The theoretical analysis of vibrating elastic structures

by

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Summary

The vibration of elastic structures is widely researched, especially in engineering and applied mathematics. Partial differential equations or systems of partial differential equations are used to model such vibrating structures. However, solutions of partial differential equations or systems of partial differential equations are seldom possible to determine explicitly.

This dissertation is a literature study on linear vibration problems. The research forms part of an ongoing research project *Vibration analysis*. It deals with modelling, theoretical analysis and finite element computation. In this project the aim is to obtain results of a theoretical as well as practical nature.

A significant part of this dissertation deals with the theory for existence of solutions and the application thereof to model problems. The existence results in a 2002 article of Van Rensburg and Van der Merwe are presented and proved in greater detail. Additional results were formulated for this purpose. Semigroup theory is used to obtain existence results.

The theory of existence is applied to various models with different types of damping. Examples with weak damping as well as boundary damping are presented. The application to each model problem is rigorous and more complete than any publication before on this topic.

In a recent article on hyperbolic heat conduction a model problem that is not well posed, is given. In one section of this dissertation we do a thorough analysis and prove that the problem does not even admit a mild solution. We also prove rigorously that solution methods in the article are valid and provide estimates for errors (not finite element method errors).

Two recent articles on error estimates for the semi-discrete and fully discrete Galerkin approximations of the general weak variational problem are a paper

by Basson and Van Rensburg published in 2013 and another by Basson, Stapelberg and Van Rensburg published in 2017. Different types of damping are considered since the properties of the solutions as well as the numerical approximations of these solutions depend on the damping. We consider weak damping as well as general damping.

In one chapter of this dissertation the finite element method (FEM) is applied to the vibration of the Timoshenko beam. The objective was to compare the standard finite element method (SFEM) using Hermite cubic basis functions to the mixed finite element method (MFEM) using piecewise linear basis functions. It should however be noted that the method used in this dissertation to compare the models is naive since we do not have the means to calculate the computational effort of each method. More numerical experiments and detailed analysis are required, but that is considered to be a project in its own right.

A modest contribution was made regarding the two-dimensional beam model. The derivation of the matrices for the MFEM should be useful for future work.

Tracking a sharp crested wave front in hyperbolic heat transfer, an article by Sieberhagen and Van Rensburg published in 2012, is studied in one of the chapters of this dissertation. The article was written for an audience not particularly interested in mathematical analysis; the contribution of this dissertation was to conduct a serious analysis of the problem and methods used.

DECLARATION

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Contents

Chapter 1

Linear vibration problems

1.1 Introduction

The vibration of elastic structures is widely researched, especially in engineering and applied mathematics. Partial differential equations or systems of partial differential equations are used to model such vibrating structures. Solutions of partial differential equations or systems of partial differential equations are seldom possible to determine explicitly, therefore it is necessary to approximate the solutions of these systems numerically. Here the finite element method (FEM) proves to be ideal.

This dissertation is a literature study on linear vibration problems. The research forms part of an ongoing research project *Vibration analysis*. It deals with modelling, theoretical analysis and finite element computation. In this project the aim is to obtain results of a theoretical as well as practical nature. All linear vibration problems have the same variational form which resembles the variational form for the wave equation. It is referred to as either the general linear vibration problem or the general linear second order hyperbolic problem. In this project results on modelling, well posedness of model problems, FEM approximation of model problems and convergence results for FEM approximations are obtained.

The research in the recent past included modelling, theoretical analysis and FEM computation for vibration problems in solid mechanics. Theorems on existence and uniqueness were formulated and proved. These results covered an arbitrary vibration problem with damping, provided that the problem is linear. It was established that damping can be classified as weak, strong or neither and the assumptions and properties of solutions are different for the three cases. (Detail is provided in Chapter 2 of this dissertation.) Articles were published on application of general existence theory, FEM theory and other applications. In addition five master's dissertations were completed.

Considering the results obtained and the flood of other publications, it appears that the linear theory is complete. However, there are important issues outstanding. For example, the "gap" between existence theory and FEM convergence theory; assumptions necessary for convergence are restrictive and are not available from existence and regularity theory. In the literature study [Sta14], this was identified as a possible research problem. Also in this study, it became clear that there is no easy explanation.

In the article [BV13], convergence results in previous publications were generalised. It transpired that the results in [VV02] are convenient to establish existence of a weak solution. Also, the estimates needed for existence theory are also required for convergence theory.

General existence results for second order hyperbolic problems can be found in [Sho77], [AKS96] and [VV02]. However, we are not aware that any of these publications are cited in articles on FEM convergence. Where publications on existence are cited, it is invariably a result from the sixties by J.L. Lions.

At this point it should be added that attempts to approximate non-existent solutions are sometimes made, see e.g. [SV12]. Inevitably, this leads to poor results. It is then usually assumed that a better numerical procedure is required.

Concerning existence, a literature study [DeV08] was done on [VV02] in 2008. Proofs were scrutinised, not only to ensure correctness of the detail, but also to investigate assumptions made. Applications received the necessary attention: These were mostly routine and not published.

Having read [DeV08], we identified a need for a thorough study of [VV02]. To fully appreciate the significance of the results it is necessary to have a thorough understanding of the proofs. It is also necessary to make a careful study of applications: How are the assumptions of the theory verified and what can be deduced regarding properties of solutions?

A study of other publications, e.g. [Sho77] and [AKS96], would be ideal. In the end we managed only a brief look at [Sho77].

As mentioned, FEM convergence results were generalised in [BV13]. But weak damping (defined in Chapter 2) was assumed. This paper was followed by [BSV17] (which appeared online in 2016). These two articles are relatively easy to follow. The exposition is clear and the notation user friendly but rigorously implemented. In this dissertation we have little to comment on the theory and mostly display results for convenience and easy reference. The treatment of applications are rather brief, so it was decided that more should be done in this department.

A study of the FEM is not complete without the discontinuous Galerkin method and the mixed FEM (MFEM). A thorough study of the first mentioned method was done in [Sta14]. In this dissertation one objective was to study the MFEM to establish the advantages and disadvantages of the method. The theory in [Arn81], [Sem94] and [FXX99] was studied in detail in [Hoo16]. In this dissertation the emphasis was on the application of the theory and numerical experiments for comparison.

Another objective was to give serious consideration to the Timoshenko beam model. It is important in practice: a relatively simple model which provides a realistic way to model many practical problems. In this dissertation the model is also important to provide examples for applications of the theory: Existence theory, FEM and MFEM theory and numerical experiments.

Comparison of beam models was done in [LVV09] and [Jan16]. It was not considered for this dissertation.

A preliminary investigation leads to the conclusion that the linear theory is not complete and that there are outstanding issues that need attention. Having said that, serious consideration was given to include some non-linear examples for which existence of solutions can be proved. However, the volume of work on the linear theory expanded faster than anticipated and the idea was abandoned.

In the rest of this chapter various model problems are presented. The (standard) FEM is explained in the following section using the one dimensional wave equation.

1.2 Transverse vibration of a string with viscous damping

The one-dimensional wave equation is the simplest example of a linear vibration problem. The partial differential equation is a mathematical model for longitudinal vibrations of a bar or transverse vibrations of a string. One may expect viscous damping in the vibrating string model due to air resistance. We use this simple problem to illustrate the scope of the dissertation. Other model problems are also presented in Chapter 1.

1.2.1 Model problem

The following assumptions are made to derive the model:

1. The string executes small transverse vibrations.
2. Tensile force $\bar{T}$ is in the direction of the tangent to the curve.
3. The string is "tightly" stretched and hence the tension T may be considered to be constant.

Let u denote the transverse displacement of the string and θ be the angle between the string and the direction of $\bar{e}_1$, that is $\tan\theta = \partial_x u$. The components of the tensile force are $H = T\cos\theta$ and $V = T\sin\theta$. For θ sufficiently small, we have $T\sin\theta \approx T\tan\theta = T\partial_x u$.

Now consider the equation of motion

$$\delta\partial_t^2 u = \partial_x V + q \tag{1.2.1}$$

where q is an external force density and δ the mass per unit length.

Substituting the constitutive equation $V = T\partial_x u$ into the equation of motion (1.2.1), results in the well known wave equation

$$\partial_t^2 u = \alpha^2\partial_x^2 u + \tilde{q} \tag{1.2.2}$$

with $\alpha^2 = \dfrac{T}{\delta}$ and $\tilde{q} = \dfrac{q}{\delta}$. The constant α represents physical properties of the string (or bar).

For viscous damping we have that $\tilde{q} = -\gamma\partial_t u + f$ where $\gamma > 0$ is the damping constant. (The book of [Inm94] can be consulted.) Let ℓ denote the length of the string.

Problem WD

Given α^2 and $\gamma > 0$, find u such that for $t \in \mathbb{R}$,

$$\begin{aligned} \partial_t^2 u &= \alpha^2\partial_x^2 u - \gamma\partial_t u + f \quad \text{in } (0,\ell), \qquad (1.2.3)\\ u(0,t) &= 0,\\ k_1 u(\ell,t) + k_2\partial_x u(\ell,t) &= 0,\\ u(x,0) &= u_0(x) \quad \text{for} \quad x \in (0,\ell),\\ \partial_t u(x,\, 0) &= u_1(x) \quad \text{for} \quad x \in (0,\ell). \end{aligned}$$

Remark The general existence theory is presented in Chapter 2, where the terms weak damping and strong damping are also defined. The damping in Problem WD is an example of weak damping.

1.2.2 Variational form

Notation We write $u \in C^k[0,\ell]$ if u has derivatives up to order k which are continuous on the closed interval $[0,\ell]$.

Consider Problem WD with $k_1 = 0$ and $k_2 = 1$. That means the string is fixed at the endpoint where $x = 0$ while the other endpoint is free to slide. To write Problem WD in variational form, multiply (1.2.3) by an arbitrary function $v \in C^1[0,\ell]$ and integrate from 0 to ℓ. Using integration by parts we find that

$$\begin{aligned} \int_0^\ell \partial_t^2 u(\cdot,t)v = -\int_0^\ell \alpha^2\partial_x u(\cdot,t)v' - \int_0^\ell \gamma\partial_t u(\cdot,t)v + \int_0^\ell f(\cdot,t)v \\ + \ \alpha^2\partial_x u(\ell,t)v(\ell) \ - \ \alpha^2\partial_x u(0,t)v(0). \end{aligned}$$

Next we introduce suitable test functions and bilinear forms.

Test functions

$$T[0,\ell] = \{v \in C^1[0,\ell] \ \big| \ v(0) = 0\}.$$

Then, since $\partial_x u(\ell, t) = 0$,

$$\int_0^\ell \partial_t^2 u(\cdot, t)v = -\int_0^\ell \alpha^2 \partial_x u(\cdot, t)v' - \int_0^\ell \gamma \partial_t u(\cdot, t)v + \int_0^\ell f(\cdot, t)v \tag{1.2.4}$$

for each $v \in T[0, \ell]$.

Define the following bilinear form and inner product.

$$b(u, v) = \int_0^\ell \alpha^2 u'v' \quad \text{for} \quad u, v \in C^1[0, \ell],$$
$$(f, g) = \int_0^\ell fg \qquad \text{for} \quad f, g \in \mathcal{L}^2(0, \ell).$$

The variational form of Problem WD can now be presented.

Problem WD-V
Find a function u such that for each t, $u(\cdot, t) \in T[0, \ell]$ and

$$(\partial_t^2 u(\cdot, t), v) + \gamma(\partial_t u(\cdot, v), v) + b(u(\cdot, t), v) = (f(\cdot, t), v) \tag{1.2.5}$$

for each $v \in T[0, \ell]$, while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Remark If $k_1 = 1$ and $k_2 = 0$, then $u(\ell, t) = 0$. The test functions must then satisfy $v(0) = v(\ell) = 0$.

The weak variational form of Problem WD is formulated in Section 3.1. The general existence theory of Chapter 2 can then be applied.

1.2.3 Semi-discrete approximation

The domain $[0, \ell]$ is divided into elements of equal length h. The admissible basis functions are numbered from $1, 2, \ldots, k$. Let S^h be the span of the basis functions $\phi_1, \phi_2, \ldots, \phi_k$.

Galerkin approximation

Problem WD-V^h
Find a function u_h such that for each t, $u_h(\cdot, t) \in S^h$ and

$$(\partial_t^2 u_h(\cdot, t), v) + \gamma(\partial_t u_h(\cdot, t), v) + b(u_h(\cdot, t), v) = (f(\cdot, t), v) \tag{1.2.6}$$

for each $v \in S^h$, while $u_h(\cdot, 0) = u_0^h$ and $\partial_t u_h(\cdot, 0) = u_1^h$.

Note that u_0^h and u_1^h are approximations of u_0 and u_1 in the finite dimensional subspace S^h and must be chosen in applications. We will use the interpolants in our applications, but other options are also possible.

Problem WD-V^h is equivalent to a system of ordinary differential equations as we show below. Since $u_h \in S^h$, we have that

$$u_h(x,t) = \sum_{j=1}^{k} u_j(t)\phi_j(x)$$

where $u_j(t) \in \mathbb{R}$ for $j = 1, 2, \ldots, k$.

Therefore Problem WD-V^h can be rewritten as

$$\sum_{j=1}^{k} u_j''(t)(\phi_j, \phi_i) + \gamma \sum_{j=1}^{k} u_j'(t)(\phi_j, \phi_i) + \alpha^2 \sum_{j=1}^{k} u_j(t)(\phi_j', \phi_i') = (f, \phi_i)$$

for $i = 1, 2, \ldots, k$.

Define the following matrices:

$$M = (M_{ij}) \quad \text{and} \quad K = (K_{ij})$$

where

$$M_{ij} = (\phi_j, \phi_i) \quad \text{and} \quad K_{ij} = \alpha^2(\phi_j', \phi_i')$$

for $i, j = 1, 2, \ldots, k$.

Problem WD-ODE

Find $\bar{u}$ such that

$$M\bar{u}'' + \gamma M\bar{u}' + K\bar{u} = \bar{F}$$

where $F_i(t) = f(x_i, t)$. The initial conditions are $\bar{u}(0) = \bar{a}$ and $\bar{u}'(0) = \bar{b}$ where $a_i = u_0(x_i)$ and $b_i = u_1(x_i)$.

The system of ordinary differential equations above can be used to simulate the motion of the string. In Chapter 4 it is shown how the general FEM theory is applied to derive error estimates.

1.2.4 Plucked string

A popular example in textbooks is the so called plucked string. (See e.g. [EP96], Example 2 in Section 8.6). Consider a stretched string of length ℓ

that is fixed at both endpoints. The string is set in motion by moving its midpoint $x = \dfrac{\ell}{2}$ aside the distance $\dfrac{1}{2}k\ell$ and then releasing it from rest at time $t = 0$. This is a special case of Problem WD with $\gamma = 0$, $f = 0$, $k_1 = 1$ and $k_2 = 0$. The initial conditions are given by $u(x, 0) = u_0(x)$ and $\partial_t u(x,\, 0) = 0$ where

$$u_0(x) = \begin{cases} kx \ \text{ for } \ 0 \leq x \leq \frac{\ell}{2} \\ k(\ell - x) \ \text{ for } \ \frac{\ell}{2} \leq x \leq \ell. \end{cases}$$

The variational form is given by (3.1.1) with $\gamma = f = 0$. In Chapter 3 we explain why u_0 is not an acceptable initial state.

1.3 Well posed problems

1.3.1 Introduction

Evans [Eva98] informally defines what it means for a problem to be well posed. The author states that a given problem for a partial differential equation is well posed if

(i) the problem has a solution,

(ii) this solution is unique and

(iii) the solution depends continuously on the data given in the problem.

There is a need to define what is meant by a solution. The conditions imposed on a solution will then determine whether the "solution" is acceptable. It is desirable that a solution is infinitely differentiable, but this is very rarely the case. Therefore the concept of a classical solution is introduced.

Classical solution (This definition differs slightly from the one in [Eva98])
A proposed solution is called a classical solution if it has partial derivatives of sufficient order for the problem at hand.

Therefore, solving a given problem for a partial differential equation in the classical sense implies that a classical solution satisfying (i)-(iii) above can be written down, or it can be proved that such a solution exists.

If a solution exists but no representation for the solution can be found, there remains the problem of calculating approximations. This is where we rely on numerical methods.

1.3.2 Examples of problems that are not well posed

What happens when someone calculates a numerical approximation for the solution of a problem that is not well posed? In this subsection we use a hyperbolic heat transfer problem to provide an example of a problem that is not well posed. Hyperbolic heat transfer is discussed extensively in Section 1.4.2. Solutions to hyperbolic heat transfer problems are often approximated using numerical techniques. Researchers have however encountered oscillatory behaviour in numerical approximations in some cases. There are a few examples from the literature that refer to these numerical oscillations, see for example [CT82], [LCPX05], [JLZ02] and [HW06]. Researchers believe that the main cause of these oscillations is the presence of sharp wave fronts, see for example [CL04]. In [JLZ02] the authors believe that numerical oscillations and the sharp discontinuity are caused by the numerical differencing of the hyperbolic equation.

However, in [SV12] it is shown that it is not the numerical methods that cause the oscillations, but rather the fact that the problem is not well posed, unless a jump condition is specified. The authors focus on the one-dimensional hyperbolic heat conduction problem as studied by [CT82] and [HW06]. They investigate the numerical oscillations encountered when approximating the solution.

The following (dimensionless) model problem was derived in [CT82], see Section 1.4.2. The dimensionless temperature is denoted by T. Note that this is a special case of the wave equation in Section 1.2.

Problem CT
Find T defined on $[0,1] \times [0,\infty)$ such that

$$\begin{aligned}
&\partial_t^2 T + 2\partial_t T - \partial_x^2 T = 0, \\
&T(0,t) = 1, \quad \partial_x T(1,t) = 0, \\
&T(\cdot,0) = \partial_t T(\cdot,0) = 0.
\end{aligned}$$

This problem is not well posed. The problem either has no solution or more than one solution if discontinuities are allowed with no jump condition. One possible "solution" is: For $t \geq 0$,

$$T(x,t) = \begin{cases} 0 & \text{for } x > t \\ 1 & \text{for } x \leq t. \end{cases}$$

Then T satisfies the partial differential equation of Problem CT, except on the line $x = t$. The boundary and initial conditions are also satisfied.

In [SV12] the authors considered an equivalent problem where both boundary conditions are zero. (This is for theoretical purposes.) Let $\theta = T - 1$ and consider the following equivalent problem.

Problem CT2
Find θ defined on $[0,1] \times [0,\infty)$ such that

$$\begin{aligned} &\partial_t^2\theta + 2\partial_t\theta - \partial_x^2\theta = 0, \\ &\theta(0,t) = \partial_x\theta(1,t) = 0, \\ &\theta(\cdot,0) = \theta_{in}(\cdot), \quad \partial_t\theta(\cdot,0) = 0. \end{aligned}$$

If $\theta_{in}(x) = -1$, then $T = \theta + 1$ is a solution of Problem CT if and only if θ is a solution of Problem CT2.

Problem CT2 is a special case of the wave equation with viscous damping. (Problem WD with $k_1 = 0$ and $k_2 = 1$). In this case $\gamma = 2$, $\alpha = 1$ and $f = 0$. Therefore the variational form does not need to be derived again.

Problem CT2-V
Find θ such that for each $t > 0$, $\theta(\cdot,t) \in T[0,1]$ and

$$(\partial_t^2\theta(\cdot,t), v) + 2(\partial_t\theta(\cdot,t), v) + b(\theta(\cdot,t), v) = 0 \quad \text{for each} \quad v \in T[0,1],$$

while $\theta(\cdot,0) = \theta_{in}(\cdot)$ and $\partial_t\theta(\cdot,0) = 0$.

Using existence theory, we will later on prove that Problem CT2 does not even have a weak solution. The weak variational form of this problem is not well posed, as proved in Chapter 3.

Remark Recall the plucked string discussed in Subsection 1.2.4. This model is another example of a problem that is not well posed. This will also be proved in Chapter 3.

1.4 The multi-dimensional wave equation

1.4.1 Model problems

Consider the wave equation in a bounded domain Ω in $\mathbb{R}^n$ ($n = 2$ or 3). Denote the boundary of Ω by $\partial\Omega$. Let Σ be a part of the boundary and denote the unit outer normal on the boundary by $\mathbf{n}$.

Problem MW
Given a function f, find u such that

$$
\begin{aligned}
\rho\partial_t^2 u &= -\text{div}F - d\partial_t u + f && \text{in } \Omega \times (0,T) && (1.4.1)\\
F &= -A\nabla u && && (1.4.2)\\
u(\cdot,t) &= 0 && \text{on } \partial\Omega - \Sigma &&\\
A\nabla u \cdot \mathbf{n} &= 0 && \text{on } \Sigma &&
\end{aligned}
$$

while $u(\cdot,0) = u_0(\cdot)$ and $\partial_t u(\cdot,0) = u_1(\cdot)$ in Ω.

The given parameters in the problem are the matrix of functions $A = (a_{ij})$ and positive functions ρ and d.

Remark Special cases of Problem MW can be obtained if $d = 0$, $f = 0$ or if Σ is empty.

Combining equations (1.4.1) and (1.4.2) results in the well known multi-dimensional wave equation

$$\rho\partial_t^2 u = \text{div}(A\nabla u) - d\partial_t u + f. \tag{1.4.3}$$

Assumptions on the parameters

1. $a_{ij} \in C(\bar{\Omega}) \cap C^1(\Omega)$, $\rho \in C(\bar{\Omega})$, and $d \in C(\bar{\Omega})$.
2. The matrix A is uniformly positive definite, i.e. there exists a positive constant α such that

   $$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2$$

   for all $x \in \bar{\Omega}$ and all $(\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$.
3. There exists positive constants k_1 and k_2 such that $k_1 \leq \rho \leq k_2$ and $d > 0$.

Examples

Vibration of a membrane

In the two dimensional case the wave equation models the vibration of a membrane [Inm94, Section 6.6]. The partial differential equation is

$$\rho\partial_t^2 w \;\;=\;\; \mu\nabla^2 w - \nu\partial_t w, \tag{1.4.4}$$

where ρ is the mass per unit area, μ is the constant tension per unit length and ν is a damping constant. For the model to be realistic, ∇w must be small.

Acoustic wave equation

The wave equation models the propagation of sound waves in the three dimensional case. The acoustic wave equation is derived from the continuity equation and the equation of motion for an ideal gas, see e.g. [Bat67]. The derivation done in [PR05, Section 1.4.2] yields the acoustic wave equation (for pressure)

$$c\partial_t^2 p - \nabla^2 p = 0.$$

Hyperbolic heat conduction

The multi-dimensional wave equation also models hyperbolic heat conduction, as is shown in the following subsection.

1.4.2 Hyperbolic heat conduction

Hyperbolic heat conduction (first introduced in Subsection 1.3.2) is interesting because sharp wave fronts are possible. For the vibrating membrane this is physically unrealistic. In this subsection we will show how the model is derived.

The classical heat equation

Consider the energy conservation law

$$\rho c_p \partial_t T = -\nabla \cdot q \tag{1.4.5}$$

where ρ is the density of the material, c_p denotes the specific heat of the material and T is the temperature.

Fourier's law of heat conduction is given by

$$q = -k\nabla T \tag{1.4.6}$$

where q denotes the heat flux and k is the thermal conductivity.

Combining equations (1.4.5) and (1.4.6) leads to the classical heat equation

$$\partial_t T = c^2 \nabla^2 T, \quad \text{where } c^2 = \frac{k^2}{\rho^2 c_p^2}.$$

Hyperbolic heat conduction

Cattaneo [Cat48] and Vernotte [Ver58] independently proposed a modification to Fourier's law, called the Cattaneo-Vernotte model. The constitutive equation for this model is given by

$$q + \tau \partial_t q = -k\nabla T, \tag{1.4.7}$$

where τ is the time delay or relaxation time. Tzou [Tzo95] extended the hyperbolic heat conduction model to the single-phase-lag model given by

$$q(x, t+\tau) = -k\nabla T(x, t)$$

where τ is the phase-lag in the heat flux q. Since the time lag is extremely short, the equation may be linearised and equation (1.4.7) is obtained.

When combined with the energy conservation law (1.4.5), this model gives rise to a (weakly) damped wave equation, known as the hyperbolic heat conduction equation. Differentiating (1.4.5) yields

$$\rho c_p \partial_t^2 T = -\partial_t \nabla \cdot q. \tag{1.4.8}$$

Taking the divergence of (1.4.7) we get

$$\nabla \cdot q + \tau \nabla \cdot \partial_t q = -k \nabla \cdot \nabla T. \tag{1.4.9}$$

Combining equations (1.4.8) and (1.4.9) we obtain

$$\tau \rho c_p \partial_t^2 T - \nabla \cdot q = k \nabla^2 T. \tag{1.4.10}$$

Now, substituting the energy conservation law (1.4.5) into (1.4.10), yields the hyperbolic heat conduction equation:

$$\frac{1}{c^2}\partial_t^2 T + \frac{1}{\alpha}\partial_t T = \nabla^2 T, \tag{1.4.11}$$

where c is the thermal wave speed, α is the thermal diffusivity, $\frac{1}{c^2} = \frac{\tau\rho c_p}{k}$ and $\tau = \frac{\alpha}{c^2}$.

Dimensionless form

In order to have a model containing only two parameters, dimensionless variables are introduced through the following transformation of parameters:

$$\begin{aligned} T^* &= \frac{T}{T_0} \\ x^* &= \frac{x}{L} \\ t^* &= \frac{t}{t_0} \\ q^* &= \frac{q t_0}{L\rho c_p T_0} \\ \tau^* &= \frac{\tau}{t_0} \\ k^* &= \frac{k t_0}{L^2 \rho c_p} \end{aligned}$$

where T_0 is a suitable reference temperature, L is the distance within a specimen, defined by $L = \sqrt{x^2 + y^2 + z^2}$ and t_0 is a suitable reference time parameter.

Substituting these dimensionless parameters and using the original notation, the energy conservation law becomes

$$\partial_t T = -\nabla \cdot q \tag{1.4.12}$$

and the constitutive equation for the Cattaneo-Vernotte model is expressed as

$$q + \tau\partial_t q = -k\nabla T. \tag{1.4.13}$$

Combining equations (1.4.12) and (1.4.13) leads to

$$\tau\partial_t^2 T + \partial_t T - k\nabla^2 T = 0. \tag{1.4.14}$$

Remark Note that a partial differential equation of the form (1.4.14) is also the model for the vibration of a membrane with viscous damping, as given in (1.4.4). Sharp crested wave fronts are possible in heat conduction, but not for a membrane. In the derivation of the vibration model it is necessary to assume that the gradient of the displacement is small. The presence of sharp crested wave fronts creates challenges for numerical approximation, as discussed in Chapter 7.

It is assumed that the part of the boundary Σ is isolated. That is, the flux q is zero at Σ. Consequently $\partial_t q = 0$ and it follows from (1.4.14) that the boundary condition on Σ is $\nabla T \cdot \mathbf{n} = 0$.

It is justifiable to let the ambient temperature be 0. Therefore the boundary condition on $\partial\Omega - \Sigma$ is $T(\cdot, t) = 0$.

The hyperbolic heat conduction model is a special case of Problem MW with $\rho = \tau$, $d = 1$, $f = 0$ and $A = kI$ where I is the identity matrix.

The one-dimensional case

The energy conservation law is given by

$$\rho c_p \partial_t T = -\partial_x q \tag{1.4.15}$$

and the constitutive equation becomes

$$q + \tau \partial_t q = -k \partial_x T. \tag{1.4.16}$$

Dimensionless form

In [CT82] scaling was done in such a way that $\tau = \dfrac{1}{2}$ and $k = \dfrac{1}{2}$. Consequently

$$\partial_t T = -\partial_x q \tag{1.4.17}$$

and

$$\partial_t q + 2q = -\partial_x T. \tag{1.4.18}$$

This yields the partial differential equation in [CT82]:

$$\partial_t^2 T + 2\partial_t T - \partial_x^2 T = 0. \tag{1.4.19}$$

1.4.3 Variational form of the multi-dimensional wave equation

Consider the three dimensional case ($n = 3$) of Problem MW. Recall that the unit outer normal vector to Ω at $\partial\Omega$ is denoted by $\mathbf{n}$.

Divergence form of Green's theorem

If $F \in C^1(\bar{\Omega})^3$, then

$$\iiint_\Omega \text{div}F \; dV = \iint_{\partial\Omega} F \cdot \mathbf{n} \; dA. \tag{1.4.20}$$

Green's formula

If $F \in C^1(\bar{\Omega})^3$ and $v \in C^1(\bar{\Omega})$, then

$$\iiint_\Omega -(\text{div}F)v \; dV = \iiint_\Omega F \cdot \nabla v \; dV - \iint_{\partial\Omega} vF \cdot \mathbf{n} \; dA. \tag{1.4.21}$$

Proof We have that

$$\text{div}(vF) = v \; \text{div}F + \nabla v \cdot F$$

and therefore

$$\iiint_\Omega -(\text{div}F)v \; dV = \iiint_\Omega F \cdot \nabla v \; dV - \iiint_\Omega \text{div}(vF) \; dV.$$

From the divergence form of Green's theorem (1.4.20) it follows that

$$\iiint_\Omega \text{div}(vF) \; dV = \iint_{\partial\Omega} vF \cdot \mathbf{n} \; dA$$

and the result follows. □

To write Problem MW in variational form, multiply equation (1.4.1) by an arbitrary function v and integrate. Using Green's formula (1.4.21) yields

$$\begin{aligned} \iiint_\Omega \rho\partial_t^2 uv \; dV &= -\iiint_\Omega \text{div} \; Fv \; dV - \iiint_\Omega d\partial_t uv \; dV + \iiint_\Omega fv \; dV \\ &= \iiint_\Omega F \cdot \nabla v \; dV - \iint_{\partial\Omega} vF \cdot \mathbf{n} \; dA \\ &\quad - \iiint_\Omega d\partial_t uv \; dV + \iiint_\Omega fv \; dV. \end{aligned} \tag{1.4.22}$$

The term $\iint_{\partial\Omega} vF \cdot \mathbf{n}\; dA$ in (1.4.22) is reduced to $\iint_{\partial\Omega-\Sigma} vA\nabla u \cdot \mathbf{n}\; dA$ since $F \cdot \mathbf{n} = -A\nabla u \cdot \mathbf{n} = 0$ on Σ.

Test functions

$$T(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega - \Sigma\}.$$

Substitute equation (1.4.2) into (1.4.22). The standard variational form of Problem MW follows.

Problem MW-VS
Find a function u such that for $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$\iiint_\Omega \rho\partial_t^2 uv\; dV + \iiint_\Omega d\partial_t uv\; dV + \iiint_\Omega A\nabla u \cdot \nabla v\; dV \quad = \quad \iiint_\Omega fv\; dV$$

for each $v \in T(\Omega)$ while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Remark For the two-dimensional case the volume integrals are replaced by area integrals and the surface integrals replaced by line integrals.

To apply the theory in Chapters 2 and 3, we need to write the variational form in terms of bilinear forms. Define the following bilinear forms.

$$\begin{aligned} b(u, v) &= \iiint_\Omega A\nabla u \cdot \nabla v\; dV, \\ c(u, v) &= \iiint_\Omega \rho uv\; dV, \\ a(u, v) &= \iiint_\Omega duv\; dV. \end{aligned}$$

The $\mathcal{L}^2$-inner product on Ω is defined by

$$(f, v)_\Omega = \iiint_\Omega fv\; dV.$$

We can now write Problem MW-VS in terms of bilinear forms.

Problem MW-VS (Alternative)
Find a function u such that for $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot, t), v) + a(\partial_t u(\cdot, t), v) + b(u(\cdot, t), v) = (f(\cdot, t), v)_\Omega \quad \text{for each } v \in T(\Omega),$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

1.5 Timoshenko beam model

1.5.1 Introduction

In 1921 Timoshenko [Tim21] improved the theory for beams. The model is known as the Timoshenko model and is generally considered to be an improvement on the better known Euler-Bernoulli model (also known as the classical or engineering beam theory) and Rayleigh models. Although the Euler-Bernoulli model is commonly used in engineering applications, it is known that the Timoshenko model is superior in predicting the transient response of a beam. The reason why this needs to be considered, is put aptly by [Wu03]: "The transient response plays an important role in many aspects of structural analysis for engineering applications". The superiority of the Timoshenko model is more pronounced for beams with a low aspect ratio (shorter beams) or beams that may have large expected deflections. See also [Inm94] or [LVV09]. In Subsection 1.5.5 we show that the Euler-Bernoulli and Rayleigh models can be derived from the Timoshenko model if additional assumptions are made.

The Timoshenko beam theory is remarkable in the sense that it is a one-dimensional theory that compares well to higher dimensional theories, see [LVV09] and [SP06]. (This is true for beam applications.) In this dissertation the theory is useful to provide examples for applications of existence theory and finite element method (FEM) convergence theory.

1.5.2 Viscous damping

The mathematical model consists of two partial differential equations, one for the deflection w of the beam and the other for the angle ϕ due to the rotation of a cross section. In this subsection we introduce the equations of motion and the constitutive equations and write the model in dimensionless form.

Equations of motion

$$\rho A \partial_t^2 w = \partial_x F - k_1 \partial_t w + q \tag{1.5.1}$$

$$\rho I \partial_t^2 \phi = F + \partial_x M - k_2 \partial_t \phi \tag{1.5.2}$$

In these equations ρ denotes the density, A the area of the cross section, I the area moment of inertia, M the moment, F the shear force and q the load. The terms $k_1\partial_t w$ and $k_2\partial_t\phi$ are the damping terms of viscous type. The viscous type damping in (1.5.2) is physically unrealistic since ϕ is the rotation of the cross-section and is not moving through any medium.

In [Sem94] and [FXX99] the authors have a damping term in (1.5.2). However we proceed with these damping terms since it does not affect the theory from a mathematical point of view. (It is a special case of weak damping.) We offer an explanation after deriving the dimensionless form. Other forms, such as material damping, were not even discussed in either [Sem94] or [FXX99].

Constitutive equations

$$F = AG\kappa^2(\partial_x w - \phi) \tag{1.5.3}$$
$$M = EI\partial_x\phi \tag{1.5.4}$$

where E and G are elastic constants and κ^2 the shear correction factor.

Dimensionless form

Denote the length of the beam by ℓ and let

$$\tau = \frac{t}{t_0} \quad \text{and} \quad \xi = \frac{x}{\ell}.$$

A convenient choice for t_0 is

$$t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}.$$

We transform the variables as follows:

$$\begin{aligned}
w^*(\xi,\tau) &= \frac{w(x,t)}{\ell}\\
\phi^*(\xi,\tau) &= \phi(x,t)\\
F^*(\xi,\tau) &= \frac{F(x,t)}{AG\kappa^2}\\
M^*(\xi,\tau) &= \frac{M(x,t)}{AG\kappa^2\ell}\\
q^*(\xi,\tau) &= \frac{q(x,t)\ell}{AG\kappa^2}
\end{aligned}$$

We introduce the following dimensionless constants

$$\alpha = \frac{A\ell^2}{I}, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}$$

and for damping

$$\mu_1 = \frac{k_1 \ell}{A\sqrt{\rho G \kappa^2}} \quad \text{and} \quad \mu_2 = \frac{k_2}{\rho A \ell^2}.$$

Using the original notation, we can now write the Timoshenko beam model with viscous damping in dimensionless form.

Problem T

Equations of motion:

$$\partial_t^2 w = \partial_x F - \mu_1 \partial_t w + q \tag{1.5.5}$$

$$\frac{1}{\alpha}\partial_t^2 \phi = F + \partial_x M - \mu_2 \partial_t \phi \tag{1.5.6}$$

Constitutive equations:

$$F = \partial_x w - \phi \tag{1.5.7}$$

$$M = \frac{1}{\beta}\partial_x \phi \tag{1.5.8}$$

Initial conditions are required for a well posed problem. Let

$$\begin{aligned} w(x,0) = w_0(x), \quad \partial_t w(x,0) = w_1(x), \\ \phi(x,0) = \phi_0(x), \quad \partial_t \phi(x,0) = \phi_1(x) \end{aligned} \tag{1.5.9}$$

for $x \in (0,1)$.

Remark The dimensionless model in [Sem94] is:

$$\partial_t^2 w + \delta \partial_t w + d^{-2}\partial_x(\phi - \partial_x w) = q, \tag{1.5.10}$$

$$\partial_t^2 \phi + \delta \partial_t \phi + d^{-2}(\phi - \partial_x w) - \partial_x^2 \phi = 0. \tag{1.5.11}$$

Equations (1.5.10) and (1.5.11) are obtained by adding inertia and damping terms to the steady state problem in [Arn81]. The resulting equations are questionable. The same model is considered in [FXX99].

The authors of [Arn81], [Sem94] and [FXX99] consider only boundary conditions where the beam is clamped at both ends:

$$w(0,t) = w(1,t) = \phi(0,t) = \phi(1,t) = 0 \quad \text{for} \quad t > 0.$$

Other boundary conditions are not even mentioned. We will not follow these articles in considering boundary conditions where the beam is clamped at both ends. We will rather consider various cases of different boundary conditions and discuss how the theory changes with different boundary conditions.

Cantilever beam

The boundary conditions for the cantilever beam are given by

$$w(0,t) = \phi(0,t) = F(1,t) = M(1,t) = 0 \quad \text{for} \quad t > 0. \tag{1.5.12}$$

Pinned-pinned beam

The boundary conditions for the pinned-pinned beam are given by

$$w(0,t) = w(1,t) = M(0,t) = M(1,t) = 0 \quad \text{for} \quad t > 0. \tag{1.5.13}$$

Remark It is possible that the beam may be tapered, thus the cross-sectional area A and area moment I may vary along the length of the beam. Consequently α and β are not constant. We may assume that these parameters are bounded above and below by positive constants. The same argument holds for the damping constants μ_1 and μ_2. These possibilities do not affect the theory.

1.5.3 Cantilever beam with boundary damping

Boundary damping (or control) is introduced in [KR87] to damp unwanted vibrations. It is modelled by

$$\begin{aligned} F(\ell,t) &= -k_1 \partial_t w(\ell,t), \\ M(\ell,t) &= -k_2 \partial_t \phi(\ell,t). \end{aligned}$$

Introduce dimensionless constants $\nu_1 = \dfrac{k_1 \ell}{TAG\kappa^2}$ and $\nu_2 = \dfrac{k_2}{AG\kappa^2 \ell T}$. Then the dimensionless form is given by

$$\begin{aligned} F(1,t) &= -\nu_1 \partial_t w(1,t), \\ M(1,t) &= -\nu_2 \partial_t \phi(1,t). \end{aligned}$$

The mathematical model consists of equations of motion (1.5.5)-(1.5.6) with $\mu_1 = \mu_2 = 0$, constitutive equations (1.5.7)-(1.5.8) and initial conditions

(1.5.9). The boundary conditions for the cantilever beam are given by

$$
\begin{aligned}
w(0,t) &= \phi(0,t) = 0, \\
F(1,t) &= -\nu_1 \partial_t w(1,t), \\
M(1,t) &= -\nu_2 \partial_t \phi(1,t).
\end{aligned}
\tag{1.5.14}
$$

The article [KR87] drew a lot of attention. Our interest lies in the fact that the damping is neither weak nor strong.

1.5.4 Variational forms

In this subsection we write the Timoshenko beam model in variational form. Multiply equation (1.5.5) by an arbitrary function v and integrate. Using integration by parts yields

$$
\begin{aligned}
\int_0^1 \partial_t^2 w v &= \int_0^1 \partial_x F v - \int_0^1 \mu_1 \partial_t w v + \int_0^1 q v \\
&= -\int_0^1 F v' - \int_0^1 \mu_1 \partial_t w v + \int_0^1 q v \\
&\quad + F(1,t)v(1) - F(0,t)v(0).
\end{aligned}
\tag{1.5.15}
$$

Similarly multiply equation (1.5.6) by an arbitrary function ψ and integrate. Again using integration by parts yields

$$
\begin{aligned}
\frac{1}{\alpha}\int_0^1 \partial_t^2 \phi \psi &= \int_0^1 F\psi + \int_0^1 \partial_x M \psi - \int_0^1 \mu_2 \partial_t \phi \psi \\
&= \int_0^1 F\psi - \int_0^1 M\psi' - \int_0^1 \mu_2 \partial_t \phi \psi \\
&\quad + M(1,t)\psi(1) - M(0,t)\psi(0).
\end{aligned}
\tag{1.5.16}
$$

The test function space for the cantilever beam is given by

$$
T[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}.
$$

We can now write Problem T in standard variational form.

Problem T-V

Find a pair of functions $\langle w, \phi\rangle$ such that for each $t > 0$, $w(\cdot,t) \in T[0,1]$, $\phi(\cdot,t) \in T[0,1]$ and the following equations hold

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v =& -\int_0^1 \partial_x w(\cdot,t)v' + \int_0^1 \phi(\cdot,t)v' \\ &- \int_0^1 \mu_1 \partial_t w(\cdot,t)v + \int_0^1 q(\cdot,t)v, \qquad (1.5.17)\\ \int_0^1 \frac{1}{\alpha}\,\partial_t^2 \phi(\cdot,t)\psi =& -\int_0^1 \frac{1}{\beta}\partial_x \phi(\cdot,t)\psi' - \int_0^1 \mu_2 \partial_t \phi(\cdot,t)\psi \\ &+ \int_0^1 \partial_x w(\cdot,t)\psi - \int_0^1 \phi(\cdot,t)\psi \qquad (1.5.18)\end{aligned}$$

for each $\langle v,\psi\rangle \in T[0,1]\times T[0,1]$, while $\langle w(\cdot,0), \phi(\cdot,0)\rangle = \langle w_0, \phi_0\rangle$ and $\langle \partial_t w(\cdot,0), \partial_t \phi(\cdot,0)\rangle = \langle w_1, \phi_1\rangle$.

Remark Equations (1.5.17)-(1.5.18) are used for the application of the FEM. For the theory we need a single variational equation. Adding equations (1.5.17)-(1.5.18), we obtain

$$\begin{aligned}&\int_0^1 \partial_t^2 w(\cdot,t)v + \frac{1}{\alpha}\int_0^1 \partial_t^2 \phi(\cdot,t)\psi \\ =\;& -\int_0^1 (\partial_x w(\cdot,t) - \phi(\cdot,t))(v' - \psi) - \frac{1}{\beta}\int_0^1 \partial_x \phi(\cdot,t)\psi' \\ & -\int_0^1 \mu_1 \partial_t w(\cdot,t)v - \int_0^1 \mu_2 \partial_t \phi(\cdot,t)\psi + \int_0^1 q(\cdot,t)v. \qquad (1.5.19)\end{aligned}$$

Pinned-pinned beam

To accommodate the boundary conditions for the pinned-pinned beam, different test functions need to be used. The test functions $T_0[0,1]$ are all functions $v \in C^1[0,1]$ for which $v(0) = v(1) = 0$. Now, $w(\cdot,t) \in T_0[0,1]$, $\phi(\cdot,t) \in C^1[0,1]$ and equations (1.5.17) and (1.5.18) hold for each $\langle v,\psi\rangle \in T_0[0,1]\times C^1[0,1]$.

Boundary damping (or control)

Using the boundary conditions (1.5.14), equations (1.5.17) and (1.5.18) become

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v &= -\int_0^1 F(\cdot,t)v' - \nu_1\partial_t w(1,t)v(1) + \int_0^1 q(\cdot,t)v,\\ \int_0^1 \frac{1}{\alpha}\,\partial_t^2\phi(\cdot,t)\psi &= -\int_0^1 \frac{1}{\beta}\,\partial_x\phi(\cdot,t)\psi' + \int_0^1 F(\cdot,t)\psi - \nu_2\partial_t\phi(1,t)\psi(1)\end{aligned}$$

for each $\langle v,\psi\rangle \in T[0,1]\times T[0,1]$.

1.5.5 Euler-Bernoulli and Rayleigh beam models

As mentioned in the introduction of this section, the Euler-Bernoulli and Rayleigh beam models are special cases of the Timoshenko beam model. In these models it is assumed that the cross-section remains perpendicular to the neutral surface, that is $\partial_x w = \phi$.

We first combine the equations of motion (1.5.5) and (1.5.6) (with $\mu_2 = 0$) to eliminate the shear force F and find that

$$\partial_t^2 w = \frac{1}{\alpha}\partial_t^2\partial_x\phi - \partial_x^2 M - \mu_1\partial_t w + q.$$

Then, assuming $\partial_x w = \phi$,

$$\partial_t^2 w = \frac{1}{\alpha}\partial_t^2\partial_x^2 w - \partial_x^2 M - \mu_1\partial_t w + q.$$

Using the constitutive equation $M = \dfrac{1}{\beta}\partial_x\phi = \dfrac{1}{\beta}\partial_x^2 w$ yields

$$\partial_t^2 w = \frac{1}{\alpha}\partial_t^2\partial_x^2 w - \frac{1}{\beta}\partial_x^4 w - \mu_1\partial_t w + q.$$

The constitutive equation for the shear force is now redundant. This model is sometimes referred to as the Rayleigh model.

The term $\dfrac{1}{\alpha}\partial_t^2\partial_x^2 w$ in the equation above is referred to as the rotary inertia term. The Euler-Bernoulli model is a special case of the Rayleigh model where the rotary inertia term is ignored.

Variational form

For the variational form for the Rayleigh beam model consider equations (1.5.15) and (1.5.16) with $\mu_2 = 0$. It is admissible to substitute ψ by v' if $v'(0) = 0$. Adding the two equations yields

$$\int_0^1 \partial_t^2 wv + \frac{1}{\alpha}\int_0^1 \partial_t^2 \phi v' = -\int_0^1 Mv'' + \int_0^1 qv - \int_0^1 \mu_1 \partial_t wv + F(1,t)v(1) \\ - F(0,t)v(0) + M(1,t)v'(1) - M(0,t)v'(0). \tag{1.5.20}$$

For a cantilever beam the test functions are

$$T_R[0,1] = \{v \in C^1[0,1] \mid v(0) = v'(0) = 0\}.$$

Substituting the assumption $\partial_x w = \phi$ and the constitutive equation $M = \dfrac{1}{\beta}\partial_x \phi = \dfrac{1}{\beta}\partial_x^2 w$, equation (1.5.20) becomes

$$\int_0^1 \partial_t^2 wv + \frac{1}{\alpha}\int_0^1 \partial_t^2 \partial_x wv' + \frac{1}{\beta}\int_0^1 \partial_x^2 wv'' + \int_0^1 \mu_1 \partial_t wv = \int_0^1 qv \tag{1.5.21}$$

for each $v \in T_R[0,1]$.

We can now write the variational form for the Rayleigh beam model.

Problem R-V
Find a function w such that for $t > 0$, $w(\cdot,t) \in T_R[0,1]$ and equation (1.5.21) holds for each $v \in T_R[0,1]$, while $w(\cdot,0) = w_0$ and $w'(\cdot,0) = w_1$.

Chapter 2

Existence theory for linear problems

As mentioned before, it is well known that a model problem may not have a solution in the classical sense. For this reason existence theory (although theoretical) is of great practical importance. It is desirable to determine whether a given problem is well posed before considering a numerical approximation for the solution.

The broad strategy in existence theory is to write a problem as an abstract problem in a Banach or Hilbert space. From this abstract equation some authors use semigroup theory to prove existence of solutions.

This chapter is mostly devoted to the article [VV02]. The proofs are given in greater detail to render them more readable. Additional propositions and lemmas are also formulated for this purpose.

2.1 Weak variational form for the Rayleigh beam model

Consider as an example the variational form of the Rayleigh beam model, Problem R-V, given in Section 1.5.5. It is repeated here for the sake of completeness.

Problem R-V

Find a function w such that for $t > 0$, $w(\cdot, t) \in T_R[0,1]$ and

$$\int_0^1 \partial_t^2 wv + \frac{1}{\alpha}\int_0^1 \partial_t^2\partial_x wv' + \frac{1}{\beta}\int_0^1 \partial_x^2 wv'' + \int_0^1 \mu_1 \partial_t wv = \int_0^1 qv$$

for each $v \in T_R[0,1]$, while $w(\cdot, 0) = w_0$ and $w'(\cdot, 0) = w_1$.

Define the following bilinear forms:

$$\begin{aligned} a(u,v) &= \int_0^1 \mu_1 uv \\ b(u,v) &= \frac{1}{\beta}\int_0^1 u''v'' \\ c(u,v) &= \int_0^1 uv + \frac{1}{\alpha}\int_0^1 u'v'. \end{aligned}$$

Then Problem R-V can be written in terms of bilinear forms.

Find a function w such that for $t > 0$, $w(\cdot, t) \in T_R[0,1]$ and

$$c(\partial_t^2 w(\cdot,t), v) + a(\partial_t w(\cdot, v), v) + b(w(\cdot,t), v) = (q(\cdot,t), v) \tag{2.1.1}$$

for each $v \in T_R[0,1]$, while $w(\cdot, 0) = w_0$ and $w'(\cdot, 0) = w_1$.

Let J be an open interval containing zero. J may be a bounded or unbounded interval. Instead of considering a function w defined on $[0,1] \times J$, we consider a function $u : J \to H^1(0,1)$. If the problem has a classical solution, then $u(t)(x) = w(x,t)$.

Definition 2.1.1. Derivative
Let t be any interior point of J. Suppose there exists an element $\phi \in Y$ such that

$$\lim_{h\to 0} \|h^{-1}(u(t+h) - u(t)) - \phi\|_Y = 0,$$

then ϕ is the derivative of u at t. $\|\cdot\|_Y$ may be the $\mathcal{L}^2$-norm or the H^1-norm.

We denote the derivative by $u'(t)$ and write $u'(t) \in Y$ to show that the derivative exists with respect to the norm of Y. The derivative (function) u' is defined as $u'(t)$ for every $t \in J$, with the second derivative defined by $u'' = (u')'$.

To construct a weak formulation of the problem, extend the definition of b to $H^2(0,1)$ and the definition of c to $H^1(0,1)$. Let $V(0,1)$ be the closure of $T_R[0,1]$ in $H^1(0,1)$ and let $\tilde{q} : t \mapsto q(\cdot,t)$. Then Problem R-V can be written in weak variational form.

Problem R-W
Find a function u such that for $t > 0$, $u(t) \in V(0,1)$, $u'(t) \in V(0,1)$, $u''(t) \in \mathcal{L}^2(0,1)$ and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (\tilde{q}(t),v)$$

for each $v \in V(0,1)$, while $u(0) = w_0$ and $u'(0) = w_1$.

Remark Problems R-V and R-W are not equivalent. A solution of Problem R-W is called a weak solution. If a solution of Problem R-W is sufficiently smooth, it will be a solution of Problem R-V.

2.2 General linear vibration problem

Let V, W and X denote real Hilbert spaces with $V \subset W \subset X$. Let a, b and c be bilinear forms where a and b are defined on V and c is defined on W. The following inner products and associated norms are introduced.

V has inner product $b(\cdot,\cdot)$ and norm $\|\cdot\|_V$,

W has inner product $c(\cdot,\cdot)$ and norm $\|\cdot\|_W$,

X has inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$.

Remark It is important to note that in many applications W and X are the same linear spaces and in these cases it can be proved that the norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent, see Chapter 3.

Notation

$u \in C(J,Y)$ if u is continuous on J with respect to the norm of Y.

$u \in C^k(J,Y)$ if $u^{(k)} \in C^k(J,Y)$.

In [VV02] the authors consider the following general weak problem.

Problem G

Given a function $f : J \to X$, find a function $u \in C(J, V)$ such that u' is continuous at 0 with respect to $\| \cdot \|_W$ and for each $t \in J$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X \text{ for each } v \in V \qquad (2.2.1)$$

while $u(0) = u_0$, $u'(0) = u_1$.

Assumptions
The following assumptions are made for the existence results of this chapter.

E1 V is dense in W and W is dense in X.

E2 There exists a constant $\beta_1 > 0$ such that $\|v\|_W \leq \beta_1 \|v\|_V$ for each $v \in V$.

E3 There exists a constant $\beta_2 > 0$ such that $\|w\|_X \leq \beta_2 \|w\|_W$ for each $w \in W$.

E4 The bilinear form a is non-negative, symmetric and bounded on V, i.e. there exists a positive constant K such that $|a(u, v)| \leq K\|u\|_V \|v\|_V$ for $u, v \in V$.

The following theorems are the main results of this chapter. These results will be used in subsequent chapters to prove the existence results for various applications. The proofs are provided in Sections 2.5 to 2.7.

Theorem 2.2.1.
Suppose Assumptions **E1**, **E2**, **E3** *and* **E4** *hold. If, for $u_0 \in V$ and $u_1 \in V$, there exists some $y \in W$ such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \text{ for each } v \in V, \qquad (2.2.2)$$

then for each $f \in C^1([0, T), X)$ there exists a unique solution

$$u \in C([0, T), V) \cap C^1([0, T), W) \cap C^1((0, T), V) \cap C^2((0, T), W)$$

for Problem G. If $f = 0$ then $u \in C^1([0, \infty), V) \cap C^2([0, \infty), W)$.

Proof See Section 2.5.

Remark In the formulation of Theorem 1 of [VV02], the condition on u_1 is that $u_1 \in W$ instead of $u_1 \in V$ as in Theorem 2.2.1. In the proof of Theorem 1 however it is assumed that that $u_1 \in V$.

Definition 2.2.1. The space E_b

$E_b = \{x \in V \mid \textit{there exists } a\, y \in W \textit{ such that } c(y,v) = b(x,v) \textit{ for each } v \in V\}$.

Two special damping cases are considered. In the first case, the following assumption is made instead of Assumption E4.

Assumption E4W
The bilinear form a is non-negative, symmetric and bounded on W, i.e. there exists a positive constant K such that $|a(u,v)| \leq K\|u\|_W\|v\|_W$ for $u, v \in W$.

If Assumption E4W holds, the damping is referred to as *weak damping.*

Theorem 2.2.2. Weak damping
Suppose Assumptions **E1***,* **E2***,* **E3** *and* **E4W** *hold. Then there exists a unique solution*

$$u \in C^1(J,V) \cap C^2(J,W)$$

for Problem G for each $u_0 \in E_b$*, each* $u_1 \in V$ *and each* $f \in C^1(J,X)$*. If* $f = 0$ *then*

$$u \in C^1((-\infty,\infty),V) \cap C^2((-\infty,\infty),W).$$

Proof See Section 2.6.

Remarks

1. Theorem 2.2.2 still applies when the bilinear form a is zero. See e.g. Chapter 3.

2. It is proved in Lemma 2.6.2 that in the case of weak damping, $u_0 \in E_b$ and $u_1 \in V$ imply that (2.2.2) holds.

Now consider the second special case. If the following additional assumption holds, the damping is referred to as *strong damping.*

Assumption E5S
The bilinear form a is positive definite on V, that is there exists a constant $K > 0$ such that $a(u,u) \geq K\|u\|_V^2$ for all $u \in V$.

The following definition is not in [VV02], but is given here for the sake of convenience.

Definition 2.2.2. Locally Hölder continuous function
Let J be an interval and $\alpha \in (0,1]$. A function $f : J \to H$ is Hölder continuous on J with exponent α if there exists a $L > 0$ such that

$$\|f(t) - f(s)\| \leq L|t-s|^{\alpha} \quad \text{for } t, s \in J.$$

The function f is locally Hölder continuous on J (with exponent α) if every $t \in J$ has a neighbourhood in which f is Hölder continuous (with exponent α).

Remark If a function $f : J \to H$ is locally Lipschitz, it is locally Hölder continuous with exponent one and if $f \in C^1(J,H)$, it is locally Lipschitz.

Theorem 2.2.3. Strong damping
Suppose assumptions **E1**, **E2**, **E3**, **E4**, *and* **E5S** *hold. Let $f : [0,T] \to W$ be Hölder continuous. Then there exists a unique solution*

$$u \in C([0,T),V) \cap C^1([0,T),W) \cap C^2((0,T),W)$$

for Problem G for any $u_0 \in V$, $u_1 \in W$. If $f = 0$, then

$$u \in C([0,\infty),V) \cap C^1([0,\infty),W) \cap C^{\infty}((0,\infty),V).$$

Proof See Section 2.7.

Although there are other existence results, e.g. in [AKS96] and [Sho77], the results from [VV02] are more convenient since it is formulated in terms of bilinear forms.

The problem in [Sho77] is to find

$$u \in C([0,\infty),V) \cap C^1((0,\infty),V) \cap C^1([0,\infty),W) \cap C^2((0,\infty),W)$$

such that $u(0) = u_0$, $u'(0) = u_1$ and

$$Cu''(t) + Bu'(t) + Au(t) = f(t) \quad \text{for all} \quad t > 0 \tag{2.2.3}$$

where $u_0 \in V$, $u_1 \in W$ and $f \in C((0,\infty),W')$ are given.

Theorem 2.2.4. Existence theorem in [Sho77]
Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume $A \in \mathcal{L}(V,V')$ and $C \in \mathcal{L}(W,W')$ are Riesz maps of V and W, respectively, and let B be linear from the subspace $\mathcal{D}(B)$ of V into V'. Assume that B is monotone and that $A+B+C : \mathcal{D}(B) \to V'$ is surjective. Then for every $f \in C^1([0,\infty),W')$ and $u_0 \in V$, $u_1 \in \mathcal{D}(B)$ with $Au_0 + Bu_1 \in W'$, there exists a unique solution $u(t)$ of (2.2.3) *(on $t \geq 0$) with $u(0) = u_0$ and $u'(0) = u_1$.*

In our view the existence theorem in [Sho77] is inconvenient to apply to problems in weak variational form.

2.3 Abstract differential equation

The aim is to write Problem G as an initial value problem for an abstract first order differential equation in the Hilbert space $V \times W$. Semigroup theory can then be applied. To start off, Problem G is written as a first order system.

$$\begin{gathered} u'(t) = w(t) \quad \text{and} \\ c(w'(t),v) + a(w(t),v) + b(u(t),v) = (f(t),v)_X \quad \text{for all} \quad v \in V. \end{gathered} \tag{2.3.1}$$

We deviate from [VV02] in constructing a bounded linear mapping from X to W. This mapping is then used to rewrite (2.3.1). The result of the next lemma is contained in the proof of Lemma 6 in [VV02].

Lemma 2.3.1.
For each $x \in X$ there exists a unique $w \in W$ such that

$$c(w,v) = (x,v)_X \quad \textit{for all } v \in W$$

and

$$\|w\|_W \leq \beta_2 \|x\|_X.$$

Proof Consider any $x \in X$. Let $\phi(v) = (x,v)_X$ for all $v \in W$. Then, from Assumption E3, ϕ is a bounded linear functional on W. It then follows from Riesz's theorem that there exists a unique $w \in W$ such that

$$c(w,v) = \phi(v) = (x,v)_X \quad \text{for all } v \in W.$$

Furthermore, from the Cauchy-Schwarz inequality and Assumption E3,

$$\|w\|_W^2 = c(w,w) = (x,w)_X \leq \|x\|_X \|w\|_X \leq \beta_2 \|x\|_X \|w\|_W.$$

□

It follows from Lemma 2.3.1 that there exists a bounded linear mapping from X to W. Use P to denote the mapping Pf of a function f, that is $(Pf)(t) = Pf(t)$ for each $t \in J$. Note that this mapping is not explicitly defined in [VV02].

Corollary 2.3.1.
For any function $f : J \to X$ there exists a unique function $g : J \to W$ such that $g(t) = Pf(t)$ for each $t \in J$.

Remark Let $f(t) = y \in \mathbb{R}$ in Problem G and let $u \in V$ be the equilibrium, that is $b(u, v) = (y, v)_X$ for all $v \in V$. Then Lemma 2.3.1 implies that there exists a unique $w \in W$ such that $c(w, v) = (y, v)_X = b(u, v)$ for all $v \in V$. Hence $u \in E_b$. Thus solutions of physically realistic equilibrium problems are contained in E_b. In [VV02] it is stated that E_b corresponds to solutions of physically realistic equilibrium problems. However, in some cases the space E_b is equal to the set of equilibrium states.

We can now rewrite equation (2.3.1):

$$c(w'(t), v) + a(w(t), v) + b(u(t), v) = c(g(t), v) \quad \text{for all} \quad v \in V \tag{2.3.2}$$

where $g : J \to W$.

To apply semigroup theory, the first order system must be written in the form

$$x' = Ax + f.$$

It is now necessary to use the bilinear forms a, b and c to define the linear operator A.

Let $H = V \times W$ and denote the element $x \in H$ and its components by $x = \langle x_1, x_2 \rangle$. The inner product $(\cdot, \cdot)_H$ is defined by

$$(x, y)_H = b(x_1, y_1) + c(x_2, y_2) \quad \text{for all} \quad x, y \in H.$$

Following [VV02], we define a mapping Λ. This mapping will be used to define an operator A, which will in turn be used to write (2.3.2) as a first order abstract differential equation. Semigroup theory can then be applied. The next proposition is given as Lemma 1 in [VV02].

Proposition 2.3.1.
Suppose $\lambda \geq 0$. For each $y \in H$ there exists a unique $x \in H$ such that

$$\lambda x_1 - x_2 = y_1 \quad \textit{and}$$
$$b(x_1, v) + a(x_2, v) + \lambda c(x_2, v) = c(y_2, v) \quad \textit{for all} \quad v \in V.$$

Proof Set $F(v) = a(y_1, v) + c(\lambda y_1 + y_2, v)$ for each $v \in V$. It follows from Assumptions E2, E4 and the Cauchy-Schwarz inequality that

$$\begin{aligned}
|F(v)| &\leq |a(y_1, v)| + |c(\lambda y_1 + y_2, v)| \\
&\leq K\|y_1\|_V\|v\|_V + (\lambda\|y_1\|_W + \|y_2\|_W)\|v\|_W \\
&\leq K\|y_1\|_V\|v\|_V + \lambda\beta_1^2\|y_1\|_V\|v\|_V + \beta_1\|y_2\|_W\|v\|_V \\
&\leq C(\|y_1\|_V + \|y_2\|_W)\|v\|_V
\end{aligned}$$

where $C = \max\{K + \lambda\beta_1^2,\ \beta_1\}$. So the linear functional F is bounded on V.

Since a is non-negative, $b(v, v) + \lambda a(v, v) + \lambda^2 c(v, v) \geq b(v, v)$. Consequently $b + \lambda a + \lambda^2 c$ defines an inner product on V. Since F is a bounded linear functional on V, it follows from Riesz's theorem that there exists a unique $x_1 \in V$ such that $b(x_1, v) + \lambda a(x_1, v) + \lambda^2 c(x_1, v) = F(v)$ for all $v \in V$. Let $x_2 = \lambda x_1 - y_1$, then $\langle x_1, x_2\rangle$ is the solution. □

Note that if $a = 0$, $b + \lambda a + \lambda^2 c$ still defines an inner product on V. Hence the result remains valid in the undamped case.

Definition 2.3.1. The mapping Λ
For $\lambda = 0$, the mapping Λ is defined on H by $\Lambda y = -x$ where $-x_2 = y_1$ and $x_1 \in V$ such that

$$b(x_1, v) + a(x_2, v) = c(y_2, v) \quad \textit{for all} \quad v \in V.$$

Remark If x is in the range of Λ, then x_1 and x_2 are in V.

Lemma 2 in [VV02] is split into the following two propositions for greater clarity.

Proposition 2.3.2.
Λ is a bounded linear operator.

Proof Let $\alpha > 0$ and let $v \in V$. Consider $-x = \Lambda(\alpha(y+z))$. The definition of Λ implies that

$$-x_2 = \alpha(y_1 + z_1) \quad \text{and} \tag{2.3.3}$$

$$b(x_1, v) = c(\alpha(y_2 + z_2), v) - a(x_2, v) \quad \text{for all } v \in V. \tag{2.3.4}$$

Also consider $-\tilde{x} = \Lambda y$ and $-\tilde{w} = \Lambda z$. By definition we have that

$$-\tilde{x}_2 - \tilde{w}_2 = y_1 + z_1 \quad \text{and} \tag{2.3.5}$$

$$b(\tilde{x}_1 + \tilde{w}_1, v) = c(y_2 + z_2, v) - a(\tilde{x}_2 + \tilde{w}_2, v) \quad \text{for all } v \in V. \tag{2.3.6}$$

It follows from (2.3.3) and (2.3.5) that $x_2 = \alpha\tilde{x}_2 + \alpha\tilde{w}_2$. Hence

$$b(\alpha(\tilde{x}_1 + \tilde{w}_1), v) = \alpha c(y_2 + z_2, v) - a(x_2, v) \quad \text{for all } v \in V. \tag{2.3.7}$$

Since b is an inner product on V, we have from (2.3.4) and (2.3.7) that $x_1 = \alpha\tilde{x}_1 + \alpha\tilde{w}_1$. Consequently

$$\Lambda(\alpha(y + z)) = -x = -\alpha\tilde{x} - \alpha\tilde{w} = \alpha\Lambda y + \alpha\Lambda z.$$

To prove boundedness, suppose $\Lambda y = -x$. It follows from the definition of the operator Λ that

$$-x_2 = y_1 \quad \text{and} \tag{2.3.8}$$
$$b(x_1, v) + a(x_2, v) = c(y_2, v) \quad \text{for all } v \in V. \tag{2.3.9}$$

Then, by Assumption E2 and equation (2.3.8)

$$\|x_2\|_W \leq \beta_1\|x_2\|_V = \beta_1\|y_1\|_V. \tag{2.3.10}$$

Also, with $v = x_1$, it follows from (2.3.9), the Cauchy-Schwarz inequality and Assumptions E2 and E4 that

$$\begin{aligned}
\|x_1\|_V^2 &= b(x_1, x_1) \\
&\leq |a(x_2, x_1)| + |c(y_2, x_1)| \\
&\leq K\|x_2\|_V\|x_1\|_V + \beta_1\|y_2\|_W\|x_1\|_V \\
&= K\|y_1\|_V\|x_1\|_V + \beta_1\|y_2\|_W\|x_1\|_V.
\end{aligned}$$

Consequently

$$\|x_1\|_V \leq K\|y_1\|_V + \beta_1\|y_2\|_W. \tag{2.3.11}$$

Combining (2.3.10) and (2.3.11) and applying the inequality $2\alpha\mu \leq \alpha^2 + \mu^2$, it follows that there exists a constant C such that

$$\|x_1\|_V^2 + \|x_2\|_W^2 \leq C(\|y_1\|_V^2 + \|y_2\|_W^2) = C\|y\|_H^2.$$

We have that

$$\|\Lambda y\|_H^2 = \|x\|_H^2 = \|x_1\|_V^2 + \|x_2\|_W^2 \leq C\|y\|_H^2.$$

Thus Λ is bounded. $\square$

Proposition 2.3.3.
The nullspace of Λ is trivial.

Proof Assume that $\Lambda y = 0$ for some $y \in H$. Thus $y_1 = 0$ and $c(y_2, v) = 0$ for all $v \in V$ from the definition of Λ. Since V is dense in W it follows that for any $w \in W$ there exists a sequence $(v_n) \in V$ such that $\|v_n - w\|_W \to 0$ as $n \to \infty$. It follows from the Cauchy-Schwarz inequality that

$$|c(y_2, v_n - w)| \leq \|y_2\|_W \|v_n - w\|_W \to 0 \text{ as } n \to \infty.$$

Since c is continuous we have that $c(y_2, v_n) \to c(y_2, w)$ as $n \to \infty$. But $c(y_2, v_n) = 0$ for each v_n, therefore $c(y_2, w) = 0$. Since this is true for any $w \in W$ and $y_2 \in W$, it follows that $y_2 = 0$. □

Remark Since the nullspace of Λ is trivial, Λ is one-to-one.

Some preparation is needed for the proof of the following proposition. The next result follows from [Kre78, Lemma 3.3-2, p.145].

If a closed subspace M is not dense in H, then it follows that for $x \in H$ but $x \notin M$, there exists a $y \in M$ such that $y \neq x$ and $x - y$ is orthogonal to M, since M is a closed subspace of H. If we let $w = x - y$, then $w \neq 0$ and

$$(w, z)_H = 0 \text{ for each } z \in M. \tag{2.3.12}$$

Notation Denote the range of Λ by $\mathcal{R}(\Lambda)$ and its closure by $\overline{\mathcal{R}(\Lambda)}$. The domain of Λ is denoted by $\mathcal{D}(\Lambda)$.

The following proposition is formulated in order to clarify the flow of ideas in [VV02].

Proposition 2.3.4.
$\mathcal{R}(\Lambda)$ is dense in H.

Proof This proposition is proved by means of a contradiction. Suppose that $\mathcal{R}(\Lambda)$ is not dense in H. Then it follows from (2.3.12) and the definition of $(\cdot,\cdot)_H$ that

$$b(w_1, z_1) + c(w_2, z_2) = (w, z)_H = 0 \text{ for each } z \in \overline{\mathcal{R}(\Lambda)} \tag{2.3.13}$$

where $w \neq 0$.

Let $y = \langle w_1, w_1 + w_2 \rangle$. Then $y \in H$ and there exists a $x \in \mathcal{R}(\Lambda)$ such that $\Lambda y = -x$. It follows from the definition of Λ that

$$x_2 = -w_1 \quad \text{and} \tag{2.3.14}$$
$$b(x_1, v) + a(x_2, v) = c(w_1 + w_2, v) \quad \text{for each } v \in V. \tag{2.3.15}$$

Rewriting equation (2.3.15) we find that for each $v \in V$

$$b(x_1, v) - c(w_2, v) = c(w_1, v) - a(x_2, v).$$

Since $w_1 \in V$, we have that

$$b(x_1, w_1) - c(w_2, w_1) = c(w_1, w_1) - a(x_2, w_1).$$

It then follows from (2.3.14) and the fact that b is symmetric that

$$b(w_1, x_1) + c(w_2, x_2) = c(w_1, w_1) + a(w_1, w_1).$$

Since $x \in \mathcal{R}(\Lambda)$, it follows from equation (2.3.13) that

$$c(w_1, w_1) + a(w_1, w_1) = 0.$$

Therefore, since a is non-negative, $c(w_1, w_1) = 0$ and consequently $w_1 = 0$.

To show that $w_2 = 0$, consider equation (2.3.13) with $w_1 = 0$. Thus $c(w_2, z_2) = 0$ for any $z \in \mathcal{R}(\Lambda)$. Since $z \in \mathcal{R}(\Lambda)$, there exists a $u \in H = V \times W$ such that $\Lambda u = -z$. From the definition of Λ, $-z_2 = u_1$. Hence $z_2 \in V$ and since $z \in \mathcal{R}(\Lambda)$ is arbitrary,

$$c(w_2, v) = 0 \quad \text{for any } v \in V.$$

But V is dense in W, hence

$$c(w_2, v) = 0 \quad \text{for any } v \in W.$$

Since $w_2 \in W$, it follows that $w_2 = 0$.

We have proved that $w = 0$, but $w \neq 0$ by the assumption that $\mathcal{R}(\Lambda)$ is not dense in H. Thus we have a contradiction, which proves the result. □

Definition 2.3.2. Operator A
Let $\mathcal{D}(A) = \mathcal{R}(\Lambda)$ *and* $A = \Lambda^{-1}$.

The next lemma follows from the definitions of the operators Λ and A.

Lemma 2.3.2.
For any $x \in \mathcal{D}(A)$, $Ax = y$ if and only if

$$x_2 = y_1 \quad \textit{and}$$
$$b(x_1, v) + a(x_2, v) = -c(y_2, v) \quad \textit{for all } v \in V.$$

Corollary 2.3.2.
$x \in \mathcal{D}(A)$ if and only if $x_2 \in V$ and there exists a $y \in W$ such that

$$b(x_1, v) + a(x_2, v) = c(y, v) \quad \textit{for all } v \in V. \tag{2.3.16}$$

Lemma 2.3.3.
For any $\lambda \geq 0$, $\mathcal{R}(\lambda - A) = H$.

Proof Let $\lambda \geq 0$. Then it follows from Proposition 2.3.1 that for any $y \in H$ there exists a unique $x \in H$ such that

$$\lambda x_1 - x_2 = y_1 \quad \text{and}$$
$$b(x_1, v) + a(x_2, v) + \lambda c(x_2, v) = c(y_2, v) \quad \text{for all} \quad v \in V.$$

Rewriting this yields

$$-x_2 = y_1 - \lambda x_1 \quad \text{and}$$
$$b(x_1, v) + a(x_2, v) = c(y_2 - \lambda x_2, v) \quad \text{for all } v \in V.$$

It follows from Lemma 2.3.2 that $Ax = \lambda x - y$ and hence $(\lambda - A)x = y$. $\square$

The following corollary and lemma correspond to Lemma 3 in [VV02].

Corollary 2.3.3.
The domain of A is dense in H.

Proof Since $\mathcal{D}(A) = \mathcal{R}(\Lambda)$ the result follows from Proposition 2.3.4. $\square$

A necessary condition for the operator A to be an infinitesimal generator of a semigroup, is that A must be a closed linear operator. This is indeed the case, as proved in the following lemma.

Lemma 2.3.4.
A is a closed linear operator.

Proof The operator Λ is by definition a closed linear operator, since it is defined on all of H and bounded by Proposition 2.3.2. Since A is the inverse of Λ, it follows that A is also a closed linear operator, since a linear operator is closed if its inverse is closed (by definition). $\square$

Lemma 2.3.5.
$(Ax, y)_H = b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2)$ *for all* $x, y \in \mathcal{D}(A)$.

Proof Consider any $x \in \mathcal{D}(A)$ and let $Ax = w$. Then it follows from Lemma 2.3.2 that

$$x_2 = w_1 \quad \text{and} \tag{2.3.17}$$
$$c(w_2, v) = -b(x_1, v) - a(x_2, v) \quad \text{for all } v \in V. \tag{2.3.18}$$

For $y \in \mathcal{D}(A)$, it follows from the definition of the inner product of H and equations (2.3.17) and (2.3.18) that

$$\begin{aligned}(Ax, y)_H = (w, y)_H &= b(w_1, y_1) + c(w_2, y_2) \\ &= b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2).\end{aligned}$$

$\square$

The following corollary is not explicitly formulated in [VV02], but is a consequence of Corollary 3 in [VV02].

Corollary 2.3.4.
The operator A *is dissipative.*

Proof It follows from Lemma 2.3.5 and the symmetry of b that $(Ax, x)_H = -a(x_2, x_2)$ for all $x \in \mathcal{D}(A)$. Since a is non-negative, the result follows. $\square$

2.4 An equivalent first order system

Consider the following initial value problem with A the linear operator constructed in the previous section.

Problem FOS-1

Given a function $F : [0, T) \to H$, find $U \in C([0, T), H)$ such that for each $t \in (0, T)$, $U(t) \in \mathcal{D}(A)$, $U'(t) \in H$ and

$$U'(t) = AU(t) + F(t), \tag{2.4.1}$$
$$U(0) = U_0. \tag{2.4.2}$$

The definition of a solution is not given in [VV02]. It is given here for completeness. We use the same definition for a solution as in [Paz83, p.105].

Definition 2.4.1.
A function U is said to be a solution of Problem FOS-1 if it satisfies equations (2.4.1)-(2.4.2) *and for each $t > 0$, $U(t) \in \mathcal{D}(A)$ and*

$$U \in C([0,T), H) \cap C^1((0,T), H).$$

The following two propositions correspond to Lemma 7 in [VV02].

Proposition 2.4.1.
Let $F(t) = \langle 0, g(t) \rangle$ for each $t \in [0,T)$. If

$$u \in C([0,T), V) \cap C^1([0,T), W)$$

is a solution of Problem G, then $U = \langle u, u' \rangle$ is a solution of Problem FOS-1 with $U_0 = \langle u_0, u_1 \rangle$. Furthermore, if

$$u \in C([0,T), V) \cap C^1([0,T), W) \cap C^1((0,T), V) \cap C^2((0,T), W)$$

then

$$U \in C([0,T), H) \cap C^1((0,T), H).$$

Proof Let u be a solution of Problem G. Then

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = c(g(t), v) \quad \text{for all } v \in V.$$

Rearranging we get that

$$b(u(t), v) + a(u'(t), v) = c(g(t) - u''(t), v) \quad \text{for all } v \in V. \tag{2.4.3}$$

Let $U(t) = \langle u(t), u'(t) \rangle$ for all $t \in [0,T)$. Then, since $u \in C([0,T), V)$ and $u' \in C([0,T), W)$, we have that $U \in C([0,T), H)$. Since $g(t) - u''(t) \in W$ for all $t \in (0,T)$, it follows from (2.4.3) and Corollary 2.3.2 that $U(t) \in \mathcal{D}(A)$. Furthermore, $U'(t) = \langle u'(t), u''(t) \rangle \in H$ since $u'(t) \in V$ and $u''(t) \in W$.

Let $Y(t) \in H$ with $Y_1(t) = U_2(t)$ and $Y_2(t) = u''(t) - g(t)$. Then

$$b(U_1(t), v) + a(U_2(t), v) = -c(Y_2(t), v) \quad \text{for all } v \in V.$$

It then follows from Lemma 2.3.2 that $AU(t) = Y(t)$. That is

$$\begin{aligned} AU(t) &= \langle U_1'(t), U_2'(t) - g(t) \rangle \\ &= \langle U_1'(t), U_2'(t) \rangle - \langle 0, g(t) \rangle \\ &= U'(t) - F(t). \end{aligned}$$

Finally, $U(0) = \langle u(0), u'(0)\rangle = \langle u_0, u_1\rangle$. Thus U is a solution of Problem FOS-1.

Furthermore, if $u \in C^1((0,T),V)$ and $u' \in C^1((0,T),W)$ it follows that $U \in C^1((0,T),H)$. $\square$

Proposition 2.4.2.
Let $F(t) = \langle 0, g(t)\rangle$ for each $t \in [0,T)$. If U is a solution of Problem FOS-1 with $U_0 = \langle u_0, u_1\rangle$, then the first component $U_1 = u$ of U is a solution of Problem G. Furthermore, if

$$U \in C^1((0,T),H)$$

then

$$u \in C([0,T),V) \cap C^1([0,T),W) \cap C^1((0,T),V) \cap C^2((0,T),W).$$

Proof Let U be a solution of Problem FOS-1. Thus $U(t) \in \mathcal{D}(A)$ and

$$\begin{aligned} AU(t) &= U'(t) - F(t) \\ &= \langle U_1'(t), U_2'(t)\rangle - \langle 0, g(t)\rangle \\ &= \langle U_1'(t), U_2'(t) - g(t)\rangle. \end{aligned}$$

It follows from Lemma 2.3.2 that

$$U_2(t) = U_1'(t) \text{ and} \tag{2.4.4}$$
$$b(U_1(t),v) + a(U_2(t),v) = -c(U_2'(t) - g(t), v) \text{ for all } v \in V. \tag{2.4.5}$$

Rewriting this we get that

$$c(U_2'(t),v) + a(U_2(t),v) + b(U_1(t),v) = c(g(t),v) \text{ for all } v \in V. \tag{2.4.6}$$

Let $u(t) = U_1(t)$. Then $u(t) \in V$ since $U(t) \in \mathcal{D}(A)$. It follows from (2.4.4) that $u'(t) = U_1'(t) = U_2(t)$. Also, since $U'(t) \in H$ it follows that $u'(t) \in V$ and $u''(t) \in W$.

It now follows from (2.4.6) that

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = c(g(t),v) \text{ for all } v \in V.$$

Furthermore, since $U \in C([0,T),H)$, we have that $u \in C([0,T),V) \cap C^1([0,T),W)$. Finally, $\langle u(0), u'(0)\rangle = U(0) = \langle u_0, u_1\rangle$, thus the initial conditions are satisfied. Consequently $U_1 = u$ is a solution of Problem G.

Furthermore, if $U \in C^1((0,T),H)$, then $\langle u', u''\rangle = U' \in C((0,T),H)$. Thus $u' \in C((0,t),V)$ and $u'' \in C((0,T),W)$. $\square$

2.5 General existence theorem

The following lemma and corollary form part of the additional results that are not explicitly formulated in [VV02]. It will be used in order to render the proof of Theorem 2.2.1 more readable.

Lemma 2.5.1.
A is the infinitesimal generator of a C_0 semigroup of contractions in H.

Proof From Corollary 2.3.3 we have that $\mathcal{D}(A)$ is dense in H. Also, from Corollary 2.3.4, A is dissipative. Furthermore, Lemma 2.3.3 implies that $\mathcal{R}(\lambda - A) = H$ for any $\lambda \geq 0$. It then follows from [Paz83, Theorem 4.3, p.14] that A is the infinitesimal generator of a C_0 semigroup of contractions in H. □

Corollary 2.5.1.
If $U_0 \in \mathcal{D}(A)$ and $F \in C^1([0,T),H)$, then there exists a unique solution $U \in C^1([0,T),H)$ for Problem FOS-1. If $F = 0$ then this solution is in $C^1([0,\infty),H)$.

Proof See [Paz83, Corollary 2.5, p.107 and Theorem 1.3, p.102].

Proof of Theorem 2.2.1

Let $U_0 = \langle u_0, u_1 \rangle$. Then $U_0 \in \mathcal{D}(A)$ by Corollary 2.3.2. If $f \in C^1([0,T),X)$, it follows that $g \in C^1([0,T),W)$ and consequently $F(t) = \langle 0, g(t) \rangle \in C^1([0,T),H)$.

It follows from Corollary 2.5.1 that there exists a unique solution U for Problem FOS-1. From Proposition 2.4.2 it follows that $u = U_1$ is a solution of Problem G with

$$u \in C([0,T),V) \cap C^1([0,T),W) \cap C^1((0,T),V) \cap C^2((0,T),W).$$

If $f = 0$ then $F = 0$ and $U \in C^1([0,\infty),H)$ from Corollary 2.5.1, which means that

$$u \in C^1([0,\infty),V) \cap C^2([0,\infty),W).$$

□

2.6 Weak damping

The definition for weak damping is given in Section 2.2. In this section Theorem 2.2.2 is proved. Throughout this section it is assumed that Assumption E4W holds, that is the bilinear form a is non-negative, symmetric and bounded with respect to the norm of W. All the results in this section are also true for the undamped case, that is $a = 0$.

Recall that J is an open interval containing zero. Consider the following initial value problem.

Problem FOS-2

Given a function $F : J \to H$, find $U \in C(J, H)$ such that for each $t \in J$, $U(t) \in \mathcal{D}(A)$, $U'(t) \in H$ and

$$U'(t) = AU(t) + F(t), \tag{2.6.1}$$

$$U(0) = U_0. \tag{2.6.2}$$

The following two propositions correspond to Lemma 7 in [VV02].

Proposition 2.6.1.
Suppose $F(t) = \langle 0, g(t)\rangle$ for each $t \in J$. If $u \in C(J, V) \cap C^1(J, W)$ is a solution of Problem G, then $U = \langle u, u'\rangle$ is a solution of Problem FOS-2 with $U_0 = \langle u_0, u_1\rangle$. Furthermore, if $u \in C^1(J, V) \cap C^2(J, W)$, then $U \in C^1(J, H)$.

Proof The proof is similar to that of Proposition 2.4.1.

Proposition 2.6.2.
Suppose $F(t) = \langle 0, g(t)\rangle$ for each $t \in J$. If U is a solution of Problem FOS-2 with $U_0 = \langle u_0, u_1\rangle$, then the first component $u = U_1$ of U is a solution of Problem G. Furthermore, if $U \in C^1(J, H)$, then $u \in C^1(J, V) \cap C^2(J, W)$.

Proof The proof is similar to that of Proposition 2.4.2.

Lemma 2.6.1.
A is the infinitesimal generator of a C_0 group in H.

Proof From Lemma 2.3.4 and Corollary 2.3.3 we have that A is a closed linear operator and the domain of A is dense in H.

Since a is bounded on W (from Assumption E4W), there exists a $K > 0$ such that

$$|a(u, u)| \leq K\|u\|_W^2 \text{ for each } u \in W.$$

For any real λ we have that $\lambda a(u,u) \geq -K|\lambda|\|u\|_W^2$ and hence

$$b(u,u) + \lambda a(u,u) + \lambda^2 c(u,u) \geq \lambda a(u,u) + \lambda^2 c(u,u) \geq |\lambda|(|\lambda| - K)\|u\|_W^2$$

for all $u \in V$. Thus the form $b + \lambda a + \lambda^2 c$ defines an inner product on V for all real λ with $|\lambda| > K$. As in the proof of Lemma 2.3.3 it follows that $\mathcal{R}(\lambda - A) = H$ for all real λ with $|\lambda| > K$.

It follows from Lemma 2.3.5 and the definition of $\|\cdot\|_H$ that

$$|(Ax,x)_H| = |a(x_2,x_2)| \leq K\|x_2\|_W^2 \leq K\|x\|_H^2 \quad \text{for all } x \in \mathcal{D}(A). \tag{2.6.3}$$

Now, from the Cauchy-Schwarz inequality and (2.6.3),

$$\begin{aligned} |\lambda|\|x\|_H^2 &= |((\lambda - A)x, x)_H + (Ax,x)_H| \\ &\leq \|(\lambda - A)x\|_H\|x\|_H + K\|x\|_H^2. \end{aligned}$$

Therefore

$$\|(\lambda - A)x\|_H \geq (|\lambda| - K)\|x\|_H \quad \text{for all } x \in \mathcal{D}(A)$$

which implies that

$$\|(\lambda - A)^{-1}y\|_H \leq (|\lambda| - K)^{-1}\|y\|_H \quad \text{for each } y \in H.$$

Hence $(\lambda - A)^{-1} \in B(H)$ for any real λ with $|\lambda| > K$. It follows from the definition of the operator norm that

$$\|(\lambda - A)^{-1}\| \leq (|\lambda| - K)^{-1}.$$

Consequently

$$\|((\lambda - A)^{-1})^n\| \leq (|\lambda| - K)^{-n} \quad \text{for each } n \in \mathbb{N}.$$

The result follows from [Paz83, Theorem 6.3, p.23]. $\square$

The following result is a characterisation of the domain of A.

Lemma 2.6.2.
$\mathcal{D}(A) = E_b \times V$

Proof Let $x \in E_b \times V$. It follows from the definition of E_b that there exists an $f \in W$ such that

$$b(x_1, v) = c(f, v) \quad \text{for all} \quad v \in V. \tag{2.6.4}$$

Consider $F(w) = -c(f, w) - a(x_2, w)$ for all $w \in W$. Then, using the Cauchy-Schwarz inequality and Assumption E4W,

$$\begin{aligned} |F(w)| &\leq |c(f, w)| + |a(x_2, w)| \\ &\leq \|f\|_W \|w\|_W + K\|x_2\|_W \|w\|_W \\ &= (\|f\|_W + K\|x_2\|_W)\|w\|_W \end{aligned}$$

for all $w \in W$. Thus F is a bounded linear functional on W. By Riesz's theorem there exists a unique $y_2 \in W$ such that

$$-c(f, w) - a(x_2, w) = c(y_2, w) \quad \text{for all} \quad w \in W.$$

This equality will also hold for any $w \in V$, thus from (2.6.4)

$$-b(x_1, w) - a(x_2, w) = c(y_2, w) \quad \text{for all } w \in V.$$

Let $y_1 = x_2$, then it follows from Lemma 2.3.2 that $Ax = y$.

To prove the converse, suppose that $x \in D(A)$ and $Ax = y$. It follows from Lemma 2.3.2 that

$$x_2 = y_1 \quad \text{and} \tag{2.6.5}$$

$$b(x_1, v) + a(x_2, v) = -c(y_2, v) \quad \text{for all} \quad v \in V. \tag{2.6.6}$$

Equation (2.6.5) implies that $x_2 \in V$. Consider $F(w) = -a(x_2, w)$ for all $w \in W$. Assumption E4W implies that $|F(w)| = |a(x_2, w)| \leq K\|x_2\|_W \|w\|_W$ for all $w \in W$. So F is a bounded linear functional on W. It follows from Riesz's theorem that there exists a unique $z \in W$ such that $c(z, w) = F(w) = -a(x_2, w)$ for all $w \in W$. But this equality will also hold for any $w \in V$, hence from (2.6.6)

$$b(x_1, w) - c(z, w) = -c(y_2, w) \quad \text{for all} \quad w \in V$$

which implies that $b(x_1, w) = c(z - y_2, w)$ for all $w \in V$. Thus $x_1 \in E_b$. $\square$

The next result is not formulated in [VV02], but is given here in order to render the proof of Theorem 2.2.2 more readable.

Theorem 2.6.1.
Let A be the infinitesimal generator of a C_0 group and $F \in C^1(\bar{J}, H)$. Then Problem FOS-2 has a unique solution $U \in C^1(J, H)$ for each $U_0 \in \mathcal{D}(A)$. If $F = 0$, then $U \in C^1((-\infty, \infty), H)$.

Proof See [Sho77, Corollary 5.3 in Chapter iii].

Proof of Theorem 2.2.2

If $f \in C^1(J, X)$, it follows that $g \in C^1(J, W)$ and consequently $F(t) = \langle 0, g(t) \rangle \in C^1(J, H)$. Let $U_0 = \langle u_0, u_1 \rangle$ where $u_0 \in E_b$ and $u_1 \in V$. Then $U_0 \in \mathcal{D}(A)$ from Lemma 2.6.2. It follows from Theorem 2.6.1 that Problem FOS-2 has a unique solution $U \in C^1(J, H)$. From Proposition 2.6.2, $u = U_1$ is a solution of Problem G with $u \in C^1(J, V) \cap C^2(J, W)$. If $f = 0$, then $F = 0$ and it follows from Theorem 2.6.1 that $u \in C^1((-\infty, \infty), V) \cap C^2((-\infty, \infty), W)$. $\square$

Remark As mentioned $\mathcal{D}(A) = E_b \times V$. From semigroup theory we have that the solution of Problem FOS-2 is in $E_b \times V$ if $\langle u_0, u_1 \rangle \in E_b \times V$.

2.7 Strong damping

2.7.1 Proof of existence result

The definition of strong damping is given in Section 2.2. Consider Problem FOS-1. If Assumptions E1, E2, E3 and E4 hold, we have from our previous results that A is the infinitesimal generator of a C_0 semigroup and Problem FOS-1 has a unique solution for $U_0 \in \mathcal{D}(A)$ and $F \in C^1([0, T), W)$. In this section we show that both these conditions may be relaxed if a is positive definite on V. To be specific, we prove the following theorem.

Theorem 2.7.1.
Suppose Assumptions **E1**, **E2**, **E3**, **E4** *and* **E5S** *hold. If $F : [0, T) \to H$ is locally Hölder continuous on $(0, T)$, then Problem FOS-1 has a unique solution U for each $U_0 \in H$. If $F = 0$, then $U \in C^\infty((0, \infty), H)$.*

Throughout this section it is assumed that Assumption E5S holds, that is there exists a constant K such that

$$a(u, u) \geq K \|u\|_V^2 \text{ for all } u \in V.$$

In order to prove Theorem 2.7.1, it is necessary to consider semigroups in a complex Hilbert space. Suppose H is a real Hilbert space. Then H may be imbedded in a complex Hilbert space, say $\widetilde{H}$. This construction can be made rigorous, see [Sch71, p.153].

Let $\widetilde{H}$ be the set of all ordered pairs $\langle x, y\rangle$ of elements in H.

Definitions

$$\begin{aligned}\langle x, y\rangle + \langle u, v\rangle &= \langle x+u, y+v\rangle \\ (\alpha + i\beta)\langle x, y\rangle &= \langle \alpha x - \beta y, \beta x + \alpha y\rangle.\end{aligned}$$

From these definitions, it follows that $\widetilde{H}$ is a complex vector space. We can convert to more convenient notation. If $\alpha = 0$, $x = 0$ and $\beta = 1$, we have that $i\langle 0, y\rangle = \langle -y, 0\rangle$. The interpretation of this is that x is the real part while y is the imaginary part. Thus for each $\tilde{x} \in \widetilde{H}$, we may write $\tilde{x} = x + iy$ with x and $y \in H$.

The inner product for $\widetilde{H}$ is defined as

$$(x+iy, u+iv)_{\widetilde{H}} = (x,u)_H + (y,v)_H + i(y,u)_H - i(x,v)_H.$$

The linear operator A (in Problem FOS-1) may be extended to $\widetilde{A}$ in the space $\widetilde{H}$:

$$\begin{aligned} D(\widetilde{A}) &= \{x + iy \mid x \in \mathcal{D}(A),\ y \in \mathcal{D}(A)\}, \\ \widetilde{A}(x+iy) &= Ax + iAy. \end{aligned}$$

Lemma 2.7.1.
If Assumptions **E4** *and* **E5S** *hold, the operator $\widetilde{A}$ is the infinitesimal generator of an analytic semigroup.*

Proof See Subsection 2.7.2.

Problem FOS-3
Given a function $G : [0,T) \to \widetilde{H}$, find $Y \in C([0,T), \widetilde{H})$ such that for each $t \in (0,T)$, $Y(t) \in \mathcal{D}(\widetilde{A})$, $Y'(t) \in \widetilde{H}$ and

$$Y'(t) = \widetilde{A}Y(t) + G(t) \text{ for } t \in (0,T), \tag{2.7.1}$$

$$Y(0) = Y_0. \tag{2.7.2}$$

Theorem 2.7.2.
If $G : [0,T) \to \widetilde{H}$ is locally Hölder continuous on $(0,T)$, then Problem FOS-3 has a unique solution Y for each $Y_0 \in \widetilde{H}$. If $G = 0$, then $Y \in C^\infty((0,\infty), \widetilde{H})$.

Proof Due to Lemma 2.7.1, it follows from [Paz83, Corollary 3.3, p.113] that the initial value problem $Y'(t) = \widetilde{A}Y(t) + G(t)$ with $Y(0) = Y_0$ has a unique solution for each $Y_0 \in \widetilde{H}$. Furthermore, if $G = 0$ it follows that $Y \in C^\infty((0,\infty), \widetilde{H})$. $\square$

Proposition 2.7.1.
If $G \in C([0,T), H)$, $Y_0 \in H$ and Y is the solution of Problem FOS-3, then $Y(t) \in H$ for each t and Y is the solution of Problem FOS-1.

Proof If $Y(t) = u(t) + iv(t)$ and Y is a solution of Problem FOS-3, then $v(t) = 0$ for all $t \geq 0$. $\square$

We can now complete the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3

Let $Y_0 = \langle u_0, u_1\rangle \in H$ and $G(t) = \langle 0, f(t)\rangle$. If f is locally Hölder continuous with respect to the norm $\|\cdot\|_W$, then G is locally Hölder continuous with respect to the norm $\|\cdot\|_H$. Therefore Theorem 2.7.1 and Proposition 2.7.1 implies that Problem FOS-1 has a unique solution Y. It follows from Proposition 2.4.2 that $u = Y_1$ is a solution of Problem G and

$$u \in C([0,T), V) \cap C^1([0,T), W) \cap C^2((0,T), W).$$

If $f = 0$, then

$$u \in C([0,\infty), V) \cap C^1([0,\infty), W) \cap C^\infty((0,\infty), V).$$

$\square$

Remark The main advantage of strong damping is that it is sufficient that $u_0 \in V$ and $u_1 \in W$.

2.7.2 Generation of an analytic semigroup

The next proposition corresponds to Lemma 10 in [VV02].

Proposition 2.7.2.
For any $w = x + iy \in D(\widetilde{A})$

$$Re(\widetilde{A}w, w)_{\widetilde{H}} = (Ax, x)_H + (Ay, y)_H$$

and

$$Im(\widetilde{A}w, w)_{\widetilde{H}} = (Ay, x)_H - (Ax, y)_H.$$

Proof

$$(\widetilde{A}(x+iy), x+iy)_{\widetilde{H}} = (Ax,x)_H + (Ay,y)_H + i(Ay,x)_H - i(Ax,y)_H.$$

□

The following two propositions are part of the additional results that are not given in [VV02].

Proposition 2.7.3.
There exists a constant K such that for any $w = x+iy \in D(\widetilde{A})$ we have

$$Re(\widetilde{A}w,w)_{\widetilde{H}} \leq -K(\|x_2\|_V^2 + \|y_2\|_V^2).$$

Proof

$$Re(\widetilde{A}w,w)_{\widetilde{H}} = (Ax,x)_H + (Ay,y)_H \text{ from Proposition 2.7.2.}$$

Then Lemma 2.3.5, implies that

$$Re(\widetilde{A}w,w)_{\widetilde{H}} = -a(x_2,x_2) - a(y_2,y_2).$$

The result follows from Assumption E5S. □

Proposition 2.7.4.
For any $w = x+iy \in D(\widetilde{A})$ we have

$$|Im(\widetilde{A}w,w)_{\widetilde{H}}| \leq \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2.$$

Proof Using the boundedness of b and applying the inequality $2\alpha\mu \leq \alpha^2 + \mu^2$, it follows that

$$2|b(u,v)| \leq 2\|u\|_V\|v\|_V \leq \|u\|_V^2 + \|v\|_V^2 \text{ for any } u,v \in V. \tag{2.7.3}$$

Now, from Lemma 2.3.5 and the symmetry of a and b,

$$\begin{aligned} Im(\widetilde{A}w,w)_{\widetilde{H}} &= (Ay,x)_H - (Ax,y)_H \\ &= b(y_2,x_1) - b(y_1,x_2) - a(x_2,y_2) \\ &\quad - (b(x_2,y_1) - b(x_1,y_2) - a(x_2,y_2)) \\ &= 2b(x_1,y_2) - 2b(x_2,y_1). \end{aligned}$$

It then follows from (2.7.3) that

$$\begin{aligned}|Im(\widetilde{A}w,w)_{\widetilde{H}}| &\le 2|b(x_1,y_2)| + 2|b(x_2,y_1)|\\ &\le \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2.\end{aligned}$$

□

Let $\widetilde{B} = \widetilde{A} - \widetilde{I}$ where $\widetilde{I}$ is the identity operator on $\widetilde{H}$. The next proposition is given as Lemma 11 in [VV02].

Proposition 2.7.5.
There exists a constant C such that

$$C\ Re(\widetilde{B}w,w)_{\widetilde{H}} + |Im(\widetilde{B}w,w)_{\widetilde{H}}| \le 0 \quad \textit{for any } w \in \mathcal{D}(\widetilde{B}) = \mathcal{D}(\widetilde{A}).$$

Proof From Propositions 2.7.3 and 2.7.4 we have that

$$\begin{aligned}Re(\widetilde{A}w,w)_{\widetilde{H}} &\le -K(\|x_2\|_V^2 + \|y_2\|_V^2) \quad \text{and}\\ |Im(\widetilde{A}w,w)_{\widetilde{H}}| &\le \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2\end{aligned}$$

for any $w = x + iy \in \mathcal{D}(\widetilde{A})$.
Since $(\widetilde{B}w,w)_{\widetilde{H}} = (\widetilde{A}w,w)_{\widetilde{H}} - \|w\|_{\widetilde{H}}^2$, we have that

$$\begin{aligned}Re(\widetilde{B}w,w)_{\widetilde{H}} &= Re(\widetilde{A}w,w)_{\widetilde{H}} - Re\|w\|_{\widetilde{H}}^2\\ &= Re(\widetilde{A}w,w)_{\widetilde{H}} - (\|x\|_H^2 + \|y\|_H^2)\\ &= Re(\widetilde{A}w,w)_{\widetilde{H}} - (\|x_1\|_V^2 + \|x_2\|_W^2 + \|y_1\|_V^2 + \|y_2\|_W^2)\\ &\le -K(\|x_2\|_V^2 + \|y_2\|_V^2) - (\|x_1\|_V^2 + \|x_2\|_W^2 + \|y_1\|_V^2 + \|y_2\|_W^2)\\ &\le -K(\|x_2\|_V^2 + \|y_2\|_V^2) - (|x_1\|_V^2 + \|y_1\|_V^2)\end{aligned}$$

and

$$\begin{aligned}|Im(\widetilde{B}w,w)_{\widetilde{H}}| &= |Im(\widetilde{A}w,w)_{\widetilde{H}}|\\ &\le \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2.\end{aligned}$$

Then, for any $C > 0$, we have that

$$\begin{aligned}&CRe(\widetilde{B}w,w)_{\widetilde{H}} + |Im(\widetilde{B}w,w)_{\widetilde{H}}|\\ \le\ &(\|x_1\|_V^2 + \|y_1\|_V^2)(1-C) + (\|x_2\|_V^2 + \|y_2\|_V^2)(1-CK).\end{aligned}$$

If we choose $C > \max\{1, K^{-1}\}$, the result follows. □

Now consider Theorem 5 in [VV02]:

Theorem 2.7.3.
A linear operator L is the infinitesimal generator of an analytic semigroup in a complex Hilbert space X if $D(L)$ is dense in X, $0 \in \rho(L)$ and there exists a constant $C > 0$ such that

$$C\ Re(Lx, x)_X + |Im(Lx, x)_X| \leq 0 \quad \textit{for any } x \in D(L). \tag{2.7.4}$$

The proof in [VV02] is complete and clear and it serves no purpose to copy it. Of importance is how to use the result.

Next, we combine Proposition 2.7.5 and Theorem 2.7.3.

Proposition 2.7.6.
The operator $\widetilde{B}$ is the infinitesimal generator of an analytic semigroup.

Proof Since $\mathcal{D}(A)$ is dense in H (Corollary 2.3.3), it follows from the definitions of $\widetilde{H}$ and $\widetilde{A}$ that $\mathcal{D}(\widetilde{B}) = \mathcal{D}(\widetilde{A})$ is dense in $\widetilde{H}$.

From Lemma 2.3.3 we have that $\mathcal{R}(\lambda - A) = H$ for $\lambda \geq 0$. It then follows that $\mathcal{R}(\lambda - \widetilde{A}) = \widetilde{H}$ and therefore $\mathcal{R}(\widetilde{A} - \widetilde{I}) = \widetilde{H}$ if we let $\lambda = 1$.

From Proposition 2.7.3 we have that

$$Re(\widetilde{A}w, w)_{\widetilde{H}} \leq 0 \ \text{ for all } w \in \mathcal{D}(\widetilde{A}).$$

Consequently

$$|(\widetilde{B}w, w)_{\widetilde{H}}| \geq -Re(\widetilde{B}w, w)_{\widetilde{H}} \geq \|w\|^2_{\widetilde{H}} \ \text{ for all } w \in \mathcal{D}(\widetilde{B}).$$

But

$$|(\widetilde{B}w, w)_{\widetilde{H}}| \leq \|\widetilde{B}w\|_{\widetilde{H}} \|w\|_{\widetilde{H}}$$

from the Cauchy-Schwarz inequality. Hence $\widetilde{B}^{-1}$ is a bounded operator and hence $0 \in \rho(\widetilde{B})$.

From Proposition 2.7.5 we have that there exists a constant C such that

$$C\ Re(\widetilde{B}w, w)_{\widetilde{H}} + |Im(\widetilde{B}w, w)_{\widetilde{H}}| \leq 0 \ \text{ for any } w \in D(\widetilde{B}).$$

The result follows from Theorem 2.7.3. □

To prove that the operator $\widetilde{A}$ is also the infinitesimal generator of an analytic semigroup, consider the following formal calculation:

$$\frac{d}{dt}\left[e^{\alpha t} T(t)x\right] = \alpha e^{\alpha t} T(t)x + e^{\alpha t} A T(t)x = (\alpha + A) e^{\alpha t} T(t)x.$$

If valid, then $e^{\alpha t}T(t)x$ is differentiable when $T(t)x$ is differentiable.

Proposition 2.7.7.
Suppose A is the infinitesimal generator of an analytic semigroup in a complex Hilbert space X and let $L = \alpha + A$. Then L is the infinitesimal generator of an analytic semigroup.

Proof Suppose $T(\cdot)$ is the analytic semigroup generated by A. Let $S(t) = e^{\alpha t}T(t)$. We will prove that $S(\cdot)$ is the analytic semigroup generated by L. For any $x \in X$ we have that

$$\begin{aligned}\|h^{-1}(S(h)x - x) - Lx\|_X = {} & \|h^{-1}(e^{\alpha h}T(h)x - x) - (\alpha + A)x\|_X \\ \leq {} & \|h^{-1}(e^{\alpha h}T(h)x - e^{\alpha h}x) - Ax\|_X \\ & + \|h^{-1}(e^{\alpha h}x - x) - \alpha x\|_X.\end{aligned}$$

But

$$\begin{aligned}& \|h^{-1}(e^{\alpha h}T(h)x - e^{\alpha h}x) - Ax\|_X \\ \leq {} & |e^{\alpha h}|\|h^{-1}(T(h)x - x) - Ax\|_X + |e^{\alpha h} - 1|\|Ax\|_X \longrightarrow 0 \text{ as } h \to 0+\end{aligned}$$

and

$$\lim_{h\to 0+} \|h^{-1}(e^{\alpha h} - 1)x - \alpha x\|_X = 0.$$

The result follows. □

Lemma 2.7.2.
The operator $\widetilde{A}$ is the infinitesimal generator of an analytic semigroup.

Proof Since $\widetilde{A} = \widetilde{B} + \widetilde{I}$, the result is a direct consequence of Proposition 2.7.6 and Proposition 2.7.7. □

2.8 Energy

In this section we derive a general energy inequality. We consider solutions of Problem G in Section 2.2 and use the same notation. It is first necessary to derive a general product rule.

Proposition 2.8.1.
Suppose J is an interval and u and v are functions defined on J with values in a Hilbert spaces H. Let β be a bounded bilinear form on H and set $\phi(t) = \beta(u(t), v(t))$ for each $t \in J$. If u and v are differentiable in H, then ϕ is differentiable and

$$\phi'(t) = \beta(u'(t), v(t)) + \beta(u(t), v'(t)).$$

Proof The usual method to prove a product rule is used. □

In the theorem below we use the notation of Section 2.2.

Theorem 2.8.1.
Suppose Assumptions **E1**, **E2**, **E3** *and* **E4** *hold and for each $t \in J$*

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = c(g(t), v) \tag{2.8.1}$$

for each $v \in V$. If

$$E(t) = \frac{1}{2}\ c(u'(t), u'(t)) + \frac{1}{2}\ b(u(t), u(t)) \tag{2.8.2}$$

for each $t \in J$, then $E'(t) \le c(g(t), v)$.

Proof It follows from Proposition 2.8.1 and equation (2.8.1) that

$$\begin{aligned} E'(t) &= c(u''(t), u'(t)) + b(u(t), u'(t)) \\ &= -a(u'(t), u'(t)) + c(g(t), u'(t)), \end{aligned}$$

since $u'(t) \in V$. The result follows since a is non-negative (Assumption E4).
□

Next we use a special case of Gronwall's inequality. A more general version of this inequality can be found in [Eva98]. Let $\eta(\cdot)$ be a non-negative function with continuous derivative on $[0, T]$. Suppose that $\eta(\cdot)$ satisfies for t the differential inequality

$$\eta'(t) \le c\eta(t) + \psi(t), \tag{2.8.3}$$

where $c > 0$ and $\psi(\cdot)$ is a non-negative, integrable function on $[0, T]$. Then

$$\eta(t) \le e^{ct}\Big[\eta(0) + \int_0^t \psi(s)\ ds\Big] \tag{2.8.4}$$

for all $0 \le t \le T$.

Theorem 2.8.2.
If u is a solution of Problem G and $E(t)$ is defined by (2.8.2)*, then*

$$E(t) \leq e^t \Big[\frac{1}{2}\|u_1\|_W^2 + \frac{1}{2}\|u_0\|_V^2 + \int_0^t \frac{1}{2}\|g(s)\|_W^2 \, ds \Big]. \tag{2.8.5}$$

Proof It follows from Theorem 2.8.1 that

$$\begin{aligned} E'(t) &\leq c(g(t), u'(t)) \\ &\leq \|g(t)\|_W \|u'(t)\|_W \\ &\leq \frac{1}{2}\|g(t)\|_W^2 + \frac{1}{2}\|u'(t)\|_W^2 \\ &\leq \frac{1}{2}\|u'(t)\|_W^2 + \frac{1}{2}\|u(t)\|_V^2 + \frac{1}{2}\|g(t)\|_W^2 \\ &= E(t) + \frac{1}{2}\|g(t)\|_W^2 \end{aligned}$$

Then Gronwall's inequality implies that

$$\begin{aligned} E(t) &\leq e^{ct} \Big[E(0) + \int_0^t \frac{1}{2}\|g(s)\|_W^2 \, ds \Big] \\ &= e^t \Big[\frac{1}{2}\|u_1\|_W^2 + \frac{1}{2}\|u_0\|_V^2 + \int_0^t \frac{1}{2}\|g(s)\|_W^2 \, ds \Big] \end{aligned}$$

□

Problem G is well posed
Suppose the assumptions for Theorems 2.2.1, 2.2.2 or 2.2.3 hold, then Problem G is well posed. Existence and uniqueness of solutions follow from one of the three theorems. Using Theorem 2.8.2, we now prove that the solution depends continuously on the data. This is not proved in [VV02] or any other publication that applies the theory of [VV02].
Suppose that u is a solution of (2.8.1) with initial values $u(0) = u_0$ and $u'(0) = u_1$. Also suppose that w is a solution of (2.8.1) with g replaced by a function h and initial values $w(0) = w_0$ and $w'(0) = w_1$. Let $y = u - w$. Then it follows from Theorem 2.8.2 that

$$E(t) \leq e^t \Big[\frac{1}{2}\|u_1 - w_1\|_W^2 + \frac{1}{2}\|u_0 - w_0\|_V^2 + \int_0^t \frac{1}{2}\|g(s) - h(s)\|_W^2 \, ds \Big].$$

Consequently $\|u'(t) - w'(t)\|^2 + \|u(t) - w(t)\|_V^2$ depends continuously on the differences $u_1 - w_1$, $u_0 - w_0$ and $g - h$.

2.9 Necessary conditions for existence

As mentioned before, the advantage of semigroup theory is that necessary conditions for existence are obtained, that is $\langle u_0, u_1 \rangle \in \mathcal{D}(A)$. In this section we investigate what happens when these conditions are not satisfied. This is not done in [VV02], but is of great importance in the application of the existence theory.

Note that a classical solution of (2.2.1) is in application a weak solution. There is thus reason to question the significance of a weak solution of Problem G for practical purposes. Nevertheless, the situation arises in some publications. In applications, some scientists often compute approximations for model problems where the solution does not even qualify to be a classical solution of the weak variational form and then ascribe poor numerical results to limitation of the numerical method. Such an example is provided in Subsection 1.3.2.

2.9.1 Mild solutions

The article by J.M. Ball [Bal77] is helpful to understand the concept of a mild solution of

$$x' = Ax + f \tag{2.9.1}$$

in the Hilbert space H.

Definition 2.9.1. Weak solution according to Ball
A function $x \in C([0,T],H)$ is a weak solution of (2.9.1) *if for every $v \in \mathcal{D}(A)$ the function $\langle x(t), v\rangle$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\langle x(t), v\rangle = \langle x(t), A^*v\rangle + \langle f(t), v\rangle \ \ a.e. \ on \ [0,T].$$

Theorem 2.9.1. Ball
Let $f \in C([0,t],H)$. There exists for each $b \in H$ a unique weak solution of (2.9.1) *on $[0,T]$ if and only if A is the infinitesimal generator of a C_0-semigroup $T(t)$ of bounded linear operators on H, and in this case the weak solution x is given by*

$$x(t) = T(t)b + \int_0^t T(t-s)f(s)\,ds \ \ for \ t \in [0,T]. \tag{2.9.2}$$

Remark Due to the theorem of Ball, Theorem 2.9.1, a weak solution of [Bal77] is the same as a mild solution according to [Paz83]. We will use the term mild solution.

First consider $f \notin C^1([0,T),X)$. In this case a classical solution for (2.9.1) is possible but not certain. Such an example can be found in [Paz83].

Now assume that $f \in C^1[0,T]$ and consider (2.9.1) with initial condition $x(0) = x_0$. If $x_0 \notin \mathcal{D}(A)$, a classical solution of (2.9.1) does not exist, since $x_0 \in \mathcal{D}(A)$ is a necessary condition. We may choose $f = 0$. Since $\mathcal{D}(A)$ is dense in H, there exists a sequence $x_{0,n} \in \mathcal{D}(A)$ such that $\|x_{0,n} - x_0\|_H \to 0$. For each of these initial values the initial value problem has a unique classical solution $x_n(t) = T(t)x_{0,n}$. These solutions converge to the mild solution $x(t) = T(t)x_0$ uniformly on bounded intervals [Paz83, Section 4.1].

2.9.2 Mild solution for Problem G

We can now investigate the solution of Problem G when the necessary conditions for existence are not satisfied.

Suppose the awkward condition (2.2.2) does not hold. Thus $\langle u_0, u_1\rangle \notin \mathcal{D}(A)$ and the result of Theorem 2.2.1 does not apply. The aim is to formulate a weak form for Problem G.

Recall that Problem G is equivalent to the system

$$\begin{aligned} u'(t) &= w(t) \quad \text{and} \\ c(w'(t),v) + a(w(t),v) + b(u(t),v) &= c(g(t),v) \quad \text{for all} \quad v \in V \end{aligned} \tag{2.9.3}$$

with $\langle u(0), w(0)\rangle = \langle u_0, u_1\rangle$.

Consequently, a weak form for the system (2.9.3) is required. This system is in turn equivalent to Problem FOS-1 in Section 2.4, when $F(t) = \langle 0, g(t)\rangle$. Since $\mathcal{D}(A)$ is dense in H, there exists a sequence $(\langle u_n^0, w_n^0\rangle)$ such that $\|u_n^0 - u_0\|_V \to 0$ and $\|w_n^0 - u_1\|_W \to 0$ as $n \to \infty$. For each of these initial conditions Problem FOS-1 has a classical solution $T(t)\langle u_n^0, w_n^0\rangle$. First note that $\langle u_n(t), w_n(t)\rangle = T(t)\langle u_n^0, w_n^0\rangle$ is a solution of the system with $\langle u_n(0), w_n(0)\rangle = \langle u_n^0, w_n^0\rangle$. It is proved in [Paz83, Theorem 2.7, p.108] that $T(t)\langle u_n^0, w_n^0\rangle$ converges to $T(t)\langle u_0, u_1\rangle$ (the mild solution) uniformly on bounded intervals.

Let $\langle u_n(t), w_n(t)\rangle = T(t)\langle u_n^0, w_n^0\rangle$. Since $u_n'(t) = w_n(t)$, it follows that $\|w_n(t) - w(t)\|_W \to 0$ and $\|u_n(t) - u(t)\|_V \to 0$. Since a and b are bounded on W and V respectively, it follows that $a(w_n(t), v) \to a(w(t), v)$ and $b(u_n(t), v) \to b(u(t), v)$ as $n \to \infty$. Consequently, for each $t \in J$,

$$c(w_n'(t), v) \to -a(w(t), v) - b(u(t), v) + c(g(t), v) \text{ for all } v \in V.$$

It follows that for all $v \in V$,

$$c(w(t), v) = -\int_0^t a(w(s), v)ds - \int_0^t b(u(s), v)ds + \int_0^t c(g(s), v)ds.$$

Finally we have the weak form for the system (2.9.3):

$$\begin{aligned} u(t) &= \int_0^t w(s)\, ds \quad \text{and} \\ \frac{d}{dt}c(w(t), v) + a(w(t), v) + b(u(t), v) &= c(g(t), v) \text{ for each } v \in V. \end{aligned}$$

If $u_0 \in V$ and $u_1 \in W$, then u is a mild solution of Problem G:

$$\frac{d}{dt}c(u'(t), v) + a(u'(t), v) + b(u(t), v) = c(g(t), v) \text{ for each } v \in V.$$

Remark (FEM) Suppose V^h is a finite dimensional subspace of V. For the Galerkin approximation u_h of u we have $u_h(t) \in V^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = c(g(t), v) \text{ for each } v \in V^h.$$

This is equivalent to a system of ordinary differential equations and the semi-discrete problem can be solved. But $u_h''(t)$ can not converge to $u''(t)$ which does not exist.

Chapter 3

Applications of linear existence theory

3.1 One-dimensional wave equation

Every textbook on partial differential equations present one or more methods to “solve” the wave equation. The precise conditions for “solutions” to be valid are not discussed in most popular textbooks.

3.1.1 Weak variational form

Consider the variational form of the one-dimensional wave equation with viscous damping, Problem WD-V, given in Subsection 1.2.2.

Define the bilinear forms c and a as follows

$$\begin{aligned} c(u,v) &= \int_0^\ell uv \quad \text{and} \\ a(u,v) &= \gamma c(u,v). \end{aligned}$$

Then Problem WD-V can be written in terms of bilinear forms.

Find a function u such that for each t, $u(\cdot,t) \in T[0,\ell]$ and

$$c(\partial_t^2 u(\cdot,t),v) + a(\partial_t u(\cdot,v),v) + b(u(\cdot,t),v) = (f(\cdot,t),v) \tag{3.1.1}$$

for each $v \in T[0,\ell]$, while $u(\cdot,0) = u_0$ and $\partial_t u(\cdot,0) = u_1$.

Remark Note that the bilinear forms a and c are defined for functions in $\mathcal{L}^2(0,\ell)$ while b is defined for functions in $H^1(0,\ell)$.

Let $V(0,\ell)$ be the closure of $T[0,\ell]$ in $H^1(0,\ell)$ and let $\tilde{f} : t \mapsto f(\cdot,t)$. Then Problem WD can be written in weak variational form.

Problem WD-W
Find a function w such that for each t, $w(t) \in V(0,\ell)$, $w'(t) \in V(0,\ell)$, $w''(t) \in \mathcal{L}^2(0,\ell)$ and

$$c(w''(t),v) + a(w'(t),v) + b(w(t),v) = (\tilde{f}(t),v)$$

for each $v \in V(0,\ell)$, while $w(0) = u_0$ and $w'(0) = u_1$.

Remark Problems WD-V and WD-W are not equivalent. If a solution of Problem WD-W is sufficiently smooth, it will be a solution of Problem WD-V.

3.1.2 Existence for the one-dimensional wave equation

In this subsection we will apply the existence theory of Chapter 2 to Problem WD-W. We first need to define the Hilbert spaces V, W and X. Let $X = \mathcal{L}^2(0,\ell)$ and in this case $W = X$ since the bilinear form c is the same as the inner product for $\mathcal{L}^2(0,\ell)$. Let $V = V(0,\ell)$. It will be shown that b is an inner product for V.

In order to apply the existence theory of Chapter 2, we need to verify that Assumptions E1, E2, E3 and E4 are valid. Poincaré type inequalities (see Appendix A.4) will be used frequently in the proofs of the results in this section. The next result follows directly from Proposition A.4.4. For any $u \in V$,

$$\|u\| \le \ell \|u'\|. \tag{3.1.2}$$

Proposition 3.1.1.
The bilinear form b is an inner product for V.

Proof It is clear from the definition of b that it is nonnegative and symmetric. It remains to show that $u = 0$ if $b(u,u) = 0$. If $b(u,u) = 0$, then $\|u'\| = 0$ from the definition of b. It follows from inequality (3.1.2) that $\|u\| = 0$. Consequently $u = 0$. $\square$

The norm associated with the inner product b is defined by $\|u\|_V = \sqrt{b(u,u)}$ for any $u \in V$.

Proposition 3.1.2.
There exists a constant β such that $\|v\|_W^2 \leq \beta^2 \|v\|_V^2$ for each $v \in V$.

Proof For any $v \in V$ we have from (3.1.2) and the definition of b that

$$\|v\|_W^2 = \|v\|^2 \leq \ell^2 \|v'\|^2 = \frac{\ell^2}{\alpha^2} b(v,v) = \frac{\ell^2}{\alpha^2} \|v\|_V^2.$$

So $\|v\|_W^2 \leq \beta^2 \|v\|_V^2$ where $\beta^2 = \dfrac{\ell^2}{\alpha^2}$. □

Proposition 3.1.3.
The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(0,\ell)$ on V.

Proof Since $\|v\|_1^2 = \|v\|^2 + \|v'\|^2$, it follows that $\|v\|_V^2 = \alpha^2 \|v'\|^2 \leq \alpha^2 \|v\|_1^2$ for all $v \in V$. Furthermore, from (3.1.2) and the definition of $\|\cdot\|_V$,

$$\|v\|_1^2 = \|v\|^2 + \|v'\|^2 \leq \ell^2 \|v'\|^2 + \|v'\|^2 = \frac{\ell^2 + 1}{\alpha^2} \|v\|_V^2.$$

Therefore there exists constants μ_1 and μ_2 such that

$$\mu_1 \|v\|_1 \leq \|v\|_V \leq \mu_2 \|v\|_1.$$

□

Proposition 3.1.4.
V is dense in W.

Proof We have that $C_0^\infty(0,\ell)$ is dense in $\mathcal{L}^2(0,\ell)$ (Theorem A.1.2 in Appendix A.1) and $C_0^\infty(0,\ell)$ is a subset of V. Hence V is dense in $\mathcal{L}^2(0,\ell) = W$. □

Proposition 3.1.5.
The bilinear form a is non-negative, symmetric and bounded on W. That is, there exists a constant K such that

$$|a(u,v)| \leq K \|u\|_W \|v\|_W \quad \textit{for} \quad u, v \in W.$$

Proof It is clear from the definition of a that it is non-negative and symmetric since $\gamma > 0$. From the Cauchy-Schwarz inequality,

$$|a(u,v)| \leq \int_0^\ell |\gamma u v| \leq \gamma \|u\| \|v\| = \gamma \|u\|_W \|v\|_W.$$

□

Let J be an open interval containing zero and recall that $X = W$.

Theorem 3.1.1.
Suppose that $\tilde{f} \in C^1(J, W)$. Problem WD-W has a unique solution

$$u \in C^1(J, V) \cap C^2(J, W)$$

for each $u_0 \in E_b$ and each $u_1 \in V$.

Proof We have proved that Assumptions E1, E2 and E4W hold. Assumption E3 holds automatically since $\| \cdot \|_W = \| \cdot \|_X$. The result follows directly from Theorem 2.2.2. $\square$

Remark If $u_0 \in E_b$ and $u_1 \in V$, then $u(t) \in E_b$ for all $t \in J$. See Remark at the end of Section 2.6.

Necessary and sufficient conditions

Consider the case where $\partial_x u(\ell, t) = 0$. Recall that we have weak damping and that sufficient conditions for existence are $\tilde{f} \in C^1(J, X)$, $u_0 \in E_b$ and $u_1 \in V$. The conditions $u_0 \in E_b$ and $u_1 \in V$ are also necessary, see Section 2.9. Suppose in this case that $\tilde{f} \in C^1(J, X)$. It is required to characterise the spaces E_b and V. Now $V \subset H^1(0, \ell)$, so u_1 must be equal to a continuous function a.e. Since V is the closure of the test functions, $u_1(0) = 0$ (in the sense of trace).

We shall prove that $E_b = H^2(0, \ell) \cap V$. We first show that $H^2(0, \ell) \cap V \subset E_b$. For u_0 to be in E_b, we need to find a function $y \in W = \mathcal{L}^2(0, \ell)$ such that $c(y, v) = b(u_0, v)$ for all $v \in V$. Suppose that $u_0 \in C^2(0, \ell) \cap T[0, \ell]$. Using integration by parts, it follows that

$$\alpha^2(u_0', v') = \alpha^2 u_0'(\ell)v(\ell) - \alpha^2(u_0'', v) \quad \text{for all} \quad v \in T[0, \ell].$$

If $u_0'(\ell) = 0$, then

$$b(u_0, v) = \alpha^2(u_0', v') = -\alpha^2(u_0'', v) = c(-\alpha^2 u_0'', v) \quad \text{for all} \quad v \in T[0, \ell]. \quad (3.1.3)$$

Since the bilinear forms b and c are both continuous on V and the test functions are by definition dense in V, it follows that (3.1.3) holds for all $v \in V$. Thus $C^2(0, \ell) \cap T[0, \ell] \subset E_b$. Now suppose that $u_0 \in H^2(0, \ell) \cap V$. Then there exists a sequence $(u_{0,n}) \subset C^2(0, \ell) \cap T[0, \ell]$ such that $\|u_{0,n} - u_0\|_2 \to 0$. Since $\| \cdot \|_V \leq K \| \cdot \|_1 \leq K \| \cdot \|_2$, (3.1.3) holds and $u_0 \in E_b$.

It now remains to prove that $E_b \subset H^2(0, \ell) \cap V$. Suppose that $u_0 \in E_b$. It follows from the definition of E_b that $u_0 \in V$. Furthermore, there exists a function $y \in W = \mathcal{L}^2(0,\ell)$ such that $b(u_0, v) = c(y, v)$ for all $v \in V$. That is, $\alpha^2(u_0', v') = (y, v)$ for all $v \in V$. Hence u_0' has a weak derivative $u_0'' = \dfrac{1}{\alpha^2} y$. Thus $u_0 \in H^2(0,\ell)$.

Remark Since $\|u_{0,n} - u_0\|_2 \to 0$, it follows that $u_0'(\ell) = 0$ (in the sense of trace).

3.1.3 Examples

Hyperbolic heat equation

Consider the problem for hyperbolic heat conduction, Problem CT2, first introduced in Subsection 1.3.2. As mentioned before, this problem is an example of a problem that is not well posed. It was shown in Subsection 1.3.2 that Problem CT2 does not have a solution. In this subsection however we will use the existence theory of Chapter 2 to prove why there does not exist even a weak solution.

Recall that Problem CT2 is a special case of Problem WD with boundary condition $\partial_x u(\ell, t) = 0$. Thus the weak variational form is just Problem WD-W with $\tilde{f} = 0$ and initial conditions $w(0) = \theta_{in}$ and $w'(0) = 0$.

If $\theta_{in}(x) = -1$ as in Section 1.3, then $\theta_{in} \notin V$ since $\theta_{in}(0) = -1 \neq 0$ and hence $\theta_{in} \notin E_b$. Consequently the result of Theorem 3.1.1 does not apply. Thus there does not exist a solution for Problem CT2. In fact, Problem CT2 does not even have a mild solution.

Plucked string

Recall from Subsection 1.2.4 that the plucked string is a special case of Problem WD with boundary condition $u(\ell, t) = 0$ and initial conditions $u(x, 0) = u_0$ and $\partial_t u(x, 0) = 0$ where

$$u_0(x) = \begin{cases} kx & \text{for } \ 0 \leq x \leq \frac{\ell}{2} \\ k(\ell - x) & \text{for } \ \frac{\ell}{2} \leq x \leq \ell. \end{cases}$$

Thus the weak variational form is Problem WD-W with $a = \tilde{f} = 0$ and initial conditions $w(0) = u_0$ and $w'(0) = 0$.

Recall from Subsection 3.1.2 that necessary conditions for existence are $u_0 \in E_b$ and $u_1 \in V$. We have proved that $E_b = H^2(0,\ell) \cap V$. For the plucked string we will show that u_0 is in V but not in $H^2(0,\ell)$, which proves that the problem is not well posed.

Using functions in $C^1(0,\ell)$, the triangular shape of the initial condition can be approximated in $H^1(0,\ell)$. We smooth out the vertex by using a polynomial to join the two remaining lines.

Let $L_n = \dfrac{\ell}{2} - \dfrac{1}{n}$ and $R_n = \dfrac{\ell}{2} + \dfrac{1}{n}$. It is not difficult to construct a polynomial p_n of degree 4 on $[L_n, R_n]$ with the following properties:

$$p_n(L_n) = p_n(R_n) = kL_n \quad \text{and} \quad p_n'(L_n) = -p_n'(R_n) = k,$$

while $|p_n'(x)| \le k$ and

$$kL_n \le p_n(x) \le \frac{k\ell}{2} \quad \text{for} \quad x \in [L_n, R_n].$$

Therefore

$$\int_0^\ell (p_n - u_0)^2 = \int_{L_n}^{R_n} (p_n - u_0)^2 \;\le\; \Big(\frac{1}{2}k\ell\Big)^2 \Big(\frac{2}{n}\Big) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\int_0^\ell (p_n' - u_0')^2 = \int_{L_n}^{R_n} (p_n' - u_0')^2 \;\le\; k^2 \Big(\frac{2}{n}\Big) \to 0 \quad \text{as} \quad n \to \infty.$$

Now define the function g_n by

$$g_n(x) = \begin{cases} u_0(x) & \text{for} \quad x < L_n \\ p_n(x) & \text{for} \quad L_n \le x \le R_n \\ u_0(x) & \text{for} \quad x > R_n. \end{cases}$$

Then $\|g_n - u_0\|_1 \to 0$ as $n \to \infty$ and hence $u_0 \in H^1(0,\ell)$. Also, since $g_n \in T[0,\ell]$, $u_0 \in V$.

Next we show that u_0 is not in $H^2(0,\ell)$. If it is, then $u_0' \in H^1(0,\ell)$ and u_0' is equal to a continuous function a.e. on $(0,\ell)$. This is clearly impossible.

As discussed in Section 2.9, we then have a solution u of Problem WD-W, the weak form. Therefore u is a mild solution and

$$\frac{d}{dt}c(u'(t),v) + a(u'(t),v) + b(u(t),v) = 0 \quad \text{for each} \quad v \in V.$$

Classical theory

Using Fourier series a formal series solution is derived for the homogeneous damped wave equation in [Wei65, Section 26] and proved that this formal solution is indeed a solution. The condition $u_0(0) = u_0(\ell) = 0$ is of course required. The interesting part is that $u_0''' \in \mathcal{L}^2(0, \ell)$ is also required if the theory of Fourier series is used. This condition can be dropped but then it is necessary to assume that $u_0''(0) = u_0''(\ell) = 0$.

3.2 The multi-dimensional wave equation

Consider the multi-dimensional wave equation discussed in Section 1.4. Let $A = \alpha I$ where α is a positive constant and let $\rho = 1$. In this section we consider the two-dimensional case, i.e. $n = 2$.

3.2.1 Weak variational form

Consider the variational form of Problem MW formulated in Subsection 1.4.3. To apply the existence theory from Chapter 2, we need to write Problem MW in weak variational form. Let $V(\Omega)$ be the closure of $T(\Omega)$ in $H^1(\Omega)$ and let $\tilde{f} : t \to f(\cdot, t)$. The weak variational form of Problem MW can now be formulated.

Problem MW-W
Find u such that for each $t > 0$, $u(t) \in V(\Omega)$, $u'(t) \in V(\Omega)$, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{f}(t), v)_\Omega \tag{3.2.1}$$

for each $v \in V(\Omega)$, while $u(0) = u_0$ and $u'(0) = u_1$.

Remark It is again the case that the variational form Problem MW-VS and the weak variational form Problem MW-W are not equivalent. If a solution of Problem MW-W is sufficiently smooth, it will be a solution of Problem MW-VS.

3.2.2 Existence for the multi-dimensional wave equation

We first define the Hilbert spaces V, W and X. Let $V = V(\Omega)$ and let $X = \mathcal{L}^2(\Omega)$. Since $\rho = 1$, the bilinear form c is just the inner product for $\mathcal{L}^2(\Omega)$. Consequently $W = X$ in this case. In order to apply the existence theory, we need to verify that Assumptions E1, E2, E3 and E4 hold.

Poincaré type inequality
To extend the proof of Theorem A.4.4 to two dimensions is highly non-trivial. Therefore we shall only present the result and refer to [Bra01, p.30] or [Eva98] for the proof.

If $\partial\Omega - \Sigma$ has positive area (or positive length for $n = 2$), it follows that for any $u \in V$ there exists a constant C such that

$$\|u\|^2 \leq Cb(u, u), \tag{3.2.2}$$

where C depends on Ω and $T(\Omega)$.

Proposition 3.2.1.
The bilinear form b is an inner product for V.

Proof It is evident from the definition of b that it is non-negative and symmetric. If $b(u, u) = 0$, it follows from (3.2.2) that $\|u\| = 0$ and hence $u = 0$. □

We denote the norm corresponding to the inner product b by $\|u\|_V = \sqrt{b(u, u)}$ for any $u \in V$.

Proposition 3.2.2.
There exists a constant β such that $\|v\|_W^2 \leq \beta^2\|v\|_V^2$ for each $v \in V$.

Proof It follows directly from (3.2.2) and the definition of $\|\cdot\|_V$ with $\beta^2 = C$. □

Proposition 3.2.3.
The norm $\|\cdot\|_V$ is equivalent to the norm $\|\cdot\|_1$ on V.

Proof It follows from the definition of $\|\cdot\|_1$ that

$$\|v\|_V^2 = \alpha\|\nabla v\|^2 \leq \alpha\|v\|_1^2.$$

Furthermore, from (3.2.2) and the definition of $\|\cdot\|_V$,

$$\|v\|_1^2 = \|v\|^2 + \|\nabla v\|^2 \leq C\|v\|_V^2 + \frac{1}{\alpha}\|v\|_V^2.$$

Therefore there exists constants μ_1 and μ_2 such that

$$\mu_1\|v\|_1 \leq \|v\|_V \leq \mu_2\|v\|_1.$$

□

Proposition 3.2.4.
V is dense in W.

Proof See the proof of Proposition 3.1.4. □

Now consider the damping term. Recall that the bilinear form a for viscous type damping is given by

$$a(u, v) = \iint_\Omega duv\; dA$$

and is thus defined on $\mathcal{L}^2(\Omega)$.

Proposition 3.2.5.
The bilinear form a is non-negative, symmetric and bounded on W. That is, there exists a constant K such that

$$|a(u, v)| \leq K\|u\|_W\|v\|_W \quad \textit{for} \quad u, v \in W.$$

Proof The symmetry of a is clear from its definition. Since $d > 0$, a is non-negative. From the Cauchy-Schwarz inequality,

$$|a(u, v)| \leq \iint_\Omega |duv| \leq d\|u\|\|v\| = d\|u\|_W\|v\|_W.$$

□

We again let J be an open interval containing zero and recall that $X = W$.

Theorem 3.2.1.
Suppose that $\tilde{f} \in C^1(J, W)$. Problem MW-W has a unique solution

$$u \in C^1(J, V) \cap C^2(J, W)$$

for each $u_0 \in E_b$ and each $u_1 \in V$.

Proof Assumptions E1, E2 and E4W are proved in the results above. Since $\|\cdot\|_X = \|\cdot\|_W$, Assumption E3 holds. The result then follows from Theorem 2.2.2. □

3.3 The Timoshenko beam model

3.3.1 Weak variational form of the Timoshenko beam model

Consider the Timoshenko beam model introduced in Section 1.5. In this section we apply the existence theory of Chapter 2 to the cantilever beam as well as the pinned-pinned beam. In order to do that, we first need to write the problem in weak variational form.

Product spaces and estimates

The following product spaces are necessary for the weak formulation of Problem T.

$$X = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1),$$

$$H^m = H^m(0,1) \times H^m(0,1).$$

Notation

An element $x \in X$ is written as $x = \langle x_1, x_2 \rangle$.

A natural inner product for the product space X is

$$(x,y)_X = (x_1,y_1) + (x_2,y_2)$$

and the corresponding norm is denoted by $\| \cdot \|_X$.

A natural inner product for the product space H^m is

$$(x,y)_{H^m} = (x_1,y_1)_m + (x_2,y_2)_m$$

and the corresponding norm is denoted by $\| \cdot \|_{H^m}$.

The following bilinear forms are defined on the product spaces.

$$\begin{aligned} c(u,v) &= \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2 && \text{for } u, v \in X, \\ b(u,v) &= \int_0^1 (u_1' - u_2)(v_1' - v_2) + \int_0^1 \frac{1}{\beta} u_2' v_2' && \text{for } u, v \in H^1 \end{aligned}$$

where the derivatives are weak derivatives. Note that the bilinear forms c and b are symmetric.

Next we derive estimates that are required for the existence and convergence theory.

The dimensionless parameters α and β are functions or constants that are bounded above and below by positive constants. Therefore there exist positive constants k_α, K_α, k_β and K_β such that

$$k_\alpha \|u\|^2 \;\leq\; \int_0^1 \frac{1}{\alpha}\, u^2 \leq K_\alpha \|u\|^2 \quad \text{for any} \;\; u \in \mathcal{L}^2(0,1) \tag{3.3.1}$$

and

$$k_\beta \|u'\|^2 \;\leq\; \int_0^1 \frac{1}{\beta}\, (u')^2 \leq K_\beta \|u'\|^2 \quad \text{for any} \;\; u \in H^1(0,1). \tag{3.3.2}$$

The proofs below depend on (3.3.1) and (3.3.2), but to avoid certain technicalities, we assume that α and β are constant.

Proposition 3.3.1.
The bilinear form c is an inner product for the space X.

Proof As mentioned above, c is a symmetric bilinear form. It is also clear that c is non-negative since $\alpha > 0$.

It follows from the definition of c and (3.3.1) that

$$c(u,u) = \|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 \geq \|u_1\|^2 + k_\alpha \|u_2\|^2 \geq \min\{1, k_\alpha\}\|u\|_X^2. \tag{3.3.3}$$

Thus $u = 0$ if $c(u,u) = 0$. □

Definition 3.3.1. Inertia space W
We refer to the vector space X equipped with the inner product c as the space W. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.

Proposition 3.3.2.
The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.

Proof It follows from (3.3.3) that

$$\|u\|_W^2 \;=\; c(u,u) \;\geq\; k^* \|u\|_X^2$$

where where $k^* = \min\{1, k_\alpha\}$. The definition of the bilinear form c and inequality (3.3.1) implies that

$$\|u\|_W^2 \;=\; c(u,u) = \|u_1\|^2 + \frac{1}{\alpha}\|u_2\|^2 \;\leq\; K^*\|u\|_X^2$$

where $K^* = \max\{1, K_\alpha\}$. $\square$

We now construct the space V. Let $V(0,1)$ be the closure of the test function space $T[0,1]$ with respect to $H^1(0,1)$ and define the space V by $V = V(0,1) \times V(0,1)$.

Proposition 3.3.3.
V is a dense subset of the inertia space W.

Proof Since $C_0^\infty(0,1)$ is dense in $\mathcal{L}^2(0,1)$ (Theorem A.1.2 in Appendix A.1), it follows that $V(0,1)$ is dense in $\mathcal{L}^2(0,1)$. Consequently V is dense in X and hence also dense in W from Proposition 3.3.2. $\square$

In [FXX99] and [Sem94] the beam is clamped at both ends. Other possible boundary conditions are not mentioned. The proofs of the following estimates are the same for a beam that is clamped at both sides and a cantilever beam. For a pinned-pinned beam the difference in the theory will be highlighted.

Poincaré type inequalities (see Appendix A.4) will be used frequently in the proofs of the following propositions: For any $u \in V(0,1)$,

$$\|u\| \leq \|u'\|. \tag{3.3.4}$$

Proposition 3.3.4.
For any u in V there exists a constant K_b such that

$$\|u_1'\|^2 + \|u_2'\|^2 \leq K_b\, b(u,u).$$

Proof It follows from inequalities (3.3.4) and (3.3.2) that

$$\begin{aligned}\|u_1'\|^2 + \|u_2'\|^2 \;&\leq\; \|u_1' - u_2\|^2 + 2\|u_2'\|^2 \\ &\leq\; K_b\, b(u,u)\end{aligned}$$

where $K_b = \max\left\{1, \dfrac{2}{k_\beta}\right\}$. $\square$

Proposition 3.3.5.
There exists a constant C_b such that

$$b(u,u) \geq C_b\|u\|_X^2 \quad \textit{for each } u \in V.$$

Proof Again from inequality (3.3.4) and the definition of the norm $\| \cdot \|_X$, we have that

$$\|u\|_X^2 \leq \|u_1'\|^2 + \|u_2'\|^2.$$

The result follows from Proposition 3.3.4 with $C_b = \dfrac{1}{K_b}$. $\square$

Corollary 3.3.1.
The bilinear form b is an inner product for V.

Proof The bilinear form b is clearly symmetric and non-negative. It follows from Proposition 3.3.5 that $u = 0$ if $b(u, u) = 0$. $\square$

Remark Proposition 3.3.5 and Corollary 3.3.1 are applicable for the cantilever beam as well as the beam that is clamped on both sides. The reason being that we use the Poincaré inequality for u_1 and u_2. Note that this is not applicable for the pinned-pinned beam, since u_2 does not have a zero.

It is proved in the appendix of [VZV09] that Proposition 3.3.5 still holds for the pinned-pinned beam. We present the proof here in greater detail in order to make it more readable.

Recall that the test function space for the pinned-pinned beam is given by

$$T_0[0,1] = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\}.$$

The closure of $T_0[0,1]$ with respect to $H^1(0,1)$ is just the space $H_0^1(0,1)$. Define the space V_0 by $V_0 = H_0^1(0,1) \times H^1(0,1)$.

Proof of Proposition 3.3.5 for the pinned-pinned beam
For the sake of convenience, let $u_1 = w$ and $u_2 = \phi$. We prove the result for $u = \langle w, \phi \rangle \in T_0(0,1) \times C^1[0,1]$. It will then also hold for $u \in V_0$ since $T_0(0,1)$ is dense in $H_0^1(0,1)$ and $C^1[0,1]$ is dense in $H^1(0,1)$. The proof is done by contradiction. Suppose the result is not true. Then there exists a sequence $(\langle w_n, \phi_n \rangle)$ such that $\|w_n\|^2 + \|\phi_n\|^2 = 1$, while

$$\|\phi_n'\|^2 + \|w_n' - \phi_n\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

We will first show that $\|\phi_n\| > \dfrac{1}{2}$ for n sufficiently large. We can use a Poincaré type inequality since $w_n \in T_0[0,1]$.

Then

$$\|w_n\| \leq \|w_n' - \phi_n\| + \|\phi_n\| < \frac{1}{4} + \|\phi_n\|.$$

for n sufficiently large, since $\|w_n' - \phi_n\| \to 0$ as $n \to \infty$.

Therefore, $\left(\|\phi_n\| + \frac{1}{4}\right)^2 + \|\phi_n\|^2 > 1$. Solving this inequality yields that $\|\phi_n\| > \frac{1}{2}$.

Next we show that $\int_0^1 \phi_n > \frac{1}{3}$ for n sufficiently large. First note that if $\phi_n(x) = 0$ for some $x \in [0,1]$, it follows by a Poincaré type inequality that $\|\phi_n\| \leq \|\phi_n'\|$. But $\|\phi_n\| > \frac{1}{2}$ and $\|\phi_n'\| \to 0$ as $n \to \infty$, which leads to a contradiction. Therefore $\phi_n(x) \neq 0$ for all $x \in [0,1]$ and we may assume without loss of generality that $\phi_n > 0$.

To prove that $\int_0^1 \phi_n > \frac{1}{3}$ for n sufficiently large, let (a,b) be any subinterval in $(0,1)$. Then, by using the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality,

$$\left|\phi_n^2(b) - \phi_n^2(a)\right| = \left|\int_a^b 2\phi_n\phi_n'\right| \leq 2\|\phi_n\|\|\phi_n'\| \leq \frac{1}{10}$$

for n sufficiently large.

Since ϕ_n is a continuous function defined on a closed and bounded interval $[0,1]$, there exist minimum and maximum values for ϕ_n. Let $\phi_{\min}$ and $\phi_{\max}$ denote the minimum and maximum values for ϕ_n respectively. Then

$$\phi_{\max} - \phi_{\min} \leq \frac{1}{10}.$$

Therefore

$$\phi_{\min}^2 = \int_0^1 \phi_{\min}^2 = \int_0^1 \phi_n^2 - \int_0^1 \left(\phi_n^2 - \phi_{\min}^2\right) > \frac{1}{4} - \frac{1}{10} > \frac{1}{9}.$$

Consequently

$$\int_0^1 \phi_n > \int_0^1 \phi_{\min} = \phi_{\min} > \frac{1}{3}.$$

We now show that $\int_0^1 w_n' > 0$ for n sufficiently large. Using properties of the integral, we obtain

$$\left|\int_0^1 w_n' - \int_0^1 \phi_n\right| \leq \int_0^1 |w_n' - \phi_n| \leq \|w_n' - \phi_n\| < \frac{1}{3},$$

for n sufficiently large. Since $\int_0^1 \phi_n > \frac{1}{3}$, it follows that $\int_0^1 w_n' > 0$.

Finally, since $\int_0^1 w_n' > 0$ and $w_n(0) = 0$, it follows from the Fundamental Theorem of Calculus that $w_n(1) = \int_0^1 w_n' + w_n(0) > 0$. This is a contradiction since $w_n \in T_0[0,1]$. $\square$

Remark It is clear that Corollary 3.3.1 will also hold for the pinned-pinned beam.

Remark From here on onward, we will refer to the space V_0 as V, since the relevant properties of V_0 are the same as for V.

Definition 3.3.2. Energy space V
We refer to the space V equipped with the inner product b as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

Proposition 3.3.6.
The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V.

Proof From the definition of the bilinear form b we have that for any $u \in V$,

$$\|u\|_V^2 = b(u,u) \leq K_1\left(\|u_1' - u_2\|^2 + \|u_2'\|^2\right) \leq K_1\|u\|_{H^1}^2$$

where $K_1 = \max\left\{1, \frac{1}{\beta}\right\}$.

Now, to prove the "reverse" inequality, note that

$$\begin{aligned} \|u\|_{H^1}^2 &= \|u_1\|_1^2 + \|u_2\|_1^2 \\ &= \|u\|_X^2 + \|u_1'\|^2 + \|u_2'\|^2. \end{aligned} \tag{3.3.5}$$

It follows from equality (3.3.5) and Proposition 3.3.4 that

$$\|u\|_{H^1}^2 \leq \|u\|_X^2 + K_b\, b(u,u). \tag{3.3.6}$$

Proposition 3.3.5 implies that

$$\|u\|_X^2 \le C_b^{-1}\; b(u,u) \quad \text{for each} \quad u \in V. \tag{3.3.7}$$

It then follows from inequalities (3.3.6) and (3.3.7) that

$$\|u\|_{H^1}^2 \le K_2 b(u,u) = K_2 \|u\|_V^2$$

for any $u \in V$, where $K_2 = C_b^{-1} + K_b$. □

Corollary 3.3.2.
The energy space V is complete.

Weak variational form

Problem T-V can now be written in weak variational form. Let $u(t) = \langle w(\cdot,t), \phi(\cdot,t)\rangle$ and, for the given function q, let $\tilde{q} : t \mapsto q(\cdot,t)$.

Problem T-W
Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{q}(t), v)_X$$

for each $v \in V$, while $u(0) = u_0 = \langle w_0, \phi_0\rangle$ and $u'(0) = u_1 = \langle w_1, \phi_1\rangle$.

Remark Problem T-W is slightly more general than the Timoshenko model discussed in Section 1.5. In practice $\tilde{q}_2(t) = 0$.

Estimates for damping terms

We consider two cases, namely weak damping and boundary damping.

We note that for the damping as in [Sem94] and [FXX99] the bilinear form a is as follows:

$$a(u,v) = \int_0^1 \mu_1\, u_1 v_1 + \int_0^1 \mu_2\, u_2 v_2 \tag{3.3.8}$$

for u and v in X.

Proposition 3.3.7.
The bilinear form a is non-negative, symmetric and bounded on W. That is, there exists a constant $K > 0$ such that

$$|a(u,v)| \leq K\|u\|_W\|v\|_W$$

for all u and v in W.

Proof It is clear from the definition of a that it is non-negative and symmetric. We have from the Cauchy-Schwarz inequality that

$$\begin{aligned} |a(u,v)| &\leq \Big|\int_0^1 \mu_1\, u_1 v_1\Big| + \Big|\int_0^1 \mu_2\, u_2 v_2\Big| \\ &\leq \mu_1\|u_1\|\|v_1\| + \mu_2\|u_2\|\|v_2\| \\ &\leq (\mu_1+\mu_2)\|u\|_X\|v\|_X. \end{aligned} \tag{3.3.9}$$

The result follows from the equivalence of the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ (Proposition 3.3.2). $\square$

It is clear that Assumption E4W is true in this case and we therefore simply assume weak damping.

Boundary damping

For boundary damping the bilinear form a_b is defined as

$$a_b(u,v) = \nu_1 u_1(1)v_1(1) + \nu_2 u_2(1)v_2(1).$$

Proposition 3.3.8.
Boundary damping is neither weak nor strong.

Proof Consider a sequence (z_n) where $z_n(x) = 0$ if $x < 1 - \dfrac{1}{n}$ and $z_n(x) = nx + 1 - n$ if $x \geq 1 - \dfrac{1}{n}$. Thus $z_n(1) = 1$ for all n. Note that

$$\|z_n\|^2 = \int_{1-\frac{1}{n}}^1 z_n^2 \leq \frac{1}{n} \to 0 \quad\text{and}\quad \|z_n'\|^2 = \int_{1-\frac{1}{n}}^1 n^2 = n \to \infty.$$

Consider the sequence $(u_n) = (\langle z_n, z_n\rangle)$. It follows from Proposition 3.3.6 that $\|u_n\|_V^2 \geq K^{-1}\|u_n\|_{H^1}^2 \geq K^{-1}\|z_n'\|^2 \to \infty$. Since $a_b(u_n,u_n) = \nu_1 + \nu_2$, it follows that

$$\frac{a_b(u_n,u_n)}{\|u_n\|_V^2} \to 0.$$

Thus a_b is not positive definite on V and Assumption E5S does not hold.

Furthermore, $\|u_n\|_W^2 = \|z_n\|^2 + \dfrac{1}{\alpha}\|z_n\|^2 \to 0$ and consequently

$$\frac{a_b(u_n, u_n)}{\|u_n\|_W^2} \to \infty.$$

Thus a_b is not bounded on V with respect to $\|\cdot\|_W$ and Assumption E4W does not hold. $\square$

Remark By proving that a_b is bounded in V, one can know whether the problem is solvable. However, it is then necessary to satisfy the condition

$$b(u_0, v) + a_b(u_1, v) = c(y, v) \quad \text{for each} \quad v \in V$$

in Theorem 2.2.1. This can be avoided if one can prove that the damping is either weak or strong. For boundary damping or control however, it is proved in Proposition 3.3.8 that the damping is neither weak nor strong.

3.3.2 Existence for the dynamic problem

In this subsection the existence and uniqueness of a weak solution for Problem T-W (Subsection 3.3.1) is considered. Problem T-W is a special case of Problem G in Chapter 2.

Theorem 3.3.1. Weak damping
Let J be an interval containing zero and suppose $\tilde{q} \in C^1(J, X)$. Then there exists a unique solution

$$u \in C^1\big(J, V\big) \cap C^2\big(J, W\big)$$

for Problem T-W for each $u_0 \in E_b$, $u_1 \in V$.

Proof We have proved that Assumptions E1, E2, E3 and E4W hold. The result follows from Theorem 2.2.2. $\square$

Remark Note that the conditions $u_0 \in E_b$ and $u_1 \in V$ are necessary and sufficient for the existence of a solution. Recall that when the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent, $u_0 \in E_b$ if and only if u_0 is a solution of the steady state problem. If $u_0 = \langle w_0, \phi_0 \rangle \in E_b$ and $\langle w_1, \phi_1 \rangle \in V$, then $\langle w(t), \phi(t) \rangle \in E_b$ for all t.

We have proved that boundary damping is neither weak nor strong.

Theorem 3.3.2. Boundary damping
If, for $u_0 \in V$ and $u_1 \in V$, there exists some $y \in W$ such that

$$b(u_0, v) + a_b(u_1, v) = c(y, v) \quad \text{for each} \quad v \in V, \tag{3.3.10}$$

then for each $\tilde{q} \in C^1([0,T), X)$ there exists a unique solution for Problem T-W.

Proof We have proved that Assumptions E1, E2 and E3 hold. The bilinear form a_b is clearly non-negative and symmetric. Now

$$\begin{aligned}
|a_b(u, v)| &\le |\nu_1 u_1(1) v_1(1)| + |\nu_2 u_2(1) v_2(1)| \\
&\le \nu_1 C_1 \|u_1\|_1 \|v_1\|_1 + \nu_2 C_2 \|u_2\|_1 \|v_2\|_1 \\
&\le C^*(\|u_1\|_1 \|v_1\|_1 + \|u_2\|_1 \|v_2\|_1) \\
&\le C^*(\|u_1\|_1 + \|u_2\|_1)(\|v_1\|_1 + \|v_2\|_1) \\
&= C^* \|u\|_{H^1} \|v\|_{H^1}
\end{aligned}$$

where $C^* = \max\{\nu_1 C_1, \nu_2 C_2\}$. From Proposition 3.3.6 there exists a $K > 0$ such that $|a_b(u, v)| \le K \|u\|_V \|v\|_V$. Thus Assumption E4 holds. The result follows from Theorem 2.2.1. $\square$

Remark Note that if $\tilde{q} \in C^1([0,T), X)$, then (3.3.10) is necessary and sufficient for the existence of a weak solution. See also Subsection 3.3.4.

3.3.3 Existence and regularity for the steady state problem

In view of the fact that the subspace E_b is important for existence in the case of weak damping, we investigate the steady state problem.

The weak variational form for the Timoshenko model is Problem T-W in Section 3.3.1. The steady state problem is the next problem, where $u''(t)$ and $u'(t)$ are zero.

Problem TE-W
Find u such that for each $t > 0$, $u(t) \in V$ and

$$b(u(t), v) = (\tilde{q}(t), v)_X \quad \text{for each } v \in V.$$

We are in a position to prove the existence of a solution, the regularity of a solution and to derive estimates.

Theorem 3.3.3. Existence Theorem
For any $\tilde{q} \in X$ there exists a unique $u \in V$ such that

$$b(u, v) = (\tilde{q}, v)_X \quad \textit{for each } v \in V.$$

Proof Define a functional f by $f(v) = (\tilde{q}, v)_X$. Clearly f is linear and, using Cauchy-Schwarz,

$$|f(v)| \leq \|\tilde{q}_1\| \|v_1\| + \|\tilde{q}_2\| \|v_2\| \leq K \|v\|_V.$$

Therefore f is in the dual of V. By Riesz's theorem there exists a unique $u \in V$ such that

$$b(u, v) = f(v) \quad \text{for each } v \in V.$$

□

Next we proof the regularity of the solution. We do not follow the exposition of [Arn81] or [FXX99] slavely.

Theorem 3.3.4.
Suppose β is differentiable, $\beta > 0$ on $[0, 1]$ and u is the solution of Problem TE-W. Then $u \in H^2$ if $\tilde{q} \in X$.

Proof Let $v = \langle z, 0 \rangle$ in Problem TE-W, where $z \in V(0, 1)$. Then

$$\int_0^1 (u_1' - u_2) z' = \int_0^1 \tilde{q}_1 z \quad \text{for each } z \in V(0, 1). \tag{3.3.11}$$

By definition $\tilde{q}_1$ is the weak derivative of $u_1' - u_2$ and $(u_1' - u_2) \in H^1(0, 1)$. Now, $u_1' = (u_1' - u_2) + u_2 \in H^1(0, 1)$. Thus $u_1 \in H^2(0, 1)$.

For $\psi \in V(0, 1)$, let $v = \langle 0, \psi \rangle$ in Problem TE-W. Then

$$\left(\frac{1}{\beta} u_2', \psi' \right) = (u_1' - u_2, \psi) + (\tilde{q}_2, \psi) \quad \text{for each } \psi \in V(0, 1). \tag{3.3.12}$$

Since $\dfrac{1}{\beta}$ is differentiable (in the usual sense), it follows from (3.3.12) that

$$\begin{aligned}
\left(u_2', \left(\frac{1}{\beta} \psi \right)' \right) &= \left(u_2', \left(\frac{1}{\beta} \right)' \psi \right) + \left(u_2', \frac{1}{\beta} \psi' \right) \\
&= \left(\left(\frac{1}{\beta} \right)' u_2', \psi \right) + (u_1' - u_2, \psi) + (\tilde{q}_2, \psi) \\
&= \left(\beta \left(\frac{1}{\beta} \right)' u_2', \frac{1}{\beta} \psi \right) + \left(\beta (u_1' - u_2), \frac{1}{\beta} \psi \right) + \left(\beta \tilde{q}_2, \frac{1}{\beta} \psi \right).
\end{aligned}$$

Thus $u_2'' = \beta\Big(\dfrac{1}{\beta}\Big)' u_2' + \beta(u_1' - u_2) + \beta\tilde{q}_2$ and $u_2 \in H^2(0,1)$. $\square$

Since the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent for this application, the space of equilibrium states is E_b. Thus Theorem 3.3.4 implies that $E_b \subset H^2$.

For weak damping we have already remarked that $\langle w(t), \phi(t)\rangle \in E_b$ for all t. Therefore $w(t)$ and $\phi(t)$ are in $H^2(0,1)$ for all t.

3.3.4 Sufficient conditions

We need to find conditions on w_0 and ϕ_0 such that condition (2.2.2) in Theorem 2.2.1 is satisfied:

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each} \quad v \in V. \tag{3.3.13}$$

Suppose

$$u_0 \in (C^2[0,1] \times C^2[0,1]) \cap (T[0,1] \times T[0,1]) \quad \text{and} \quad u_1 \in V.$$

Let v and ψ be arbitrary test functions and denote $\langle v, \psi\rangle$ by z. Using integration by parts, we obtain

$$\begin{aligned} b(u_0, z) &= \int_0^1 (w_0' - \phi_0)(v' - \psi) + \int_0^1 \frac{1}{\beta}\phi_0'\psi' \\ &= -\int_0^1 (w_0' - \phi_0)'v + \big(w_0'(1) - \phi_0(1)\big)v(1) \\ &\quad - \int_0^1 (w_0' - \phi_0)\psi - \int_0^1 \frac{1}{\beta}\phi_0''\psi + \frac{1}{\beta}\phi_0'(1)\psi(1). \end{aligned}$$

Recall that

$$a(u_1, z) = \nu_1 w_1(1)v(1) + \nu_2\phi_1(1)\psi(1).$$

Suppose

$$w_0'(1) - \phi_0(1) = -\nu_1 w_1(1) \quad \text{and} \tag{3.3.14}$$

$$\frac{1}{\beta}\phi_0'(1) = -\nu_2\phi_1(1). \tag{3.3.15}$$

Using the boundary conditions on w_0, w_1, ϕ_0 and ϕ_1, we obtain for any $z \in T[0,1] \times T[0,1]$ that

$$b(u_0, z) + a(u_1, z) = -\int_0^1 (w_0' - \phi_0)'v - \int_0^1 \Big[\frac{1}{\beta}\phi_0'' + (w_0' - \phi_0)\Big]\psi.$$

Consequently

$$b(u_0, z) + a(u_1, z) = c(y, z) \tag{3.3.16}$$

where

$$\begin{aligned} y_1 &= -(w_0' - \phi_0)' \;\in\; \mathcal{L}^2(0,1) \;\text{ and} \\ y_2 &= -\frac{\alpha}{\beta}\phi_0'' - (w_0' - \phi_0) \;\in\; \mathcal{L}^2(0,1). \end{aligned}$$

Since the bilinear forms a, b and c are continuous with respect to the norm of V, this result will also be true for any $z \in V$.

Now consider $u_0 \in H^2 \cap V$. Then there exists a sequence

$$(u_{0,n}) \subset (C^2(0,1) \times C^2(0,1)) \cap (T(0,1) \times T(0,1))$$

such that $\|u_{0,n} - u_0\|_2 \to 0$. Since $\|\cdot\|_V \leq K_1\|\cdot\|_1 \leq K_1\|\cdot\|_2$, (3.3.16) holds.

We conclude that the sufficient conditions for (3.3.13) are that w_0 and ϕ_0 is each in $H^2(0,1) \cap V(0,1)$ and equations (3.3.14) and (3.3.15) are satisfied.

Chapter 4

Applications of the general finite element convergence results

The general second order hyperbolic equation in variational form is presented in Chapter 2 where conditions for the existence of a unique solution are also given. In this chapter we consider error estimates for the semi-discrete Galerkin approximation of Problem G. Important considerations are the influence of damping on the properties of the solutions as well as the numerical approximations of these solutions. We consider two cases, namely general damping and weak damping.

The general existence results in Chapter 2 are used. This is more convenient for the finite element method (FEM) as one can compare conditions for existence to conditions for convergence. The general theory is from [BV13] and [BSV17] (available online in 2016). Nothing is added to the theory of these articles since it is complete with user friendly notation. The proofs are comprehensive and are therefore not repeated in this dissertation. Our focus was on the application of the theory.

To illustrate the application of the theory, we consider the multi-dimensional wave equation with weak damping in Section 4.4 and the Timoshenko beam model with boundary damping in Section 4.5.

4.1 General Galerkin approximation

Consider the general linear vibration problem, Problem G, formulated in Chapter 2. Suppose that Assumptions E1, E2, E3 and E4 hold. In this section no assumption (apart from E4) is made regarding the damping term.

The semi-discrete problem for Problem G can now be formulated. Suppose that S^h is a finite dimensional subspace of V. The Galerkin finite element approximation of Problem G is referred to as Problem G^h.

Problem G^h
Given a function $f : J \to X$, find a function $u_h \in C^2[0,T]$ such that for each $t \in (0,T)$

$$c\,(u_h''(t), v) + a\,(u_h'(t), v) + b\,(u_h(t), v) = (f(t), v)_X \tag{4.1.1}$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

The initial values u_0^h and u_1^h are elements of S^h and must be chosen appropriately in applications in order to be as close as possible to u_0 and u_1.

Remark Although the symbol h has no meaning at this stage, it is customary to use the notation. In applications h is related to the diameter of the elements.

In the rest of this section we consider the error for the semi-discrete approximation u_h. To begin we introduce the projection operator.

4.1.1 Projection

Subtracting the equation in Problem G^h from the equation in Problem G, yields

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e_h(t), v) = 0 \text{ for all } v \in S^h. \tag{4.1.2}$$

A projection is used in order to obtain an estimate for the discretization error $e_h(t) = u(t) - u_h(t)$. The projection operator P_h is defined by

$$b(u - P_h u, v) = 0 \text{ for each } v \in S^h.$$

Notation If no confusion is possible P_h will be written as P. The symbol P is also used to denote the projection Pu of a function u, that is, $(Pu)(t) = Pu(t)$ for each $t \in (0,T)$.

The idea of the projection is to split the error such that

$$e_h(t) = e_p(t) + e(t) \tag{4.1.3}$$

where $e_p(t) = u(t) - Pu(t)$ and $e(t) = Pu(t) - u_h(t)$. Consequently

$$\|u(t) - u_h(t)\|_V \leq \|e_p(t)\|_V + \|e(t)\|_V. \tag{4.1.4}$$

In Subsection 4.1.2 approximation theory is used to obtain estimates for the norm of $e_p(t)$. The challenge is to find an estimate for $e(t)$, the difference between the projection $Pu(t)$ and the Galerkin approximation $u_h(t)$.

In [BV13] it is proved that if $u \in C^1(J, V)$ then $Pu \in C^1(J)$ and $(Pu)'(t) = Pu'(t)$. Thus if u is a solution of Problem G, it follows from the existence theorem (Theorem 2.2.1) that $u \in C^1((0,T), V)$ and hence $Pu \in C^1(0,T)$. However, the existence of $(Pu)''$ is required for the convergence analysis.

Assumption D1
The solution u of Problem G has the property that $(Pu) \in C^2(0,T)$.

Assumption D2
The solution u of Problem G satisfies $u \in C^1([0,T), V) \cap C^2((0,T), V)$.

Remark Note that the fact that $(Pu)' = Pu'$ is used, but not that $(Pu)'' = Pu''$.

Proposition 4.1.1.
If $u \in C^2((0,T), W)$ and u satisfies Assumption D1, then

$$e_p \in C^2(J, W).$$

Proposition 4.1.2.
If the solution u of Problem G satisfies Assumption D1, then for each $t \in (0,T)$,

$$c(e_p''(t), v) + a(e_p'(t), v) + c(e''(t), v) + a(e'(t), v) + b(e(t), v) = 0$$

for all $v \in S^h$.

4.1.2 Projection error

It is necessary to estimate the projection errors e_p, e_p' and e_p'' in order to prove existence results. Interpolation error estimates are used for this.

Definition 4.1.1. Generalised interpolation operator

$$\Pi u = \sum_{k=1}^{n} \phi_k(u) w_k$$

where $\{w_k\}$ *is a basis for* S^h *and* ϕ_k *are linear functionals.*

The authors of [BSV17] suppose that h is a parameter related to the dimension n of S^h and $h \to 0$ as $n \to \infty$. This is done in order to formulate an assumption regarding the error when an element of V is approximated by an element of S^h.

Assumption D3
There exists a subspace $H(V,k)$ of V, an interpolation operator Π and positive constants C_Π and α (depending on V and k) such that for $u \in H(V,k)$

$$\|u - \Pi u\|_V \leq C_\Pi h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ is a norm or semi-norm associated with $H(V,k)$.

The generalised interpolation operator and Assumption D3 were introduced in [BSV17]. Since P is an orthogonal projection onto S^h, we have the following result.

Proposition 4.1.3.
There exists a subspace $H(V,k)$ *of* V *and positive constants* C_Π *and* α *(depending on* V *and* k*) such that for* $u \in H(V,k)$

$$\|u - Pu\|_V \leq C_\Pi h^\alpha \|u\|_{H(V,k)},$$

where $\|\cdot\|_{H(V,k)}$ *is a norm or semi-norm associated with* $H(V,k)$.

Proposition 4.1.3 is a trivial consequence of Assumption D3 and will be used in general theory.

The problem still remains to estimate $e(t)$. Estimates for $e(t)$ are in terms of $e_p(t)$. Different methods are used for different types of damping.

4.2 General damping

The theory in this section is from [BSV17].

4.2.1 Fundamental estimate

The following "energy" expression for e is given in [BSV17].

$$\begin{aligned} E(t) &= \frac{1}{2}c(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) \\ &= \frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_V^2. \end{aligned} \tag{4.2.1}$$

The next result is derived from Proposition 4.1.2.

Lemma 4.2.1.
If the solution u of Problem G satisfies Assumption D1, then for any $t \in (0, T)$,

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \tag{4.2.2}$$

Using equation (4.2.2) an estimate for $E(t)$ in terms of projection errors will be obtained. That will then yield an estimate for $e(t)$ since $\|e'(t)\|_W^2 + \|e(t)\|_V^2 = 2E(t)$.

Note that we have from Assumption E4 that

$$a(e_p'(t), e'(t)) \leq K\|e_p'(t)\|_V \|e'(t)\|_V,$$

but $\|e'(t)\|_V$ is not bounded by $E(t)$. Thus the following lemma is needed to estimate the last term of equation (4.2.2). It is what the authors consider to be the fundamental estimate.

Lemma 4.2.2.
If the solution u of Problem G satisfies Assumption D2, then for any $t \in (0, T)$

$$\|e(t)\|_V + \|e'(t)\|_W \leq \sqrt{24e^{3t}}K_T, \quad \textit{where}$$

$$\begin{aligned} K_T = \int_0^T \|e_p''(\cdot)\|_W + 3K \max \|e_p'(t)\|_V + 3K \int_0^T \|e_p''(\cdot)\|_V + \|e'(0)\|_W \\ + \sqrt{1+K}\|e(0)\|_V + \sqrt{K}\|e_p'(0)\|_V. \end{aligned}$$

4.2.2 Convergence and error estimates

The following lemma is a direct consequence of Lemma 4.2.2.

Lemma 4.2.3.
Assume that the solution u of Problem G satisfies Assumption D2. Then, for any $t \in (0,T)$,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\quad \leq \|u(t) - Pu(t)\|_V + \|u'(t) - Pu'(t)\|_W + \sqrt{24e^{3t}}K_T, \quad \textit{where}
\end{aligned}$$

$$\begin{aligned}
K_T = \int_0^T \|u'' - Pu''\|_W + 3K \max \|u'(t) - Pu'(t)\|_V + 3K \int_0^T \|u'' - Pu''\|_V \\
+\|Pu_1 - u_1^h\|_W + \sqrt{1+K}\|Pu_0 - u_0^h\|_V + \sqrt{K}\|u_1 - Pu_1\|_V.
\end{aligned}$$

Projection errors
To prove the convergence results, it is now necessary to consider the projection errors e_p' and e_p''. Recall Proposition 4.1.3. Substituting these estimates for the projection errors yields the following result.

Notation $u^{(k)} \in \mathcal{L}^2\big(J;Y\big)$ if $u^{(k)}(t) \in Y$ for each t and $\displaystyle\int_J \|u^{(k)}\|_Y^2 < \infty$.

Theorem 4.2.1.
Suppose Assumption D3 holds for the space V. Assume also that the solution u of Problem G satisfies Assumption D2 and that $u'' \in \mathcal{L}^2([0,T]; H(V,k))$. Then, for any $t \in (0,T)$,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + \beta_1\|u'(t)\|_{H(V,k)}\right) + \sqrt{24e^{3t}}C_\Pi h^\alpha \left[\int_0^T \beta_1 \|u''(\cdot)\|_{H(V,k)}\right. \\
&\quad \left. +3K \max \|u'(t)\|_{H(V,k)} + 3K \int_0^T \|u''(\cdot)\|_{H(V,k)}\right] \\
&\quad + \sqrt{24e^{3t}}\left[\|Pu_1 - u_1^h\|_W + \sqrt{1+K}\|Pu_0 - u_0^h\|_V + \sqrt{K}\|u_1 - Pu_1\|_V\right].
\end{aligned}$$

Remark It is clear that the error depends on u_0^h and u_1^h, but u_0^h and u_1^h are not given. As mentioned in Section 4.1, u_0^h and u_1^h must be chosen in applications. One possibility is to use the interpolants of the initial conditions u_0 and u_1. Then the following consequence of Theorem 4.2.1 is obtained.

Corollary 4.2.1.
Suppose Assumption D3 holds for the space V and u_0 and u_1 are in $H(V,k)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume also that the solution u of Problem G satisfies Assumption D2 and $u'' \in \mathcal{L}^2([0,T]; H(V,k))$. Then,

$$\begin{aligned}
&\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\
&\leq C_\Pi h^\alpha \left(\|u(t)\|_{H(V,k)} + \beta_1\|u'(t)\|_{H(V,k)}\right) + \sqrt{24e^{3t}} C_\Pi h^\alpha \left[\int_0^T \beta_1 \|u''(\cdot)\|_{H(V,k)}\right. \\
&\quad +3K \max \|u'(t)\|_{H(V,k)} + 3K \int_0^T \|u''(\cdot)\|_{H(V,k)} \\
&\quad \left. + \, 2\beta_1\|u_1\|_{H(V,k)} + 2\sqrt{1+K}\|u_0\|_{H(V,k)} + \sqrt{K}\|u_1\|_{H(V,k)}\right]
\end{aligned}$$

for $t \in (0,T)$.

4.3 Weak damping

In this section we study the theory in [BV13] where it is assumed that the bilinear form a satisfies Assumption E4W.

4.3.1 Fundamental estimate

For general damping, Proposition 4.1.2 was used to derive an “energy” inequality and from there the fundamental estimate. For weak damping, [BV13] did not use this proposition but derived the next result directly from (4.1.2).

Proposition 4.3.1.

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e(t), v) = 0 \quad \textit{for all} \quad v \in S^h.$$

Using this result, the fundamental estimate below is obtained with fewer steps and it is not necessary to consider $(Pu)''$. In the process, Assumption D1 is not required, but in [BV13] this assumption is erroneously stated in the results. The proofs are however correct.

Lemma 4.3.1.
If u is the solution of Problem G, then for each $t \in [0, T]$,

$$\|e(t)\|_W \leq \sqrt{2}\Big(\|e(0)\|_W + 3T\|e_h'(0)\|_W + 3\int_0^T \|e_p'(t)\|_W + 3KT\|e_h(0)\|_W + 3K\int_0^T \|e_p(t)\|_W\Big).$$

4.3.2 Convergence and error estimates

In this subsection we present results for the error estimates of the semi-discrete approximation as well as the fully discrete approximation. The following error estimate for the semi-discrete approximation follows from Lemma 4.3.1.

Theorem 4.3.1.
If u is the solution of Problem G, then for each $t \in [0, T]$,

$$\begin{aligned} \|u(t) - u_h(t)\|_W \leq \; & \|e_p(t)\|_W + \sqrt{2}\Big(\|Pu_0 - u_0\|_W \\ & + 3T\|u_1 - u_1^h\|_W + 3\int_0^T \|e_p'(t)\|_W \\ & + (1 + 3KT)\|u_0 - u_0^h\|_W + 3K\int_0^T \|e_p(t)\|_W\Big). \end{aligned} \tag{4.3.1}$$

Remark The errors on the right-hand side of (4.3.1) need to be considered in applications.

Error estimate for the fully discrete approximation

The theorem below is from [BV13, Section 6].

Theorem 4.3.2.
If $f \in C^2([0, T]; X)$, then

$$\begin{aligned} \|u_h(t_k) - u_k^h\|_W \leq \; & 7T^2\tau^2 \max \|u_h^{(4)}\|_W + 7T\tau^2 \max \|u_h'''\|_W \\ & + \sqrt{2K}\,\tau^4 \max \|u_h'''\|_W \end{aligned}$$

for each $t_k \in (0, T)$.

Estimates for $\|u_h'''\|_W$, $\|u_h^{(4)}\|_W$ and $\|u_h'''\|_V$ can be obtained in terms of the initial data and the forcing function.

Finally, error estimates for the fully discrete approximation of the solution of Problem G are obtained by combining Theorems 4.3.1 and 4.3.2 and using the triangle inequality.

4.4 Multi-dimensional wave equation

In Section 3.2 existence of a solution for the weak variational problem of the multi-dimensional wave equation with weak damping was proved. In this section the finite element approximation theory is applied. The two-dimensional case with triangle elements is considered. Using piecewise linear basis functions, a finite dimensional subspace S^h of V is constructed.

Galerkin approximation
Find a function $u_h \in C^2(0,T) \cap C^1[0,T]$ such that

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

The Galerkin approximation is a special case of Problem G^h.

Notation We use the notation $|\cdot|_k$ for the seminorm of order k, i.e. $|u|_k = \sum_{|\alpha|=k} \|\partial^\alpha u\|$.

It can be proved that Assumptions D1 and D2 hold, depending on the properties of the boundary of Ω, the parameters, initial values u_0 and u_1 and the function f. For the problem under discussion, we have that $H(V,k) = H^k(\Omega) \cap V$.

Interpolation results are provided in Appendix B. Instead of Assumption D3, the following result is true for piecewise linear basis functions: If $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\|_W \leq C_\Pi T h^2 |u|_2. \tag{4.4.1}$$

Consider application of Theorem 4.2.1. The terms on the right-hand side of

(4.3.1) can be estimated using (4.4.1), e.g.

$$\begin{aligned}\|Pu_0 - u_0\|_W &\leq \beta_1 \|Pu_0 - u_0\|_V \\ &\leq \|\Pi u_0 - u_0\|_V \\ &\leq C_\Pi T h^2 |u_0|_2.\end{aligned}$$

4.5 Timoshenko beam

4.5.1 Galerkin approximation

Consider the weak variational form of the Timoshenko beam model, Problem T-W, given in Subsection 3.3.1. Construct a finite dimensional subspace S^h of V using piecewise linear or piecewise Hermite cubic basis functions.

The Galerkin approximation of Problem T-W is denoted by Problem T-W^h and is again a special case of Problem G^h.

Problem T-W^h
Find $u_h \in C^2(0,T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (\tilde{q}(t), v)_X$$

for each $v \in S^h$, while $u_h(0) = u_0^h = \langle w_0^h, \phi_0^h \rangle$ and $u_h'(0) = u_1^h = \langle w_1^h, \phi_1^h \rangle$.

4.5.2 Interpolation on the product space V

Consider Assumption D3. For this problem we have that

$$H(V,k) = H^k \cap V.$$

Recall that we use the notation $|\cdot|_k$ for the seminorm of order k, i.e. $|u|_k = \|u^{(k)}\|$. For $u \in H^k$ we define $|u|_{H^k}$ by

$$|u|_{H^k}^2 = |u_1|_k^2 + |u_2|_k^2.$$

Instead of Assumption D3 we have a more specific estimate which depends on S^h and the interpolation operator Π. The projection errors depend on the choice of basis functions. First suppose that we use piecewise linear basis functions.

Proposition 4.5.1.
There exists a constant K_b such that for each $u \in H^2 \cap V$

$$\|\Pi u - u\|_V \leq K_b C_\Pi h |u|_{H^2}.$$

Now suppose that piecewise Hermite cubic basis functions are used.

Proposition 4.5.2.
For $k \geq 2$ and $u \in H^k \cap V$,

$$\|\Pi u - u\|_V \leq K_b C_\Pi h^{\alpha-1} |u|_{H^\alpha}, \tag{4.5.1}$$

where $\alpha = k$ for $2 \leq k \leq 4$ and $\alpha = 4$ for $k > 4$.

4.5.3 Error estimate

We use piecewise Hermite cubic basis functions in the following result. It is assumed that $u \in C^2([0,T], V)$ and the estimate (4.5.1) is used.

Application of Corollary 4.2.1

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$ and suppose that $u'' \in \mathcal{L}^2([0,T], H^2 \cap V)$. Then,

$$\begin{aligned} &\|u(t) - u_h(t)\|_V + \|u'(t) - u_h'(t)\|_W \\ &\quad \leq K_b C_\Pi h \big[|\, u(t)|_{H^2} + \beta_1 |\, u'(t)|_{H^2} + 2|\, u_0|_{H^2} + 2\beta_1 |\, u_1|_{H^2} \\ &\qquad + \beta_1 \int_0^t (|u''(\cdot)|_{H^2} + K|u'(\cdot)|_{H^2}) \,\big] \end{aligned}$$

for $t \in [0,T]$.

Chapter 5

Finite element approximation for Timoshenko beam vibration

It is proved in Chapter 4 that optimal error estimates are obtained when the standard finite element method (SFEM) is applied to the Timoshenko beam model. Although piecewise linear basis functions are theoretically admissible test functions, it is known that locking occurs. However, piecewise Hermite cubic basis functions prove to be efficient, as stated in [ZVV04].

Alternatively, the mixed finite element method (MFEM) can be used. In [Sem94] optimal convergence rates for the mixed method are proved for sufficiently smooth solutions, using estimates from [Arn81]. Numerical results are also provided to confirm the results. This is done for the Timoshenko beam that is clamped at both ends. Since the results depend on estimates, we suspect that the theory holds for the pinned-pinned beam as well. From the results in [Arn81] we expect that the MFEM should yield a more accurate approximation for the same number of elements. [Sem94] did however not compare the results for the two different finite element methods.

Our objective is to compare numerical results obtained when the SFEM with cubic basis functions is used to the results obtained when the MFEM with piecewise linear basis functions is used. As far as we know, this has not been done. We consider the undamped dynamic problem and suppose that $q = 0$.

It is proved in previous chapters that the convergence and existence results are the same for beams with different boundary conditions. In this chapter we shall consider the beam with pinned-pinned boundary conditions, since the eigenfunctions have a simple representation given by

$$[w\ \phi]^T = [\sin k\pi x\ A_k \cos k\pi x]^T$$

where k is an integer and A_k a constant. In contrast to this, the eigenfunctions for the cantilever beam have the far more complicated representation

$$\begin{aligned}\begin{bmatrix} u(x) \\ \phi(x) \end{bmatrix} =& A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda+\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda+\mu^2}{\mu} \sinh \mu x \end{bmatrix} \\ &+ C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\lambda-\omega^2}{\omega} \sin \omega x \end{bmatrix}.\end{aligned}$$

These eigenfunctions were derived in [VV06].

5.1 Mixed finite element method

We start off with a brief overview of the theory of the MFEM for the Timoshenko beam.

5.1.1 Mixed variational form

For the MFEM the same procedure is followed as for the standard method in Subsection 1.5.4, except that the constitutive equation (1.5.7) is not substituted after using integration by parts. Instead it is multiplied by an arbitrary function $g \in C^1[0,1]$ and integrated.

$$\int_0^1 Fg = \int_0^1 (\partial_x w - \phi) g \ \text{ for each } \ g \in C^1[0,1].$$

Recall that the test function space for the pinned-pinned beam is given by

$$T_0[0,1] = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\}.$$

The mixed variational form for the Timoshenko beam can now be formulated.

Problem T-VM

Find a pair of functions $\langle w, \phi\rangle$ and a function F such that for each $t > 0$, $w(\cdot,t) \in T_0[0,1]$, $\phi(\cdot,t) \in C^1[0,1]$, $F(\cdot,t) \in C^1[0,1]$ and the following equations hold

$$\int_0^1 \partial_t^2 w(\cdot,t)v = -\int_0^1 F(\cdot,t)v' - \int_0^1 \mu_1 \partial_t w(\cdot,t)v + \int_0^1 q(\cdot,t)v, \quad (5.1.1)$$

$$\frac{1}{\alpha}\int_0^1 \partial_t^2 \phi(\cdot,t)\psi = \int_0^1 F(\cdot,t)\psi - \frac{1}{\beta}\int_0^1 \partial_x \phi(\cdot,t)\psi' - \int_0^1 \mu_2 \partial_t \phi(\cdot,t)\psi, \quad (5.1.2)$$

$$\int_0^1 F(\cdot,t)g = \int_0^1 (\partial_x w(\cdot,t) - \phi)g, \quad (5.1.3)$$

for each $\langle v, \psi\rangle \in T_0[0,\ell] \times C^1[0,\ell]$ and each $g \in C^1[0,1]$, while $\langle w(\cdot,0), \phi(\cdot,0)\rangle = \langle w_0, \phi_0\rangle$ and $\langle \partial_t w(\cdot,0), \partial_t \phi(\cdot,0)\rangle = \langle w_1, \phi_1\rangle$.

We need to add equations (5.1.1) and (5.1.2) for the theory:

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v + \frac{1}{\alpha}\int_0^1 \partial_t^2 \phi(\cdot,t)\psi = &-\int_0^1 (v' - \psi)F(\cdot,t) - \frac{1}{\beta}\int_0^1 \partial_x \phi(\cdot,t)\psi' \\ &- \int_0^1 \mu_1 \partial_t w(\cdot,t)v - \int_0^1 \mu_2 \partial_t \phi(\cdot,t)\psi \\ &+ \int_0^1 q(\cdot,t)v\end{aligned}$$

Recall the bilinear forms

$$\begin{aligned} c(u,v) &= \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2 \quad \text{and} \\ a(u,v) &= \int_0^1 \mu_1 u_1 v_1 + \int_0^1 \mu_2 u_2 v_2 \quad \text{(for weak damping).}\end{aligned}$$

Define the new bilinear forms

$$\begin{aligned} b_1(g,v) &= \int_0^1 (v_1' - v_2)g \quad \text{and} \\ b_2(u,v) &= \int_0^1 u_2' v_2'.\end{aligned}$$

Using the same product spaces as defined in Section 3.3, we can now write Problem T-VM in weak mixed variational form. Let $u(\cdot,t) = \langle w(\cdot,t), \phi(\cdot,t)\rangle$, $\nu = \langle v, \psi\rangle$ and $\tilde{q} : t \mapsto \langle q(\cdot,t), 0\rangle$.

Problem T-WM
Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$, $F(t) \in C^1[0,1]$ and

$$c(u''(t), \nu) + a(u'(t), \nu) + \frac{1}{\beta} b_2(u(t), \nu) + b_1(F(t), \nu) = (\tilde{q}(t), \nu)_X,$$
$$(F(t), g) - b_1(g, u(t)) = 0$$

for all $\nu \in V$ and $g \in C^1[0,1]$, while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_1 = \langle w_1, \phi_1 \rangle$.

Problem T-WM is equivalent to Problem T-W since $C^1[0,1]$ is dense in $\mathcal{L}^2(0,1)$. Thus existence is not an issue.

5.1.2 Convergence for the semi-discrete approximation of the mixed finite element method

Galerkin approximation

We consider only C_0 piecewise linear basis functions. Let V^h be a finite dimensional subspace of V and let X^h be a finite dimensional subspace of $C^1[0,1]$.

Problem T-WM^h
Find $u_h \in C^2(0,T)$ such that u_h' is continuous at 0 and for each $t > 0$, $u_h(t) \in V^h$, $F(t) \in X^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + \frac{1}{\beta} b_2(u_h(t), v) + b_1(F_h(t), v) = (\tilde{q}(t), v)_X,$$
$$c(F_h(t), g) - b_1(g, u_h(t)) = 0$$

for all $v \in V^h$ and $g \in X^h$, while $u_h(0) = u_0^h = \langle w_0^h, \phi_0^h \rangle$ and $u_h'(0) = u_1^h = \langle w_1^h, \phi_1^h \rangle$.

Convergence and error estimate

The optimal convergence rate for the semi-discrete approximation of the MFEM can be obtained under the same approximations as for the SFEM. The following error estimate is from [Sem94]:

If u is a solution of Problem T-WM and a satisfies Assumption E4W, then for $t \in [0, T]$,

$$\begin{aligned} \|u(t) - u_h(t)\| + \|F(t) - F_h(t)\| \; \leq \; & Ch\big[\|u'(t)\|_2 + \|F'(t)\|_1\big] \\ & + \sqrt{2}Ch\Big(\|u(0)\|_2 + \|F(0)\|_1 \\ & + 3\int_0^\tau \big[\|u'(t)\|_2 + \|F'(t)\|_1\big] \\ & + 3\int_0^\tau \big[\|u(t)\|_2 + \|F(t)\|_1\big]\Big). \end{aligned}$$

Remark Theoretically the order of convergence is the same for the SFEM and the MFEM (when the same basis functions are used). In practice however, locking occurs in the SFEM when piecewise linear basis functions are used, as mentioned in the beginning of this chapter.

5.2 Semi-discrete approximation

We now consider the undamped Timoshenko beam model and let $q = 0$.

5.2.1 Standard finite element method

The standard variational form for the Timoshenko beam model is given in Subsection 1.5.4.

To construct the finite dimensional subspace V^h of V, divide the interval $[0, 1]$ into n elements of equal length and number the basis functions δ_i from 1 to $N = 2n + 2$. Cubic basis functions are used as mentioned before.

Denote $\delta_1 = \delta_0^{(1)}, \delta_2 = \delta_0^{(2)}, \delta_3 = \delta_1^{(1)}, ..., \delta_{2n+1} = \delta_n^{(1)}, \delta_{2n+2} = \delta_n^{(2)}$.

Let S_0^h be the span of the basis functions $\{\delta_2, \delta_3, ..., \delta_{2n}, \delta_{2n+2}\}$. δ_1 and δ_{2n+1} are omitted since $\delta_1(0) \neq 0$ and $\delta_{2n+1}(1) \neq 0$. Let S^h be the span of the basis functions $\{\delta_1, \delta_2, ..., \delta_{2n+2}\}$. Then $V^h = S_0^h \times S^h$.

Problem T-V^h
Find w_h and ϕ_h such that for each $t > 0$, $w_h(\cdot, t) \in S_0^h$, $\phi_h(\cdot, t) \in S^h$ and the following equations hold

$$\int_0^1 \partial_t^2 w_h(\cdot,t)v = -\int_0^1 \partial_x w_h(\cdot,t)v' + \int_0^1 \phi_h(\cdot,t)v',$$
$$\int_0^1 \frac{1}{\alpha}\,\partial_t^2 \phi_h(\cdot,t)\psi = -\int_0^1 \frac{1}{\beta}\partial_x \phi_h(\cdot,t)\psi' + \int_0^1 \partial_x w_h(\cdot,t)\psi - \int_0^1 \phi_h(\cdot,t)\psi$$

for each $\langle v, \psi\rangle \in S_0^h \times S^h$, while $\langle w_h(\cdot,0), \phi_h(\cdot,0)\rangle = \langle w_0^h, \phi_0^h\rangle$ and $\langle \partial_t w(\cdot,0), \partial_t \phi(\cdot,0)\rangle = \langle w_1^h, \phi_1^h\rangle$.

If $(u,v) = \displaystyle\int_0^1 uv$, Problem T-V^h can be rewritten.

Find w_h and ϕ_h such that for each $t > 0$, $w_h(\cdot, t) \in S_0^h$, $\phi_h(\cdot, t) \in S^h$ and the following equations hold

$$(\partial_t^2 w_h(\cdot,t), v) + (\partial_x w_h(\cdot,t), v') - (\phi_h(\cdot,t), v') = 0,$$
$$(\partial_t^2 \phi_h(\cdot,t), \psi) + \frac{1}{\gamma}(\partial_x \phi_h(\cdot,t), \psi') - \alpha(\partial_x w_h(\cdot,t), \psi) + \alpha(\phi_h(\cdot,t), \psi) = 0$$

for each $\langle v, \psi\rangle \in S_0^h \times S^h$, while $\langle w_h(\cdot,0), \phi_h(\cdot,0)\rangle = \langle w_0^h, \phi_0^h\rangle$ and $\langle \partial_t w_h(\cdot,0), \partial_t \phi_h(\cdot,0)\rangle = \langle w_1^h, \phi_1^h\rangle$.

Define the following matrices

$$K_{ij} = (\delta_j', \delta_i'),$$
$$M_{ij} = (\delta_j, \delta_i),$$
$$L_{ij} = (\delta_j, \delta_i').$$

We also use sub-matrices. If A is a $N \times N$ matrix, then

> A_0 is matrix A with rows 1 and $2n+1$ and columns 1 and $2n+1$ deleted.
>
> A_R is matrix A with rows 1 and $2n+1$ deleted.

Then Problem T-V^h can be written as a system of ordinary differential equations.

$$\begin{aligned} M_0\bar{w}'' + K_0\bar{w} - L_R\bar{\phi} &= 0, \\ M\bar{\phi}'' + \frac{1}{\gamma}K\bar{\phi} - \alpha L_R^T\bar{w} + \alpha M\bar{\phi} &= 0 \end{aligned} \tag{5.2.1}$$

with $\bar{w}(0) = \bar{a}$ and $\bar{\phi}(0) = \bar{b}$, where $a_i = w_0(x_i)$ and $b_i = \phi_0(x_i)$.

5.2.2 Mixed finite element method

Piecewise linear basis functions are used for the MFEM as mentioned before. The interval $[0, 1]$ is divided into n elements of equal length and the basis functions δ_i are numbered from 1 to $N = n + 1$.

Let S_0^h be the span of the basis functions $\{\delta_2, \delta_3, ..., \delta_n\}$. δ_1 and δ_{n+1} are omitted since $\delta_1(0) \neq 0$ and $\delta_{n+1}(1) \neq 0$. Let S^h be the span of the basis functions $\{\delta_1, \delta_2, ..., \delta_{n+1}\}$.

The mixed variational form, Problem T-VM, is given in Subsection 5.1.1. Again using the notation $(u, v) = \displaystyle\int_0^1 uv$, the Galerkin approximation for the mixed variational form can be formulated.

Problem T-VM^h
Find w_h, ϕ_h and F_h such that for each $t > 0$, $w_h(\cdot, t) \in S_0^h$, $\phi_h(\cdot, t) \in S^h$, $F_h(\cdot, t) \in S^h$ and the following equations hold

$$\begin{aligned}
(\partial_t^2 w_h(\cdot, t), v) + (F_h(\cdot, t), v') &= 0 \quad \text{for each } v \in S_0^h, \\
\frac{1}{\alpha}(\partial_t^2 \phi_h(\cdot, t), \psi) - (F_h(\cdot, t), \psi) + \frac{1}{\beta}(\partial_x \phi_h(\cdot, t), \psi') &= 0 \quad \text{for each } \psi \in S^h, \\
(F_h(\cdot, t), g) - (\partial_x w_h(\cdot, t), g) + (\phi_h(\cdot, t), g) &= 0 \quad \text{for each } g \in S^h
\end{aligned}$$

while $\langle w_h(\cdot, 0), \phi_h(\cdot, 0)\rangle = \langle w_0^h, \phi_0^h\rangle$ and $\langle \partial_t w_h(\cdot, 0), \partial_t \phi_h(\cdot, 0)\rangle = \langle w_1^h, \phi_1^h\rangle$.

We use the same matrices as defined for the SFEM. Then Problem T-VM^h can be written as the following system of ordinary differential equations.

$$\begin{aligned}
M_0 \bar{w}'' + L_R \bar{F} &= 0, \\
M \bar{\phi}'' - \alpha M \bar{F} + \frac{1}{\gamma} K \bar{\phi} &= 0, \\
M \bar{F} - L_R^T \bar{w} + M \bar{\phi} &= 0
\end{aligned} \tag{5.2.2}$$

with $\bar{w}(0) = \bar{a}$ and $\bar{\phi}(0) = \bar{b}$, where $a_i = w_0(x_i)$ and $b_i = \phi_0(x_i)$.

5.3 Numerical results

This section contains the numerical results obtained. As mentioned before, our emphasis is on the comparison of the results for the SFEM and the MFEM using piecewise linear and piecewise Hermite cubic basis functions respectively. We considered the dynamic problem for the undamped Timoshenko beam with pinned-pinned boundary conditions.

The initial conditions are given by $u(x,0) = u_0$ and $\phi(x,0) = \phi_0$ where

$$u_0 = \frac{1}{2}x^2 - \frac{\beta}{24}x^4 + \frac{\beta}{12}x^3 - \frac{1}{2}x - \frac{\beta}{24}x$$

and

$$\phi_0 = -\frac{\beta}{6}x^3 + \frac{\beta}{4}x^2 - \frac{\beta}{24}.$$

(The pair $\langle u_0, \phi_0 \rangle \in E_b$, see Section 3.3.)

For our experiments we used $\gamma = \dfrac{\beta}{\alpha} = 0.25$ and $\alpha = 1200$. The approximations for the deflection w and the angle of rotation ϕ at a final dimensionless time of 2 were investigated.

Our approach was to solve the system of ordinary differential equations as accurately as possible so that the finite difference scheme used does not contribute to the errors obtained and therefore it does not matter which one was used. (We used central differences.)

The ordinary differential equations used for the SFEM and the MFEM are given in (5.2.1) and (5.2.2) respectively.

Since cubic basis functions are used in the SFEM, four equations are solved on each element, namely the deflection, angle of rotation as well as their spatial derivatives. While with the MFEM only three equations are solved on each element, namely the deflection, angle of rotation and the shear force. Thus in order to compare the two methods, more elements need to be used in the MFEM than in the SFEM.

5.3.1 Deflection

Since the point $x = 0.5$ is where we expect the largest deviation to take place, numerical results for that point is presented in the following table to demonstrate convergence. Since we do not have the exact solution for this problem, the convergence is analysed by using the absolute difference.

Table 1 Deflection at $x = 0.5$ for SFEM

# elements	Result	Abs. difference	Ratio
4	-1.7911	-	-
8	-1.7924	0.0013	-
16	-1.7926	0.0002	0.1538
32	-1.7926	0	0
64	-1.7926	0	0

Table 2 Deflection at $x = 0.5$ for MFEM

# elements	Result	Abs. difference	Ratio
8	-1.7723	-	-
16	-1.7869	0.0146	-
32	-1.7912	0.0043	0.2945
64	-1.7922	0.001	0.2326
128	-1.7925	0.0003	0.3000

It is clear from Table 1 that the average ratio with which the deflection at $x = 0.5$ for the SFEM converges is 0.1538, while from Table 2 the average ratio is 0.2757. Hence it can be concluded that the SFEM with cubic basis functions has a higher rate of convergence than the MFEM with piecewise linear basis functions.

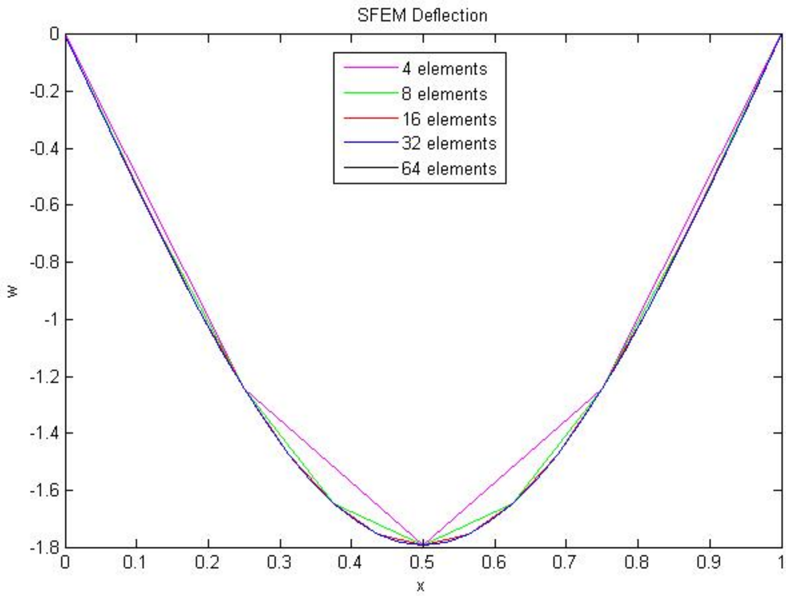

Fig. 1 SFEM deflection

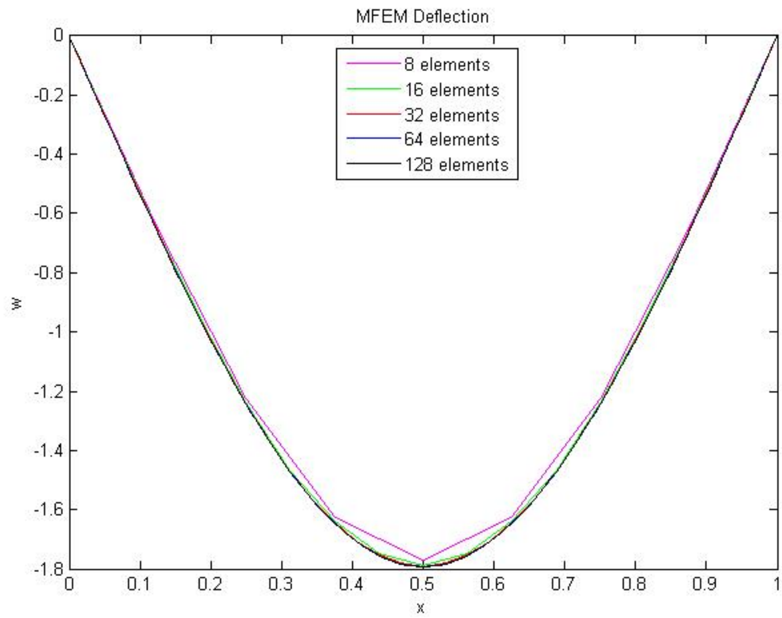

Fig. 2 MFEM deflection

Remark Note that there are no conclusions made from the graphs. It is merely presented for completeness.

5.3.2 Angle of rotation

Since the largest deviation takes place at the two endpoints, that is where $x = 0$ and $x = 1$, results at $x = 1$ will be given in the following tables to illustrate convergence.

Table 3 Angle of rotation at $x = 1$ for SFEM

# elements	Result	Abs. difference	Ratio
4	5.2522	-	-
8	5.2692	0.017	-
16	5.2696	0.0004	0.02353
32	5.2695	0.0001	0.25
64	5.2695	0	0

Table 4 Angle of rotation at $x = 1$ for MFEM

# elements	Result	Abs. difference	Ratio
8	5.2053	-	-
16	5.2370	0.0317	-
32	5.2587	0.0217	0.6845
64	5.2673	0.0086	0.3963
128	5.2690	0.0017	0.1977

From Table 3 we see that the average ratio with which the angle of rotation for the SFEM decreases is 0.1368 while from Table 4 we get that the average ratio for the MFEM is 0.4262. Again this indicates that the SFEM with cubic basis functions has a higher rate of convergence that the MFEM with piecewise linear basis functions.

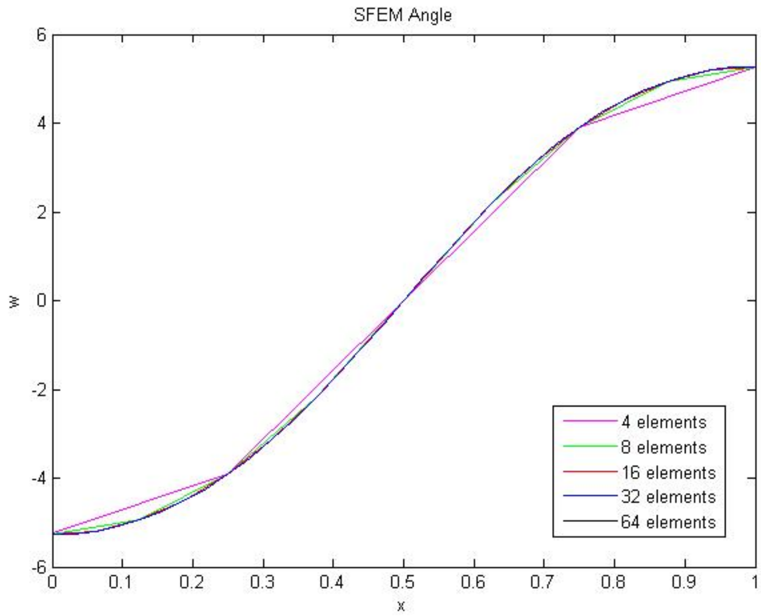

Fig. 3 SFEM angle

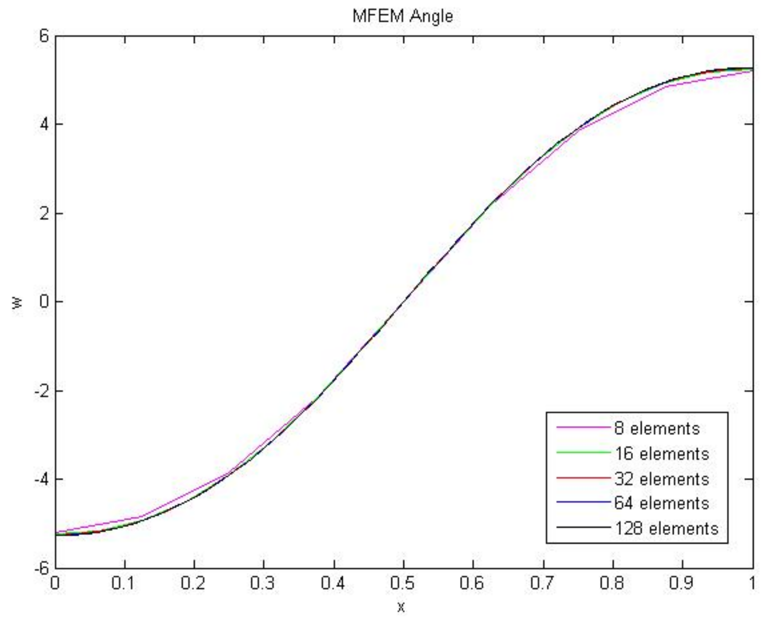

Fig. 4 MFEM angle

5.4 Conclusion

The intention in this chapter was to apply the theory of the FEM to examples. We became interested in how the MFEM compares to the SFEM. The only comparison between the two methods was found in [Arn81] and in the unpublished technical report by Basson and Van Rensburg. [Arn81] did a thorough comparison of the SFEM and the MFEM, but for the static problem. Although [Sem94] did numerical experiments, no comparison was made between the two methods. In the technical report, Basson and Van Rensburg compared the MFEM to the SFEM using C_0 piecewise linear basis functions for both. We were curious as to what would happen when the SFEM with Hermite cubic basis functions is compared to the MFEM with piecewise linear basis functions.

It can be concluded from the numerical experiments that the SFEM with cubic basis functions has a higher rate of convergence than the MFEM with piecewise linear basis functions.

It should however be noted that the method used to compare the two finite element methods in this chapter is rather naive. We do not have the criteria or means that enable us to calculate the computational cost of each method. It was also never the intention to do so. But an educated guess would be that for the mixed method three functions needed to be evaluated using piecewise linear basis functions while for the standard method four functions had to be approximated using Hermite cubic basis functions, which does suggest that the computational effort for the standard method will be higher than that of the mixed method.

More numerical experiments and detailed analysis are required, but that is considered to be a project in its own right.

Chapter 6

Two-dimensional model for a beam

The article [LVV09] is mentioned in Section 1.5 as one of the sources where the Timoshenko beam model is compared to higher dimensional theories. In this chapter we consider this article, more specifically the case where the Timoshenko theory is compared to a two-dimensional theory.

The authors of [LVV09] used the finite element method (FEM) with Hermite piecewise bicubic basis functions to compare the two models. It is concluded that the Timoshenko beam model is remarkably accurate in comparison to the two-dimensional model for a beam.

The intention in this chapter was to investigate how the two beam models would compare should the mixed finite element method (MFEM) be used. In this case, piecewise bilinear basis functions can be used. It became clear that this task was too ambitious and is rather an opportunity for future research. However, a valuable start is made in this chapter.

6.1 Equation of motion

We start off by giving a short review of linear elasticity. A three-dimensional elastic body with density ρ is considered. The equation of motion in the theory of linear elasticity is given by

$$\rho\partial_t^2 u = \text{div } T + Q, \tag{6.1.1}$$

where $u(x,t)$ is the displacement of a point x in the reference configuration at time t, T is the stress tensor and Q is an external body force or density force. This equation of motion is derived from the conservation law of momentum, see [Fun65, Sections 5.5 and 5.7] or [AF80, p.125].

The stress components in the matrix representation of T are denoted by σ_{ij}. Recall that the components of the vector div T are given by

$$[\text{div } T]_i = \partial_1\sigma_{i1} + \partial_2\sigma_{i2} + \partial_3\sigma_{i3} \quad \text{for } i = 1,\ 2,\ 3.$$

Constitutive equations

The infinitesimal strain $\mathcal{E} = (e_{ij})$ is given by

$$e_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right). \tag{6.1.2}$$

In [LVV09] Hooke's law for homogeneous isotropic materials is used. For the case of plane stress, the constitutive equations are

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(e_{11} + \nu e_{22}\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(e_{22} + \nu e_{11}\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{1+\nu}\ e_{12}, \end{aligned}$$

where E is Young's modulus and ν Poisson's ratio. See [Fun65, Section 9.1].

Although this is not done in [LVV09], the constitutive equations can be written in terms of the components of the displacement u by using the infinitesimal strain (6.1.2).

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(\partial_1 u_1 + \nu\partial_2 u_2\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(\partial_2 u_2 + \nu\partial_1 u_1\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{2(1+\nu)}\left(\partial_1 u_2 + \partial_2 u_1\right). \end{aligned}$$

Remark Substitution of the constitutive equations above into the equation of motion (6.1.1) yields a system of partial differential equations for the components of u. We will however not make use of this system of partial differential equations.

6.1.1 Dimensionless form

As mentioned above, the length of the beam is denoted by ℓ. To obtain a model in dimensionless form, let

$$\tau = \frac{t}{t_0} \quad \text{and} \quad \xi_i = \frac{x_i}{\ell}.$$

Recall from Section 1.5 that a convenient choice for t_0 is

$$t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}.$$

The variables are transformed as follows:

$$u^*(\xi,\tau) = \frac{u(x,t)}{\ell} \quad \text{and} \quad \sigma^*_{ij}(\xi,\tau) = \left(\frac{1}{G\kappa^2}\right)\sigma_{ij}(x,t).$$

Recall also from Section 1.5 the dimensionless constant $\gamma = \dfrac{G\kappa^2}{E}$.

Free vibration is considered in [LVV09], that is $Q = 0$. We will however first present the case where $Q \neq 0$ and then consider special cases. Using the original notation, the equation of motion and constitutive equations are now presented in dimensionless form.

Equation of motion

$$\partial_t^2 u = \text{div } T + Q \tag{6.1.3}$$

where

$$\text{div } T = \begin{bmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} \\ 0 \end{bmatrix}.$$

Constitutive equations

$$\begin{aligned} \sigma_{11} &= \frac{1}{\gamma(1-\nu^2)}\big(\partial_1 u_1 + \nu\partial_2 u_2\big), \\ \sigma_{22} &= \frac{1}{\gamma(1-\nu^2)}\big(\partial_2 u_2 + \nu\partial_1 u_1\big), \\ \sigma_{12} = \sigma_{21} &= \frac{1}{2\gamma(1+\nu)}\big(\partial_1 u_2 + \partial_2 u_1\big). \end{aligned} \tag{6.1.4}$$

6.2 Cantilever beam

In Section 6.1 a two-dimensional model is proposed. In this section we will introduce the dynamic problem for the cantilever beam and then discuss two special cases, namely the equilibrium problem and the eigenvalue problem. Consider a cantilever beam as illustrated in Fig. 1 below.

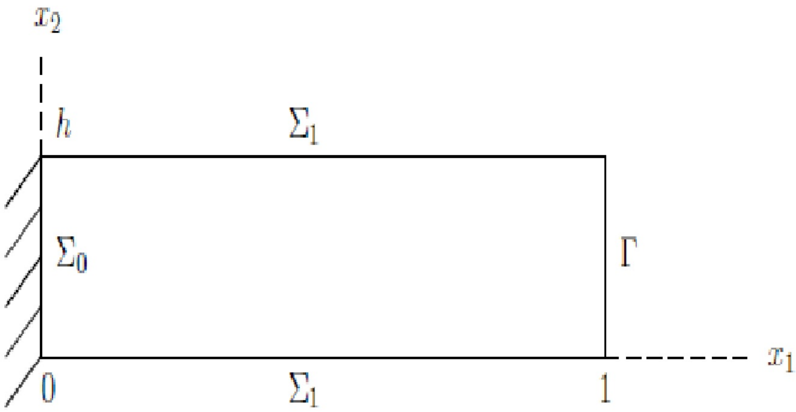

Fig. 1

Boundary conditions

For this problem it is assumed that the beam is fixed rigidly to the support at $x_1 = 0$. The reference configuration Ω is the rectangle where

$$0 \leq x_1 \leq 1 \quad \text{and} \quad 0 \leq x_2 \leq h.$$

It is assumed that the boundary condition on Σ_0 is zero displacement, see [Fun65, Section 7.7]. The two parts of Σ_1 (where $x_2 = 0$ or $x_2 = h$) are stress free. For the general problem we assume that a (dimensionless) vertical force F is applied at Γ (where $x_1 = 1$). This leads to $Te_1 = pe_2$ where p is a function of t and x_2 and $F(t) = \displaystyle\int_0^h p(x_2, t)\, dx_2$.

The boundary conditions remain the same for the equilibrium problem, however, for the eigenvalue problem we have that $Te_1 = 0$ on Γ.

Dynamic problem

$$
\begin{aligned}
\partial_t^2 u &= \text{div } T + Q && \text{in } \Omega, \\
u &= 0 && \text{on } \Sigma_0, \\
Te_2 &= 0 && \text{on } \Sigma_1, \\
Te_1 &= pe_2 && \text{on } \Gamma
\end{aligned}
\tag{6.2.1}
$$

with the constitutive equations given by (6.1.4).

Equilibrium problem

$$
\begin{aligned}
\text{div } T + Q &= 0 && \text{in } \Omega, \\
u &= 0 && \text{on } \Sigma_0, \\
Te_2 &= 0 && \text{on } \Sigma_1, \\
Te_1 &= pe_2 && \text{on } \Gamma
\end{aligned}
\tag{6.2.2}
$$

with the constitutive equations given by (6.1.4).

Eigenvalue problem

$$
\begin{aligned}
\partial_t^2 u &= \text{div } T && \text{in } \Omega, \\
u &= 0 && \text{on } \Sigma_0, \\
Te_2 &= 0 && \text{on } \Sigma_1, \\
Te_1 &= 0 && \text{on } \Gamma
\end{aligned}
\tag{6.2.3}
$$

with the constitutive equations given by (6.1.4).

6.3 Variational forms

To write the model in variational form, multiply both sides of the equation of motion (6.1.3) by an arbitrary vector valued function ϕ and integrate over the reference configuration Ω.

$$
\iint_\Omega (\partial_t^2 u) \cdot \phi \, dA = \iint_\Omega (\text{div } T) \cdot \phi \, dA + \iint_\Omega Q \cdot \phi \, dA. \tag{6.3.1}
$$

It is assumed that T is symmetric. Then

$$
\text{div } (T\phi) = (\text{div } T) \cdot \phi + T : \Phi, \tag{6.3.2}
$$

where

$$\Phi = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$$

and $T : \Phi$ is the trace of the tensor $T\Phi$.

It follows from the divergence theorem (Subsection 1.4.3) and the symmetry of T that

$$\iint_\Omega \text{div } (T\phi)\, dA = \int_{\partial\Omega} T\phi \cdot n\, ds = \int_{\partial\Omega} Tn \cdot \phi\, ds. \tag{6.3.3}$$

Substituting (6.3.2) into (6.3.3) yields the Green formula

$$\iint_\Omega (\text{div } T) \cdot \phi\, dA = -\iint_\Omega T : \Phi\, dA + \int_{\partial\Omega} Tn \cdot \phi\, ds. \tag{6.3.4}$$

Combining (6.3.1) and the Green formula (6.3.4) yields

$$\iint_\Omega (\partial_t^2 u) \cdot \phi\, dA = -\iint_\Omega T : \Phi\, dA + \int_{\partial\Omega} Tn \cdot \phi\, ds + \iint_\Omega Q \cdot \phi\, dA \tag{6.3.5}$$

for any sufficiently smooth vector field ϕ.

Test functions

To define the space of test functions $T(\Omega)$, recall that the boundary of Ω consists of the two parts Σ and Γ. The test functions ϕ must satisfy the forced boundary conditions on Σ. Thus

$$T(\Omega) = \{\phi \in C^1(\bar{\Omega})^2 \ \mid\ \phi = 0 \text{ on } \Sigma_0\}.$$

Since Σ_1 is stress free, it follows that $Tn \cdot \phi = 0$ on Σ_1. Consequently

$$\int_{\partial\Omega} Tn \cdot \phi\, ds = \int_\Gamma Te_1 \cdot \phi\, ds = \int_\Gamma pe_2 \cdot \phi\, ds \quad \text{for each } \phi \in T(\Omega).$$

Note that $T : \Phi = \sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2$. Substituting the constitutive equations (6.1.4) yields

$$\begin{aligned} T : \Phi = \ & \frac{\partial_1\phi_1}{\gamma(1-\nu^2)}(\partial_1 u_1 + \nu\partial_2 u_2) \ + \ \frac{\partial_1\phi_2 + \partial_2\phi_1}{2\gamma(1+\nu)}(\partial_1 u_2 + \partial_2 u_1) \\ & + \ \frac{\partial_2\phi_2}{\gamma(1-\nu^2)}(\partial_2 u_2 + \nu\partial_1 u_1). \end{aligned}$$

Define the following bilinear form b.

$$\begin{aligned} b(u,\phi) = & \frac{1}{\gamma(1-\nu^2)} \iint_\Omega \big(\partial_1 u_1 \partial_1 \phi_1 + \partial_2 u_2 \partial_2 \phi_2 + \nu(\partial_1 u_1 \partial_2 \phi_2 + \partial_2 u_2 \partial_1 \phi_1)\big)\, dA \\ & + \frac{1}{2\gamma(1+\nu)} \iint_\Omega (\partial_1 u_2 + \partial_2 u_1)(\partial_1 \phi_2 + \partial_2 \phi_1)\, dA \end{aligned}$$

The variational form is to find u such that for each $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$\iint_\Omega \partial_t^2 u \cdot \phi \, dA = -b(u,\phi) + \int_\Gamma p e_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA \tag{6.3.6}$$

for each $\phi \in T(\Omega)$.

Equilibrium problem

The space of test functions remains the same as for the dynamic problem. For the equilibrium problem (6.3.6) becomes

$$-b(u,\phi) + \int_\Gamma p e_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA \;=\; 0 \;\text{ for each }\; \phi \in T(\Omega).$$

Eigenvalue problem

Separation of variables in the dynamic problem leads to the variational form for the eigenvalue problem. Equation (6.3.6) becomes

$$b(u,\phi) = \lambda \iint_\Omega u \cdot \phi \, dA \;\text{ for each }\; \phi \in T(\Omega).$$

In [LVV09] Hermite piecewise bicubic basis functions are used. These functions are constructed by using a product of the Hermite piecewise cubic functions, hence the name bicubics, see [SF73, p.88-89] for detail. One immediate concern is that sixteen local basis functions are used on each rectangle element.

6.4 Mixed finite element method

In many applications the stresses are more important than displacements. Therefore it would be computationally efficient to use the MFEM rather than the SFEM with bicubic basis functions. To explain, consider the FEM for the Timoshenko beam model in Chapter 5. For the SFEM, the shear force F need to be approximated using the constitutive equation $F = \partial_x w - \phi$. It should however be noted that although w and ϕ can be approximated accurately, it does not imply that the approximation for $\partial_x w$ is accurate. This is where the MFEM is advantageous. For the MFEM, the shear force F is approximated together with w and ϕ (see (5.2.2)). Therefore no further calculations are needed to obtain the approximation for F. In the two-dimensional beam model we need accurate approximations for these stress components.

6.4.1 Mixed variational form

The MFEM for the two-dimensional beam model is not done in [LVV09]. For the MFEM of the two-dimensional beam model, we consider equation (6.3.5) and recall that $T : \Phi = \sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2$. Using the same test functions as before, we get that

$$\begin{aligned} \iint_\Omega \partial_t^2 u \cdot \phi \, dA = & - \iint_\Omega (\sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2) \, dA \\ & + \int_\Gamma p e_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA \end{aligned} \tag{6.4.1}$$

for each $\phi \in T(\Omega)$.

Instead of substituting the constituve equations (6.1.4) into (6.4.1), we write the strain as the vector $\sigma = [\, \sigma_{11} \ \ \sigma_{12} \ \ \sigma_{22} \,]^T$. We multiply this strain vector by an arbitrary function $\psi \in \mathcal{L}^2(\Omega)^3$ and integrate.

$$\begin{aligned} \iint_\Omega \sigma \cdot \psi \, dA \;=\; & \iint_\Omega \Big(\frac{\psi_1}{\gamma(1-\nu^2)}(\partial_1 u_1 + \nu \partial_2 u_2) + \frac{\psi_2}{2\gamma(1+\nu)}(\partial_1 u_2 + \partial_2 u_1) \\ & + \frac{\psi_3}{\gamma(1-\nu^2)}(\partial_2 u_2 + \nu \partial_1 u_1) \Big) \, dA. \end{aligned}$$

The mixed variational form is to find $u(\cdot, t) \in T(\Omega)$ and $\sigma \in \mathcal{L}^2(\Omega)^3$ such that

$$\iint_\Omega \partial_t^2 u \cdot \phi \, dA = -\iint_\Omega (\sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2) \, dA \\ + \int_\Gamma pe_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA \quad \text{for each } \phi \in T(\Omega) \tag{6.4.2}$$

and

$$\iint_\Omega \sigma \cdot \psi \, dA = \iint_\Omega \Big(\frac{\psi_1}{\gamma(1-\nu^2)}(\partial_1 u_1 + \nu\partial_2 u_2) + \frac{\psi_2}{2\gamma(1+\nu)}(\partial_1 u_2 + \partial_2 u_1) \\ + \frac{\psi_3}{\gamma(1-\nu^2)}(\partial_2 u_2 + \nu\partial_1 u_1) \Big) \, dA \quad \text{for each } \psi \in \mathcal{L}^2(\Omega)^3. \tag{6.4.3}$$

Again we consider the two special cases, namely the equilibrium problem and the eigenvalue problem.

Equilibrium problem

For the equilibrium problem, equations (6.4.2) and (6.4.3) become

$$-\iint_\Omega (\sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2) \, dA \\ + \int_\Gamma pe_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA = 0 \quad \text{for each } \phi \in T(\Omega)$$

and

$$\iint_\Omega \sigma \cdot \psi \, dA = \iint_\Omega \Big(\frac{\psi_1}{\gamma(1-\nu^2)}(\partial_1 u_1 + \nu\partial_2 u_2) + \frac{\psi_2}{2\gamma(1+\nu)}(\partial_1 u_2 + \partial_2 u_1) \\ + \frac{\psi_3}{\gamma(1-\nu^2)}(\partial_2 u_2 + \nu\partial_1 u_1) \Big) \, dA \quad \text{for each } \psi \in \mathcal{L}^2(\Omega)^3.$$

Eigenvalue problem

Recall that for the eigenvalue problem $p = 0$ and $Q = 0$. The mixed variational form for the eigenvalue problem is obtained from separation of variables in the dynamic problem. Equations (6.4.2) and (6.4.3) become

$$\iint_\Omega (\sigma_{11}\partial_1\phi_1 + \sigma_{12}(\partial_1\phi_2 + \partial_2\phi_1) + \sigma_{22}\partial_2\phi_2) \, dA = \lambda \iint_\Omega u \cdot \phi \, dA$$

for each $\phi \in T(\Omega)$ and

$$\iint_\Omega \Big(\frac{\psi_1}{\gamma(1-\nu^2)}(\partial_1 u_1 + \nu \partial_2 u_2) + \frac{\psi_2}{2\gamma(1+\nu)}(\partial_1 u_2 + \partial_2 u_1) + \frac{\psi_3}{\gamma(1-\nu^2)}(\partial_2 u_2 + \nu \partial_1 u_1)\Big)\, dA \;=\; \mu \iint_\Omega \sigma \cdot \psi \, dA$$

for each $\psi \in \mathcal{L}^2(\Omega)^3$.

6.4.2 Semi-discrete approximation

In [LVV09] Hermite piecewise bicubic basis functions are used. These functions are constructed by using a product of the Hermite piecewise cubic functions, hence the name bicubics. See [SF73, p.88-89] for detail. The reference configuration Ω is divided into rs rectangular elements, where r and s denote the number of intervals on the x_1-axis and x_2-axis respectively. The total number of nodes for this grid is $N = (r+1)(s+1)$ and hence the number of bicubic basis functions is $4N$.

We will however consider piecewise bilinear functions. The reference configuration Ω is again divided into rs rectangular elements. Number the nodes from 1 to N. The interior nodes together with the nodes at Σ_1 and Γ are numbered from 1 to k while the nodes at Σ_0 are numbered from $k+1$ to N.

The approximate solutions are denoted by u^h and σ^h and the components are expressed as linear combinations of the basis functions δ_j.

$$u^h \;=\; \begin{bmatrix} u_1^h & u_2^h \end{bmatrix}^T \;=\; \begin{bmatrix} \sum_{j=1}^{k} u_{1j}\delta_j & \sum_{j=1}^{k} u_{2j}\delta_j \end{bmatrix}^T$$

and

$$\sigma^h \;=\; \begin{bmatrix} \sigma_{11}^h & \sigma_{12}^h & \sigma_{22}^h \end{bmatrix}^T \;=\; \begin{bmatrix} \sum_{j=1}^{N} \sigma_{11j}\delta_j & \sum_{j=1}^{N} \sigma_{12j}\delta_j & \sum_{j=1}^{N} \sigma_{22j}\delta_j \end{bmatrix}^T .$$

Then the set with elements

$$[\delta_1 \; 0]^T, \quad [\delta_2 \; 0]^T, \quad \dots \quad [\delta_k \; 0]^T,$$
$$[0 \; \delta_1]^T, \quad [0 \; \delta_2]^T, \quad \dots \quad [0 \; \delta_k]^T$$

is a basis for

$$S_0^h = \left\{ \left[\sum_{j=1}^{k} u_{1j}\delta_j \quad \sum_{j=1}^{k} u_{2j}\delta_j \right]^T \;\middle|\; u_{1j} \text{ and } u_{2j} \in \mathbb{R} \right\}$$

while the set with elements

$$\begin{gathered}
[\,\delta_1 \; 0 \; 0\,]^T, \; [\,\delta_2 \; 0 \; 0\,]^T, \; \dots \; [\,\delta_N \; 0 \; 0\,]^T, \\
[\,0 \; \delta_1 \; 0\,]^T, \; [\,0 \; \delta_2 \; 0\,]^T, \; \dots \; [\,0 \; \delta_N \; 0\,]^T, \\
[\,0 \; 0 \; \delta_1\,]^T, \; [\,0 \; 0 \; \delta_2\,]^T, \; \dots \; [\,0 \; 0 \; \delta_N\,]^T
\end{gathered}$$

is a basis for

$$S^h = \left\{ \left[\sum_{j=1}^{N} \sigma_{11j}\delta_j \quad \sum_{j=1}^{N} \sigma_{12j}\delta_j \quad \sum_{j=1}^{N} \sigma_{22j}\delta_j \right]^T \;\middle|\; \sigma_{11j}, \; \sigma_{12j} \text{ and } \sigma_{22j} \in \mathbb{R} \right\}.$$

Galerkin approximation

Find $u^h(\cdot, t) \in S_0^h$ and $\left[\sigma_{11}^h \;\; \sigma_{12}^h \;\; \sigma_{22}^h \right]^T \in S^h$ such that

$$\begin{aligned}
\iint_\Omega \partial_t^2 u^h \phi \, dA = &- \iint_\Omega (\sigma_{11}^h \partial_1 \phi_1 + \sigma_{12}^h (\partial_1 \phi_2 + \partial_2 \phi_1) + \sigma_{22}^h \partial_2 \phi_2) \, dA \\
&+ \int_\Gamma p e_2 \cdot \phi \, ds + \iint_\Omega Q \cdot \phi \, dA
\end{aligned} \tag{6.4.4}$$

for each $\phi \in S_0^h$ and

$$\begin{aligned}
\iint_\Omega (\sigma_{11}^h \psi_1 + \sigma_{12}^h \psi_2 + \sigma_{22}^h \psi_3) \, dA \;=\; & \iint_\Omega \Big(\frac{\psi_1}{\gamma(1-\nu^2)} (\partial_1 u_1^h + \nu \partial_2 u_2^h) \\
& + \frac{\psi_2}{2\gamma(1+\nu)} (\partial_1 u_2^h + \partial_2 u_1^h) \\
& + \frac{\psi_3}{\gamma(1-\nu^2)} (\partial_2 u_2^h + \nu \partial_1 u_1^h) \Big) \, dA
\end{aligned} \tag{6.4.5}$$

for each $\psi \in S^h$.

Equations (6.4.4) and (6.4.5) can be written as a system of linear equations, as we show below.

Substitute $\phi = [\,\phi_1\ \phi_2\,]^T$ with

$$[\,\delta_1\ 0\,]^T,\ [\,\delta_2\ 0\,]^T,\ \dots\ [\,\delta_k\ 0\,]^T$$

and

$$[\,0\ \delta_1\,]^T,\ [\,0\ \delta_2\,]^T,\ \dots\ [\,0\ \delta_k\,]^T$$

in (6.4.4). Also substitute $\psi = [\,\psi_1\ \psi_2\ \psi_3\,]^T$ with

$$[\,\delta_1\ 0\ 0\,]^T, [\,\delta_2\ 0\ 0\,]^T,\ \dots\ [\,\delta_N\ 0\ 0\,]^T,$$

$$[\,0\ \delta_1\ 0\,]^T,\ [\,0\ \delta_2\ 0\,]^T,\ \dots\ [\,0\ \delta_N\ 0\,]^T$$

and

$$[\,0\ 0\ \delta_1\,]^T,\ [\,0\ 0\ \delta_2\,]^T,\ \dots\ [\,0\ 0\ \delta_N\,]^T$$

in (6.4.5).

Using the notation $\big(f, g\big) = \iint_\Omega fg$, the following system of linear equations is obtained.

$$\begin{aligned}
&(\partial_t^2 u_1^h, \delta_i) = -(\sigma_{11}^h, \partial_1\delta_i) - (\sigma_{12}^h, \partial_2\delta_i) + (Q_1, \delta_i),\\
&(\partial_t^2 u_2^h, \delta_i) = -(\sigma_{12}^h, \partial_1\delta_i) - (\sigma_{22}^h, \partial_2\delta_i) + \int_\Gamma p\delta_i\ ds + (Q_2, \delta_i) \quad \text{for } i = 1, 2, \dots k.\\
&(\sigma_{11}^h, \delta_i) = \frac{1}{\gamma(1-\nu^2)}(\partial_1 u_1^h + \nu\partial_2 u_2^h, \delta_i),\\
&(\sigma_{12}^h, \delta_i) = \frac{1}{2\gamma(1+\nu)}(\partial_1 u_2^h + \partial_2 u_1^h, \delta_i),\\
&(\sigma_{22}^h, \delta_i) = \frac{1}{\gamma(1-\nu^2)}(\partial_2 u_2^h + \nu\partial_1 u_1^h, \delta_i) \qquad \text{for } i = 1, 2, \dots N.
\end{aligned} \tag{6.4.6}$$

6.4.3 Matrix formulation

Recall that the components of the approximate solutions u^h and σ^h are expressed as a linear combination of the piecewise bilinear basis functions. That is

$$u_1^h(x) = \sum_{j=1}^{k} u_{1j}\delta_j(x)\,, \quad u_2^h(x) = \sum_{j=1}^{k} u_{2j}\delta_j(x)\,,$$

$$\sigma_{11}^h = \sum_{j=1}^{N} \sigma_{11j}\delta_j\,, \quad \sigma_{12}^h = \sum_{j=1}^{N} \sigma_{12j}\delta_j \quad \text{and} \quad \sigma_{22}^h = \sum_{j=1}^{N} \sigma_{22j}\delta_j\,.$$

The matrix formulation for the dynamic problem is obtained by substituting the components of u^h and σ^h above into the system of linear equations (6.4.6).

The following matrices are defined.

$$M = (M_{ij})_{N\times N} \quad \text{where} \quad M_{ij} = (\delta_j,\ \delta_i)$$

and

$$L^q = (L^q_{ij})_{N\times N} \quad \text{where} \quad L^q_{ij} = (\delta_j,\ \partial_q \delta_i) \quad \text{for} \quad q = 1, 2.$$

We also use sub-matrices. Consider an $N \times N$ matrix A. The following notation is introduced:

> A_0 is obtained from A by deleting rows $k+1, \ldots, N$ and columns $k+1, \ldots, N$.
>
> A_R is obtained from A by deleting rows $k+1, \ldots, N$.

We define a load vector c with components

$$c_i = \int_\Gamma p\delta_i \, ds = \int_0^h p(x_2)\delta_i(1, x_2)\, dx_2 \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Then the system of linear equations (6.4.6) can be written as the following system.

$$\begin{aligned}
M_0 u_1'' &= -L_R^1 \sigma_{11} - L_R^2 \sigma_{12} + M_R q_1 \\
M_0 u_2'' &= -L_R^1 \sigma_{12} - L_R^2 \sigma_{22} + M_R q_2 + c \\
M\sigma_{11} &= \frac{1}{\gamma(1-\nu^2)}\left[(L_R^1)^T u_1 + \nu (L_R^2)^T u_2\right] \\
M\sigma_{12} &= \frac{1}{2\gamma(1+\nu)}\left[(L_R^2)^T u_1 + (L_R^1)^T u_2\right] \\
M\sigma_{22} &= \frac{1}{\gamma(1-\nu^2)}\left[\nu (L_R^1)^T u_1 + (L_R^2)^T u_2\right]
\end{aligned} \tag{6.4.7}$$

Equilibrium problem

Recall that for the equilibrium problem $\partial_t^2 u = 0$. The system of equations (6.4.7) then becomes

$$\begin{aligned}
-L_R^1\sigma_{11} - L_R^2\sigma_{12} + M_R q_1 &= 0 \\
-L_R^1\sigma_{12} - L_R^2\sigma_{22} + M_R q_2 + c &= 0 \\
M\sigma_{11} &= \frac{1}{\gamma(1-\nu^2)}\left[(L_R^1)^T u_1 + \nu(L_R^2)^T u_2\right] \\
M\sigma_{12} &= \frac{1}{2\gamma(1+\nu)}\left[(L_R^2)^T u_1 + (L_R^1)^T u_2\right] \\
M\sigma_{22} &= \frac{1}{\gamma(1-\nu^2)}\left[\nu(L_R^1)^T u_1 + (L_R^2)^T u_2\right]
\end{aligned} \tag{6.4.8}$$

Eigenvalue problem

The system of equations (6.4.7) for the eigenvalue problem is

$$\begin{aligned}
L_R^1\sigma_{11} + L_R^2\sigma_{12} &= \lambda M_0 u_1 \\
L_R^1\sigma_{12} + L_R^2\sigma_{22} &= \lambda M_0 u_2 \\
\frac{1}{\gamma(1-\nu^2)}\left[(L_R^1)^T u_1 + \nu(L_R^2)^T u_2\right] &= \mu M\sigma_{11} \\
\frac{1}{2\gamma(1+\nu)}\left[(L_R^2)^T u_1 + (L_R^1)^T u_2\right] &= \mu M\sigma_{12} \\
\frac{1}{\gamma(1-\nu^2)}\left[\nu(L_R^1)^T u_1 + (L_R^2)^T u_2\right] &= \mu M\sigma_{22}
\end{aligned} \tag{6.4.9}$$

6.5 Example

It should be clear that the system of equations is rather complex and not easy to program. For the sake of clarity we consider a special case where only four basis functions are used. That is, the reference configuration is divided into only one rectangular element. The nodes are numbered as explained in Subsection 6.4.2. Thus $k = 2$ and $N = 4$.

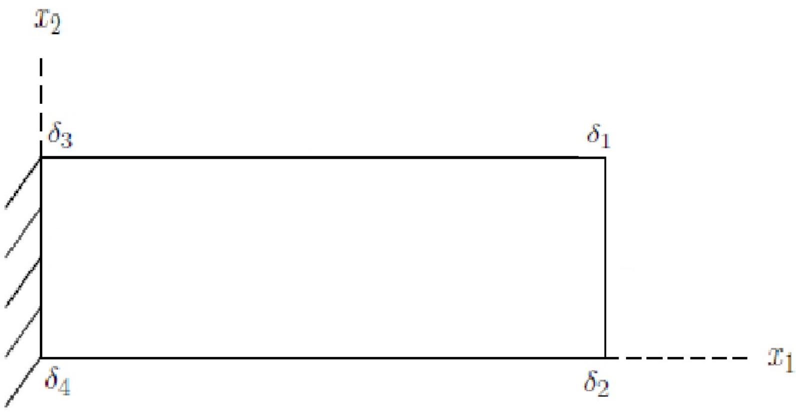

Fig. 2

The set with elements

$$[\delta_1\ 0]^T, \quad [\delta_2\ 0]^T,$$
$$[0\ \delta_1]^T, \quad [0\ \delta_2]^T$$

is a basis for

$$S_0^h = \left\{ \left[\sum_{j=1}^{2} u_{1j}\delta_j \quad \sum_{j=1}^{2} u_{2j}\delta_j \right]^T \;\middle|\; u_{1j} \;\text{ and }\; u_{2j} \;\in \mathbb{R} \right\}$$

while the set with elements

$$[\,\delta_1\ 0\ 0\,]^T, \; [\,\delta_2\ 0\ 0\,]^T, \; [\,\delta_3\ 0\ 0\,]^T, \; [\,\delta_4\ 0\ 0\,]^T,$$
$$[\,0\ \delta_1\ 0\,]^T, \; [\,0\ \delta_2\ 0\,]^T, \; [\,0\ \delta_3\ 0\,]^T, \; [\,0\ \delta_4\ 0\,]^T,$$
$$[\,0\ 0\ \delta_1\,]^T, \; [\,0\ 0\ \delta_2\,]^T, \; [\,0\ 0\ \delta_3\,]^T, \; [\,0\ 0\ \delta_4\,]^T$$

is a basis for

$$S^h = \left\{ \left[\sum_{j=1}^{4} \sigma_{11j}\delta_j \quad \sum_{j=1}^{4} \sigma_{12j}\delta_j \quad \sum_{j=1}^{4} \sigma_{22j}\delta_j \right]^T \;\middle|\; \sigma_{11j}, \;\; \sigma_{12j} \;\text{ and }\; \sigma_{22j} \;\in \mathbb{R} \right\}.$$

The approximate solution is given by

$$u_1^h(x) = u_{11}\delta_1(x) + u_{12}\delta_2(x)$$
$$u_2^h(x) = u_{21}\delta_1(x) + u_{22}\delta_2(x)$$

$$\sigma_{11}^h = \sigma_{11,1}\delta_1 + \sigma_{11,2}\delta_2 + \sigma_{11,3}\delta_3 + \sigma_{11,4}\delta_4$$

$$\sigma_{12}^h = \sigma_{12,1}\delta_1 + \sigma_{12,2}\delta_2 + \sigma_{12,3}\delta_3 + \sigma_{12,4}\delta_4$$

$$\sigma_{22}^h = \sigma_{22,1}\delta_1 + \sigma_{22,2}\delta_2 + \sigma_{22,3}\delta_3 + \sigma_{22,4}\delta_4$$

Matrices

M is a 4×4 matrix.

M_0 is a 2×2 matrix obtained from M by deleting rows 3 and 4 and columns 3 and 4.

L^q is a 4×4 matrix (for $q = 1, 2$).

L_R^q is a 2×4 matrix obtained from L^q by deleting rows 3 and 4.

Equilibrium problem

The MFEM yielded acceptable results for the equilibrium problem. These results are not of any importance in themself. Comparison to the SFEM and computation of eigenvalues and eigenfunctions are left for future work.

Chapter 7

Tracking a sharp crested wave front in hyperbolic heat transfer

7.1 Introduction

Recall the model problem for hyperbolic heat transfer, Problem CT. As mentioned in Chapter 1, researchers have encountered oscillatory behaviour in the solution when this problem is approximated using numerical techniques. In [SV12] it is shown that it is not the numerical techniques that cause these oscillations, but rather the fact that the problem is not well posed, as proved in Section 3.1, unless a jump condition is specified.

Recall that the equivalent problem Problem CT2 is studied in [SV12]. In this problem it is necessary that the initial condition, although smooth, must be arbitrary close to a discontinuous function. This is necessary to accommodate the heat pulse in the original problem. The end result is that a sharp crested wave front is present in the solution and it is almost impossible to track this front without generating oscillations; at least with standard numerical methods. In [SV12] the problem is broken up into three auxiliary problems to be solved by different methods.

The article [SV12] was written for an audience more interested in answers than mathematical analysis. In this chapter we analyse the method in [SV12]. Our aim is to investigate the applicability (or not) of the existence and convergence theory.

7.2 Initial conditions for Problem CT2

Problem CT2 is formulated in Chapter 1, but is repeated here for completeness.

Problem CT2
Find θ defined on $[0,1] \times [0,\infty)$ such that

$$\begin{aligned}
&\partial_t^2\theta + 2\partial_t\theta - \partial_x^2\theta = 0, \\
&\theta(0,t) = \partial_x\theta(1,t) = 0, \\
&\theta(x,0) = \theta_{in}(x), \quad \partial_t\theta(x,0) = 0.
\end{aligned}$$

As mentioned in Chapter 1, if $\theta_{in}(x) = -1$, then $T = \theta + 1$ is a solution of Problem CT if and only if θ is a solution of Problem CT2. But we proved in Section 3.1 that there does even not exist a mild solution for Problem CT2 if $\theta_{in}(x) = -1$.

Problem CT2 has a unique classical solution if θ_{in} satisfies the following conditions [Wei65]:

$$1.\ \ \theta_{in} \in C^2[0,1] \text{ and} \tag{7.2.1}$$

$$2.\ \ \theta_{in}(0) = \theta'_{in}(1) = \theta''_{in}(0) = 0. \tag{7.2.2}$$

In order to overcome the problem of the inadmissable initial value, the authors of [SV12] formulate an initial smooth temperature distribution that approximates $\theta_{in}(x) = -1$. Let $0 < \delta < 1$ and define θ_δ such that $\theta_\delta \in C^2[0,1]$,

$$\theta_\delta(0) = \theta'_\delta(1) = \theta''_\delta(0) = 0 \tag{7.2.3}$$

and

$$\begin{aligned}
-1 \leq \theta_\delta(x) \leq 0 \quad &\text{for} \quad 0 \leq x < \delta, \\
\theta_\delta(x) = -1 \quad &\text{for} \quad \delta \leq x \leq 1.
\end{aligned}$$

Thus θ_δ satisfies conditions (7.2.1) and (7.2.2) for the existence of a classical solution. Also, for δ arbitrary small, $\theta_\delta(x) = -1$ for $x \in [\delta, 1]$. An example for such a θ_δ is presented in [SV12]. Now let $\theta_{in}(x) = \theta_\delta(x)$. Then, provided that δ is small, a sufficiently smooth initial value which is a close approximation for the inadmissable initial value is obtained. Bear in mind that heating is due to a heat pulse with a lazer (see [SV12]). It is therefore necessary that δ be as small as possible.

7.3 Auxiliary problems and solution for Problem CT2

Tracking the sharp crested wave front remains a challenge. The approach in [SV12] is to formulate a set of three auxiliary problems in order to construct a solution for Problem CT2. The problems are labelled A1 to A3.

Problem A1

$$\begin{aligned}
&\partial_t^2 u = \partial_x^2 u, \\
&u(0,t) = \partial_x u(1,t) = 0, \\
&u(x,0) = \theta_{in}(x), \\
&\partial_t u(x,0) = \theta_{in}(x).
\end{aligned} \tag{7.3.1}$$

For any sufficiently smooth θ_{in} that satisfies the second condition for the existence of a classical solution given in (7.2.2), the well-known method of D'Alembert can be used to construct an exact classical solution for Problem A1.

Now consider the second auxiliary problem.

Problem A2

$$\begin{aligned}
&\partial_t^2 w + 2\partial_t w - \partial_x^2 w + w = 0, \\
&w(0,t) = \partial_x w(1,t) = 0, \\
&w(x,0) = \theta_{in}(x), \quad \partial_t w(x,0) = 0.
\end{aligned} \tag{7.3.2}$$

The solution for Problem A1 is used to obtain an exact solution for Problem A2. Let $w(x,t) = e^{-t}u(x,t)$ where u is the solution of Problem A1 as obtained above. Then let $u(x,t) = e^t w(x,t)$. Consequently

$$\begin{aligned}
&\partial_t u = e^t(w + \partial_t w), \\
&\partial_t^2 u = e^t(w + 2\partial_t w + \partial_t^2 w) \quad \text{and} \\
&\partial_x^2 u = e^t \partial_x^2 w.
\end{aligned}$$

It is clear that w is a solution of (7.3.2) if and only if u is a solution of (7.3.1).

Also, $w(0,t) = e^{-t}u(0,t) = 0$ and $\partial_x w(1,t) = e^{-t}\partial_x u(1,t) = 0$. Furthermore,

$$\begin{aligned}
&w(x,0) = u(x,0) = \theta_{in}(x) \quad \text{and} \\
&\partial_t w(x,0) = \partial_t u(x,0) - u(x,0) = 0.
\end{aligned}$$

Thus w is a solution of Problem A2 if and only if u is a solution of Problem A1.

Consider the last auxiliary problem.

Problem A3

$$
\begin{aligned}
&\partial_t^2 v + 2\partial_t v - \partial_x^2 v = f,\\
&v(0,t) = \partial_x v(1,t) = 0,\\
&v(x,0) = \partial_t v(x,0) = 0.
\end{aligned}
$$

Now, if v is the solution of Problem A3 where $f = w$, the solution of Problem A2, then $\theta = v + w$ is the solution of Problem CT2.

In the next two sections a solution of Problem A1 is derived and a numerical approximation for the solution of Problem A3 is computed.

7.4 Solution of Problem A2

In this section we have nothing to add to the method in [SV12], but present it here for completeness. The authors explained in detail how to solve Problem A1. After that one simply multiply the solution by e^{-t}.

The problem is solved for two cases after which the two solutions are just added to get a solution for Problem A1.

In the first case a solution u_1 for Problem A1 is obtained with $\partial_t u_1(x,0) = 0$. An extension g_1 of θ_{in} is constructed such that $g_1 \in C^2(-\infty,\infty)$, g_1 is periodic with period 4 and

$$
g_1(x) = \begin{cases} 1 \;\text{ for } \; -2 \le x < 0, \\ 0 \;\text{ for } x = 0, \\ -1 \;\text{ for } 0 < x < 2. \end{cases}
$$

Note that

$$
g_1(x) = -g_1(-x) \;\text{ and }\; g_1(x) = g_1(2-x).
$$

Then it follows from D'Alembert's method that

$$
u_1(x,t) = \frac{1}{2} g_1(x+t) + \frac{1}{2} g_1(x-t)
$$

is a solution of the wave equation. Furthermore, u_1 satisfies the boundary and initial conditions, except that $\partial_t u(x,0) = 0$.

In the second case a solution u_2 for Problem A1 is obtained with $u_2(x,0) = 0$. A function g_2 is constructed such that $g_2' = g_1$ with $g_2(0) = 0$. It follows from the definition of g_1 that

$$g_2(x) = g_2(-x) \quad \text{and} \quad g_2'(x) = g_2'(2-x).$$

Note that g_2 is periodic with period 4 and

$$g_2(x) = \begin{cases} x & \text{for } -2 \leq x < 0, \\ -x & \text{for } 0 \leq x < 2. \end{cases}$$

Again by D'Alembert's method it follows that the function

$$u_2(x,t) = \frac{1}{2} g_2(x+t) - \frac{1}{2} g_2(x-t)$$

is a solution of the wave equation. It can be verified easily that u_2 satisfies the boundary and initial conditions except that $u(x,0) = 0$.

Now add u_1 and u_2 to obtain the exact solution for Problem A1:

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) \\ &= \frac{1}{2} g_1(x+t) + \frac{1}{2} g_1(x-t) + \frac{1}{2} g_2(x+t) - \frac{1}{2} g_2(x-t). \end{aligned} \tag{7.4.1}$$

7.5 Tracking the wave front

The following two cases are considered:

$$\theta_{in}(x) = \theta_\delta(x) \quad \text{and} \quad \theta_{in}(x) = \theta_0(x) = -1.$$

If $\theta_{in}(x) = \theta_\delta(x)$ for any $\delta > 0$, then Problem A1 has a classical solution, say u_δ. If $\delta = 0$, then Problem A1 has no solution but interestingly D'Alembert's method can still be applied, which yields u_0 as in (7.4.1). Clearly

$$u_\delta(x,t) \to u_0(x,t) \quad \text{as} \quad \delta \to 0. \tag{7.5.1}$$

Thus u_0 may be used as an approximation for the solution u_δ, since δ is assumed to be extremely small.

The function $w_\delta(x,t) = e^{-t}u_\delta(x,t)$ is the solution for Problem A2. The function $w_0(x,t) = e^{-t}u_0(x,t)$ is not a solution of Problem A2, but from (7.5.1)

$$w_\delta(x,t) = e^{-t}u_\delta(x,t) \to e^{-t}u_0(x,t) = w_0(x,t) \ \text{ as } \ \delta \to 0.$$

Thus, since δ is extremely small, w_0 may be used as an approximation for the solution w_δ of Problem A2.

Recall that a solution for Problem CT2 is obtained by adding the solutions of Problems A2 and A3. There are two possible solutions v_δ and v_0 for Problem A3 depending on the choice $f = w_\delta$ or $f = w_0$. In [SV12] $f = w_0$ was used and we show in Section 7.6 that the approximation is more than acceptable.

Remark Since the initial values for Problem A3 are zero, there is no prominent wave front.

Problem A3

The solution for Problem A3 can be approximated using any numerical technique. In [SV12] it is approximated using the finite element method (FEM). Problem A3 is a special case of Problem WD in Section 1.2 where $\gamma = 2$, $\alpha^2 = 1$, $\ell = 1$, $k_1 = 0$ and $k_2 = 1$. Also, $v(x,0) = \partial_t v(x,0) = 0$. Consequently Problem WD-ODE in Subsection 1.2.3 reduces to

$$M\bar{v}'' + 2M\bar{v}' + K\bar{v} = \bar{F} \tag{7.5.2}$$

where $F_i(t) = w_0(x_i,t)$ and $\bar{v}(0) = \bar{v}'(0) = \bar{0}$.

In [SV12] the authors use a special case of the Newmark scheme, since this scheme is theoretically unconditionally stable. The semi-discrete problem (7.5.2) then becomes

$$(\delta t)^{-2}M[\bar{v}_{k+1} - 2\bar{v}_k + \bar{v}_{k-1}] + (\delta t)^{-1}M[\bar{v}_{k+1} - \bar{v}_{k-1}] + \frac{1}{4}K[\bar{v}_{k+1} + 2\bar{v}_k + \bar{v}_{k-1}]$$
$$= \frac{1}{4}[\bar{F}(t_{k+1}) + 2\bar{F}(t_k) + \bar{F}(t_{k-1})]. \tag{7.5.3}$$

Recall that $w_0 = e^{-t}u_0$ is used as an approximation for $f = e^{-t}u_\delta$ in Problem A3. It follows that $F_i(t) = e^{-t}\displaystyle\int_0^1 u_0(\cdot,t)\phi_i$ for $i = 1, 2, \ldots, n$.

If $u_0(x,t)$ is approximated by its interpolant $\sum_{j=1}^{n} u_0(x_j,t)\phi_j(x)$, it implies that $\bar{F}(t) = e^{-t}M\bar{u}(t)$ where $u_j(t) = u_0(x_j,t)$. Substituting this into the semi-discrete problem (7.5.2), yields

$$M\bar{v}'' + 2M\bar{v}' + K\bar{v} = e^{-t}M\bar{u}(t) \text{ with } \bar{v}(0) = \bar{v}'(0) = \bar{0}.$$

The Newmark scheme (7.5.3) for the FEM approximation then becomes

$$(\delta t)^{-2}M[\bar{v}_{k+1} - 2\bar{v}_k + \bar{v}_{k-1}] + (\delta t)^{-1}M[\bar{v}_{k+1} - \bar{v}_{k-1}] + \frac{1}{4}K[\bar{v}_{k+1} + 2\bar{v}_k + \bar{v}_{k-1}]$$
$$= \frac{1}{4}e^{-t_k}M[e^{-\delta t}\bar{u}(t_{k+1}) + 2\bar{u}(t_k) + e^{\delta t}\bar{u}(t_{k-1})].$$

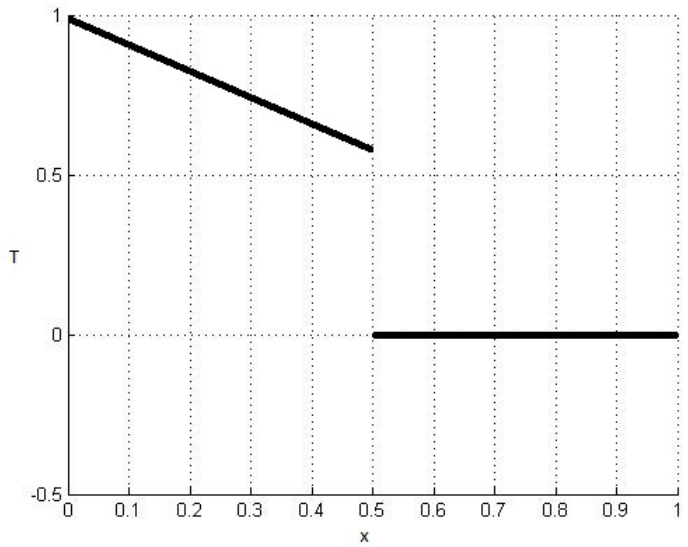

Fig. 1 Result for Problem CT: t = 0.5.

Numerical experiments

In [SV12] numerical experiments were performed in order to study the convergence of the solution for Problem A3. Problem CT was solved for $t = 0.5$, $t = 0.9$ and $t = 1.1$ respectively. In this dissertation we show only the temperature profile for $t = 0.5$ in Fig. 1.

It was clear from the graphs that no oscillations occur. To ensure that this is indeed the case, the authors of [SV12] examined the actual values of v as a function of x for the case $t = 0.5$. The results were given in a table which we do not include, but no oscillations could be observed.

The authors stated that experiments showed that the results are reliable for at least three significant digits. Thus the solution strategy used in [SV12] yields results that are free of oscillations.

7.6 Analysis

As mentioned, Problem A3 is a special case of Problem WD. Consider the weak variational form, Problem WD-W in Section 3.1:

Problem WD-W
Find a function w such that for $t > 0$, $w(t) \in V(0,\ell)$, $w'(t) \in V(0,\ell)$, $w''(t) \in \mathcal{L}^2(0,\ell)$ and

$$c(w''(t), v) + a(w'(t), v) + b(w(t), v) = (\tilde{f}(t), v)$$

for each $v \in V(0,\ell)$, while $w(0) = u_0$ and $w'(0) = u_1$.

In this application the initial conditions are zero and $\tilde{f}(t)(x) = w_0(x,t) = e^{-t}u_0(x,t)$.

We claim that a classical solution for the weak variational form is not guaranteed since $\tilde{f} \notin C^1([0,T), \mathcal{L}^2(0,1))$. To see this, consider the differentiability of $\tilde{u}$ where $\tilde{u}(t)(x) = u(x,t)$.

First,

$$\tilde{u}_1(t+h)(x) - \tilde{u}_1(t)(x) = \begin{cases} 0 & x \leq t \\ 1 & t < x < t+h \\ 0 & x \geq t+h \end{cases}$$

It follows that

$$\int_0^1 [h^{-1}(\tilde{u}_1(t+h) - \tilde{u}_1(t))]^2 = h^{-2} \int_t^{t+h} d\xi = h^{-1}.$$

Clearly the limit as $h \to 0$ does not exist. On the other hand, using a similar argument, we find that

$$\tilde{u}_2'(t) = 1 - t.$$

Now,

$$\tilde{f}(t+h) - \tilde{f}(t) = e^{-t}[\tilde{u}(t+h) - \tilde{u}(t)].$$

It follows that

$$\lim_{h\to 0}\int_0^1 [h^{-1}(\tilde{f}(t+h) - \tilde{f}(t))]^2 = \infty.$$

However, it can be proved that $\tilde{f} \in C([0,T), \mathcal{L}^2(0,1))$ and a mild solution for Problem WD-W exists. It is possible that this solution could be a classical solution.

An approximation v_0 for the solution of Problem A3 with $f = w_0$ is calculated instead of the solution v_δ with $f = w_\delta$. From Section 2.8 we have the following error (in energy):

$$\begin{aligned}
&\|v_\delta'(t) - v_0'(t)\|^2 + \|v_\delta(t) - v_0(t)\|_V^2 \\
&\quad \le \int_0^t \|w_0(\cdot) - w_\delta(\cdot)\|^2 \\
&\quad \le e^{-t}\int_0^t \|u_0 - u_\delta\|^2 \\
&\quad \le 2\delta t e^{-t}.
\end{aligned}$$

Now the solution of Problem CT2 with initial value θ_δ is $v_\delta(x,t) + e^{-t}u_\delta(x,t)$ and the "solution" of the problem with initial value θ_0 is $v_0(x,t) + e^{-t}u_0(x,t)$. The error is the sum of two errors. One is bounded in energy by $2\delta t e^{-t}$ and the other error is $e^{-t}|u_\delta(x,t) - u_0(x,t)|$. It converges pointwise to zero and

$$\|\tilde{w}_0(t) - \tilde{w}_\delta(t)\| \le \delta e^{-t}.$$

Lastly, we consider the convergence of the finite element approximation to the solution v_0 of Problem A3 where $f = w_0$. Since v_0 is not necessarily a classical solution of the weak variational problem, Theorem 4.3.1 on the convergence of the semi-discrete approximation cannot be applied. However, convergence cannot be ruled out. Theorem 4.3.2 on the convergence of the fully discrete approximation of the semi-discrete approximation, is also not applicable since it is required that $\tilde{f} \in C^2([0,T], \mathcal{L}^2(0,1))$.

We can conclude with the following remark. The theory for FEM convergence does not apply to Problem A3 but numerical experiments suggest that the approximation does converge to the solution.

Chapter 8

Conclusion

8.1 Overview

In the introduction it was mentioned that this dissertation is part of a research project on *Vibration Analysis*. A brief account of previous work was given and a motivation was provided for the selection of material in this dissertation. The objectives in the introduction became more or less the chapters of the dissertation.

In the first chapter we introduced various models for linear vibration problems. These models include the transverse vibration of a string with viscous damping and the multi-dimensional wave equation. The hyperbolic heat transfer model is also included since it resembles a mechanical vibration problem. The Timoshenko beam model with viscous damping and boundary damping were derived and boundary conditions discussed. Different types of damping are considered since the existence and convergence results differ for each type of damping. All the models for vibration problems are written in variational form.

Well posedness is discussed and the (informal) definition of what it means for a problem to be well posed, according to by L.C. Evans [Eva98], is stated. Examples of problems that are not well posed are given. In some cases researchers have encountered oscillatory behaviour in numerical approximations for the solution of this model problem. These oscillations are usually ascribed to limitation of the numerical method, whereas it may be a consequence of the problem not being well posed. The authors of [SV12] cite articles where the authors try to remove oscillations in the numerical approximation, not realising that the problem is not well posed.

In Chapter 2 we consider the theory of existence of a solution to a general linear vibration problem, called Problem G. This theory can be applied to the weak variational form of a model problem. If one is considering a problem in variational form, it is convenient to use results on existence formulated in terms of bilinear forms. For that reason we thoroughly investigated the article of Van Rensburg and Van der Merwe [VV02] published in 2002, where semigroup theory is used to obtain existence results. Semigroup theory has certain advantages that is not mentioned in the article. The advantage of semigroup theory is that necessary and sufficient conditions for existence are obtained. We investigate what happens when these conditions are not satisfied. The concept of a mild solution is discussed and the formulation of J.M. Ball [Bal77] given. We use this to derive a variational form satisfied by a weak solution of the general variational form.

The next chapter contains the application of the existence theory to the model problems introduced in Chapter 1. In order to do so, each model is written in weak variational form. The one-dimensional wave equation is considered first. The precise conditions for "solutions" of the one-dimensional wave equation to be valid are not discussed in most popular textbooks. We investigate these necessary and sufficient conditions for existence. Classical conditions for existence are compared to the conditions required by the "modern" existence theory in Chapter 2. The existence theory is also applied to the multi-dimensional wave equation with weak damping.

For the Timoshenko beam model, application of the existence theory is done on the dynamic problem with weak damping as well as boundary damping. Positive definiteness is proved in the energy space for the pinned-pinned beam. This is not done in any of the publications that we considered except in an appendix of [VZV09]. (It was necessary to elaborate on the proof.) It is also proved that boundary damping is neither weak nor strong. The steady state problem is also analysed to prove existence and regularity. Furthermore this regularity result implies higher regularity for the dynamic problem in the case of weak damping.

In Chapter 4 we consider error estimates for the semi-discrete and fully discrete Galerkin approximations of the general weak variational problem. The influence of damping on the properties of the solutions as well as the numerical approximations of these solutions are important considerations. We consider two cases, namely general damping and weak damping. An additional assumption on the regularity of solutions is required for the case with general damping. The theory in this chapter is from an article by Basson and Van Rensburg [BV13] published in 2013 and a follow up article by Basson, Stapelberg and Van Rensburg [BSV17] published in 2017 (available online in 2016).

Nothing is added to the theory of these articles since it is complete with user friendly notation. Our focus was on the application of the theory. We consider the multi-dimensional wave equation with weak damping and the Timoshenko beam model with boundary damping in order to exemplify the application of the theory. For the Timoshenko beam, piecewise linear basis functions as well as piecewise Hermite cubic basis functions are considered. The general existence results in Chapter 2 are used. This is more convenient for the finite element method (FEM) as one can compare conditions for existence to conditions for convergence.

The FEM is applied to the vibration of a Timoshenko beam in one of the chapters of this dissertation. Although piecewise linear basis functions are theoretically admissible test functions, it is well known that locking occurs and as a result approximations are worse than predicted by the theory. However, piecewise Hermite cubic basis functions prove to be very efficient, as stated in the article of Zietsman, Van Rensburg and Van der Merwe [ZVV04] published in 2004. Alternatively, the mixed finite element method (MFEM) can be used. From the literature it is expected that the MFEM should yield more accurate results than the standard finite element method (SFEM). The author D.N. Arnold of an article published in 1981 [Arn81] compares the MFEM to the SFEM, but only for the steady state problem. There is no comparison between the standard and mixed methods done in [Sem94], the article by B. Semper published in 1994. In an unpublished technical report, Basson and Van Rensburg compared the MFEM to the SFEM using C_0 piecewise linear basis functions for both. We were curious as to what would happen when the SFEM with Hermite cubic basis functions is compared to the MFEM with piecewise linear basis functions. The Timoshenko beam model is written in mixed variational form as well as weak mixed variational form. An error estimate from [Sem94] is presented. Numerical results for the dynamic problem are presented and the absolute difference is used to analyse the convergence of the two methods respectively.

Chapter 6 is devoted to a two-dimensional model for a beam. The article by Labuschagne, Van Rensburg and Van der Merwe [LVV09] published in 2009 is considered. The dynamic problem for the cantilever beam is formulated. Two special cases, namely the equilibrium problem and the eigenvalue problem are also discussed. The standard variational forms for the respective problems are derived to show that even this complex problem is a special case of the general second order hyperbolic problem. The focus is however on the MFEM, which was not considered in the 2009 article. In the chapter we explain why the MFEM for this problem should receive serious attention. The mixed formulation for the semi-discrete problem is derived and the necessary matrices introduced. What remains is to develop the necessary programs for numerical experiments.

Tracking a sharp crested wave front in hyperbolic heat transfer is an article by Sieberhagen and Van Rensburg [SV12] published in 2012, discussed in one of the chapters of this dissertation. The article was written for an audience not particularly interested in mathematical analysis; the contribution of this dissertation was to conduct a serious analysis of the problem and methods used.

8.2 Results

A significant part of this dissertation deals with the theory for existence of solutions and the application thereof to model problems. The existence results in the 2002 article of Van Rensburg and Van der Merwe are presented and proved in greater detail. Instead of the vague reference to "In applications E_b will correspond to solutions of physically realistic equilibrium problems", we provide a precise characterisation of this important subspace of the energy space. Additional results were formulated for this purpose. Semigroup theory is used to obtain existence results. This has certain advantages that were not mentioned in the 2002 article. In this dissertation these advantages are mentioned and discussed.

The theory of existence is applied to various models with different types of damping. Examples with weak damping as well as boundary damping are presented. In this chapter valuable contributions are made. First, the application to each model problem is rigorous and more complete than any publication before on this topic.

In a recent article on hyperbolic heat conduction a model problem that is not well posed, is given. This statement is motivated using D'Alembert's method. In Section 7.6 we do a thorough analysis and prove that the problem does not even admit a mild solution. We also prove rigorously that solution methods in the article are valid and provide estimates for errors (not finite element method errors).

Two recent articles on error estimates for the semi-discrete and fully discrete Galerkin approximations of the general weak variational problem are [BV13] by Basson and Van Rensburg published in 2013 and [BSV17] by Basson, Stapelberg and Van Rensburg published in 2017 (available online 2016). Different types of damping are considered since the properties of the solutions as well as the numerical approximations of these solutions depend on the damping. We consider weak damping as well as general damping. We had little to add to the theory. The contribution of this dissertation is rather in the application of the theory.

In Chapter 5 the finite element method (FEM) is applied to the vibration of the Timoshenko beam. The objective was to compare the standard finite element method (SFEM) using Hermite cubic basis functions to the mixed finite element method (MFEM) using piecewise linear basis functions. It is found that the SFEM with Hermite cubic basis functions has a higher rate of convergence than the MFEM with piecewise linear basis functions. It should however be noted that the method used in this dissertation to compare the models is rather naive since we do not have the means to calculate the computational effort of each method. However, since three functions needed to be evaluated using piecewise linear basis functions while four functions had to be approximated using Hermite cubic basis functions, we can make the educated assumption that the computational effort for the SFEM will be higher than that of the MFEM.

A modest contribution was made regarding the two-dimensional beam model. The derivation of the matrices for the MFEM should be useful for future work.

8.3 Future work

A study to compare the article [VV02] to the relevant chapter in [Sho77] will be valuable. It may even lead to original work. Next, the two-dimensional beam problem is worthy of further research. First, numerical experiments should be carried out regarding the computation of eigenvalues and eigenfunctions with the MFEM. Next, the (standard) FEM and MFEM should be compared.

Consider the "gap" between existence theory and FEM convergence theory: assumptions necessary for convergence are restrictive and are not available from existence and regularity theory. To formulate the research problem and make the notion of a gap precise is already a challenge.

Lastly, the most important future work in our view concern non-linear models. Consider two examples.

First, the undamped one dimensional wave equation:

$$\rho A \partial_t^2 u = \partial_x F + p, \tag{8.3.1}$$

$$F = EA\partial_x u \tag{8.3.2}$$

where u is the longitudinal displacement. It is well known that (8.3.2) is an approximation and in some cases not a very good approximation. Instead, one may consider

$$F = EA\partial_x u - g(\partial_x u).$$

The function g is increasing and not linear.

Next the damped string vibration introduced in Section 1.2 where viscous damping term $-\gamma\partial_t u$ appear. It is well known in mechanics that damping due to motion through a medium is only approximately proportional to the velocity. An improvement on Equation (1.2.3) could be

$$\partial_t^2 u = \alpha^2 \partial_x^2 u - \gamma(\partial_t u)^\nu + f$$

where $\nu \geq 0$ need not even be an integer.

Appendix A

Sobolev spaces

A.1 The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$ and denote its closure by $\bar{\Omega}$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω.

Theorem A.1.1.
The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with inner product

$$(f, g) = \int_\Omega fg = \int_\Omega fg \; d\mu$$

where μ is the n-dimensional Lebesgue measure.

Notation Since $(\cdot, \cdot)$ has the properties of an inner product, a norm for $\mathcal{L}^2(\Omega)$ is denoted by $\| \cdot \|$.

Notation

1. $C^m(\bar{\Omega})$: The class of functions with continuous derivatives up to order m in $\bar{\Omega}$.
2. $C^\infty(\bar{\Omega})$: Functions in $C^m(\bar{\Omega})$ for each m.
3. $C^\infty(\Omega)$: Functions in $C^m(\Omega)$ for each m.

Definition Support of a function
The support of a function f is the closure of the set $\{x \in \Omega \mid f(x) \neq 0\}$ and denoted by $\mathrm{supp}(f)$.

Definition $C_0^\infty(\Omega)$
A function f is in $C_0^\infty(\Omega)$ if it is in $C^\infty(\Omega)$ and $\text{supp}(f) \subset \Omega$.

Remark If a function $f \in C_0^\infty(\Omega)$, then the distance between $\text{supp}(f)$ and $\partial\Omega$ (the boundary of Ω) is positive. This fact is important for more than one proof.

Theorem A.1.2.
$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

A.2 Sobolev spaces

Suppose Ω is an open subset of $\mathbb{R}^n$.

Notation Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$, then

$$\begin{aligned} \partial^\alpha &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n} \quad \text{and} \\ |\alpha| &= \alpha_1 + \alpha_2 \cdots \alpha_n. \end{aligned}$$

Remark $|\alpha|$ denotes the order of the derivative.

Theorem A.2.1. *If $u \in \mathcal{L}^2(\Omega)$, then there exists at most one $w \in \mathcal{L}^2(\Omega)$ such that*

$$(u, \partial^\alpha\phi) = (w, \phi) \;\; \forall\, \phi \in C_0^\infty(\Omega).$$

Definition Weak Derivative
Suppose $u \in \mathcal{L}^2(\Omega)$ and there exists a $v \in \mathcal{L}^2(\Omega)$ such that

$$(u, \partial^\alpha\phi) = (-1)^{|\alpha|}(v, \phi) \;\; \forall\, \phi \in C_0^\infty(\Omega),$$

then v is called the weak derivative $D^\alpha u$ of u. The set of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m is denoted by $W^m(\Omega)$.

Theorem A.2.2.
The set $W^m(\Omega)$ is a vector space.

Definition Inner product for $W^m(\Omega)$
For functions u and v in $W^m(\Omega)$,

$$(u, v)_m = \sum_{|\alpha| \le m} (\partial^\alpha u, \partial^\alpha v) \quad \text{for} \quad m = 0, 1, \ldots$$

Theorem A.2.3.
The function $(\cdot,\cdot)_m$ *is an inner product for* $W^m(\Omega)$.

Definition Norm for $W^m(\Omega)$
For any function $u \in W^m(\Omega)$

$$\|u\|_m = \sqrt{(u,u)_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Theorem A.2.4.
The function $\|\cdot\|_m$ *is a norm for* $W^m(\Omega)$.

The vector space $W^m(\Omega)$ with inner product $(\cdot,\cdot)_m$ is called a Sobolev space.

Definition The space $H^m(\Omega)$
$H^m(\Omega)$ is the closure of $C^m(\bar{\Omega})$ in $W^m(\Omega)$ - with respect to the norm of $W^m(\Omega)$. A closed subset of a complete space is complete, hence $H^m(\Omega)$ is a Hilbert space.

A.3 Fundamental properties of Sobolev spaces

Definition Star shaped
Suppose $\Omega \subset \mathbb{R}^n$. If there exists a $p \in \Omega$ such that for any $x \in \Omega$ the set $\{\tau p + (1-\tau)x \mid \tau \in [0,1]\} \subset \Omega$, then Ω is called star shaped.

Definition Cone property
The set Ω satisfy the cone condition if there exists a cone with fixed dimensions such that if its vertex is placed at any point in Ω it can be orientated in such a way that it is a subset of Ω.

Assumption Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

Remark It is not necessary for Ω to be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Theorem A.3.1.
The Sobolev space $H^m(\Omega)$ *is complete.*

Theorem A.3.2.
$C^m(\bar{\Omega})$ *is dense in* $H^m(\Omega)$ *with respect to the norm of* $H^m(\Omega)$.

Remark A function in $H^m(\Omega)$ can be approximated by a function in $C^m(\bar{\Omega})$: If $u \in H^m(\Omega)$, then for any $\epsilon > 0$ there exists a $\phi \in C^m(\bar{\Omega})$ such that $\|u - \phi\|_m < \epsilon$.

Theorem A.3.3.
If Ω is star shaped, then $H^m(\Omega) = W^m(\Omega)$.

Theorem A.3.4. *Sobolev's lemma*
Suppose Ω is star shaped. If $u \in H^m(\Omega)$ and $r < m - \frac{n}{2}$, then there exists a $\phi \in C^r(\bar{\Omega})$ such that $u = \phi$ a.e and there exists a constant K such that

$$\|\partial^\alpha \phi\|_{\sup} \leq K\|u\|_m \quad \textit{for} \quad |\alpha| \leq r.$$

Proof: See [OR76, Theorem 3.10, p.80].

Remark One-dimensional case
If $n = 1$ in Theorem A.3.4, then $u \in C^{p-1}(\bar{\Omega})$ and $\|u^{(k)}\|_{\sup} \leq \|u\|_p$ for $k \leq m - 1$.

A.4 Inequalities

The one-dimensional case

Proposition A.4.1.
Consider any $u \in C^1[0, \ell]$. For any two points x and y in $[0, \ell]$,

$$|u(x)| \leq \sqrt{\ell}\|u'\| + |u(y)|.$$

Proof For any f and $g \in \mathcal{L}^2(\Omega)$ and $x > y$, the Cauchy-Schwartz inequality holds

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right).$$

Let $g = 1$. It then follows that that

$$\left(\int_y^x f\right)^2 \leq \left(\int_y^x f^2\right)(x - y) \leq \ell\|f\|^2.$$

Consequently

$$\Big|\int_y^x f\Big| \leq \sqrt{\ell}\|f\| \qquad \text{(A.4.1)}$$

for each $f \in \mathcal{L}^2(0,\ell)$. Since $u(x) - u(y) = \displaystyle\int_y^x u'$, it follows from the reverse triangle inequality and equation (A.4.1) that

$$\begin{aligned} |u(x)| &\leq \left| \int_y^x u' \right| + |u(y)| \\ &\leq \sqrt{\ell}\|u'\| + |u(y)|, \end{aligned}$$

□

Proposition A.4.2.
For any $u \in C^1[0,\ell]$ with a zero in $[0,\ell]$ we have

$$\|u\|_{\sup} \leq \sqrt{\ell}\|u'\|.$$

Proof Suppose $u(y) = 0$, then it follows from Proposition A.4.1 that $|u(x)| \leq \sqrt{\ell}\|u'\|$. Since $\sqrt{\ell}\|u'\|$ is an upper bound for $|u|$, the result follows. □

Proposition A.4.3.
For any $u \in C^1[0,\ell]$ with a zero in $[0,\ell]$ we have

$$\|u\| \leq \ell\|u'\|.$$

Proof We use Proposition A.4.2,

$$\|u\|^2 = \int_0^\ell \big(u(x)\big)^2 dx \leq \ell\|u\|_{\sup}^2 \leq \ell^2\|u'\|^2.$$

□

Proposition A.4.4.
Let $T[0,\ell] = \{u \in C^1[0,\ell] \bigm| u(0) = 0\}$. Denote the closure of $T[0,\ell]$ in $H^1(0,\ell)$ by $V(0,\ell)$. For any $u \in V(0,\ell)$,

$$\|u\| \leq \ell\|u'\|.$$

Proof Since $C^1[0,\ell]$ is dense in $V(0,\ell)$, it follows that there exists a sequence $\{u_n\} \in C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Then, since $\big|\|u_n\|_1 - \|u\|_1\big| \leq \|u_n - u\|_1$, it follows that $\|u_n\|_1 \to \|u\|_1$ as $n \to \infty$. Consequently $\|u_n\| \to \|u\|$ and $\|u_n'\| \to \|u'\|$ as $n \to \infty$.

The result follows from Proposition A.4.3 by taking the limits. □

Proposition A.4.5.
For any $u \in C^1[0,\ell]$ *there exists a constant* $K_\ell > 0$ *such that*

$$|u(\ell)| \leq K_\ell \|u\|_1.$$

Proof If u has a zero in $[0,\ell]$, then it follows from Proposition A.4.1 that $|u(\ell)| \leq \sqrt{\ell}\|u'\|$. If u does not have a zero, suppose $u > 0$ on $[0,\ell]$. Let m be the minimum of u on $[0,\ell]$. Let $w(x) = u(x) - m$. Then

$$|u(\ell)| \leq |w(\ell)| + m \leq \sqrt{\ell}\|w'\| + m \tag{A.4.2}$$

by Proposition A.4.2. But $w' = u'$ and $\|u\|^2 = \displaystyle\int_0^\ell u^2 \geq m^2\ell$.

Therefore, $m \leq \dfrac{1}{\sqrt{\ell}}\|u\|$ and it follows from Equation (A.4.2) that

$$|u(\ell)| \leq \sqrt{\ell}\|u'\| + \frac{1}{\sqrt{\ell}}\|u\|$$

which implies the result. If $u < 0$, then $-u > 0$ and the result follows. $\square$

A.5 Trace

We consider only the one-dimensional case. The idea is that a linear functional defined on $C^1[0,\ell]$ can be extended to $H^1(0,\ell)$. If $f \in C^1[0,\ell]$ and f has a zero in $[0,\ell]$, then we have from Proposition A.4.2 that $|f(\ell)| \leq \sqrt{\ell}\|f'\|$. If f does not have a zero in $[0,\ell]$, then it follows from Proposition A.4.5 that $|f(\ell)| \leq K_\ell\|f\|_1$ where K_ℓ depends on ℓ. By taking limits, it follows that these inequalities hold for any $f \in H^1(0,\ell)$. Now define a linear functional $\gamma_\ell f$ on $C[0,\ell]$ by $\gamma_\ell f = f(\ell)$. The inequalities above imply that the functional is bounded on $C^1[0,\ell]$ with respect to the norm of $H^1(0,\ell)$, and hence it can be extended to $H^1(0,\ell)$. Similarly a linear functional $\gamma_0 f = f(0)$ defined on $C^1[0,\ell]$ can be extended to $H^1(0,\ell)$. We shall denote $\gamma_\ell f$ by $f(\ell)$ and $\gamma_0 f$ by $f(0)$.

Appendix B

Interpolation

The results in this appendix are simplified versions of the theory in [OR6, Chpater 6], [OC83, Chapter 4] and [SF73, Chapter 5].

B.1 The one dimensional case

Theorem B.1.1 below is formulated as a special case of a general result. This result may be found in [SF73], [OC83] and [OR76].

Notation
We will use $\widehat{C}$ to denote a generic constant. Also denote the interpolation operator on an element by Π_e and the interpolation operator on the entire domain by Π. (The definitions are given in [SF73], [OC83] and [OR76]).

Theorem B.1.1.
Suppose there exists an integer k such that for each element

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1$$

for the interpolation operator Π. *Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(I)$ we have*

$$|\Pi u - u|_{m,I} \leq \widehat{C} h^{k-m} |u|_{k,I} \quad \textit{for} \quad m = 0,\ 1,\ \ldots, k.$$

The interpolation operator is denoted by Π_L for piecewise linear basis functions and by Π_c for Hermite cubics.

Corollary B.1.1. Hermite cubic basis functions
There exists a constant $\widehat{C}_c$ *such that if* $u \in H^k(I)$ *for*

a) $2 \le k \le 4$, *then*

$$\|u - \Pi_c u\|_m \le \widehat{C}_c h^{k-m} |u|_k, \quad m = 0, 1, \ldots, k.$$

b) $k > 4$, *then*

$$\|u - \Pi_c u\|_m \le \widehat{C}_c h^{4-m} |u|_4, \quad m = 0, 1, \ldots, 4.$$

Proof It is clear that $s(\Pi_c) = 1$ and it can be shown that $r(\Pi_c) = 3$. Consequently Theorem B.1.1 is applicable with $k = 2,\ 3$ or 4. □

Corollary B.1.2. Piecewise linear basis functions
There exists a constant $\widehat{C}_L$ *such that if* $u \in H^k(I)$ *for* $k \ge 2$, *then*

$$\|\Pi_L u - u\|_1 \le \widehat{C}_L h |u|_2$$

Proof It is clear that $s(\Pi_L) = 1$ and it can be shown that $r(\Pi_L) = 1$. Consequently Theorem B.1.1 is applicable with $k = 2$. □

B.2 The two-dimensional case

Theorem B.2.1 below is formulated as a special case of a general result. As mentioned before, this result may be found in [SF73], [OC83] and [OR76]. In the theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e.

Theorem B.2.1. *Suppose there exists an integer* k *such that for each element*

$$s(\Pi_e) + 2 \le k \le r(\Pi_e) + 1$$

for the interpolation operator Π. *Then there exists a constant* $\widehat{C}$ *such that for any* $u \in H^k(\Omega)$ *we have*

$$|\Pi u - u|_{m,\Omega} \le \widehat{C} h^{k-m} |u|_{k,\Omega} \quad \textit{for} \quad m = 0,\ 1,\ \ldots, k.$$

The constant $\widehat{C}$ depends on the shape of the elements in the finite element mesh. We shall only consider triangle elements. The interpolation operator is denoted by Π_L.

Corollary B.2.1. Piecewise linear basis functions on triangle elements
If $k \geq 2$, then there exists a constant $\widehat{C}$ such that for any $u \in H^k(\Omega)$ we have

$$|\Pi_L u - u|_{m,\Omega} \leq \widehat{C} h^{2-m} |u|_{k,\Omega} \quad \text{for} \quad m = 0,\ 1,\ 2.$$

Bibliography

[AF80] R.J. Atkin & N. Fox, *An introduction to the theory of elasticity*, Longman, London and New York, 1980.

[AKS96] K.T. Andrews, K.L. Kuttler & M. Shillor, Second order evolution equations with dynamic boundary conditions, *J. Math. Anal. Appl.*, 197:781-795, 1996.

[Arn81] D.N. Arnold, Discretization by finite elements of a model parameter dependent problem, *Numer. Math.*, 37:405-421, 1981.

[Bal77] J.M. Ball, Strongly continuous semigroups, weak solutions and the variation of Constants formula, *Proc. Amer. Math. Soc.*, 63(2):370-373, 1977.

[Bat67] G.K. Batchelor, *An introduction to fluid dynamics*, Cambridge University Press, 1967.

[BV13] M. Basson & N.F.J. van Rensburg, Galerkin finite element approximation of general linear second order hyperbolic equations, *Numerical Functional Analysis and Optimization*, 34(9):976-1000, 2013.

[Bra01] D. Braess, *Finite elements*, Cambridge University Press, ii edition, 2001.

[BSV17] M. Basson, B. Stapelberg & N.F.J Van Rensburg. Error estimates for semi-discrete and fully discrete Galerkin finite element approximations of the general linear second order hyperbolic equation, *Numer. Func. Anal. Opt.*, 38(4):466485, 2017.

[Cat48] M.C. Cattaneo, Sulla conduzione de calore (on the conduction of heat), *Atti del Semin. Mat. e Fis. Univ. Modena*, 3(3):83-101, 1948.

[CL04] H. Chen & K. Liu, Numerical analysis for the hyperbolic heat conduction problem under pulsed surface disturbance, *Appl. Math. Comput.* 159:887-901, 2004.

[Cow66] G.R. Cowper, The shear coefficient in Timoshenko's beam theory, Trans of ASME, *Journal of Applied Mathematics*, 33:335-340, 1966.

[CT82] G.F. Carey & M. Tsai, Hyperbolic heat transfer with reflection, *Num. Heat Transfer*, 5:309-327, 1982.

[DeV09] M. de Villiers, *Existence theory for linear vibration models of elastic bodies*, MSc dissertation, University of Pretoria, 2009.

[EP96] C.H. Edwards & D.E. Penney, *Elementary differential equations with boundary value problems*, Pearson Education, Inc., Upper Saddle River, New Jersey, 1996.

[Eva98] L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, RI, 1998.

[Fun65] Y.C.Fung, *Foundations of solid mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

[FXX99] Min-fu Feng, Xiao-ping Xie & Hau-xing Xiong, Semi-discrete and fully discrete partial projection finite element methods for the vibrating Timoshenko beam, *Journal of Computational Mathematics*, 17(4):353-368, 1999.

[Hoo16] A. Hoogkamer, *The mixed finite element method for second order partial differential equations of hyperbolic type*, MSc dissertation, University of Pretoria, 2016.

[HW06] C. Huang & H. Wu, An inverse hyperbolic heat conduction equation problem in estimating surface heat flux by the conjugate gradient method, *J. Phys. D: Appl. Phys.*, 39:4087-4096, 2006.

[Inm94] D.J. Inman, *Engineering vibration*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1994.

[Jan16] L. Janse van Rensburg, *Comparison of different models for the vibration of beams and plates*, MSc dissertation, University of Pretoria, 2016.

[JLZ02] F. Jiang, D. Liu & J. Zhou, Non-Fourier heat conduction phenomena in porous material heated by microsecond laser pulse, *Microscale Thermophys. Eng.* 6:331-346, 2002.

[Kre78] E. Kreyzig, *Introductory functional analysis with applications*, John Wiley & Sons Inc, 1978.

[KR87] J.U. Kim, Y. Renardy, Boundary control of the Timoshenko beam, *SIAM Journal Control and Optimization*, 25(6):1417-1429, November 1987.

[LCPX05] J. Li, P. Cheng, G.P. Peterson & J.Z. Xu, Rapid transient heat conduction in multilayer materials with pulsed heating boundary, *Num. Heat Transfer A*, 47:633-652, 2005.

[LVV09] A. Labuschagne, N.F.J. van Rensburg & A.J. van der Merwe, Comparison of linear beam theories, *Mathematical and Computer Modelling*, 49:20-30, 2009.

[OC83] J.T. Oden & G.F. Carey, *Finite elements: Mathematical aspects*, Prentice-Hall International, New Jersey, 1983.

[OR76] J.T. Oden & J.N. Reddy, *An introduction to the mathematical theory of finite elements*, John Wiley and Sons, 1976.

[Paz83] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New-York, 1983.

[PR05] Y. Pinchover & J. Rubinstein, *An introduction to partial differential equations*, Cambridge University Press, 2005.

[Sem94] B. Semper, Semi-discrete and fully discrete Galerkin methods for the vibrating Timoshenko beam, *Comput. Methods Appl. Mech. Engrg.*, 117:353-360, 1994.

[SF73] G. Strang & G.J. Fix, *An analysis of the finite element method*, Prentice-Hall, New Jersey, 1973.

[Sho77] R.E. Showalter, *Hilbert space methods for partial differential equations*, Pitman, London, 1977.

[SP06] N.G. Stephen & S. Puchegger, On the valid frequency range of Timoshenko beam theory, *Journal of Sound and Vibration*, 297:1082-1087, 2006.

[Sta14] B. Stapelberg, *Convergence analysis of finite element approximations of solutions of hyperbolic equations*, MSc dissertation, University of Pretoria, 2014.

[SV12] R.H. Sieberhagen & N.F.J. van Rensburg, Tracking a sharp crested wave front in hyperbolic heat transfer, *Appl. Math. Model.,* 36:3399-3410, 2012.

[Tim21] S.P. Timoshenko. Lxvi. On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine Series*, 6(41):744-746, 1921.

[Tzo95] D.Y. Tzou, A unified field approach for heat conduction from macro-to micro-scales, *ASME Trans. J. Heat Transfer*, 117(9):8-16, 1995.

[Ver58] P. Vernotte, Les paradoxes de la théorie continue de l'équation de la chaleur, *C.R. Acad. Sci. Paris Ser. I Math.*, 246:3154, 1958.

[VV02] N.F.J. van Rensburg & A.J. van der Merwe, Analysis of the solvability of linear vibration models, *Applicable Analysis*, 81(5):1143-1159, 2002.

[VV06] N.F.J. van Rensburg & A.J. van der Merwe, Natural frequencies and modes of a Timoshenko beam, *Wave Motion*, 44:58-69, 2006.

[VVR10] N.F.J. van Rensburg, A.J. van der Merwe & A. Roux, Waves in a vibrating solid with boundary damping, *Wave Motion*, 47:663:675, 2010.

[VZV09] N.F.J. van Rensburg, L. Zietsman & A.J. van der Merwe, Solvability of Reissner-Mindlin-Timoshenko plate-beam vibration model, *IMA Journal of Applied Mathematics*, 74:149-162, 2009.

[Wei65] H.F. Weinberger, *A first course in partial differential equations*, Lexington, Mass.: Xerox, 1965.

[Wu03] S.R. Wu, A priori error estimates for explicit finite element for linear elasto-dynamics by Galerkin method and central difference method, *Computer Methods in Applied Mechanics and Engineering*, 192:5329-5353, 2003.

[ZVV04] L. Zietsman, N.F.J. van Rensburg & A.J. van der Merwe, A Timoshenko beam with tip body and boundary damping, *Wave Motion*, 39(3):199-211, 2004.