Designs, Codes and Cryptography (2025) 93:3463–3479
https://doi.org/10.1007/s10623-025-01647-3

Primitive rank 3 groups, binary codes, and 3-designs

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Received: 22 January 2025 / Revised: 30 April 2025 / Accepted: 2 May 2025 /
Published online: 20 May 2025

Abstract
Let G be a primitive rank 3 permutation group acting on a set of size v. Binary codes of length v globally invariant under G are well-known to hold PBIBDs in their A_w codewords of weight w. The parameters of these designs are $\left(A_w, v, w, \frac{wA_w}{v}, \lambda_1, \lambda_2\right)$. When $\lambda_1 = \lambda_2 = \lambda$, the PBIBD becomes a 2-$(v, w, \lambda)$ design. We obtain computationally 111 such designs when G ranges over $\mathrm{L}_2(8){:}3$, $\mathrm{U}_4(2)$, $\mathrm{U}_3(3){:}2$, A_8, $\mathrm{S}_6(2)$, $\mathrm{S}_4(4)$, $\mathrm{U}_5(2)$, M_{11}, M_{22}, HS, $\mathrm{G}_2(4)$, $\mathrm{S}_8(2)$, $\mathrm{O}_{10}^{+}(2)$, and $\mathrm{O}_{10}^{-}(2)$ in the notation of the Atlas. Included in the counting are 2-designs which are held by nonzero weight codewords of the binary adjacency codes of the triangular and square lattice graphs, respectively. The 2-designs in this paper can be obtained neither from Assmus–Mattson theorem, nor by the classical 2-tra nsitivity (or 2-homogeneity) argument of the automorphism group of the code. Further, the extensions of the codes that hold 2-designs sometimes hold 3-designs. We thus obtain nine self-complementary 3-designs on 16 (4), 28, 36 (2), 56, 176 points respectively. The design on 176 points is invariant under the Higman–Sims group.

Keywords PBIBD · BIBD · Rank 3 groups

Mathematics Subject Classification 05B05 · 05E30 · 94B15

1 Introduction

Partially Balanced Incomplete Block designs (shortly PBIBDs) were introduced by Bose and Nair as a relaxation of Balanced Incomplete Block designs (BIBDs, also known as 2-designs) [9]. Their motivation was statistics, but the concept of association schemes they introduced then gathered later an interest of its own in the field now called Algebraic Combinatorics

Dedicated to Jennifer D. Key, for her endearing commitment to Mathematics and for her contributions to coding theory and its connections to design theory and algebraic graph theory.

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[7, 11]. For a common reference for PBIBD and association schemes see [5]. Given a set X of size v equipped with a two-class association scheme $(X, (R_0, R_1, R_2))$, a PBIBD D is a collection of size b of k-subsets of X called blocks such that a pair $x, y \subseteq X$, with $x \neq y$ is included in exactly λ_i blocks iff $x R_i y$ for $i = 1, 2$. When $\lambda_1 = \lambda_2 = \lambda$, the PBIBD becomes a 2-design. When C is a binary code invariant under a rank 3 group acting on its coordinate places by permutation, the words of given weight of C hold a PBIBD. In that situation if O_i for $i = 1, 2, 3$ are the three orbits of G on $X \times X$, the relations R_i are defined by $x R_i y$ iff $(x, y) \in O_i$. Assuming O_1 to be the trivial orbit, and $|X|$ to be even, the graphs (X, R_2) and (X, R_3) are strongly regular, complementary of each other [20]. Recently, this technique was used to construct a 2-design held by the codewords of weight 9 and 10 of the quadratic residue code of length 41 [32]. The group G in that context is a one point stabilizer of $\mathrm{PSL}(2, 41)$ acting on $\mathbb{Z}_{41}$ [32, Theorem 5]. We remark that our method is different in two ways from Tonchev's approach in [34, 35]. We do not assume that G acts transitively on the codewords of weight w which would correspond to a base block in [34]. We do not suppose that the blocks of weight w are the incidence vectors of subgraphs of (X, R_1) that are either the neighborhood of a point or that neighborhood union that point [34], a condition that restricts w to be either $|R_1|$ or $|R_1| + 1$. It should be noted that codes holding designs of different strengths at different weights are of current interest in algebraic combinatorics [4, 21, 30], motivated by an analogue of Lehmer's conjecture in number theory [29]. In particular the existence of such designs cannot be explained by transitivity arguments or the Assmus–Mattson theorem, two tools which construct designs of the same strength on every weight class [18]. In complete analogy with [8], the 2-designs we construct here can sometimes be embedded into 3-designs by considering the extension codes. Our interest in this problem was primarily stimulated by [32] where the authors presented parameters of PBIBDs obtained from the binary codes of some rank 3 graphs. In particular, the authors considered the adjacency codes, i.e., codes defined by the row span of the adjacency matrix of a strongly regular rank 3 graph. Our strategy is more general in that we extend this to the submodules (regarded as codes) of the rank 3 permutation modules over $\mathbb{F}_2$ defined by the primitive rank 3 permutation action of a finite nonsolvable group G on various geometric objects. Thus, this paper is a contribution to the search of nontrivial 2- and 3-designs with small parameters and large automorphism groups acting on.

For a survey of designs held by codes we refer the reader to the recent book [18].

The paper is organized as follows: in Sect. 2 we outline the background and notation and give a brief overview on the motivation for the study carried in the paper. In Sect. 3 we describe the construction method used to obtain codes (submodules) invariant under the action of a rank 3 group, and give our results on the 2- and 3-designs invariant under rank 3 groups of almost simple and grid types in Sects. 4, 5 and 6, respectively.

2 Preliminary

2.1 Permutation groups

Denote by G a permutation group acting on a finite set X. The size of X is called the degree of G. The group G is transitive if it has only one orbit on X. It is t-transitive if it has only one orbit on the ordered t-tuples of distinct elements of X. It is t-homogeneous if it has only one orbit on $\binom{X}{t}$. It has rank ρ if it is transitive and if any one-point stabilizer of G has ρ orbits on $X \times X$ including the diagonal. In this work we concentrate on the case $\rho = 3$ because of

Table 1 Notation used for some rank 3 groups

Symbol	Name
$L_n(q)$	Projective special linear group in dimension n over $\mathbb{F}_q$
$U_n(q^2)$	Projective special unitary group in dimension n over $\mathbb{F}_{q^2}$
$O_n^+(q)$	Orthogonal group of plus type in dimension n over $\mathbb{F}_q$
$O_n^-(q)$	Orthogonal group of minus type in dimension n over $\mathbb{F}_q$
$S_n(q)$	Projective symplectic group in dimension n over $\mathbb{F}_q$
S_n	Symmetric group on n letters
A_n	Alternating group on n letters
$G_2(q)$	Adjoint Chevalley group of type G_2 over $\mathbb{F}_q$
M_d	Mathieu group of degree d for $d \in \{11, 22\}$
HS	Higman–Sims group

its connections with strongly regular graphs. A graph is rank 3 if it is a Cayley graph on a rank 3 group with generating set one of the two nontrivial orbits.

The study of rank 3 permutation groups dates back to the paper [20] of Donald Higman. Rank 3 groups can be either primitive or imprimitive. A subset $\mathcal{B}$ of X is a block for G if for all $g \in G$ either

$$\mathcal{B}^g = \mathcal{B} \quad \text{or} \quad \mathcal{B}^g \cap \mathcal{B} = \varnothing.$$

The action of G is primitive on X if it is transitive and all blocks are trivial, that is either $|\mathcal{B}| = 1$ or $\mathcal{B} = X$. Also G is said to be imprimitive on X if G is transitive on X and G preserves some nontrivial block of X.

The primitive rank 3 permutation groups have been completely classified (in [6, 13, 22, 27, 28]) and those of imprimitive type have been dealt with in [17].

Below, in Result 1 we collect the information on the classification of finite primitive rank 3 permutation groups and in Table 1 we give the names of the individual groups and the classes of groups that we will consider in the paper followed by the notation used in the Atlas [15] for each group. This choice of groups and rank 3 graphs was motivated by the need to extend the results of [32] and include binary codes of moderate dimensions which are amenable to computational calculations.

A primitive rank 3 permutation group G has a unique minimal normal subgroup S, called its socle and denoted here by $S = \mathrm{soc}(G)$. The socle S can be a non-abelian simple group, a direct product of two isomorphic non-abelian simple groups, or elementary abelian. When S is elementary abelian, G is said to be of affine type; and when S is a direct product of two non-abelian simple groups, G is said to be of product action type.

We are interested in situations where the group S is a non-abelian simple group and G is thus of almost simple type. An almost simple group is a group G containing a non-abelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(G)$. We are also interested in the situation where S is a direct product of two non-abelian simple groups.

Result 1 *If G is a primitive rank 3 permutation group on a finite set X of size v then one of the following holds*:

(a) Almost simple type: $S \trianglelefteq G \leq \mathrm{Aut}(S)$, *where* $S = \mathrm{soc}(G)$ *is a non-abelian simple group*;

Table 2 Some finite primitive rank 3 group of almost simple type of degree v

Group	Degree	Parameters of SRG
$\mathrm{P\Gamma L}(2,8) \cong \mathrm{L}_2(8){:}3$	36	(36, 14, 7, 4)
A_9	120	(120, 56, 28, 24)
$\mathrm{S}_6(2)$	63	(63, 30, 13, 15)
	120	(120, 56, 28, 24)
A_8	35	(35, 16, 6, 8)
$\mathrm{O}_5(3)$	27	(27, 16, 10, 8)
	36	(36, 15, 6, 6)
	40	(40, 12, 2, 4)
	40	(40, 12, 2, 4)
$\mathrm{U}_4(2)$	45	(45, 12, 3, 3)
$G_2(2) \cong \mathrm{U}_3(3){:}2$	36	(36, 14, 4, 6)
$\mathrm{S}_4(4)$	85	(85, 20, 3, 5)
	120	(120, 51, 18, 24)
	136	(136, 60, 24, 28)
$\mathrm{U}_5(2)$	176	(176, 40, 12, 8)
$\mathrm{G}_2(4)$	2016	(2016, 975, 462, 480)
M_{11}	55	(55, 18, 9, 4)
M_{22}	176	(176, 70, 18, 34)
HS	100	(100, 22, 0, 6)
$\mathrm{S}_8(2)$	255	(255, 126, 61, 63)
$\mathrm{O}_{10}^{+}(2)$	496	(496, 240, 120, 112)
$\mathrm{O}_{10}^{-}(2)$	528	(528, 255, 126, 120)

(b) Product action type: $S \times S \trianglelefteq G \leq S_0 \wr Z_2$, *where* S_0 *is a* 2*-transitive group of degree* v_0, *with* $S \trianglelefteq S_0 \leq \mathrm{Aut}(S)$, S *non-abelian simple, and* $v = {v_0}^2$;

(c) Affine type: $G = SG_0$, *where* S *is an elementary abelian* p*-group acting regularly on a vector space* V, G_0 *is an irreducible subgroup of* $\mathrm{GL}(v, p)$ *and* G_0 *has exactly* 2 *orbits on the nonzero vectors of* V.

The groups in case (a) were classified by Bannai [6] for $S \cong A_n$, by Kantor and Liebler [22] if S is classical, and by Liebeck and Saxl [27] if S is sporadic or of exceptional Lie type. Case (b) follows from the classification of 2-transitive groups [13] and case (c) was done by Liebeck [28].

The first and second columns of Table 2 list the individual finite primitive rank 3 permutation groups of degree v that will be considered in this paper and the third column lists the parameters of the corresponding strongly regular graphs. For the cases where S is an alternating group, see Sect. 5, and where S is a direct product of two non-abelian simple groups, see Sect. 6.

2.2 Association schemes

An s class association scheme (X, R) on a finite set X is determined by a partition of $X \times X$ into $s + 1$ equivalence classes R_i, $i = 0, \ldots,$ satisfying the following axioms.

(1) $x R_0 y$ if and only if $x = y$,

(2) xR_ky if and only if yR_kx,
(3) If xR_ky the number of $z \in X$ such that xR_iz and yR_jz is a number p_{ij}^k which does not depend on the choice of x and y.

Note that this implies that the graph (X, R_i) is regular of constant valence p_{ii}^0. In this work we will limit ourselves to the case $s = 2$ when the graphs (X, R_2) and (X, R_3) are known to be *strongly regular* (SRG for short). For such a regular graph there are constants c, d such that every pair forming an edge (resp. non connected) has c (resp. d) common neighbours. A SRG on a points and degree b has parameters (a, b, c, d). Thus, in terms of association schemes, we have $b = p_{11}^0, c = p_{11}^1, d = p_{11}^2$.

2.3 Designs

A BIBD or 2-design is a collection of k-sets of a v-set X such that every pair of elements of X is contained in λ blocks. The parameters are compactly written as t-(v, k, λ).

A PBIBD is a collection of k-sets (called *blocks*) of a v-set X [assumed equipped with a two class association scheme structure (X, R)] such that every pair of R_i-related elements is contained in λ_i blocks for $i = 1, 2$. Each point of a BIBD is contained in r blocks. This parameter is called the *replication* number and satisfies $bk = vr$, as flag counting shows easily. The parameters of a PBIBD are written $(b, v, k, r, \lambda_1, \lambda_2)$.

2.4 Codes

The set of vectors of length n over $\mathbb{F}_2$ is equipped with the *Hamming weight* $w(u)$ which counts the number of nonzero components of a vector u. The *Hamming distance* of two vectors u, v is then $d(u, v) = w(u + v)$. A *binary code* of *length* n is an $\mathbb{F}_2$-subspace of $\mathbb{F}_2^n$. Its *minimum distance* is the minimum Hamming weight of its codewords. Its elements are called *codewords*. Here k denotes its *dimension* as a vector space over $\mathbb{F}_2$. Its parameters are written $[n, k, d]$. The number of codewords of weight w is denoted by A_w and the series $(A_w)_w$ is called *the weight distribution* of the code. By *adjacency code* of a graph we will denote the row span of the adjacency matrix of that graph.

2.5 Codes and designs

The Assmus–Mattson theorem [2] connects codes to designs in the following way. If C is a binary $[n, k, d]$ code such that the weight distribution of its dual code contains at most $d - t$ nonzero weights less than $n - t$, then the codewords of C of given weight are the characteristic vectors of the blocks of a t-design. A more general theorem was found recently [33]. We say then that the codewords of weight w *hold* a t-design.

Another way to show that a code holds designs is to use permutation groups. Let $Perm(C)$ be the group of coordinate permutations that leave C invariant. If $Perm(C)$ is t-homogeneous or t-transitive then the codewords of C of given weight hold a t-design for all weights of C.

Caveat: in this text, all designs are obtained from codewords of given weight in a code.

3 The codes

Note that in the sequel, given a finite primitive rank 3 permutation group G acting on a finite set X we may define $\mathbb{F}_2 X = \left\{ \sum_{x \in X} g_x x \mid g_x \in \mathbb{F}_2 \right\}$ as a vector space over $\mathbb{F}_2$ with basis X. Extending the G-action on X linearly, $\mathbb{F}_2 X$ becomes an $\mathbb{F}_2 G$-module, called an $\mathbb{F}_2 G$-permutation module with permutation basis X. The $\mathbb{F}_2$-vector space $\mathbb{F}_2 X$ is equipped with a non-degenerate symmetric bilinear form

$$\langle \mathbf{g}, \mathbf{h} \rangle = \left\langle \sum_{x \in X} g_x x, \sum_{x \in X} h_x x \right\rangle = \sum_{x \in X} g_x h_x, \quad \forall \mathbf{g} \quad \text{and} \quad \mathbf{h} \in \mathbb{F}_2 X$$

called the standard inner product on $\mathbb{F}_2 X$. For any $a \in G$ and any $\mathbf{g} = \sum_{x \in X} g_x x$ and $\mathbf{h} = \sum_{x \in X} h_x x \in \mathbb{F}_2 X$, we have

$$\begin{aligned} \langle a(\mathbf{g}), a(\mathbf{h}) \rangle &= \left\langle a\left(\sum_{x \in X} g_x x \right), a\left(\sum_{x \in X} h_x x \right) \right\rangle \\ &= \left\langle \sum_{x \in X} g_x a x, \sum_{x \in X} h_x a x \right\rangle = \sum_{x \in X} g_x h_x \\ &= \langle \mathbf{g}, \mathbf{h} \rangle. \end{aligned}$$

So, the standard inner product on the vector space $\mathbb{F}_2 X$ is G-invariant in the following sense:

$$\langle a(\mathbf{g}), a(\mathbf{h}) \rangle = \langle \mathbf{g}, \mathbf{h} \rangle, \quad \forall a \in G, \quad \forall \mathbf{g}, \mathbf{h} \in \mathbb{F}_2 X.$$

Recall that the action of G on X has rank 3. So, the conjugacy class of the maximal subgroups isomorphic to the stabilizer G_x of a point in X generates a rank 3 permutation module over $\mathbb{F}_2$. We shall consider this $\mathbb{F}_2$-module and a chain of all its invariant submodules under the action of G. The sections that follow present the calculations on these modules. The vectors in each submodule form a code, over $\mathbb{F}_2$, whose length is the dimension of the permutation module and whose dimension is the dimension of the submodule. The weight enumerators of the submodules are therefore also the weight enumerators of these codes which are invariant under the action of G.

Remark 1 For $x \in {\mathbb{F}_2}^n$ and a permutation $\sigma \in S_n$ we set

$$\sigma x = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}). \tag{1}$$

Let C be a linear code over $\mathbb{F}_2$ of length n and let $G \leq \mathrm{Aut}(C)$. If the action of G on C is defined by Eq. (1) then the code C becomes an $\mathbb{F}_2 G$-module. Note that the ambient space $\mathbb{F}_2^n$ is also an $\mathbb{F}_2 G$-module with respect to the same action of G. We formulate the fact that C is an $\mathbb{F}_2 G$-module as follows.

Result 2 *Let C be an $[n, k, d]_2$ code and let $G \leq \mathrm{Aut}(C)$. Then C is a k-dimensional submodule of the ambient space $\mathbb{F}_2^n$, considered as an $\mathbb{F}_2 G$-module.*

In the setting of this paper, $\mathbb{F}_2 X$ will be identified with $\mathbb{F}_2^v$ and each submodule $\mathbb{F}_2 G$ of $\mathbb{F}_2^v$ will be regarded as a linear code over $\mathbb{F}_2$ invariant under G. So, in Sect. 4 we shall be considering rank 3 permutation modules over $\mathbb{F}_2$ whose submodules (i.e., binary codes) hold 2-designs with point set X.

Table 3 Some 2-designs invariant under primitive rank 3 groups of almost simple type

G	C	Aut(C)	$\mathcal{D}$ = 2-(v, w, λ)	r	Aut($\mathcal{D}$)
$L_2(8){:}3$	$[36, 7, 16]_2$	$S_6(2)$	2-(36, 16, 12)	28	$S_6(2)$
	$[36, 9, 14]_2$	$L_2(8){:}3$	2-(36, 16, 12)	28	$S_6(2)$
	$[36, 15, 8]_2$	$S_6(2)$	2-(36, 8, 6)	30	$S_6(2)$
	$[36, 15, 8]_2$	$S_6(2)$	2-(36, 12, 99)	314	$S_6(2)$
	$[36, 15, 8]_2$	$S_6(2)$	2-(36, 14, 624)	1680	$S_6(2)$
	$[36, 15, 8]_2$	$S_6(2)$	2-(36, 16, 1482)	3388	$S_6(2)$
	$[36, 15, 8]_2$	$S_6(2)$	2-(36, 18, 1632)	3360	$S_6(2)$
	$[36, 19, 6]_2$	$L_2(8){:}3$	2-(36, 6, 2)	14	$L_2(8){:}3$
	$[36, 19, 6]_2$	$L_2(8){:}3$	2-(36, 10, 288)	1120	$L_2(8){:}3$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 6, 8)	56	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 8, 42)	210	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 10, 1152)	4480	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 12, 8217)	26,145	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 14, 33176)	89,320	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 16, 83964)	195,916	$S_6(2)$
	$[36, 21, 6]_2$	$S_6(2)$	2-(36, 18, 137088)	282,240	$S_6(2)$

4 The designs

In Table 3 (respectively Table 4) we list 2-designs obtained from the binary codes invariant under the individual rank 3 primitive permutation groups of almost simple type given in Table 2 and using the construction described in Result 2 which we implemented in Magma [10]. The permutation groups were found either directly from the Magma database of primitive groups (primitive groups of degree < 4096), or by downloading the generators from the Atlas [36] webpage.

The PBIBDs and 2-designs obtained from the binary codes defined by row span of the adjacency matrices of triangular graphs, and of the square lattice graphs are examined in Sects. 5 and 6, respectively.

We note that the construction presented in [32] is restricted to determining PBIBDs from the so-called binary adjacency codes, i.e., binary codes defined by the adjacency matrices of strongly regular graphs. The ideas discussed in Sect. 3 generalize the construction given in [32] since the PBIBDs, in particular the 2-designs that we found are held by nonzero weight codewords of submodules (regarded as codes) including their duals of the rank 3 permutation modules associated with a rank 3 action of G, of which the adjacency code is a special case. In this paper we use the six-parameters notation $\left(A_w, v, w, \frac{wA_w}{v}, \lambda_1, \lambda_2\right)$ instead of the four parameters notation used in [32].

We summarize the discussion in the previous section and present the following result obtained from the computational searches on the submodules (codes) of the relevant permutation modules defined over $\mathbb{F}_2$.

Proposition 1 *Let G be a finite primitive permutation group acting rank 3 on a set X of degree v as listed in Table 2. Let $\mathbb{F}_2X$ be the permutation module of degree v over $\mathbb{F}_2$ associated with the rank 3 action of G on X, and let $C \leq \mathbb{F}_2X$ be a binary code admitting G as a*

Table 4 Some 2-designs invariant under primitive rank 3 groups of almost simple type (Table 3 continued)

G	C	Aut(C)	$\mathcal{D}$ = 2-(v, w, λ)	r	Aut($\mathcal{D}$)
A_8	$[35, 14, 8]_2$	S_8	2-(35, 18, 864)	1728	S_8
	$[35, 20, 6]_2$	S_8	2-(35, 18, 72576)	145,152	S_8
	$[35, 28, 4]_2$	S_8	2-(35, 18, 18238176)	36,476,352	S_8
$O_5(3)$	$[27, 21, 3]_2$	$O_5(3)$:2	2-(27, 13, 70080)	151,840	$O_5(3)$:2
	$[36, 20, 6]_2$	$O_5(3)$:2	2-(36, 18, 68544)	141,120	$O_5(3)$:2
	$[40, 15, 10]_2$	$O_5(3)$:2	2-(40, 14, 126)	378	$O_5(3)$:2
$U_4(2)$	$[45, 14, 12]_2$	$U_4(2)$:2	2-(45, 12, 8)	32	$U_4(2)$:2
	$[45, 20, 8]_2$	$U_4(2)$:2	2-(45, 12, 80)	320	$U_4(2)$:2
A_9	$[120, 9, 56]_2$	$S_8(2)$	2-(120, 56, 55)	119	$S_8(2)$
$S_6(2)$	$[63, 7, 31]_2$	$L_6(2)$	2-(63, 31, 15)	31	$L_6(2)$
	$[63, 20, 16]_2$	$S_6(2)$	2-(63, 32, 116432)	232,864	$S_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 16, 120)	496	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 24, 25760)	69,440	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 28, 96768)	222,208	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 32, 232144)	464,288	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 36, 125440)	222,208	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 40, 43680)	69,440	$L_6(2)$
	$[63, 21, 16]_2$	$L_6(2)$	2-(63, 48, 376)	496	$L_6(2)$
	$[63, 21, 15]_2$	$S_6(2)$	2-(63, 31, 108480)	224,192	$S_6(2)$
	$[63, 21, 15]_2$	$S_6(2)$	2-(63, 31, 109155)	225,587	$S_6(2)$
	$[63, 27, 12]_2$	$S_6(2)$	2-(63, 32, 13636304)	27,272,608	$S_6(2)$
$G_2(2) \cong U_3(3)$:2	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 4, 9)	105	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 6, 728)	5096	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 8, 20952)	104,760	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 10, 284112)	1,104,880	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 12, 2047188)	6,513,780	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 14, 8572616)	23,080,120	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 16, 21740664)	50,728,216	$S_6(2)$
	$[36, 29, 4]_2$	$S_6(2)$	2-(36, 18, 34449888)	70,926,240	$S_6(2)$

permutation group of automorphisms acting rank 3 *on its coordinate positions. Let $\mathcal{D}$ be a PBIBD held by the support of some nonzero weight codewords of C. If $\mathcal{D}$ is a* 2*-design then the parameters of $\mathcal{D}$ and C are as given in Tables* 3 *and* 4*.*

5 Triangular graphs

For any n the triangular graph $T(n)$ is defined to be the line graph of the complete graph K_n. It is a strongly regular graph on $v = \binom{n}{2}$ vertices, i.e. on the pairs of letters $\{a, b\}$ where $a, b \in \{1, \ldots, n\}$. In [32] the authors determined the parameters of the PBIBDs obtained from the adjacency codes of the triangular graphs, and found no occurrences of 2-designs held by the nonzero weight codewords. In this section we confirm the results obtained in

Table 4 continued

G	C	Aut(C)	$\mathcal{D}$ = 2-(v, w, λ)	r	Aut($\mathcal{D}$)
$S_4(4)$	$[85, 17, 21]_2$	$L_4(4)$:2	2-(85, 21, 5)	21	$L_4(4)$:2
	$[85, 17, 21]_2$	$L_4(4)$:2	2-(85, 32, 496)	1344	$L_4(4)$:2
	$[85, 17, 21]_2$	$L_4(4)$:2	2-(85, 37, 4440)	10,360	$L_4(4)$:2
	$[85, 17, 21]_2$	$L_4(4)$:2	2-(85, 40, 8320)	17,920	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 24, 1104)	4032	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 29, 3480)	10,440	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 32, 71920)	194,880	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 33, 135168)	354,816	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 36, 345600)	829,440	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 37, 483960)	1,129,240	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 40, 1150240)	2,477,440	$L_4(4)$:2
	$[85, 25, 21]_2$	$L_4(4)$:2	2-(85, 41, 1259520)	2,644,992	$L_4(4)$:2
	$[120, 18, 40]_2$	$S_4(4)$:2	2-(120, 56, 13255)	28,679	$S_4(4)$:2
	$[120, 19, 40]_2$	$S_4(4)$	2-(120, 56, 27335)	59,143	$S_4(4)$
	$[120, 25, 24]_2$	$S_4(4)$:2	2-(120, 56, 1623875)	3,513,475	$S_4(4)$:2
	$[136, 9, 64]_2$	$S_8(2)$	2-(136, 64, 56)	120	$S_8(2)$
	$[136, 18, 48]_2$	$S_4(4)$:2	2-(136, 64, 13496)	28,920	$S_4(4)$:2
	$[136, 19, 48]_2$	$S_4(4)$	2-(136, 64, 27832)	59,640	$S_4(4)$
	$[136, 20, 36]_2$	$S_4(4)$:2	2-(136, 64, 42168)	90,360	$S_4(4)$:2
	$[136, 25, 32]_2$	$S_4(4)$:2	2-(136, 64, 1599640)	3,427,800	$S_4(4)$:2
$U_5(2)$	$[176, 24, 40]_2$	$U_5(2)$:2	2-(176, 56, 352)	1120	$U_5(2)$:2
$G_2(4)$	$[2016, 13, 992]_2$	$S_{12}(2)$	2-(2016, 992, 991)	2015	$S_{12}(2)$

[32] and extend the calculations to include the parameters of additional PBIBDs, see [31]. By doing so we find examples of 2-designs held by some codewords of the binary codes of the rank 3 permutation modules of dimension $v = \binom{n}{2}$ which are listed in Table 6.

An alternative way to approach the graphs that we will be examining is through the primitive rank 3 action of the simple alternating group A_n, $n \geq 5$ on the 2-subsets, $X^{\{2\}}$, of an n-element set X. The orbits of the stabilizer in A_n of a 2-element subset $A = \{x, y\}$ consist of $\{A\}$ and one orbit of length $2(n-2)$ and another of length $\binom{n-2}{2}$.

The triangular graphs correspond to the graphs obtained by rank 3 action of an almost simple group when the socle is an alternating group as described in Result 1(a), see for example [12, Theorem 11.3.1].

In our study we take for binary codes the $\mathbb{F}_2 A_n$ invariant submodules over $\mathbb{F}_2$ of the permutation module $\mathbb{F}_2 X^{\{2\}}$ of degree $v = \binom{n}{2}$. Now, if C denotes the binary code of $T(n)$ then C is a $\left[\binom{n}{2}, n-1, n-1\right]_2$ code for n odd, and C is a $\left[\binom{n}{2}, n-2, 2(n-2)\right]_2$ code for n even. If n is even then C is a self-orthogonal doubly-even code, while $C \oplus C^{\perp} = \mathbb{F}_2^n$ if n is odd. It follows from [24, Lemma 3.2] that $C^{\perp} = \left[\binom{n}{2}, \frac{n^2-3n+2}{2}, 3\right]_2$ if $n \geq 5$ and odd, and $\left[\binom{n}{2}, \frac{n^2-3n+4}{2}, 3\right]$ if $n \geq 6$ and even. Further, we have that the all-one vector $\mathbf{1} \in C^{\perp}$ for any n, see [24, Lemma 3.3] (Table 5).

Table 5 Some 2-designs invariant under primitive rank 3 groups of almost simple type (Table 4 continued)

G	C	Aut(C)	$\mathcal{D}$ = 2-(v, w, λ)	r	Aut($\mathcal{D}$)
M_{11}	$[55, 11, 10]_2$	S_{11}	2-(55, 27, 78)	162	S_{11}
M_{22}	$[176, 22, 50]_2$	HS	2-(176, 50, 14)	50	HS
	$[176, 22, 50]_2$	HS	2-(176, 56, 110)	350	HS
	$[176, 22, 50]_2$	HS	2-(176, 64, 540)	1500	HS
	$[176, 22, 50]_2$	HS	2-(176, 66, 780)	2100	HS
	$[176, 22, 50]_2$	HS	2-(176, 70, 2760)	7000	HS
	$[176, 22, 50]_2$	HS	2-(176, 72, 2556)	6300	HS
	$[176, 22, 50]_2$	HS	2-(176, 78, 37752)	85,800	HS
	$[176, 22, 50]_2$	HS	2-(176, 80, 124030)	274,750	HS
	$[176, 22, 50]_2$	HS	2-(176, 82, 99630)	215,250	HS
	$[176, 22, 50]_2$	HS	2-(176, 86, 87720)	180,600	HS
	$[176, 22, 50]_2$	HS	2-(176, 88, 210540)	423,500	HS
HS	$[100, 21, 32]_2$	HS	2-(100, 36, 525)	1485	HS:2
HS	$[100, 21, 32]_2$	HS	2-(100, 40, 14560)	36,960	HS:2
$S_8(2)$	$[255, 9, 127]_2$	$S_8(2)$	2-(255, 127, 63)	127	$L_8(2)$
$O^+_{10}(2)$	$[496, 11, 240]_2$	$S_{10}(2)$	2-(496, 240, 239)	495	$S_{10}(2)$
$O^-_{10}(2)$	$[528, 11, 256]_2$	$S_{10}(2)$	2-(528, 256, 240)	496	$S_{10}(2)$

Table 6 Some 2-designs held by nonzero weight codewords of binary adjacency codes of the triangular graph

G	C	Aut(C)	$\mathcal{D}$ = 2-(v, w, λ)	r	Aut($\mathcal{D}$)
A_5	$[10, 5, 4]_2$	S_6	2-(10, 4, 2)	6	S_6
A_6	$[15, 5, 7]_2$	A_8	2-(15, 7, 3)	7	A_8
	$[15, 10, 3]_2$	S_6	2-(15, 7, 39)	91	S_6
	$[15, 10, 3]_2$	S_6	2-(15, 7, 48)	112	S_6
	$[15, 11, 3]_2$	A_8	2-(15, 3, 1)	7	A_8
		A_8	2-(15, 4, 6)	28	A_8
		A_8	2-(15, 5, 16)	56	A_8
		A_8	2-(15, 6, 40)	112	A_8
		A_8	2-(15, 7, 87)	203	A_8
A_7	$[21, 15, 3]_2$	S_7	2-(21, 5, 12)	60	S_7
A_8	$[28, 7, 12]_2$	$S_6(2)$	2-(28, 12, 11)	27	$S_6(2)$
	$[28, 21, 4]_2$	$S_6(2)$	2-(28, 4, 5)	45	$S_6(2)$
		$S_6(2)$	2-(28, 6, 240)	1296	$S_6(2)$
		$S_6(2)$	2-(28, 8, 3542)	13,662	$S_6(2)$
		$S_6(2)$	2-(28, 10, 24640)	73,920	$S_6(2)$
		$S_6(2)$	2-(28, 12, 82423)	202,311	$S_6(2)$
		$S_6(2)$	2-(28, 14, 151840)	315,360	$S_6(2)$
A_{10}	$[45, 10, 9]_2$	S_{10}	2-(45, 21, 70)	154	S_{10}
A_{11}	$[55, 11, 10]_2$	S_{11}	2-(55, 27, 78)	162	S_{11}

In what follows we state results on the PBIBDs obtained by examining the binary adjacency codes of the triangular graphs (and their dual codes) and recall that the notation $\left(A_w, v, w, \frac{wA_w}{v}, \lambda_1, \lambda_2\right)$ is used for the PBIBDs obtained from the codewords of nonzero weight of the codes.

Proposition 2 *Let C denote the binary adjacency code of the triangular graph. For $n \geq 7$ and odd (respectively for $n \geq 8$ and even) the codewords of weight $2(n-2)$ hold a $\left(\binom{n}{2}, \binom{n}{2}, 2(n-2), 2(n-2), n-2, 4\right)$ PBIBD.*

Proof For the case $n \geq 7$ and odd it follows from [19, Theorem 4.1] that the number of codewords of weight $2(n-2)$ equals $\binom{n}{2}$. Further, these codewords span the code. Now, as in the proof on [32, Proposition 1], we note that the support $s(x)$ of the row $r(x)$ indexed by x are the vertices of the graph at distance 1 from x. Take a pair in position $\{u, v\}$. Then the number of x such that $\{u, v\} \subset s(x)$ equals the number of x's at distance 1 from both u and v. Furthermore, the number of vertices at distance 1 depends by strong regularity on the distance of a vertex u to v which equals $n-2$ or 4.

The case $n \geq 8$ and even can be dealt with by noticing that the adjacency code C of $T(n)$ has a basis of minimum-weight vectors which are the characteristic vectors of subsets of the edge set of the complete graph K_n. Moreover, these are also the rows of the adjacency matrix of $T(n)$, see for example [24, Lemma 3.6]. Furthermore, there are precisely $\binom{n}{2}$ codewords of minimum weight in C. The remaining details of the proof follow as in the above case. □

Remark 2 The values of λ_1 and λ_2 are interchanged when $n = 5$. Notice that the codewords of weight 6 in the adjacency code of the triangular graph $T(5)$, i.e., $[10, 4, 4]_2$ code, hold a $(10, 10, 6, 6, 4, 3)$ PBIBD.

For $n = 6$ the codewords of weight 8 in the $[15, 4, 8]_2$ code hold a $(15, 15, 8, 8, 4, 4)$ PBIBD. This is in fact a 2-$(15, 8, 4)$ design invariant under the alternating group A_8.

Other instances of 2-designs invariant under the alternating group A_6 are listed in rows 2 - 9 of Table 6.

Proposition 3 *For $n \geq 7$ and odd the codewords of minimum weight $n-1$ in the binary code of adjacency code $C = \left[\binom{n}{2}, n-1, n-1\right]_2$ of the triangular graph hold a $\left(n, \binom{n}{2}, n-1, 2, 1, 0\right)$* PBIBD.

Proof We first note that the parameters of the PBIBD designs indicated in the proposition were deduced from the tables given in [32, Sect. 4.2]. Observe that the parameters of λ and μ appear interchanged in the said tables.

It was shown in [24, Lemma 3.6] that for $n \geq 5$ and odd, the adjacency code $C = \left[\binom{n}{2}, n-1, n-1\right]_2$ of the triangular graph is spanned by the codewords of weight $n-1$, and there are n of these. Now, the PBIBD that we are examining in the proposition has $v = \binom{n}{2}$ points, blocks of size $k = n-1$, and $b = n$ blocks. The replication number $r = 2$ follows by using $bk = vr$. Since each two points are on either 1 or 0 blocks, then the blocks must meet exactly once. The reader will notice that the PBIBD thus obtained is the dual structure, i.e. the complete graph on n vertices. □

Observe that when $n = 5$ we have $v = 10$ points, $b = 5$ blocks, $k = 4$ points per block, $r = 2$ blocks per point, and two points are on 1 or 0 blocks (same for blocks, but they must meet for this case). Thus, we obtain a (5, 10, 4, 2, 0, 1) PBIBD. As in Remark 2, here too we note that the values of λ_1 and λ_2 are interchanged.

The following proposition concerns the codewords of minimum weight in the dual binary code of the adjacency code of $T(n)$.

Proposition 4 *For $n \geq 7$, the codewords of minimum weight in the dual binary code of the adjacency code of the triangular graph hold a* $\left(\binom{n}{2}, \binom{n}{3}, 3, n-2, 1, 0\right)$ PBIBD.

Proof Recall that $C^{\perp} = \left[\binom{n}{2}, \frac{n^2-3n+2}{2}, 3\right]_2$ if $n \geq 5$ and odd, and $C^{\perp} = \left[\binom{n}{2}, \frac{n^2-3n+4}{2}, 3\right]$ if $n \geq 6$ and even, and the number of words of weight 3 is $\binom{n}{3}$ except if $n = 6$ when there are more words of this weight, see [24, Lemma 3.2]. That $\lambda_1 = 1$ and $\lambda_2 = 0$ follows at once as any two points are either on one block or none. □

We remark that for $n = 5$, the codewords of minimum weight 3 in the dual code of $T(5)$ hold a (10, 10, 3, 3, 0, 1) PBIBD, and for $n = 6$ we have a (35, 15, 3, 7, 1, 1) PBIBD held by minimum weight codewords of the dual code of $T(6)$ which gives rise to a 2-(15, 3, 1) design.

For $n = 5$, we note that the PBIBD obtained is the complement of $T(5)$. It follows from [24, Lemma 3.2] that there are precisely 35 codewords of weight 3 in $C^{\perp}$ when $n = 6$. Hence, there are 35 blocks, i.e., number of codewords of weight 3 in the code, $v = \binom{6}{2} = 15$, block size 3 and replication number $r = 7$. Since $\lambda_1 = \lambda_2 = 1$ we obtain a (35, 15, 3, 7, 1, 1) PBIBD. This 2-(15, 3, 1) design is the design of points and lines in the projective geometry PG(3, 2). By a theorem of Key and Shult [23], the automorphism group PSL(4, 2) of this design acts 2-transitively on the points.

We now ask for which positive integer values of n can we construct 2-designs for the adjacency codes of the triangular graph $T(n)$? Proposition 5 and the results given in Table 6 constitute an attempt to provide an exhaustive answer to this question.

Proposition 5 *Suppose that C is the code defined by the binary row span of the adjacency matrix of the triangular graph. If the parameters of C are as given in Table 6 then the codewords in C of given weight for some nonzero weight hold 2-designs.*

6 Lattice graphs

The *complete bipartite graph* $K_{n,n}$ on $2n$ vertices, $A \cup B$, where $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, with n^2 edges, has for line graph, the *lattice graph* $L_2(n)$, which has vertex set the set of ordered pairs $\{(a_i, b_j) \mid 1 \leq i, j \leq n\}$, where two pairs are adjacent if and only if they have a common coordinate. Thus two edges of $K_{n,n}$ are at distance 1 or 2 depending on their number of common vertices. $L_2(n)$ is a strongly regular graph. The binary adjacency code $C_2(L_2(n))$ of the lattice graph is a $[n^2, 2(n-1), 2(n-1)]_2$ code. Note that lattice graph corresponds to the graph obtained by the primitive rank 3 permutation groups of product action type as described in Result 1(b), see remarks that follow after [12, Table 11.1.1] for relevant comments.

In [32, Proposition 1] the authors proved the following result for the minimum weight codewords of the binary adjacency code of the lattice graph.

Result 3 *For $n \geq 5$, the minimum weight codewords of the binary adjacency code of the lattice graph hold a* $(n^2, n^2, 2(n-1), 2(n-1), n-2, 2)$ PBIBD.

Examining the information given on the tables of [32, Sect. 4.3] we state and prove a result similar to Result 3 for the codewords of weight $2n$ in the binary adjacency code of the lattice graph.

Proposition 6 *For $n \geq 5$, the codewords of weight $2n$ in $C_2(L_2(n))$ hold a $(n^2 - n, n^2, 2n, 2(n-1), n, 2)$ PBIBD.*

Proof We start by noting that for $n \geq 5$ and odd the codewords of minimum weight $2(n-1)$ span $C_2(L_2(n))$ and that this also holds for the codewords of weight $2n$. For $n \geq 6$ and even, it follows from [26, Proposition 7] that the codewords of weight $2n$ in $C_2(L_2(n))$ span a self-orthogonal doubly even subcode, say $C_2(\widetilde{L}_2(n))$ of codimension 1 in $C_2(L_2(n))$, with parameters $[n^2, 2n-3, 2n]_2$. As a direct application of [19, Theorem 4.2] we deduce that the number of codewords of weight $2n$ in $C_2(L_2(n))$ and $C_2(\widetilde{L}_2(n))$, respectively equals $n^2 - n$. The PBIBD that we are examining in the proposition has $v = n^2$ points, blocks of size $k = 2n$, and $b = n^2 - n$ blocks. The replication number $r = 2(n-1)$ follows by using $bk = vr$. A codeword of weight $2n$ corresponds to the edges of a subgraph $K_{n,2}$ of $K_{n,n}$ by [19, Sect. 4.2] with $m_1 = n$ and $m_2 = 2$. Depending if they intersect or not as edges of $K_{n,n}$ they are together incident with n or 2 blocks. □

We now examine the existence of PBIBDs held by codewords of minimum weight in the dual code $C_2(L_2(n))^\perp$ of $C_2(L_2(n))$ for $n \geq 5$. We know from [25, Lemma 2] that $C_2(L_2(n))^\perp$ is a $[n^2, (n-1)^2+1, 4]_2$ code, and by [25, Lemma 3] we have that together with the all-one vector $\mathbf{1}$ a sequence $\mathcal{V}$ of weight-4 vectors $v(i, j; k, l)$ can be found to form a basis for $C_2(L_2(n))^\perp$ when n is odd; and that $\mathcal{V}$ together with a certain vector $w(\sigma)$ where $\sigma = (1, 2, \ldots, n) \in S_n$, the symmetric group on n letters, form a basis for $C_2(L_2(n))^\perp$ when n is even. In what follows we show that for either choice of the parity of n the above sequence $\mathcal{V}$ of weight-4 codewords holds a PBIBD.

Proposition 7 *If $n \geq 5$, the codewords of minimum weight 4 in $C_2(L_2(n))^\perp$ hold a $\left(\frac{n^2(n-1)^2}{4}, n^2, 4, (n-1)^2, n-1, 1\right)$ PBIBD.*

Proof We know from [25, Eq. (6)] that when $\sigma \in S_n$ is of the form $\sigma = (i,k)(j,l)$, where $k, l \in \{1, 2, \ldots, n\}$ and $k \neq l$, a weight-4 vector $v(\{i, j\}; \{k, l\}) = w(i, j; \sigma)$ is in $C_2(L_2(n))^\perp$. Conveniently here we choose the equivalent notation given in [26, Eq. (16)] for these weight-4 vectors $v = v((a_i, b_j), (a_k, b_l))$ and note for the sake of the reader that for $p = 2$, [26, Lemma 2, Proposition 2] shows that $v \in C_2(L_2(n))^\perp$ for either choice of the parity of n. Now, let $\mathcal{L}_n$ be the set of supports of the vectors $v((a_i, b_j), (a_k, b_l))$. Then $(\mathcal{P}_n, \mathcal{L}_n)$ with $\mathcal{P}_n = A \times B$, the vertex set of $L_2(n)$ is a 1-$(n^2, 4, r)$ design having $\frac{(n(n-1))^2}{4}$ blocks and $r = (n-1)^2$.

Suppose that $u \in C_2(L_2(n))$ and $\text{Supp}(u) = \mathcal{A}$, where $|\mathcal{A}| = a$. Let $P \in \mathcal{A}$. As in [26, Proposition 1] we count the number of blocks of $\mathcal{L}_n$ through P and another point Q. Suppose $P = (a_i, b_j)$. Then

(1) if $Q = (a_i, b_k)$ then $P, Q \in \text{Supp}(v((a_i, b_j), (a_l, b_k))$ for all $l \neq i$, giving $n-1$ such blocks;
(2) if $Q = (a_l, b_j)$ then P, Q are on $n-1$ blocks again;
(3) if $Q = (a_l, b_k)$ where $l \neq i, k \neq j$, then $P, Q \in \text{Supp}(v((a_i, b_j), (a_l, b_k))$ only, giving one block.

Thus we have the result. □

Table 7 The value of λ, μ for $[16, 6, 6]_2$ code invariant under $S_4 \wr 2$

w	6	8	10	16
A_w	16	30	16	1
λ	2	7	6	1
μ	2	7	6	1

We note that the above results were proved for $n \geq 5$. The inclusion of $n = 4$ suggests that the codewords of any given nonzero weight in $C_2(L_2(4))$ hold 2-designs. A similar statement can be made for the codewords of any given nonzero weight in the dual code $C_2(L_2(4))^{\perp}$.

We now prove the following results

Corollary 1 *The codewords of a given nonzero weight in $C_2(L_2(4))$ hold a 2-design. In particular, the codewords of weight 8 hold a 3-design.*

Proof Magma [10] calculations give the weight distribution of the binary adjacency code $C_2(L_2(4)) = [16, 6, 6]_2$ of the lattice graph $L_2(4)$ as follows:

It can be deduced from Table 7 that the codewords of minimum weight 6 hold a $(16, 16, 6, 6, 2, 2)$ PBIBD. This is in fact a 2-$(16, 6, 2)$ design invariant under 2^4:S_6 acting 2-transitively on points. Designs with these parameters were studied by Assmus and Salwach [3].

The codewords of weight 8 in $C_2(L_2(4))$ span a binary self-orthogonal doubly even code $C_2(\widetilde{L}_2(4))$ (see the proof of Proposition 6, for the notation for the code) with parameters $[16, 5, 8]_2$ and hold a $(30, 16, 8, 15, 7, 7)$ PBIBD. This is in fact a 3-$(16, 8, 7)$ design on which 2^4:A_8 acts 3-transitively on points. □

Corollary 2 *The codewords of any given nonzero weight in $C_2(L_2(4))^{\perp}$ hold 2-designs. In particular, the codewords of weight 8 hold a 3-design.*

Proof Using Magma [10], it can be shown that the codewords of minimum weight in the code $C_2(L_2(4))^{\perp} = [16, 10, 4]_2$ hold a $(60, 16, 4, 15, 3, 3)$ PBIBD. This is a 2-$(16, 4, 3)$ design with $r = 15$. Furthermore, the designs 2-$(16, 6, 32)$ with $r = 96$ and 2-$(16, 8, 91)$ with $r = 195$ and their complementary designs are held by nonzero weight codewords of $C_2(L_2(4))^{\perp}$. It can also be shown that these designs are invariant under the group 2^4:S_6 which acts 2-transitively on points. The 2-$(16, 8, 91)$ is in fact a 3-$(16, 8, 39)$ design. □

Corollary 3 *The codewords of any given nonzero weight in $C_2(\widetilde{L}_2(4))^{\perp}$ hold 2-designs.*

Proof The designs with parameters 2-$(16, 4, 7)$, 2-$(16, 6, 56)$ and 2-$(16, 8, 203)$ and their complementary designs are held by the nonzero weight codewords of the code $C_2(\widetilde{L}_2(4))^{\perp} = [16, 11, 4]_2$. These are in fact 3-$(16, 4, 1)$, 3-$(16, 6, 16)$ and 3-$(16, 8, 87)$ designs on which the group 2^4:A_8 acts 3-transitively on points. □

Remark 3 (1) We note that some designs and codes listed in Tables 3 and 4 are invariant under the rank 3 action of more than one group. For example, the designs with parameters 2-$(36, 16, 12)$, 2-$(36, 8, 6)$, 2-$(36, 12, 99)$, 2-$(36, 14, 624)$, 2-$(36, 16, 1452)$, 2-$(36, 18, 1632)$, and a few others listed in Table 3 as invariant under a primitive rank 3 representation of $L_2(8)$:3 of degree 36 are also invariant under primitive rank 3 representations of the groups $U_4(2)$:2 and $G_2(2) \cong U_3(3)$:2 of the same degree. These occur since $L_2(8)$:3, $U_4(2)$:2 and $G_2(2)$ are subgroups (maximal) of $S_6(2)$. Similarly, the 2-(120, 56, 55) design is invariant under the alternating group A_9 and $S_6(2)$, respectively. Observe that this occurs since $A_9 \leq S_6(2)$. To avoid duplications we list only one instance of the design. We encourage the reader to consult [31], where these instances (and others) are listed. In [31] we list the parameters of many PBIBDs invariant under primitive rank 3 permutation groups including the parameters of the 2-designs presented in this paper;

(2) Some of the 2-designs that we found appear in the book [14, I.1.3] and are listed as follows:

(a) 2-(16, 4, 7) with $r = 35$ No 818;
(b) 2-(36, 16, 12) design with $r = 28$ No 550;
(c) 2-(36, 8, 6) with $r = 30$ No 617;
(d) 2-(36, 6, 2) with $r = 14$ No 103;
(e) 2-(45, 12, 8) with $r = 32$ No 706;
(f) 2-(63, 31, 15) with $r = 31$ No 681;
(g) 2-(85, 21, 5) with $r = 21$ No 309.

Remark 4 As said above, the extensions of the codes that hold 2-designs sometimes hold 3-designs. In particular, the instances of 3-designs occur as follows:

(a) $G = L_2(8){:}3$, 3-(36, 18, 768) with automorphism group isomorphic to the symplectic group $S_6(2)$.
(b) $G = L_2(8){:}3$, 3-(36, 18, 64512) with automorphism group isomorphic to the symplectic group $S_6(2)$.
(c) $G = U_4(2)$, 3-(28, 14, 70080) with automorphism group isomorphic to the symplectic group $S_6(2)$. This design also occurs for $G = A_8$ of degree 28, see Table 6.
(d) $G = M_{11}$, 3-(56, 28, 78) with automorphism group isomorphic to the symmetric group S_{11}. This design also occurs for $G = A_{11}$ of degree 55, see Table 6.
(e) $G = M_{22}$, 3-(176, 88, 104060) with automorphism group isomorphic to the Higman–Sims group HS.
(f) $G = S_4 \wr 2$, 3-(16, 8, 39) design with automorphism group isomorphic to the group $2^4{:}S_6$.
(g) $G = S_4 \wr 2$, 3-(16, 4, 1) design with automorphism group isomorphic to the group $2^4{:}A_8$.
(h) $G = S_4 \wr 2$, 3-(16, 6, 16) design with automorphism group isomorphic to the group $2^4{:}A_8$.
(i) $G = S_4 \wr 2$, 3-(16, 8, 87) design with automorphism group isomorphic to the group $2^4{:}A_8$.

All the 3-designs listed above are self-complementary.

Designs invariant under $L_2(8)$ with parameters 1-(36, 18, 42) were given in [16]. In general, 3-designs invariant under $PSL(2, q)$ are obtained by the action of this group on $q + 1$ points. This is not the situation presented in items (a) and (b) of Remark 4.

We refer the reader to the homepage of the first author for calculations concerning the parameters of rank 3 PBIBD held by the nonzero weight codewords of binary codes of moderate dimensions (see [31]), including those presented in the above tables.

Acknowledgements The authors thank Jennifer D. Key for comments on the contents of this paper. The second author acknowledges research support by the DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) which enabled him to visit the University of Pretoria in 2023. This work is based on the research supported by the National Research Foundation of South Africa (Grant Number CPRR23041894647).

Funding Open access funding provided by Télécom Paris.

Data Availability Data can be provided upon request to the authors.

Competing interests The authors declare no competing interests.

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