Formative assessment feedback on the mathematical competence of students

by

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Many thanks to

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- *Friends, colleagues and family for their encouragement and*
- *My husband for always believing in me.*

Correction does much, but encouragement does more.

Johann Wolfgang von Goethe

DECLARATION

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract

The aim of the study reported on in this dissertation is to develop a formative feedback instrument on the mathematical competence (competence instrument) of students and to examine students' experiences of the use of such an instrument. The study was conducted amongst first-year students of the University of Pretoria enrolled for a Finite Mathematics module.

A quantitative approach in the form of a non-experimental research design, together with questionnaires completed by the students, was used in the study. A mathematical competence assessment instrument was developed in terms of a five-point radial chart, each radial related to a mathematical competency. The five fundamental competencies (or skills) were identified from literature (Kilpatrick, 2001; Kilpatrick et al., 2001; Niss, 2003; OECD, 2003; Sáenz, 2008; Schoenfeld, 2007) for constituting mathematical ability or competence. A rubric was designed to assess formative tests written by students and the five-point radial chart was used to report the mathematical competence of the students as assessed by using the rubric. Students completed questionnaires on their experience of the competence instrument at the end of the study.

Student performance in formative assessments where the competence instrument was used correlated well with the performance in the examination. The competence instrument proved to be valid in assessing the mathematical ability of students. Students found the instrument helpful in identifying their lack of mathematical skills, but did not know how to improve on all the mathematical skills or competencies.

When conducting formative assessment, the importance of providing students with performance standards is essential, but students need to be informed on how to close the gap between their actual performance and the expected performance (Nicol and Macfarlane-Dick, 2006; Black & Wiliam, 1998). The mathematical competence instrument has potential in realising the objectives of formative assessment, but more research needs to be done on the implementation of the instrument in guiding students to improved performance. Future research based on the study, can take the form of the utilisation of the competence instrument in the assessment of students in other mathematics modules.

Keywords: formative assessment, feedback, mathematical competence, competencies, metacognition

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Chapter 1: Background

In order to enhance self-regulated learning (students taking control of their own learning), Nicol and Macfarlane-Dick (2006) emphasise the importance of formative assessment and feedback in higher education. Formative assessment refers to the use of assessments to expose students' thinking and provide timely feedback in order to augment and quicken students' learning (Black & Wiliam, 1998; Nicol & Macfarlane-Dick, 2006). The term feedback implies that information on recent past performances in assessments is provided or fed back to the students (Nicol & Macfarlane-Dick, 2006).

Problem statement

At the Department of Mathematics and Applied Mathematics of the University of Pretoria undergraduate students in most modules write weekly tutorial tests (10 to 20 marks) on the content covered in the previous week. These tutorial tests are written at the end of a two- or three-hour tutorial session in which students had the opportunity to engage with the relevant content. The tests are marked by the lecturer or teaching assistant and are handed back to the students during the following week's tutorial session. A memorandum discussion follows in which students receive feedback about common mistakes that have been made. The test marks contribute a small percentage to the student's final mark and aim to serve the purpose of formative assessment.

As a lecturer of first year students at the Department of Mathematics and Applied Mathematics of the University of Pretoria, my experience is that opportunities created for feedback, such as memorandum discussions or written comments on scripts, are not adequately utilised by students. Students immediately inform themselves of their acquired mark and in some instances question the marking for the sake of an improved mark, but rarely does it appear that students endeavour to utilise the feedback or learn from their mistakes. Furthermore it is evident that *students do not necessarily lack the appropriate content knowledge*, but *lack mathematical skills* such as manipulation of formulas or the proper use of mathematical notation. In other words, students lack mathematical abilities that are not explicitly outlined as the learning outcomes of a semester module, but which definitely influence their current and future performance in mathematics.

In the words of Wiliam (1999, p. 11) “feedback to learners should focus on what they need to do to improve” and “we should aim to reduce the amount of ego-involving feedback we give to learners” and focus more on feedback involving the task itself.

The following questions come to mind:

What is mathematical ability?

What are the skills associated with mathematical ability?

A study of the relevant literature (Kilpatrick, 2001; Kilpatrick, Swafford & Findell, 2001; Niss, 2003; The Organisation for Economic Co-operation and Development [OECD], 2003; Sáenz, 2008; Schoenfeld, 2007) led me to the term *mathematical competence* that is used to collectively describe the skills that constitute mathematical ability. The concept of mathematical competence as explicated by the relevant literature is discussed in chapter 2.

Aim of the study

For first year students the transition from school mathematics to formal mathematics is not an easy process. It is imperative that students, especially first year students, are informed of what it means to be able to do mathematics and are given timely and meaningful feedback about their mathematical competence. Apart from focusing students’ attention on their lack of mathematical skills the process of feedback has to incorporate opportunities to show students how they can improve these skills. Only by getting students to be involved in their own learning, can the ideal of self-regulated learning be realised.

The aim of this study is to develop a formative feedback instrument on the mathematical competence (competence instrument) of students and to examine the students' experience of the use of such an instrument.

Structure of dissertation

The dissertation consists of six chapters. The background to the study and the problem statement essentially form the contents of in chapter 1. The relevant literature is reviewed in chapter 2. Chapter 3 contains a detailed description of the theoretical framework on which the study was based. The following two chapters, chapter 4 and 5 contain the research approach, research methodology and findings of the study. In Chapter 6 the study is concluded and the relevance of the study for future research purposes discussed.

Ethical considerations

The Ethics Committee of the University of Pretoria granted the necessary permission to perform the study (see Addendum B), since the study conforms to the Ethics Committee requirements. The students were informed that the completion of the questionnaire is voluntary and that they are under no obligation to participate in the study.

All relevant information was handled confidentially and no individual may be identified in any publication that may result from the study.

Limitations

The study was conducted amongst first year students enrolled for the finite mathematics module at the University of Pretoria in 2013, spanning one semester. The population of the study consisted of 111 first year students and the instrument developed to give feedback about the mathematical competence of the students was incorporated to assess three of the six tutorial tests written by all 111 students enrolled for the module. Due to time constraints the instrument was not used to assess the mathematical competence of students enrolled for other modules or for the same module in the following year.

The competence instrument was developed and utilised to serve a purpose in the context of a specific mathematics module. The instrument was only implemented for one semester amongst a small population.

Regardless of these factors the generalisability of the study is enhanced by the substantive engagement with theory and the triangulation between multiple research methods (Stange, Miller, Crabtree, O'Connor & Zylanski, 1994)..

Relevance of the study

The aim of the study, to develop an instrument to give feedback on the mathematical competence of students, was realised for a first year module in finite mathematics. The principles in terms of competencies on which the instrument is based are relevant to mathematics in general, but the instrument will possibly need to be adapted somewhat to be used for assessment purposes in other mathematics modules.

Future research

Future research based on the study can take the form of refinement and utilisation of the competence instrument in the assessment of students in other mathematics modules and/ or guiding the students to improve their mathematical competence.

Chapter 2: Literature Review

Formative assessment and feedback

From the perspective that teaching and learning must be interactive, the general term *assessment* includes all the activities undertaken in a classroom, by teachers and students, that generate information that can be used to guide learning and teaching activities. The term *formative assessment* refers to assessment where “the evidence is used to adapt teaching to meet student needs” (Black & Wiliam, 1998, p. 82) or assessment that “provides information to be used as feedback to modify the teaching and learning activities in which they (the teacher and students) are engaged” (Black & Wiliam, 1998, p. 7).

Two consecutive actions on the part of the student lie at the centre of formative assessment (Black & Wiliam, 1998, p. 20). The first is the realisation that there exists a gap between the anticipated objective and the student’s performance and the second is the action taken by the student to close the gap. Even though it is evident that in the case of both actions the responsibility lies with the student, it will only be accomplished by a complex, but effective interaction between student, teacher and learning activities. Well-planned and well-executed learning activities should be used by teachers to train students in self-assessment, so that students are able to understand the goals of their learning and understand what it is they have to do to achieve these learning goals and in the process abandon an attitude of mere acceptance of current performance. According to Black and Wiliam (2009, p. 22) it is the fundamental task of the teacher to “engineer” interactive learning opportunities so that learners can become more proficient and more practical in managing and advancing their own learning.

Helping students to take control of their own learning by teaching *metacognitive* strategies is an important aspect of active learning, where the responsibility for learning is placed on the student (Bransford, Brown & Cocking, 2000). Metacognition can be explained as an individual's consciousness of his/ her thinking or in the words of Bransford et al. (2000, p. 12) "metacognition refers to people's abilities to predict their performance on various tasks and to monitor their current levels of mastery and understanding". Formative assessment and *feedback* is an important part of such a metacognitive approach to learning, since giving feedback on students' performance helps them to assess and monitor their current level of mastery and to develop skills of self-assessment or self-regulation. Nicol and Macfarlane-Dick (2006, p. 199) refer to "self-regulation as the degree to which students can regulate aspects of their thinking, motivation and behaviour during learning". Formative assessment and feedback is a compass by which students can navigate their learning and by means of which the teacher can steer his/her instructions toward the development of students who are competent in mathematics.

Nicol and Macfarlane-Dick (2006, p. 205) identify seven principles of effective feedback practice that enhances self-regulation in higher education.

Good feedback practice:

1. Helps clarify what good performance is (goals, criteria, expected standards);
2. Facilitates the development of self-assessment (reflection) in learning;
3. Delivers high-quality information to students about their learning;
4. Encourages teacher and peer dialogue around learning;
5. Encourages positive motivational beliefs and self-esteem;
6. Provides opportunities to close the gap between current and desired performance;
7. Provides information to teachers that can be used to help shape teaching.

To elaborate on the seven principles of good feedback practice quoted from Nicol and Macfarlane-Dick (2006):

1. Help make clear what the expected performance is and highlight standards, criteria and goals by which performance will be measured. If students understand what is expected of them, then they will be able to distinguish between actual and expected performance.
2. Facilitate the development of self-assessment or self-regulation in individual students by providing opportunities for practice and reflection on performance as measured against standards or criteria. The emphasis is on the word facilitate, hence feedback can be given by peers or by the students themselves.
3. Provide students with good quality external feedback from the teacher who is the central spring from which students can measure their own performance against standards and goals. Quality feedback focuses not just on strengths and shortcomings but also provides the student with direction on how to better his/her performance. To quote Nicol and Macfarlane-Dick about quality feedback (2006, p. 208):

   > Good quality external feedback is information that helps students troubleshoot their own performance and self-correct: that is, it helps students take action to reduce the discrepancy between their intentions and the resulting effects.

4. Encourage discussions between students and students and teacher. The emphasis is on dialogue, to provide the student with a better understanding of what is expected of him/her and inform the teacher of possible misconceptions.
5. Motivate and encourage positive views and self-esteem i.e. a productive disposition (Kilpatrick, Swafford & Findell, 2001), by focussing on learning goals (task-involved goals) rather than goals irrelevant to the process of learning or that are ego-involved goals (e.g. passing the test, looking good). Wiliam (1999, p. 10) notes that “the quality of feedback (in mathematics) is a powerful influence on the way that learners attribute their successes and failures”, in other words is their performance attributed

"to themselves (ego-involvement) or to the effort that they are doing (task-involvement)". Feedback should encourage students to believe that their ability in mathematics is "incremental rather than fixed" in other word that their ability can be altered as a result of effort and training. From the document *Adding it up* (Kilpatrick et al., 2001, p. 132):

> Students who view ability as expandable in response to experience and training are more likely to seek out challenging situations and learn from them.

6. Provide opportunities to close the gap between actual and expected performance or to close the feedback cycle. This implies using the feedback to produce work of a better quality. In higher education students are not provided with enough opportunity to use feedback to better their performance, because they are always moving on to the next assessment or assignment. In the words of Wiliam (1999, p. 10):

> Feedback is a necessary first step, but feedback is formative only if the information fed back to the learner is used by the learner in improving performance.

7. Provide information to teachers to help contour their teaching strategies. Feedback enhances teaching and learning and effective feedback should serve both. Heritage, Kim, Vendlinski and Herman (2009, p. 24) reason that "for assessment to be formative, action must be taken to close the gap based on the evidence elicited", but argue that "action is dependent on the teacher's knowledge of how learning develops in the domain and on their pedagogical content knowledge". Their study found that "teacher knowledge is critical to effective formative assessment".

Wiliam (1999, p. 9) stresses that feedback (in mathematics) should be timely and "scaffolded" meaning that only the appropriate information should be given when needed.

Feedback conditions

The *conditions of feedback* should be such that each student receives information on his/ her performance, not just a measure of performance in the form of marks, but a report of where performance is adequate and where to improve. Black and Wiliam (1998a) encourages moving away from the idea of providing students with marks: “Feedback has been shown to improve learning when it gives each pupil specific guidance on strengths and weaknesses, preferably without any overall marks” (Black & Wiliam, 1998a, p. 87).

Ruth Butler (1987) investigated different kinds of feedback conditions and its influence on motivation, interest and performance. In this study 200 sixth and seventh grade pupils from four Israeli schools were randomly assigned to four experimental groups. The pupils worked on a variety of divergent thinking tasks in three separate sessions. After the first session the work was collected and four different kinds of feedback were given to the four experimental groups at the beginning of the second session (two days later): One experimental group was given comments, a second was given grades, a third was given praise and the fourth group was given no feedback at all on their individual performance.

A third session, three hours later, was handled in a similar way. Comparisons of the work done in the first session to the work done in the second and third sessions yielded the following result: The quality of work of students who were given feedback in the form of comments improved considerably, whereas the quality of work of students who were given grades or praise did not improve much more than those who were given no feedback at all.

After session three pupils were asked to rate the degree to which various *ego-and task-involved factors* affected their effort and performance. Examples of ego-and task-involved factors are given in table 1 (adapted from Wiliam, 1999, p. 9).

Table 1: Ego-and task-involved factors affecting effort and performance of students (adapted from Wiliam, 1999, p. 9)

Examples of factors that attribute to	Ego-involved factors	Task-involved factors
Effort	• Desire to do better than others • Desire to not do worse than others	• Interest • Desire to improve past performance
Success	• Ability • Competition with peers	• Interest • Effort • Experience from session 1

The results confirmed that individual comments generated higher task-involved views and lower ego-involved views, whereas grades and praise generated no significant increase in task-involved views, but a significant increase in ego-involved views. No feedback showed no significant influence on task- or ego-involved views.

Diagnostic tests to measure mathematical ability

The assessment of the mathematical ability of students, especially students making the transition between school and university (first-year students) necessitates the discussion of diagnostic assessment. Diagnostic assessment is assessment used to give feedback about the knowledge and abilities of students, before a learning program is commenced (Heck & Van Gastel, 2006). The purpose of such an assessment is to provide information for learning and is, in a sense, the first step to formative assessment.

In their article on classroom formative assessment, Black and Wiliam (1998b) review literature on mastery learning of which effective formative feedback is an important component. The following aspects of formative assessment are outlined as essential: students should be given timely feedback on their current performance as measured against a desired goal (the ‘mastery level’), such feedback should be diagnostic and students should be given the opportunity to discuss with their peers how to remedy any weaknesses. In order for the formative assessment of the mathematical ability of students to have meaning, a baseline or diagnostic assessment of student ability needs to be considered.

Heck and Van Gastel (2006) in an attempt to support first year students, suggest the implementation of diagnostic assessment of the basic mathematical abilities of incoming students. They developed a diagnostic pre-test and post-test approach with 100 students at the Faculty of Science of the University of Amsterdam. The pre-test was administered immediately upon arrival at the university and the students were given feedback on their level of performance. During the first four weeks of the Calculus 1 course, they had a session of two hours each week to practise basic mathematics and in the fifth week they wrote the second diagnostic test. Special attention was then given to the low-achieving students in the form of remedial teaching or tutoring in small groups. Even though not all of the students reached the expected level of mathematical ability, overall the study showed substantial progress and the researchers were able to identify where the students are lacking in mathematical techniques (algebraic manipulation) and computational skills.

Analysis of a questionnaire completed by the students indicated that students "appreciated that they were confronted with the mathematical abilities as desired by universities, and were informed of their own level" (Heck & Van Gastel, 2006, p. 942).

After observing that 23 countries outperformed the United States in mathematical literacy in the Programme for International Student Assessment (PISA) test of 2003, Shute and Underwood (2006) focused their study on approaches to diagnose the mathematical problem solving abilities of 15-year old students. They combined technology with practice and feedback to help students improve their mathematical problem solving skills.

Their study focused on algebra and the skills involved in this domain e.g. simplifying expressions, solving linear equations or graphing equations and the design of a Mathematics Intervention Module to assess these skills based on an evidence-centred design approach. The evidence-centred design framework ensures that relevant evidence of knowledge and abilities are consistently collected and inferred in a meaningful way, based on the aims of the assessment (Mislevy, Almond & Lukas, 2003). In the study of Shute and Underwood (2006) feedback was given about the errors that students made in the form of information provided on reasons for incorrect answers. After three attempts accompanied each time by more detailed feedback, the correct answer was provided together with an explanation.

Mathematical competence

What is mathematical competence?

Mathematical competence is the term referred to by researchers (Kilpatrick, 2001; Kilpatrick, et al., 2001; Niss, 2003; OECD, 2003; Sáenz, 2008; Schoenfeld, 2007) in mathematics education to the following questions:

> What does it mean to master mathematics? (Niss, 2003, p. 120)
>
> What does it mean for a student to be proficient in mathematics? (Schoenfeld, 2007, p. 31)

In an attempt to improve the teaching and learning of mathematics, researchers (Kilpatrick, 2001; Kilpatrick et al., 2001; OECD, 2003) have tried to define successful mathematics learning and many possible terms have been used to represent the characteristics of such learning e. g. *mathematical literacy, numeracy, mastery of mathematics, mathematical proficiency* and *mathematical competence.*

From the perspective of these researchers (Kilpatrick, 2001; Kilpatrick et al., 2001; Niss, 2003; OECD, 2003; Sáenz, 2008; Schoenfeld, 2007) the mastering of mathematics is not only about mathematical content (knowing mathematics), but also about the mathematical manipulation skills and the mathematical thinking skills that form the basis of doing mathematics. To quote Sáenz (2008, p. 124) the mathematical literacy or competence notion in PISA, of the OECD, (2003) "provides valuable information on the (mathematical) skills of students". To evaluate the mathematical literacy of 15-year olds with the purpose of improving teaching, PISA is coordinated by the OECD and the study has been conducted every three years since 2000. According to PISA (OECD, 2003, p.15) "mathematical literacy (or competence) is concerned with the capacities of students to analyse, reason and communicate efficiently as they pose, formulate, solve and interpret mathematical problems in a variety of situations". Mathematics is not just about knowing and doing; it is about analysing, reasoning, communicating, posing, formulating, solving and interpreting.

Being mathematically literate or competent implies much more than just knowing, in the same way that being literate in a language implies much more than just being able to read and write. A student is mathematically competent if he/she has the ability, knowledge and skill(s) to do mathematics successfully (Niss 2003).

The question remains:

What ís mathematical competence?

Niss (2011, p. 17) gives the following definition of mathematical competence: "Possessing mathematical competence means having knowledge of, understanding, doing and using mathematics and having a well-founded opinion about it, in a variety of situations and contexts where mathematics plays or can play a role." The Danish KOM project (KOM: Competencies and the Learning of Mathematics) was initiated by the Ministry of Education in Denmark with the purpose of transforming Danish mathematics education from school to university level (Niss, 2003). Niss as the director of the Danish KOM project is also a member of the mathematics expert group of the OECD's Pisa project. Niss (2003) identifies eight clearly recognisable and distinct competencies or components of mathematical competence summarised for this study as follows:

1. **Thinking mathematically** involves asking mathematical questions and knowing the types of answers that mathematics has to offer; comprehending and managing concepts in mathematics as well as abstracting and generalising concepts or properties of concepts and distinguishing between different types of mathematical statements e.g. definitions and theorems.
2. **Posing and solving mathematical problems** involves identifying, posing and specifying different types of mathematical problems and solving them.
3. **Mathematical modelling** involves understanding and analysing existing mathematical models and performing the modelling process such as mathematizing and solving related problems.
4. **Reasoning mathematically** involves developing or following formal and informal mathematical arguments or uncovering basic ideas in a given line of argument e.g. a mathematical proof.
5. **Representing mathematical entities** involves comprehending and employing different representations of mathematical entities and translating between different representations.
6. **Handling mathematical symbols and formalisms** involves handling expressions containing mathematical symbols and formulae.
7. **Communicating in, with, and about mathematics** involves understanding mathematical text or mathematical discourse and expressing oneself in the language of mathematics.
8. **Making use of aids and tools** involves knowing the existence and properties of mathematical tools and aids and being able to use it thoughtfully.

All these competencies involve intellectual or physical processes, so the emphasis is on what the students can do. All eight competencies are closely related and might even intersect, but are clearly distinguishable. All competencies have an analytical and a creative characteristic e. g. understanding different mathematical representations (analytical) and representing mathematical objects (creative).

In the PISA (OECD, 2003) construct of mathematical literacy, three components are distinguished: the situations or contexts, mathematical content and the competencies. Competencies are the cognitive processes "which should be activated to connect the real world in which problems arise with mathematics and to solve the problems posed" (OECD, 2003, p. 30). The seven basic PISA competencies that constitute mathematical literacy are described below:

1. **Thinking and reasoning.** This involves posing and answering mathematical questions, knowing the kinds of answers that mathematics offer to such questions, distinguishing between different kinds of statements and understanding and handling the extent and limits of mathematical concepts.
2. **Argumentation.** This involves knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning, following and developing mathematical arguments.
3. **Communication.** This involves expressing oneself, in a variety of ways, on matters with mathematical content.
4. **Modelling.** This basically involves translating a real life problem into mathematical structures and other actions involved with a mathematical model e.g. validating, reflecting, analysing, communicating and monitoring the model and its results.
5. **Problem posing and solving.** This involves posing and formulating different kinds of mathematical problems and solving them in a variety of ways.
6. **Representation.** This involves decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations and the interrelationship between the various representations; choosing and switching between different forms of representation, according to situation and purpose.

7. **Using symbolic, formal and technical language and operations.** This involves decoding and interpreting symbolic and formal language and understanding its relationship to the natural language and translating from natural language to symbolic/formal language.

Schoenfeld (2007) refers to the document *Adding it up* (Kilpatrick et al., 2001, p. 5) in describing "five interwoven strands of mathematical proficiency":

1. **Conceptual understanding**: Comprehension of mathematical concepts
2. **Procedural fluency**: Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. **Strategic competence:** Ability to formulate, represent, and solve mathematical problems
4. **Adaptive reasoning:** Capacity for logical thought, reflection, explanation, and justification
5. **Productive disposition:** Habitual inclination to see mathematics as sensible, useful and worth wile, coupled with a belief in diligence and one's own efficacy.

The use of the term 'interwoven' is to illustrate that the different strands of mathematical proficiency are interdependent and work together in order to constitute a proficiency to do mathematics.

According to the Principles and Standards for School Mathematics guidelines produced by the National Council of Teachers of Mathematics (NCTM) in 2000, for the teaching of mathematics from preschool through twelfth grade in the United States and Canada, competency in mathematics is encapsulated by five content standards and five process standards (NCTM, 2000, p. 3-4). Three of the five process standards relate to the three of the five strands as described by Kilpatrick et al. (2001), but instead of procedural fluency and a productive disposition, the NCTM (2000) mentions oral and written communication and mathematical representation.

Table 2: Content and process standards that encapsulate competency in mathematics according to the NCTM (2000)

1. Content	**2. Process**
1.1 Number and operations	2.1 **Problem solving**
1.2 Algebra	2.2 **Reasoning and proof**
1.3 Geometry	2.3 **Making connections of mathematical ideas**
1.4 Measurement	2.4 **Oral and written communication**
1.5 Data analysis and probability	2.5 **Use of mathematical representation**

The National Assessment of Educational Progress (NAEP) was initiated in 1969 and provides continuous information on the academic performance of fourth, eighth and twelve grade students in various learning areas in the United States. The NAEP 1996 Mathematics Framework for the assessment of mathematics categorise questions on content strands, mathematical ability and mathematical power (Reese, Miller, Mazzeo & Dossey, 1997). Mathematical strands describe the nature of the content (knowledge or procedure) involved in successfully handling a question, mathematical abilities refer to processes such as conceptual understanding, procedural fluency and problem solving, while mathematical power refers to processes or a student's ability to reason, communicate and make connections across the mathematical strands. From a diagrammatic representation of the NAEP 1996 mathematics assessment framework (see figure 1) it is evident that mathematical power or the processes such as reasoning, communicating and making connections form the foundation and mathematical abilities such as conceptual understanding, procedural fluency and problem solving form the supporting structure that together with the content strands (represented by pillars) produce a meaningful three-dimensional representation of what is meant by mathematical proficiency.

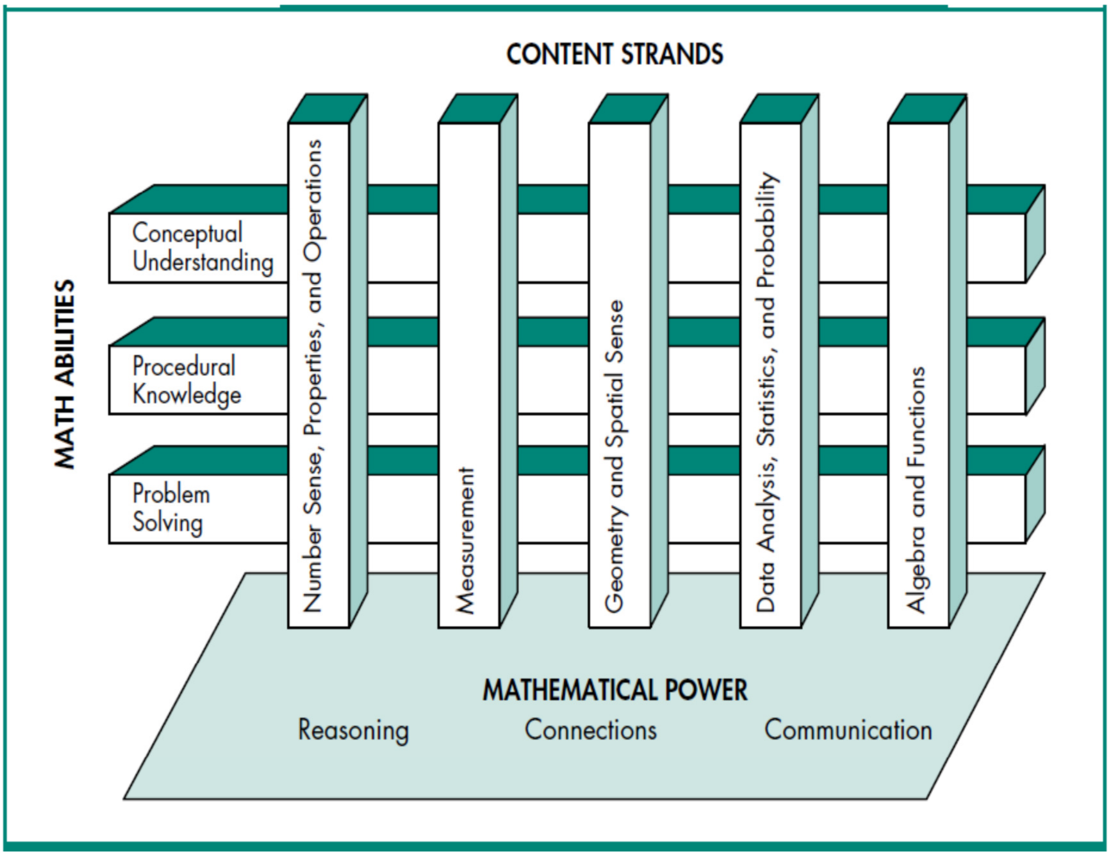

Figure 1: The NAEP 1996 mathematics assessment framework

(Reese et al., 1997, p. 3)

As a project of the International Association for the Evaluation of Educational Achievement (IEA) the Trends in International Mathematics and Science Study (TIMSS) is conducted every four years and assesses attainment in mathematics and science at the grade four and eight levels. TIMSS is an international comparative study of performance in mathematics (and science) and is aimed at improving teaching and learning. The mathematics assessment framework for TIMSS (Mullis, Martin, Ruddock, O'Sullivan and Preuschoff, 2009) is organised around content and cognitive abilities. The content domains differ for the fourth and the eighth grade, but the cognitive domains are the same for both grades:

Table 3: Content and cognitive domains of the TIMSS mathematics framework (Mullis et al., 2009)

Content Domains		
Grade six	**Grade 8**	**Cognitive Domains**
Number	Number	Knowing
Geometric shapes and measures	Algebra	Applying
	Geometry	Reasoning
Data display	Data and chance	

The cognitive domains describe the abilities expected of students as they engage with mathematical content (Mullis et al., 2009, p. 42-44):

Table 4: The mathematical (or cognitive) abilities expected of students according to the TIMSS mathematics framework

Domain	To be able to	What?
Knowing	Recall	Recall definitions; terminology; number properties; geometric properties and notation
	Recognise	Recognise mathematical objects e.g., shapes, numbers, expressions, and quantities. Recognise mathematical entities that are mathematically equivalent (e.g., equivalent familiar fractions, decimals and percentages; different orientations of simple geometric figures).
	Compute	Carry out algorithmic procedures for +, −, ×, ÷, or a combination of these with whole numbers, fractions, decimals and integers. Approximate numbers to estimate computations. Carry out routine algebraic procedures.
	Retrieve	Retrieve information from graphs, tables, or other sources; read simple scales.
	Measure	Use measuring instruments; choose appropriate units of measurement.
	Classify/ Order	Classify/group objects, shapes, numbers, and expressions according to common properties; make correct decisions about class membership; and order numbers and objects by attributes.
Applying	Select	Select an efficient/appropriate operation, method, or strategy for solving problems where there is a known procedure, algorithm, or method of solution.
	Represent	Display mathematical information and data in diagrams, tables, charts, or graphs, and generate equivalent representations for a

		given mathematical entity or relationship.
	Model	Generate an appropriate model, such as an equation, geometric figure, or diagram for solving a routine problem.
	Implement	Implement a set of mathematical instructions (e.g., draw shapes and diagrams to given specifications)
	Solve routine problems	Solve standard problems similar to those encountered in class. The problems can be in familiar contexts or purely mathematical
Reasoning	Analyse	Determine, describe, or use relationships between variables or objects in mathematical situations, and make valid inferences from given information.
	Generalise/ Specialise	Extend the domain to which the result of mathematical thinking and problem solving is applicable by restating results in more general and more widely applicable terms.
	Integrate/ Synthesise	Make connections between different elements of knowledge and related representations, and make linkages between related mathematical ideas. Combine mathematical facts, concepts, and procedures to establish results, and combine results to produce a further result.
	Justify	Provide a justification by reference to known mathematical results or properties.
	Solve non-routine problems	Solve problems set in mathematical or real life contexts where students are unlikely to have encountered closely similar items, and apply mathematical facts, concepts, and procedures in unfamiliar or complex contexts.

Synthesis

Table 5 shows a final synthesis of the different expositions of mathematical competence. An analysis of table 5 led me to identify eight distinct and necessary abilities that constitute mathematical competence as follows:

1. Knowing mathematics
2. Doing mathematics
3. Thinking and reasoning mathematics
4. Modelling and solving mathematical problems
5. Representing mathematical entities
6. Communicating and using mathematical symbols
7. Using aids and tools

8. Having beliefs of efficacy in mathematics

The Common Core State Standards

The Common Core State Standards Initiative (CCSSI) is the most recent initiative aimed at transforming education in the United States. The initiative specifies the knowledge and skills that learners of a school subject (e.g. mathematics) are expected to have mastered at the end of each grade. In 2009, the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) initiated the development of the Standards and in 2010 the Standards were released for mathematics (NGA & CCSSO, 2010). The Standards for Mathematical Practice describe diversities of skill that mathematics educators should aim to develop in their learners and are based on the process standards of the NCTM and the strands of mathematical proficiency specified in the document *Adding It Up* (NGA & CCSSO, 2010). The Standards indicate that eight principles of mathematical practice to be taught in schools (NGA & CCSSO, 2010, p.6-8):

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Table 5: Synthesis of different expositions of mathematical competence

Name	**Abilities that constitute mathematical competence**	Danish KOM project: Mathematical competence	PISA: Mathematical competence	Strands of mathematical proficiency	Principles and Standards for School Mathematics: Process	TIMSS: Cognitive domains
Author		Niss	OECD	National Research Council [NRC] (Kilpatrick et al.)	NCTM	Mullis et al.
Year		2003	2003	2001	2000	2009
Level of relevance		School to university	15-year olds	Pre-K to eighth grade	Preschool through twelfth grade	Grades four and eight
Abilities/ Competencies	**Knowing mathematics**	Thinking mathematically i.e. understanding and handling a given concept	Thinking mathematically i.e. understanding and handling the extent and limits of mathematical concepts	Conceptual understanding: comprehension of mathematical concepts	Knowledge of content Making connections	Knowing-to be able to recall, recognise, retrieve, measure, classify/ order
	Doing mathematics			Procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately		Knowing-to be able to calculate Applying-to be able to select, implement, solve routine problems
	Reasoning mathematics	Reasoning mathematically i.e. following and assessing chains of argument	Reasoning mathematically i.e. posing and answering mathematical questions Argumentation involves knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning, following and developing mathematical arguments.	Adaptive reasoning i.e. capacity for logical thought, reflection, explanation, and justification	Reasoning and proof	Reasoning-to be able to analyse, generalise/ specialise, integrate/ synthesise, justify

Abilities/ Competencies	**Modelling and solving mathematical problems**	Posing and solving mathematical problems Doing mathematical modelling	Problem posing and solving Modelling	Strategic competence: ability to formulate, represent, and **solve mathematical problems**	Problem solving	Reasoning-**solve non-routine problems** Applying-**generate an appropriate model,** such as an equation, geometric figure, or diagram for solving a problem
	Representing mathematical entities	Representing mathematical entities (objects and situations) i.e. understanding and utilising different representations	Representation involves decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations	Strategic competence: ability to formulate, **represent,** and solve mathematical problems	Use of mathematical representation	Applying-display mathematical information and data in diagrams, tables, charts, or graphs, and generate equivalent **representations** for a given mathematical entity or relationship
	Communicating and using mathematical symbols	Communication Using symbolic, formal and technical language and operations	Communication Handling mathematical symbols and formalisms	*See section on developing competence with mathematical symbols	Oral and written communication	Reasoning-determine, describe, or use relationships between variables or objects in mathematical situations, and make valid inferences from given information
	Using aids and tools	Making use of aids and tools				
	Beliefs about mathematics			Productive disposition**:** a belief in one's own efficacy		

Measuring mathematical competence

Any mathematical action entails the involvement of one or a number of mathematical competencies and different activities involve different combinations of competencies. An effective assessment of a student's mathematical competencies has to be based on observing the existence and array of mathematical competencies as revealed in a variety of mathematical activities. No single one activity can be used to judge a student's mastering of mathematical competencies.

An individual's mathematical competency has three dimensions (Niss, 2003, p. 10):

- The *degree of coverage* as the extent to which the person masters the characteristic aspects of the competence.
- The *radius of action* indicates the spectrum of contexts and situations in which the person can activate the competence.
- *The technical level* indicates how conceptually and technically advanced the entities and tools are with which the person can activate the competence.

Niss (2003) compares mathematical competence to a three-dimensional box where each of the dimensions has to be nonzero in order for mathematical competence (the box) to have any volume. If any dimensions have measure zero, then the volume of the box is zero. Also equal volumes can be obtained with different combinations of the three measures.

The following is a possible school task at the secondary level, taken from Niss (2011, p. 14-16). The purpose of the example is to illustrate a breakdown of mathematical competencies involved in a particular activity.

Illustrative example

Consider a rectangular piece of cardboard, of dimensions b and 2b. We want to produce an open box by folding up sides from the cardboard (see figure 2). How large a volume can we obtain by this process? If we want to obtain at least a certain volume, what dimensions should the cardboard have?

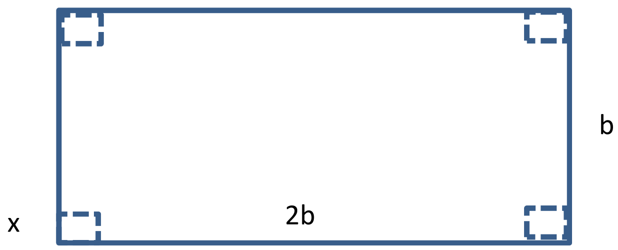

Figure 2: Cardboard of dimensions b and 2b

This is a somewhat stylised applied mathematics problem. To answer the questions implied by it, mathematical modelling is needed which consists of the following components (Niss, 2011, p. 14): mathematising the problem, specification in mathematical terms, representing the problem graphically and answering the questions.

1. Mathematising the problem

   On each side of the cardboard bands of equal width, x, need to be marked off in order for the corners of the box to be cut and folded upwards to produce an open box. The resulting box will have the volume

   $$V(x) = x(2b - 2x)(b - 2x) = 2x(b - x)(b - 2x)$$

   The question in the problem can be translated into mathematics: Are there one or more feasible values of x for which $V(x)$ attains a maximal value?

2. Specification in mathematical terms

   Based on the natural fact that the volume of the box is nonnegative ($V \geq 0$), the feasible values of x satisfy the inequality $0 < x < \frac{b}{2}$, even though V as a polynomial is defined for all $x \epsilon R$. We are now ready to specify the mathematical questions demanded by the problem:

   Question 1: Is there a value of x in the feasible interval $I = \left[0, \frac{b}{2}\right]$ for which V is maximal on I?

   Question 2: If so, what is the value?

   Question 3: Is there more than one such value of x on I?

Question 4: If so, is one value better than the others if additional criteria are introduced?

Question 5: What should b be in order to obtain a minimum volume?

3. Representing V by its graph

We observe that $V(0) = V\left(\frac{b}{2}\right) = V(b) = 0$ and that the polynomial has no other roots. For negative x the polynomial $V(x)= 2x(b-x)(b-2x)$ is negative. In the feasible interval $I = \left[0, \frac{b}{2}\right]$, we have $x > 0,\ b - x > 0$ and $b - 2x > 0$, hence $V(x) > 0$ on I. For $x \epsilon \ (\frac{b}{2}, b)$, $V(x)$ is negative, since $b - x > 0$ and $b - 2x < 0$. For $x > b$, $V(x)$ is positive as both $b - x$ and $b - 2x$ are negative. From this analysis it is possible to sketch the graph of V (see figure 3) on the interval $[0, \frac{b}{2}]$.

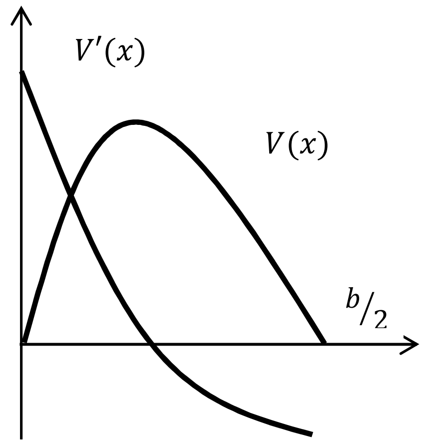

Figure 3: The graph of the function $V(x)$ and $V'(x)$

4. Answering the applicable questions

4.1 V is a polynomial and continuous on the interval $I = \left[0, \frac{b}{2}\right]$, hence attains a maximum value on the interval. Since $V(0) = V\left(\frac{b}{2}\right)$, V is positive on I and the maximal value is reached on I, the answer to Question 1 is "yes".

4.2 The polynomial $V(x)= 2x(b-x)(b-2x)$ can be rewritten as

$V(x)= 2x(b^2 - bx - 2bx + 2x^2)= 2b^2x - 6bx^2 + 4x^3$

By doing some calculus we find the derivative of $V(x)$:

$V'(x)= 2b^2 - 12bx + 12x^2 = 2(6x^2 - 6bx + b^2)$.

By equating the derivative to zero and solving the equation, the x value for the maximal value of V can be determined.

It follows that

$$V'(x) = 0 \text{ for } x = b\frac{3\pm\sqrt{3}}{6}.$$

Since $x = b\frac{3+\sqrt{3}}{6}$ does not lie in the interval $I = [0, \frac{b}{2}]$, it is non-feasible, whereas $x = b\frac{3-\sqrt{3}}{6}$ is feasible, since $b\frac{3-\sqrt{3}}{6} < \frac{b}{2}$.

Therefore there is only one feasible solution, i. e. $x_0 = b\frac{3-\sqrt{3}}{6}$.

The value of V in x_0 is $2b\frac{3-\sqrt{3}}{6}\left[b - b\frac{3-\sqrt{3}}{6}\right]\left[b - 2b\frac{3-\sqrt{3}}{6}\right]$

$$V(x_0) = 2b\frac{3-\sqrt{3}}{6}\left[b - b\frac{3-\sqrt{3}}{6}\right]\left[b - 2b\frac{3-\sqrt{3}}{6}\right]$$

$$= b\frac{3-\sqrt{3}}{3}\left[b - b\frac{3-\sqrt{3}}{6}\right]\left[b - b\frac{3-\sqrt{3}}{3}\right]$$

$$= b^3\frac{3-\sqrt{3}}{3}\left[1 - \frac{3-\sqrt{3}}{6}\right]\left[1 - \frac{3-\sqrt{3}}{3}\right] = \frac{\sqrt{3}}{9}b^3 > 0$$

A maximum point for $V(x)$ have been found. As a result of the fact established in 4.1 that $V(x)$ has a maximum on I the maximum point has to be x_0, since $x_0 \in I$.

Furthermore it can be argued that since $V''(x)$=$-12b + 24x = 12(2x - b)$ such that $V''(x_0)$=$-12b + 24x_0 = 12\left(2b\frac{3-\sqrt{3}}{6} - b\right) = -4b\sqrt{3} < 0$, which implies that x_0 is a maximum point for V.

4.3 Answering Questions 3 and 4:

There is no other maximum point for V on I.

4.4 Answering Question 5:

The maximum value that can be obtained is $\frac{\sqrt{3}}{9}b^3$. If we want to obtain a certain given minimum volume, V_0, this can be obtained by choosing

$$b = (3\sqrt{3}V_0)^{1/3}.$$

The entire problem has now been solved. It is now necessary to examine what competencies were involved in the completion of the task. In order to successfully complete this task one has (Niss, 2011, p. 17)

- to perform *mathematical modelling,* to mathematize the given problem or treat the 'real' problem in mathematical terms.
- to *think mathematically,* to know the kinds of mathematical questions to ask and possible answers to expect e.g. Questions 1 to 5 and to *reason mathematically* or formulate and justify statements, solutions and arguments e.g. answering Questions 1 to 5.

- to *know mathematics.*
- to formulate, pose and *solve mathematical problems*.
- to make use of different *mathematical representations* e.g. in this case verbal, graphical and symbolic representations.
- to handle *mathematical symbols and formalism.*
- to *communicate* about mathematics.
- to make use of *mathematical tools and aids* e.g. a calculator to get an approximation of the maximal value $\frac{\sqrt{3}}{9}b^3$.

Developing mathematical competence

Conceptual understanding and procedural fluency

Conceptual understanding and procedural understanding are, according to Usiskin (2012) different aspects of understanding mathematics. He analyses the meaning of mathematical understanding or specifically the understanding of mathematical concepts from the viewpoint of the student. A concept in mathematics does not only involve associated skills, but also “properties, uses, representations and history” (Usiskin, 2012, p. 15). He refers to the phenomenon identified by educators where students have the ability to “do mathematics” without having a proper understanding (Usiskin, 2012, p. 2).

According to Usiskin (2012, p. 1) “five relatively independent dimensions of mathematical understanding exist: the skill-algorithm dimension, the property-proof dimension, the use-application (modelling) dimension, the representation-metaphor dimension, and the history-culture dimension”. These dimensions can be mastered independently, but together constitute a proper understanding of mathematics. A brief discussion of the five dimensions follows. The skill-algorithm dimension refers to the how of doing mathematical problems or the procedural skills. The property-proof dimension involves understanding the mathematical properties that justify the procedure. The use-application dimension describes an ability to apply concepts and to know when to apply concepts. To be able to represent a concept is to have representation-metaphor understanding of a concept. The fifth dimension, the history-culture dimension involves understanding how and why concepts developed. In order to enhance mathematical understanding all five dimensions need to be considered when designing learning activies.

Developing competence with mathematics through symbols

Hiebert (1988) presents a theory of how competence with written mathematical symbols develops, in order to provide a foundation for the improvement of *symbolic competence* in students. The theory deals with competence in manipulating the regular written symbols of mathematics and applies to all written symbols in mathematics. In 2001 Tall, Gray, Bin Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas and Yusof attempted to build a theory of how students use symbols in mathematics as part of a study into why some students are more successful in mathematics than others. These researchers (Tall et al., 2001) consider the incoherence that occurs between conceptual understanding and procedural fluency in the development of some students' mathematical ability.

Hiebert (1988, p. 335) identifies five cognitive processes that influence competence with written mathematical symbols. These processes must all be engaged and in sequence as mentioned in order to develop symbolic competence. This means the first process needs to be engaged in order for the second process to develop successfully and the first and second in order for the third to develop and so forth:

1. *Connecting* individual symbols with referents;
2. *Developing* symbol manipulation procedures;
3. *Elaborating* procedures for symbols;
4. *Routinising* the procedures for manipulating symbols; and
5. *Using* the symbols and rules as referents for building more abstract symbol systems.

The first cognitive process refer to the connection that is made between a written mathematical symbol and the appropriate quantity or operations on quantities that it refers to (referent) e.g. connecting the symbol 5 with five fingers or connecting the symbol + with adding. The connections between individual numeric symbols and familiar quantities provide bridges between the symbols and the referents, so that the quantities become conceptual entities.

The second cognitive process refers to the development of symbol procedures or manipulations. These procedures imply the mapping of actions on referents to actions on symbols. "So the connecting process focuses on action in the referent world, and the developing process considers how such actions can be mirrored with symbols" (Hiebert, 1988, p. 339). In the development process the conceptual knowledge associated with referents play an essential role.

The first two cognitive processes provide the symbol system with significance or a scaffold that supports the use of symbols. The last three processes move the student away from a concern for the connections of symbols with referents towards the fact that symbols can be manipulated without regard to their referents. The scaffolds of quantitative meaning must be removed in order for symbolic competence to develop further. In the words of Hiebert (1988, p. 341): "Symbols and rules are extracted from particular contexts and are considered objects in their own right."

The fourth cognitive process routinising symbol procedures imply that when procedures are well-practised, the symbol system becomes more efficient. The fifth cognitive process involves the building of more abstract systems on the earlier familiar systems. "Students can transfer meaning from a familiar symbol system to a new, more abstract system if they have established meanings for the familiar symbols (the first two processes have been thoroughly engaged), and if they recognise a mapping between the systems so that the familiar symbols and rules can serve as referents for the new system" (Hiebert, 1988, p. 344).

Gray and Tall (2001, p. 66) refer to embodied objects instead of referents as the "mental conception of a physical object in the world as perceived through the senses" in their study on the relationship between embodied objects and symbolic *procepts*. To explain what is meant by a procept, a distinction needs to be made between *procedure* and *process*. The term procedure refers to a step-by-step set of rules where the rules are followed in a certain sequence. A process represents all different procedures that are relevant in a certain situation. A procept provide a link between mental concepts and processes. Symbols act as prompts for processes and concepts representing both the process and the outcome of the process and hence are referred to as procepts by Gray and Tall (2001) (see figure 4). "We believe that procepts are at the root of human ability to manipulate mathematical ideas in arithmetic, algebra and other theories involving manipulable symbols; they allow the biological brain to switch effortlessly from *doing a process to thinking about a concept* in a minimal way" (Gray & Tall, 2001, p. 85).

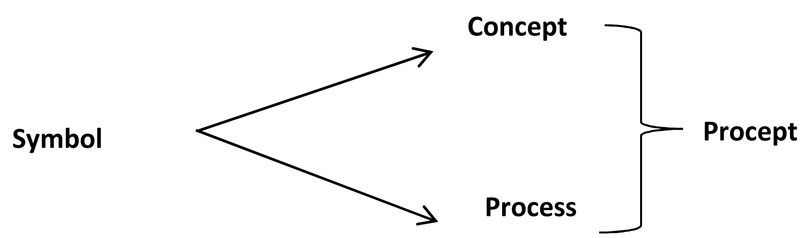

Figure 4: Symbol as a pivot between concept and process

The twofold use of the symbol as process and concept develop by first becoming familiar with procedures, then processes, then routinising of processes in increasingly refined ways, so the student progresses from doing computations to developing flexibility and efficiency with mathematics and then to thinking about mathematics symbolically. This use of the symbol provides the student with a connection between the procedural and the conceptual and a range of possibilities (rather abilities) or a spectrum of performance (see figure 5).

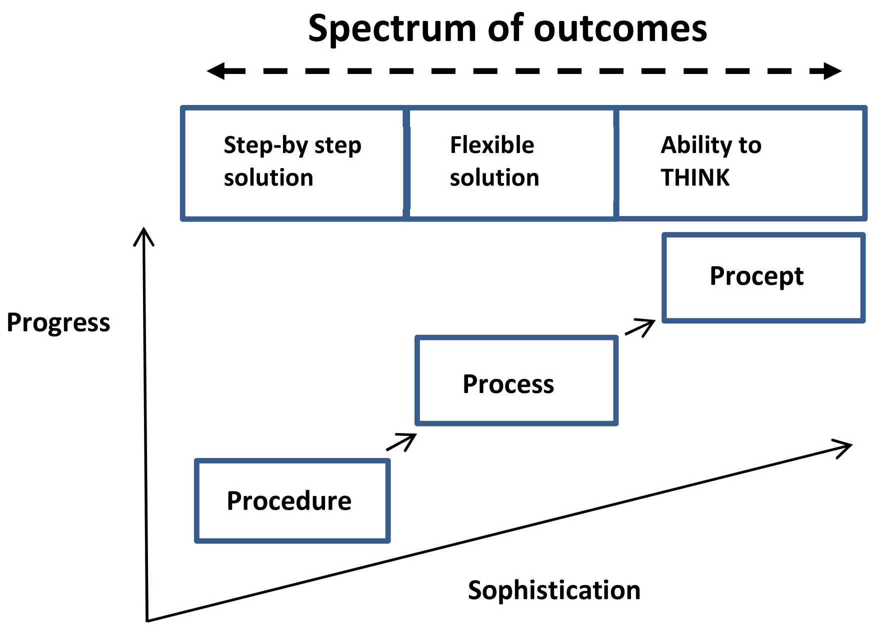

Figure 5: A spectrum of performance

Gray and Tall (2001) reason that the successes and failures of students are rooted in the use of symbols as procepts and in the case of students that experience difficulties with mathematics, discontinuities surfaced in the development of their symbolic competence.

Learning to think mathematically

Bransford et al. (2000) look at how experts (people who have developed expertise in a particular area) differ from novices in order to get insight into the nature of thinking and problem solving. By studying the characteristics of successful learning in the mathematics classroom it becomes more evident what abilities students need to develop in order to improve their mathematical thinking skills. The key principles of experts' knowledge (Bransford et al., 2000, p. 31) are:

1. Experts notice arrays of meaningful information that are not noticed by novices.
2. Experts have acquired a rich organisation of knowledge that mirror a deep understanding of the subject material.
3. The knowledge attained by experts is embedded in context to which it is applicable.
4. Experts have the ability to effortlessly retrieve relevant information.
5. Even though experts have a vast knowledge base, it is not guaranteed that an expert will be a good teacher.
6. Experts' knowledge combined by an ability to adjust and the desire to improve make for successful learning.

Schoenfeld (2007, p. 60) note that it is not only important that students have knowledge, but being able to use that knowledge appropriately and efficiently is an essential constituent of mathematical proficiency:

> Thus, in addition to possessing a substantial amount of specialized knowledge, mathematicians possess other things as well. Good problem solvers are flexible and resourceful…they also have a willingness to pit themselves against difficult mathematical challenges under the assumption that they will be able to make progress on them.

Problem solving strategies

Strategic competence is described as one of the strands of mathematical proficiency:

> Strategic competence is the ability to formulate, represent and solve mathematical problems (Kilpatrick et al., 2001, p. 5)

Problem solving is the ability to utilise all knowledge and skills to deal with new or unfamiliar tasks or non-routine problems in a productive way (Schoenfeld 1992). Students should be able to use and apply the mathematical knowledge they have.

In 1945 Pólya opened up the study of problem solving strategies with the first edition of *How to solve it:*

> Pólya described powerful problem solving strategies such as making use of analogy, making generalizations, re-starting or re-formulating a problem, exploiting the solutions of related problems, exploiting symmetry, and working backwards (Schoenfeld, 2000, p. 64).

Schoenfeld (1992) notes that these strategies are not trivial and no general rules can be deduced. The characterisations of Pólya's strategies can be recognised when used, but lack detail and instructions. Schoenfeld suggests that when attempting to teach problem solving strategies the teacher must accommodate the different approaches of students, be ready to intervene when necessary and be comfortable with discovering though without not always knowing the answer. Schoenfeld (1992) also stresses the importance of teaching students metacognitive skills, to be able to reflect on one's own performance and adjust one's problem solving strategies accordingly.

Knowledge base

As noted by Bransford et al. (2000, p. 38) experts notice meaningful patterns of information and have an organised knowledge base. Their knowledge is chunked or organised around core concepts that guide their thinking:

> Knowing more means having more conceptual chunks in memory, more features or relations defining each chunk, more interrelations among chunks and efficient methods of retrieval of information.

The knowledge of experts also mirror context of applicability. It is important to note that the knowledge base of students may contain misconceptions, misremembered facts or facts that have not been properly accessed or used.

> They come to formal education with pre-existing knowledge, skills, beliefs and concepts that influence what they notice about the environment and how they organise and interpret. This, in turn affect their abilities to remember, reason, solve problems and acquire new knowledge
> (Bransford et al., 2000, p. 10).

Schoenfeld (1992, p. 44) uses the analogy of a library to illustrate what is meant by a knowledge base. With regard to the contents of a library he asks two questions: “What's in it, and how do you gain access to the contents?” When considering knowledge the considerations are to what the students know (the content) and how that knowledge is arranged.

Aspects of the content relevant for skilled performance in a domain according to Schoenfeld (1992, p. 45) include

- Informal and intuitive knowledge about the domain e.g. perceptual biases in geometry
- facts, definitions, and algorithmic procedures
- routine procedures and relevant competencies e.g. proof techniques
- knowledge about the rules of discourse in the domain e.g. understanding the rules intrinsic to Euclidean geometry

Students begin as novices; their knowledge base consists of a set of disconnected facts, they lack understanding and knowledge is without a conceptual framework (Bransford et al., 2000, p. 38). Learning activities (and assessments) should be designed to reveal misconceptions and create opportunity for developing a better understanding of concepts and their connections.

Beliefs about mathematics

The fifth strand of mathematical proficiency is described in *Adding It Up:*

> Productive disposition is a habitual inclination to see mathematics as sensible, useful and worth wile, coupled with a belief in diligence and one’s own efficacy (Kilpatrick et al., 2001, p. 5)

As mentioned earlier students bring their own beliefs to the classroom. Typical beliefs of students about the nature of mathematics are (Schoenfeld, 1992, p. 69):

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem -- usually the rule the teacher has most recently demonstrated to the class.

- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.
- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has little or nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention.

These beliefs of students have to be challenged by teachers. Other beliefs that influence mathematical behaviour are teacher beliefs and societal beliefs and will not be discussed for the sake of this study.

From the Principles and Standards for School Mathematics (NCTM, 2000, p. 4):

> By solving mathematical problems, students acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that serve them well outside the mathematics classroom.

Chapter summary

The purpose of formative assessment in the classroom is to provide students and teachers with continuous information regarding students' progress. Teachers use the information to adapt classroom activities in order to enhance students' learning, while students use the information to measure their individual performance against set standards of performance. As such, feedback about individual performance is an integral part of formative assessment. Students need to develop metacognitive strategies in their learning or the ability to monitor and modify their own performance in accordance with performance criteria.

Students bring misconceptions and practices to the mathematics classroom that could impede the process of learning. When provided with grades students tend to focus on scores and to measure their performance against peers, instead of against standards of performance.

It is the responsibility of the teacher to create learning activities that assist students to focus their attention on their own performance and progress and not on ego-involved factors such as the performance of peers or their own perceived ability.

Mathematical competence is the collective term used to describe the ensemble of behaviours associated with being adept at mathematics. Different researchers (Niss, 2003; OECD, 2003; Kilpatrick et al., 2001; NCTM, 2000; Mullis et al., 2009) provide different expositions of mathematical competence. An analysis of the different expositions generated the following interpretation of mathematical competence:

To be competent in mathematics is to know mathematics and mathematical concepts, to be fluent in mathematical procedures (solving routine problems), to adaptively reason mathematics, to have problem solving and modelling skills relevant to non-routine problems, to be able to represent mathematical entities and translate between representations, to communicate and use symbols in a mathematical meaningful way, to be able to employ aids and tools and to have beliefs about mathematics that are beneficial to successful performance in mathematics.

These abilities or competencies are interrelated and symbiotic in the development of mathematical competence or proficiency.

Chapter 3: Theoretical Framework

Revision of the relevant literature inspired me to develop an appropriate formative feedback instrument used to provide individual students with feedback about their mathematical competence and the competencies that constitute mathematical competence. Designing such an instrument not only to provide information, but also to initiate interaction between teacher and student in order to stimulate students to take ownership of their own learning , necessitated some deliberation as to what form the instrument should take. What follows is the theoretical framework underlying the development of such a formative assessment feedback instrument.

Black and Wiliam (2009, p. 11) analyse any formative interaction as constituting

- A contingency *plan* devised by the teacher to advance learning.
- The developmental *interactions* directed by the teacher within the context of the plan.
- The cognitive processes of students of which the teacher can only observe or interpret *indirect evidence*.

The main purpose of the planned formative feedback instrument has already been stated and such an instrument would serve as part of a developmental interaction plan aimed at developing the mathematical competence of students. The objective of stimulating students to own their own learning through interaction and to use the feedback instrument as facilitator of such an interaction, led to the following considerations.

- Feedback should be such as to distract students' attention from merely observing scores as is many times the case with traditional testing and scoring of tests and to help them focus their attention on where improvement is needed.

- The feedback instrument would have to provide a holistic measure of a student's mathematical competence and a meaningful and informative explanation has to be provided as to how to the measure was attained.

In order for students to be assessed on competencies and to be given appropriate and meaningful feedback the following was also considered:

- Feedback about the mathematical competencies of students has to be communicated in a language that could be comprehended by first year students
- The feedback instrument has to be concise, but reveal essential information about the mathematical competence of a student
- The feedback instrument has to be relevant at university level
- The feedback instrument has to be relevant considering the varied content in a first year finite mathematics course
- The feedback instrument has to be relevant considering the definition of mathematical competence as referenced from the literature
- The feedback instrument has to appeal to the students, preferably visually, and motivate students to analyse the feedback from the instrument and discover the meaning behind the attained score.

The different definitions of mathematical competence as from the literature enlightened me to the appropriate framework for the planned study. Eight distinct and necessary abilities to constitute mathematical competence were identified in the previous chapter (Table 1, Chapter 2) and are summarised as follows:

1. Knowing mathematics
2. Doing mathematics
3. Thinking and reasoning mathematics

4. Modelling and solving mathematical problems
5. Representing mathematical entities
6. Communicating and using mathematical symbols
7. Using aids and tools
8. Beliefs about mathematics

The study was aimed at first year students of mathematics, so the exposition of mathematical ability described by Niss (2003) as relevant to students of mathematics at university level was adopted. I decided to represent the mathematical competency "thinking mathematically" termed by Niss (2003) as two distinct competencies, knowing mathematics and doing mathematics, based on the fact that "conceptual understanding" and "procedural fluency" are regarded as two separate strands of mathematical competence (or proficiency) as described by Kilpatrick et al. (2001).

Keeping in mind that the feedback instrument has to be simple and concise to be effective, I decided to condense the eight competencies in the feedback instrument into five essential and relevant abilities (with references to the eight competencies above):

1. Knowing mathematics (1) is to know and understand mathematical concepts and the relevant connections between different mathematical concepts, to the extent that the strengths and limitations of such concepts are evident.
2. Communicating mathematics (6) is to be able to use mathematical symbols and language appropriately. To be able to communicate in and about mathematics or to be able to write down answers to problems, using the appropriate mathematical symbols and language.
3. Doing and reasoning mathematics (2, 3 and 7) is to be able to apply mathematical knowledge in the solving of routine problems. To be able to perform mathematical

procedures fluently and to reason mathematics or generate and interpret mathematical arguments. To be able to use mathematical aids and tools, when doing mathematics.

4. Representing mathematics (5) is to be able to understand and utilise representations of mathematical entities and to be able to translate between different kinds of mathematical representations.

5. Modelling and solving mathematical problems (4) is to be able to solve non-routine problems by asking and solving mathematical questions generated by non-routine problems.

The decision to only refer to five essential competencies was based on the following:

Beliefs about mathematics are important but not essential to mathematical ability

- Measuring beliefs about mathematics objectively (without consulting students about their beliefs) could prove to be unreliable

The consideration that the feedback instrument should be simple and accessible inspired me to make use of a five-point radial chart (see figure 6). Every radian of the five-point radial chart represents one of the five identified competencies, expressed in easily understandable terms as follows:

1. How you know mathematics
2. How you write mathematics
3. How you reason mathematics
4. How you illustrate mathematics
5. How you solve problems

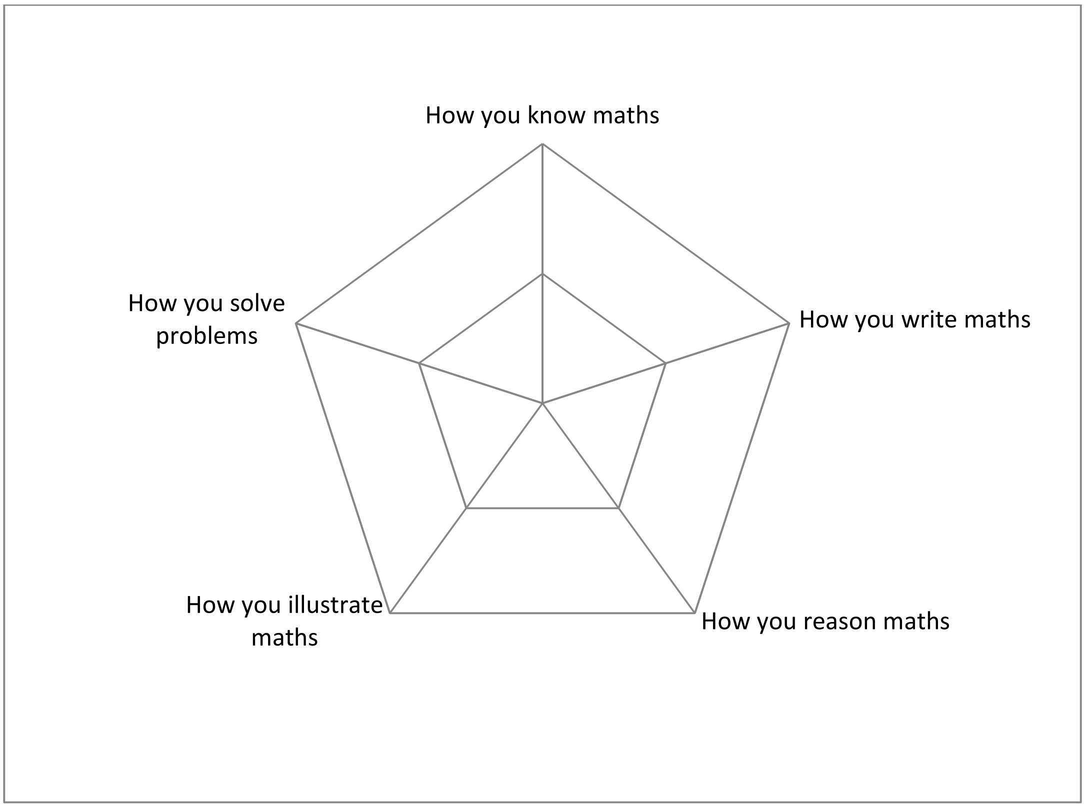

Figure 6: Feedback instrument

In the words of Black and Wiliam (2009, p. 27) two intricate steps are involved in a teacher's feedback decisions: "The two steps involved: the diagnostic, in interpreting the student contribution in terms of what it reveals about the student' thinking and motivation, and the prognostic, in choosing the optimum response, both involve complex decisions".

Keeping in mind that mathematical competencies are cognitive processes of which only secondary evidence is available, the following was decided on in order to provide a meaningful and representative diagnosis of each of the five competencies:

- A maximum score of 2 is associated with every radian (or competency).
- Assessment of written tests involves the lecturer allocating a score of 0, 1 or 2 to each of the five radians (or competencies).

- The scoring of competencies is based on whether the student provided evidence of no competence (a score of 0), some competence (a score of 1) or a complete competence (a score of 2)., for example reasoning mathematics in the written test.
- A student's completed test paper is compared to a complete memorandum and based on the student's performance in all the relevant questions a decision is made about whether to allocate a 0, 1 or 2 to each radian.
- A decision is made beforehand as to which competency is most evident from which question. Not all questions provide sufficient evidence of all the competencies.
- A rubric (figure 7) provides a summary of the criteria used to analyse the mistakes made by students and score the relevant competencies.
- A maximum score of 10 is associated with each assessment of a test.

What follows is an example of possible mistakes made by students on a particular question and the resultant effect on competency scores.

	Allocated Score		
	0	**1**	**2**
How you know mathematics	No or poor knowledge of concepts	Evidence of knowledge of concepts; also evidence of misconceptions	Comprehensive knowledge of concepts
How you write mathematics	No or poor use of mathematical symbols and no communication in mathematical language	Use of mathematical symbols out of context and/or incoherent communication in mathematical language. Traces of writing ability.	Expert use of mathematical symbols and communication in mathematical language
How you reason mathematics	No or poor skill in posing and answering mathematical questions, no solving of routine problems or carrying out mathematical procedures.	Incomplete answers to mathematical questions, incomplete solutions of routine problems and/or poor coherence in carrying out procedures or reasoning mathematics.	Competent in posing and answering mathematical questions and solving routine problems; skilled in carrying out mathematical procedures.
How you illustrate mathematics	No or poor use of mathematical representation	Deficiency in use of mathematical representation	Skilled in (multiple) use of mathematical representations
How you solve problems	No or irrelevant questions posed and no or irrelevant solution(s) to non-routine problem(s)	Incomplete questions generated and/or solution(s) to non-routine problem(s)	Competent in modelling and solving non-routine problems

Figure 7: A rubric used to score competencies

An important aspect of the assessment that has been mentioned is that the student's response is judged holistically. If the student, for example, generally writes mathematics well but is prone to the odd error a mark of 1 is awarded for how you write mathematics.

Illustrative example

Question

A factory needs two raw materials, say, E and F. The probability of not having an adequate supply of material E is 0.06, whereas the probability of not having an adequate supply of material F is 0.04. A study shows that the probability of having a shortage of both E and F is 0.02. What is the probability of the factory being short of either material E or F?

Solution

Event E: Not having an adequate supply of material E

Event F: Not having an adequate supply of material F

$P(E) = 0.06$

$P(F) = 0.04$

$P(E \cap F) = 0.02$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$\Rightarrow P(E \cup F) = 0.06 + 0.04 - 0.02$

$\Rightarrow P(E \cup F) = 0.08$

Excerpts of students' solutions are provided to illustrate scoring.

Student A

Solution

$P(E) = 0.06$

$P(F) = 0.04$

$P(E \cap F) = 0.02$

$P(E \cap F) = P(E). P(F) - P(E \cup F)$

	Score	Comment
How you know mathematics	0	Poor knowledge of concepts i.e. formulas

Student B

Solution

$P(E \cup F)' = P(E') + P(F') - P(E \cup F)'$

	Score	Comment
How you reason mathematics	0	Poor skill in posing and answering mathematical questions i.e. reasoning mathematics

Student C

Solution

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(E \cap F) = P(E \cup F) - P(E) - P(F)$$

	Score	Comment
How you reason mathematics	1	Incoherence in carrying out procedures

Student D

Solution

$$P(E \cap F) = P(E).P(F) = 0.94 \times 0.96 = 0.90$$

	Score	Comment
How you solve problems	0	Irrelevant solution to a non-routine problem

Chapter 4: Research Methodology

In the previous chapters the theoretical framework and the literature supporting the current study were discussed. In doing so the relevance of the study was illuminated, through emphasising the fundamental theoretical principles underlying the study. The purpose of this chapter is to discuss how the study was conducted. What follows is a detailed description of the approach of the current study (research paradigm), the strategy of the research (action research) and the methods used to collect data and to interpret data (research design). The chapter ends with a discussion of the reliability and validity of the current study.

The research paradigm

A paradigm is a collection of beliefs about the essential features of reality that defines a particular world-view (Maree, 2007). Ontology is the study of existence or reality and scientific research is always founded on a specific ontological view or view of reality (Slevitch, 2011). The quantitative or positivist approach to research or quantitative paradigm is rooted in the ontological belief that one reality exist and that such reality exists independent of the observer and can be described or studied objectively

(Slevitch, 2011, p. 76):

> The ontological position of the quantitative paradigm holds that objective reality exists independent of human perception. It also postulates that the ultimate truth exists and there is only one truth.

A qualitative paradigm is based on the view that multiple realities exist and that reality is determined by the participants or that reality is emergent. Such reality is dependent on people's values and beliefs and does not exist independently (Slevitch, 2011). In the words of Maree (2007, p. 54):

> 'Reality' as portrayed by qualitative researchers therefore tends to follow the constructivist cue that reality is a social construction; it accepts that the researcher cannot be separated from the research; and it asserts that research findings are *created* rather than *discovered*.

The aim of qualitative research is to attain a better comprehension of occurrences as experienced by the participants and the emphasis is on detailed descriptions and understanding. The aim of quantitative research is to represent a single reality as accurately as possible.

Quantitative paradigm

The current study is based on a quantitative paradigm, since the objective is to realistically and accurately measure an entity that is the mathematical ability or competence of a student and to report on the process and findings and not the varying experiences or constructions of the human participants.

Whereas ontology is the study of reality, epistemology is the study of the knowledge of reality and the ontological beliefs of a researcher influence the epistemological approach of the researcher or the researcher's perception of what knowledge entails (Slevitch, 2011).

A researcher's methodology is the framework of theories and beliefs that underpins the researchers' approach to research and is influenced by the epistemological and ontological beliefs of the researcher. The methodology implied by a certain ontological perspective determines the methods of sampling, measurement, data-collection and data-analysis best suited to conducting the research (Mouton, 1996).

Action research

The aim of the study is to develop an instrument to measure and provide formative feedback about a student's mathematical competence. This is grounded in the ambition of gaining knowledge about and assisting in the improvement of students' performance in mathematics. The study is intended to develop and improve educational practice and according to Koshy (2005) such an approach qualifies as action research. The strategy of probing and reflecting on findings in order to enhance the quality of practice is what action research entails (Koshy, 2005, p. 16):

> My own belief is that action research is all about developing the act of knowing through observation, listening, analysing, questioning and being involved in constructing one's own knowledge. The new knowledge and experiences inform the researcher's future direction and influences action.

Maree (2007, p. 124) refers to action research as "a strategy" and states that "action research centres on the interaction between research methodology and solving local, practical problems".

In the words of Sagor (2009, p. 7) action research is about "wisdom honed through a disciplined analysis of action".

Koshy (2005) refers to different researchers' models of action research and identifies the cyclical nature of action research in which the observations and findings generated by a certain action are reflected upon and as a result actions are adjusted and the process of observation and reflection repeats itself until the desired goal is attained. The cycle of evaluation is circumstantial to a real-life problem, evolving and aimed at improving practice (Koshy, 2005).

Sagor (2009, p. 10) stresses the importance of teachers' involvement in action research and refers to four 'habits of mind' or consecutive actions that are fundamental to action research. These four stages of action research are goal setting, developing theories, action and data collection, and reflection and action planning. Maree (2007) stresses the fact that a practical problem is the driving force behind the action research process. He identifies the important phases of action research as reflection, research, planning and action. Melrose (2001) identify the important phases as planning, action, observation and reflection and states that (Melrose, 2001, p. 166) "action research is best envisaged as a spiral of cycles through time, with the focus of each cycle sometimes shifting a little within the general knowledge field while understanding and improvement occurs."

The fundamental stages of the action research process are represented in figure 8, as my interpretation of the four stages of Sagor (2009) and the important phases identified by Maree (2007) and Melrose (2001). The cyclical process of goal setting, theory development and research, planning, action and reflection is represented as spiralling around the core vision of solving a real-life problem. In the process of attempting to solve practical problems, smaller problems or goals emerge that influence the direction that the research and consecutive action plans take, but the overall objective remains intact and is reached by attaining consecutive smaller goals.

Challenges

When trying to ascertain students' mathematical competence the challenge proved to be the analysis of students' responses in tutorial and other tests in order to get an overall picture of the mistakes that students make and linking their mistakes to the relevant competencies constituting mathematical competence as elucidated by the literature, in other words developing the theoretical framework of the study. The implementation of the radial chart (see Chapter 3) was judged and improved each time that the instrument was used to assess the students' mathematical competence.

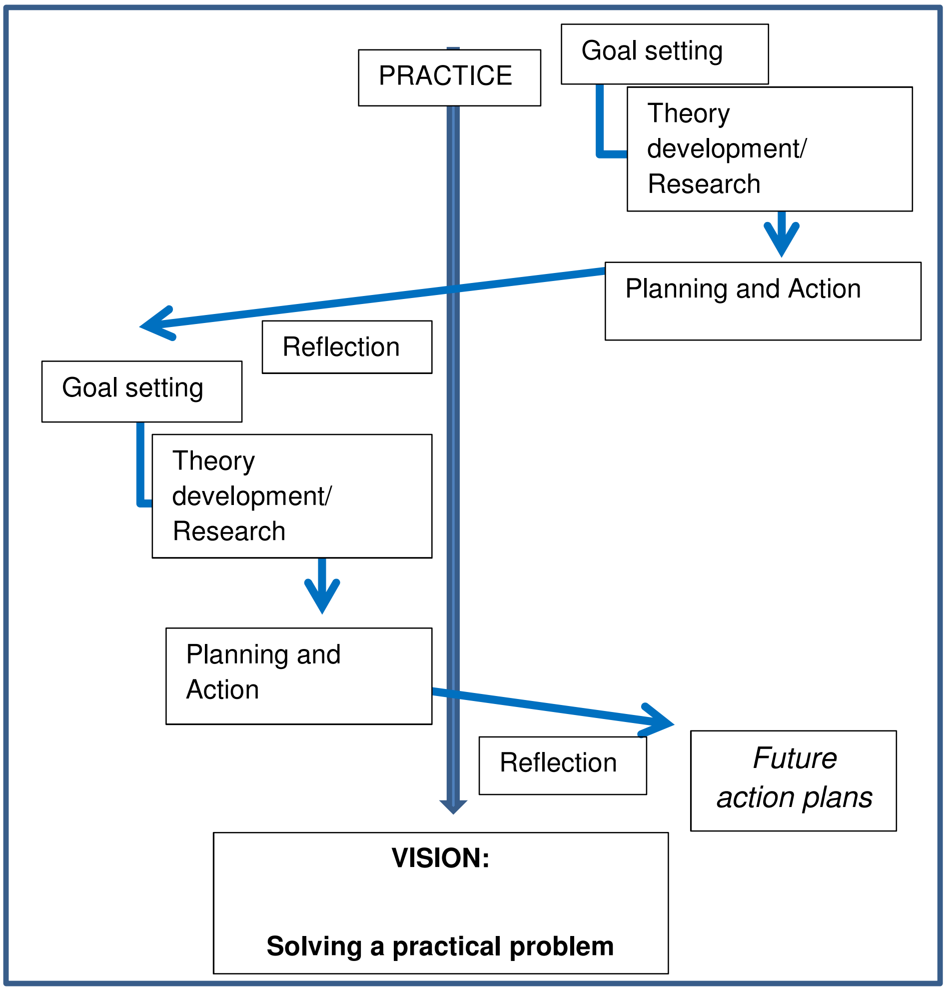

Figure 8: Diagrammatical interpretation by the researcher of the important stages of the action research process according to Sagor (2009) and Maree (2007)

Another challenge (or goal) that emerged was educating students on how to interpret the feedback instrument, how to connect the provided feedback to a specific incompetence or inability and how to improve the related competency and hence his/her mathematical competence. This task proved to be an even bigger challenge than developing the instrument and reflecting on previous use of the instrument informed me each time on how to improve the practice of providing feedback and providing information on how to advance competencies and mathematical competence as a whole. Information on improvement was provided by having discussions with students about possible mistakes that were made in the tests, how these mistakes influenced their assessment and how mistakes could be avoided in future.

Each instance of using the instrument proved to be part of a developmental and cyclical process, characterised by the important phases of goal setting, theory development, action and reflection as explained in figure 8.

Research design

Action research as a research approach does not prescribe a certain research design. In the words of Maree (2007, p. 130) “action research privileges flexible research designs”.

Action research is aimed at improving practice and quantitative and/ or qualitative methods can be employed to reach the desired goal (Maree, 2007). In order to strengthen the relevance of action research, multiple methods can be employed to generate different data sets that are integrated and interpreted together to improve practice (Melrose, 2001).

Mouton (1996, p. 108) stresses the fact that the research problem is the most important determinant of the research design:

> The structure and particular logic of a research design is determined by the formulation of the research problem. The degree of structure in our design will be a direct function of the research goals that we have set ourselves.

The goal of developing an instrument to measure and give feedback about mathematical competence of students is embedded in the vision of assisting the improvement of students’ mathematical performance.

The research question led me to conduct the study in two phases, an implementation and evaluation phase. Both phases were employed to judge the effectiveness of the mathematical competence measuring instrument for improving mathematical performance.

The implementation phase was aimed at judging the effectiveness of the measuring instrument empirically and the evaluation phase was aimed at exploring the experiences of the participants (the students).

Phase 1: The implementation phase

There are two main classes into which quantitative research designs can be classified i.e. experimental and non-experimental designs.

Three elements constitute a classical experimental research design: manipulation, randomisation and control (Maree, 2007). Manipulation is the introduction of an independent variable, randomisation implies the random application of such a variable to the participants and control implies control of the irrelevant variables that may influence the study (Welman, Kruger & Mitchell, 2005).

The implementation phase of the research design of the current study is a non-experimental research design, since an independent variable was not introduced to some of the participants (no control group was present) and other variables could have influenced the findings (Maree, 2007).

As part of the implementation phase a mathematical competence feedback instrument was developed and used to assess the mathematical competence of the students. All the students experienced the introduction of and consecutive evaluations of using the competence instrument. No manipulation took place in the sense that an independent variable or treatment (e.g. competence instrument) was randomly introduced to some of the students whereas the other students were not exposed to the variable or treatment (e.g. competence instrument). The results of the study could have been influenced by other variables such as students' own attempts at improving their mathematical performance.

Sampling, pilot and data collection strategies

The population of the study consisted of the 111 students enrolled for the Finite Mathematics (WTW 154) semester module at the University of Pretoria in 2013. Since the whole population was used in the study, no specific method of sampling was used for the implementation phase.

The students wrote six tutorial tests of which the first three were graded in the traditional manner and a mark was either given out of 10 or out of 15. After the feedback instrument had been developed tutorial test 4 was assessed by using the feedback instrument.

With tutorial test 4, the competence instrument as a radial chart was piloted, the first cycle of the action research. Each student was given feedback about his/her mathematical competence in the form of a radial chart with no traditional marks attached. Although a mark was assigned to each student, this mark was not conveyed to the student but only entered on a class list by the lecturer. Test paper responses were corrected and mistakes indicated but only a radial chart evaluation was stapled to each test, not indicating a score. It was observed how the students experienced the new way of giving feedback on performance in order to refine the instrument for future use. After conducting the pilot it became evident that students experienced a need to have a score attached to any form of assessment.

Hence before assessing tutorial test 5 it was decided to convert the assessment to a mark out of 10 (see Chapter 3) for every tutorial test (tutorial tests 4, 5 and 6) that was assessed using the instrument. The feedback instrument was utilised to provide students with feedback about their mathematical competence in tutorial tests 4, 5 and 6, but the method of calculating the mark out of 10 was also made known to the students. In this manner three different data sets were collected (by me) for tutorial tests 4, 5 and 6.

Data analysis

The three quantitative data sets generated by the use of the feedback instrument in assessing tutorial tests 4, 5 and 6 were added together to produce a mark out of 30 and was converted to a percentage.

These percentages attained by students and the percentages attained by students in the final examination were represented by a scatter plot in order to analyse the relationship between the two data sets. The Pearson correlation coefficient was calculated to summarize the relationship between the two data sets and was also calculated for different performance groups: students that attained a percentage between 50 and 75, less than 50 percent and 75% or more on the competence instrument.

The average percentages of the three tutorial tests (tutorial test 4, 5 and 6) were also calculated and analysed.

Phase 2: The evaluation phase

The research design employed in the evaluation phase of the study is the survey research design. For the evaluation phase of the study, quantitative data was collected by administering a questionnaire. The questionnaire was aimed at gathering information about students' experiences of the mathematical competence assessment instrument.

Sampling, pilot and data collection strategies

After the three tutorial test assessments (tutorial tests 4, 5 and 6) had been completed a questionnaire (see Addendum A) on students' experiences of the feedback instrument was distributed to all students. The non-probability sampling method of convenience sampling was employed in the evaluation phase of the study in the sense that the questionnaire was distributed to all the students that attended lectures during the last week of the semester. Of the 111 students, 56 (N=56) returned the completed questionnaire of which not all students had answered all questions (see figures 12, 13, 15 and 16 in Chapter 5).

The questionnaire consisted of 13 closed-ended Likert scale and 5 open-ended questions on whether and how the students found the feedback instrument useful.

Typical closed-ended question included in the questionnaire:

Answer the following questions based on your opinion and experience of the new way of assessment, the mathematical competence feedback model:

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
The feedback was easy to interpret					
I understood the meaning of each category					

An open-ended question included in the questionnaire:

1. Thinking back, and comparing your results for the last three tutorial tests, do you think your mathematical competence improved?

	Yes
	No

2. In what way did your mathematical competence improve/ not improve?

____
____
____

The questionnaire was not piloted.

Data analysis

The student responses on the questionnaire were encoded and frequencies of the coded responses were calculated. The quantitative data generated by the questionnaire was represented by bar-charts and in the instance of question 8 a pie-chart.

Reliability and validity

An important aspect of the design of a study is to consider and attempt to minimise all factors that can undermine the integrity of the study. McMillan and Schumacher (2010, p. 114) refer to the credibility of a study as the "extent to which the results approximate reality." In order for a study to be credible, opportunities for error need to be recognised and controlled.

Melrose (2001) explains what is meant by rigour in research and especially its meaning in action research and how it can be improved. For research to be rigorous the research has to be representative of reality. Synonyms for rigour identified by Melrose (2001, p. 163) are "quality" or "validity". McMillan and Schumacher (2010, p. 116) clarify the term validity in research to be "the degree to which scientific explanations of phenomena match reality."

When considering rigour or validity of action research, Melrose (2001, p. 164) stresses the fact that the value of action research lies in the change or improvement induced by the cycle of actions and not the research as such, but provides guidelines for improving the validity of action research conducted for academic purposes.

Maree (2007) states the importance of validity and reliability of the instruments used in quantitative research. The reliability of an instrument is explained by Maree (2007. p. 147) as the "consistency" of the instrument or the ability of the instrument to generate the same results when administered more than once.

What follows is a discussion of how rigour or validity and reliability were considered in the two phases of the current study.

The implementation phase

The instrument used to measure the mathematical competence of the students was designed and refined after a thorough study of the relevant literature had been conducted. The validity of the instrument is enhanced by the design being grounded in the literature. The perceived correlation between the percentages attained from using the competence instrument and the final examination percentages also enhances the validity of the measuring instrument (see chapter 5).

When considering the reliability of the measuring instrument, the allocation of marks to each of the five radians (or competencies) needs to be discussed. It was decided that a competency will only be measured by allocating a 0, 1 or 2 in the following matter: evidence of no competence (a score of 0), some competence (a score of 1) or a complete competence (a score of 2). The decision was made in order to ensure that the instrument will generate reliable results when used to assess different tests. By only distinguishing between no competence, some competence and complete competence, I could ensure that the students were assessed in a consistent and controlled manner.

The evaluation phase

Maree (2007) stresses the importance of a standardised questionnaire, where the validity and reliability of the questionnaire have been considered.

According to Melrose (2001) the validity of a study is influenced by ethical considerations. The participants of the current study were informed of the purpose of the questionnaire and the completion of the questionnaire was optional and not enforced.

The face validity of a questionnaire refers to the extent to which the instrument appears to be valid when scrutinised by experienced scientists (Maree, 2007). The questionnaire used in the current study was designed in collaboration with the study supervisors, which strengthens the validity of the questionnaire.

The questionnaire's reliability or consistency in examining the students' experience of the mathematical competence instrument was considered. In order to impartially measure the reliability of a questionnaire, the Cronbach alpha coefficient can be calculated (Tavakol, 2011; Maree, 2007). Cronbach's alpha is used to measure the internal consistency of an instrument or questionnaire, which refers to the "extent to which the items in a question or test measure the same concept" (Tavakol, 2011 p. 53). Question 3 of the questionnaire consisted of eight Likert-scale sub-questions aimed at exploring student's experience of the competence instrument. Cronbach's alpha was calculated to determine whether the eight sub-questions of Question 3 are interrelated. Because question 3.4 was not well formulated (as pointed out by an independent statistical evaluator), the Cronbach alpha coefficient was calculated for seven of the eight sub-questions, excluding question 3.4. A Cronbach alpha coefficient of 0.8505 was calculated for Question 3 (question 3.4 excluded). Tavakol (2011) indicates that acceptable values for Cronbach's alpha range between 0.7 and 0.95 and according to Maree (2007) acceptable values range between 0.7 and 0.9. Since the Cronbach alpha coefficient of 0.8505 falls within the range of acceptable values it can be concluded that the items constituting question 3 are well related, which enhances the reliability of the questionnaire.

Two factors that could have undermined the validity of the questionnaire are the absence of a pilot study and the subject effect. The purpose of a pilot study is to determine whether the respondents understand the questions and to be able to make adjustments to questions before the questionnaire is officially administered (Maree, 2007). Since no pilot study was conducted, the reliability of the questionnaire could have been compromised and as a result the validity. Subject effect refers to the phenomenon where the subjects of a study adapt their behaviour in order to positively influence the study (McMillan & Schumacher, 2010). Since the subjects of the current study (the students of WTW 154) were taught by me, some of the responses might have been aimed at pleasing the lecturer.

Even though a non-probability sampling method was used to select the sample for the evaluation phase of the study, the sample of 56 was big enough to represent the population of 111 and hence the validity of the study was not jeopardised by the sampling method.

Chapter 5: Findings and Interpretation

The statistical analysis and interpretation of data generated by the two phases of the study yielded informative results. The results of the two phases, the implementation phase and the evaluation phase, will be represented and discussed separately in this chapter and then triangulated to conclude the findings of the study.

The implementation phase

To illustrate how marks for a tutorial test were obtained the instrument results of two students (in tutorial test 6) are presented in figure 9 (student A) and 10 (student B). Student A (figure 9) showed complete competence in knowing mathematics (score of 2), some competence in writing, reasoning and illustrating mathematics (score of 1 each) and no competence in solving problems (score of 0). When converted to a mark, student A attained a mark of 5 out of 10 for tutorial test 6.

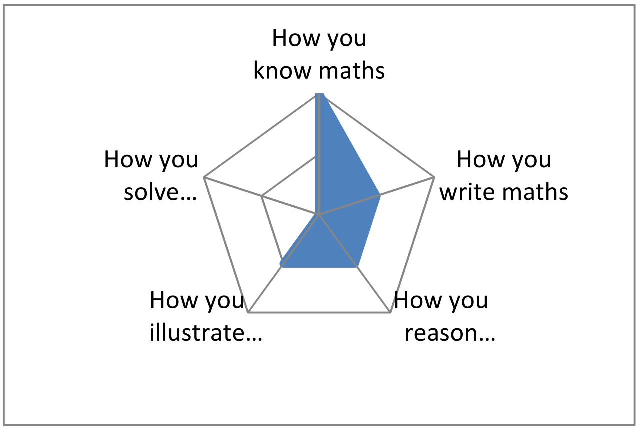

Figure 9: Feedback on the mathematical competence of student A

Student B (figure 10) showed complete competence in knowing mathematics, writing and illustrating mathematics (score of 2 each) and some competence in reasoning mathematics and solving problems (score of 1 each). Student B attained a mark of 8 out of 10 for tutorial test 6.

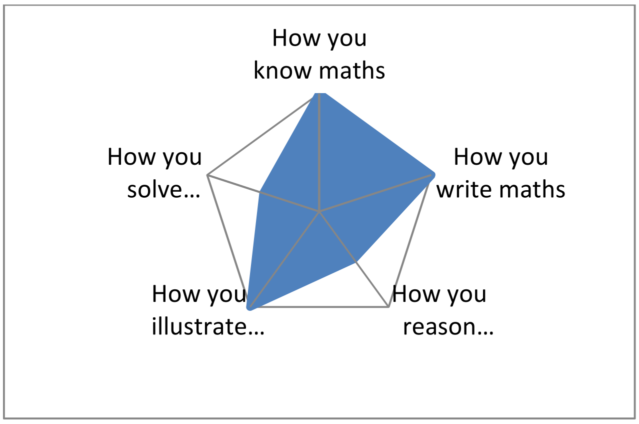

Figure 10: Feedback on the mathematical competence of student B

The average percentages of the three tutorial tests were calculated. In table 6 the average percentages of the three tutorial tests (tutorial test 4, 5 and 6) are represented. For every student the average of the percentages of the three tutorial tests were calculated to constitute a competence instrument percentage.

Table 6: The average percentages of the three tutorial tests

Tutorial test assessed by using competence instrument	**Average percentage**
Tutorial test 4	37.7%
Tutorial test 5	74.7%
Tutorial test 6	75%

It is clear from table 6 that the average percentage of the tests assessed by using the competence instrument was low for tutorial test 4, but higher and almost the same for the other two tutorial tests (tutorial tests 5 and 6). The increase from test 4 to test 5 and 6 could be due to the fact that the difficulty level of the tests varied or due to the fact that the students became aware after the pilot (tutorial test 4) of how their marks were attained and/or compromised in tutorial test 4 and attempted to improve their performance in tutorial test 5 (e.g. use of notation). The fact that the average of tutorial test 6 does not show much improvement from tutorial test 5 can be interpreted as the students not knowing how to (further) improve on their mathematical competence. Usiskin (2012) refers to modelling and problem solving as higher-order activities or advanced skills that are difficult to teach. The significant increase of average percentages from tutorial test 4 to tutorial test 5, compared to no significant increase from tutorial test 5 to tutorial test 6 can be explained by considering Usiskin's view of higher-order activities. To improve use of mathematical notation and communication in mathematics does not acquire as much skill as being able to model and solve non-routine problems. Students found it easier to adjust their writing skills than to improve their problem solving skills for example.

The scatter plot representation (figure 11) of the average of the competence instrument percentages attained by the students against the examination percentages attained by the same students shows a strong positive linear correlation The Pearson correlation coefficient of 0.418 calculated for the two data sets, confirms a strong linear correlation at a one percent significance level. This could be interpreted as an indication of the measuring instrument measuring roughly the same skills as the examination paper when assessing the mathematical performance of students.

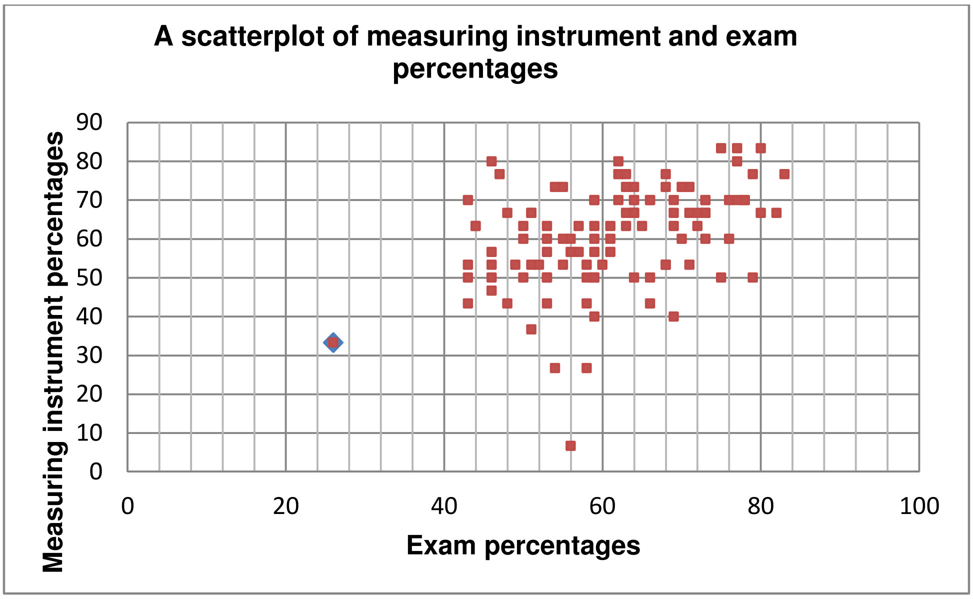

Figure 11: A scatter plot representation of the competence instrument percentages attained by students against examination percentages attained by the same students

The Pearson correlation coefficients calculated for different performance groups are summarised in table 7. Note that the classification into three performance bands is based on examination percentages.

From table 7 it appears that the correlation between the two data sets is significant for the students in the less than 50% performance group at a 10 percent significance level. The correlation is also significant for the students in the between 50% and 75% performance group, in this case at a 1 percent significance. Surprisingly, there appears to be little correlation for the students in the 75% or more performance group with no significance. The lack of significant correlation for the high performance group can possibly be explained by referring to metacognition and what is meant by a proper understanding of mathematics. Metacognition is the ability to think about and adjust learning in order to have a better understanding. Students in the high performance group have the ability to do mathematics and obtain high marks, but on a multidimentional level (Usiskin, 2012) these students lack a proper understanding of mathematics, they lack metacognitive skill or the ability to assess their own understanding.

Table 7: The Pearson correlation coefficient calculated for different performance groups

Performance group (based on examination percentages)	Pearson correlation coefficient	Significance level
75 % or more (n=12)	0.11732293	No significance
Between 50 % and 75% (n=87)	0.347808747	Significant at 1% level
Less than 50 % (n=12)	0.466409907	Significant at 10% level

The evaluation phase

The statistical analysis of the student responses on the questionnaire showed that most students (62%) found the assessment to be useful in identifying their mathematical strengths and shortcomings (figure 12).

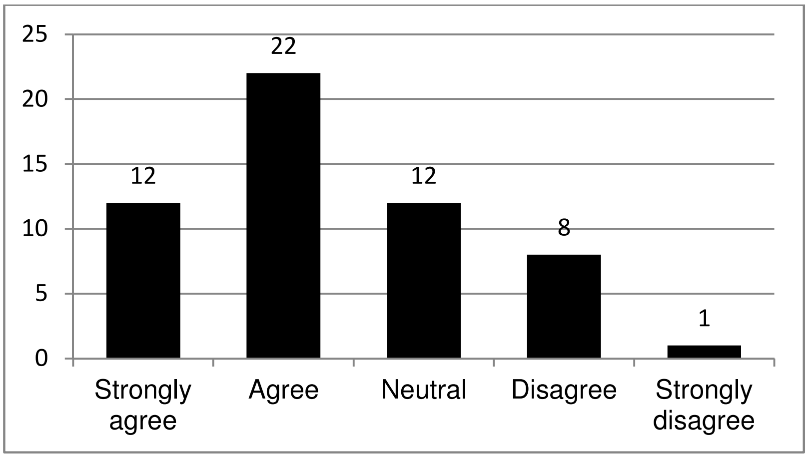

Figure 12: Student responses to the statement “I could identify my mathematical strengths and shortcomings”

The majority of students (74%) answered the question "Do you think your mathematical competence improved?' with a "yes" (see figure 13).

In a follow-up question students were asked "In what way did your mathematical competence improve/not improve?" The responses of the students with improved mathematical competence are represented by a pie-chart in figure 14. From the pie-chart it is evident that most of these students noted an improvement in their mathematical writing (33%) and reading, understanding and application skills (27%). The other skills of improvement mentioned by students were reasoning (5%), illustrating (11%), problem solving (16%) and motivation and effort (8%).

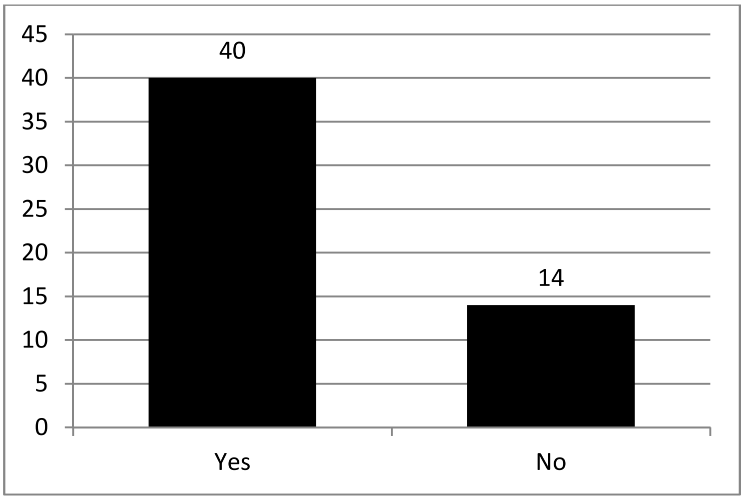

Figure 13: Student responses to the question "Do you think your mathematical competence improved?"

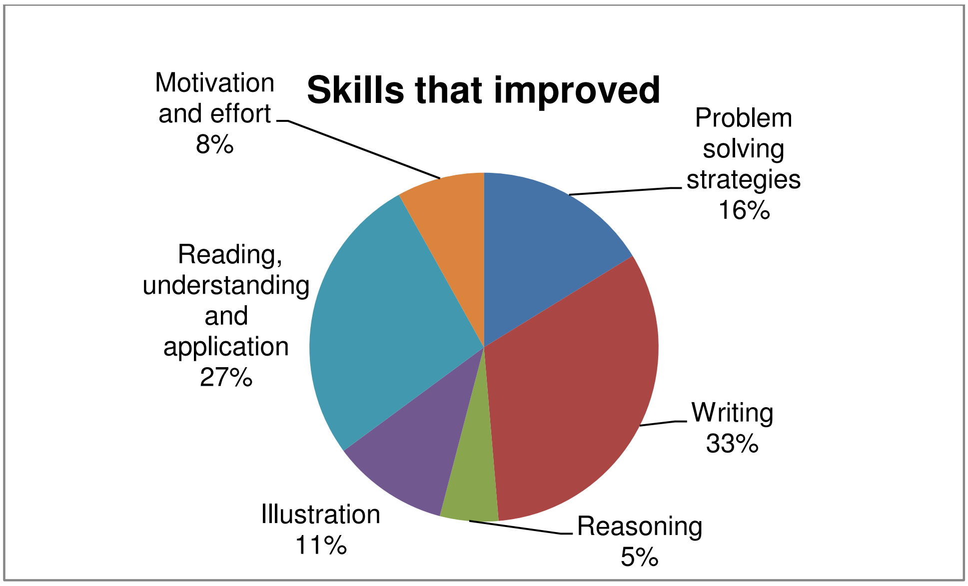

Figure 14: Student responses to the question “In what way did your mathematical competence improve/ not improve?”

Although there appears to be a fairly even spread in the responses to the statements “The feedback helped direct my focus, when preparing for tests” (figure 15), it is noticeable that more students agree (or strongly agree) than disagree (or strongly disagree), 40% against 22%. More students saw the benefit in using the instrument for preparing for the test than did not. The same pattern of response is observed on the statement “The feedback on my mathematical competence made a difference in my approach to answering questions” where the spread again leans somewhat to the left (figure 16). In this case, again, more students agree (or strongly agree) than disagree (or strongly disagree), 44% against 31%, that feedback made a difference in their approach to answering questions.

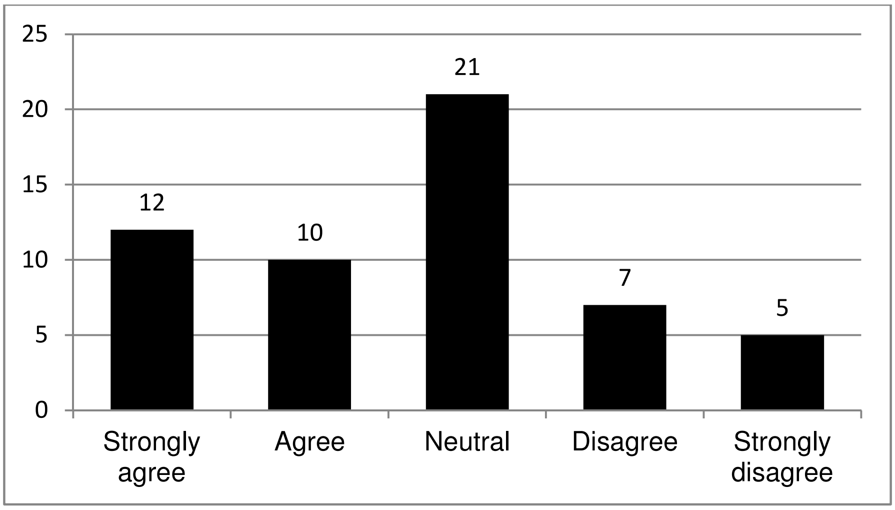

Figure 15: Student responses to the statement "The feedback helped direct my focus, when preparing for tests"

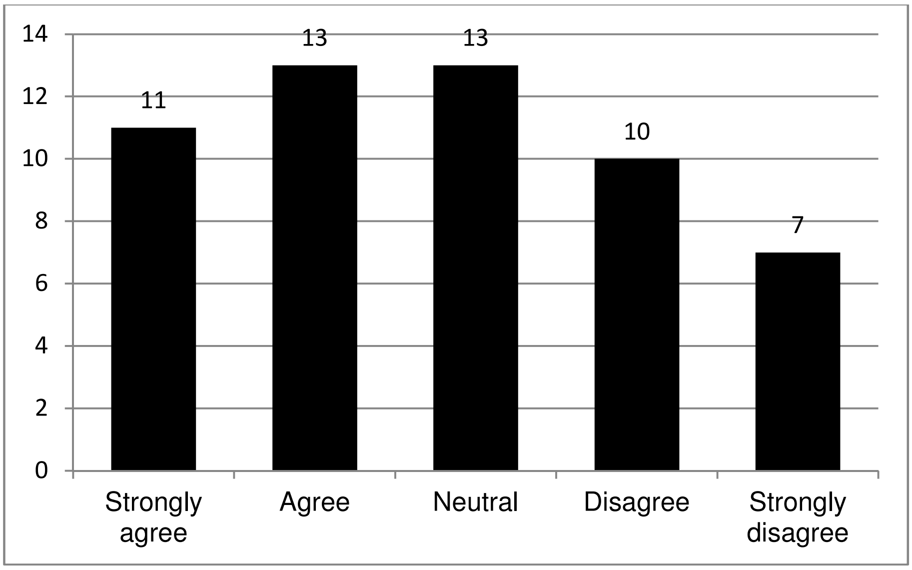

Figure 16: Student responses to the statement "The feedback on my mathematical competence made a difference in my approach to answering questions"

When asked what the students did not like about the assessment, most students indicated that they experienced the interpretation of the assessment to be difficult while some students expressed their preference of traditional marking of tests (figure 17). The fact that most students indicated that they either do not like the assessment or that they found the interpretation difficult, can be interpreted as an indication that students need more guidance in understanding, and utilising the competence instrument to assist their performance in mathematics. This finding could also be indicative of the paradigm that students want to pass a module and are not necessarily intent on improving their mathematical competence. One should caution against judging the value of the feedback instrument simply from student preferences. This word of caution is supported by the findings shown in figure 13 where the majority of students indicated that their mathematical competence improved, despite not liking the instrument (figure 17).

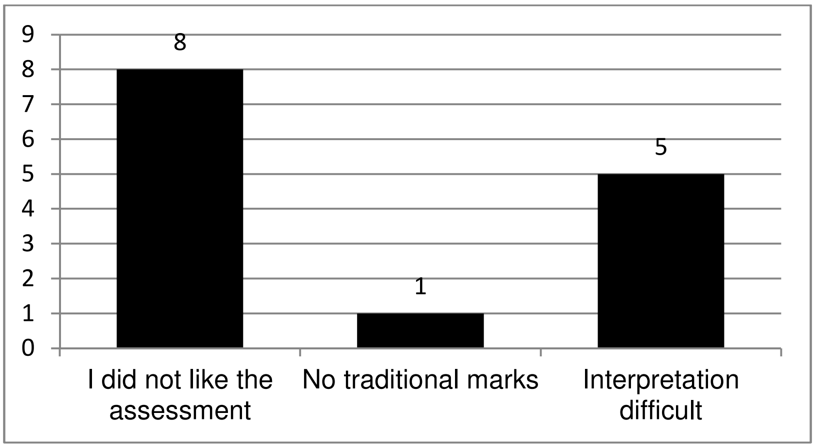

Figure 17: Student responses to the question

"What did you not like?'

Triangulation

The assessment averages showed some improvement (table 6) and the questionnaire responses of students to the question "Do you think your mathematical competence improved?" confirmed that their mathematical competence showed improvement from tutorial test 4 to tutorial test 6 (figure 13). Collectively the students noted some or other improvement in all five the competencies as translated by me in terms of the radial chart terminology (see figure 18), but also mentioned motivation and effort as skills inspired to be improved by the assessment.

The strong positive linear correlation between the competence instrument percentages and the examination percentages attained by the students' show that the competence instrument is valid in assessing the students. Closer examination of linear correlation coefficients for different performance groups confirmed this, except in the instance of the high performance group (75% or more). It appears that besides attaining good marks some students still lack proper understanding and skill in mathematics.

Student responses to questions of the questionnaire discussed (figures 12 to 18), confirm the conclusion that students could identify their mathematical shortcomings or lack of skills but did not always know how to correct or improve these skills.

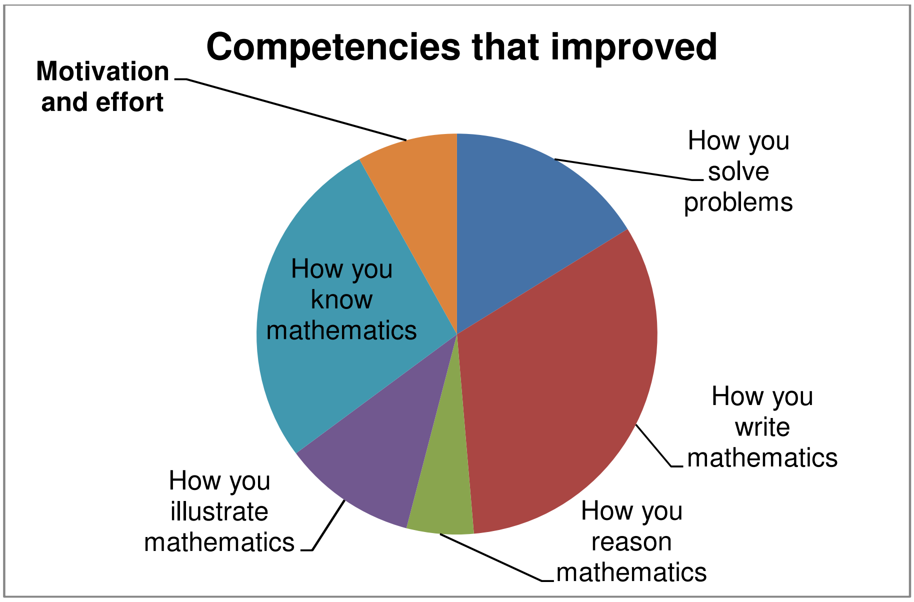

Figure 18: Interpretation of the mathematical competencies as experienced by students to have improved

Some students undervalued the relevance of the instrument and indicated that they did not like the assessment or prefer traditional marking. The reason could be that these students did not know how to utilise the instrument or it could be that the competence instrument challenges the ego-involved pursuit of marks evident in some students' approach to learning.

Chapter 6: Conclusions

In chapter 2 seven principles of effective feedback practice that enhance self-regulation in students (Nicol & Macfarlane-Dick, 2006) were discussed. A summary of the seven principles of effective feedback is given for discussion purposes:

1. Help make clear what good performance is.
2. Facilitate the development of self-assessment.
3. Provide students with good quality external feedback.
4. Encourage discussions between students and students and students and teacher.
5. Motivate and encourage positive views and self-esteem.
6. Provide opportunities to close the gap between actual and expected performance.
7. Provide information to teachers to help contour their teaching strategies.

In this study I developed a formative feedback instrument, with which to measure and provide feedback about the mathematical competence of a student. To conclude the effectiveness of the feedback instrument, the use of the instrument is mapped against the seven principles of Nicol and Macfarlane-Dick (2006).

1. The five radians of the measuring instrument provided students with a clear and concise map of what mathematical competence entails. From the design of the measuring instrument it was evident that one has to master multiple mathematical skills in order to be judged mathematically competent. Students were clearly informed of set standards and what was expected of them.

2. By utilising the measuring instrument to assess three consecutive tutorial tests, multiple opportunities were provided for reflection on performance as measured against set standards.

3. From the findings of the study it is evident that students could identify their mathematical shortcomings or lack of skills but did not necessarily know how to correct or improve these skills. Students were not provided with enough direction on how to better their performance.

4. The introduction of a new way of assessment and the concise design of the measuring instrument encouraged dialogue between students and students and students and teacher.

5. The development of the measuring instrument was inspired by a desire to redirect students' attention away from goals irrelevant to the process of learning e.g. attaining good marks, to goals that encourage growth and development. Students noted that the measuring instrument was useful, but indicated that they prefer their performance to be associated with marks.

6. Even though the average percentages of students for the three tutorial tests showed improvement in the mathematical competence of the students, not enough opportunities were provided for students to explore how to close the gap between actual and expected performance.

7. The design of the measuring instrument was an on-going process and each opportunity of using the instrument provided me with valuable information to help contour the instrument and related teaching strategies.

Measured against the seven principles of effective feedback practices, it is evident that the implementation of a mathematical competence instrument in the assessment of student performances largely complies with these principles. . The goal with which the instrument was developed as well as the instrument itself, are not at fault but there is opportunity for improvement with regards the implementation. The challenge of developing the competence instrument enveloped me to the extent that not enough attention was given to how to assist students in completing the feedback cycle.

The competence instrument was aimed at reducing ego-involved feedback and providing feedback about the task involved in improving mathematical ability (Black & Wiliam, 1998) and in this respect succeeded. With the competence instrument the students' focus was redirected from merely observing marks to being aware of their individual mathematical competence or lack thereof. The competence instrument did succeed in refocusing students' attention, but its value in providing task-involved feedback can be improved upon.

Diagnostic assessment is assessment used to give feedback about the knowledge and abilities of students before a learning program (Heck & Van Gastel, 2006). The competence instrument was not utilised in the current study for the purpose of diagnostic assessment, but could be used to assess students' mathematical competence before they start with first year mathematics.

Based on the literature (Kilpatrick et al., 2001; NCTM, 2000; Niss, 2003; OECD, 2003; Mullis et al., 2009) I identified eight distinctive competencies that constitute mathematical ability (chapter 2):

1. Knowing mathematics
2. Doing mathematics
3. Thinking and reasoning mathematics
4. Modelling and solving mathematical problems
5. Representing mathematical entities

6. Communicating and using mathematical symbols
7. Using aids and tools
8. Beliefs about mathematics

For the sake of the study the eight competencies were incorporated into five essential competencies that constitute mathematical ability or competence (see chapter 3):

1. Knowing mathematics
2. Communicating mathematics
3. Doing, thinking and reasoning mathematics
4. Representing mathematics
5. Modelling and solving mathematical problems

Niss (2003) noted that the competencies are clearly discernible, but also interrelated or “interwoven” (Kilpatrick et al., 2001). Analysing students’ solutions in order to assess their mathematical competence necessitated a breakdown of the mathematical competencies involved in answering a test question (see illustrative example, Chapter 3). In some instances the decision had to be made as to what competency is most evident in a certain question and based on the decision the relevant question was used to assess the most prominent competency. When using the instrument to assess students’ mathematical ability the design of a rubric is imperative and further refinement of the instrument could involve the design of a rubric to assess all the competencies involved in answering a specific question.

The measuring instrument proved to be valid in the assessment of students’ mathematical competence (see chapter 5) and students indicated that utilising the instrument positively influenced all five of the relevant competencies (see figure 16), especially mathematical knowledge and communication skills.

Students also indicated an improvement in their motivation and effort. Kilpatrick et al. (2001) refer to a "productive disposition", or the belief in one's own mathematical ability, as a competency relevant and essential to mathematical ability. In order for students to be able to do mathematics, they have to believe that they are capable of doing mathematics. In chapter 1 it was noted that formative assessment or feedback should motivate students to believe that their mathematical ability or competence can be transformed as a result of effort and preparation.

Niss (2003) compares mathematical competence to a three-dimensional box and Usiskin (2012) refers to "five relatively independent dimensions of mathematical understanding". No single one activity can be used to judge a student's mastering of mathematical competencies and effective assessment of a student's mathematical competence has to be based on observing every mathematical competency as revealed in a variety of mathematical activities (Niss, 2003). The findings of the study led me to the conclusion that one should avoid regarding an examination paper as the ultimate tool when assessing students' mathematical ability or competence.

The value of the study lies essentially in reconsidering the formative assessment process, resulting in the development of an instrument that measures students' mathematical competence in an innovative way, an instrument that is decisively rooted in principles quoted from literature. The characterising trait of the instrument is that it steers away from the correct or incorrect approach of allocating marks. The instrument assists in making students aware of the competencies involved in mastering mathematics and also informs as to where their strengths and weaknesses lie.

As mentioned in the literature review (chapter 2), two consecutive actions on the part of the student form the centre of formative assessment (Black & Wiliam, 1998, p. 20). The first is the realisation that there exists a gap between the anticipated objective and the student's performance and the second is the action taken by the student to close the gap. The competence instrument was perhaps more successful in the first of the two actions. A realisation was created by the feedback instrument of a gap in the mathematical competence of the student but the student then appeared not to be equipped with the know-how of the action that is necessary for closing the gap. The lesson learned here is that students need not only to be assisted in rethinking their fixation on marks, they also need to be taught how to interpret feedback on competence and how to take action from there. The instrument as implemented here has proven therefore in a sense more successful for diagnostic purposes rather than for formative purposes.

Helping students to take control of their own learning by teaching metacognitive strategies is an important aspect of active learning, where the responsibility for learning is placed on the student (Bransford, Brown and Cocking, 2000). Metacognition refers to the abilities of students to keep track of their own performance, measured against set standards, and to be able to adjust and realign performance against set standards. Formative assessment enhances self-regulation and Nicol and Macfarlane-Dick (2006, p. 199) refer to "self-regulation as the degree to which students can regulate aspects of their thinking, motivation and behaviour during learning". The competence instrument successfully informed students of set standards of performance. The visual appearance and concise character of the instrument encouraged students to question how the assessment was conducted and how results were attained. In this sense the instrument succeeded in stimulating self-regulation, inspiring students to think about their performance. Schoenfeld (1992) also stresses the importance of teaching students metacognitive skills, to be able to reflect on one's own performance and adjust one's problem solving strategies accordingly. The objective of stimulating students to own their own learning was realised in the sense that students became aware and could reflect on their own performance. Being taught strategies by which to improve their performance was where the implementation of the instrument fell short, but future research can address this issue.

Kilpatrick et al. (2001, p. 5) refer to a productive disposition as “a habitual inclination to see mathematics as sensible, useful and worth wile, coupled with a belief in diligence and one’s own efficacy”. Bransford et al. (2000) note that an expert in a particular area, be it mathematics, is characterised by the “desire to improve”. Schoenfeld (2007) describes good problem solvers to “have a willingness to pit themselves against difficult mathematical challenges under the assumption that they will be able to make progress on them”. The competence instrument inspired and motivated students to improve. When asked “In what way did your mathematical competence improve/ not improve?” some students responded that their motivation and effort improved. In the words of Kilpatrick et al. (2001, p. 132) “students who view ability as expandable in response to experience and training are more likely to seek out challenging situations and learn from them”.

In conclusion the competence instrument proved to be valuable in assessing the mathematical competence of students and motivating students to improve. Besides focusing students’ attention on their own ability and potential, the competence instrument’s value lies in informing students of what it means to be able to do mathematics.

Future research could address what needs to be done to ensure that the feedback loop is closed, in other words how students can improve their mathematical competence. Future research could also involve adapting the competence instrument to assess the mathematical competence of students in other mathematics modules.

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Addendum A: Questionnaire

Dear student,

The questionnaire that you are about to complete is voluntary and anonymous. The data will be used for research purposes in a project for improving assessment in the BSc Four Year Programme.

Your contribution is highly appreciated and we thank you for taking time to complete the questionnaire.

Please answer all questions as accurately as possible from your own perspective.

1. In what category do you classify your performance in WTW 154?

	80-100%
	70-80%
	60-70%
	50-60%
	40-50%
	Less than 40%

2. Answer the following questions as truthfully as possible:

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
A pass in WTW 154 is good enough for me					
I want to qualify for transfer to another Faculty					

3. Answer the following questions based on your opinion and experience of the new way of assessment, the mathematical competence feedback model:

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
The feedback was easy to interpret					
I understood the meaning of each category					
I spent time analysing my results					
I had to approach the lecturer to where I made mistakes					
I could relate my results back to the relevant question(s)					
I could identify my mathematical strengths/ shortcomings					
The feedback helped direct my focus, when preparing for tests					
The feedback on my mathematical competence made a difference in my approach to answering questions					

4. Thinking back, and comparing your results for the last three tutorial tests, do you think your mathematical competence improved?

	Yes
	No

5. In what way did your mathematical competence improve/ not improve?

6. Why do you think this is?

7. Did you find being assessed on your mathematical competence useful?

	Yes
	No

8. If you answered yes, explain why did you find it useful?

   If you answered no, explain why you didn't find it useful?

9. Comments

10. What did you like?

11. What did you not like?

Addendum B: Ethical clearance

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

ETHICS COMMITTEE
Faculty of Natural and Agricultural Sciences

03 June 2013
Prof A Harding
Department of Mathematics
University of Pretoria
Pretoria
0002

Dear Prof Harding

EC130514-044: Formative assessment feedback on the mathematical competence of students

This protocol conforms to the requirements of the NAS Ethics Committee.

Kind regards

Prof NH Casey
Chairman: Ethics Committee

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