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Non-solvable groups with few vanishing elements

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ABSTRACT

Let G be a finite group with a trivial solvable radical. Let $\mathrm{Vo}(G)$ denote the set of the orders of vanishing elements of G, $\mathrm{Vo}_{pp}(G)$ denote the set of the orders of vanishing elements of G that are prime powers. Non-solvable groups which satisfy the one prime hypothesis on $\mathrm{Vo}(G)$ are studied. It is shown that G is almost simple or G has the socle $A_5 \times A_5$. We also consider groups G such that $|\mathrm{Vo}(G)\backslash \mathrm{Vo}_{pp}(G)| \leqslant 2$. This is a generalization of groups in which $\mathrm{Vo}(G) = \mathrm{Vo}_{pp}(G)$. Lastly, we study groups G such that $|\mathrm{Vo}(G)| \leqslant 5$. This is analogous to a corresponding problem on the size of character degrees set of non-solvable groups.

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1. Introduction

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G and let $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. An element $g \in G$ is a vanishing element of G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. A classical theorem of Burnside [18, Theorem 3.15] states that $\chi(g) = 0$ for every nonlinear $\chi \in \mathrm{Irr}(G)$. Let $\mathrm{Vo}(G)$ denote the set of the orders of vanishing elements of G, $\mathrm{Vo}_{pp}(G)$ be the subset of $\mathrm{Vo}(G)$ consisting of orders of vanishing elements that are prime powers and $\mathrm{Vo}_c(G) = \mathrm{Vo}(G)\backslash \mathrm{Vo}_{pp}(G)$.

We shall recall the definition of the m-prime hypothesis: Fix a positive integer m. Let X be a non-empty finite subset of positive integers associated to the group G. Then the group G is said to satisfy the m-prime hypothesis on X if, for distinct numbers $a, b \in X\backslash\{1\}$, the greatest common divisor $\gcd(a, b)$ is divisible by at most m primes counting multiplicity. Many authors have studied finite groups that satisfy the m-hypothesis on X where X is either the character degrees set of G, the set of conjugacy class sizes of G or character codegrees set of G (see for example [1, 2, 19] for latest articles on these problems). In this article, we study finite non-solvable groups that satisfy the one-prime hypothesis on $\mathrm{Vo}(G)$. Note that the one-prime hypothesis on $\mathrm{Vo}(G)$ is a generalization of the problem studied in [22]. Our results show that $G/\mathrm{Sol}(G)$ is either almost simple or has $A_5 \times A_5$ as its socle. We describe the socle of each $G/\mathrm{Sol}(G)$. Here $\mathrm{Sol}(G)$ denotes the solvable radical of G.

Theorem A. Let G be a finite group with $\mathrm{Sol}(G) = 1$. If G satisfies the one-prime hypothesis for $\mathrm{Vo}(G)$, then either $A_5 \times A_5$ is the socle of G or G is an almost simple group with socle S and one of the following holds:

(a) $S \in \{J_1, A_7, A_8\}$;
(b) $S \cong {}^2\mathrm{B}_2(2^{2k+1})$, where $2^{2k+1} - 1 \in \{r, r^2rs\}$, $2^{2k+1} - 2^{k+1} + 1 \in \{t, t^2tu\}$ and $2^{2k+1} + 2^{k+1} + 1 \in \{v, v^2vw\}$ for distinct primes r, s, t, u, v, w.

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(c) S is isomorphic to some group in Table 1.

Take note that $A_5 \times A_5$ is an exception indeed. This is because $\mathrm{Vo}(A_5 \times A_5) = \{2, 3, 5, 6, 10, 15\}$ and so satisfies the one-hypothesis on the set of orders of vanishing of a group.

Finite groups G such that $\mathrm{Vo}(G) = \mathrm{Vo}_{pp}(G)$ were studied in [20, 27]. We consider a generalization of this property for non-solvable groups. In our next result, at most two elements of $\mathrm{Vo}(G)$ are allowed to not be prime powers:

Theorem B. Let G be a finite group with $\mathrm{Sol}(G) = 1$. If $|\mathrm{Vo}_c(G)| \leqslant 2$, then G is almost simple and one of the following holds:

(a) $S \in \{M_{11}, M_{12}, M_{22}, A_7, A_8, G_2(3)\}$;
(b) $S \cong {}^2B_2(2^{2k+1})$, with $|\mathrm{Vo}_c(H)| \leqslant 2$, where $H = H_1 \cup H_2 \cup H_3$, and H_i is a cyclic subgroup of order $2^{2k+1} - 1, 2^{2k+1} - 2^{k+1} + 1$ and $2^{2k+1} + 2^{k+1} + 1$.
(c) S is isomorphic to some group in Table 1.

Many authors have studied finite groups in with restrictions on the number of vanishing conjugacy classes. In particular, it was shown in [24] that if a group has three vanishing conjugacy classes then the group is solvable. Moreover, non-solvable groups with six vanishing conjugacy classes have been classified in [25]. In [21], the second author studied groups such that there exists only one vanishing conjugacy class with elements divisible by a fixed prime. We continue in this direction by studying non-solvable groups with a few orders of vanishing elements.

Table 1. Groups of Lie type with few orders of vanishing elements.

S	Conditions	A	B	C
$\mathrm{PSL}_2(q)$	$q \in \{5, 7, 8, 9, 11\}$	✓	✓	✓
$\mathrm{PSL}_2(q)$	$q = 16$	✓	×	✓
$\mathrm{PSL}_n(q)$	$(n, q) \in \{(2, 243), (3, 3)\}$	✓	✓	×
$\mathrm{PSL}_n(q)$	$(n, q) \in \{(2, 127), (2, 37), (2, 23), (2, 25), (3, 7)\}$	×	✓	×
$\mathrm{PSL}_3(q)$	$q \in \{4, 16\}$	✓	×	×
$\mathrm{PSU}_3(q)$	$q \in \{4, 8\}$	✓	✓	×
$\mathrm{PSU}_3(q)$	$q \in \{3, 5\}$	×	✓	×
$\mathrm{PSp}_4(q)$	$q = 3$	×	✓	×
$\mathrm{PSp}_4(q)$	$q = 4$	✓	×	×
$\mathrm{PSL}_2(2^f)$	$f > 3, 2^f - 1 = s$ and $2^f + 1 = 3r$	✓	✓	✓
$\mathrm{PSL}_2(3^f)$	$3^f + 1 = 4r$ and $3^f - 1 = 2s$	✓	✓	✓
$\mathrm{PSL}_2(q)$	$q = 2^{2^a} + 1, a > 2$ and $q + 1 = 3r$	×	✓	×
$\mathrm{PSL}_2(q)$	$q = 4r \pm 1 = 2st \mp 1$	✓	✓	×
$\mathrm{PSL}_2(q)$	$q = rs - 1$ and $q = tu + 1$	✓	✓	×
$\mathrm{PSL}_2(q)$	$q = 2 \cdot 3^b \pm 1$ with $(q \pm 1)/2 \in \{2r, 4r, 2r^2\}$	×	✓	×
$\mathrm{PSU}_3(2^f)$	f odd, $q^2 = 3rs + 1$ and $q^2 - q + 1 \in \{3t, 3t^2, 3tu\}$	✓	×	×

Theorem C. Let G be a finite group with $\mathrm{Sol}(G) = 1$. If $|\mathrm{Vo}(G)| \leqslant 5$, then G is almost simple and one of the following holds:

(a) $S \in \{A_7, {}^2B_2(8)\}$;
(b) S is isomorphic to some group in Table 1.

Note that this is an analogue of the work in [23] and [14] in which the authors studied non-solvable groups such that $|\mathrm{cd}(G)| \leqslant 6$. Their results show that $G/\mathrm{Sol}(G)$ is almost simple with a socle isomorphic to either $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(4)$ or ${}^2B_2(q^2)$. Our results points to some evidence that the more the prime divisors of orders of non-abelian composition factors of a group G are, the bigger the numbers $|\mathrm{Vo}_c(G)|$ and $|\mathrm{Vo}(G)|$.

Our proof relies on the classification of finite simple groups and the existence of p-defect zero characters in finite simple groups for almost all prime divisors p of the order of the group. We could have considered the converse of our results. However, the infinite families make the work of considering all almost simple groups too technical. In Section 2, we collect some results on vanishing elements of simple groups and on the spectra of finite simple groups. In the last section, we prove our results by reducing the problems to almost simple groups and then look at the finite groups on a case by case.

Table 1 has the information concerning which groups have our required properties for Theorems A–C. Take note that a tick in the column A, B, C means that a group satisfies the property in Theorems A, B, C, respectively

2. Preliminaries

In this section we shall list some properties of vanishing elements, number theory results and orders of simple groups of Lie type.

A non-linear irreducible character χ of G is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a result of Brauer (see [18, Theorem 8.17]), if χ is an irreducible character of p-defect zero of G, then $\chi(g) = 0$ whenever p divides the order of g in G. The existence of p-defect zero characters is guaranteed in finite simple groups G for all primes $p \geqslant 5$ dividing $|G|$ as the following result shows:

Lemma 2.1. [12, Corollary 2.2] *Let G be a non-abelian finite simple group and p be a prime. If G is a finite group of Lie type, or if* $p \geqslant 5$, *then there exists* $\chi \in \mathrm{Irr}(G)$ *of p-defect zero.*

Lemma 2.2. [5, Lemma 2.2] *Let G be a finite group, N a normal subgroup of G and p a prime. If N has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element in G.*

Lemma 2.3. [3, Lemma 5] *Let G be a finite group, and* $N = S_1 \times \cdots \times S_k$ *a minimal normal subgroup of G, where every* S_i *is isomorphic to a non-abelian simple group S. If* $\theta \in \mathrm{Irr}(S)$ *extends to* $\mathrm{Aut}(S)$, *then* $\varphi = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ *extends to G.*

The next result from number theory is useful in our proofs.

Lemma 2.4. [16, Lemma 1.3] *Let q be a power of a prime. Then the following hold:*

(a) $q^2 - 1$ *has at most two different prime divisors if and only if* $q \in \{2, 3, 4, 5, 7, 8, 9, 17\}$.
(b) $q^4 - 1$ *has at most three different prime divisors if and only if* $q \in \{2, 3, 4, 5, 7, 9\}$.

Lemma 2.5. *Let G be an almost simple group whose socle N is a simple group of Lie type. Suppose that N contains an element of order a and let r, s, t be distinct primes.*

(i) If $a \in \{rst, r^3s\}$*, then the following holds:*

(a) There exist vanishing elements $x, y \in N$ *of* G *such that* $\mathrm{ord}(x) \neq \mathrm{ord}(y)$ *and either* $r^2 \mid \gcd(\mathrm{ord}(x), \mathrm{ord}(y))$ *or* $rs \mid \gcd(\mathrm{ord}(x), \mathrm{ord}(y))$.

(b) $|\mathrm{Vo}_c(G)| \geqslant 3$.

(c) $|\mathrm{Vo}(G)| \geqslant 6$.

(ii) If $a = r^2s$*, then (a) and (c) above hold.*

Proof. It is easy to show that (a) holds in both cases. If $a = rst$, then G has vanishing elements of orders rst, rs, st, rt, r, s, t by Lemmas 2.2 and 2.3, and so (b) and (c) also hold. If $a = r^3s$, then G has vanishing elements of orders $r^3s, r^2s, rs, r^2, r, s$, which means that (b) and (c) hold. The result then follows. □

2.1. Classical groups

Let $[a_1, a_2, \ldots, a_n]$ denote the lowest common multiple of the positive integers $a_1, a_2, \ldots, a_n$. By Lemmas 2.2 and 2.3, we only need to consider the spectra of simple groups of Lie type to prove our results. The spectra of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ are well known and we list some of the orders below:

Lemma 2.6. *Let* $n \geqslant 2$ *and let* q *be a power of a prime* p, $\epsilon \in \{\pm 1\}$. *Let* $G \cong \mathrm{PSL}_n^{\epsilon}(q)$ *and* $d = \gcd(n, q-1)$. *Then the divisors of the numbers below are orders of some of the elements of* G*:*

(i) $[q^{n_1} - \epsilon^{n_1}, q^{n_2} - \epsilon^{n_2}]/\gcd(n/\gcd(n_1, n_2), q - \epsilon)$, $n_1 + n_2 = n$;

(ii) $p^t(q^{n_1} - \epsilon^{n_1})/d$ *for* $t, n_1 > 0$ *such that* $p^{t-1} + 1 + n_1 = n$;

(iii) $p^t[q^{n_1} - \epsilon^{n_1}, \ldots, q^{n_s} - \epsilon^{n_s}]$, *where we have that* $s \geqslant 2$, $t, n_i > 0$ *such that* $p^{t-1} + 1 + n_1 + \cdots + n_s = n$.

Proof. This follows from [7, Corollary 3]. □

The next set of results list some of the orders of elements of symplectic and orthogonal groups.

Lemma 2.7. *Let* q *be a power of* 2 *and let* $G \cong \mathrm{Sp}_{2n}(q) \cong \Omega_{2n+1}(q)$, *where* $n \geqslant 2$. *Then the divisors of the following numbers are orders of some of the elements of* G*:*

(i) $[q^{n_1} + \epsilon_1, q^{n_2} + \epsilon_2, \ldots, q^{n_s} + \epsilon_s]$ *for all* $s \geqslant 1$, $\epsilon_j \in \{\pm 1\}$ *and* $n_i > 0$ *with* $n_1 + n_2 + \cdots + n_s = n$;

(ii) $2^t[q^{n_1} + \epsilon_1, q^{n_2} + \epsilon_2, \ldots, q^{n_s} + \epsilon_s]$, *where* $s \geqslant 1$, $t \geqslant 2$, $\epsilon_j \in \{\pm 1\}$ *and* $n_i > 0$ *with* $2^{t-2} + 1 + n_1 + n_2 + \cdots + n_s = n$.

Proof. This follows from [8, Corollary 3]. □

Lemma 2.8. *Let* $G \cong \mathrm{PSp}_{2n}(q)$, *where* $n \geqslant 2$ *and let* q *be a power of an odd prime* p. *Then the divisors of the following numbers are orders of some of the elements of* G*:*

(i) $[q^{n_1} + \epsilon_1, q^{n_2} + \epsilon_2, \ldots, q^{n_s} + \epsilon_s]$ *for all* $s \geqslant 2$, $\epsilon_j \in \{\pm 1\}$, $1 \leqslant i \leqslant s$ *and* $n_i > 0$ *with* $n_1 + n_2 + \cdots + n_s = n$;

(ii) $p^t[q^{n_1} + \epsilon_1, q^{n_2} + \epsilon_2, \ldots, q^{n_s} + \epsilon_s]$, *where* $s \geqslant 1$, $\epsilon_j \in \{\pm 1\}$, $1 \leqslant i \leqslant s$ *and* $t, n_i > 0$ *with* $p^{t-1} + 1 + 2n_1 + 2n_2 + \cdots + 2n_s = 2n$.

Proof. This follows from [8, Corollary 2]. □

Lemma 2.9. *Let* q *be a power of an odd prime* p *and let* $G \cong \Omega_{2n+1}(q)$, *where* $n \geqslant 3$. *Then the divisors of the following numbers are orders of some of the elements of* G*:*

(i) $p^t(q^{n_1} \pm 1)/2$ for all t and n_1 with $p^{t-1} + 1 + 2n_1 = 2n$;
(ii) $p^t[q^{n_1} + \epsilon_1, q^{n_2} + \epsilon_2, \dots, q^{n_s} + \epsilon_s]$ for all $s \geqslant 2$, $\epsilon_i \in \{\pm 1\}$ and $n_i > 0$ with $p^{t-1} + 1 + 2n_1 + 2n_2 + \cdots + 2n_s = 2n$.

Proof. This follows from [8, Corollary 6]. □

Lemma 2.10. *Let q be a power of 2 and let $G \cong \Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$. Then the divisors of the following numbers are orders of some of the elements of G:*

(i) $[q^{n_1} + 1, q^{n_2} + 1, \dots, q^{n_r} + 1, q^{n_{r+1}} - 1, q^{n_{r+2}} - 1, \dots, q^{n_s} - 1]$ for all $s \geqslant 1$, $r \geqslant 1$ and $n_i > 0$ with $n_1 + n_2 + \cdots + n_s = n$, where r is even if $\epsilon = +$ and odd if $\epsilon = -$;
(ii) $2^t[q^{n_1} + 1, q^{n_2} + 1, \dots, q^{n_r} + 1, q^{n_{r+1}} - 1, q^{n_{r+2}} - 1, \dots, q^{n_s} - 1]$ for all $s \geqslant 1$, $r \geqslant 1$ and $n_i > 0$ with $2^{t-2} + 2 + n_1 + n_2 + \cdots + n_s = n$.

Proof. This follows from [8, Corollary 4]. □

Lemma 2.11. *Let q be a power of an odd prime p and let $G \cong P\Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$, $\epsilon \in \{\pm 1\}$ and $\gcd(4, q^n - \epsilon) = 4$. For $t \geqslant 1$, let $n(t) = (p^{t-1} + 3)/2$. Then divisors of the following numbers are orders of some of the elements of G:*

(i) $p^t[q^{n_1} + 1, q^{n_2} + 1, \dots, q^{n_r} + 1, q^{n_{r+1}} - 1, q^{n_{r+2}} - 1, \dots, q^{n_s} - 1]$ for all $s \geqslant 2$, $r \geqslant 1$ and $n_i > 0$ with $n(t) + n_1 + n_2 + \cdots + n_s = n$;
(ii) $p[q \pm 1, (q^{n-2} - \epsilon)/2]$.

Proof. This follows from [8, Corollary 9]. □

2.2. Exceptional groups

We now consider exceptional groups of Lie type.

Lemma 2.12. *Let $G \cong G_2(q), q \geqslant 3$ and let q be a power of a prime p. Then divisors of the following numbers are orders of some of the elements of G:*

(i) $8, 12, 2(q \pm 1), q^2 - 1, q^2 \pm q + 1$ for $p = 2$;
(ii) $p^2, p(q \pm 1), q^2 - 1, q^2 \pm q + 1$ for $p = 3, 5$;
(iii) $p(q \pm 1), q^2 - 1, q^2 \pm q + 1$ for $p > 5$.

Proof. This follows from [26, Lemma 1.4]. □

Lemma 2.13. *Let $G \cong {}^3D_4(q)$, with q a power of a prime p. Then the divisors of the following numbers are orders of some of the elements of G:*

(i) $p(q^3 \pm 1)$;
(ii) $4(q^2 \pm q + 1)$ and 8 if $p = 2$;
(iii) p^2 if $p \in \{3, 5\}$.

Proof. This follows from [13, Theorem 3.2]. □

2.3. Simple groups with few prime divisors

We need some information about simple groups S such that $|\pi(S)| \leqslant 5$. These have been classified in [15], [6], and [17]. We list some of them here for ease of reference.

Theorem 2.14. *Let S be a finite simple group. Then the following holds:*

(a) If $|\pi(S)| = 3$*, then S is isomorphic to*

$$A_5, A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), \mathrm{PSL}_2(17), \mathrm{PSL}_3(3), \mathrm{PSU}_3(3) \text{ and } \mathrm{PSU}_4(2);$$

(b) If $|\pi(S)| = 4$ *and S is not isomorphic to* $\mathrm{PSL}_2(q)$*, then S is isomorphic to*

$$A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, \mathrm{PSL}_3(4), \mathrm{PSL}_3(8), \mathrm{PSL}_3(5), \mathrm{PSL}_3(7), \mathrm{PSL}_3(17), \mathrm{PSL}_4(3), \mathrm{PSO}_5(4), \mathrm{PSO}_5(8), \mathrm{PSO}_5(5), \mathrm{PSO}_5(7), \mathrm{PSO}_7(2), \mathrm{P}\Omega_8^+(2), \mathrm{G}_2(3), \mathrm{PSU}_3(4), \mathrm{PSU}_3(8), \mathrm{PSU}_3(9), \mathrm{PSU}_3(5), \mathrm{PSU}_3(7), \mathrm{PSU}_4(3), \mathrm{PSU}_5(2), {}^2\mathrm{B}_2(8), {}^2\mathrm{B}_2(32), {}^3\mathrm{D}_4(2), {}^2\mathrm{F}_4(2)';$$

(c) If $|\pi(S)| = 5$*, then the following holds:*

- *(i) If* $S \cong \mathrm{PSL}_2(q)$*, then* $|\pi(q^2-1)| = 4$*;*
- *(ii) If* $S \cong \mathrm{PSL}_3(q)$*, then* $|\pi[(q^2-1)(q^3-1)]| = 4$*;*
- *(iii) If* $S \cong \mathrm{PSU}_3(q)$*, then* $|\pi[(q^2-1)(q^3+1)]| = 4$*;*
- *(iv) If* $S \cong {}^2\mathrm{B}_2(2^{2m+1})$*, then* $|\pi[(2^{2m+1}-1)(2^{2m+1}-2^{k+2}+1)(2^{2m+1}+2^{k+2}+1)]| = 4$.

In part (b) of the result above, we excluded the case when $S \cong \mathrm{PSL}_2(q)$. We list this case in the following result.

Theorem 2.15. *Let* $S \cong \mathrm{PSL}_2(q)$ *and* $r, s > 3$ *be distinct prime divisors of* $|S|$*. If* $|\pi(S)| = 4$*, then one of the following holds:*

(a) $q \in \{16, 25, 31, 49, 81, 127, 243, 97^2, 577^2\}$*;*
(b) $q = 2^f, f > 3\ 2^f + 1 = 3r$ *and* $s = 2^f - 1$*;*
(c) $q = 3^f, f > 2, 4r = 3^f + 1$ *and* $2s = 3^f - 1$*;*
(d) $q = 2^{2^a} + 1$*, is a prime,* $a > 2$ *and* $3r = 2^{2^a-1} + 1$*, where* $2^a - 1$ *is a prime;*
(e) $q = 2^a 3^b + \delta$*, is a prime,* $r = 2^{a-1}3^b + \delta$ *where* $ab > 0$*,* $\delta \in \{\pm 1\}$*;*
(f) $q = 2 \cdot 3^{2^a} + 1$*, is a prime,* $3^{2^a} + 1 = 2r$ *where* $a > 0$*;*
(g) $q = 2 \cdot 3^a + 1$*, is a prime,* $3^a + 1 = 4r$*, where* $a > 0$*;*
(h) $q = 2 \cdot 3^a - 1$*, is a prime,* $3^a - 1 = 2r$*, where* $a > 0$.

3. Main results

We are now in a position to prove our main results. We will start with considering simple groups of Lie type with large rank, alternating groups of large degree and sporadic simple groups with large orders. Let $\mathfrak{S} = \{M_{11}, M_{12}, M_{22}, J_1\} \cup \{A_n, 5 \leqslant n \leqslant 10\} \cup \{\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSp}_4(q), {}^2B_2(q^2), G_2(3)\}$.

Theorem 3.1. *Suppose G is an almost simple group with socle S. If* $S \notin \mathfrak{S}$*, then the following holds:*

(a) There exist vanishing elements $x, y \in S$ *of G such that* $\mathrm{ord}(x) \nmid \mathrm{ord}(y)$ *and either* $p^2 \mid \gcd(\mathrm{ord}(x), \mathrm{ord}(y))$ *or* $pq \mid \gcd(\mathrm{ord}(x), \mathrm{ord}(y))$ *for some distinct primes p and q.*
(b) $|\mathrm{Vo}_c(G)| \geqslant 3$.
(c) $|\mathrm{Vo}(G)| \geqslant 6$.

Proof. Let S be a sporadic simple group. Suppose that S is isomorphic to one of the following groups: $HS, Ru, Suz, O'N, Co_1, Co_2, Co_3, Fi_{22}, HN, Ly, Th, Fi_{23}, J_4, Fi_{24}', B$ or M. Then S has elements of order 10, 15 and 20 by inspection using the Atlas [9]. These are vanishing elements of G by Lemmas 2.3 and 2.1. Hence (a) and (b) follows. Using the argument in [20, Proposition 3.1], we have that $|\mathrm{Vo}(G)| \geqslant 6$, with the exception of $Suz, O'N$ and J_4. Let us consider the exceptions. We have that these groups have

Table 2. Orders of elements of some sporadic groups.

S	$V \subseteq \text{Vo}(G)$	S	$V \subseteq \text{Vo}(G)$
J_2	$\{3, 5, 6, 10, 12\}$	M_{23}	$\{3, 4, 6, 8, 15\}$
${}^2F_4(2)'$	$\{2, 4, 6, 8, 10, 12\}$	J_3	$\{3, 5, 6, 9, 12, 15\}$
M_{24}	$\{3, 6, 10, 12, 15, 21\}$	M^cL	$\{5, 7, 10, 14, 15, 30\}$
He	$\{5, 7, 10, 14, 21, 28\}$		

elements of orders 5, 7, and 11, and hence are vanishing elements of G by Lemmas 2.3 and 2.1. The result then follows.

For the rest of the sporadic groups or ${}^2F_4(2)'$, it can be shown that there is a subset V of $\text{Vo}(G)$ that satisfies the properties (a)–(c). The appropriate V is given in Table 2.

Suppose that S is an alternating group A_n, $n \geqslant 11$. By Lemma 2.1, S has irreducible characters of 5-defect zero, 7-defect zero and 11-defect zero. Note that S has elements of order 5, 7, 10, 11, 15, 20. Using Lemma 2.2, we have that these elements are vanishing elements of G and the result follows.

Suppose that $S \cong \text{PSL}_n(q)$, where $n \geqslant 4$, q is a power of a prime p and $d = \gcd(n, q-1)$. Firstly, assume that $q = 2$. Since $\text{PSL}_4(2) \cong A_8$ is contained in $\mathfrak{S}$ and considering the character table of $\text{PSL}_5(2)$ in the Atlas [9] we may assume that $n \geqslant 6$. Using Lemma 2.6(ii), let $t = 2$. Then $n - 3 \geqslant 3$. Hence $4(q^{n-3} - 1)/d = 4(q^{n-3} - 1)$ is an element order of S, which means that there exist x and y such that $\text{ord}(x) \neq \text{ord}(y)$ and $4 \mid \gcd(\text{ord}(x), \text{ord}(y))$. This is (a). Let $t = 1$ in Lemma 2.6(ii). Then $n - 2 \geqslant 4$ and $2(q^{n-2} - 1)$ is an element order of S. It follows that $4(q^{n-3} - 1)$, $2(q^{n-3} - 1)$ and $2(q^{n-2} - 1)$ are three distinct composite orders and also that $|\text{Vo}(G)| \geqslant 6$. Assume that $q \geqslant 3$. Suppose that n is even. Let us consider the case when $n = 4$. Using the Atlas[9], we may assume that $q \geqslant 4$. Let $t = 1$ in Lemma 2.6(ii). Then $n_1 = n - 2 \geqslant 2$. We have that S has an element of order $p \cdot (q-1)/d \cdot (q+1)$. If $(q-1)/d > 1$, then we are done. Hence we may assume that $q - 1 = d$. This means that $q - 1$ divides 4 which implies that either $q = 3$ or $q = 5$. We have already considered $\text{PSL}_4(3)$. For $q = 5$, let $t = 1$ and $n_1 = 2$ in Lemma 2.6(ii). Then S has an element order $5(5^2 - 1)/4 = 2 \cdot 3 \cdot 5$ and the result follows by Lemma 2.5. Suppose that $n \geqslant 6$. If $t = 1$ in Lemma 2.6(ii), then $(n-2)/2 \geqslant 2$ and S has an element of order $p(q^{n-2} - 1)/d = p \cdot (q^{(n-2)/2} - 1)/d \cdot (q^{(n-2)/2} + 1)$. Suppose that $n \geqslant 5$ is odd. Then we consider two cases: when $t = 1$ in Lemma 2.6(ii) and so $n_1 \geqslant 3$, and when $t = 1$, $n_1 = n - 3 \geqslant 2$ is even and $n_2 = n_s = 1$ in Lemma 2.6(iii). It follows that S has elements of orders $p(q^{n_1-1})/d = p \cdot (q-1)/d \cdot (q^{n_1-1} + q^{n_1-2} + \cdots + q + 1)$ and $p(q^{n_1} - 1) = p(q^{n_1/2} - 1)(q^{n_1/2} + 1)$, respectively. Our result then follows.

Suppose that $S \cong \text{PSU}_n(q)$, where $n \geqslant 4$, q is a power of a prime p and $d = \gcd(n, q+1)$. Assume that $q = 2$. Note that $\text{PSU}_4(2) \cong \text{PSp}_4(3)$ which is in $\mathfrak{S}$. Using the character tables of $\text{PSU}_5(2)$ and $\text{PSU}_6(2)$ in the Atlas [9], we may assume that $n \geqslant 7$. Then we consider divisors of orders of elements of S such that $t = 1$ and $t = 2$ in Lemma 2.6(ii). We have that elements of orders $2(2^{n-2} - 1)$ and $4(2^{n-3} - 1)$. It follows that (a)–(c) of our theorem holds. We may assume that $q \geqslant 3$. Suppose that n is even. We first consider when $n = 4$. Using the character table of $\text{PSU}_4(3)$ in the Atlas [9], we may assume that $q \geqslant 4$. Let $t = 1$ in Lemma 2.6(ii). Then $n_1 = 2$ and so we have elements of order $p \cdot (q-1) \cdot (q+1)/d$. If $d \neq q + 1$, then (a)–(c) follows. Let $d = q + 1$. Then $q + 1$ divides 4, which means $q = 3$, a case we have already considered. Suppose that $n \geqslant 6$ and $q \geqslant 3$. If we pick $t = 1$ in Lemma 2.6(ii), then $n_1 = n - 2 \geqslant 4$ and $p(q^{(n-2)/2} - 1)(q^{(n-2)/2} + 1)/d$ is the order of an element of S. The result then follows. Suppose that $n \geqslant 5$ is odd and let $t = 1$ in Lemma 2.6(ii). Then $n_1 = n - 2 \geqslant 3$ is odd and $p(q^{n-2} + 1)/d$ is an order of an element of S. If $d \neq q + 1$, then the result follows. If $d = q + 1$, then $q + 1 \mid n$ and so q is even, that is, $p = 2$. If we let $t = 2$ in Lemma 2.6(ii), then $n_1 = n - 3 \geqslant 2$ and properties (a) - (c) hold.

Let $S \cong \text{Sp}_{2n}(q) \cong \Omega_{2n+1}(q)$, where $n \geqslant 3$ and q be a power of 2. Suppose that $q = 2$. Then considering the character tables of $\text{PSp}_6(2)$ and $\text{PSp}_8(2)$ in the Atlas [9], we may assume that $n \geqslant 5$. Choose $t = 3$ and $n_1 = n - 3 \geqslant 2$ in Lemma 2.7(ii). Then S has an elements of orders $8(q^{n-3} + 1)$ and $8(q^{n-3} - 1)$, and the result follows. We may assume that $q \geqslant 4$. Choose that $t = 2$ and $n_1 = n - 2 \geqslant 1$

Table 3. Orders of elements of some exceptional groups of Lie type.

S	Orders	S	Orders
$F_4(q)$	$p(q^2+1)(q+1)$	$E_7(q)$	q^6-1, if $p=2$ $p(q^5-1)$, if $p \neq 2$
$E_6(q)$	$p(q^6-1)/d(q-1)$	$E_8(q)$	$(q+1)(q^2+q+1)(q^5-1)$
${}^2E_6(q)$	$p(q^6-1)/d(q+1)$		

in Lemma 2.7(ii). Then S has an elements of orders $4(q^{n-2}+1)$ and $4(q^{n-2}-1)$. It follows that S satisfy properties (a)–(c).

Let $S \cong \mathrm{PSp}_{2n}(q)$, where $n \geqslant 3$ and q be a power of an odd prime p. Suppose that $q=3$. Using the character table of $\mathrm{PSp}_6(3)$ in the Atlas, we may assume that $n \geqslant 4$. Choose $t=2$ and $n_1=n-2 \geqslant 2$ in Lemma 2.8(ii). Then $9(3^{2n-4}-1)$ and $9(3^{2n-4}+1)$ are orders of elements of S. Hence the result follows. Suppose that $q \geqslant 5$ and $n \geqslant 3$. Choose $t=1$ and $n_1=n-1 \geqslant 2$ in Lemma 2.8(ii). Then either $p(q^{n-1}+1)$ and $p(q^{n-1}-1)$ gives the right orders.

Let $S \cong \Omega_{2n+1}(q)$, where $n \geqslant 3$ and q be a power of an odd prime p. First assume that $q=3$. Then we may assume that $n \geqslant 4$ by using the character table of $\Omega_7(3)$ in the Atlas. Choose $t=2$ in Lemma 2.9(i). Then $n_1=n-2 \geqslant 2$ and $9(3^{n-2} \pm 1)/2$ are orders of elements of S and the result follows. Suppose that $q \geqslant 5$. Choose $t=1$ in Lemma 2.9(i). Then $n_1=n-1 \geqslant 3$ and $3(q^{n-1} \pm 1)/2$ are orders of elements of S. Also choose $t=1$, $n_1=n-2$ and $n_2=n_s=1$ in Lemma 2.9(ii). We have that $3[q^{n-2} \pm 1, q \pm 1]$ are also orders of elements of S. The result then follows.

Let $S \cong \Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$ and q be a power of 2. Choose $t=2$ and $n_1=n-3 \geqslant 1$ in Lemma 2.10(ii). Then we have an element of order $4(q^{n-3}+1)$ and so (a) follows. Suppose that $\epsilon=+$. Then choose $n_1=n-2 \geqslant 2$, $n_r=n_2=1$ and $n_3=n_s=1$ in Lemma 2.10(i). We have elements of composite orders $[q^{n_1}-1, q+1, q-1]$ and the result follows. Suppose that $\epsilon=-$. Choose $n_r=n_1=n-2$ and $n_2=2$. Then we have elements of order $[q^{n-2}+1, q^2-1]$. We have our result.

Let q be a power of an odd prime p and let $S \cong \mathrm{P}\Omega^{\epsilon}_{2n}(q)$, where $n \geqslant 4$, $\epsilon \in \{\pm 1\}$ and $(4, q^n-\epsilon)=4$. Choose $t=1$ and $n_1=n-2 \geqslant 2$ in Lemma 2.11(i). We have that $n(t)=2$ and so $n(t)+n_1=n$. We also consider elements with orders in Lemma 2.11(ii). It follows that we have elements with divisors of orders $p^2(q^2+1)$ and $p[q \pm 1, (q^{n-2}-\epsilon)/2]$. Hence our result follows.

Suppose that $S \cong G_2(q)$, where $q \geqslant 4$ is a power of a prime p. If $p=2$, then by Lemma 2.12(i) and we have that G has elements of order 8 and 12, so G has property (a). The other properties hold if we consider the set in Lemma 2.12(i). Suppose that p is odd. Using the character table of $G_2(5)$ in the Atlas [9], we may assume that $q \geqslant 7$. Then by Lemma 2.12(ii, iii), we have that q^2-1 and $p(q \pm 1)$ will give us the right orders. Let $S \cong {}^3D_4(q)$, with q a power of a prime p. Then our result easily follows from Lemma 2.13. If $S \cong {}^2F_4(q^2)$, then our result follows by [11, Lemma 3]. Let $S \cong {}^2G_2(q^2)$, $q^2=3^{2k+1} \geqslant 27$ since ${}^2G_2(3)' \cong \mathrm{PSL}_2(8)$. By [4, Lemma 4], divisors of $q^2-1=3^{2k+1}-1$ are orders of elements of S. Since this number is divisible by 4, we conclude that S satisfies properties (a)–(c).

Suppose $S \in \{F_4(q), E_6(q), {}^2E_6(q), E_7(q), E_8(q)\}$. Then Table 3 has the appropriate orders of vanishing elements.

□

Theorem 3.2. *Let G be a finite group with a trivial solvable radical and such that A_5 is not a composition factor of G. Suppose that one of the following hold:*

(a) G satisfies the one prime hypothesis for $\mathrm{Vo}(G)$*;*
(b) $|\mathrm{Vo}_c(G)| \leqslant 2$;
(c) $|\mathrm{Vo}(G)| \leqslant 5$.

Then G is an almost simple group.

Proof. We first show that G has a unique non-abelian normal subgroup N. Suppose that G has two minimal normal subgroups N_1 and N_2. Note that $N_i \ncong A_5$ for each i. By [10, Main Theorem], S has an element of order rs or r^2 for distinct primes r and s. Note that N_i has a p_i-defect zero character θ_i for each i, $i = 1, 2$ by Lemma 2.1. Then $\theta_1 \times \theta_2$ is of p_i-defect zero for $i = 1, 2$. If p_i is either equal to r or s, then $N_1 \times N_2$, has elements of either order rst or orders r^2t and r^2u for some distinct primes t and u not equal to p_i since $|\pi(N_i)| \geqslant 3$ by [18, Theorem 3.10]. If p_i does not divide rs, then $N_1 \times N_2$, has elements of either order p_irs or orders p_ir^2 and p_it for some prime t distinct from the other primes. Therefore (a)–(c) are not satisfied. Hence we may assume that G has a unique minimal normal subgroup N.

(a) Let $N = S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$ is a non-abelian simple group.. It is sufficient to show that $k = 1$. Suppose that $k \geqslant 2$. If $k \geqslant 3$, N has a vanishing element of G whose order is divisible by three distinct primes and we are done. Hence we may assume that $k = 2$. Note that S has at least one irreducible character of p-defect zero for some prime $p \geqslant 5$ by Lemma 2.1. By Lemma 2.2, p-singular elements of N are vanishing elements of G. Since $S \ncong A_5$, we have that S has a element of composite order g_i say, using [10, Main Theorem]. If the order of g_i is divisible by three primes then the result follows by Lemma 2.5, so we may assume that $\mathrm{ord}(g_i) = r^2$ or $\mathrm{ord}(g_i) = rs$ for distinct primes r and s. If $r = p$ or $s = p$, then we can have an element whose order is either r^2t or rst, where t is a distinct prime divisor of $|S|$. If $r \neq p$ and $s \neq p$, then we also have an element of order r^2p or rsp. Hence our result follows in both cases.

We will now look at (b) and (c). If $|\pi(S)| \geqslant 5$, then S has at least 3 distinct prime divisors p_1, p_2 and p_3 such that $p_i \geqslant 5$. Hence S has vanishing elements of G of order p_i by Lemmas 2.1 and 2.2, we have that p_i-singular elements in N are vanishing elements of G. Note that N has elements of order $2p_i$ for each i. This means that G has at least three vanishing elements with composite orders that are not of prime power order. Also note that $|\mathrm{Vo}(G)| \geqslant 6$. We may consider simple groups S such that $|\pi(S)| \leqslant 4$. These have been listed in Theorems 2.14 and 2.15. In particular, S either a simple group of Lie type or one of the following: $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2$. Suppose that S is a group of Lie type. Using Lemma 2.1, it follows that S has at least three primes p_i such that every p_i-singular element of N is a vanishing element. We have that p_1p_2, p_2p_3, and p_1p_3 are orders of elements of N. Hence both (b) and (c) follow from this. We may assume that S is one $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2$.

Suppose that $S \cong A_7$. Then S has elements of order 5 and 7. Note that S has an irreducible character θ that is extendible to $\mathrm{Aut}(S)$. We also have that θ vanishes on an element of order 4. Hence $\theta \times \theta$ is extendible to G by Lemma 2.3. It follows that G has elements of order 12, 20, 28. This means that (b) and (c) hold. Suppose that $S \cong A_n$, $8 \leqslant n \leqslant 10$. Then S elements of order 5, 7, and 15 which are vanishing elements of G. It follows that N has elements of order 10, 14, and 30.

Suppose that $S \cong M_{11}, M_{12}$ or $S \cong J_2$. Using the character tables in the Atlas [9], we have that S has at least three irreducible characters of p_i-defect zero for p_i, $i = 1, 2, 3$. The argument on groups of Lie type adapted to sporadic simple groups gives us our result. □

Suppose that G has a trivial solvable radical. By Theorem 3.2, G is an almost simple group with the exception of A_5. Note that the group $A_5 \times A_5$ satisfies the one prime hypothesis on $\mathrm{Vo}(A_5 \times A_5)$, but we have that $|\mathrm{Vo}(A_5 \times A_5)| = 6$ and $|\mathrm{Vo}_c(A_5 \times A_5)| = 3$. Hence the exception in Theorem 3.2 only applies to (a). Now, using Theorem 3.1, it is sufficient to consider the following groups: $M_{11}, M_{12}, M_{22}, J_1 A_n, n \leqslant 10, \mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSp}_4(q), {}^2\mathrm{B}_2(q^2), G_2(3)$. We shall start by investigating the following infinite families:

$$\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSp}_4(q).$$

Theorem 3.3. *Let $G \in \{\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSp}_4(q)\}$. Suppose that one of the following holds:*

(A) G satisfies the one-prime hypothesis on $\mathrm{Vo}(G)$*;*
(B) $|\mathrm{Vo}_c(G)| \leqslant 2$;
(C) $|\mathrm{Vo}(G)| \leqslant 5$.

Then G is isomorphic to one of the groups in Table 1.

Proof. Suppose that $S \cong \mathrm{PSL}_2(q)$. Suppose that $|\pi(S)| \geqslant 6$. Then it means that $|\pi(q^2-1)| \geqslant 5$. Note that S has cyclic subgroups of orders $(q \pm 1)/\gcd(2, q-1)$. Then either $|\pi((q+1)/\gcd(2, q-1))| \geqslant 3$ or $|\pi((q-1)/\gcd(2, q-1))| \geqslant 3$. Using Lemma 2.5, we have that G does not satisfy the one prime hypothesis on $\mathrm{Vo}(G)$ and also that $|\mathrm{Vo}_c(G)| \geqslant 3$. Hence we may assume that $|\pi(S)| \leqslant 5$ for (A) and (B). For (C), suppose that $|\pi(S)| \geqslant 5$. Then it means that S only has elements of prime order, contradicting [10]. Hence we may assume that $|\pi(S)| \leqslant 4$.

We start by considering when $|\pi(S)| = 3$. It is sufficient to only consider $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(8)$. Using the character tables in the Atlas, we have that both $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(8)$ satisfy (A), (B), and (C).

We may assume that $|\pi(S)| = 4$. Using Theorem 2.15, we shall consider the cases one by one. Let $q \in \{16, 25, 31, 49, 81, 127, 243, 97^2, 577^2\}$. Suppose that $q \in \{16, 25, 31\}$. Then using the character tables in the Atlas [9], we have our result for (A). For (B), we have that $S \cong \mathrm{PSL}_2(16), \mathrm{PSL}_2(25)$, or $\mathrm{PSL}_2(31)$. For (C), we have that only $S \cong \mathrm{PSL}_2(16)$ has the property. Let $q = 49$ or $q = 81$. Then $(q-1)/2 \in \{2^3 \cdot 3, 2^3 \cdot 5\}$ and so G does not satisfy any of the properties in (A), (B), or (C) by Lemma 2.5. Let $q = 127$. Then $(q+1)/2 = 64 = 2^6$ and so does not satisfy (A). Let us consider (B). Then $(q-1)/2 = 63 = 3^2 \cdot 7$ and S satisfies the property. Let $q = 243$. Then $f = 5$. Then $(q-1)/2 = 121 = 11^2$ and $(q+1)/2 = 122 = 2 \cdot 61$. This means that $G \cong \mathrm{PSL}_2(243)$ satisfies properties (A) and (B). If $q \in \{97^2, 577^2\}$, then $(q-1)/2 \in \{2^5 \cdot 3 \cdot 7^2, 2^6 \cdot 3^2 \cdot 17^2\}$ and so G does not satisfy property (A) or (B). For (C), let us consider q, where $q \in \{127, 243, 97^2, 577^2\}$. Then by finding $(q \pm 1)/2$, it follows that S at least two composite orders and so $|\mathrm{Vo}(G)| \geqslant 6$.

Let $q = 2^f, f > 3$ $s = 2^f - 1$ and $3r = 2^f + 1$. Then S satisfies the properties (A), (B), (C) since $\mathrm{Vo}(S)$ has exactly one element which is of composite order. Let $q = 3^f, f > 2$, $(3^f + 1)/2 = 2r$ and $(3^f - 1)/2 = s$. Then S satisfy properties (A), (B), and (C) since, again, $\mathrm{Vo}(S)$ has exactly one element which is composite. Let $q = 2^{2^a} + 1$, where $a > 2$ and $(q+1)/2 = 2^{2^a - 1} + 1 = 3r$ for some prime r. Then $(q-1)/2 = 2^{2^a - 1}$ and so G does not satisfy property (A) and (C). For (B), S satisfies the property.

Let $q = 2^a 3^b \pm 1$. If $(a, b) \in \{(1, 1), (1, 2), (2, 1), (2, 2)$, then $S \in \{\mathrm{PSL}_2(5), \mathrm{PSL}_2(7), \mathrm{PSL}_2(17), \mathrm{PSL}_2(19), \mathrm{PSL}_2(11), \mathrm{PSL}_2(13), \mathrm{PSL}_2(37)\}$. We have already considered $\mathrm{PSL}_2(5)$ and $\mathrm{PSL}_2(7)$. Then using the character tables in the Atlas [9] with the exception of $\mathrm{PSL}_2(37)$, we have that $S \cong \mathrm{PSL}_2(11), \mathrm{PSL}_2(13)$ are the new groups that satisfy (A) and (C). For (B), it follows that $S \cong \mathrm{PSL}_2(11), \mathrm{PSL}_2(13), \mathrm{PSL}_2(17), \mathrm{PSL}_2(19)$ satisfy the property. Note that $\mathrm{PSL}_2(37)$ has elements of order 18 and so does not satisfy (A) and (C) but does satisfy (B).

Hence $a > 2$ or $b > 2$. Since $q = 2^a 3^b - 1$ or $q = 2^a 3^b + 1$, it follows that $(q+1)/2 = 2^{a-1} 3^b$ or $(q-1)/2 = 2^{a-1} 3^b$ and G does not satisfy (A) hypothesis since $a - 1 \geqslant 2$ or $b \geqslant 3$. This deals with all the remaining cases. Let us consider this case for (B). Hence $a > 2$ or $b > 2$. Suppose that $a > 2$. Let us consider $(a, b) = (3, 1)$ first. Then $S \cong \mathrm{PSL}_2(23)$ or $\mathrm{PSL}_2(25)$. Using the character tables in the Atlas [9], it follows that both $\mathrm{PSL}_2(23)$ and $\mathrm{PSL}_2(25)$ satisfy (B). Suppose that $a \geqslant 4$. Then S has an element with an order divisible by $24 = 2^3 \cdot 3$ and using Lemma 2.5, our result follows. Suppose that $a \geqslant 2$ and $b \geqslant 3$. Then $(q \pm 1)/2$ is divisible by $2 \cdot 27$ and our result follows. Hence we may assume that $a = 1$ and $b \geqslant 3$. Let us consider $q = 2 \cdot 3^b + 1$. Then $(q-1)/2 = 3^b$ and $(q+1)/2 = 3^b + 1$. If $3^b + 1$ is divisible by $2rs$, $4r^2$ or $2r^3$ for distinct primes r and s, then S does not satisfy the hypothesis. Hence $3^b + 1 \in \{2r, 4r, 2r^2\}$. Suppose that $q = 2 \cdot 3^b - 1$. Then $(q+1)/2 = 3^b$ and $(q-1)/2 = 3^b - 1$. Using the same argument, we have that $3^b - 1 \in \{2r, 4r, 2r^2\}$. Hence our result follows. This deals with the rest of the cases in Theorem 2.15. We now consider property (C). Since $q = 2^a 3^b - 1$ or $q = 2^a 3^b + 1$, it follows that $(q+1)/2 = 2^{a-1} 3^b$ or $(q-1)/2 = 2^{a-1} 3^b$ and G does not satisfy the hypothesis since $a - 1 \geqslant 2$ or $b \geqslant 3$. This also deals with all the remaining cases in Theorem 2.15. Our result the follows.

Suppose that $|\pi(S)| = 5$ and that q is odd. Then $(q \pm 1)/2 = 2r$ and $(q \mp 1)/2 = st$ for some distinct primes r, s, t. This means that $q = 4r \pm 1 = 2st \mp 1$. Suppose that q is even. Then $q + 1 = rs$ and $q - 1 = tu$ for some primes r, s, t, u. Hence S satisfies properties (A) and (B).

From now on we shall now consider (A) and (B) and then consider (C) in the last paragraph.

Suppose that $S \cong \mathrm{PSL}_3(q)$. Using the character tables in the Atlas, we may assume that $S \cong \mathrm{PSL}_3(q)$, where $q \geqslant 11$. Let us consider when $n_1 = 2$ and $n_2 = 1$ in Lemma 2.6(i). Then we have that S has an element of order $(q^2 - 1)/\gcd(3, q-1)$. Suppose that q is odd. Then 8 divides $(q^2 - 1)/\gcd(3, q-1)$.

If q^2-1 is divisible by at most two primes, then $q=17$ by Lemma 2.4 and so $q^2-1=16\cdot 18$. Our result follows. Otherwise q^2-1 has at least the prime divisors and we are done using Lemma 2.5. Hence S does not satisfy properties (A) and (B).

Suppose that q is even. Let us consider $(q^2-1)/\gcd(3,q-1)$, which is an order of an element in S. Suppose that $\gcd(3,q-1)=1$. If q^2-1 has at least three prime divisors, then the result follows from Lemma 2.5. We may assume that q^2-1 has at most two prime divisors. By Lemma 2.4, $q\in\{2,4,8\}$. We have already considered these cases and hence our result follows. We may assume that $\gcd(3,q-1)=3$. Then 3 divides $q-1$ which means that $q=2^f$, where f is even, that is, $f=2f_1$ for some integer f_1. This means $(q^2-1)/\gcd(3,q-1)=(2^{4f_1}-1)/\gcd(3,q-1)$. If $2^{4f_1}-1$ is divisible by at least four prime divisors, then the result follows from Lemma 2.5. Therefore $2^{4f_1}-1$ is divisible by at most three prime divisors. This means that $2^{f_1}\in\{2,4\}$ and so $q\in\{4,16\}$. It is sufficient to consider $S\cong \mathrm{PSL}_3(16)$. By considering all the orders of elements of S using [7, Corollary 3], we have that S satisfies (A) but does not satisfy (B).

Suppose that $S\cong \mathrm{PSU}_3(q)$. Using the character tables in the Atlas, we may assume that $\mathrm{PSU}_3(q)$, where $q\geqslant 13$. Now, considering $n_1=2$ and $n_2=1$ in Lemma 2.6 (i), we obtain that S has an element of order $(q^2-1)/\gcd(3,q+1)$. Suppose that q is odd. Then 8 divides $(q^2-1)/\gcd(3,q+1)$. If q^2-1 is divisible by at most two primes, then $q=17$ by Lemma 2.4 and so $q^2-1=16\cdot 18$. Otherwise q^2-1 at least the prime divisors and the results follows using Lemma 2.5. Suppose that q is even, $q\geqslant 16$. Suppose also that $\gcd(3,q+1)=1$. If $(q^2-1)/\gcd(3,q+1)=q^2-1$ has at least three primes, then the result follows by Lemma 2.5. We may assume that q^2-1 has at most two prime divisors. Using Lemma 2.4, it follows that $q\in\{2,4,8\}$, which we have considered above. Suppose that 3 divides $q+1$. This means $q=2^f$, where f is odd. If $(q^2-1)/3$ has at least three primes, then the result follows by Lemma 2.5. We may assume that $(q^2-1)/3$ has at most two prime divisors. If 3 divides $(q^2-1)/3$, then it means that q^2-1 has at most two prime divisors and so we may assume that $q^2-1=3rs$ for distinct primes r,s. Note that $(q^3+1)/3(q+1)=(q^2-q+1)/3$ is an order of an element of S. Hence $(q^2-q+1)/3\in\{t,t^2,tu\}$.

Suppose that $S\cong \mathrm{PSp}_4(q)$. Using the character tables in the Atlas, we may assume that $q\geqslant 7$. Let q be even. Note that q^2-1 is an order of an element of S by Lemma 2.7(i). If q^2-1 has at least three prime divisors, then the result follows by Lemma 2.5. This means that it is sufficient to consider when $q=8$ by Lemma 2.4. Then $q^2-1=3^2\cdot 7$ and hence this does not satisfy (A) by Lemma 2.5. We need to consider an additional order for (B). Let $t=1$ and $n_1=n_s=3$ in Lemma 2.7(ii). Then $p(q^3-1)=p(q-1)(q^2+q+1)$ and the result follows.

Let q be odd. Suppose that q^2-1 has at least three prime divisors. Then $[q-1,q+1]$, which is an order of S by Lemma 2.8(i), also has at least three prime divisors and the result follows by Lemma 2.5. Hence it is enough to assume that $q\in\{7,9,17\}$. In all these cases $[q-1,q+1]$ is divisible by $2^3\cdot r$ for some odd prime r and hence the result follows from Lemma 2.5.

We now consider property (C). Suppose that $S\in\{\mathrm{PSL}_3(q),\mathrm{PSU}_3(q),\mathrm{PSp}_4(q)\}$ with $|\pi(S)|\leqslant 4$. Using Theorem 2.14, it is sufficient to consider the following groups: $\mathrm{PSL}_3(3),\mathrm{PSU}_3(3),\mathrm{PSL}_3(8)$, $\mathrm{PSL}_3(5),\mathrm{PSL}_3(7),\mathrm{PSL}_3(17),\mathrm{PSU}_3(4),\mathrm{PSU}_3(8),\mathrm{PSU}_3(9)$,
$\mathrm{PSU}_3(5),\mathrm{PSU}_3(7)$. Suppose that S is not isomorphic to $\mathrm{PSL}_3(17)$. Using the character tables in the Atlas [9], we have that the only group G satisfying the hypothesis is ${}^2\mathrm{B}_2(8)$. Suppose that S is isomorphic to $\mathrm{PSL}_3(17)$. $q^2-1=18\cdot 16$ and it is out. □

Remark 1. Using Theorems 3.2, 3.1, and 3.3, it is sufficient to consider the following cases: $S\cong M_{11},M_{12},M_{22},J_1A_n,n\leqslant 10$, $G_2(3)$ or ${}^2\mathrm{B}_2(q)$.

Proof of Theorem A. Let $S\cong M_{11},M_{12},M_{22},J_1A_n,n\leqslant 10$ or $G_2(3)$. By using the character tables in the Atlas, we have that the only groups S with the property are: J_1, A_5, A_6, A_7, and A_8.

Let $S\cong{}^2\mathrm{B}_2(q)$. Using the character tables in the Atlas, we may assume that $q>32$. Then $2^{2k+1}-1\in\{r,r^2,rs\}$, $2^{2k+1}-2^{k+1}+1\in\{t,t^2,tu\}$ and $2^{2k+1}+2^{k+1}+1\in\{v,v^2,vw\}$ for distinct primes r,s,t,u,v,w since these terms are pairwise coprime orders of elements in S. We have our result by Remark 1. □

Proof of Theorem B. Let $S \cong M_{11}, M_{12}, M_{22}, J_1, A_n, n \leqslant 10, G_2(3)$. By using the character tables in the Atlas, we have that the only groups G with the property are: M_{11}, M_{12}, M_{22}, A_5, A_6, A_7, A_8, and $G_2(3)$. □

Proof of Theorem C. Let $S \cong M_{11}, M_{12}, M_{22}, J_1 A_n, n \leqslant 10$ or $G_2(3)$. By using the character tables in the Atlas, we have that the only groups G with the property are: A_5, A_6, and A_7.

Suppose that $S \cong^2 \mathrm{B}_2(q^2)$. It is sufficient to consider ${}^2\mathrm{B}_2(8)$ and ${}^2\mathrm{B}_2(32)$ since $|\pi(S)| \leqslant 4$. Using the character tables in the Atlas, we have that the only group G satisfying the hypothesis is ${}^2\mathrm{B}_2(8)$. We have our result by the Remark 1. □

Examples. We shall list some examples of the general cases that are in Table 1:

(a) Let $\mathrm{PSL}_2(2^f)$ with $f > 3$, $2^f - 1 = s$ and $2^f + 1 = 3r$. Then $\mathrm{PSL}_2(32)$ and $\mathrm{PSL}_2(128)$ are examples.
(b) $\mathrm{PSL}_2(3^f)$, with $3^f + 1 = 4r$ and $3^f - 1 = 2s$. Then $\mathrm{PSL}_2(27)$ and $\mathrm{PSL}_2(2187)$ are examples.
(c) $\mathrm{PSL}_2(q)$, with $q = 2^{2^a} + 1$, $a > 2$ and $(q+1)/2 = 3r$. Then are not examples of these. The only Fermat's prime numbers that are known do not satisfy this property.
(d) $\mathrm{PSL}_2(q)$, $q = 4r \pm 1 = 2st \mp 1$. Then $\mathrm{PSL}_2(29)$, $\mathrm{PSL}_2(67)$, and $\mathrm{PSL}_2(71)$ are examples.
(e) $\mathrm{PSL}_2(q)$ with $q = rs - 1$ and $q = tu + 1$. Then $\mathrm{PSL}_2(2048)$ is an example.
(f) $\mathrm{PSL}_2(q)$, $q = 2 \cdot 3^b \pm 1$ with $(q \pm 1)/2 \in \{2r, 4r, 2r^2\}$. Then $q \in \{19, 53, 163, 487\}$ are examples which satisfy the properties.
(g) $\mathrm{PSU}_3(2^f)$, f odd, $q^2 = 3rs + 1$ and $q^2 - q + 1 \in \{3t, 3t^2, 3tu\}$. Then $\mathrm{PSU}_3(32)$ and $\mathrm{PSU}_3(128)$ are examples.

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