



A new kind of polynomials for finite groups

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Abstract

Let Property X be a certain property of some finite groups; for instance, nilpotent, supersolvable, solvable et cetera. The Thompson-like problem asks whether for two finite groups G_1 and G_2 of the same order type, does G_2 always satisfy Property X if G_1 satisfies Property X ? Thompson-like problem has been solved where Property X is nilpotent. In this paper, we will introduce a new kind of polynomials for finite groups, which we shall call ‘the order polynomials’ and use it to propose another way of solving the Thompson-like problem, where Property X is nilpotent. Furthermore, we examine Thompson-like problem when Property X is supersolvable, and give infinitely many counterexamples to show that the answer to the problem is in the negative there.

Keywords Thompson’s problem · Same order type · Finite groups · Nilpotent · Supersolvable · Solvable · Polynomials

Mathematics Subject Classification 20D15 · 20D60 · 20F16 · 05E16

1 Introduction

All groups considered in this paper are finite. Let G be a finite group. The set of prime divisors of $|G|$ is denoted by $\pi(G)$. The set of all element orders of G is denoted by $\pi_e(G)$. Given $i \in \pi_e(G)$, we write $m_i(G) := |\{g \in G : o(g) = i\}|$. Two finite groups G_1 and G_2 are said to be of the *same order type* if and only if $\pi_e(G_1) = \pi_e(G_2)$ and $m_t(G_1) = m_t(G_2)$ for all $t \in \pi_e(G_1)$.

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When two groups G_1 and G_2 are of the same order type, it is clear that $|G_1| = |G_2|$. Vasil’ev et al. [12] proved that if G is any finite group and S is a finite simple group such that $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$, then $G \cong S$. The well-known Thompson’s problem (given as Problem 12.37 of [8]) asks whether for two finite groups G_1 and G_2 of the same order type, is G_2 necessarily solvable if G_1 is solvable? The answer to this question is known for nilpotent groups; see [10].

Let Property X be a certain property of some finite groups; for instance, nilpotent, supersolvable, solvable et cetera. The Thompson-like problem asks whether for two finite groups G_1 and G_2 of the same order type, does G_2 always satisfy Property X if G_1 satisfies Property X ? Thompson-like problem has been solved where Property X is nilpotent; see [10]. In this paper, we will introduce a new kind of polynomials, which we shall call ‘the order polynomials’ and use it to propose another way of solving the Thompson-like problem, where Property X is nilpotent. Furthermore, we shall show that the answer to Thompson-like problem is in the negative when Property X is supersolvable.

2 Order polynomials and some properties

In this section, we will introduce a special kind of polynomials, which we shall call ‘the order polynomials’. We then study some properties of it.

Definition Let G be a finite group. We denote the *Order polynomial* of G by $\mathcal{P}(G, x)$, and define it as

$$\mathcal{P}(G, x) := \sum_{g \in G} x^{\circ(g)}. \tag{1}$$

Remark It is clear that $\mathcal{P}(G, 1) = |G|$. Upon differentiating (1) and substituting 1 for x , we have that

$$\mathcal{P}'(G, 1) = \left[\frac{d}{dx} \mathcal{P}(G, x) \right] \Big|_{x=1} = \sum_{g \in G} \circ(g).$$

Definition Let G_1 and G_2 be two finite groups. Suppose $\pi_e(G_1) = \{a_0, a_1, a_2, \dots, a_k\}$ and $\pi_e(G_2) = \{b_0, b_1, b_2, \dots, b_l\}$, where $a_0 = 1 = b_0$. Then $\mathcal{P}(G_1, x) = m_{a_0}x^{a_0} + m_{a_1}x^{a_1} + \dots + m_{a_k}x^{a_k} = \sum_{i=0}^k m_{a_i}x^{a_i}$, where m_{a_i} is the number of elements of order a_i in G_1 . Similarly, $\mathcal{P}(G_2, x) = m_{b_0}x^{b_0} + m_{b_1}x^{b_1} + \dots + m_{b_l}x^{b_l} = \sum_{j=0}^l m_{b_j}x^{b_j}$, where m_{b_j} is the number of elements of order b_j in G_2 . We define $\mathcal{P}(G_1, x) \times \mathcal{P}(G_2, x)$ as follows:

$$\mathcal{P}(G_1, x) \times \mathcal{P}(G_2, x) = \sum_{i=0}^k \sum_{j=0}^l m_{a_i} m_{b_j} x^{[a_i, b_j]},$$

where $[a_i, b_j]$ is the least common multiple of a_i and b_j .

Example 1 We know that $\pi_e(C_6) = \{1, 2, 3, 6\}$, with $m_1 = 1, m_2 = 1, m_3 = 2$ and $m_6 = 2$. One can easily use GAP [7] to obtain that $\pi_e(A_5) = \{1, 2, 3, 5\}$, with $m_1 = 1, m_2 = 15, m_3 = 20$ and $m_5 = 24$. Therefore, $\mathcal{P}(A_5, x) \times \mathcal{P}(C_6, x) = x + 15x^2 + 20x^3 + 24x^5 + x^2 + 15x^2 + 20x^6 + 24x^{10} + 2x^3 + 30x^6 + 40x^3 + 48x^{15} + 2x^6 + 30x^6 + 40x^6 + 48x^{30} = x + 31x^2 + 62x^3 + 24x^5 + 122x^6 + 24x^{10} + 48x^{15} + 48x^{30}$.

Definition For a nontrivial finite group G , we say $\mathcal{P}(G, x)$ is *decomposable* if there exist nontrivial groups A and B such that $\mathcal{P}(G, x) = \mathcal{P}(A, x) \times \mathcal{P}(B, x)$. If $\mathcal{P}(G, x)$ is not decomposable, then $\mathcal{P}(G, x)$ is *indecomposable*.

Example 2 There are subgroups H_1 and H_2 of $A_5 \times C_6$ such that $H_1 \cong A_5$ and $H_2 \cong C_6$. From Example 1, we know that $\mathcal{P}(A_5, x) = x + 15x^2 + 20x^3 + 24x^5$ and $\mathcal{P}(C_6, x) = x + x^2 + 2x^3 + 2x^6$. Using Magma [5] or GAP [7], we see that $\pi_e(A_5 \times C_6) = \{1, 2, 3, 5, 6, 10, 15, 30\}$, with $m_1 = 1, m_2 = 31, m_3 = 62, m_5 = 24, m_6 = 122, m_{10} = 24, m_{15} = 48$ and $m_{30} = 48$. So, $\mathcal{P}(A_5 \times C_6, x) = x + 31x^2 + 62x^3 + 24x^5 + 122x^6 + 24x^{10} + 48x^{15} + 48x^{30}$. Since $\mathcal{P}(A_5 \times C_6, x)$ gotten here is the same as $\mathcal{P}(A_5, x) \times \mathcal{P}(C_6, x)$ gotten from Example 1, we conclude that $\mathcal{P}(A_5 \times C_6, x)$ is decomposable.

Proposition 3 Let G be a finite group and $G \neq 1$. If $G = A \times B$, then $\mathcal{P}(G, x) = \mathcal{P}(A, x) \times \mathcal{P}(B, x)$.

Proof Let $\pi_e(A) = \{a_0 = 1, a_1, \dots, a_n\}$, $\pi_e(B) = \{b_0 = 1, b_1, \dots, b_r\}$. Then

$$\pi_e(A \times B) = \{1, a_1, \dots, a_n, b_1, \dots, b_r, [a_1, b_1], \dots, [a_1, b_r], \dots, [a_n, b_1], \dots, [a_n, b_r]\}.$$

The rest of the proof follows from $m_{c_k} x^{c_k} = \sum_{[a_i, b_j]=c_k} m_{a_i} x^{a_i} m_{b_j} x^{b_j}$. □

Remark 4 The converse of Proposition 3 is not necessarily true; i.e., it is possible that $\mathcal{P}(G, x) = \mathcal{P}(A, x) \times \mathcal{P}(B, x)$, where $G \not\cong A \times B$. Let $G := \langle a, b \mid a^4 = b^2 = (aba)^2 = (ba^{-1})^4 = (baba^{-1})^2 = 1 \rangle \cong (C_4 \times C_2) \times C_2 \cong \text{SmallGroup}(16, 3)$. Here, $\pi_e(G) = \{1, 2, 4\}$, with $m_1 = 1, m_2 = 7$ and $m_4 = 8$. We define group $H := \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle \cong (C_4 \times C_2) \times C_2 \cong \text{SmallGroup}(16, 10)$. Here, $\pi_e(H) = \{1, 2, 4\}$, with $m_1 = 1, m_2 = 7$ and $m_4 = 8$. It is clear that $\mathcal{P}(H, x) = x + 7x^2 + 8x^4 = \mathcal{P}(G, x)$. By Proposition 3, we have that $\mathcal{P}(H, x) = \mathcal{P}(C_4 \times C_2, x) \times \mathcal{P}(C_2, x)$. Thus, $\mathcal{P}(G, x) = \mathcal{P}(C_4 \times C_2, x) \times \mathcal{P}(C_2, x)$. But $G \not\cong (C_4 \times C_2) \times C_2$.

Remark 5 Observe that

$$\int_0^1 \mathcal{P}(G, x) dx = \sum_{g \in G} \frac{1}{\circ(g) + 1}$$

and

$$\int_0^1 \frac{1}{x} \mathcal{P}(G, x) dx = \sum_{g \in G} \frac{1}{\circ(g)}.$$

The latter “ $\sum_{g \in G} \frac{1}{\circ(g)}$ ” has been studied by some authors; for instance, see [4].

3 Nilpotency on Thompson-like problem via Order polynomials

In 1895, Frobenius [6] proved that if n divides the order of G , then the number of solutions of $x^n = 1$ is a multiple of n . He conjectured that if the number of solutions of $x^n = 1$ is exactly n , then the set of solutions form a characteristic subgroup of G . His conjecture was proved in 1991 (see [9]). We shall refer to it as the Frobenius theorem. Considering this fact, we now present the following key theorem.

Theorem 6 *Let G and H be the same order type groups and $G = A \times B$ with $\gcd(|A|, |B|) = 1$, then there exist normal subgroups C and D of H such that $H = C \times D$, where C is of the same order type as A , and D is of the same order type as B .*

Proof As $|A| = \gcd(|A|, |G|)$, the number of solutions of $\{x \in G : x^{|A|} = 1\}$ is exactly $|A|$. By the Frobenius theorem and the fact that every characteristic subgroup is normal, we conclude that H has a normal subgroup C of size $|A|$. Similarly, as $|B| = \gcd(|B|, |G|)$, we deduce that H has a normal subgroup D of size $|B|$. Now, C and D are normal in H , $\gcd(|A|, |B|) = 1$, $|C| = |A|$, $|D| = |B|$ and $|G| = |A||B| = |H|$. So, $|H| = |C||D|$ and $\gcd(|C|, |D|) = 1$. Therefore H is the direct product of C and D . Thus, C has the same order type as A , and D has the same order type as B . \square

Theorem 7 *Let G be a nilpotent group of order $n = p_1^{k_1} \cdots p_t^{k_t}$, where each p_i is a prime number for $i \in \{1, \dots, t\}$. Then $\mathcal{P}(G, x) = \mathcal{P}(\text{Syl}_{p_1}(G), x) \times \cdots \times \mathcal{P}(\text{Syl}_{p_t}(G), x)$.*

Proof Since G is nilpotent, we have that $G = \text{Syl}_{p_1}(G) \times \cdots \times \text{Syl}_{p_t}(G)$. By induction and Proposition 3 therefore, $\mathcal{P}(G, x) = \mathcal{P}(\text{Syl}_{p_1}(G), x) \times \cdots \times \mathcal{P}(\text{Syl}_{p_t}(G), x)$. \square

Corollary 8 *If G is a nilpotent group which is not a p -group, then $\mathcal{P}(G, x)$ is decomposable.*

Corollary 9 *Let G and H be two groups having the same order type. If G is a nilpotent group, then H is nilpotent too.*

Proof By hypothesis, $|G| = |H|$, $\pi_e(G) = \pi_e(H)$ and $m_t G = m_t(H)$ for every $t \in \pi_e(G)$. So, $\mathcal{P}(G, x) = \mathcal{P}(H, x)$. Let G be a nilpotent group of order $p_1^{k_1} \times \cdots \times p_t^{k_t}$, where each p_i is a prime number for $i \in \{1, \dots, t\}$. The proof is easy when G is a p -group. Suppose G is not a p -group. Then

$$\mathcal{P}(G, x) = \mathcal{P}(\text{Syl}_{p_1}(G), x) \times \cdots \times \mathcal{P}(\text{Syl}_{p_t}(G), x) = \mathcal{P}(H, x).$$

By Theorem 6 and induction, we deduce that for each $i \in \{1, \dots, t\}$, there exists normal subgroup H_i with $\mathcal{P}(\text{Syl}_{p_i}(G), x) = \mathcal{P}(H_i, x)$. Thus, $H = H_1 \times \cdots \times H_t$. Since H_i is nilpotent for each $i \in \{1, \dots, t\}$ and the direct product of nilpotent groups is nilpotent, we conclude that H is nilpotent. \square

4 Supersolvability counterexamples on Thompson-like problem

Corollary 9 above shows that one can answer Thompson’s question affirmatively if asked it specifically for just two finite nilpotent groups; i.e., that the answer to Thompson-like problem is positive (in affirmation) when Property X is nilpotent. Next, we give counterexamples to show that one cannot answer Thompson’s question affirmatively if asked specifically for just two supersolvable groups. Let

$$G_1 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1, (ab)^4 = (bc)^2 = (bd)^2 = 1, ac = ca, ad = da, dc = cd \rangle$$

and

$$G_2 = \langle x, y, z \mid x^2 = y^2 = z^3 = 1, (xy)^4 = (xz)^2 = (xyz^2)^4 = 1, xyzyxy = z, (yz^2)^2(yz)^2 = 1 \rangle.$$

Note that $G_1 \cong \text{SmallGroup}(72,35)$ and $G_2 \cong \text{SmallGroup}(72,40)$. One can use Magma [5] or GAP [7] to see that G_1 and G_2 have the same order type; in particular, each has twenty-one involutions, eight elements of order 3, eighteen elements of order 4 and twenty-four elements of order 6. But G_1 is supersolvable while G_2 is not supersolvable. Using G_1 and G_2 above, one can see that given $n \in \mathbb{N}$, the two groups

$$G_{72nA} := C_n \times G_1$$

and

$$G_{72nB} := C_n \times G_2$$

are of the same order type, where G_{72nA} is supersolvable and G_{72nB} is not supersolvable. This tells us that there are infinitely many finite groups that can be used to show that the answer to Thompson-like problem is in the negative when Property X is supersolvable.

A final question we will address here is whether the Thompson-like problem is true when Property X is replaced with any of solvable or simple? The latter case (for simple groups) is settled by Observation 6 of [1] and the main theorem of [12]. Observation 6 of [1] implies that there are no non-isomorphic simple groups S and G with the same order type. On the other hand, the main theorem of [12] is that if G is any finite group and S is a finite simple group such that $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$, then $G \cong S$. For the former case (for solvable groups), Piwek [11] gave a counterexample to [8, Problem 12.37]. An interested reader may see [2, 3], where another polynomial (Sylow polynomial) was defined for finite groups. Moreover, some differential and integral notions were introduced in [3] to group theory, and used to give various characterizations of the smallest nonsolvable group in [2, 3].

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References

1. Anabanti, C.S.: A question of Malinowska on sizes of finite nonabelian simple groups in relation to involution sizes. *C. R. Math. Acad. Sci. Paris* **358**(11–12), 1135–1138 (2020)
2. Anabanti, C.S., Asboei, A.K.: On the integral of the Sylow polynomial of a finite simple group. *Siberian Mathematical Journal* **65**(6), 1402–1406 (2024)
3. Anabanti, C.S., Asboei, A.K.: Two characterizations of the smallest non-solvable group. *Communications in Algebra* **52**(12), 5138–5142 (2024)
4. Azad, M.B., Khosravi, B., Rashidi, H.: On the sum of the inverses of the element orders in finite groups. *Communications in Algebra* **51**(2), 694–698 (2023)
5. Bosma, Wieb, Cannon, John, Playoust, Catherine: The Magma algebra system. I. The user language. *J. Symbolic Comput.* **24**, 235–265 (1997)
6. Frobenius, G.: Verallgemeinerung des Sylow'schen Satzes *Sitzungsberichte der Königl. Preuß. Akad. der Wissenschaften zu Berlin*, 981–993 (1895)
7. The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.12.2; 2022, <https://www.gap-system.org>
8. Khukhro, E.I., Mazurov, V.D.: Unsolved Problems in Group Theory. The Kourovka Notebook, e-prints, July 2023. Available at [arXiv:1401.0300v27](https://arxiv.org/abs/1401.0300v27)
9. Iiyori, N., Yamaki, H.: On a conjecture of Frobenius. *Bulletin (New Series) of the American Mathematical Society* **25**(2), 413–416 (1991)
10. Shen, R.: On same order type groups. *SAÜ. Fen Bilimleri Dergisi* **15**, 156–158 (2011)
11. Piwek, P.: Solvable and non-solvable finite groups of the same order type, [arXiv:2403.02197v2](https://arxiv.org/abs/2403.02197v2) [math.GR]; <https://doi.org/10.48550/arXiv.2403.02197>
12. Vasil'ev, A.V., Grechkoseeva, M., Mazurov, V.D.: Characterization of the finite simple groups by spectrum and order. *Algebra i Logic* **48**(6), 385–409 (2009)

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