.

We have that Assumptions E1, E2 and E3 are satisfied. For existence we also need Assumption E4.

Proposition 5.2.2. *The bilinear form a is nonnegative, symmetric and bounded on V .*

Proof. The bilinear form a is clearly symmetric. Since $a(u, u) = \iint_{\Sigma} \beta_1 (\Gamma u)^2 dS$ and $\beta_1 \geq c_1 > 0$, it follows that a is nonnegative.

For any u and v in V , we have from the Schwartz inequality and Theorem A.5.3 in Appendix A that

$$\begin{aligned} |a(u, v)| &= \left| \iint_{\Sigma} \beta_1 (\Gamma u \Gamma v) dS \right| \\ &\leq \max \beta_1 \|\Gamma u\|_{\Sigma} \|\Gamma v\|_{\Sigma} \\ &\leq \max \beta_1 K^2 \|u\|_1 \|v\|_1. \end{aligned}$$

From the equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_1$ it follows that there exists a constant K_a such that

$$|a(u, v)| \leq K_a \|u\|_V \|v\|_V.$$

□

Existence of a unique solution now follows from Theorem 2.2.1, depending on the smoothness of $\tilde{f}$, see Section 5.1.

Theorem 5.2.1. *Suppose $\tilde{f} \in C^1(J, \mathcal{L}^2(\Omega))$ and $g = 0$. Problem WBDW has a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W)$$

if for $u_0 \in V$ and $u_1 \in V$ there exists some $y \in W$ such that

$$b_B(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each } v \in V. \quad (5.2.1)$$

Proof. Assumptions E1, E2, E3 and E4 hold. We use Theorem 2.2.1. All the Assumptions are satisfied and the initial condition is satisfied by (5.2.1). Therefore Problem WBDW has a unique solution. □

Remark Existence of a solution for the problem in [Dup73] is uncertain. It is not considered in [Dup73].

5.2.5 Convergence for the semi-discrete approximation

Consider the Galerkin finite element approximation of Problem WBDW on a three-dimensional domain. A finite dimensional subspace S^h of $V(\Omega)$ is constructed, e.g. using piecewise linear basis functions on tetrahedron elements. (In Section 5.1 we considered the two-dimensional case.)

Consider the special case of Problem G^h . Recall that we denote the mapping $t \rightarrow f(\cdot, t)$ by $\tilde{f}$ and the mapping $t \rightarrow g(\cdot, t)$ by $\tilde{g}$.

Problem WBDW^h

Find a function $u_h \in C^2(0, T)$ such that u'_h is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b_B(u_h(t), v) = (\tilde{f}(t), v)_\Omega + (\tilde{g}(t), v)_\Sigma,$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u'_h(0) = u_1^h$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{f}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$ and $\tilde{g}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Sigma)$.

We need to investigate whether the Assumptions used in Section 3.3 (general damping) hold for this problem in order to apply the theorems.

Interpolation (See Appendix B)

As in Section 5.1, we have that

$$H(V, k) = H^k(\Omega) \cap V(\Omega),$$

and Assumption A3 is replaced by a more specific estimate which depends on S^h and the interpolation operator Π . Suppose Ω is a three-dimensional interval and suppose we use tetrahedron elements.

For piecewise linear basisfunctions (Corollary B.4.1):

If $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\|_1 \leq \hat{C}_t h |u|_2.$$

From Proposition 5.2.1 we have that the norms $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent. Therefore we will use the estimate

$$\|\Pi u - u\|_V \leq \hat{C} h |u|_2. \quad (5.2.2)$$

Remark In [Dup73] Assumption (2.11) p.882 is used:

$$\inf_{\chi \in M_h} \|V - \chi\|_{H^s(\Omega)} \leq C h^{\ell-s} \|V\|_{H^\ell(\Omega)}.$$

No reference is given for the estimate.

Error estimates

Assumption 2 is still needed. In the applications below, we assume that $u \in C^2([0, T], V)$ and use estimate (5.2.2).

Application of Corollary 3.3.1

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and assume that $u'' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \widehat{C}h(|u(t)|_2 + C_b|u'(t)|_2) \\ & \quad + \sqrt{24e^{3t}}\widehat{C}h \left[\int_0^T C_b|u''(\cdot)|_2 + 3K_a \max |u'(t)|_2 + 3K_a \int_0^T |u''(\cdot)|_2 \right. \\ & \quad \left. + 2C_b|u_1|_2 + 2\sqrt{1+K_a}|u_0|_2 + \sqrt{K_a}|u_1|_2 \right]. \end{aligned}$$

for $t \in [0, T]$.

As mentioned in 5.1, in some cases we have that Assumption A4 is satisfied and therefore it is possible to use the Aubin-Nitsche trick to obtain **higher order $\mathcal{L}^2$ estimates** for the projection error.

Application of Corollary 3.3.2

Assume that $K\|Pu_0 - u_0^h\|_V + \|Pu_1 - u_1^h\|_W \leq C_0h^2$ and $u'' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\| & \leq \widehat{C}^2h^2\widehat{c}_b|u(t)|_2 + \sqrt{24e^{3t}}C_b\widehat{C}^2h^2\widehat{c}_b \left(\int_0^T C_b|u''(\cdot)|_2 \right. \\ & \quad \left. + 3K_a \max |u'(t)|_2 + 3K_a \int_0^T |u''(\cdot)|_2 + \sqrt{K_a}|u_1|_2 \right) + \sqrt{24e^{3t}}C_bKh^2. \end{aligned}$$

If we choose the initial values as $u_0^h = Pu_0$ and $u_1^h = Pu_1$ (as some authors do), we apply Corollary 3.3.3 and obtain the estimate:

$$\begin{aligned} \|u(t) - u_h(t)\| & \leq \widehat{C}^2h^2\widehat{c}_b|u(t)|_2 + \sqrt{24e^{3t}}C_b\widehat{C}^2h^2\widehat{c}_b \left(\int_0^T C_b|u''(\cdot)|_2 \right. \\ & \quad \left. + 3K_a \max |u'(t)|_2 + 3K_a \int_0^T |u''(\cdot)|_2 + \sqrt{K_a}|u_1|_2 \right). \end{aligned}$$

Dupont's error estimates

Dupont ([Dup73]) used the following assumption as initial condition:

$$\|(U - W)(0)\|_1 + \|\frac{\partial(U-W)}{\partial t}(0)\| \leq C_1 h^k.$$

Theorem 2 [Dup73]

Let u be the solution to D73W, and let U be its Galerkin approximation. Suppose that $u, \frac{\partial u}{\partial t} \in L^\infty(H^k(\Omega))$ and that $\frac{\partial^2 u}{\partial t^2} \in \mathcal{L}^2(H^k(\Omega))$. Suppose that $U(0)$ and $\frac{\partial U}{\partial t}(0)$ are taken so that $\|(U - W)(0)\|_1 + \|\frac{\partial(U-W)}{\partial t}(0)\| \leq C_1 h^k$, with C_1 independent of h . Then there exists a constant C_D independent of h such that

$$\|\frac{\partial}{\partial t}(U - u)\|_{L^\infty(\mathcal{L}^2(\Omega))} + \|U - u\|_{L^\infty(\mathcal{L}^2(\Omega))} \leq C_D h^k.$$

Remarks

1. The constant C_D actually depends on T .
2. The notation $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$ used in the article may be misleading. These derivatives are defined in Section 2.1.

5.2.6 Fully discrete

Since this problem has boundary damping, we apply Section 4.5 (central differences) and use the algorithm in Problem G^h -D-Central.

Problem WBDW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N - 1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ + b_B(u_k^h, v) = (f(t_k), v)_\Omega + (g(t_k), v)_\Sigma, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Application of Theorem 4.5.1

Recall that in Section 4.5 the fully discrete result (Theorem 4.4.1) is combined with the energy estimate from Section 3.3 (Corollary 3.3.1). We apply this result to our problem, using the inequality (5.2.2). Note that Assumption A2 is necessary for the energy estimate.

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and assume $u \in C^2([0, T], V)$ (Assumption A2). If $u'' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$, $f \in C^2([0, T], \mathcal{L}^2(\Omega))$, $g \in C^2([0, T], \mathcal{L}^2(\Sigma))$ and the sequence $\{u_k^h\}$ is a solution of Problem $WBDW^h$ -D, then

$$\begin{aligned} \|u(t_k) - u_k^h\|_V &\leq \widehat{C}h(|u(t)|_2 + C_b|u'(t)|_2) \\ &\quad + \sqrt{24e^{3t}}\widehat{C}h \left[\int_0^T C_b|u''(\cdot)|_2 + 3K_a \max |u'(t)|_2 \right. \\ &\quad \left. + 3K_a \int_0^T |u''(\cdot)|_2 + 2C_b|u_1|_2 + 2\sqrt{1+K_a}|u_0|_2 + \sqrt{K_a}|u_1|_2 \right] \\ &\quad + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\| + \tau \max \|u_h'''(s)\| \right). \end{aligned}$$

for each $t_k \in (0, T)$.

As mentioned in Section 5.1, Proposition 2.5.3 can now be used to obtain an estimate for $\|u_h^{(4)}\|$ in terms of the functions f and g and the initial conditions $u_h(0) = d^h$ and $u_h'(0) = v^h$.

Remark Other options for combining a semi-discrete and fully discrete estimate are possible, see Section 5.1.

5.3 Euler-Bernoulli and Rayleigh beam models

5.3.1 Model problem

In this subsection we present the equations of motion and constitutive equations and write them in dimensionless form. Boundary conditions are also formulated.

Equations of motion

The equations of motion for the deflection of the beam, w , are given by

$$\rho A \partial_t^2 w = \partial_x V + q, \quad (5.3.1)$$

$$\rho I \partial_t^2 \partial_x w = V + \partial_x M. \quad (5.3.2)$$

In these equations ρ denotes the density, A the area of the cross section, I the area moment of inertia, M the moment, V the shear force and q the load.

Parameters The beam may be tapered and the cross-sectional area A and area moment of inertia I vary along the length of the beam.

Combining the equations of motion yields

$$\rho A \partial_t^2 w = \partial_x (\rho I \partial_t^2 \partial_x w) - \partial_x^2 M + q.$$

Constitutive equation The constitutive equation is given by

$$M = EI \partial_x^2 w, \quad (5.3.3)$$

where E is an elastic constant.

Combining (5.3.1), (5.3.2) and (5.3.3) yields the model without shear due to Rayleigh:

$$\rho A \partial_t^2 w = \partial_x (\rho I \partial_t^2 \partial_x w) - \partial_x^2 (EI \partial_x^2 w) + q.$$

Remark This is a special case of the Timoshenko beam, see Section 6.2.

The well known Euler-Bernoulli model is obtained by neglecting the rotary inertia term, $\rho I \partial_t^2 \partial_x w$, in the equation above.

Dimensionless form

The length of the beam is denoted by ℓ . Set $\tau = \frac{t}{T}$, $\xi = \frac{x}{\ell}$ and $w^*(\xi, \tau) = \frac{w(x, t)}{\ell}$, where T is to be specified.

The dimensionless forms of the shear force and moment are

$$V^*(\xi, \tau) = \frac{V(x, t)}{AE}, \quad M^*(\xi, \tau) = \frac{M(x, t)}{AE\ell} \quad \text{and} \quad q^*(\xi, \tau) = \frac{q(x, t)\ell}{AE}.$$

A convenient choice for T is

$$T = \ell \sqrt{\frac{\rho}{E}}.$$

We introduce the dimensionless constant

$$\alpha = \frac{A\ell^2}{I}.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

$$\partial_t^2 w = \partial_x V + q, \quad (5.3.4)$$

$$\frac{1}{\alpha} \partial_t^2 \partial_x w = V + \partial_x M, \quad (5.3.5)$$

$$M = \frac{1}{\alpha} \partial_x^2 w. \quad (5.3.6)$$

Remarks

1. The Euler-Bernoulli model is the special case where the term $\frac{1}{\alpha} \partial_t^2 \partial_x w$ is neglected in Equation (5.3.5).
2. It is also possible to derive dimensionless equations for a nonuniform beam. Then α is a function that must be bounded above and below by positive constants.

We consider a cantilever beam, but other boundary conditions can also be accommodated.

Problem R

The model consists of

the equations of motion (5.3.4) and (5.3.5),

the constitutive equation (5.3.6) and

the boundary conditions $w(0, t) = \partial_x w(0, t) = V(1, t) = M(1, t) = 0$.

The initial conditions are $w(\cdot, 0) = w_0$ and $\partial_t w(\cdot, 0) = w_d$.

5.3.2 Variational form

To obtain the variational form, multiply the equation of motion (5.3.4) by an arbitrary function v and integrate. Using integration by parts we have

$$\begin{aligned} \int_0^1 \partial_t^2 w v &= - \int_0^1 V v' - V(0)v(0) + \int_0^1 q v \\ &= - \frac{1}{\alpha} \int_0^1 \partial_t^2 \partial_x w v' + \int_0^1 \partial_x M v' - V(0)v(0) + \int_0^1 q v. \end{aligned}$$

Doing integration by parts again,

$$\begin{aligned} \int_0^1 \partial_t^2 wv + \frac{1}{\alpha} \int_0^1 \partial_t^2 \partial_x wv' &= - \int_0^1 Mv'' + \int_0^1 qv \\ &\quad - M(0)v'(0) - V(0)v(0). \end{aligned} \quad (5.3.7)$$

Test functions

A function v is a test function if $v \in C^2[0, 1]$ and $v(0) = v'(0) = 0$. The space of these test functions is denoted by $T[0, 1]$.

Substituting the constitutive equation, Equation (5.3.7) becomes

$$\int_0^1 \partial_t^2 wv + \frac{1}{\alpha} \int_0^1 \partial_t^2 \partial_x wv' = -\frac{1}{\alpha} \int_0^1 \partial_x^2 wv'' + \int_0^1 qv, \quad (5.3.8)$$

for each $v \in T[0, 1]$,

Bilinear forms

$$\begin{aligned} (u, v) &= \int_0^1 uv \\ b(u, v) &= \int_0^1 \frac{1}{\alpha} u''v'' \\ c(u, v) &= \int_0^1 uv + \int_0^1 \frac{1}{\alpha} u'v' \end{aligned}$$

Problem RV

Find a function w such that for each $t > 0$, $w(\cdot, t) \in T[0, 1]$ and

$$c(\partial_t^2 w(\cdot, t), v) + b(w(\cdot, t), v) = (q(\cdot, t), v)$$

for each $v \in T[0, 1]$, with initial conditions $w(\cdot, 0) = w_0$ and $\partial_t w(\cdot, 0) = w_d$.

Other boundary conditions may be accommodated by changing the set of test functions.

5.3.3 Weak variational form

The spaces V , W and X are defined as follows: V is the closure of the test functions in the Sobolev space $H^2(0, 1)$, W is the closure of the test functions in $H^1(0, 1)$ and X is $\mathcal{L}^2(0, 1)$.

The bilinear forms are defined as follows:

$$\begin{aligned} \text{For } u \text{ and } v \text{ in } H^2(0, 1) \quad b(u, v) &= \int_0^1 \frac{1}{\alpha} u'' v'' \\ \text{and for } u \text{ and } v \text{ in } W \quad c(u, v) &= \int_0^1 uv + \int_0^1 \frac{1}{\alpha} u' v'. \end{aligned}$$

In these definitions the derivatives are weak derivatives. Note that the bilinear forms are symmetric.

Problem RW

Find u such that for each $t > 0$, $u(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (q(\cdot, t), v) \quad \text{for each } v \in V,$$

while $u(0) = u_0$ and $u'(0) = u_d$.

5.3.4 Existence and properties of a solution

To apply the existence results from Chapter 2 we need Assumptions E1, E2 and E3. We prove these assumptions in this subsection.

Since α is a constant or a function that is bounded above and below by positive constants, it follows that there exist positive constants K_i such that

$$K_1 \|\phi\|^2 \leq \int_0^1 \frac{1}{\alpha} \phi^2 \leq K_2 \|\phi\|^2. \quad (5.3.9)$$

Proposition 5.3.1. *There exists constants $\tilde{K}_1$ and $\tilde{K}_2$ such that*

$$\tilde{K}_1 \|u\|_1^2 \leq c(u, u) \leq \tilde{K}_2 \|u\|_1^2 \quad \text{for each } u \in W.$$

Proof. It follows directly from the definition of c and (5.3.9). □

Corollary 5.3.1. *The bilinear form c is an inner product for the space W .*

Proof. It follows from Proposition 5.3.1 since the bilinear form c is clearly symmetric. □

Definition Inertia space W

We refer to the vector space W equipped with the inner product c as the inertia space. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$.

Remarks

1. The norms $\|\cdot\|_W$ and $\|\cdot\|_1$ are equivalent.
2. For the Euler-Bernoulli model, the bilinear form c is defined as $c(u, v) = \int_0^1 uv$ and the space W is the same as the space $X = \mathcal{L}^2$. Now $\|\cdot\|_W$ is equivalent to the $\mathcal{L}^2$ norm.

Proposition 5.3.2. *V is a dense subset of W and W is dense in X .*

Proof. W is dense in $\mathcal{L}^2(0, 1)$ since $C_0^\infty(0, 1)$ is dense in $\mathcal{L}^2(0, 1)$ and $C_0^\infty(0, 1) \subset T(0, 1)$.

From the definitions of V and W , $T(0, 1) \subset V$ and $T(0, 1)$ is dense in W . So V is dense in W . $\square$

Proposition 5.3.3. *There exists a constant C_b such that*

$$\|u\|_W^2 \leq C_b b(u, u) \text{ for each } u \in V.$$

Proof. From (5.3.9)

$$\|u\|_W^2 \leq \|u\|^2 + K_2 \|u'\|^2$$

and from the well known Poincaré type inequality $\|u\| \leq \|u'\|$ and $\|u'\| \leq \|u''\|$ (since u is in V). Combining these inequalities we find that

$$\|u\|_W^2 \leq \|u\|^2 + K_2 \|u''\|^2 \leq (1 + K_2) \|u''\|^2.$$

From the definition of b it is obvious that $b(u, u) \geq \beta \|u''\|^2$ for some constant β . Consequently $\|u\|_W^2 \leq \beta^{-1}(1 + K_2)b(u, u)$. $\square$

Corollary 5.3.2. *The bilinear form b is an inner product for V .*

Definition Energy space V

We refer to the space V equipped with the inner product b as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

Recall the definition of E_b :

$$E_b = \{x \in V \mid \text{there exists a } y \in W \text{ such that } c(y, v) = b(x, v) \text{ for all } v \in V\}.$$

Theorem 5.3.1. *Let J be an open interval containing zero. There exists a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W),$$

for Problem RW for each $u_0 \in E_b$, $u_1 \in V$ and each $q \in C^1(J; X)$. If $q = 0$ then $u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W)$.

Proof. We apply Theorem 2.2.2. The Assumptions E1, E2 and E3 hold from Propositions 5.3.2, 5.3.3 and 5.3.1, and $a = 0$ (no damping). $\square$

5.3.5 Convergence for the semi-discrete approximation

A finite dimensional subspace S^h of V is constructed, using piecewise Hermite cubic basis functions (see Appendix B). The Galerkin approximations for the problem is the following special case of Problem G^h .

Problem RW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + b(u_h(t), v) = (q(\cdot, t), v) \quad \text{for each } v \in S^h,$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

Existence of a unique solution follows from Theorem 2.3.1, provided that $\tilde{q}$ is continuous w.r.t. the norm of $\mathcal{L}^2(0, 1)$.

Despite the result in Section 4.3 it is still necessary to consider convergence of the semi-discrete approximation as different algorithms may be used for the fully discrete approximation.

Interpolation

Consider Assumption A3. The space $H(V, k)$ is now $H^k(0, 1) \cap V$ and we choose the interpolation operator for Hermite cubic basis functions, denoted by Π_c .

For Hermite cubic basis functions:

There exists a constant $\hat{C}_c$ such that if $u \in H^k(0, 1)$ for $k \geq 3$,

$$\|u - \Pi_c u\|_2 \leq \hat{C}_c h^\alpha |u|_{\alpha+2}, \quad (5.3.10)$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

From the definition of the bilinear form b , we have that $\|\cdot\|_V \leq K_b \|\cdot\|_2$, hence the estimate (5.3.10) holds for the energy norm. We use the constant $\widehat{C}$ for convenience:

$$\|u - \Pi_c u\|_V \leq \widehat{C} h^\alpha |u|_{\alpha+2} \quad (5.3.11)$$

Energy error estimates

Since no damping is a special case of weak damping, we apply Section 3.2. In each of the applications below we assume $u \in C^2([0, T], V)$ (Assumption A2) and use the inequality (5.3.11).

Application of Corollary 3.2.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and suppose that $u'' \in \mathcal{L}^2([0, T], H^k(0, 1) \cap V)$ with $k \geq 3$. Then, for any $t \in [0, T]$

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \widehat{C} h^\alpha [|u(t)|_{\alpha+2} + C_b |u'(t)|_{\alpha+2} + 2|u_0|_{\alpha+2} + 2C_b |u_d|_{\alpha+2} \\ & \quad + C_b \int_0^t (|u''(\cdot)|_{\alpha+2} + k|u'(\cdot)|_{\alpha+2})], \end{aligned}$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Application of Theorem 3.2.3

If we use Theorem 3.2.3, we do not have to make the assumptions on u' and u'' . Since $H^3(0, 1)$ is dense in V , Assumption A5 is satisfied. If we choose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_d - u_d^h\|_W = 0.$$

Therefore, by Theorem 3.2.3,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_V = 0 \quad \text{for } t \in [0, \tau].$$

It is sometimes possible to use the Aubin-Nitsche trick to obtain **higher order estimates** for the projection error. Note that the following is an inertia norm estimate.

Application of Corollary 3.2.2

As mentioned in Subsection 5.1, in some cases we have that Assumption A4 is satisfied and we can apply Corollary 3.2.2.

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0, T], H^k(0, 1) \cap V)$ for $k \geq 3$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \widehat{C}^2 h^{2\alpha} \widehat{c}_b |u(t)|_{\alpha+2} + C_b C_0 h^{2\alpha} \\ &\quad + C_b \widehat{C}^2 h^{2\alpha} \widehat{c}_b \left(\int_0^t |u''(\cdot)|_{\alpha+2} + k \int_0^t |u'(\cdot)|_{\alpha+2} \right), \end{aligned}$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

H^1 -norm error estimates

If less stringent assumptions are made on the smoothness of the solution, we can still apply the results of Section 3.4 to obtain error estimates in the inertia norm (H^1 -norm in this case).

Application of Corollary 3.4.3

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If the solution u of Problem RW satisfies $u' \in \mathcal{L}^2([0, T], H^k(0, 1) \cap V)$ for $k \geq 3$, then

$$\begin{aligned} \|u(t) - u_h(t)\|_1 &\leq C_b \widehat{C} h^\alpha |u(t)|_{\alpha+2} + \sqrt{2} C_b \widehat{C} h^\alpha \left(3 \int_0^T |u'(\cdot)|_{\alpha+2} \right. \\ &\quad \left. + 3K_a \int_0^T |u(\cdot)|_{\alpha+2} + (2 + 3K_a T) |u_0|_{\alpha+2} + 3T |u_d|_{\alpha+2} \right), \end{aligned}$$

for each $t \in [0, T]$, where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Remark For the Euler-Bernoulli model, this is a $\mathcal{L}^2$ error estimate.

5.3.6 Fully discrete

Suppose one choose to use the algorithm in Problem G^h -D-Baker.

Problem RW^h -D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \dots, N-1$,

$$\begin{aligned}\delta_t u_k^h &= v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) &= \frac{1}{2}([q(t_k) + q(t_{k+1})], \psi)\end{aligned}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Application of Theorem 4.3.1

Recall that in Section 4.3 the fully discrete result (Theorem 4.2.1) is combined with the inertia norm estimate from Section 3.4 (Corollary 3.4.3). We apply this result to our problem, using the inequality (5.3.11).

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If $u'(t) \in \mathcal{L}^2([0, T], H^k(0, 1) \cap V)$ for $k \geq 3$, $\tilde{q} \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem RW^h -D, then

$$\begin{aligned}\|u(t_k) - u_k^h\|_1 &\leq C_b \hat{C} h^\alpha |u(t_k)|_{\alpha+2} + \sqrt{2} C_b \hat{C} h^\alpha \left(3 \int_0^T |u'(\cdot)|_{\alpha+2} \right. \\ &\quad \left. + 3K_a \int_0^T |u(\cdot)|_{\alpha+2} + (2 + 3K_a T) |u_0|_{\alpha+2} + 3T |u_d|_{\alpha+2} \right) + 7T^2 \tau^2 \max \|u_h^{(4)}\|_1 \\ &\quad + 7T \tau^2 \max \|u_h'''\|_1 + \sqrt{2K_a} \tau^4 \max \|u_h'''\|_1,\end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Recall that an estimate for $\|u_h^{(4)}\|_1$ in terms of the function q and the initial conditions $u_h(0) = d^h$ and $u_h'(0) = v^h$ can be obtained using Proposition 2.5.3.

Remark There are other options for combining a semi-discrete and fully discrete estimate, see Section 5.1.

Chapter 6

The Timoshenko beam

6.1 Introduction

In 1921 Timoshenko introduced an improved theory for beams. The model consists of two partial differential equations - one for the deflection w of the beam and the other for the angle ϕ due to the rotation of a cross section. It became known as the Timoshenko model and is generally considered to be an improvement on the Euler-Bernoulli and Rayleigh models. In fact, these models can be derived from the Timoshenko theory (see Sections 5.3 and 6.2).

Numerous articles for a clamped Timoshenko beam with boundary damping (or control) can be found in the literature. A paper often cited is [KR87]. A slightly different model can be found in a recent article [VVR10], where the issue of a series representation for a solution of a linear vibration model problem is considered. There are serious concerns on the validity of the series representation for the beam with boundary damping. A number of references are given in [VVR10]. For the Timoshenko beam with modal damping, the series solution may be used as a numerical method. For boundary damping, which is nonmodal, the validity of this method is not only in doubt but it is also not practical. Therefore a numerical method to simulate the motion is necessary.

Finite element approximation for the vibration of a Timoshenko beam is considered in [Sem94] and [FXX99]. The model problem in these two articles contains viscous type damping terms, see Section 6.3. The parameters are

chosen in such a way that the damping is modal. In a realistic model with viscous damping, the damping is nonmodal.

We consider 5 model problems in this Chapter, for each of these problems we will show how the general theory can be applied.

Lastly, it should be noted that finite differences for a Timoshenko beam with boundary damping is considered in [LWH08].

6.2 Equations of motion and constitutive equations

For realistic model problems we need to consider the equations of motion and constitutive equations. In this section we present the equations of motion and constitutive equations and write them in dimensionless form. Boundary conditions for the various model problems are also formulated.

We consider the case where there is no damping first. The equations of motion are the same as for Euler-Bernoulli and Rayleigh (see Section 5.4).

Equations of motion

$$\rho A \partial_t^2 w = \partial_x V + q, \quad (6.2.1)$$

$$\rho I \partial_t^2 \phi = V + \partial_x M. \quad (6.2.2)$$

In these equations ρ denotes the density, A the area of the cross section, I the area moment of inertia, M the moment, V the shear force and q the load. (Recall that w is the transverse displacement and ϕ the angle of rotation.)

Constitutive equations

$$M = EI \partial_x \phi, \quad (6.2.3)$$

$$V = AG \kappa^2 (\partial_x w - \phi), \quad (6.2.4)$$

where E and G are elastic constants and κ^2 the shear correction factor. For more detail, see e.g. [Tim37, pp.337-338], [Inm94, pp.337-338], [New89, pp.394-395] and [Fun65, p323-324].

Parameters The beam may be tapered such that the cross-sectional area A and area moment of inertia I vary along the length of the beam.

The **partial differential equations** are derived from the equations of motion and the constitutive equations.

$$\begin{aligned}\rho A \partial_t^2 w &= \partial_x (AG\kappa^2 (\partial_x w - \phi)), \\ \rho I \partial_t^2 \phi &= AG\kappa^2 (\partial_x w - \phi) + \partial_x (EI \partial_x \phi).\end{aligned}$$

We do not use them.

Remark For the Euler-Bernoulli and Rayleigh models it is assumed that cross sections remain perpendicular to the neutral plane. Consequently $\theta = \phi$, where θ is the angle of rotation of the neutral plane. If θ is small, then $\theta \approx \tan \theta = \partial_x w$ so (6.2.3) becomes $M = EI \partial_x^2 w$ as in Section 5.4.

Dimensionless form

The length of the beam is denoted by ℓ . Set $\tau = \frac{t}{T}$, $\xi = \frac{x}{\ell}$, $w^*(\xi, \tau) = \frac{w(x, t)}{\ell}$ and $\phi^*(\xi, \tau) = \phi(x, t)$, where T is to be specified. We introduce the dimensionless constants

$$\alpha = \frac{A\ell^2}{I} \quad \text{and} \quad \beta = \frac{AG\kappa^2 \ell^2}{EI}.$$

The dimensionless forms of the shear force and moment are

$$V^*(\xi, \tau) = \frac{V(x, t)}{AG\kappa^2}, \quad M^*(\xi, \tau) = \frac{M(x, t)}{AG\kappa^2 \ell} \quad \text{and} \quad q^*(\xi, \tau) = \frac{q(x, t)\ell}{AG\kappa^2}.$$

A convenient choice for T is

$$T = \ell \sqrt{\frac{\rho}{G\kappa^2}}.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

$$\partial_t^2 w = \partial_x V + q, \tag{6.2.5}$$

$$\frac{1}{\alpha} \partial_t^2 \phi = V + \partial_x M, \tag{6.2.6}$$

$$M = \frac{1}{\beta} \partial_x \phi, \tag{6.2.7}$$

$$V = \partial_x w - \phi. \tag{6.2.8}$$

It is also possible to derive dimensionless equations for a nonuniform beam. Then α and β are functions required to be bounded above and below by positive constants.

6.3 Damping

We consider 2 types of damping, viscous damping and boundary damping.

6.3.1 Viscous damping

For viscous damping Equation (6.2.1) changes to $\rho A \partial_t^2 w = \partial_x V + q - k \partial_t w$. The dimensionless form is

$$\partial_t^2 w = \partial_x V - \mu \partial_t w + q, \quad (6.3.1)$$

where $\mu = \frac{k\ell}{A\sqrt{\rho G \kappa^2}}$.

The damping in [Sem94] and [FXX99] is unrealistic. To obtain the model, the second equation of motion (6.2.2) should change to

$$\rho I \partial_t^2 \phi = V + \partial_x M - k_1 \partial_t \phi.$$

The dimensionless form is

$$\frac{1}{\alpha} \partial_t^2 \phi = V + \partial_x M - \mu_1 \partial_t \phi, \quad (6.3.2)$$

where $\mu_1 = \frac{k_1}{\rho A \ell^2}$. This is physically unrealistic but we proceed since it does not affect the theory from a mathematical point of view.

Remark Note that the damping is not modal. For the model in [Sem94] the damping is modal:

$$\partial_t^2 w + \delta \partial_t w + d^{-2} \partial_x (\phi - \partial_x w) = q, \quad (6.3.3)$$

$$\partial_t^2 \phi + \delta \partial_t \phi + d^{-2} (\phi - \partial_x w) = 0. \quad (6.3.4)$$

The dimensionless form of (6.3.3) and (6.3.4) is obtained by adding inertia and damping terms to the steady state problem in [Arn81]. The same model is considered in [FXX99].

Boundary conditions

Various homogeneous boundary conditions may be considered. In [FXX99] and [Sem94] the beam is clamped at both ends, other possibilities are not mentioned. We consider three cases: Clamped at both ends, clamped at one

end and fixed (pinned) at both ends. There are a number of other possibilities as well. The variational forms are virtually the same, see Section 6.4.

Problem TV1 (Clamped at both ends)

The mathematical model consists of

the equations of motion (6.3.1) and (6.3.2),

the constitutive equations (6.2.7) and (6.2.8) and

the boundary conditions $w(0, t) = \phi(0, t) = w(1, t) = \phi(1, t) = 0$.

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

Problem TV2 (Cantilever)

The same as Problem TV1, but with boundary conditions

$$w(0, t) = \phi(0, t) = V(1, t) = M(1, t) = 0.$$

Problem TV3 (Fixed at both ends but not clamped)

The same as Problem TV1, but with boundary conditions

$$w(0, t) = w(1, t) = M(0, t) = M(1, t) = 0.$$

6.3.2 Boundary damping or control

The boundary damping (or control) is modelled by (see e.g. [KR87])

$$\begin{aligned} V(\ell, t) &= -k_1 \partial_t w(\ell, t), \\ M(\ell, t) &= -k_2 \partial_t \phi(\ell, t). \end{aligned}$$

In dimensionless form they are $V(1, t) = -\gamma_1 \partial_t w(1, t)$ and $M(1, t) = -\gamma_2 \partial_t \phi(1, t)$, where $\gamma_1 = \frac{k_1 \ell}{TAG\kappa^2}$ and $\gamma_2 = \frac{k_2}{AG\kappa^2 \ell T}$.

Now consider a cantilever beam with boundary damping.

Problem TB1

The mathematical model consists of

the equations of motion (6.2.5) and (6.2.6),

the constitutive equations (6.2.7) and (6.2.8) and

the boundary conditions

$$w(0, t) = \phi(0, t) = 0, \quad (6.3.5)$$

$$V(1, t) = -\gamma_1 \partial_t w(1, t) \text{ and} \quad (6.3.6)$$

$$M(1, t) = -\gamma_2 \partial_t \phi(1, t). \quad (6.3.7)$$

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

Lastly we consider boundary damping for a pinned-pinned beam. This problem was considered in [VVR10].

Problem TB2

The mathematical model consists of

the equations of motion (6.2.5) and (6.2.6),

the constitutive equations (6.2.7) and (6.2.8) and

the boundary conditions

$$w(0, t) = w(1, t) = M(0, t) = 0 \text{ and } M(1, t) = -\gamma_2 \partial_t \phi(1, t).$$

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

6.4 Variational form

6.4.1 Viscous type damping

Consider Problem TV1. Multiplying (6.3.1) and (6.3.2) with arbitrary functions v and ψ in $C^1[0, 1]$ respectively, integrating and using integration by

parts yields

$$\begin{aligned}
\int_0^1 \partial_t^2 w(\cdot, t) v + \mu \int_0^1 \partial_t w(\cdot, t) v &= - \int_0^1 V(\cdot, t) v' + [V(x, t) v(x)]_0^1 + \int_0^1 q(\cdot, t) v \\
\int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + \mu_1 \int_0^1 \partial_t \phi(\cdot, t) \psi &= \int_0^1 V(\cdot, t) \psi - \int_0^1 M(\cdot, t) \psi' + [M(x, t) \psi(x)]_0^1.
\end{aligned}$$

Adding and substituting the constitutive equations we have that

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + \mu \int_0^1 \partial_t w(\cdot, t) v + \mu_1 \int_0^1 \partial_t \phi(\cdot, t) \psi \\
&= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) + \int_0^1 q(\cdot, t) v \\
&\quad + V(1, t) v(1) - V(0, t) v(0) + M(1, t) \psi(1) - M(0, t) \psi(0). \tag{6.4.1}
\end{aligned}$$

Test functions $T_1[0, 1]$ A function v is a test function if $v \in C^1[0, 1]$ and $v(0) = v(1) = 0$. The space of these test functions is denoted by $T_1[0, 1]$.

Problem TV1V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_1[0, 1]$, $\phi(\cdot, t) \in T_1[0, 1]$ and

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + \mu \int_0^1 \partial_t w(\cdot, t) v + \mu_1 \int_0^1 \partial_t \phi(\cdot, t) \psi \\
&= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) \\
&\quad + \int_0^1 q(\cdot, t) v. \tag{6.4.2}
\end{aligned}$$

for each pair $\langle v, \psi \rangle \in T_1[0, 1] \times T_1[0, 1]$.

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

Other boundary conditions are accommodated by changing the set of test functions.

Test functions $T_2[0, 1]$ A function v is a test function if $v \in C^1[0, 1]$ and $v(0) = 0$. The space of these test functions is denoted by $T_2[0, 1]$.

For Problem TV2, substituting the boundary conditions $V(1, t) = M(1, t) = 0$, we have that for $v \in T_2[0, 1]$ and $\psi \in T_2[0, 1]$, Equation (6.4.1) again reduces to Equation (6.4.2).

Problem TV2V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_2[0, 1]$, $\phi(\cdot, t) \in T_2[0, 1]$ and (6.4.2) holds for each pair $\langle v, \psi \rangle \in T_2[0, 1] \times T_2[0, 1]$.

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

Lastly, with the boundary conditions $M(0, t) = M(1, t) = 0$ and for $v \in T_1[0, 1]$, we also arrive at Equation (6.4.2).

Problem TV3V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_1[0, 1]$, $\phi(\cdot, t) \in C^1[0, 1]$ and (6.4.2) holds for each pair $\langle v, \psi \rangle \in T_1[0, 1] \times C^1[0, 1]$.

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

6.4.2 Boundary damping

Consider equation (6.4.1) with $\mu = \mu_1 = 0$.

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi \\ = & - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t))(v' - \psi) + \int_0^1 q(\cdot, t) v \\ & + V(1, t) v(1) - V(0, t) v(0) + M(1, t) \psi(1) - M(0, t) \psi(0). \end{aligned} \quad (6.4.3)$$

Using the boundary conditions for Problem TB1 and the test functions $T_2[0, 1]$, we obtain the variational form.

Problem TB1V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_2[0, 1]$, $\phi(\cdot, t) \in T_2[0, 1]$

and

$$\begin{aligned}
& \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi \\
&= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) \\
&\quad - \gamma_1 \partial_t w(1, t) v(1) - \gamma_2 \partial_t \phi(1, t) \psi(1) + \int_0^1 q(\cdot, t) v. \tag{6.4.4}
\end{aligned}$$

for each pair $\langle v, \psi \rangle \in T_2[0, 1] \times T_2[0, 1]$.

Problem TB2V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_1[0, 1]$, $\phi(\cdot, t) \in C^1[0, 1]$ and

$$\begin{aligned}
& \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi \\
&= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) \\
&\quad - \gamma_2 \partial_t \phi(1, t) \psi(1) + \int_0^1 q(\cdot, t) v. \tag{6.4.5}
\end{aligned}$$

for each pair $\langle v, \psi \rangle \in T_1[0, 1] \times C^1[0, 1]$.

6.5 Product spaces

The following product spaces are necessary for the weak formulation.

$$X = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1).$$

$$H^m = H^m(0, 1) \times H^m(0, 1).$$

Notation

An element $y \in X$ is written as $y = \langle y_1, y_2 \rangle$.

A natural inner product for X is

$$(x, y)_X = (x_1, y_1) + (x_2, y_2),$$

and the corresponding norm is denoted by $\|\cdot\|_X$.

The natural inner product for the product space H^m is

$$(x, y)_{H^m} = (x_1, y_1)_m + (x_2, y_2)_m$$

and the corresponding norm is denoted by $\|\cdot\|_{H^m}$.

We define the following bilinear forms:

$$\begin{aligned} \text{For } u \text{ and } v \text{ in } H^1 \quad b(u, v) &= \int_0^1 \frac{1}{\beta} u'_2 v'_2 + \int_0^1 (u'_1 - u_2)(v'_1 - v_2) \\ \text{and for } u \text{ and } v \text{ in } X \quad c(u, v) &= \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2. \end{aligned}$$

In these definitions the derivatives are weak derivatives. Note that the bilinear forms are symmetric.

Since α and β are constants or functions that are bounded above and below by positive constants, it follows that there exist positive constants K_i such that

$$K_1 \|u'\|^2 \leq \int_0^1 \frac{1}{\beta} (u')^2 \leq K_2 \|u'\|^2 \quad \text{and} \quad (6.5.1)$$

$$K_3 \|u\|^2 \leq \int_0^1 \frac{1}{\alpha} u^2 \leq K_4 \|u\|^2, \quad (6.5.2)$$

for any u in $H^1(0, 1)$. As a consequence we obtain the following result.

Proposition 6.5.1. *The bilinear form c is an inner product for the space X .*

Proof. The bilinear form c is clearly symmetric and $c(u, u) \geq \|u_1\|^2 + K_3 \|u_2\|^2$ by Equation (6.5.2). $\square$

Definition Inertia space W

We refer to the vector space X equipped with the inner product c as the space W . The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$.

Proposition 6.5.2. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof. It follows directly from (6.5.1) and (6.5.2). $\square$

Let $V_1(0, 1)$ be the closure of $T_1[0, 1]$ and $V_2(0, 1)$ be the closure of $T_2[0, 1]$ in the Sobolev space $H^1(0, 1)$, then $V_1(0, 1)$ and $V_2(0, 1)$ are Hilbert spaces.

Energy subspaces V_i are required for the weak formulation of the Problems. They are defined as:

$$V_1 = V_1(0, 1) \times V_1(0, 1),$$

$$V_2 = V_2(0, 1) \times V_2(0, 1) \text{ and}$$

$$V_3 = V_1(0, 1) \times H^1(0, 1).$$

Proposition 6.5.3. *V_i is a dense subset of W for $i = 1, 2, 3$.*

Proof. $V_i(0, 1)$ is dense in $\mathcal{L}^2(0, 1)$ since $C_0^\infty(0, 1)$ is dense in $\mathcal{L}^2(0, 1)$. $\square$

We have that the bilinear form b is symmetric and nonnegative, but more is needed.

Proposition 6.5.4. *There exists a constant C_b such that*

$$b(u, u) \geq C_b \|u\|_X^2 \text{ for each } u \in V_i.$$

Proof. Consider $i = 1, 2$. From (6.5.1) and (6.5.2)

$$\|u\|_X^2 \leq \|u_1\|^2 + K_4 \|u_2\|^2$$

and from the well known Poincaré type inequality $\|u_1\| \leq \|u'_1\|$ and $\|u_2\| \leq \|u'_2\|$ (since u_1 and u_2 are in $V_i(0, 1)$). Combining these inequalities we find that

$$\begin{aligned} \|u\|_X^2 \leq \|u'_1\|^2 + K_4 \|u'_2\|^2 &\leq \|u'_1 - u_2\|^2 + \|u_2\|^2 + K_4 \|u'_2\|^2 \\ &\leq \|u'_1 - u_2\|^2 + (K_4 + 1) \|u'_2\|^2. \end{aligned} \quad (6.5.3)$$

From (6.5.1) we have

$$b(u, u) \geq \|u'_1 - u_2\|^2 + K_1 \|u'_2\|^2. \quad (6.5.4)$$

The result follows from (6.5.3) and (6.5.4).

To prove the analogue for the space V_3 is not trivial. The proof can be found in the Appendix of [VZV09]. $\square$

Corollary 6.5.1. *The bilinear form b is an inner product for V_i , for $i = 1, 2, 3$.*

Remark The result in Proposition 6.5.4 is crucial and it holds whenever the boundary conditions correspond to a situation where rigid motion of the beam is prevented.

Definition Energy spaces V_i

We refer to the space V_i equipped with the inner product b as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

6.6 Weak variational forms and existence of solutions

In this section we show how the existence and uniqueness of a weak solution for each model problem follows from the general results in Section 2.2. The results may not be new but the proofs are.

Notation V_i is denoted by V in this section.

6.6.1 Viscous damping

We define the following bilinear form:

$$\text{For } u \text{ and } v \text{ in } X \quad a_V(u, v) = \mu \int_0^1 u_1 v_1 + \mu_1 \int_0^1 u_2 v_2.$$

Note that the bilinear form a_V is symmetric. In realistic models the damping is nonmodal. (For modal damping it is required that $\frac{\mu_1}{\mu} = \frac{1}{\alpha}$.)

Problem TVW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a_V(u'(t), v) + b(u(t), v) = (q(t), v_1) \quad \text{for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d \rangle$.

Remark For the model in [Sem94] and [FXX99], $a_V = kc$ for some constant k and the damping is modal. This fact is used in their proofs.

Existence and properties of a solution

Proposition 6.6.1. *There exists a constant K_V such that, for all u and v in V ,*

$$a_V(u, v) \leq K_V \|u\|_W \|v\|_W$$

Proof.

$$\begin{aligned} |a_V(u, v)| &\leq |\mu(u_1, v_1)| + |\mu_1(u_2, v_2)| \leq \max\{\mu, \mu_1\}(u, v)_X \\ &\leq \max\{\mu, \mu_1\} \|u\|_X \|v\|_X. \end{aligned}$$

The result now follows by Proposition 6.5.2. $\square$

This problem is a special case of the general linear vibration problem considered in Chapter 2. Note that Assumptions E1, E2, E3 and E4W have been established. Recall the definition of E_b in Section 2.2.

Theorem 6.6.1. *Let J be an interval containing zero and suppose $q \in C^1(J, \mathcal{L}^2(0, 1))$. Then there exists a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W)$$

for Problem TVW for each $u_0 \in E_b$ and $u_1 \in V$.

Proof. Let $f(t) = \langle q(\cdot, t), 0 \rangle$ for the forcing term. Note that $f \in C^1(J, X)$ if $q \in C^1(J, \mathcal{L}^2(0, 1))$. Assumptions E1, E2, E3 and E4W hold. The result follows from Theorem 2.2.2. $\square$

6.6.2 Boundary damping

We use V_1 for Problem TB1 and V_3 for Problem TB2.

We define the **bilinear form** $a_B(u, v)$ for Problem TB1:

$$\text{For } u \text{ and } v \text{ in } V \quad a_B(u, v) = \gamma_1 \Gamma u_1 \Gamma v_1 + \gamma_2 \Gamma u_2 \Gamma v_2.$$

The bilinear form a_B for Problem TB2 is the special case where $\gamma_1 = 0$.

The theory for Problems TB1 and TB2 is the same, therefore we refer to Problem TBW in this section, with the understanding that it could be the weak formulation of either Problem TB1 or TB2.

Problem TBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a_B(u'(t), v) + b(u(t), v) = (q(t), v_1) \quad \text{for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d \rangle$.

Existence and properties of a solution

Proposition 6.6.2. *There exists a constant K_B such that, for all u and v in V ,*

$$|a_B(u, v)| \leq K_B \|u\|_V \|v\|_V.$$

Proof. From Theorem A.5.2

$$\begin{aligned} |a_B(u, v)| &\leq \gamma_1 |\Gamma u_1| |\Gamma v_1| + \gamma_2 |\Gamma u_2| |\Gamma v_2| \\ &\leq \gamma_1 \|u_1\|_1 \|v_1\|_1 + \gamma_2 \|u_2\|_1 \|v_2\|_1. \end{aligned} \quad (6.6.1)$$

As in the proof of Proposition 6.5.4, we have

$$\|u_1\|^2 \leq \|u'_1\|^2 \leq \|u'_1 - u_2\|^2 + \|u_2\|^2 \leq Kb(u, u)$$

and

$$\|u_2\|^2 \leq \|u'_2\|^2 \leq Kb(u, u).$$

It follows that

$$\|u_1\|_1 \leq \|u_1\| + \|u'_1\| \leq 2K \|u\|_V \quad \text{and} \quad \|u_2\|_1 \leq 2K \|u\|_V. \quad (6.6.2)$$

Combining these inequalities with Equation (6.6.1) yields the result. $\square$

Theorem 6.6.2. *Suppose $q \in C^1([0, T], \mathcal{L}^2)$. If $u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that*

$$b(u_0, v) + a_B(u_d, v) = c(y, v) \quad \text{for each } v \in V,$$

then Problem TBW has a unique solution,

$$u \in C([0, T]; V) \cap C^1([0, T]; W) \cap C^1((0, T); V) \cap C^2((0, T); W).$$

Proof. Let $f(t) = \langle q(\cdot, t), 0 \rangle$ for the forcing term. Note that $f \in C^1(J, X)$ if $q \in C^1(J, \mathcal{L}^2(0, 1))$. The product spaces satisfy Assumptions E1, E2 and E4 and Assumption E3 holds from Proposition 6.6.2. The result follows from Theorem 2.2.1. $\square$

The following proposition will be used to apply interpolation results.

Proposition 6.6.3. *The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V .*

Proof. From the definition of $b(u, u)$ and (6.5.1) we have that

$$b(u, u) \leq K_2 \|u'_2\|^2 + \|u'_1\|^2 + \|u_2\|^2 \leq K_b \|u\|_{H^1}^2, \quad (6.6.3)$$

for each $u \in H^1$. It remains to show that $\|\cdot\|_{H^1}$ is bounded by $\|\cdot\|_V$. From Proposition 6.5.4 we have $b(u, u) \geq C_b \|u\|_X^2$. Using this estimate and (6.5.4) we arrive at

$$2b(u, u) \geq \|u'_1\|^2 + \|u'_2\|^2$$

and hence

$$3b(u, u) \geq C_b \|u\|_{H^1}^2.$$

□

6.7 Convergence for the semi-discrete approximation

6.7.1 Finite dimensional subspaces

As a first step, construct finite dimensional subspaces $S_i^h(0, 1)$ of $V_i(0, \ell)$, using piecewise linear or piecewise Hermite cubic basis functions. The finite dimensional subspace for $H^k(0, 1)$ is denoted by $S^h(0, 1)$. Next we define finite dimensional subspaces for the energy spaces V_i :

$$S_1^h: S_1^h(0, 1) \times S_1^h(0, 1) \text{ for } V_1$$

$$S_2^h: S_2^h(0, 1) \times S_2^h(0, 1) \text{ for } V_2$$

$$S_3^h: S_1^h(0, 1) \times S^h(0, 1) \text{ for } V_3.$$

Consider the Galerkin finite element approximation of Problems TVW and TBW. The Galerkin approximations for the problems are the same and we use the notation a with the understanding that it is either a_V or a_B .

Let S^h be a finite dimensional subspace of V . Consider the special case of Problem G^h .

Problem TW^h

Find a function $u_h \in C^2(0, T)$ such that u'_h is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u''_h(t), v) + a(u'_h(t), v) + b(u_h(t), v) = (q(t), v_1) \quad \text{for each } v \in S^h,$$

while $u_h(0) = u_0^h = \langle w_0^h, \phi_0^h \rangle$ and $u'_h(0) = u_d^h = \langle w_d^h, \phi_d^h \rangle$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $q(t)$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$.

6.7.2 Interpolation on the product space V

Consider Assumption A3. For this problem we have that

$$H(V, k) = H^k \cap V.$$

Instead of Assumption A3 we have a more specific estimate which depends on S^h and the interpolation operator Π .

The projection errors depend on the choice of basis functions. Suppose we use **piecewise linear** basis functions. Πu for $u \in H^k$ is defined by

$$\Pi u = \langle \Pi_L u_1, \Pi_L u_2 \rangle,$$

where Π_L is the interpolation operator for piecewise linear basis functions (see Appendix B).

We use the notation $|\cdot|_k$ for the seminorm of order k , i.e. $|u|_k = \|u^{(k)}\|$. For $u \in H^k$ we define $|u|_{H^k}$ by

$$|u|_{H^k}^2 = |u_1|_k^2 + |u_2|_k^2.$$

Proposition 6.7.1. *There exists a constant K_b such that for each $u \in H^2 \cap V$*

$$\|\Pi u - u\|_V \leq K_b \widehat{C}_L h |u|_{H^2}.$$

Proof.

$$\|\Pi u - u\|_{H^1}^2 = \|\Pi_L u_1 - u_1\|_1^2 + \|\Pi_L u_2 - u_2\|_1^2.$$

From Appendix B we have that

$$\|\Pi_L v - v\|_1 \leq \widehat{C}_L h |v|_2 \quad \text{for } v \in H^2(0, 1).$$

Therefore

$$\|\Pi u - u\|_{H^1}^2 \leq \widehat{C}_L^2 h^2 (|u_1|_2^2 + |u_2|_2^2) \leq \widehat{C}_L^2 h^2 |u|_{H^2}^2.$$

The result follows from Equation (6.6.3). $\square$

Now consider **piecewise Hermite cubic** basis functions. The interpolation operator Π is defined on H^2 by $\Pi u = \langle \Pi_c u_1, \Pi_c u_2 \rangle$ where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions. (See Appendix B.)

Proposition 6.7.2. *For $k \geq 2$ and $u \in H^k \cap V$,*

$$\|\Pi u - u\|_V \leq K_b \widehat{C}_c h^{\alpha-1} |u|_{H^\alpha}, \quad (6.7.1)$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

The proof is similar to the proof of Proposition 6.7.1.

6.7.3 Viscous damping. Energy estimates.

Problem TVW is a model with weak damping. In this subsection we apply Section 3.2. The theorems that may be applied depend on the regularity of the solution which depends on the given data.

We use piecewise hermite cubic basis functions in the following results. (With piecewise linear basis functions locking is a problem, see Section 6.8.)

In each of the applications below we assume $u \in C^2([0, T], V)$ and use the estimate (6.7.1).

Application of Corollary 3.2.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and suppose that $u'' \in \mathcal{L}^2([0, T], H^2 \cap V)$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\ & \leq K_b \widehat{C}_c h [|u(t)|_{H^2} + C_b |u'(t)|_{H^2} + 2|u_0|_{H^2} + 2C_b |u_d|_{H^2} \\ & \quad + C_b \int_0^t (|u''(\cdot)|_{H^2} + k|u'(\cdot)|_{H^2})], \end{aligned}$$

for $t \in [0, T]$.

Application of Theorem 3.2.3

If we do not make the assumptions on u' and u'' , we may still use Theorem 3.2.3. Since $H^2(0, 1)$ is dense in $H^1(0, 1)$, H^2 is dense in H^1 . By Proposition 6.6.3, H^2 is then dense in V and Assumption A5 is satisfied. If we choose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_d - u_d^h\|_W = 0,$$

and by Theorem 3.2.3,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_V = 0 \quad \text{for } t \in [0, \tau].$$

As mentioned in Section 5.1, it is sometimes possible to use the Aubin-Nitsche trick to obtain **higher order estimates** for the projection error. Note that the following is an inertia norm estimate.

Application of Corollary 3.2.2

Recall that in some cases we have that, for a given $f \in \mathcal{L}^2(\Omega)$ the solution u of $b(u, v) = (f, v)$ for all $v \in V$ is in $H^2(\Omega) \cap V$ and $\|u\|_2 \leq C\|f\|$, i.e. Assumption A4 is satisfied. As mentioned before, regularity theory for partial differential equations is beyond the scope of this thesis.

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^2$ and $u'' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq K_b^2 \widehat{C}_c^2 h^2 \widehat{C}_b |u(t)|_{H^2} + C_b C_0 h^2 \\ &\quad + C_b K_b^2 \widehat{C}_c^2 h^2 \widehat{C}_b \left(\int_0^t |u''(\cdot)|_{H^2} + k \int_0^t |u'(\cdot)|_{H^2} \right). \end{aligned}$$

6.7.4 Viscous damping. Inertia norm estimate.

In this subsection we apply the results of Section 3.4, where we require less regularity of the solution u .

Application of Corollary 3.4.3

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If the solution u of Problem TV satisfies

$u' \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$, then

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_b K_b \widehat{C}_c h |u(t)|_{H^2} + \sqrt{2} C_b K_b \widehat{C}_c h \left(3 \int_0^T |u'(\cdot)|_{H^2} \right. \\ &\quad \left. + 3K_a \int_0^T |u(\cdot)|_{H^2} + (2 + 3K_a T) |u_0|_{H^2} + 3T |u_d|_{H^2} \right), \end{aligned}$$

for each $t \in [0, T]$.

6.7.5 Boundary damping

In this subsection we apply the results for general damping (Section 3.3) and assume Assumption A2.

Application of Corollary 3.3.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$ with $k \geq 2$. Then, for any $t \in [0, T]$

$$\begin{aligned} &\|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ &\leq K_b \widehat{C}_c h^{\alpha-1} [|u|_{H^\alpha} + C_b |u'|_{H^\alpha}] \\ &\quad + \sqrt{24e^{3t}} K_b \widehat{C}_c h^{\alpha-1} \left[C_b \int_0^T |u''|_{H^\alpha} + 3K_a \max |u'|_{H^\alpha} + 3K_a \int_0^T |u''|_{H^\alpha} \right. \\ &\quad \left. + 2C_b |u_d|_{H^\alpha} + 2\sqrt{1 + K_a} |u_0|_{H^\alpha} + \sqrt{K_a} |u_d|_{H^\alpha} \right], \end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

As mentioned before, Assumption A4 is sometimes satisfied, in these cases it is possible to use the Aubin-Nitsche trick to obtain **higher order estimates** for the projection error.

Application of Corollary 3.3.2

Assume that $K \|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^{2\alpha-2}$ and $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$ with $k \geq 2$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq K_b^2 \widehat{C}_c^2 h^{2\alpha-2} \widehat{c}_b |u(t)|_\alpha + \sqrt{24e^{3t}} C_b K_b^2 \widehat{C}_c^2 h^{2\alpha-2} \widehat{c}_b \left(\int_0^T C_b |u''(\cdot)|_\alpha \right. \\ &\quad \left. + 3K_a \max |u'(t)|_\alpha + 3K_a \int_0^T |u''(\cdot)|_\alpha + \sqrt{K_a} |u_d|_\alpha \right) \\ &\quad + \sqrt{24e^{3t}} C_b K h^{2\alpha-2}, \end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Remark Results on the regularity of weak solutions for the Timoshenko beam equations are not common knowledge and further study of the literature is required.

6.8 Fully discrete approximation

Since viscous damping is weak damping, we can use the results in Section 4.3. For the boundary damping Section 4.5 is used.

Viscous damping

Consider the algorithm in Problem G^h -D-Baker.

Problem TVW^h -D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \dots, N-1$,

$$\begin{aligned} \delta_t u_k^h &= v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + a(v_{k+\frac{1}{2}}, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) &= \frac{1}{2}([q(t_k) + q(t_{k+1})], \psi) \end{aligned}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Application of Theorem 4.3.1

As mentioned in Section 5.1, the fully discrete result is combined with the inertia norm estimate in this case.

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If $u'(t) \in \mathcal{L}^2([0, T], H^2(\Omega) \cap V)$, $\tilde{q} \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem TVW^h -D, then

$$\begin{aligned} &\|u(t_k) - u_k^h\|_W \\ &\leq C_b K_b \widehat{C}_c h |u(t)|_{H^2} + \sqrt{2} C_b K_b \widehat{C}_c h \left(3 \int_0^T |u'(\cdot)|_{H^2} \right. \\ &\quad \left. + 3 K_a \int_0^T |u(\cdot)|_{H^2} + (2 + 3 K_a T) |u_0|_{H^2} + 3 T |u_d|_{H^2} \right) \\ &\quad + 7 T^2 \tau^2 \max \|u_h^{(4)}\|_W + 7 T \tau^2 \max \|u_h'''\|_W + \sqrt{2 K_a} \tau^4 \max \|u_h'''\|_W, \end{aligned}$$

for each $t_k \in (0, T)$.

Boundary damping

We consider the algorithm for central differences.

Problem TBW^h -D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a_B((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ + b(u_k^h, v) = (q(t_k), v_1), \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Application of Theorem 4.5.1

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and assume $u \in C^2([0, T], V)$ (Assumption A2). If $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$ with $k \geq 2$, $q \in C^2([0, T], \mathcal{L}^2(\Omega))$ and the sequence $\{u_k^h\}$ is a solution of Problem TBW^h -D, then

$$\begin{aligned} \|u(t_k) - u_k^h\|_V \leq & K_b \widehat{C}_c h^{\alpha-1} [|u|_{H^\alpha} + C_b |u'|_{H^\alpha}] \\ & + \sqrt{24e^{3t}} K_b \widehat{C}_c h^{\alpha-1} \left[C_b \int_0^T |u''|_{H^\alpha} + 3K_a \max |u'|_{H^\alpha} \right. \\ & + 3K_a \int_0^T |u''|_{H^\alpha} + 2C_b |u_d|_{H^\alpha} + 2\sqrt{1+K_a} |u_0|_{H^\alpha} + \sqrt{K_a} |u_d|_{H^\alpha} \Big] \\ & + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\|_W + \tau \max \|u_h'''(s)\|_W \right). \end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Remarks

1. As mentioned before, Proposition 2.5.3 can be used to obtain an estimate for $\|u_h^{(4)}\|$ in terms of the function q and the initial conditions $u_h(0) = d^h$ and $u_h'(0) = v^h$.
2. Other options for combining a semi-discrete and fully discrete estimate are possible, see Section 5.1.

6.8.1 Remarks

Other publications

In [FXX99] and [Sem94] the method of Baker [Bak76] is used but the damping is modal and of viscous type. As in [Bak76], the existence of a fourth order time derivative for the exact solution is assumed. They also assume that the initial approximations are projections.

Locking

In theory, piecewise linear basis functions are suitable for the Timoshenko beam. However, due to the phenomenon of “locking”, the rate of convergence in practice is not as predicted by the theory. In [Sem94] the mixed finite element method is used as a remedy and references are made to other methods. A different mixed finite element method is used in [FXX99]. However, for the Timoshenko beam locking can be avoided by using cubics according to [Bra01] (see p.302). This fact is confirmed in [VVR10].

Chapter 7

Beam with a damping tip body

7.1 Introduction

In the article [AS02] the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. The Euler-Bernoulli theory for a beam with Kelvin-Voigt damping is used. The beam is clamped at one end and the tip body is attached to the other end. The authors state that “the problem contains more complicated boundary conditions” than problems “treated previously” (and provide references). Actually the model is more realistic: the fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account.

The existence of a unique solution for the model problem is established in [AS02]. To obtain the result, the problem is written as an abstract differential equation and an abstract existence result from a previous paper [Kut86] is applied.

In this chapter we present a complete analysis of the model. Firstly we derive a correct linear approximation for the model problem. Next we also prove the existence of a unique solution. Our approach differs from that in [AS02]: we write the model problem in variational form and use results from [VV02] where general linear vibration models in variational form are considered. Finally we present convergence analysis of the Galerkin approximation.

Note that the damping is nonmodal. As explained in Section 6.1 simulation is necessary.

The work in Sections 7.1 to 7.6 was carried out in collaboration with M de Villiers and NFJ van Rensburg and published in the article [BDV14].

7.2 Model problem

The Euler-Bernoulli model for the transverse vibration of a beam is derived from the equation of motion for the beam deflection w (see Section 5.4)

$$\rho A \partial_t^2 w = \partial_x V + f \quad (7.2.1)$$

and the relation

$$\partial_x M = -V, \quad (7.2.2)$$

where f is a load.

The usual constitutive equation is $M = EI \partial_x^2 w$, where E is an elastic constant (Young's modulus) and I the area moment of inertia. Due to Kelvin-Voigt damping it changes to

$$M = EI \partial_x^2 w + \lambda \partial_t \partial_x^2 w, \quad (7.2.3)$$

where λ denotes the damping parameter. The partial differential equation (which we do not use) is

$$\rho A \partial_t^2 w = -EI \partial_x^4 w + \lambda \partial_t \partial_x^4 w + f.$$

The constitutive equation for the moment M and the relation between the moment and shear force V are also used to model the interface conditions.

The left endpoint of the beam is clamped where the boundary conditions are the usual

$$w(0, t) = \partial_x w(0, t) = 0. \quad (7.2.4)$$

The interface conditions at the other endpoint are determined by the interaction between the beam and the rigid body. This is explained in [AS02] in some detail. It is necessary to consider the equations of motion for the rigid body carefully when deriving these conditions.

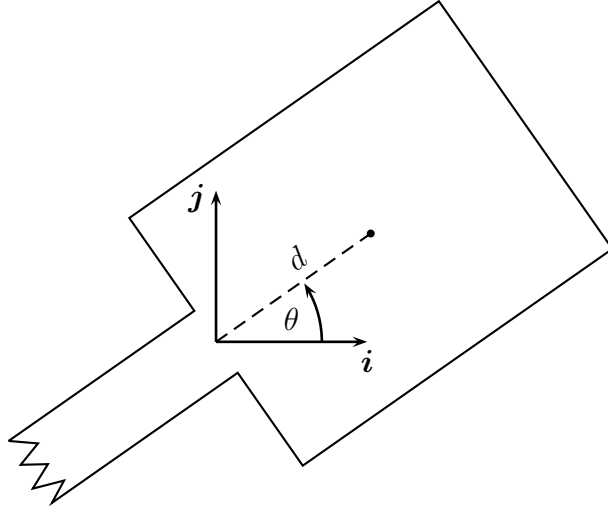


Figure 7.1: The tip body at the end of the beam

The position of the center of mass of the tip body relative to the endpoint of the beam is

$$d \cos \theta \mathbf{i} + d \sin \theta \mathbf{j},$$

where θ is the angle of the neutral plane with the horizontal (see Figure 7.1). Therefore the velocity $\mathbf{v}_C$ and acceleration $\mathbf{a}_C$ of the center of mass are given by

$$\begin{aligned} \mathbf{v}_C &= \partial_t w(\ell, t) \mathbf{j} - d\dot{\theta} \sin \theta \mathbf{i} + d\dot{\theta} \cos \theta \mathbf{j}, \\ \mathbf{a}_C &= \partial_t^2 w(\ell, t) \mathbf{j} - d\ddot{\theta} \sin \theta \mathbf{i} + d\ddot{\theta} \cos \theta \mathbf{j} - d\dot{\theta}^2 \cos \theta \mathbf{i} - d\dot{\theta}^2 \sin \theta \mathbf{j}, \end{aligned}$$

where ℓ denotes the length of the beam. For the linear approximation it is assumed that the term $\dot{\theta}^2 \sin \theta \mathbf{j}$ may be neglected, $\dot{\theta} \approx \partial_t \partial_x w(\ell, t)$, $\ddot{\theta} \approx \partial_t^2 \partial_x w(\ell, t)$ and $\cos \theta \approx 1$. Using these approximations, we have the following expressions for the vertical components of the velocity and acceleration:

$$\partial_t w(\ell, t) + d\partial_t \partial_x w(\ell, t) \quad \text{and} \quad \partial_t^2 w(\ell, t) + d\partial_t^2 \partial_x w(\ell, t).$$

In [AS02] the term $d\partial_t \partial_x w(\ell, t)$ in the expressions for the vertical component of the velocity is neglected. This should not be done and we motivate our point of view in the next section where we discuss the decay of energy for the system.

In the equations below γ and γ^* denote damping parameters, m the mass and J the moment of inertia of the rigid body. Using Newton's second law for the motion of the center of mass we have

$$\begin{aligned} m\partial_t^2 w(\ell, t) + md \partial_t^2 \partial_x w(\ell, t) \\ = -V(\ell, t) + f_B(t) - \gamma \partial_t w(\ell, t) - \gamma d \partial_t \partial_x w(\ell, t), \end{aligned} \quad (7.2.5)$$

where $f_B(t)$ is an external force that may act on the rigid body, e.g. gravity. Taking moments about the center of mass we have

$$J\partial_t^2 \partial_x w(\ell, t) = -M(\ell, t) + dV(\ell, t) - d\gamma^* \partial_t \partial_x w(\ell, t). \quad (7.2.6)$$

Following [AS02] we combine Equations (7.2.5) and (7.2.6) and find that

$$\begin{aligned} md \partial_t^2 w(\ell, t) + (J + md^2) \partial_t^2 \partial_x w(\ell, t) \\ = -M(\ell, t) - \gamma d \partial_t w(\ell, t) - (\gamma d + \gamma^*) d \partial_t \partial_x w(\ell, t) + df_B(t) \end{aligned} \quad (7.2.7)$$

It is convenient to rewrite Equations (7.2.5) and (7.2.7):

$$\begin{aligned} V(\ell, t) &= -m\partial_t^2 w(\ell, t) - md \partial_t^2 \partial_x w(\ell, t) \\ &\quad - \gamma \partial_t w(\ell, t) - \gamma d \partial_t \partial_x w(\ell, t) + f_B(t), \end{aligned} \quad (7.2.8)$$

$$\begin{aligned} M(\ell, t) &= -md \partial_t^2 w(\ell, t) - (J + md^2) \partial_t^2 \partial_x w(\ell, t) \\ &\quad - \gamma d \partial_t w(\ell, t) - (\gamma d + \gamma^*) d \partial_t \partial_x w(\ell, t) + df_B(t). \end{aligned} \quad (7.2.9)$$

Problem BeamDB

The mathematical model consists of

the Equations of motion (7.2.1) and (7.2.2),

the Constitutive equation (7.2.3) for the beam and

the Boundary conditions (7.2.4) and the Interface conditions (7.2.8) and (7.2.9).

Initial conditions $w(\cdot, 0) = w_0$ and $\partial_t w(\cdot, 0) = w_1$ need to be specified.

7.3 Variational form

Multiply Equation (7.2.1) by an arbitrary smooth function v and integrate. Using integration by parts and Equation (7.2.2) yields

$$\begin{aligned}
& \int_0^\ell \rho A (\partial_t^2 w(\cdot, t)) v \\
&= - \int_0^\ell V(\cdot, t) v' + [V(x, t) v(x)]_0^\ell + \int_0^\ell f(\cdot, t) v \\
&= - \int_0^\ell M(\cdot, t) v'' + [V(x, t) v(x)]_0^\ell + [M(x, t) v'(x)]_0^\ell + \int_0^\ell f(\cdot, t) v
\end{aligned}$$

Test functions A function v is a test function if $v \in C^2[0, \ell]$ and $v(0) = v'(0) = 0$. The space of test functions is denoted by $T[0, \ell]$.

It follows that

$$\begin{aligned}
& \int_0^\ell \rho A (\partial_t^2 w(\cdot, t)) v \\
&= - \int_0^\ell M(\cdot, t) v'' + V(\ell, t) v(\ell) + M(\ell, t) v'(\ell) + \int_0^\ell f(\cdot, t) v
\end{aligned}$$

for each $v \in T[0, \ell]$.

We now use the constitutive equation $M = EI \partial_x^2 w + \lambda \partial_t \partial_x^2 w$ and the interface conditions (7.2.8) and (7.2.9) to derive the variational form of the model problem.

It is convenient to introduce the following bilinear forms

$$\begin{aligned}
\bar{b}(u, v) &= \int_0^\ell EI u'' v'', \\
\bar{c}(u, v) &= \int_0^\ell \rho A u v + m u(\ell) v(\ell) + m d u'(\ell) v(\ell) + m d u(\ell) v'(\ell) \\
&\quad + (J + m d^2) u'(\ell) v'(\ell), \\
\bar{a}(u, v) &= \int_0^\ell \lambda u'' v'' + \gamma u(\ell) v(\ell) + \gamma d u'(\ell) v(\ell) + \gamma d u(\ell) v'(\ell) \\
&\quad + (\gamma d + \gamma^*) d u'(\ell) v'(\ell).
\end{aligned}$$

We now have the variational form of the model problem.

Problem BeamDBV

Find w such that for each $t > 0$, $w(\cdot, t) \in T(0, \ell)$ and

$$\begin{aligned} \bar{c}(\partial_t^2 w(\cdot, t), v) + \bar{a}(\partial_t w(\cdot, t), v) + \bar{b}(w(\cdot, t), v) \\ = (f(\cdot, t), v) + f_B(t)v(\ell) + df_B(t)v'(\ell) \end{aligned}$$

for each $v \in T(0, \ell)$ with $w(\cdot, 0) = w_0$ and $\partial_t w(\cdot, 0) = w_1$.

Remark Note that the bilinear forms $\bar{a}$, $\bar{b}$ and $\bar{c}$ are symmetric. The additional term $\gamma du'(\ell)v(\ell)$ in the definition of $\bar{a}$ is necessary for symmetry. Problem BeamDBV may be used to compute finite element approximations.

Mechanical energy

The mechanical energy (kinetic energy plus elastic potential energy) of the system is

$$E(t) = \frac{1}{2} \bar{c}(\partial_t w(\cdot, t), \partial_t w(\cdot, t)) + \frac{1}{2} \bar{b}(w(\cdot, t), w(\cdot, t)).$$

Using the symmetry of $\bar{b}$ and $\bar{c}$ and assuming that w is sufficiently smooth, we have

$$E'(t) = \bar{c}(\partial_t^2 w, \partial_t w) + \bar{b}(w, \partial_t w) = -\bar{a}(\partial_t w, \partial_t w)$$

for the homogeneous case. It is obvious that $\bar{b}(u, u)$ is nonnegative and it is not difficult to show that $\bar{a}(u, u)$ and $\bar{c}(u, u)$ are nonnegative.

$$\begin{aligned} \bar{c}(u, u) &= \int_0^\ell \rho A u^2 + m[u(\ell)]^2 + 2mdu'(\ell)u(\ell) + (J + md^2)[u'(\ell)]^2 \\ &= \int_0^\ell \rho A u^2 + m[u(\ell) + du'(\ell)]^2 + J[u'(\ell)]^2 \geq 0. \\ \bar{a}(u, u) &= \int_0^\ell \lambda(u'')^2 + \gamma[u(\ell)]^2 + 2\gamma du(\ell)u'(\ell) + (\gamma d + \gamma^*)d[u'(\ell)]^2 \\ &= \int_0^\ell \lambda(u'')^2 + \gamma[u(\ell) + du'(\ell)]^2 + \gamma^*d[u'(\ell)]^2 \geq 0. \end{aligned}$$

As a result $E'(t) \leq 0$. This result is to be expected from Physics. The fact that $\bar{a}$ is symmetric and $\bar{a}(u, u)$ nonnegative, is due to the additional term.

7.4 Weak variational form

Consider Problem BeamDBV. We start off as usual by considering the closure of the space of test functions. Let $V(0, \ell)$ be the closure of $T[0, \ell]$ in $H^2(0, \ell)$,

then $V(0, \ell)$ is a Hilbert space (being a closed subspace of a Hilbert space).

We require the **trace operator** which we denote by Γ , see Appendix A.5. Clearly $\Gamma u'$ is defined for $u \in H^2(0, \ell)$.

The following product spaces are necessary for the abstract problem.

Product spaces

$$X = \mathcal{L}^2(0, \ell) \times \mathbb{R} \times \mathbb{R}$$

$$H^m = H^m(0, \ell) \times \mathbb{R} \times \mathbb{R}$$

$$V_P = V(0, \ell) \times \mathbb{R} \times \mathbb{R}$$

$$V = \{v \in V_P \mid v_2 = \Gamma v_1, v_3 = \Gamma v'_1\}.$$

An element $y \in X$ is written as $y = \langle y_1, y_2, y_3 \rangle$. An obvious inner product for X is

$$(u, v)_X = \int_0^\ell u_1 v_1 + u_2 v_2 + u_3 v_3,$$

and we denote the corresponding norm by $\|\cdot\|_X$.

Definition Bilinear forms

For u and v in X

$$c(u, v) = \int_0^\ell \rho A u_1 v_1 + m u_2 v_2 + m d(u_3 v_2 + u_2 v_3) + (J + m d^2) u_3 v_3;$$

and for u and v in H^2

$$b(u, v) = \int_0^\ell E I u_1'' v_1'' \quad \text{and}$$

$$a(u, v) = \int_0^\ell \lambda u_1'' v_1'' + \gamma u_2 v_2 + \gamma d(u_3 v_2 + u_2 v_3) + (\gamma d + \gamma^*) d u_3 v_3.$$

Remarks

1. The bilinear forms a , b and c are symmetric.
2. For u and v in V , $a(u, v) = \bar{a}(u_1, v_1)$, $b(u, v) = \bar{b}(u_1, v_1)$ and $c(u, v) = \bar{c}(u_1, v_1)$.

3. It is essential to use product spaces since the bilinear form $\bar{c}$ is not defined on $\mathcal{L}^2(0, \ell)$.

For the weak variational form of the model problem we need to show that the bilinear forms c and b are inner products for X and V respectively. We use the well known Poincare type inequalities given in the proposition below. The boundedness of Γ is also required.

Proposition 7.4.1.

$$\begin{aligned} \|u\| &\leq \ell \|u'\| \leq \ell^2 \|u''\|, \\ |\Gamma u| &\leq \sqrt{\ell} \|u'\| \text{ for each } u \in V(0, \ell). \end{aligned}$$

Proof. Using the fundamental theorem of calculus, the inequalities are easy to prove for the space of test functions $T[0, \ell]$. Since $V(0, \ell)$ is the closure of $T[0, \ell]$ with respect to the norm of $H^2(0, \ell)$ the result follows. $\square$

Proposition 7.4.2. *There exists a constant K such that*

$$c(u, u) \geq K(u, u)_X \text{ for each } u \in X.$$

Proof. It is sufficient to show that there exists a constant K_1 such that $m(u_2 + du_3)^2 + Ju_3^2 \geq K_1(u_2^2 + u_3^2)$ for each $(u_2, u_3) \in \mathbb{R}_2$. From a well known inequality

$$m(u_2 + du_3)^2 \geq m(1 - \epsilon)u_2^2 + m(1 - \frac{1}{\epsilon})d^2u_3^2.$$

By the definition of the moment of inertia of a rigid body, $J \geq \beta md^2$ for some $\beta > 0$. Now choose $0 < \epsilon < 1$ and $\epsilon < (1 + \beta)^{-1}$ and the desired result follows. $\square$

Corollary 7.4.1. *The bilinear form c is an inner product for the space X .*

Definition Inertia space

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$. We refer to the vector space X equipped with this norm as the inertia space and denote it by W .

Proposition 7.4.3. *There exists a constant K such that*

$$b(u, u) \geq K\|u_1\|_2^2 \text{ for each } u \in V.$$

Proof. We use Proposition 7.4.1.

$$\begin{aligned}
\|u_1\|_2^2 &= \|u_1\|^2 + \|u_1'\|^2 + \|u_1''\|^2 \\
&\leq \|u_1''\|^2 (\ell^2 + \ell + 1) \\
&= \frac{\ell^2 + \ell + 1}{EI} b(u, u).
\end{aligned}$$

□

Corollary 7.4.2. *The bilinear form b is an inner product for V .*

Proof. Clearly $b(u, u) = 0$ implies that $u_1 = 0$ and therefore $u_2 = \Gamma u_1 = 0$ and $u_3 = \Gamma u_1' = 0$. □

Definition Energy space

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

We proceed to determine a weak variational form of the model problem. Let $f_X(t) = \langle f(\cdot, t), f_B(t), df_B(t) \rangle$, $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3} \rangle$ and $u_d = \langle w_1, \tilde{u}_{d,2}, \tilde{u}_{d,3} \rangle$, with $\tilde{u}_{0,2}$, $\tilde{u}_{0,3}$, $\tilde{u}_{d,2}$ and $\tilde{u}_{d,3}$ arbitrary.

Problem BeamDBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f_X(t), v)_X \quad \text{for each } v \in V$$

with $u(0) = u_0$ and $u'(0) = u_d$.

Remark It is natural to think that $\tilde{u}_{0,2} = \Gamma w_0$, etc. are the correct initial conditions. This is discussed in Section 7.6.

7.5 Auxiliary results

We need the results of this section to apply the existence theorem in Section 2.2.

Proposition 7.5.1. *The space V is a dense subset of X .*

Proof. Consider any $y \in W$. Since $C_0^\infty(0, \ell)$ is dense in $\mathcal{L}^2(0, \ell)$ there exists a sequence $\{\phi_n\} \subset C_0^\infty(0, \ell)$ such that $\|\phi_n - y_1\| \rightarrow 0$.

It is not difficult to construct sequences $\{\eta_n\}$ and $\{\zeta_n\}$ in $H^2(0, \ell)$ with the following properties

$$\begin{aligned}\Gamma\eta_n &= 1, \quad \Gamma\eta'_n = 0 \quad \text{and} \quad \|\eta_n\| \rightarrow 0, \\ \Gamma\zeta_n &= 0, \quad \Gamma\zeta'_n = 1 \quad \text{and} \quad \|\zeta_n\| \rightarrow 0.\end{aligned}$$

Now, let $v_n = \phi_n + y_2\eta_n + y_3\zeta_n$, then $v_n \in V(0, \ell)$, $\Gamma v_n = y_2$ and $\Gamma v'_n = y_3$.

Consequently $u_n = \langle v_n, v_n(\ell), v'_n(\ell) \rangle \in V$ and $\|u_n - y\|_X \rightarrow 0$. $\square$

Proposition 7.5.2. *There exists a constant K such that*

$$b(u, u) \geq Kc(u, u) \quad \text{for each } u \in V.$$

Proof. We use Proposition 7.4.1.

$$\begin{aligned}c(u, u) &= \int_0^\ell \rho A u_1^2 + m(u_2 + d u_3)^2 + J u_3^2 \\ &= \int_0^\ell \rho A u_1^2 + m(\Gamma u_1 + d \Gamma u'_1)^2 + J(\Gamma u'_1)^2 \\ &= \rho A \|u_1\|^2 + m(\Gamma u_1 + d \Gamma u'_1)^2 + J(\Gamma u'_1)^2 \\ &\leq \rho A \ell^4 \|u''_1\|^2 + 2m(\Gamma u_1)^2 + 2md^2(\Gamma u'_1)^2 + J(\Gamma u'_1)^2 \\ &\leq \|u''_1\|^2 (\rho A \ell^4 + 2m\ell^3 + 2md^2\ell + J\ell) \\ &\leq \|u_1\|_2^2 (\rho A \ell^4 + 2m\ell^3 + 2md^2\ell + J\ell)\end{aligned}$$

Now apply Proposition 7.4.3. $\square$

Proposition 7.5.3. *There exists a constant K such that*

$$|a(u, v)| \leq K \|u\|_V \|v\|_V.$$

for each u and v in V .

Proof. We use Proposition 7.4.1.

$$\begin{aligned}|a(u, v)| &\leq \lambda \|u''_1\| \|v''_1\| + \gamma |u_2 v_2| + \gamma d |u_3 v_2| + \gamma d |u_2 v_3| \\ &\quad + (\gamma d + \gamma^*) d |u_3 v_3| \\ &= \lambda \|u''_1\| \|v''_1\| + \gamma |\Gamma u_1 \Gamma v_1| + \gamma d |\Gamma u'_1 \Gamma v_1| + \gamma d |\Gamma u_1 \Gamma v'_1| \\ &\quad + (\gamma d + \gamma^*) d |\Gamma u'_1 \Gamma v'_1| \\ &\leq \|u_1\|_2 \|v_1\|_2 (\lambda + \gamma \ell^3 + 2\gamma d \ell^2 + (\gamma d + \gamma^*) d \ell).\end{aligned}$$

Now use Proposition 7.4.3. □

The result above is true for $\lambda \geq 0$. If $\lambda > 0$, the bilinear form a is positive definite on V and this has implications for existence results.

Proposition 7.5.4.

$$a(u, u) \geq \frac{\lambda}{EI} \|u\|_V^2$$

Proof.

$$a(u, u) \geq \lambda \|u_1''\|^2 = \frac{\lambda}{EI} \|u\|_V^2.$$

□

7.6 Existence

Theorem 2.2.1 is applied to the case where $\lambda = 0$ and Theorem 2.2.3 to the case where $\lambda > 0$. Note that the Assumptions E1, E2, E3 and E4 are satisfied due to Propositions 7.5.1, 7.5.2, 7.4.2 and 7.5.3 respectively. In the formulation of the theorems we denote the function $t \rightarrow f(\cdot, t)$ by $\tilde{f}$.

Theorem 7.6.1. *Suppose $\lambda = 0$ and*

- (a) $\tilde{f} \in C^1([0, \tau], \mathcal{L}^2(0, \ell))$ and $f_B \in C^1([0, \tau], \mathbb{R})$,
- (b) $u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that

$$b(u_0, v) + a(u_d, v) = c(y, v) \text{ for each } v \in V.$$

Then Problem PW has a unique solution

$$u \in C([0, \tau], V) \cap C^1([0, \tau], W) \cap C^1((0, \tau), V) \cap C^2((0, \tau), W).$$

Proof. Clearly $f_X \in C^1([0, \tau], X)$. The result follows from Theorem 2.2.1. □

Now consider the case $\lambda > 0$. The bilinear form a satisfies Assumption E5S (Proposition 7.5.4).

Theorem 7.6.2. *Suppose $\lambda > 0$ and*

(a) $\tilde{f}$ is locally Hölder continuous on $[0, \tau)$ with respect to the norm of $\mathcal{L}^2(0, \ell)$ and f_B is locally Hölder continuous on $[0, \tau)$,

(b) $u_0 \in V$ and $u_d \in W$.

Then Problem PW has a unique solution

$u \in C([0, \tau), V) \cap C^1([0, \tau), W) \cap C^2((0, \tau), W)$.

If $\tilde{f} = f_B = 0$, then $u \in C([0, \infty); V) \cap C^1([0, \infty); W) \cap C^\infty((0, \infty); V)$.

Proof. Clearly f_X is locally Hölder continuous on $[0, \tau)$ with respect to the norm $\|\cdot\|_X$ and the bilinear form a is positive definite by Proposition 7.5.4. The result follows from Theorem 2.2.3. $\square$

Sufficient conditions for existence

The case $\lambda > 0$ is trivial.

If $\lambda = 0$ sufficient conditions on $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3} \rangle$ and $u_d = \langle w_1, \tilde{u}_{d,2}, \tilde{u}_{d,3} \rangle$, are required to satisfy Condition (b) in Theorem 7.6.1. It is obviously necessary to assume that u_0 and u_d are in V which implies that $u_0 = \langle w_0, \Gamma w_0, \Gamma w'_0 \rangle$ and $u_d = \langle w_1, \Gamma w_1, \Gamma w'_1 \rangle$. Suppose $w_0 \in C^4[0, \ell]$ and $w_1 \in C^2[0, \ell]$. From the definition of the bilinear form b , using integration by parts we obtain

$$b(u, v) = \int_0^\ell EI w_0^{(4)} v_1 - EI w_0'''(\ell) v_1(\ell) + EI w_0''(\ell) v_1'(\ell).$$

From the definition of a , we have

$$\begin{aligned} & b(u, v) + a(u, v) \\ &= \int_0^\ell EI w_0^{(4)} v_1 - EI w_0'''(\ell) v_1(\ell) + EI w_0''(\ell) v_1'(\ell) \\ & \quad + \gamma w_1(\ell) v_1(\ell) + \gamma d w_1'(\ell) v_1(\ell) + \gamma d w_1(\ell) v_1'(\ell) + (\gamma d + \gamma^*) d w_1'(\ell) v_1'(\ell). \end{aligned}$$

Therefore

$$b(u, v) + a(u, v) = \int_0^\ell EI w_0^{(4)} v_1 \quad \text{for each } v \in V$$

if and only if

$$\begin{aligned} 0 &= -EI w_0'''(\ell) v_1(\ell) + EI w_0''(\ell) v_1'(\ell) \\ & \quad + \gamma w_1(\ell) v_1(\ell) + \gamma d w_1'(\ell) v_1(\ell) + \gamma d w_1(\ell) v_1'(\ell) + (\gamma d + \gamma^*) d w_1'(\ell) v_1'(\ell) \end{aligned}$$

for each $v \in V$. Since $v_1(\ell)$ and $v'_1(\ell)$ are arbitrary, it follows that

$$\begin{aligned} -EIw_0'''(\ell) + \gamma w_1(\ell) + \gamma dw_1'(\ell) &= 0; \\ EIw_0''(\ell) + \gamma dw_1(\ell) + (\gamma d + \gamma^*)dw_1'(\ell) &= 0. \end{aligned}$$

Therefore a sufficient condition for existence is $w_0 \in C^4[0, \ell]$, $w_1 \in C^2[0, \ell]$ and

$$\begin{aligned} w_0(0) &= w_0'(0) = w_1(0) = w_1'(0) = 0; \\ V(\ell, 0) &= -EIw_0'''(\ell) = -\gamma w_1(\ell) - \gamma dw_1'(\ell); \\ M(\ell, 0) &= EIw_0''(\ell) = -\gamma dw_1(\ell) - (\gamma d + \gamma^*)dw_1'(\ell). \end{aligned}$$

The conditions for the shear force $V(\ell, 0)$ and bending moment $M(\ell, 0)$ have an interesting physical interpretation. Comparing them to Equations (7.2.5) and (7.2.7) we see that the force and moment at the endpoint must match the force and moment due to damping.

7.7 Galerkin approximation

Consider the dimensionless form of the model problem, see Section 5.3. To start, construct a finite dimensional subspace $S^h(0, 1)$ of $V(0, 1)$, using piecewise Hermite cubic basis functions. Let S^h be the subspace of $S^h(0, 1) \times \mathbb{R} \times \mathbb{R}$ where $u_2 = u(1)$ and $u_3 = u'(1)$ for each $u \in S^h$. Then S^h is a finite dimensional subspace of V . Consider the special case of Problem G^h .

Problem BeamDBW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f_X(t), v)_X \quad \text{for each } v \in S^h$$

with $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$. The functions u_0^h and u_d^h are suitable approximations for u_0 and u_d in S^h .

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{f}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$.

7.7.1 Interpolation estimates for the one-dimensional hybrid models

An interpolation operator on the product spaces H^k can now be defined.

Definition

$$\Pi u = \langle \Pi_c u_1, u_2, u_3 \rangle \quad \text{for } u \in H^k.$$

We have that $u - \Pi u = \langle u_1 - \Pi_c u_1, 0, 0 \rangle$, and from the definition of the bilinear form b (see Section 7.4), we have that $\|u\|_V$ and $\|u_1\|_2$ are equivalent. Therefore

$$\|u - \Pi u\|_V^2 \leq C \|u_1 - \Pi_c u_1\|_2^2,$$

and we use the results from Appendix B. For Hermite cubic basis functions: If $u \in H^k(0, 1)$ with $k \geq 3$, then there exists a constant $\widehat{C}_c$ such that,

$$\|u - \Pi_c u\|_2 \leq \widehat{C}_c h^\alpha |u|_{\alpha+2},$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

Using the result above, we obtain an estimate for the interpolation operator Π :

If $u \in H^k \cap V$ with $k \geq 3$, then

$$\|u - \Pi u\|_V \leq \widehat{C} h^\alpha |u|_{\alpha+2}, \quad (7.7.1)$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

The above inequality replaces Assumption A3. Assumption 2 is still needed.

7.7.2 Convergence for the semi-discrete approximation

We now apply the results from Section 3.3 (general damping). In the applications below, we assume that $u \in C^2([0, T], V)$ and use estimate (7.7.1).

Application of Corollary 3.3.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0, T], H^k \cap V)$ with

$k \geq 3$. Then, for $t \in [0, T]$,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \widehat{C}h^\alpha (|u_1(t)|_{\alpha+2} + C_b|u'_1(t)|_{\alpha+2}) \\ & \quad + \sqrt{24e^{3t}}\widehat{C}h^\alpha \left[\int_0^T C_b|u''_1(\cdot)|_{\alpha+2} + 3K_a \max |u'_1(t)|_{\alpha+2} + 3K_a \int_0^T |u''_1(\cdot)|_{\alpha+2} \right. \\ & \quad \left. + 2C_b|u_{d,1}|_{\alpha+2} + 2\sqrt{1+K_a}|u_{0,1}|_{\alpha+2} + \sqrt{K_a}|u_{d,1}|_{\alpha+2} \right], \end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

As mentioned in Section 5.1, in some cases we have that Assumption A4 is satisfied and therefore it is possible to use the Aubin-Nitsche trick to obtain **higher order estimates** in the inertia norm for the projection error.

Application of Corollary 3.3.2

Assume that $K\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0, T], H^k \cap V)$ with $k \geq 3$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W & \leq \widehat{C}^2h^{2\alpha}\widehat{C}_b|u_1(t)|_{\alpha+2} + \sqrt{24e^{3t}}C_b\widehat{C}^2h^{2\alpha}\widehat{C}_b \left(\int_0^T C_b|u''_1(\cdot)|_{\alpha+2} \right. \\ & \quad \left. + 3K_a \max |u'_1(t)|_{\alpha+2} + 3K_a \int_0^T |u''_1(\cdot)|_{\alpha+2} + \sqrt{K_a}|u_{d,1}|_{\alpha+2} \right) \\ & \quad + \sqrt{24e^{3t}}C_bKh^{2\alpha}, \end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

7.7.3 Fully discrete

Consider the algorithm for central differences.

Problem BeamDBW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ + b(u_k^h, v) = (f_X(t_k), v)_X, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Application of Theorem 4.5.1

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and assume $u \in C^2([0, T], V)$. If $u'' \in \mathcal{L}^2([0, T], H^k \cap V)$ with $k \geq 3$, $f \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem BeamDBW^h-D, then

$$\begin{aligned} \|u(t_k) - u_k^h\|_V \leq & \widehat{C}h^\alpha (|u_1(t)|_{\alpha+2} + C_b|u_1'(t)|_{\alpha+2}) + \sqrt{24e^{3t}}\widehat{C}h^\alpha \left[\int_0^T C_b|u_1''(\cdot)|_{\alpha+2} \right. \\ & + 3K_a \max |u_1'(t)|_{\alpha+2} + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} \\ & + 2C_b|u_{d,1}|_{\alpha+2} + 2\sqrt{1+K_a}|u_{0,1}|_{\alpha+2} + \sqrt{K_a}|u_{d,1}|_{\alpha+2} \Big], \\ & + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\|_W + \tau \max \|u_h'''(s)\|_W \right). \end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Recall that Proposition 2.5.3 can be used to obtain an estimate for $\|u_h^{(4)}\|$ in terms of the function f and the initial conditions $u_h(0) = d^h$ and $u_h'(0) = v^h$, and that other options for combining a semi-discrete and fully discrete estimate are possible, see Section 5.1.

Chapter 8

Hybrid models

8.1 Vertical slender structure

8.1.1 Introduction

In this section we consider a hybrid Rayleigh beam model. The model problem is in [LVV05] where the origin of the problem is discussed and references are given. The modeling is carried out in great detail in [Lab06], where the finite element method is used for a comprehensive modal analysis. Existence results can be found in [Lab06] and [LVV09a].

Unwanted vibrations often occur in mechanical structures. The following example is described in [New89]:

“Because of their inherent low damping, free-standing welded steel structures are prone to oscillate in the wind. This may cause the chimney to fail due to metal fatigue. One method of artificially increasing the damping is to mount the chimney on a resilient foundation incorporating bearing pads made of a high-damping material.”

In [New84], Newland discusses efforts to compute natural frequencies using numerical approximations for a beam model. He stated that the results compared poorly with experimental results and did not consider hybrid models since they increased the complexity of finite element analysis considerably. As an alternative he proposed lumped parameter system models.

In [Lab06] and [LVV05] models that correspond to Newland's models [New89, p 129-132] and [New84] are presented. The slender structure (e.g. a steel chimney) is modelled as a Euler-Bernoulli or a Rayleigh beam mounted vertically and gravity is taken into account. Modelling the behavior of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations.

[Lab06] and [LVV05] constructed four mathematical models to match those of Newland and showed that the interface conditions and additional equations can be accommodated in the variational form. Consequently the finite element method can be used. In [Lab06] convergence of finite element approximations for eigenvalue problems is considered.

Note that subsections 8.1.2 to 8.1.6 of this thesis contain no new results. It is necessary to present the material here for completeness and readability.

8.1.2 Vertical Rayleigh beam incorporating the effect of gravity

From Section 5.3 we have the relevant equations of motion for the Rayleigh and Euler-Bernoulli theories in dimensionless form. It is necessary to adapt the model if gravity is taken into account. The axial force due to gravity is given by

$$S(x) = -\rho Ag(\ell - x).$$

As a consequence we have a moment density (measured in Newton)

$L = -S\partial_x w$ and equation (5.3.5) changes to (8.1.2) below. With $\mu = \frac{\rho g \ell}{E}$ and using the original notation, the dimensionless moment density is given by

$$L(x, t) = \mu(1 - x)\partial_x w(x, t).$$

Equations of motion

$$\partial_t^2 w = \partial_x V + q, \tag{8.1.1}$$

$$\frac{1}{\alpha} \partial_t^2 \partial_x w = V + \partial_x M + L, \tag{8.1.2}$$

where

$$L(x, t) = \mu(1 - x)\partial_x w(x, t). \tag{8.1.3}$$

The relevant **constitutive equation** is given in Section 5.4:

$$M = \frac{1}{\alpha} \partial_x^2 w. \quad (8.1.4)$$

The vertical slender structure is modelled as a vertical beam with $x = 0$ at the ground level. The boundary conditions at the top present no problem and we have that

$$M(1, t) = V(1, t) = 0.$$

8.1.3 The dynamics of the foundation block and resilient seating

To formulate the boundary conditions at the base, it is necessary to consider the equations of motion for the resilient seating and foundation block. Both are modelled as rigid bodies connected to linear elastic springs and linear damping mechanisms. We consider only the most comprehensive model: Model 4 in [LVV05] referred to as Problem VR 4 in [Lab06].

Figure 1: Simplified sketch of the system

The springs and damping mechanisms in the sketch are schematic.

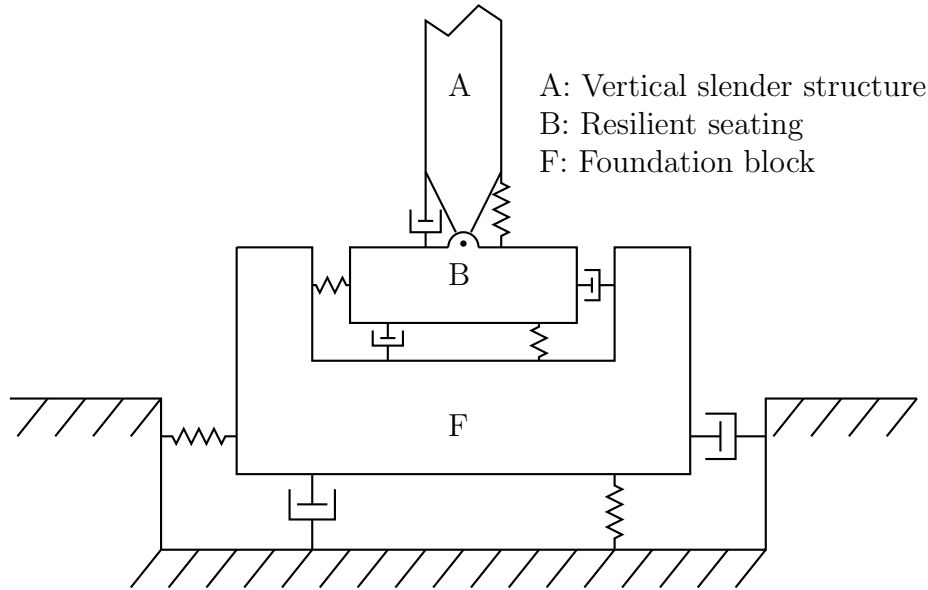
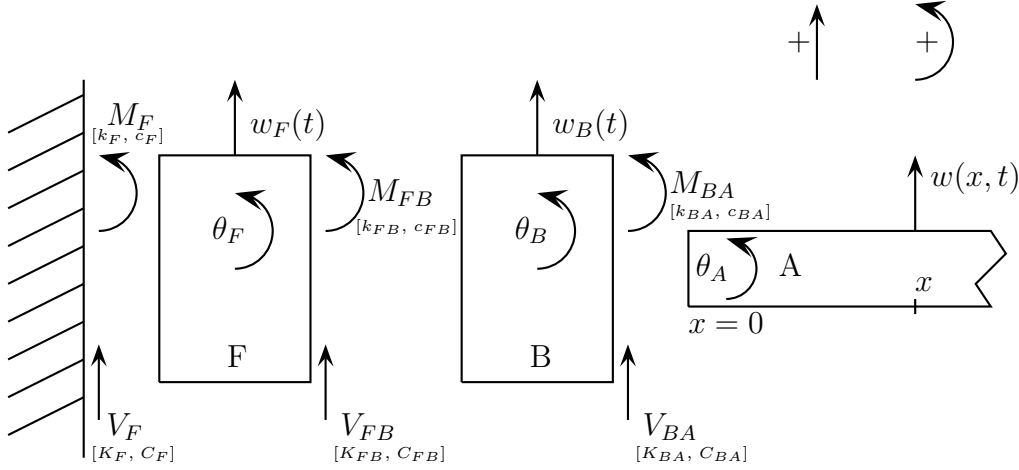


Figure 2: Displacements, angles of rotation, moments and forces

Convention: Moments and forces are denoted by the action of right on left. For instance, M_{FB} denotes the moment exerted by B on F.



Equations of motion

$$\begin{aligned}
 m_F \ddot{w}_F &= V_{FB} - V_F, \\
 m_B \ddot{w}_B &= V_{BA} - V_{FB}, \\
 I_F \ddot{\theta}_F &= M_{FB} - M_F, \\
 I_B \ddot{\theta}_B &= M_{BA} - M_{FB}.
 \end{aligned}$$

Constitutive equations

$$\begin{aligned}
 V_F &= K_F w_F + C_F \dot{w}_F, \\
 V_{FB} &= K_{FB} (w_B - w_F) + C_{FB} (\dot{w}_B - \dot{w}_F), \\
 M_F &= k_F \theta_F + c_F \dot{\theta}_F, \\
 M_{FB} &= k_{FB} (\theta_B - \theta_F) + c_{FB} (\dot{\theta}_B - \dot{\theta}_F).
 \end{aligned}$$

Interface conditions

Let $\theta_A(t)$ denote the rotation of the end point of the vertical structure.

$$\begin{aligned} M_{BA}(t) &= k_{BA}(\theta_A(t) - \theta_B(t)) + c_{BA}(\dot{\theta}_A(t) - \dot{\theta}_B(t)), \\ M_{BA}(t) &= M(0, t), \\ V_{BA}(t) &= V(0, t), \\ w_B(t) &= w(0, t), \\ \theta_B(t) &\neq \theta_A(t) \quad (\text{in general}). \end{aligned}$$

The following assumption is made for $\theta_A(t)$:

$$\theta_A(t) = \partial_x w(0, t) \text{ for the Rayleigh models.}$$

Dimensionless constants

The dimensionless constants for the foundation block and the resilient seating are

$$m^* = \frac{m}{\ell \rho A}, \quad \text{and} \quad I^* = \frac{I}{\ell^3 \rho A}.$$

The different elastic and damping constants are

$$K^* = \frac{K \ell^3}{EI}, \quad k^* = \frac{k \ell}{EI}, \quad C^* = \frac{C \ell^3}{EI t_0} \quad \text{and} \quad c^* = \frac{c \ell}{EI t_0}.$$

Returning to the original notation, all the constants in the equations of motion for the foundation block and resilient seating and the equations for the interface conditions, are replaced by the corresponding dimensionless constants.

Problem VR

Equations of motion: (8.1.1), (8.1.2) and (8.1.3).

Constitutive equation: (8.1.4).

Boundary conditions at $x = 1$: $M(1, t) = V(1, t) = 0$.

The interface conditions:

$$\begin{aligned} m_B \partial_t^2 w(0, t) &= V(0, t) - K_{FB} \left(w(0, t) - w_F(t) \right) \\ &\quad - C_{FB} \left(\partial_t w(0, t) - \dot{w}_F(t) \right), \end{aligned} \quad (8.1.5)$$

$$\begin{aligned} I_B \ddot{\theta}_B(t) &= k_{BA} \left(\partial_x w(0, t) - \theta_B(t) \right) + c_{BA} \left(\partial_t \partial_x w(0, t) - \dot{\theta}_B(t) \right) \\ &\quad - k_{FB} \left(\theta_B(t) - \theta_F(t) \right) - c_{FB} \left(\dot{\theta}_B(t) - \dot{\theta}_F(t) \right), \end{aligned} \quad (8.1.6)$$

$$\begin{aligned} M(0, t) &= k_{BA} \left(\partial_x w(0, t) - \theta_B(t) \right) \\ &\quad + c_{BA} \left(\partial_t \partial_x w(0, t) - \dot{\theta}_B(t) \right), \end{aligned} \quad (8.1.7)$$

$$\begin{aligned} m_F \ddot{w}_F(t) &= K_{FB} \left(w(0, t) - w_F(t) \right) + C_{FB} \left(\partial_t w(0, t) - \dot{w}_F(t) \right) \\ &\quad - K_F w_F(t) - C_F \dot{w}_F(t), \end{aligned} \quad (8.1.8)$$

$$\begin{aligned} I_F \ddot{\theta}_F(t) &= k_{FB} \left(\theta_B(t) - \theta_F(t) \right) + c_{FB} \left(\dot{\theta}_B(t) - \dot{\theta}_F(t) \right) \\ &\quad - k_F \theta_F(t) - c_F \dot{\theta}_F(t). \end{aligned} \quad (8.1.9)$$

The diagrams and equations in this subsection are from [LVV05] and [Lab06].

8.1.4 Variational form

Use equation (5.3.7) from Subsection 5.3.2 and add a term for the moment density L :

$$\begin{aligned} \int_0^1 \partial_t^2 w v + \frac{1}{\alpha} \int_0^1 \partial_t^2 \partial_x w v' &= - \int_0^1 M v'' + \int_0^1 q v - M(0) v'(0) - V(0) v(0) \\ &\quad + \int_0^1 L v'. \end{aligned} \quad (8.1.10)$$

From Equation (8.1.3) we have

$$(L(\cdot, t), v') = \mu \int_0^1 (1 - x) \partial_x w(x, t) v'(x) dx.$$

Combining the results above, we obtain a general variational form.

$$\begin{aligned} & (\partial_t^2 w(\cdot, t), v) + \frac{1}{\alpha} (\partial_t^2 \partial_x w(\cdot, t), v') = -\frac{1}{\beta} (\partial_x^2 w(\cdot, t), v'') \\ & + \mu \int_0^1 (1-x) \partial_x w(x, t) v'(x) dx - V(0, t) v(0) - M(0, t) v'(0). \end{aligned} \quad (8.1.11)$$

For Problem VR there are no restrictions on the space of test functions and the solution w must satisfy Equation (8.1.11) for an arbitrary function $v \in C^2[0, 1]$.

Following [LVV09a] and [Lab06], we define the following **bilinear forms** for Problem VR.

$$\begin{aligned} c_A(u, v) &= (u, v) + \frac{1}{\alpha} (u', v') + m_B u(0) v(0), \\ b_A(u, v) &= \frac{1}{\beta} (u'', v'') - \mu \int_0^1 (1-x) u'(x) v'(x) dx. \end{aligned}$$

Equations (8.1.11), (8.1.5) and (8.1.7) yield the following equation in terms of the bilinear forms.

$$\begin{aligned} c_A(\partial_t^2 w(\cdot, t), v) &= -b_A(w(\cdot, t), v) + K_{FB}(w_F(t) - w(0, t))v(0) \\ &\quad - C_{FB}(\partial_t w(0, t) - \dot{w}_F(t))v(0) \\ &\quad - c_{BA}(\partial_t \partial_x w(0, t) - \dot{\theta}_B(t))v'(0) \\ &\quad + k_{BA}(\theta_B(t) - \partial_x w(0, t))v'(0) \\ &\quad + (q(\cdot, t), v), \end{aligned} \quad (8.1.12)$$

for each $v \in C^2[0, 1]$.

We derived the variational form of Problem VR.

Problem VRV

Find w , θ_B , w_F and θ_F such that for each $t > 0$, $w(\cdot, t) \in C^2[0, 1]$,

Equation (8.1.12) holds for each $v \in C^2[0, 1]$ and

Equations (8.1.6), (8.1.8) and (8.1.9) hold.

8.1.5 Weak variational form

For the analysis of the vibration problem we consider the weak variational form. For the weak variational form we use c_A and b_A as defined in Subsection 8.1.4, but with weak derivatives.

Let $y(t) = w(\cdot, t)$ and let $\dot{y}(t)$ denote the derivative w.r.t. the norm of $\mathcal{L}^2(0, 1)$ or $H^m(0, 1)$ (see Section 2.1). We use the dot for the time derivative here to avoid confusion.

With the new notation, Equation (8.1.12) becomes

$$\begin{aligned}
c_A(\ddot{y}(t), v) = & -b_A(y(t), v) \\
& - K_{FB} \left(\Gamma y(t) - w_F(t) \right) \Gamma v \\
& - C_{FB} \left(\Gamma(\dot{y}(t)) - \dot{w}_F(t) \right) \Gamma v \\
& - k_{BA} \left(\Gamma \left[(y(t))' \right] - \theta_B(t) \right) (\Gamma v') \\
& - c_{BA} \left(\Gamma \left[(\dot{y}(t))' \right] - \dot{\theta}_B(t) \right) (\Gamma v') \\
& + (q(\cdot, t), v).
\end{aligned} \tag{8.1.13}$$

Remark Note that $\partial_t w(0, t)$ is replaced by $\Gamma(\dot{y}(t))$ and not $d_t(\Gamma y)(t)$. This is necessary for the weak variational form of the problem, see the remark at the end of subsection 8.1.6.

The weak variational form of Problem VR is to find functions y, θ_B, w_F and θ_F such that (8.1.13) holds for any $v \in H^2(0, 1)$ and equations (8.1.6), (8.1.8) and (8.1.9) are satisfied.

In [Lab06], Chapter 5, it is shown how to construct the relevant matrices and natural frequencies and modes of vibration are calculated.

For the finite element analysis, we reformulate the problem.

$$\begin{aligned}
c_A(\ddot{y}(t), v_1) + I_B \ddot{\theta}_B(t) v_2 + m_F \ddot{w}_F(t) v_3 + I_F \ddot{\theta}_F(t) v_4 \\
= -b_A(y(t), v_1) - K_{FB} \left(\Gamma y(t) - w_F(t) \right) \left(\Gamma v_1 - v_3 \right) \\
- C_{FB} \left(\Gamma(\dot{y}(t)) - \dot{w}_F(t) \right) \left(\Gamma v_1 - v_3 \right) \\
- k_{BA} \left(\Gamma[(y(t))'] - \theta_B(t) \right) \left(\Gamma v_1' - v_2 \right) \\
- c_{BA} \left(\Gamma[(\dot{y}(t))'] - \dot{\theta}_B(t) \right) \left(\Gamma v_1' - v_2 \right) \\
- k_{FB} \left(\theta_B(t) - \theta_F(t) \right) \left(v_2 - v_4 \right) \\
- c_{FB} \left(\dot{\theta}_B(t) - \dot{\theta}_F(t) \right) \left(v_2 - v_4 \right) \\
- K_F w_F(t) v_3 - C_F \dot{w}_F(t) v_3 - k_F \theta_F(t) v_4 - c_F \dot{\theta}_F(t) v_4 \\
+ (q(\cdot, t), v_1), \tag{8.1.14}
\end{aligned}$$

for each $v_1 \in H^2(0, 1)$ and any real numbers v_2, v_3 and v_4 .

Next we construct the product spaces for the weak form of the variational problem.

Product spaces

Define the product spaces

$$X = \mathcal{L}^2(0, 1) \times \mathbb{R}^3, \quad W = H^1(0, 1) \times \mathbb{R}^3 \quad \text{and} \quad V = H^2(0, 1) \times \mathbb{R}^3.$$

Bilinear forms

For u and v in W ,

$$\begin{aligned}
c(u, v) &= c_A(u_1, v_1) + I_B u_2 v_2 + m_F u_3 v_3 + I_F u_4 v_4, \\
a(u, v) &= C_{FB} (\Gamma u_1 - u_3) (\Gamma v_1 - v_3) + c_{BA} (\Gamma u_1' - u_2) (\Gamma v_1' - v_2) \\
&\quad + c_{FB} (u_2 - u_4) (v_2 - v_4) + C_F u_3 v_3 + c_F u_4 v_4.
\end{aligned}$$

For u and v in V ,

$$\begin{aligned}
b(u, v) &= b_A(u_1, v_1) + K_{FB} (\Gamma u_1 - u_3) (\Gamma v_1 - v_3) \\
&\quad + k_{BA} (\Gamma u_1' - u_2) (\Gamma v_1' - v_2) + k_{FB} (u_2 - u_4) (v_2 - v_4) \\
&\quad + K_F u_3 v_3 + k_F u_4 v_4.
\end{aligned}$$

Note that a , b and c are all symmetric.

We are now ready to reformulate the weak variational form of Problem VR.

Problem VRW

Find u such that for each $t > 0$, $u'(t) \in V$, $u''(t) \in W$, and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (q(\cdot, t), v_1) \quad \text{for each } v \in V \quad (8.1.15)$$

while $u(0) = u_0$ and $u'(0) = u_d$.

Remarks

1. The vector $\langle u_1(t), u_2(t), u_3(t), u_4(t) \rangle$ represent $\langle y(t), \theta_B(t), w_F(t), \theta_F(t) \rangle$ and Equations (8.1.14) and (8.1.15) are the same.
2. No confusion can arise now if time derivatives are denoted by primes.

8.1.6 Existence and properties of a solution

To apply the existence theorem (Theorem 2.2.1), we need Assumptions E1, E2, E3 and E4. In this subsection we present results from [Lab06] and [LVV09a] without proof.

Stability condition

In the following results, we assume throughout that the inequalities below hold for the physical constants.

$$1 > 2\mu, \quad k_{BA} > 4\mu, \quad k_{FB} > 8\mu \quad \text{and} \quad k_F > 8\mu.$$

These inequalities are necessary to ensure the stability of the equilibrium and are physically realistic, as can be seen in the applications in [LVV05] and [Lab06].

Proposition 8.1.1. *The bilinear form c is an inner product for the space W .*

Definition Inertia space W

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$.

Proposition 8.1.2. *There exists a constant C_c such that*

$$\|\cdot\|_X \leq C_c \|\cdot\|_W.$$

Proposition 8.1.3. *V is a dense subset of W and W is dense in X .*

Proposition 8.1.4. *There exists a constant C_b such that*

$$b(u, u) \geq C_b \|u\|_W^2 \text{ for each } u \in V.$$

Corollary 8.1.1. *The bilinear form b is an inner product for V .*

Definition Energy space V

The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

Proposition 8.1.5. *There exists a constant K_{bc} such that*

$$\|u\|_X^2 \leq K_{bc} b(u, u)$$

for each $u \in V$.

Proposition 8.1.6. *There exists a constant K_{ba} such that for any $u \in V$ and $v \in V$,*

$$|a(u, v)| \leq K_{ba} \|u\|_V \|v\|_V.$$

Theorem 2.2.1 may now be applied. Note that the Assumptions E1, E2, E3 and E4 are satisfied due to Propositions 8.1.1 to 8.1.6 and Corollary 8.1.1. In the formulation of the theorem we denote the function $t \rightarrow q(\cdot, t)$ by $\tilde{q}$.

Theorem 8.1.1. *Suppose*

- (a) *the stability condition is satisfied,*
- (b) *$\tilde{q} \in C^1([0, \tau), \mathcal{L}^2(0, 1))$,*
- (c) *$u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that*

$$b(u_0, v) + a(u_d, v) = c(y, v) \text{ for each } v \in V.$$

Then Problem VRW has a unique solution

$$u \in C([0, \tau), V) \cap C^1([0, \tau), W) \cap C^1((0, \tau), V) \cap C^2((0, \tau), W).$$

Proof. Clearly $\langle \tilde{q}, 0, 0, 0 \rangle \in C^1([0, \tau), X)$. The result follows from Theorem 2.2.1. $\square$

Remark $y = u_1 \in C^1((0, T); H^2(0, 1))$ and we find for example that $(\dot{y}(t))' = d_t(y'(t))$ and $d_t[\Gamma y(t)] = \Gamma \dot{y}(t)$.

8.1.7 Finite element method

As mentioned in the introduction (Subsection 8.1.1), the finite element method was used for modal analysis in [LVV05] and [Lab06] and simulation of dynamical problems was not done.

Let $S^h(0, 1)$ denote the subspace of $H^2(0, 1)$ consisting of piecewise Hermite cubic basis functions. Define $S^h = S^h(0, 1) \times \mathbb{R}^3$, then S^h is a finite dimensional subspace of V .

Problem VRW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (q(\cdot, t), v_1) \quad \text{for each } v \in S^h$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{q}$ is continuous w.r.t. the norm of $\mathcal{L}^2(0, 1)$.

As before, we consider Assumption A3. For this problem we have that $H(V, k) = H^k \cap V$.

Interpolation

An interpolation operator on the product spaces $H^k(0, 1) \times \mathbb{R}^3$ can now be defined.

Definition

$$\Pi u = \langle \Pi_c u_1, u_2, u_3, u_4 \rangle \quad \text{for } u \in H^k(0, 1) \times \mathbb{R}^3,$$

where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions (see Appendix B).

We have that

$$u - \Pi u = \langle u_1 - \Pi_c u_1, 0, 0, 0 \rangle \tag{8.1.16}$$

and we use the following result.

Proposition 8.1.7. *There exists a constant $\tilde{C}_b$ such that*

$$\|u - \Pi u\|_V^2 \leq \tilde{C}_b \|u_1 - \Pi_c u_1\|_2^2.$$

Proof. Consider $v = \langle v_1, 0, 0, 0 \rangle$ where $v_1 \in H^k(0, 1)$. From the definition of b we have

$$\begin{aligned} b(v, v) &= b_A(v_1, v_1) + K_{FB}(\Gamma v_1)^2 + k_{BA}(\Gamma v_1')^2 \\ &\leq \frac{1}{\beta}(v_1'')^2 - \mu \int_0^1 (1-x)(v_1')^2 dx + K_\ell K_{FB} \|v_1\|_1^2 + K_\ell k_{BA} \|v_1'\|_1^2 \\ &\leq \tilde{C}_b \|v_1\|_2^2, \end{aligned}$$

where $\tilde{C}_b$ depends on the constants β , μ , K_ℓ , K_{FB} and k_{BA} . The result now follows from (8.1.16). $\square$

The results from Appendix B can be used. For the piecewise Hermite cubic functions:

If $u \in H^k(0, 1)$ for $k \geq 3$, then

$$\|u - \Pi_c u\|_2 \leq \hat{C}_c h^\alpha |u|_{\alpha+2},$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Combining the result above with Proposition 8.1.7, we obtain the following estimate for the interpolation operator Π :

If $u \in H^k(0, 1) \times \mathbb{R}^3$ for $k \geq 3$

$$\|u - \Pi u\|_V \leq \hat{C} h^\alpha |u_1|_{\alpha+2}, \quad (8.1.17)$$

where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$ and the constant $\hat{C}$ depends on $\tilde{C}_b$ and $\hat{C}_c$.

8.1.8 Convergence of the semi-discrete approximation

We now apply Section 3.3 (general damping). We use estimate (8.1.17) and assume $u \in C^2([0, T], V)$ (Assumption A2) in the applications below.

Application of Corollary 3.3.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$

with $k \geq 3$. Then, for $t \in [0, T]$,

$$\begin{aligned}
& \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\
& \leq \widehat{C}h^\alpha (|u_1(t)|_{\alpha+2} + C_b|u'_1(t)|_{\alpha+2}) \\
& \quad + \sqrt{24e^{3t}}\widehat{C}h^\alpha \left[\int_0^T C_b|u''_1(\cdot)|_{\alpha+2} + 3K_a \max |u'_1(t)|_{\alpha+2} + 3K_a \int_0^T |u''_1(\cdot)|_{\alpha+2} \right. \\
& \quad \left. + 2C_b|u_d|_{\alpha+2} + 2\sqrt{1+K_a}|u_0|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2} \right],
\end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

As mentioned in Section 5.1, in some cases we have that Assumption A4 is satisfied and therefore it is possible to use the Aubin-Nitsche trick to obtain **higher order estimates** in the inertia norm for the projection error.

Application of Corollary 3.3.2

Assume that $K\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$ with $k \geq 3$. Then, for any $t > 0$,

$$\begin{aligned}
\|u(t) - u_h(t)\|_W & \leq \widehat{C}^2h^{2\alpha}\widehat{C}_b|u_1(t)|_{\alpha+2} + \sqrt{24e^{3t}}C_b\widehat{C}^2h^{2\alpha}\widehat{C}_b \left(\int_0^T C_b|u''_1(\cdot)|_{\alpha+2} \right. \\
& \quad \left. + 3K_a \max |u'_1(t)|_{\alpha+2} + 3K_a \int_0^T |u''_1(\cdot)|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2} \right) \\
& \quad + \sqrt{24e^{3t}}C_bKh^{2\alpha},
\end{aligned}$$

where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

8.1.9 Fully discrete

Consider the algorithm for central differences.

Problem VRW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$\begin{aligned}
c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\
+ b(u_k^h, v) = (q(t_k), v_1),
\end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Application of Theorem 4.5.1

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume $u \in C^2([0, T], V)$. If $u'' \in \mathcal{L}^2([0, T], H^k(\Omega) \cap V)$ with $k \geq 3$, $q \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem VRW^h-D, then

$$\begin{aligned} & \|u(t_k) - u_k^h\|_V \\ & \leq \widehat{C}h^\alpha (|u_1(t)|_{\alpha+2} + C_b|u_1'(t)|_{\alpha+2}) \\ & \quad + \sqrt{24e^{3t}}\widehat{C}h^\alpha \left[\int_0^T C_b|u_1''(\cdot)|_{\alpha+2} + 3K_a \max |u_1'(t)|_{\alpha+2} \right. \\ & \quad \left. + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} + 2C_b|u_d|_{\alpha+2} + 2\sqrt{1+K_a}|u_0|_{\alpha+2} + \sqrt{K_a}|u_d|_{\alpha+2} \right], \\ & \quad + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \left\| u_h^{(4)}(s) \right\|_W + \tau \max \|u_h'''(s)\|_W \right). \end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = 1$ if $k = 3$ and $\alpha = 2$ if $k \geq 4$.

Recall that an estimate for $\|u_h^{(4)}\|$ in terms of the data can be found. See Section 5.1 for a discussion on other possibilities.

8.2 Timoshenko cantilever beam with tip body and boundary damping

8.2.1 The model problem

In [ZVV04] the finite element method is used to compare natural frequencies and modes of vibration. For concerns regarding modal analysis, see Section 6.1. A numerical method to simulate the vibration is called for.

The equations of motion and constitutive equations (in dimensionless form) are in Section 6.2. For convenience they are repeated here.

$$\partial_t^2 w = \partial_x V + q, \quad (8.2.1)$$

$$\frac{1}{\alpha} \partial_t^2 \phi = V + \partial_x M, \quad (8.2.2)$$

$$M = \frac{1}{\beta} \partial_x \phi, \quad (8.2.3)$$

$$V = \partial_x w - \phi. \quad (8.2.4)$$

The boundary damping (or control) is considered in Subsection 6.3.2. With a tip body, we have the following interface conditions.

$$\begin{aligned} m\partial_t^2 w(\ell, t) &= -V(\ell, t) - k_1 \partial_t w(\ell, t), \\ I_m \partial_t^2 \phi(\ell, t) &= -M(\ell, t) - k_2 \partial_t \phi(\ell, t). \end{aligned}$$

Let $m^* = \frac{m}{\rho A \ell}$ and $I_m^* = \frac{I_m}{\ell \rho A I}$, then the interface conditions in dimensionless form are $m\partial_t^2 w(1, t) = -V(1, t) - \gamma_1 \partial_t w(1, t)$ and $I_m \partial_t^2 \phi(1, t) = -M(1, t) - \gamma_2 \partial_t \phi(1, t)$, where $\gamma_1 = \frac{k_1 \ell}{T A G \kappa^2}$ and $\gamma_2 = \frac{k_2}{A G \kappa^2 \ell T}$.

Returning to the original notation we present the model problem.

Problem TBB

The mathematical model consists of

the equations of motion (8.2.1) and (8.2.2),

the constitutive equations (8.2.3) and (8.2.4),

the boundary conditions $w(0, t) = \phi(0, t) = 0$ and

the interface conditions

$$m\partial_t^2 w(1, t) = -V(1, t) - \gamma_1 \partial_t w(1, t) \quad \text{and} \quad (8.2.5)$$

$$I_m \partial_t^2 \phi(1, t) = -M(1, t) - \gamma_2 \partial_t \phi(1, t). \quad (8.2.6)$$

The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

8.2.2 Variational form

Consider equation (6.4.3).

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi \\ = & - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t))(v' - \psi) + \int_0^1 q(\cdot, t) v \\ & + V(1, t) v(1) - V(0, t) v(0) + M(1, t) \psi(1) - M(0, t) \psi(0). \end{aligned} \quad (8.2.7)$$

Multiply (8.2.5) by $v(1)$, (8.2.6) by $\psi(1)$ and add them to (8.2.7):

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + m \partial_t^2 w(1, t) v(1) + I_m \partial_t^2 \phi(1, t) \psi(1) \\ = & - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t))(v' - \psi) + \int_0^1 q(\cdot, t) v \\ & - V(0, t) v(0) - M(0, t) \psi(0) - \gamma_1 \partial_t w(1, t) v(1) - \gamma_2 \partial_t \phi(1, t) \psi(1). \end{aligned}$$

Using the test functions $T_2[0, 1]$ (Section 6.4.1), we obtain the variational form.

Problem TBBV

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_2[0, 1]$, $\phi(\cdot, t) \in T_2[0, 1]$ and

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + m \partial_t^2 w(1, t) v(1) + I_m \partial_t^2 \phi(1, t) \psi(1) \\ = & - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t))(v' - \psi) \\ & - \gamma_1 \partial_t w(1, t) v(1) - \gamma_2 \partial_t \phi(1, t) \psi(1) + \int_0^1 q(\cdot, t) v. \end{aligned}$$

for each pair $\langle v, \psi \rangle \in T_2[0, 1] \times T_2[0, 1]$.

8.2.3 Product spaces

It is necessary to adapt the product spaces in Section 6.5. Note that existence of solutions was not considered in [ZVV04].

$$X_B = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1) \times \mathbb{R}^2.$$

$$H_B^m = H^m(0, 1) \times H^m(0, 1) \times \mathbb{R}^2.$$

Notation

An element $y \in X$ is written as $y = \langle y_1, y_2, y_3, y_4 \rangle$.

A natural inner product for X_B is

$$(x, y)_{X_B} = (x_1, y_1) + (x_2, y_2) + x_3 y_3 + x_4 y_4,$$

and the corresponding norm is denoted by $\|\cdot\|_{X_B}$.

The natural inner product for the product space H_B^m is

$$(x, y)_{H_B^m} = (x_1, y_1)_m + (x_2, y_2)_m + x_3 y_3 + x_4 y_4$$

and the corresponding norm is denoted by $\|\cdot\|_{H_B^m}$.

Recall that for u and v in H^1

$$b(u, v) = \int_0^1 \frac{1}{\beta} u'_2 v'_2 + \int_0^1 (u'_1 - u_2)(v'_1 - v_2),$$

where the derivatives are weak derivatives.

We define the following bilinear form: For u and v in X

$$c_B(u, v) = \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2 + m u_3 v_3 + I_m u_4 v_4.$$

Note that the bilinear forms are symmetric. Also note that b is the same as in Section 6.5 and $c_B = c + m u_3 v_3 + I_m u_4 v_4$. As a consequence we obtain the following result.

Proposition 8.2.1. *The bilinear form c is an inner product for the space X .*

Proof. Use Proposition 6.5.1 and $c_B \geq c$. □

Definition Inertia space W_B

We refer to the vector space X_B equipped with the inner product c_B as the space W_B . The norm $\|\cdot\|_{W_B}$ is defined by $\|u\|_{W_B} = \sqrt{c_B(u, u)}$.

The energy space for this problem is

$$V_B = \{y \in V_2 \times \mathbb{R}^2 \mid y_3 = \Gamma y_1, y_4 = \Gamma y_2\},$$

where V_2 is defined in Section 6.5. (See Appendix A.5 on trace.)

Proposition 8.2.2. *V_B is a dense subset of W_B .*

Proof. V_B is dense in W_B since V_2 is dense in W . □

Proposition 8.2.3. *There exists a constant C_B such that*

$$b(u, u) \geq C_B \|u\|_{X_B}^2 \text{ for each } u \in V_B.$$

Proof. From Proposition 6.5.4, $b(u, u) \geq C_b \|u\|_X^2$ for each $u \in V_2$. Now,

$$|u_3| = |\Gamma u_1| \leq \|u_1\|_1 \text{ and } |u_4| \leq \|u_2\|_1$$

if $u \in V_B$ and $\|u_1\|_1^2 + \|u_2\|_1^2 \leq \frac{3}{C_b} b(u, u)$ by Proposition 6.6.3. Therefore

$$2b(y, y) \geq \frac{C_b}{3} \|y\|_{X_B}^2$$

for $y \in V_B$. □

Corollary 8.2.1. *The bilinear form b is an inner product for V_B .*

Definition Energy space V_B

We refer to the space V_B equipped with the inner product b as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

8.2.4 Weak variational form and existence

In this section we show how the existence and uniqueness of a weak solution for the model problem follow from Section 2.2. We are not aware of any existence results for this problem.

Note that the bilinear form a_B for Problem TBB is the same as for Problem TB1.

In the formulation of the model problem we denote the mapping $t \rightarrow \langle q(\cdot, t), 0, 0, 0 \rangle$ by q_X and as always, the mapping $t \rightarrow q(\cdot, t)$ by $\tilde{q}$.

Problem TBBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a_B(u'(t), v) + b(u(t), v) = (q_X(t), v)_{X_B} \text{ for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d \rangle$.

For convenience we restate Proposition 6.6.2.

Proposition 8.2.4. *There exists a constant K_B such that, for all u and v in V ,*

$$|a_B(u, v)| \leq K_B \|u\|_V \|v\|_V.$$

Theorem 2.2.1 is applied. Note that the Assumptions E1, E2, E3 and E4 are satisfied due to Propositions 8.2.1 to 8.2.5 and Corollary 8.2.1.

Theorem 8.2.1. *Suppose $\tilde{q} \in C^1([0, T], \mathcal{L}^2(0, 1))$. If $u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that*

$$b(u_0, v) + a_B(u_d, v) = c(y, v) \text{ for each } v \in V,$$

then Problem TBW has a unique solution,

$$u \in C([0, T]; V) \cap C^1([0, T]; W) \cap C^1((0, T); V) \cap C^2((0, T); W).$$

Proof. Clearly $\tilde{f} \in C^1([0, \tau], X)$. The result follows from Theorem 2.2.1. $\square$

8.2.5 Convergence of the semi-discrete approximation

For $i = 1, 2$, the space $S_i^h(0, 1)$ consists of piecewise Hermite cubic basis functions (see Section 6.7). The functions in $S_i^h(0, 1)$ are zero at $x = 0$. Let S^h be the subspace of $S_1^h(0, 1) \times S_2^h(0, 1) \times \mathbb{R}^2$ such that $u_3 = u_1(1)$ and $u_4 = u_2(1)$ for each $u \in S^h$.

Problem TBBW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a_B(u_h'(t), v) + b(u_h(t), v) = (q_X, v)_{X_B} \text{ for each } v \in S^h,$$

while $u_h(0) = u_0^h = \langle w_0^h, \phi_0^h \rangle$ and $u_h'(0) = u_d^h = \langle w_d^h, \phi_d^h \rangle$.

The existence of a unique solution follows from Theorem 2.3.1 provided that q_X is continuous w.r.t. the norm of X , which requires $\tilde{q}$ to be continuous w.r.t. the norm of $\mathcal{L}^2(0, 1)$.

In [ZVV04] it is shown how to construct the relevant matrices for the semi-discrete problem and derive the system of ordinary differential equations. However, the paper is concerned with modal analysis and the system of ordinary differential equations is not considered.

Convergence of the finite element approximations is not considered in [ZVV04].

Interpolation

To start, note that $H(V, k) = H_B^k \cap V$ for the Timoshenko beam with tip body and boundary damping.

The interpolation operator Π is defined on the product spaces H_B^k by

$$\Pi u = \langle \Pi_c u_1, \Pi_c u_2, u_3, u_4 \rangle \quad \text{for } u \in H_B^k,$$

where Π_c is the usual interpolation operator for piecewise Hermite cubic basis functions. A **seminorm** for H_B^m is defined by

$$|u|_{k, H_B^m} = \sqrt{|u_1|_k^2 + |u_2|_k^2},$$

with $|\cdot|_k$ the seminorm in $H^m(\Omega)$.

Proposition 8.2.5. *Since $u \in H_B^k$, with $k \geq 2$,*

$$\|u - \Pi u\|_V \leq \hat{C} h^{\alpha-1} |u|_{\alpha, H_B^m}, \quad (8.2.8)$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Proof. Since $u_i \in H^k(0, 1)$ for $i = 1, 2$,

$$\|u_i - \Pi_c u_i\|_1 \leq \hat{C}_c h^{\alpha-1} |u_i|_{\alpha}, \quad (8.2.9)$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$ (see Appendix B).

From the definition of Π we have that $u - \Pi u = \langle u_1 - \Pi_c u_1, u_2 - \Pi_c u_2, 0, 0 \rangle$, therefore

$$\|u - \Pi u\|_V^2 \leq K_b (\|u_1 - \Pi_c u_1\|_1^2 + \|u_2 - \Pi_c u_2\|_1^2), \quad (8.2.10)$$

by (6.6.3). We combine (8.2.10) and (8.2.9) to obtain the result, using the definition of the semi-norm above. $\square$

In the applications below we apply the results for general damping (Section 3.3). We assume that the solution u satisfies Assumption A2 and use the estimate (8.2.8).

Application of Corollary 3.3.1

Let $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume that $u'' \in \mathcal{L}^2([0, T], H_B^k \cap V)$ with $k \geq 2$. Then, for any $t \in [0, T]$,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\|_W \\ & \leq \widehat{C} h^{\alpha-1} (|u(t)|_{\alpha, H_B^k} + C_b |u'(t)|_{\alpha, H_B^k}) \\ & \quad + \sqrt{24e^{3t}} \widehat{C} h^{\alpha-1} \left[\int_0^T C_b |u''(\cdot)|_{\alpha, H_B^k} + 3K_a \max |u'(t)|_{\alpha, H_B^k} + 3K_a \int_0^T |u''(\cdot)|_{\alpha, H_B^k} \right. \\ & \quad \left. + 2C_b |u_d|_{\alpha, H_B^k} + 2\sqrt{1+K_a} |u_0|_{\alpha, H_B^k} + \sqrt{K_a} |u_d|_{\alpha, H_B^k} \right], \end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

As discussed before (see Section 5.1), Assumption A4 is satisfied in some cases and a **higher order estimate** can be obtained.

Application of Corollary 3.3.2

Assume that $K\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^{2\alpha}$ and $u'' \in \mathcal{L}^2([0, T], H_B^k(\Omega) \cap V)$ with $k \geq 2$. Then, for any $t > 0$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W & \leq \widehat{C}^2 h^{2\alpha} \widehat{c}_b |u_1(t)|_{\alpha+2} + \sqrt{24e^{3t}} C_b \widehat{C}^2 h^{2\alpha} \widehat{c}_b \left(\int_0^T C_b |u_1''(\cdot)|_{\alpha+2} \right. \\ & \quad \left. + 3K_a \max |u_1'(t)|_{\alpha+2} + 3K_a \int_0^T |u_1''(\cdot)|_{\alpha+2} + \sqrt{K_a} |u_d|_{\alpha+2} \right) \\ & \quad + \sqrt{24e^{3t}} C_b K h^{2\alpha}, \end{aligned}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Remarks

1. Results on the regularity of weak solutions for the Timoshenko beam equations are not common knowledge and further study of the literature is required.
2. For discussion on the phenomenon of “locking”, see Section 6.8.

8.2.6 Fully discrete

Consider the algorithm for central differences.

Problem TBBW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 1, 2, \dots, N-1$,

$$\begin{aligned} c(\tau^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a_B((2\tau)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ + b(u_k^h, v) = (q_X(t_k), v)_{X_B}, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\tau v^h$.

Application of Theorem 4.5.1

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ and assume $u \in C^2([0, T], V)$. If $u'' \in \mathcal{L}^2([0, T], H_B^k(\Omega) \cap V)$ with $k \geq 2$, $q \in C^2([0, T], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem TBBW^h-D, then

$$\begin{aligned} & \|u(t_k) - u_k^h\|_V \\ & \leq \widehat{C}h^{\alpha-1}(|u(t)|_{\alpha, H_B^k} + C_b|u'(t)|_{\alpha, H_B^k}) \\ & \quad + \sqrt{24e^{3t}}\widehat{C}h^{\alpha-1} \left[\int_0^T C_b|u''(\cdot)|_{\alpha, H_B^k} + 3K_a \max |u'(t)|_{\alpha, H_B^k} \right. \\ & \quad + 3K_a \int_0^T |u''(\cdot)|_{\alpha, H_B^k} + 2C_b|u_d|_{\alpha, H_B^k} + 2\sqrt{1+K_a}|u_0|_{\alpha, H_B^k} \\ & \quad \left. + \sqrt{K_a}|u_d|_{\alpha, H_B^k} \right] + C_T R^{-\frac{1}{2}} \left(\tau^2 \max \|u_h^{(4)}(s)\|_W + \tau \max \|u_h'''(s)\|_W \right). \end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

Recall that an estimate for $\|u_h^{(4)}\|$ can be found using Proposition 2.5.3.

Chapter 9

Multi-dimensional elastodynamics and structures

Multi-dimensional elastodynamics

The Reissner-Mindlin plate model (undamped) is considered in Section 9.1 and linear three-dimensional elasto-dynamics (undamped) in Section 9.3.

Where the Timoshenko beam model consists of two partial differential equations, there are three in the Reissner-Mindlin model.

Structures

In this chapter we also consider systems where elastic bodies interact. Numerous examples can be found in [LLS94], for example networks of linked beams and a plate-beam system. The book is of a theoretical nature with a number of numerical examples, but there is no theory on the finite element method, except for references, for example [Hug87].

In Section 9.2 we present a plate-beam model from a recent article [LVV09b].

Finally, we mention a recent article [LH09] where the importance of vibration analysis of structures in engineering is stressed. The authors consider a system consisting of two plates.

9.1 Reissner-Mindlin plate

9.1.1 Equations of motion

Consider small transverse vibration of a thin plate with thickness h and density ρ . The reference configuration for the plate is a domain Ω in the plane. The transverse displacement of $\mathbf{x}$ at time t is denoted by $w(\mathbf{x}, t)$. The angle between a “material line” and a perpendicular to the plane is $\psi(\mathbf{x}, t)$ and the angle between the projection of the material line in the plane and the unit vector $\mathbf{e}_1$ is $\phi(\mathbf{x}, t)$ (see [Rei88, Sec 3.2, Sec 3.5]). For a linear model $\boldsymbol{\psi}$ is approximated by

$$\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T = [\psi \cos \phi \ \psi \sin \phi]^T.$$

Then the equations of motion (see [Min51] and [Rei88, p.152]) are given by

$$\rho h \partial_t^2 w = \operatorname{div} \mathbf{Q} + q, \quad (9.1.1)$$

$$\rho I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} \mathbf{M} - \mathbf{Q}, \quad (9.1.2)$$

where $I = \frac{h^3}{12}$ is the length moment of inertia. $\mathbf{Q}$ represents a force density, $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a moment density and q an external load on the plate.

9.1.2 Constitutive equations

The following assumptions are made for small curvature and small partial derivatives (see [Rei88, p.61] and [Min51]).

$$\mathbf{Q} = \kappa^2 G h (\nabla w + \boldsymbol{\psi}), \quad (9.1.3)$$

where G is the shear modulus and κ^2 a correction factor.

$$\mathbf{M} = \frac{1}{2} D \begin{bmatrix} 2(\partial_1 \psi_1 + \nu \partial_2 \psi_2) & (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) \\ (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) & 2(\partial_2 \psi_2 + \nu \partial_1 \psi_1) \end{bmatrix}. \quad (9.1.4)$$

D is a measure of stiffness of the plate and is given by

$$D = \frac{EI}{1 - \nu^2},$$

where E is Young’s modulus and ν Poisson’s ratio.

The equations of motion and the constitutive equations above are known as the **Reissner-Mindlin** plate model. The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [Rei88, p.152] and [Min51]). However, it is more convenient to derive the variational form directly.

9.1.3 Dimensionless form

To derive the dimensionless form, let

$$\tau = \frac{t}{t_0}, \quad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell},$$

where ℓ is a suitable length and t_0 must still be specified.

The dimensionless variables, with $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, are

$$\begin{aligned} w^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{\ell}\right) w(\mathbf{x}, t), & \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) &= \boldsymbol{\psi}(\mathbf{x}, t), \\ \mathbf{Q}^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{\ell G \kappa^2}\right) \mathbf{Q}(\mathbf{x}, t), & M^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{\ell^2 G \kappa^2}\right) M(\mathbf{x}, t) \\ \text{and} \quad q^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{G \kappa^2}\right) q(\mathbf{x}, t). \end{aligned}$$

Now, introduce the dimensionless constants

$$h^* = \frac{h}{\ell}, \quad I^* = \frac{(h^*)^3}{12} \quad \text{and} \quad \beta = \frac{\ell^3 G \kappa^2}{EI}.$$

It is convenient to choose $t_0 = \ell \sqrt{\frac{\rho}{G \kappa^2}}$. The equations of motion and constitutive equations in dimensionless form are presented below (using the original notation).

Reissner-Mindlin plate model

$$h \partial_t^2 w = \operatorname{div} \mathbf{Q} + q, \tag{9.1.5}$$

$$I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \mathbf{Q}, \tag{9.1.6}$$

$$\mathbf{Q} = h(\nabla w + \boldsymbol{\psi}), \tag{9.1.7}$$

$$M = \frac{1}{2\beta(1-\nu^2)} \begin{bmatrix} 2(\partial_1 \psi_1 + \nu \partial_2 \psi_2) & (1-\nu)(\partial_1 \psi_2 + \partial_2 \psi_1) \\ (1-\nu)(\partial_1 \psi_2 + \partial_2 \psi_1) & 2(\partial_2 \psi_2 + \nu \partial_1 \psi_1) \end{bmatrix}. \tag{9.1.8}$$

9.1.4 Model problem

Problem RM

The reference configuration for the plate is the domain Ω .

The model consists of

the equations of motion (9.1.5) and (9.1.6),

the constitutive equations (9.1.7) and (9.1.8) and

the boundary conditions $w = \psi_1 = \psi_2 = 0$.

The initial conditions are

$$w(\cdot, 0) = w_0, \quad \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_0, \quad \partial_t w(\cdot, 0) = w_1 \quad \text{and} \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1.$$

The plate in Problem RM is clamped, but other boundary conditions are possible, see the next subsection.

This model problem is also considered in [Wu05] and [Wu06].

9.1.5 Variational form

This subsection is not new. Problem RM is a special case of the plate-beam model problem in [Lab06], [LVV09b] and [VZV09]. (The variational form is not derived in [Wu05] or [Wu06].)

Using integration by parts, we obtain the following Green formulas: For any scalar valued function v

$$\iint_{\Omega} (\operatorname{div} \mathbf{Q}) v \, dA = - \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v \, ds. \quad (9.1.9)$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T$,

$$\iint_{\Omega} \operatorname{div} M \cdot \boldsymbol{\phi} \, dA = - \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA + \int_{\partial\Omega} M\mathbf{n} \cdot \boldsymbol{\phi} \, ds. \quad (9.1.10)$$

where $\Phi = \begin{bmatrix} \partial_1 \phi_1 & \partial_2 \phi_1 \\ \partial_1 \phi_2 & \partial_2 \phi_2 \end{bmatrix}$ and “tr” denotes the trace of the matrix.

Multiply equation (9.1.5) by an arbitrary scalar valued function v and integrate. Using (9.1.9), we find that

$$h \iint_{\Omega} \partial_t^2 w v \, dA + \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA - \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v \, ds = \iint_{\Omega} q v \, dA. \quad (9.1.11)$$

From equation (9.1.6) we have

$$I \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} = \operatorname{div} \mathbf{M} \cdot \boldsymbol{\phi} - \mathbf{Q} \cdot \boldsymbol{\phi},$$

where $\boldsymbol{\phi}$ is an arbitrary vector valued function. Using (9.1.10) we find that

$$\begin{aligned} I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA - \int_{\partial\Omega} M \mathbf{n} \cdot \boldsymbol{\phi} \, ds \\ + \iint_{\Omega} \mathbf{Q} \cdot \boldsymbol{\phi} \, dA = 0. \end{aligned} \quad (9.1.12)$$

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$:

$$T_1(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega\},$$

$$T_2(\Omega) = \{\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T \mid \phi_1, \phi_2 \in C^1(\bar{\Omega}), \ \boldsymbol{\phi} = 0 \text{ on } \partial\Omega\}.$$

Remark Other boundary conditions can be accommodated using different spaces of test functions.

From (9.1.11)

$$h \iint_{\Omega} \partial_t^2 wv \, dA + \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA = \iint_{\Omega} qv \, dA, \quad (9.1.13)$$

for each $v \in T_1(\Omega)$.

From (9.1.12)

$$I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA + \iint_{\Omega} \mathbf{Q} \cdot \boldsymbol{\phi} \, dA = 0, \quad (9.1.14)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Define a bilinear form b_B by

$$\begin{aligned} b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA \\ &= \frac{1}{\beta(1-\nu^2)} \iint_{\Omega} ((\partial_1 \psi_1 + \nu \partial_2 \psi_2) \partial_1 \phi_1 + (\partial_2 \psi_2 + \nu \partial_1 \psi_1) \partial_2 \phi_2) \, dA \\ &\quad + \frac{1}{2\beta(1+\nu)} \iint_{\Omega} (\partial_1 \psi_2 + \partial_2 \psi_1) (\partial_1 \phi_2 + \partial_2 \phi_1) \, dA, \end{aligned}$$

for each $\boldsymbol{\psi}, \boldsymbol{\phi}$ in $H^1(\Omega)^2$.

Finally, the second variational equation is given by

$$I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} dA + b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h \iint_{\Omega} (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi} dA = 0, \quad (9.1.15)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Problem RMV

Find w and $\boldsymbol{\psi}$ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\boldsymbol{\psi}(\cdot, t) \in T_2(\Omega)$ and Equations (9.1.13) and (9.1.15) hold for each $v \in T_1(\Omega)$ and each $\boldsymbol{\phi} \in T_2(\Omega)$. The initial conditions are

$$\begin{aligned} w(\cdot, 0) &= w_0, & \boldsymbol{\psi}(\cdot, 0) &= \boldsymbol{\psi}_0, \\ \partial_t w(\cdot, 0) &= w_1 & \text{and} & \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1. \end{aligned}$$

The variational form above is used for computational purposes, but for theoretical purposes we consider the weak form of the variational problem in the next section.

9.1.6 Weak variational form

As mentioned in the previous subsection, Problem RM is a special case of the plate-beam model considered in [Lab06], [VZV09] and [LVV09b]. We omit proofs in this subsection since they can be found in [Lab06] and [VZV09].

Add Equations (9.1.13) and (9.1.15):

$$\begin{aligned} & h \iint_{\Omega} \partial_t^2 w v dA + I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} dA \\ & = -b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) - h \iint_{\Omega} (\nabla w + \boldsymbol{\psi})(\nabla v + \boldsymbol{\phi}) dA + \iint_{\Omega} q v dA \end{aligned} \quad (9.1.16)$$

To make the formulation precise, we introduce product spaces.

Product spaces

$$\begin{aligned} X &= \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)^2 \\ H^k &= H^k(\Omega) \times H^k(\Omega)^2 \\ T &= T_1(\Omega) \times T_2(\Omega) \end{aligned}$$

We define V as the closure of T in H^1 .

Bilinear forms For u and v in V , define

$$\begin{aligned} c(u, v) &= h(u_1, v_1)_\Omega + I(u_2, v_2)_{0,2}^\Omega \text{ and} \\ b(u, v) &= b_B(u_2, v_2) + h(\nabla u_1 + u_2, \nabla v_1 + v_2)_{0,2}^\Omega. \end{aligned}$$

A natural inner product for X is

$$(x, y)_X = (x_1, y_1)_0 + (x_2, y_2)_{0,2},$$

where $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_{0,2}$ denote the inner products for $\mathcal{L}^2(\Omega)$ and $\mathcal{L}^2(\Omega)^2$ respectively. (See Appendix A.2 on Sobolev spaces of vector valued functions.) Denote the corresponding norm by $\|\cdot\|_X$.

Proposition 9.1.1. *Let u and v be any elements of X . Then there exist positive constants K_i such that*

$$K_1\|u\|_X \leq c(u, u) \leq K_2\|u\|_X^2.$$

Proposition 9.1.2. *The bilinear form c is an inner product for the space X .*

Definition Inertia space

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W .

Let $f(t) = \langle q(\cdot, t), \mathbf{0} \rangle$.

Problem RMW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (f(t), v)_X \text{ for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \boldsymbol{\psi}_0 \rangle$ and $u'(0) = u_d = \langle w_1, \boldsymbol{\psi}_1 \rangle$.

Before we discuss existence and convergence we need the following results.

Proposition 9.1.3. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proposition 9.1.4. *V is a dense subset of W .*

Theorem 9.1.1. *There exists a constant K_b such that*

$$|b(u, v)| \leq K_b \|u\|_{H^1} \|v\|_{H^1},$$

for each $u \in V$.

Theorem 9.1.2. *There exists a constant C_b such that*

$$b_B(u, u) \geq C_b \|u\|_{1,2}^2 \text{ for each } u \in V.$$

Corollary 9.1.1. *The bilinear form b is an inner product for V .*

Definition Energy space

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u, u)}$.

Corollary 9.1.2. *The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V .*

We have shown that Assumptions E1, E2 and E3 hold. Since the vibration is undamped, E4 is automatically satisfied.

Theorem 9.1.3. *Suppose $\tilde{q} \in C^1(J, \mathcal{L}^2(\Omega))$. Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W),$$

for Problem RMW for each $u_0 \in E_b$ and $u_d \in V$.

Proof. Since $\tilde{q} \in C^1(\mathcal{L}^2(\Omega))$, $f \in C^1(J, X)$ and the result follows from Theorem 2.2.2. $\square$

Remark In Section 2 of [Wu05] the plate vibration problem (Problem Wu05) is formulated and the weak variational form is stated (but not derived). This is Equation (2.6). Estimates and the existence theorem with references to other articles are given.

9.1.7 Finite element method

The finite element approximation is considered in [Wu05]. From the introduction to the article [Wu05], we conclude that this is the first result on convergence of the finite element method for the vibration of a Reissner-Mindlin plate. Wu formulated it as follows: “The static analysis is the foundation of the dynamic analysis. On the other hand, the a priori estimates for hyperbolic problems are developed in [Dup73] and [Bak76]. The method is extended to elasto-dynamic problems by explicit finite elements [Wu03].”

Let S_i^h denote a subspace of $H^2(\Omega)$ consisting of piecewise Hermite bicubic functions that vanish on the boundary of Ω . Define $S^h = S_1^h \times S_2^h \times S_3^h$, then S^h is a finite dimensional subspace of V .

Problem RMW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + b(u_h(t), v) = (q(\cdot, t), v_1) \quad \text{for each } v \in S^h \quad (9.1.17)$$

while $u_h(0) = u_0^h = \langle w_0^h, \psi_0^h \rangle$ and $u_h'(0) = u_d^h = \langle w_1^h, \psi_1^h \rangle$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{q}$ is continuous w.r.t. the norm of $\mathcal{L}^2(0, 1)$.

Interpolation

For simplicity assume that the domain Ω is a rectangle. We must define an interpolation operator on $H^k = H^k(\Omega) \times H^k(\Omega)^2$. To **illustrate** how this is done, we use piecewise bilinear basis functions on rectangular elements. Πu for $u \in H^k$ is defined by: $\Pi u = \langle \Pi_b u_1, \Pi_B u_2 \rangle$ where Π_b and Π_B are defined in Appendix B.

Proposition 9.1.5. *There exists a $\hat{C} > 0$ such that*

$$\|\Pi u - u\|_{H^1} \leq \hat{C}h \|u\|_2 \text{ for each } u \in H^2.$$

Proof. From Appendix B we have that

$$\|\Pi_b f - f\|_{1,\Omega} \leq \hat{C}h |f|_{2,\Omega} \text{ for } f \in H^2(\Omega) \text{ and}$$

$$\|\Pi_B f - f\|_{1,2} \leq \hat{C}h |f|_{2,2} \text{ for } f \in H^2(\Omega)^2.$$

Therefore, from the definition of Π , we have

$$\begin{aligned} \|\Pi u - u\|_{H^1}^2 &= \|\Pi_b u_1 - u_1\|_{1,\Omega}^2 + \|\Pi_B u_2 - u_2\|_{1,2}^2 \\ &\leq \hat{C}^2 h^2 |u_1|_{2,\Omega}^2 + \hat{C}^2 h^2 |u_2|_{2,2}^2. \end{aligned}$$

The result follows. □

Using the result above and equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ (Corollary 9.1.2), we obtain the following estimate: For $u \in H^2$,

$$\|\Pi u - u\|_V \leq \hat{C}h \|u\|_2. \quad (9.1.18)$$

9.1.8 Convergence

For this problem we have that $H(V, k) = H^k \cap V$. Since no damping is present, it follows that the results for weak damping in Sections 3.2 and 3.4 are valid.

Energy error estimates

In each of the applications below we assume $u \in C^2([0, T], V)$ and use the estimate (9.1.18).

Application of Corollary 3.2.1

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and suppose that $u'' \in \mathcal{L}^2([0, T], H^2 \cap V)$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_V + \|u'(t) - u'_h(t)\| \\ & \leq \widehat{C}h [|u(t)|_2 + C_b |u'(t)|_2 + 2|u_0|_2 + 2C_b |u_d|_2 \\ & \quad + C_b \int_0^t (|u''(\cdot)|_2 + k|u'(\cdot)|_2)], \end{aligned}$$

for $t \in [0, T]$.

Application of Theorem 3.2.3

If we do not make the assumptions on u' and u'' , we may still use Theorem 3.2.3. Since $H^2(0, 1)$ is dense in $H^1(0, 1)$, H^2 is dense in H^1 . By Corollary 9.1.2, H^2 is dense in V and Assumption A5 is satisfied. If we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\| = 0.$$

Therefore, by Theorem 3.2.3,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_V = 0 \quad \text{for } t \in [0, T].$$

It is sometimes possible to use the Aubin-Nitsche trick to obtain **higher order estimates** for the projection error. Note that the following is an inertia norm error estimate.

Application of Corollary 3.2.2

In some cases we have that Assumption A4 is satisfied.

Assume that $\|Pu_0 - u_0^h\|_V + \|Pu_d - u_d^h\|_W \leq C_0 h^2$ and $u'' \in \mathcal{L}^2([0, T], H^2 \cap V)$. Then, for any $t > 0$,

$$\|u(t) - u_h(t)\|_W \leq \widehat{C}^2 h^2 \widehat{c}_b |u(t)|_2 + C_b C_0 h^2 + C_b \widehat{C}^2 h^2 \widehat{c}_b \left(\int_0^t |u''(\cdot)|_2 + k \int_0^t |u'(\cdot)|_2 \right).$$

Inertia norm error estimates

We apply the result from Section 3.4.

Application of Corollary 3.4.3

Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem RMW satisfies $u' \in \mathcal{L}^2([0, T], H^2 \cap V)$, then

$$\begin{aligned} \|u(t) - u_h(t)\| &\leq C_b \widehat{C} h |u(t)|_2 + \sqrt{2} C_b \widehat{C} h \left(3 \int_0^T |u'(\cdot)|_2 \right. \\ &\quad \left. + 3K_a \int_0^T |u(\cdot)|_2 + (2 + 3K_a T) |u_0|_2 + 3T |u_d|_2 \right), \end{aligned}$$

for each $t \in [0, T]$.

Remark Theoretically convergence follows from the discussion above, but in practice locking may be a problem as observed in the numerical experiments in [Wu05]. However, with piecewise bicubic basis functions satisfactory results are obtained (see [LVV09b]).

Fully discrete

Suppose one choose to use the algorithm in Problem G^h -D-Baker.

Problem RMW^h-D

Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \dots, N-1$,

$$\begin{aligned} \delta_t u_k^h &= v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) &= \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \end{aligned}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Application of Theorem 4.3.1

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If $u'(t) \in \mathcal{L}^2([0, T], H^2 \cap V)$, $\tilde{q} \in C^2([0, T], \mathcal{L}^2(\Omega))$ and the sequence $\{u_k^h\}$ is a solution of Problem RMW^h -D, then

$$\begin{aligned}
& \|u(t_k) - u_k^h\|_W \\
& \leq C_b \hat{C} h |u(t)|_2 + \sqrt{2} C_b \hat{C} h \left(3 \int_0^T |u'(\cdot)|_2 \right. \\
& \quad \left. + 3K_a \int_0^T |u(\cdot)|_2 + (2 + 3K_a T) |u_0|_2 + 3T |u_d|_2 \right), \\
& \leq C_b \hat{C} h^\alpha |u(t_k)|_{\alpha+2} + \sqrt{2} C_b \hat{C} h^\alpha \left(3 \int_0^T |u'(\cdot)|_{\alpha+2} \right. \\
& \quad \left. + 3K_a \int_0^T |u(\cdot)|_{\alpha+2} + (2 + 3K_a T) |u_0|_{\alpha+2} + 3T |u_d|_{\alpha+2} \right) + 7T^2 \tau^2 \max \|u_h^{(4)}\|_W \\
& \quad + 7T \tau^2 \max \|u_h'''\|_W + \sqrt{2K_a} \tau^4 \max \|u_h'''\|_W,
\end{aligned}$$

for each $t_k \in (0, T)$, where $\alpha = 1$ for $k = 3$ and $\alpha = 2$ for $k \geq 4$.

Recall that an estimate for $\|u_h^{(4)}\|_W$ in terms of the data can be obtained using Proposition 2.5.3.

Remark There are other options for combining a semi-discrete and fully discrete estimate, see Section 5.1.

9.2 Reissner-Mindlin-Timoshenko plate-beam system

9.2.1 Model problem

In this subsection we present the plate-beam system in [LVV09b] and [Lab06].

Consider small transverse vibration of a thin rectangular plate supported by identical beams at two opposing sides and rigidly supported at the remaining sides. The beams are supported at their endpoints. It is assumed that the plate remains in contact with the beams and supporting structure at all times.

We may assume that the model is in dimensionless form (see Section 9.1). The reference configuration for the plate is the rectangle Ω , where $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq a$. The plate is rigidly supported at $x_1 = 0$ and $x_1 = 1$. These sections of the boundary of Ω are denoted by Σ_0 and Σ_1 respectively. The plate is supported by beams at $x_2 = 0$ and $x_2 = a$ and these sections are denoted by Γ_0 and Γ_1 respectively.

Equations of motion

In the mathematical model the Reissner-Mindlin plate theory and the Timoshenko beam theory are used. On the rectangle Ω , the equations of motion (9.1.5) and (9.1.6) are satisfied and on Γ_0 and Γ_1 , the two sets of equations of motion are given by adapted versions of (6.2.1) and (6.2.2):

$$\rho_b A \partial_t^2 w_b = \partial_x V + P, \quad (9.2.1)$$

$$\rho_b I \partial_t^2 \phi_b = V + \partial_x M + L. \quad (9.2.2)$$

In the equations above L represents a moment density transmitted from the plate to the beam and P a force density transmitted from the plate to the beam.

Boundary conditions on Σ_0 and Σ_1

On these sections of the boundary, the conventional homogeneous boundary conditions for a rigidly supported plate are used, i.e.

$$w = 0, \quad \psi_2 = 0 \quad \text{and} \quad M \mathbf{n} \cdot \mathbf{n} = 0, \quad (9.2.3)$$

where $\mathbf{n}$ is the unit exterior normal (see [Rei88, p.66]).

Interface conditions on Γ_0 and Γ_1

On Γ_0 and Γ_1 the interaction between the plate and the beams is considered.

$$w_b(x_1, t) = w(x_1, 0, t) \text{ on } \Gamma_0, \quad w_b(x_1, t) = w(x_1, a, t) \text{ on } \Gamma_1, \quad (9.2.4)$$

$$\phi_b(x_1, t) = -\psi_1(x_1, 0, t) \text{ on } \Gamma_0, \quad \phi_b(x_1, t) = -\psi_1(x_1, a, t) \text{ on } \Gamma_1. \quad (9.2.5)$$

The interface conditions for the force densities and moment densities on Γ_0 and Γ_1 are given by

$$\mathbf{Q} \cdot \mathbf{n} = -P, \quad (9.2.6)$$

$$M \mathbf{n} \cdot \boldsymbol{\tau} = L, \quad (9.2.7)$$

$$M \mathbf{n} \cdot \mathbf{n} = 0, \quad (9.2.8)$$

where $\boldsymbol{\tau}$ is the unit tangent oriented in such a way that Ω is on the left hand side of $\boldsymbol{\tau}$. For a detailed explanation of the moments $M\mathbf{n} \cdot \mathbf{n}$ and $M\mathbf{n} \cdot \boldsymbol{\tau}$, see [Rei88, p.66].

Conditions at the endpoints of Γ_0 and Γ_1

At the endpoints of Γ_0 and Γ_1 we have the obvious boundary conditions for the beams, namely

$$w_b = 0 \quad \text{and} \quad M_b = 0. \quad (9.2.9)$$

Dimensionless form

The dimensionless form for the plate model has been derived in Subsection 9.1.3. For the beam equations it has to be recalculated using the scaling of the plate model and

$$\tau = \frac{t}{t_0} \quad \text{and} \quad \xi_1 = \frac{x_1}{\ell}.$$

Also set

$$\begin{aligned} w_b^* &= \left(\frac{1}{\ell}\right) w_b, \quad \phi_b^* = \phi_b, \\ P^* &= \left(\frac{1}{\ell G \kappa^2}\right) P, \quad V^* = \left(\frac{1}{\ell^2 G \kappa^2}\right) V, \\ M_b^* &= \left(\frac{1}{\ell^3 G \kappa^2}\right) M_b \quad \text{and} \quad L^* = \left(\frac{1}{\ell^2 G \kappa^2}\right) L. \end{aligned}$$

Note that the parameters of the plate are used for the scaling. Choosing $t_0 = \ell \sqrt{\frac{\rho}{G \kappa^2}}$ as in Subsection 9.1.3 and using the original notation for the corresponding dimensionless quantities, the dimensionless beam model is given by

$$\eta_1 \partial_t^2 w_b = \partial_1 V + P, \quad (9.2.10)$$

$$\eta_1 \partial_t^2 \phi_b = \alpha_b (\partial_1 M_b + V + L), \quad (9.2.11)$$

$$V = \eta_2 (\partial_1 w_b - \phi_b), \quad (9.2.12)$$

$$\beta_b M_b = \eta_2 \partial_1 \phi_b. \quad (9.2.13)$$

The dimensionless constants α_b and β_b are

$$\alpha_b = \frac{A_b \ell^2}{I_b}, \quad \beta_b = \frac{A_b G_b \kappa_b^2 \ell^2}{E_b I_b}.$$

The two additional dimensionless constants η_1 and η_2 express ratios for the material properties and the geometrical properties of the plate and the beams:

$$\eta_1 = \left(\frac{\rho_b}{\rho} \right) \left(\frac{A_b}{\ell^2} \right) \quad \text{and} \quad \eta_2 = \left(\frac{G_b}{G} \right) \left(\frac{\kappa_b^2}{\kappa^2} \right) \left(\frac{A_b}{\ell^2} \right).$$

The boundary conditions on Σ_0 and Σ_1 and the interface conditions for force and moment densities remain unchanged.

The mathematical model

The vibration problem for the plate-beam system is given by the following equations.

Problem RMT

Equations of motion for the plate: (9.1.5) and (9.1.6) on Ω .

Constitutive equations for the plate: (9.1.7) and (9.1.8) on Ω .

Boundary conditions: (9.2.3) on Σ_0 and Σ_1 .

Equations of motion for the beams: (9.2.10) and (9.2.11) on Γ_0 and Γ_1 .

Constitutive equations for the beams: (9.2.12) and (9.2.13) on Γ_0 and Γ_1 .

Interface conditions: (9.2.4) to (9.2.8) on Γ_0 and Γ_1 .

Endpoint conditions: (9.2.9) at the endpoints of Γ_0 and Γ_1 .

Initial conditions: $\langle w(0), \boldsymbol{\psi}(0), w_{b0}(0), \psi_{b0}(0), w_{b1}(0), \psi_{b1}(0) \rangle$ and $\langle \partial_t w(0), \partial_t \boldsymbol{\psi}(0), \partial_t w_{b0}(0), \partial_t \psi_{b0}(0), \partial_t w_{b1}(0), \partial_t \psi_{b1}(0) \rangle$ to be specified.

9.2.2 Variational form

Writing the model problem in variational form is partially done in Subsection 9.1.5. We may use Equations (9.1.11) and (9.1.12). However, different spaces of test functions must be used.

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$, with

$$T_1(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1\},$$

$$T_2(\Omega) = \{\phi = [\phi_1 \ \phi_2]^T \mid \phi_1, \phi_2 \in C^1(\bar{\Omega}), \phi_2 = 0 \text{ on } \bar{\Sigma}_0 \text{ and } \bar{\Sigma}_1\}.$$

Forces and force densities

It follows from Equation (9.2.10) (first equation of motion for the beams), using integration by parts, that

$$\eta_1 \int_0^1 \partial_t^2 w_{b0} v_0 dx + \int_0^1 V_0 \partial_x v_0 dx = \int_0^1 P_0 v_0 dx \quad (9.2.14)$$

for each v_0 in $C^1[0, 1]$ with $v_0(0) = v_0(1) = 0$ and

$$\eta_1 \int_0^1 \partial_t^2 w_{b1} v_1 dx + \int_0^1 V_1 \partial_x v_1 dx = \int_0^1 P_1 v_1 dx \quad (9.2.15)$$

for each v_1 in $C^1[0, 1]$ with $v_1(0) = v_1(1) = 0$. The subscripts “0” and “1” are used to distinguish between quantities associated with the two different beams.

To accommodate Equation (9.2.4) (interface condition for w_{b0} and w_{b1}), choose $v_0(x_1) = v(x_1, 0)$ and $v_1(x_1) = v(x_1, a)$, where a denotes the dimensionless width of the plate.

The fact that $v = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and that $\mathbf{Q} \cdot \mathbf{n} = -P$ on both Γ_0 and Γ_1 (interface condition (9.2.6)), result in some cancelations when adding Equations (9.1.11), (9.2.14) and (9.2.15). We have

$$\begin{aligned} \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n}) v ds &= \int_{\Gamma_0} (\mathbf{Q}_0 \cdot \mathbf{n}) v ds - \int_{\Gamma_1} (\mathbf{Q}_1 \cdot \mathbf{n}) v ds \\ &= - \int_{\Gamma_0} P_0 v ds - \int_{\Gamma_1} P_1 v ds \\ &= - \left[\int_0^1 P_0 v dx_1 \right]_{x_2=0} - \left[\int_0^1 P_1 v dx_1 \right]_{x_2=a} \end{aligned}$$

for each $v \in T_1(\Omega)$.

From Equation (9.2.4) (interface condition), the remaining integrals on Γ_0 and Γ_1 can be expressed in terms of w . Therefore,

$$\begin{aligned}
& h_p \iint_{\Omega} \partial_t^2 w v \, dA + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\
& + \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA + \left[\int_0^1 V_0 \partial_1 v \, dx_1 \right]_{x_2=0} + \left[\int_0^1 V_1 \partial_1 v \, dx_1 \right]_{x_2=a} \\
& = \iint_{\Omega} q v \, dA
\end{aligned} \tag{9.2.16}$$

for each $v \in T_1(\Omega)$.

The next step is to substitute Equations (9.1.7) (constitutive equation for $\mathbf{Q}$) and (9.2.12) (constitutive equations for V_0 and V_1), expressed in terms of w and ψ_1 , into (9.2.16). (Interface conditions (9.2.4) and (9.2.5) are used.) This leads to

$$\begin{aligned}
& h_p \iint_{\Omega} \partial_t^2 w v \, dA + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 \partial_t^2 w v \, dx_1 \right]_{x_2=a} \\
& + h_p \iint_{\Omega} (\nabla w + \boldsymbol{\psi}) \cdot \nabla v \, dA \\
& + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \partial_1 v \, dx_1 \right]_{x_2=0} + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \partial_1 v \, dx_1 \right]_{x_2=a} \\
& = \iint_{\Omega} q v \, dA
\end{aligned} \tag{9.2.17}$$

for each $v \in T_1(\Omega)$.

Moments and moment densities

It follows from Equation (9.2.11) (second equation of motion for the beams) and using integration by parts, that

$$\frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b0} \chi_0 \, dx + \int_0^1 M_{b0} \partial_x \chi_0 \, dx - \int_0^1 (V_0 + L_0) \chi_0 \, dx = 0 \tag{9.2.18}$$

for each $\chi_0 \in C^1[0, 1]$ and

$$\frac{\eta_1}{\alpha_b} \int_0^1 \partial_t^2 \phi_{b1} \chi_1 \, dx + \int_0^1 M_{b1} \partial_x \chi_1 \, dx - \int_0^1 (V_1 + L_1) \chi_1 \, dx = 0 \tag{9.2.19}$$

for each $\chi_1 \in C^1[0, 1]$. (M_{b0} and M_{b1} are zero at the endpoints of the beams.)

The functions χ_0 and χ_1 in Equations (9.2.18) and (9.2.19) must satisfy the conditions $\chi_0(x_1) = -\phi_1(x_1, 0)$ and $\chi_1(x_1) = -\phi_1(x_1, a)$ in order to accommodate the interface condition (9.2.5) for ϕ_{b0} and ϕ_{b1} .

Hence Equation (9.2.18) becomes

$$\begin{aligned} \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 dx_1 \right]_{x_2=0} - \left[\int_0^1 M_{b0} \partial_1 \phi_1 dx_1 \right]_{x_2=0} \\ + \left[\int_0^1 (V_0 + L_0) \phi_1 dx_1 \right]_{x_2=0} = 0 \end{aligned} \quad (9.2.20)$$

for each $\phi \in T_2(\Omega)$, and Equation (9.2.19) becomes

$$\begin{aligned} \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 dx_1 \right]_{x_2=a} - \left[\int_0^1 M_{b1} \partial_1 \phi_1 dx_1 \right]_{x_2=a} \\ + \left[\int_0^1 (V_1 + L_1) \phi_1 dx_1 \right]_{x_2=a} = 0 \end{aligned} \quad (9.2.21)$$

for each $\phi \in T_2(\Omega)$.

As before, adding Equations (9.1.12), (9.2.20) and (9.2.21), some cancelation of terms occur. Note that $\phi = (\phi \cdot \mathbf{n})\mathbf{n} + (\phi \cdot \boldsymbol{\tau})\boldsymbol{\tau}$ and consequently,

$$\int_{\partial\Omega} M\mathbf{n} \cdot \phi ds = \int_{\partial\Omega} \left((\phi \cdot \mathbf{n})M\mathbf{n} \cdot \mathbf{n} + (\phi \cdot \boldsymbol{\tau})M\mathbf{n} \cdot \boldsymbol{\tau} \right) ds$$

The natural boundary condition on $\partial\Omega$ is $M\mathbf{n} \cdot \mathbf{n} = 0$. From the definition of the test functions, $\phi_2 = 0$ on $\bar{\Sigma}_0$ and $\bar{\Sigma}_1$ and therefore $\phi \cdot \boldsymbol{\tau} = 0$ on Σ_0 and Σ_1 .

On Γ_0 and Γ_1 the interface conditions (9.2.7) and (9.2.8) are used. It follows that

$$\begin{aligned} \int_{\partial\Omega} M\mathbf{n} \cdot \phi ds &= \int_{\Gamma_0} L_0 \phi_1 ds + \int_{\Gamma_1} (-L_1)(-\phi_1) ds \\ &= \left[\int_0^1 L_0 \phi_1 dx_1 \right]_{x_2=0} + \left[\int_0^1 L_1 \phi_1 dx_1 \right]_{x_2=a}. \end{aligned}$$

Consequently,

$$\begin{aligned}
& I_p \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_{\Omega} \text{tr}(M\Phi) \, dA + \iint_{\Omega} \mathbf{Q} \cdot \boldsymbol{\phi} \, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& - \left[\int_0^1 M_{b0} \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} - \left[\int_0^1 M_{b1} \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \left[\int_0^1 V_0 \phi_1 \, dx_1 \right]_{x_2=0} + \left[\int_0^1 V_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& = 0
\end{aligned} \tag{9.2.22}$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

To obtain the second variational equation we substitute for $\mathbf{Q}$, M , V_1 , V_2 , M_{b0} and M_{b1} into (9.2.22) using Equations (9.1.7) and (9.1.8) for $\mathbf{Q}$ and M , (9.2.12) for V_1 and V_2 and (9.2.13) for M_{b0} and M_{b1} .

Now, define a bilinear form b_B by

$$\begin{aligned}
b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_{\Omega} \text{tr}(M\Phi) \, dA \\
&= \frac{1}{\beta_p(1-\nu^2)} \iint_{\Omega} \left((\partial_1 \psi_1 + \nu \partial_2 \psi_2) \partial_1 \phi_1 + (\partial_2 \psi_2 + \nu \partial_1 \psi_1) \partial_2 \phi_2 \right) \, dA \\
&\quad + \frac{1}{2\beta_p(1+\nu)} \iint_{\Omega} (\partial_1 \psi_2 + \partial_2 \psi_1) (\partial_1 \phi_2 + \partial_2 \phi_1) \, dA.
\end{aligned}$$

for each $\boldsymbol{\psi}, \boldsymbol{\phi}$ in $H^1(\Omega)^2$.

The second variational equation is given by

$$\begin{aligned}
& I_p \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h_p \iint_{\Omega} (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi} \, dA \\
& + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=0} + \frac{\eta_1}{\alpha_b} \left[\int_0^1 \partial_t^2 \psi_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=0} + \frac{\eta_2}{\beta_b} \left[\int_0^1 \partial_1 \psi_1 \partial_1 \phi_1 \, dx_1 \right]_{x_2=a} \\
& + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \phi_1 \, dx_1 \right]_{x_2=0} + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1) \phi_1 \, dx_1 \right]_{x_2=a} \\
& = 0
\end{aligned} \tag{9.2.23}$$

for each $\phi \in T_2(\Omega)$.

Problem RMTV

Find w and ψ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\psi(\cdot, t) \in T_2(\Omega)$ and Equations (9.2.17) and (9.2.23) hold for each $v \in T_1(\Omega)$ and each $\phi \in T_2(\Omega)$.

The variational form above is used for computational purposes (see [Lab06] and [LVV09b]), but for theoretical purposes the weak form of the variational problem is necessary (see [Lab06] and [VZV09]).

The finite element method was used for modal analysis in [LVV05] and [Lab06] and simulation of dynamical problems was not done.

9.2.3 Weak variational form

For $I = (0, 1)$, the space $T(I)$ is defined as

$$T(I) = \{v \in C^1(\bar{I}) \mid v(0) = v(1) = 0\}.$$

Trace operators are defined in Appendix A.5. At this stage we are dealing with smooth functions and Γ_0 and Γ_1 simply map a function onto its value at the boundary. Therefore

$$\Gamma_0 v = v(\cdot, 0) \quad \text{and} \quad \Gamma_1 v = v(\cdot, a).$$

In order to formulate the weak variational form of Problem RMT, we start by rewriting Equations (9.2.17) and (9.2.23) in terms of inner products. The

notation is explained in Appendix A.

$$\begin{aligned}
& h_p \left(\partial_t^2 w(\cdot, t), v \right)_\Omega + \eta_1 \left(\Gamma_0(\partial_t^2 w(\cdot, t)), \Gamma_0 v \right)_I + \eta_1 \left(\Gamma_1(\partial_t^2 w(\cdot, t)), \Gamma_1 v \right)_I \\
& + h_p \left(\nabla w(\cdot, t) + \boldsymbol{\psi}(\cdot, t), \nabla v \right)_{0,2}^\Omega + \eta_2 \left(\Gamma_0(\partial_1 w(\cdot, t) + \psi_1(\cdot, t)), \Gamma_0(\partial_1 v) \right)_I \\
& + \eta_2 \left(\Gamma_1(\partial_1 w(\cdot, t) + \psi_1(\cdot, t)), \Gamma_1(\partial_1 v) \right)_I = \iint_\Omega qv \, dA. \tag{9.2.24}
\end{aligned}$$

$$\begin{aligned}
& I_p \left(\partial_t^2 \boldsymbol{\psi}(\cdot, t), \boldsymbol{\phi} \right)_{0,2}^\Omega + b_B(\boldsymbol{\psi}(\cdot, t), \boldsymbol{\phi}) + h_p \left(\nabla w(\cdot, t) + \boldsymbol{\psi}(\cdot, t), \boldsymbol{\phi} \right)_{0,2}^\Omega \\
& + \frac{\eta_1}{\alpha_b} \left(\Gamma_0(\partial_t^2 \psi_1(\cdot, t)), \Gamma_0 \phi_1 \right)_I + \frac{\eta_1}{\alpha_b} \left(\Gamma_1(\partial_t^2 \psi_1(\cdot, t)), \Gamma_1 \phi_1 \right)_I \\
& + \frac{\eta_2}{\beta_b} \left(\Gamma_0(\partial_1 \psi_1(\cdot, t)), \Gamma_0(\partial_1 \phi_1) \right)_I + \frac{\eta_2}{\beta_b} \left(\Gamma_1(\partial_1 \psi_1(\cdot, t)), \Gamma_1(\partial_1 \phi_1) \right)_I \\
& + \eta_2 \left(\Gamma_0(\partial_1 w(\cdot, t) + \psi_1(\cdot, t)), \Gamma_0 \phi_1 \right)_I \\
& + \eta_2 \left(\Gamma_1(\partial_1 w(\cdot, t) + \psi_1(\cdot, t)), \Gamma_1 \phi_1 \right)_I = 0. \tag{9.2.25}
\end{aligned}$$

The next step is to define product spaces.

Product spaces

$$\begin{aligned}
X &= \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)^2 \times_{n=1}^4 \mathcal{L}^2(I), \\
H^1 &= H^1(\Omega) \times H^1(\Omega)^2 \times_{n=1}^4 H^1(\Omega) \\
S &= T_1(\Omega) \times T_2(\Omega)^2 \times_{n=1}^2 (T(I) \times C^1(\bar{I})) \\
T &= \{v \in S \mid \gamma_0 v_1 = v_3, \gamma_1 v_1 = v_5, \gamma_0(v_2 \cdot \mathbf{e}_1) = -v_4, \gamma_1(v_2 \cdot \mathbf{e}_1) = -v_6\}
\end{aligned}$$

The following table explains the relationship between the functions used in Equations (9.2.24) and (9.2.25) and Equations (9.2.26) and (9.2.27) to follow. Note that for $v \in C^1(\bar{\Omega})$ and $i = 0, 1$, $\gamma_i(\partial_1 v) = (\gamma_i v)'$ – the derivative with respect to the variable x_1 .

$u_1(t)$	$w(\cdot, t)$	v_1	v
$u_2(t)$	$\boldsymbol{\psi}(\cdot, t)$	v_2	$\boldsymbol{\phi}$
$u_3(t)$	$\Gamma_0(w(\cdot, t)) = \Gamma_0 u_1(t)$	v_3	$\Gamma_0 v = \Gamma_0 v_1$
$u_4(t)$	$-\Gamma_0 \psi_1(\cdot, t) = -\Gamma_0(u_2(t) \cdot \mathbf{e}_1)$	v_4	$-\Gamma_0 \phi_1 = -\Gamma_0(v_2 \cdot \mathbf{e}_1)$
$u_5(t)$	$\Gamma_1(w(\cdot, t)) = \Gamma_1 u_1(t)$	v_5	$\Gamma_1 v = \Gamma_1 v_1$
$u_6(t)$	$-\Gamma_1 \psi_1(\cdot, t) = -\Gamma_1(u_2(t) \cdot \mathbf{e}_1)$	v_6	$-\Gamma_1 \phi_1 = -\Gamma_1(v_2 \cdot \mathbf{e}_1)$

In the new notation, Equations (9.2.24) and (9.2.25) become

$$\begin{aligned}
& h_p(\ddot{u}_1(t), v_1)_\Omega + \eta_1(\ddot{u}_3(t), v_3)_I + \eta_1(\ddot{u}_5(t), v_5)_I \\
& + h_p(\nabla u_1(t) + u_2(t), \nabla v_1)_{0,2}^\Omega + \eta_2(u'_3(t) - u_4(t), v'_3)_I \\
& + \eta_2(u'_5(t) - u_6(t), v'_5)_I = \iint_\Omega qv \, dA
\end{aligned} \tag{9.2.26}$$

$$\begin{aligned}
\text{and} \quad & I_p(\ddot{u}_2(t), v_2)_{0,2}^\Omega + b_B(u_2(t), v_2) + h_p(\nabla u_1(t) + u_2(t), v_2)_{0,2}^\Omega \\
& + \frac{\eta_1}{\alpha_b}(-\ddot{u}_4(t), -v_4)_I + \frac{\eta_1}{\alpha_b}(-\ddot{u}_6(t), -v_6)_I \\
& + \frac{\eta_2}{\beta_b}(-u'_4(t), -v'_4)_I + \frac{\eta_2}{\beta_b}(-u'_6(t), -v'_6)_I \\
& + \eta_2(u'_3(t) - u_4(t), -v_4)_I + \eta_2(u'_5(t) - u_6(t), -v_6)_I \\
& = 0
\end{aligned} \tag{9.2.27}$$

Bilinear forms

For u and v in T , define

$$\begin{aligned}
c(u, v) &= h_p(u_1, v_1)_\Omega + I_P(u_2, v_2)_{0,2}^\Omega + \eta_1(u_3, v_3)_I \\
&+ \frac{\eta_1}{\alpha_b}(u_4, v_4)_I + \eta_1(u_5, v_5)_I + \frac{\eta_1}{\alpha_b}\eta_1(u_6, v_6)_I, \\
b_\Gamma(u, v) &= \eta_2(u'_3 - u_4, v'_3 - v_4)_I + \eta_2(u'_5 - u_6, v'_5 - v_6)_I \\
&+ \frac{\eta_2}{\beta_b}(u'_4, v'_4)_I + \frac{\eta_2}{\beta_b}(u'_6, v'_6)_I, \\
b_\Omega(u, v) &= b_B(u_2, v_2) + h_p(\nabla u_1 + u_2, \nabla v_1 + v_2)_{0,2}^\Omega, \\
b(u, v) &= b_\Omega(u, v) + b_\Gamma(u, v).
\end{aligned}$$

By adding Equations (9.2.26) and (9.2.27), we arrive at the following variational equation:

$$c(\ddot{u}(t), v) + b(u(t), v) = (q, v_1) \quad \text{for each } v \in T.$$

We are now ready to consider the weak variational form. We define V as the closure of T in H^1 . Note that all the bilinear forms are defined for elements

of V , except for b_Γ . For u and v in V , define

$$b_\Gamma(u, v) = \lim_{n \rightarrow \infty} b_\Gamma(u_n, v_n),$$

with $\{u_n\}$ and $\{v_n\}$ sequences in T such that $u_n \rightarrow u$ and $v_n \rightarrow v$.

As a consequence, the bilinear forms b and b_Γ are now defined on V .

Inertia space

The space X with the norm induced by the inner product c is the inertia space W .

Energy space

A norm on V is defined by $\|u\|_V = \sqrt{b(u, u)}$ and is called the energy norm. The space V with norm $\|\cdot\|_V$ called the energy space.

Problem RMTW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in X$ and

$$c(u''(t), v) + b(u(t), v) = (q, v_1) \text{ for each } v \in V,$$

while

$$\begin{aligned} u(0) &= u_0 = \langle w(0), \boldsymbol{\psi}(0), w_{b0}(0), \psi_{b0}(0), w_{b1}(0), \psi_{b1}(0) \rangle \text{ and} \\ u'(0) &= u_d = \langle \partial_t w(0), \partial_t \boldsymbol{\psi}(0), \partial_t w_{b0}(0), \partial_t \psi_{b0}(0), \partial_t w_{b1}(0), \partial_t \psi_{b1}(0) \rangle. \end{aligned}$$

It is proved in [VZV09] that assumptions E1, E2 and E3 are satisfied. Since there is no damping, Assumption E4 is automatically satisfied.

Theorem 9.2.1. *Suppose $\tilde{q} \in C^1(J, \mathcal{L}^2(\Omega))$. Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W),$$

for Problem RMTW for each $u_0 \in E_b$ and $u_d \in V$.

Proof. Since $\tilde{q} \in C^1(J, \mathcal{L}^2(\Omega))$, $\langle q(\cdot, t), \mathbf{0}, 0, 0, 0, 0 \rangle \in C^1(J, X)$ and the result follows from Theorem 2.2.2. $\square$

9.2.4 Finite element method

The finite element approximation is considered in [Lab06] and [LVV09b], where the finite element method was used for modal analysis. As mentioned

before, simulation of dynamic problems was not done.

Let S_1^h denote the subspace of $H^2(\Omega)$ consisting of all piecewise Hermite bicubic functions and S_2^h denote the subspace of $H^2(\Omega)$ consisting of piecewise Hermite bicubic functions that vanish on Σ_0 and Σ_1 . Finally, let S_3^h be the subspace of $H^2(0, 1)$ consisting of piecewise Hermite cubics that are zero at the endpoints of $[0, 1]$. Define S^h as the subspace of $S_1^h \times S_2^h \times_{n=1}^4 S_3^h$, such that $u_3 = \Gamma_0 u_1$, $u_4 = \Gamma_0 u_2$, $u_5 = \Gamma_1 u_1$ and $u_6 = \Gamma_1 u_2$.

Problem RMTW^h

Find a function $u_h \in C^2(0, T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + b(u_h(t), v) = (q(\cdot, t), v_1) \quad \text{for each } v \in S^h \quad (9.2.28)$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

The existence of a unique solution follows from Theorem 2.3.1 provided that $\tilde{q}$ is continuous w.r.t. the norm of $\mathcal{L}^2(\Omega)$.

Interpolation

For simplicity assume that the domain Ω is a rectangle. We must define an interpolation operator on H^k (see Subsection 9.2.3). Πu for $u \in H^k$ is defined by: $\Pi u = \langle \Pi_{bc} u_1, \Pi_{Bc} u_2, \Pi_c u_3, \Pi_c u_4, \Pi_c u_5, \Pi_c u_6, \rangle$ where Π_{bc} , Π_{Bc} and Π_c are defined in Appendix B.

Proposition 9.2.1. *There exists a $\hat{C} > 0$ such that*

$$\|\Pi u - u\|_{H^1} \leq \hat{C} h \|u\|_2 \quad \text{for each } u \in H^2.$$

Proof. From the definition of Π , we have

$$\|\Pi u - u\|_{H^1}^2 = \|\Pi_{bc} u_1 - u_1\|_{1,\Omega}^2 + \|\Pi_{Bc} u_2 - u_2\|_{1,2}^2 + \sum_{i=3}^6 \|\Pi_c u_i - u_i\|_1^2$$

The result follows from the estimates for Π_{bc} , Π_{Bc} and Π_c in Appendix B. $\square$

Using the result above and equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ (see [VZV09]), we obtain the following estimate: For $u \in H^2$,

$$\|\Pi u - u\|_V \leq \hat{C} h \|u\|_2. \quad (9.2.29)$$

9.2.5 Convergence

For this problem we have that $H(V, k) = H^k \cap V$. Since no damping is present, the results for weak damping in Sections 3.2 and 3.4 are valid. We use the estimate (9.2.29) and all the applications mentioned below will be the same as in Section 9.1.

For energy error estimates we assume $u \in C^2([0, T], V)$. We apply Corollary 3.2.1 and if we do not make the assumption that $u'' \in \mathcal{L}^2([0, T], H^2 \cap V)$, we may still apply Theorem 3.2.3. If Assumption A4 is satisfied we obtain a higher order estimate by applying Corollary 3.2.2.

For inertia norm error estimates (Section 3.4), we apply Corollary 3.4.3.

Fully discrete

The algorithm in Problem G^h -D-Baker can be used and the application of Theorem 4.3.1 is the same as in Section 9.1.

9.3 Multi-dimensional elasto-dynamics

Linear vibration problem

Consider the case of so called small vibrations of a three-dimensional elastic body. The assumption is that the local displacements are small. The equation of motion is derived from the conservation law for momentum

$$\rho \partial_t^2 \mathbf{u} = \operatorname{div} \mathbf{T} + \rho \mathbf{b},$$

where $\mathbf{u}$ is the displacement, ρ the density, $\mathbf{T}$ the stress tensor and $\mathbf{b}$ the body force (density). It is possible to derive a system of partial differential equations using Hooke's law

$$\mathbf{T} = \frac{E}{1+\nu} \boldsymbol{\varepsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)} \operatorname{tr}(\boldsymbol{\varepsilon}) I,$$

where $\boldsymbol{\varepsilon}$ is the strain tensor, E Young's modulus and ν Poisson's ratio.

Linear vibration problem Suppose $\Omega \subset E_3$ is the reference configuration for a solid executing small vibrations. The boundary of Ω consists of two parts Σ and Γ . The problem is to find $\mathbf{u}$ such that

The equation of motion is satisfied in Ω ;

Hooke's law is satisfied in Ω ;

The specified displacement for $\mathbf{u}$ is satisfied on Σ ;

The specified traction $\mathbf{T}n$ is satisfied on Γ .

Variational form

It is not necessary to consider the system of partial differential equations with corresponding boundary conditions as it is possible to derive the variational form directly from the equation of motion. The derivation is the same as for the Reissner-Mindlin plate and the details are given in Section 9.1.

The **variational form** of the vibration problem is to find a displacement $\mathbf{u}$ such that the boundary condition on Σ is satisfied and

$$\begin{aligned} & \int_{\Omega} \rho \partial_t^2 \mathbf{u} \cdot \mathbf{v} \, dV \\ = & \int_{\Omega} c \operatorname{tr}(\boldsymbol{\mathcal{E}}V) + k \operatorname{tr}(\boldsymbol{\mathcal{E}})\operatorname{tr}(V) \, dV + \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} \, dV + \int_{\Gamma} \mathbf{T}n \cdot \mathbf{v} \, dS, \end{aligned}$$

for each $\mathbf{v}$ which is zero on Σ . The symbol tr denotes the trace of the tensor (matrix) and V is the Jacobi matrix for $\mathbf{v}$ (also denoted by $\nabla \mathbf{v}$).

If the bilinear form b is defined by

$$b(\mathbf{u}, \mathbf{v}) = \int_{\Omega} c \operatorname{tr}(\boldsymbol{\mathcal{E}}V) + k \operatorname{tr}(\boldsymbol{\mathcal{E}})\operatorname{tr}(V) \, dV,$$

then the variational form of the problem is similar to that for the wave equation. The finite element approximation for this problem is considered in [Wu03] and [Wu06].

Weak variational form and existence of solutions

The two-dimensional vibration problem is similar to Problem RM in Section 9.1 and the three-dimensional case differs slightly. Note that the undamped case is considered. The abstract formulation is the same and the theory depends on the estimate in Theorem 9.1.2 (Korn's inequality).

Galerkin approximation

Apart from the obvious differences between two-dimensional and three-dimensional elements, the theory is the same as in Section 9.1.

Appendix A

Sobolev spaces

A.1 The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$ and denote its closure by $\bar{\Omega}$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω .

Theorem A.1.1. *The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with inner product*

$$(f, g) = \int_{\Omega} fg = \int_{\Omega} fg \, d\mu$$

where μ is the n -dimensional Lebesgue measure.

Proof. See [Rud76, Theorem 3.11, p.69]. □

Notation Unless otherwise stated the norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$.

Definition Set $S_f = \text{closure}\{x \in \Omega \mid f(x) \neq 0\}$. Then $C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) \mid S_f \subset \Omega\}$.

Remark If $f \in C_0^\infty(\Omega)$, then the distance between S_f and the boundary of Ω is positive.

Theorem A.1.2. $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

Proof. See [Ada75, Theorem 2.13, p.28]. □

A.2 Sobolev spaces

The one-dimensional case

Suppose Ω is a bounded open interval. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition For f, g in $H^m(\Omega)$,

$$[f, g]_m = (f^{(m)}, g^{(m)}) \quad \text{and} \\ |f|_m = \sqrt{[f, f]_m} \quad \text{for } m = 0, 1, \dots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

Definition Inner product

For f and g in $H^m(\Omega)$, $(f, g)_m = \sum_{k=0}^m (f^{(k)}, g^{(k)})$ for $m = 0, 1, \dots$

The bilinear form $(\cdot, \cdot)_m$ has all the properties of an inner product.

Definition Norm

For f in $H^m(\Omega)$, $\|f\|_m = \sqrt{(f, f)_m}$ for $m = 0, 1, \dots$

The higher dimensional case

Suppose Ω is an open subset of $\mathbb{R}^n$. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Notation Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Remark $|\alpha|$ denotes the order of the derivative.

Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^2$.

Definition For f, g in $H^m(\Omega)$,

$$[f, g]_m = \sum_{|\alpha|=m} \partial^\alpha f \partial^\alpha g \quad \text{and} \\ |f|_m = \sqrt{[f, f]_m}.$$

Definition The inner product for $H^m(\Omega)$ is defined by

$$(f, g)_m = \sum_{k=0}^m [f, g]_k \quad \text{for } m = 0, 1, \dots$$

Definition The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f, g)_m} \quad \text{for } m = 0, 1, \dots$$

Sobolev spaces of vector valued functions

Definition

$$u \in \mathcal{L}^2(\Omega)^2 \quad \text{if } u_i \in \mathcal{L}^2(\Omega) \quad \text{for } i = 1, 2.$$

$$u \in \mathcal{L}^2(\Gamma)^2 \quad \text{if } u_i \in \mathcal{L}^2(\Gamma) \quad \text{for } i = 1, 2.$$

$$u \in H^k(\Omega)^2 \quad \text{if } u_i \in H^k(\Omega) \quad \text{for } i = 1, 2.$$

$$[u, v]_{m,2} = [u_1, v_1]_m + [u_2, v_2]_m \quad \text{for } u \in H^m(\Omega)^2 \quad \text{and } v \in H^m(\Omega)^2.$$

$$|u|_{m,2} = \sqrt{[u, u]_{m,2}} \quad \text{for } u \in \mathcal{L}^2(\Omega)^2.$$

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|_{m,2}^\Omega$ and $\|\cdot\|_{m,2}^\Gamma$.

Definition The inner product for $H^m(\Omega)^2$ is defined by

$$(f, g)_{m,2} = \sum_{k=0}^m [f, g]_{k,2} \quad \text{for } m = 0, 1, \dots$$

Definition The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f, g)_{m,2}} \quad \text{for } m = 0, 1, \dots$$

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

A.3 Fundamental properties of Sobolev spaces

Assumption Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

Remark

It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$.

Theorem A.3.1. *The space $H^m(\Omega)$ is complete.*

Proof. See [Ada75, Theorem 3.2, p.45]. □

Theorem A.3.2. *$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.*

Proof. See [OR76, Theorem 2.10, p.53]. □

Remark

A function in $H^m(\Omega)$ can be approximated by a function in $C^m(\bar{\Omega})$: if $u \in H^m(\Omega)$ then for any $\varepsilon > 0$ there exists a $\phi \in C^m(\bar{\Omega})$ such that $\|u - \phi\|_m < \varepsilon$.

Theorem A.3.3. *Sobolev's lemma*

Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > m + n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \leq \|u\|_p \quad \text{for } |\alpha| \leq m.$$

Proof. See [OR76, Theorem 3.10, p.80]. □

Remark One-dimensional case

If $n = 1$ in Theorem A.3.3, then $u \in C^{p-1}(\bar{\Omega})$ and $\|u^{(k)}\|_{\sup} \leq \|u\|_p$ for $k \leq m - 1$.

A.4 Inequalities

The one-dimensional case

Proposition A.4.1. *Consider any $u \in C^1[0, \ell]$. For any two points x and y in $[0, \ell]$,*

$$|u(x)| \leq \sqrt{\ell} \|u'\| + |u(y)|.$$

Proof. For any f and $g \in \mathcal{L}^2(\Omega)$ we have the Cauchy-Schwartz inequality

$$\left(\int_y^x fg \right)^2 \leq \left(\int_y^x f^2 \right) \left(\int_y^x g^2 \right)$$

for $x > y$. By choosing $g = 1$ we find that

$$\left(\int_y^x f \right)^2 \leq \left(\int_y^x f^2 \right) (x - y) \leq \ell \|f\|^2. \quad (\text{A.4.1})$$

Hence $\left| \int_y^x f \right| \leq \sqrt{\ell} \|f\|$ for each $f \in \mathcal{L}^2(0, \ell)$. Since $u(x) - u(y) = \int_y^x u'$,

$$\begin{aligned} |u(x)| &\leq \left| \int_y^x u' \right| + |u(y)| \\ &\leq \sqrt{\ell} \|u'\| + |u(y)|, \end{aligned}$$

from Equation (A.4.1). □

Proposition A.4.2. *For any $u \in C^1[0, \ell]$ with a zero in $[0, \ell]$ we have*

$$\|u\|_{\sup} \leq \sqrt{\ell} \|u'\|.$$

Proof. Suppose $u(y) = 0$, then $|u(x)| \leq \sqrt{\ell} \|u'\|$ by Proposition A.4.1. The result follows from the fact that $\sqrt{\ell} \|u'\|$ is an upper bound for $|u|$. □

Proposition A.4.3. *For any $u \in C^1[0, \ell]$ with a zero in $[0, \ell]$ we have*

$$\|u\| \leq \ell \|u'\|.$$

Proof. We use Proposition A.4.2,

$$\|u\|^2 = \int_0^\ell (u(x))^2 dx \leq \ell \|u\|_{\sup}^2 \leq \ell^2 \|u'\|^2.$$

□

Proposition A.4.4. *Let $T[0, \ell] = \{u \in C^1[0, \ell] \mid u(0) = 0\}$. Denote the closure of $T[0, \ell]$ in $H^1(0, \ell)$ by $V(0, \ell)$. For any $u \in V(0, \ell)$,*

$$\|u\| \leq \ell \|u'\|.$$

Proof. There exist a sequence $\{u_n\} \in C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$. Since $|\|u_n\|_1 - \|u\|_1| \leq \|u_n - u\|_1$, $\|u_n\|_1 \rightarrow \|u\|_1$ as $n \rightarrow \infty$. Consequently $\|u_n\| \rightarrow \|u\|$ and $\|u'_n\| \rightarrow \|u'\|$ as $n \rightarrow \infty$.

The result follows from Proposition A.4.3 by taking the limits. $\square$

Proposition A.4.5. *For any $u \in C^1[0, \ell]$ there exists a constant $K_\ell > 0$ such that*

$$|u(\ell)| \leq K_\ell \|u\|_1.$$

Proof. If u has a zero in $[0, \ell]$, then $|u(\ell)| \leq \sqrt{\ell} \|u'\|$ by Proposition A.4.1. If u does not have a zero, suppose $u > 0$ on $[0, \ell]$ and let m be the minimum of u on $[0, \ell]$. Let $w(x) = u(x) - m$, then

$$|u(\ell)| \leq |w(\ell)| + m \leq \sqrt{\ell} \|w'\| + m \quad (\text{A.4.2})$$

by Proposition A.4.2. But $w' = u'$ and $\|u\|^2 = \int_0^\ell u^2 \geq m^2 \ell$.

Therefore, from Equation (A.4.2),

$$|u(\ell)| \leq \sqrt{\ell} \|u'\| + \frac{1}{\sqrt{\ell}} \|u\|$$

which implies the result. If $u < 0$, then $-u > 0$ and we have the result. $\square$

A.5 Trace

The one-dimensional case

From Theorem A.3.2 we have that for each $u \in H^1(0, \ell)$, there exists a sequence $\{u_n\} \subset C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

Theorem A.5.1. *Suppose $u \in H^1(0, \ell)$. Then, there exists a unique real number u_ℓ such that for any sequence $\{u_n\} \subset C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} u_n(\ell) = u_\ell$.*

Proof. Let $\{u_n\}$ be any sequence in $C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$. The sequence $\{u_n\}$ is Cauchy in $H^1(0, \ell)$ so by Proposition A.4.5 $\{u_n(\ell)\}$ is Cauchy in $\mathbb{R}$. Thus $\lim_{n \rightarrow \infty} u_n(\ell)$ exists. This limit is independent of

the sequence $\{u_n\}$. Indeed, if $\{v_n\}$ is another sequence in $C^1[0, \ell]$ such that $\|v_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$ then

$$\|v_n - u_n\|_1 \leq \|v_n - u\|_1 + \|u - u_n\|_1 \rightarrow 0$$

as $n \rightarrow \infty$. So by Proposition 2.4.4 we have that $|v_n(\ell) - u_n(\ell)| \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Definition Trace operator Γ_ℓ

For each $u \in H^1(0, \ell)$ let

$$\Gamma_\ell u = u_\ell.$$

Theorem A.5.2. *The mapping Γ_ℓ is linear and bounded. In fact*

$$|\Gamma_\ell u| \leq K_\ell \|u\|_1.$$

Proof. The linearity follows from the properties of limits. We have that,

$$\frac{|u_n(\ell)|}{\|u_n\|_1} \leq K_\ell \text{ for all } n$$

and by considering the limits we have that

$$|\Gamma_\ell u| \leq K_\ell \|u\|_1.$$

This proves that Γ_ℓ is bounded. $\square$

Remark A trace operator Γ_0 can be defined similarly at zero.

The two-dimensional and three-dimensional cases

Suppose Ω is an open convex subset of $\mathbb{R}_n$ with a piecewise smooth boundary $\partial\Omega$ and $\Sigma \subset \partial\Omega$ is connected. It is then possible to define $\mathcal{L}^2(\Sigma)$.

Theorem A.5.3. *Suppose $u \in H^1(\Omega)$. Then there exists a unique $u_\Sigma \in \mathcal{L}^2(\Sigma)$ such that for any sequence $\{u_n\} \subset C^1(\bar{\Omega})$ where $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} u_n = u_\Sigma$.*

Proof. See [OR76, pp.141-142]. $\square$

Definition The trace operator Γ

For each $u \in H^1(\Omega)$ let

$$\Gamma u = u_\Sigma.$$

Theorem A.5.4. *The mapping Γ is linear and bounded. In fact*

$$\|\Gamma u\|_{\Sigma} \leq C_s \|u\|_1.$$

Proof. See [OR76, pp.141-142].

□

Appendix B

Interpolation

The results in this appendix are simplified versions of the theory in [OR76, Chapter 6], [OC83, Chapter 4] and [SF73, Chapter 5].

B.1 The one dimensional case

The interval $I = [0, \ell]$ is partitioned into n elements referred to as a finite element mesh. Using this mesh a finite dimensional space S^h is defined. An interpolation operator Π is a transformation from $H^k(I)$ to S^h .

It is first necessary to define interpolation on an element.

Definition Interpolation on the element $I_e = [a, b]$.

Suppose there are n basis functions that are nonzero on the element I_e , denoted by ψ_i . Let $(x_1, x_2, \dots, x_n)$ be an ordered n -tuple of points in I_e – not necessarily distinct. Suppose $(\beta_1, \beta_2, \dots, \beta_n)$ is an ordered n -tuple of nonnegative integers – each less than or equal to m . For a function in $u \in C^m(I_e)$ we define the interpolant Π_e by

$$\Pi_e u = \sum_{i=1}^n D^{\beta_i} u(x_i) \psi_i.$$

Examples

1. Hermite cubic basis functions:

There are four basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = x_3 = a$, $x_2 = x_4 = b$, $\beta_1 = \beta_2 = 0$ and $\beta_3 = \beta_4 = 1$.

$$\begin{aligned}\Pi_e u &= \sum_{i=1}^4 D^{\beta_i} u(x_i) \psi_i \\ &= u(a) \psi_1 + u(b) \psi_2 + u'(a) \psi_3 + u'(b) \psi_4.\end{aligned}$$

2. Piecewise linear basis functions:

There are two basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = a$, $x_2 = b$ and $\beta_1 = \beta_2 = 0$.

$$\Pi_e u = \sum_{i=1}^2 D^{\beta_i} u(x_i) \psi_i = u(a) \psi_1 + u(b) \psi_2.$$

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element I_e is denoted by $[f]_{I_e}$.

Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{I_e} = \Pi_e [u]_{I_e} \text{ for each element } I_e.$$

Notation

$\mathcal{P}_j(I_e)$: the set of polynomials on I_e of degree at most j is denoted by $\mathcal{P}_j(I_e)$.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e .

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e .

Remark

If $k \geq s(\Pi_e) + 1$, then the interpolation operator Π_e is defined on $H^k(I_e)$.

Recall that $|\cdot|_k$ denotes the seminorm of order k , i.e.

$$|u|_k = \|u^{(k)}\|.$$

(See Appendix A.)

The interpolation error

Theorem B.1.1 below is formulated as a special case of a general result. This result may be found in [SF73, p.144], [OC83, p.76] and [OR76, p.279].

We will use $\widehat{C}$ to denote a generic constant.

Theorem B.1.1. *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1$$

for the interpolation operator Π . Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(I)$ we have

$$|\Pi u - u|_{m,I} \leq \widehat{C} h^{k-m} |u|_{k,I} \quad \text{for } m = 0, 1, \dots, k.$$

The interpolation operator is denoted by Π_L for piecewise linear basis functions and by Π_c for Hermite cubics.

Corollary B.1.1. *Hermite cubic basis functions.*

There exists a constant $\widehat{C}_c$ such that if $u \in H^k(I)$ for

a) $2 \leq k \leq 4$, then

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{k-m} |u|_k, \quad m = 0, 1, \dots, k.$$

b) $k > 4$, then

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{4-m} |u|_4, \quad m = 0, 1, \dots, 4.$$

Proof. It is clear that $s(\Pi_c) = 1$ and it can be shown that $r(\Pi_c) = 3$. Consequently Theorem B.1.1 is applicable with $k = 2, 3$ or 4 . $\square$

Corollary B.1.2. *Piecewise linear basis functions*

There exists a constant $\widehat{C}_L$ such that if $u \in H^k(I)$ for $k \geq 2$, then

$$\|\Pi_L u - u\|_1 \leq \widehat{C}_L h |u|_2$$

Proof. It is clear that $s(\Pi_L) = 1$ and it can be shown that $r(\Pi_L) = 1$. Consequently Theorem B.1.1 is applicable with $k = 2$. $\square$

B.2 The two-dimensional case

Recall that the domain Ω is partitioned into n elements. An interpolation operator Π is a transformation from $H^k(\Omega)$ to S^h , the finite dimensional subspace.

It is first necessary to define interpolation on an element.

Definition Interpolation on the element Ω_e .

Suppose there are n basis functions that are nonzero on the element Ω_e , denoted by ψ_i . Let $(x_1, x_2, \dots, x_n)$ be an ordered n -tuple of points in Ω_e – not necessarily distinct. Suppose $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is an ordered n -tuple of multi-indices with $|\alpha_i| \leq m$. For a function in $u \in C^m(\Omega_e)$ we define the interpolant $\Pi_e u$ by

$$\Pi_e u = \sum_{i=1}^n \partial^{\alpha_i} u(x_i) \psi_i.$$

Examples

1. For piecewise linear basis functions on the triangle Ω_e :

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3,$$

where a, b and c are the vertices of the triangle.

2. For piecewise bilinear basis functions on a quadrilateral Ω_e :

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3 + w(d)\psi_4,$$

where a, b, c and d are the vertices of Ω_e .

3. For Hermite cubic basis functions on a triangle Ω_e :

There are 10 basis functions which are not zero on Ω_e . If a, b and c are the vertices of Ω_e and d the centroid, let $x_i = a$ for $i = 1, 2, 3$, $x_i = b$ for $i = 4, 5, 6$, $x_i = c$ for $i = 7, 8, 9$ and $x_{10} = d$. The multi-indices are $\alpha_i = (0, 0)$ for $i = 1, 4, 7, 10$, $\alpha_i = (1, 0)$ for $i = 2, 5, 8$ and $\alpha_i = (0, 1)$ for $i = 3, 6, 9$.

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element Ω_e is denoted by $[f]_{\Omega_e}$.

Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{\Omega_e} = \Pi_e[u]_{\Omega_e} \quad \text{for each element } \Omega_e.$$

Notation

$\mathcal{P}_j(\Omega_e)$: the set of polynomials on Ω_e of degree at most j .

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e .

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e .

Recall that for a two-dimensional domain Ω , $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix A.)

Theorem B.2.1 below is formulated as a special case of a general result. As mentioned before, this result may be found in [SF73, p.144], [OC83, p.76] and [OR76, p.279]. In the theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e .

Theorem B.2.1. *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 2 \leq k \leq r(\Pi_e) + 1$$

for the interpolation operator Π . Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have

$$|\Pi u - u|_{m,\Omega} \leq \widehat{C} h^{k-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, \dots, k.$$

Remark The constant $\widehat{C}$ depends on the shape of the elements in the finite element mesh.

Corollary B.2.1. *Piecewise linear basis functions on triangle elements. The interpolation operator is denoted by Π_Δ . If $k \geq 2$, then there exists a constant $\widehat{C}_\Delta$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_\Delta u - u|_{m,\Omega} \leq \widehat{C}_\Delta h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

Corollary B.2.2. *Piecewise bilinear basis functions on rectangle elements. The interpolation operator is denoted by Π_b . If $k \geq 2$, then there exists a constant $\widehat{C}_b$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_b u - u|_{m,\Omega} \leq \widehat{C}_b h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

Corollary B.2.3. *Hermite cubic basis functions. The interpolation operator is denoted by Π_c . If $k \geq 2$, then there exists a constant $\widehat{C}_c$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_c u - u|_{m,\Omega} \leq \widehat{C}_c h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

Corollary B.2.4. *Hermite bicubic basis functions. The interpolation operator is denoted by Π_{bc} . If $k \geq 2$, then there exists a constant $\widehat{C}_c$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_{bc} u - u|_{m,\Omega} \leq \widehat{C}_c h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

B.3 Vector-valued functions

If an interpolation operator Π is defined on $H^k(\Omega)$ we may define one on $H^k(\Omega)^2$. For $u = \langle u_1, u_2 \rangle \in H^k(\Omega)^2$, we define

$$\Pi_2 u = \langle \Pi u_1, \Pi u_2 \rangle.$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2.$$

(See Appendix A.)

Lemma B.3.1. *If $\|\Pi v - v\|_m \leq \widehat{C} h^{k-m} |v|_k$ for $v \in H^k(\Omega)$, then*

$$\|u - \Pi_2 u\|_{m,2} \leq \widehat{C} h^{k-m} |u|_{k,2} \quad \text{for } u \in H^k(\Omega)^2.$$

Proof.

$$|u - \Pi_2 u|_{m,2}^2 = |u_1 - \Pi u_1|_m^2 + |u_2 - \Pi u_2|_m^2$$

□

For piecewise bilinear basis functions on rectangles, let $\Pi_B u = \langle \Pi_b u_1, \Pi_b u_2 \rangle$.

Corollary B.3.1. *If $k \geq 2$, then there exists a constant $\hat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$|u - \Pi_B u|_{m,2} \leq \hat{C}h|u|_{2,2} \text{ for } m = 0, 1, 2.$$

Proof. The result follows from Corollary B.2.2 and Lemma B.3.1. $\square$

For piecewise bicubic basis functions on rectangles, let $\Pi_{Bc} u = \langle \Pi_{bc} u_1, \Pi_{bc} u_2 \rangle$.

Corollary B.3.2. *If $k \geq 2$, then there exists a constant $\hat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$|u - \Pi_{Bc} u|_{m,2} \leq \hat{C}h|u|_{2,2} \text{ for } m = 0, 1, 2.$$

B.4 Three-dimensional case

Theorem B.2.1 is also valid for the three-dimensional case.

Corollary B.4.1. *Piecewise linear basis functions on tetrahedron elements. The interpolation operator is denoted by Π_t . If $k \geq 2$, then there exists a constant $\hat{C}_t$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_t u - u|_{m,\Omega} \leq \hat{C}_t h^{2-m} |u|_{2,\Omega} \text{ for } m = 0, 1, 2.$$

Appendix C

Assumptions

Assumptions on the abstract spaces and bilinear forms

E1 V is dense in W and W is dense in X .

E2 There exists a constant C_b such that $\|v\|_W \leq C_b \|v\|_V$ for each $v \in V$.

E3 There exists a constant C_c such that $\|v\|_X \leq C_c \|v\|_W$ for each $v \in W$.

E4 The bilinear form a is nonnegative, symmetric and bounded on V , i.e.

$$|a(u, v)| \leq K_a \|u\|_V \|v\|_V.$$

E4W The bilinear form a is nonnegative, symmetric and bounded on W , i.e.

$$|a(u, v)| \leq K_a \|u\|_W \|v\|_W.$$

Other assumptions

Assumption A1

The solution u of Problem G has the property that $(Pu) \in C^2(J)$.

Assumption A2

The solution u of Problem G satisfies $u \in C^2(J, V)$.

Assumption A3

There exists a subspace $H(V, k)$ of V , an interpolation operator Π and positive constants C_Π and α (depending on V and k) such that for $u \in H(V, k)$:

$$\|u - \Pi u\|_V \leq C_\Pi h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V,k)$.

Assumption A4

$E_b \subset H(V,k)$ and for any $u \in E_b$ we have $\|u\|_{H(V,k)} \leq \widehat{c}_b \|y\|_W$, where

$$b(u, v) = (y, v) \quad \forall v \in V.$$

Assumption A5

$H(V,k)$ is a dense subset of V .

Assumption I *Inverse property of the subspace $S^h(\Omega)$*

$$\|u\|_V \leq C_I h^{-\lambda} \|u\|_W$$

Appendix D

Gronwall and Young's inequalities

Proposition D.1.1. *If there exists a constant $c > 0$ such that $\phi' \leq c\phi$, then*

$$\phi(t) \leq e^{ct}\phi(0) \quad \text{for } t > 0.$$

Proof.

$$\frac{d}{dt}(\phi(t) e^{-ct}) = e^{-ct}(\phi'(t) - c\phi(t)) \leq 0.$$

Therefore $\phi(t) e^{-ct} - \phi(0) \leq 0$ and the result follows. $\square$

Lemma D.1.1. *Gronwall's inequality*

Suppose $\psi \in C^1[0, T]$. If there exist positive constants c and K such that

$$\psi(t) \leq c \int_0^t \psi + K$$

then

$$\psi(t) \leq Ke^{ct} \quad \text{for each } t \in [0, T].$$

Proof. Let $\phi(t) = c \int_0^t \psi + K$, then $\phi' = c\psi$ and $\phi(0) = K$. Since $\phi' \leq c\phi$, we have

$$\phi(t) \leq e^{ct}K \quad \text{from Proposition D.1.1.}$$

The result follows from the fact that $\psi \leq \phi$. $\square$

Lemma D.1.2. *Discrete version of the Gronwall inequality*

If $\phi(t)$ and $\psi(t)$ are nonnegative functions with $\psi(t)$ nondecreasing and

$$\phi(R\tau) \leq \psi(R\tau) + C\tau \sum_{n=0}^{R-1} \phi(n\tau)$$

then

$$\phi(R\tau) \leq \psi(R\tau)e^{CR\tau}, \quad \text{where } C \geq 0.$$

The following simple inequality is very useful.

Lemma D.1.3. *Young's inequality*

$$ab \leq \frac{1}{2} (\varepsilon^2 a^2 + \varepsilon^{-2} b^2)$$

Proof. $(\varepsilon a - \varepsilon^{-1} b)^2 \geq 0$ implies that

$$\varepsilon^2 a^2 + \varepsilon^{-2} b^2 \geq 2ab.$$

□

Appendix E

Non-standard notation

The following non-standard notation is used in this thesis.

$\tilde{f}$: The mapping $t \rightarrow f(\cdot, t)$.

$u^{(k)} \in \mathcal{L}^2(J; Y)$: $u^{(k)}(t) \in Y$ for each t and $\int_J \|u^{(k)}\|_Y^2 < \infty$.

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