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Riesz bases and the series representation of solutions of linear partial differential equations

by
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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Title Riesz bases and the series representation of solutions of linear partial differential equations
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Summary

The vibration of an elastic body (or system of elastic bodies) is modelled by a partial differential equation or system of partial differential equations. Modal analysis is widely used in engineering to study vibration problems. This approach is based on the idea that eigenvalues and eigenvectors can be used to construct a series solution for a model problem. The validity of this series representation is investigated in this dissertation.

In the literature, linear vibration problems are converted to abstract first order linear differential equations. The series representation is valid if the generalized eigenvectors of the dynamics generator form a basis for the state space. However, the dynamics generator is non-normal when damping is present. There is no general spectral theory for such operators so special cases are considered in the literature. Moreover, there has been virtually no progress for problems that are two or three dimensional.

Various one dimensional linear vibration problems are considered in this dissertation, namely the wave equation with viscous, Kelvin-Voigt (material) or boundary damping and the Euler-Bernoulli beam with Kelvin-Voigt damping or boundary control.

The vibration problems considered are carefully formulated in the general framework of bilinear forms. The transition from a system of partial differential equations to an abstract first order differential equation is made rigorous. This is usually glossed over in the literature.

The general form of the dynamics generator of a linear vibration problem is found and it is proved that this dynamics generator is non-normal. In each case, the equivalence between weak and classical forms of the eigenvalue problems is considered and sufficient conditions for existence are found. For the wave equation with constant viscous damping, a Riesz basis is explicitly constructed and used to prove that the the energy of the solution decays exponentially.

An introduction to the spectral theory of nonselfadjoint operators and a summary of the disparate methods and results used in the literature are given. Then the various approaches taken by different authors to prove the Riesz basis property are presented.

First, a method using quadratically close sequences is discussed. An abstract result on the Riesz basis property is proved and applied to a beam problem.

Next, methods involving operator pencils and Krein spaces are discussed and a theorem on the Riesz basis property (due to Jacob, Trunk and Winklmeier) is generalized and made more directly applicable.

The third method, which is based on the biorthogonality of the eigenvectors is then presented. This dissertation concludes with an overview of the literature studied and comments on the similarities and differences between the methods encountered.

Contents

Chapter 1

Modal Analysis

1.1 Introduction

1.1.1 The problem

Let us begin by considering the following quotes from [Ze, p.317 & 229]. The "general superposition principle dates back to a famous paper by Daniel Bernoulli in 1753." Bernoulli made the following observation, "A mechanical system of n degrees of freedom possesses exactly n eigensolutions. A membrane is, however, a system with an infinite number of degrees of freedom. This system will, therefore, have an infinite number of eigenoscillations."

As the title indicates, this dissertation deals with the series representation of solutions of linear partial differential equations. More specifically, those that arise in mathematical models for the vibration of elastic bodies.

Modal analysis of vibration problems is widely used in engineering and "has grown into a large industry" [In, p.366], so it is important to know the limits of its applicability. According to [VLV10], "It is important to note that modal analysis is based on the fact that eigenvalues and eigenfunctions can be used to construct a series solution for a model problem."

This poses the crucial question of whether the series solution of a model problem is valid. This is the main topic of this dissertation. The authors of [VLV10] remark that "Without damping, the associated eigenvalue problem is symmetric or selfadjoint." There is a well-developed mathematical theory for this case but in general, the eigenvalue problem is "nonselfadjoint". Shubov states in [Sh97] that "at the present time there is no general spectral theory of nonselfadjoint operators" and Dunford and Schwartz declare that "the problem of extending the spectral theory of selfadjoint operators to non-normal operators is one of the most important unsolved problems in the theory of linear operations [*sic*]" in [DS, p.1925].

The challenge is to determine when the series representation of a solution of a partial differential equation is valid, knowing that it is so for some cases.

After reading [VLV10] and browsing through the relevant literature, the decision was taken to focus on the articles [CZ94], [GY01] and [JTW08].

The authors of each of these papers took a different approach for the specific problem they dealt with. This soon lead into esoteric functional analysis which was a problem as the material in "standard" books on functional analysis turned out to be insufficient. All the authors cited [GK], so the relevant parts of that book were consulted. Even the first two volumes of [DS] did not contain the required material, but fortunately Volume III did. For some of the basic theory it was necessary to consult the relatively old papers [KL78(I)] and [KL78(II)] by Krein and Langer.

Various problems were encountered in the study of [CZ94], [GY01] and [JTW08] as the authors of these articles made many errors and omissions. It was necessary to study other papers (though not in great detail) to gain some perspective on the situation. The articles [CZ95], [Gu01], [Gu02], [JT07], [La81], [Rao97], [Sh02], [Tr10] and [ZZ03] were relevant. Almost all of the errors found have been corrected and omissions and unclear arguments have been clarified in this dissertation.

In Chapters 1 and 2, the model problems, approaches and results of [CZ94], [GY01] and [JTW08] are framed in the setting proposed by Van Rensburg and Van Der Merwe in [VV02]. The expanded version of this article that appeared in [De] was very helpful in this process.

The theory needed in [CZ94], [GY01] and [JTW08] is presented in Chapters 3 and 4. Each of the three different approaches is discussed in turn in Chapters 5, 6 and 7.

For ease of reference, all relevant definitions have been collected in Appendix A.

1.1.2 Spring-mass systems

Consider a system consisting of two identical masses and three identical springs with the outer springs fixed. (Examples can be found in almost every book on differential equations.)

Mathematical model

$$M\bar{x}'' + K\bar{x} = \bar{0}$$

where

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The vector function $\bar{x}(t) = e^{rt}\bar{v}$ is a solution of the system if and only if

$$r^2 m\bar{v} + K\bar{v} = \bar{0}.$$

Now $r^2 = -\frac{k}{m}\lambda$ where λ is an eigenvalue of $\frac{1}{k}K$. The vector $\bar{v}$ is the corresponding eigenvector. We find that $r = \pm\omega i$. Therefore there are two natural angular frequencies ω_1 and ω_2. The corresponding eigenvectors are referred to as the natural modes of vibration. The general solution is $\bar{x} = \bar{x}_1 + \bar{x}_2$ where $\bar{x}_i(t) = c_i \sin(\omega_i t) + d_i \cos(\omega_i t)$.

If damping is present the model changes:

$$M\bar{x}'' + C\bar{x}' + K\bar{x} = 0. \tag{1.1}$$

The quadratic eigenvalue problem is now

$$r^2 m\bar{v} + rC\bar{v} + K\bar{v} = 0.$$

The special cases $C = \alpha I$ and $C = \beta K$ are significant. In these cases, the eigenvalues can still be obtained from the eigenvalues of K and the eigenvectors are the same as for the undamped case. The damping is referred to as *modal*. In general, the damping is not modal and the quadratic eigenvalue problem can not be simplified. It is possible to write the system (1.1) as a first order system.

$$\begin{bmatrix} \bar{x}' \\ m\bar{y}' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

This is a system of the form $\bar{w}' = A\bar{w}$. In the presence of damping we find that the matrix A is not normal.

Unwanted vibrations frequently occur in structures consisting of a system of elastic bodies. It is often the case that natural damping is not sufficient to suppress these vibrations, which may lead to damage of the structure. Engineers create damping devices that artificially suppress vibrations when necessary. These devices are placed at the endpoints or joints (see e.g. [Ne]).

An interesting example of a practical problem may be found in [Ne, Chapter 5].

In Example 5.2, the author considers "the vibration of a tall, free-standing industrial chimney." Such a structure is "prone to oscillate in the wind". "When the vortex-shedding frequency is close to one of the chimney's natural frequencies ... vibrations will build up" which if "continued long enough, will cause the chimney to fail by metal fatigue".

In Example 5.3 the author considers the situation where "the chimney is mounted on a resilient pad to increase its damping". To simplify the problem, the chimney is modelled as a series of rigid beams joined by springs. The mathematical model is of the form (1.1). The effect of the damping is then determined by calculating the eigenvalues and eigenvectors for the quadratic eigenvalue problem.

Remark One may question the use of this model as it is physically discrete. However, if a continuum model is used, there is at present no certainty that modal analysis is justified. This will become clear in Chapters 3 to 7.

1.2 The wave equation

We use the wave equation to introduce the subject and provide a concrete case to ground the abstract theory upon. The wave equation is:

$$\partial_x^2 u(x,t) = a^2 \partial_t^2 u(x,t), \quad 0 < x < \ell,\ t > 0$$

subject to various boundary and initial conditions. The parameter $\alpha = a^2$ is considered to be a positive real constant in Chapters 1 and 2.

The wave equation is one of the simplest examples of a linear vibration problem for an elastic body. It is a model for the transverse vibration of a tightly stretched string or longitudinal vibration in a bar [In, Section 6.3] though most of the literature refers to the vibrating string.

Three types of damping are considered in this dissertation - viscous damping, Kelvin-Voigt (material) damping and boundary damping. These different types of damping provide examples of weak damping, strong damping and damping that can neither be classified as weak or strong respectively. The type of damping in a given problem has significant impact on the manner in which the existence of a solution is established and the validity of the series representation of this solution.

1.2.1 Model problems

We consider the wave equation with three types of damping - viscous damping, Kelvin-Voigt damping and boundary damping. In the first two cases the damping is represented by a term in the partial differential equation:

$$\begin{aligned}
&\textbf{Undamped} && \partial_t^2 = a^2\partial_x^2, && (1.2)\\
&\textbf{Viscous damping} && \partial_t^2 u = a^2\partial_x^2 u - c\partial_t u, && (1.3)\\
&\textbf{Kelvin}-\textbf{Voigt damping} && \partial_t^2 u = a^2\partial_x^2 u + q\partial_t\partial_x^2 u,, && (1.4)
\end{aligned}$$

where the parameters a^2, c and q are positive constants.

It is convenient to consider these equations in dimensionless form. Let $\xi = x/\ell$, $u^*(\xi,\tau) = u(x,t)/\ell$, $2c^* = cT$, $2q^* = q/(a\ell)$ and $\tau = t/T$ where $T = \ell/a$. Then the wave equations (1.2), (1.3) and (1.4) become (returning to the original notation)

$$\begin{aligned}
&\textbf{Undamped} && \partial_t^2 u(x,t) = \partial_x^2 u(x,t), && (1.5)\\
&\textbf{Viscous damping} && \partial_t^2 u = \partial_x^2 u - 2c\partial_t u, && (1.6)\\
&\textbf{Kelvin}-\textbf{Voigt damping} && \partial_t^2 u = \partial_x^2 u + 2q\partial_t\partial_x^2 u, && (1.7)
\end{aligned}$$

for $0 < x < 1$ and $t > 0$. The conditions on the endpoints of the bar or string also influence the vibration of the body. Consider the boundary conditions

$$\begin{aligned}
&\textbf{Fixed}-\textbf{Fixed} && u(0,t) = u(1,t) = 0, && (1.8)\\
&\textbf{Fixed}-\textbf{Free} && u(0,t) = \partial_x u(1,t) = 0, && (1.9)\\
&\textbf{Free}-\textbf{Free} && \partial_x u(0,t) = \partial_x u(1,t) = 0. && (1.10)
\end{aligned}$$

The equations (1.8), (1.9) and (1.10) represent the situation where the body is fixed at both ends, fixed at one end and free at the other and free at both ends respectively.

Remark From a purely mathematical point of view it is legitimate to consider the boundary conditions (1.10). However, free-free boundary conditions may not make sense as a mathematical model. In the derivation of the wave equation for the vibration of a string it is assumed that the "slope" $\partial_x u$ is small [We, Section 1]. If the string is free at both ends then it may rotate, in which case the model is not valid. In the case of longitudinal vibrations of a bar these boundary condition mean that the bar is free to execute a rigid body translation. See also Section 2.6.

For boundary damping we consider the undamped wave equation with damping at the boundary $x = \ell$:

$$\partial_x u(\ell, t) = -\gamma \partial_t u(\ell, t).$$

In dimensionless form, with $k = \gamma a$, we have

$$\partial_x u(1, t) = -k \partial_t u(1, t). \tag{1.11}$$

Remark (Limiting Cases)
If $k = 0$, the right end is free and if "$k = \infty$", (or rather $1/k = 0$) the right end is fixed. This is a useful observation that is put to use in Section 5.3.

Remark In applications, we need to consider well-posed problems for a unique solution to exist. Therefore, we consider initial conditions for the boundary value problem. It is necessary to prescribe the initial position and initial velocity of every point:

$$u(x, 0) = u_0(x) \tag{1.12}$$

$$\text{and } \partial_t u(x, 0) = v_0(x). \tag{1.13}$$

1.2.2 Separation of variables

We now consider the method of separation of variables to solve the wave equation. This approach is well-known, so we use it to compare other approaches.

Undamped Case

Consider a possible solution of the form $u(x, t) = X(x)T(t)$ for the undamped wave equation (1.5) with boundary conditions (1.8). The partial differential equation (1.5) has a solution of the form $u(x, t) = X(x)T(t)$ if and only if

$$\frac{X''}{X} \quad \text{and} \quad \frac{T''}{T}$$

are both constant. This yields the eigenvalue problem

$$X'' = \lambda X, \quad X(0) = X(1) = 0. \tag{1.14}$$

The eigenvalues are $\lambda_k = -k^2\pi^2$ with corresponding eigenfunctions $X_k(x) = \sin k\pi x$.

Since $T'' - \lambda T = 0$, we have modal solutions (that each satisfy the partial differential equation and boundary conditions) of the form

$$T_k(t)X_k(x) = [A_k \sin(k\pi t) + B_k \cos(k\pi t)] \sin(k\pi x)$$

and hence a formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} [A_k \sin(k\pi t) + B_k \cos(k\pi t)] \sin(k\pi x).$$

We may also consider the wave equation with fixed-free boundary conditions given by Equation (1.9). In this case, $X'(1) = 0$ and the eigenvalues are $\lambda_k = -(k-1/2)^2\pi^2$ with corresponding eigenfunctions $X_k = \sin(k-1/2)\pi x$.

Viscous Damping

Consider separation of variables for the wave equation with viscous damping (1.6) and fixed-fixed boundary conditions (1.8). The partial differential equation (1.6) has a solution of the form $u(x,t) = X(x)T(t)$ if and only if

$$T''X = TX'' - 2cT'X.$$

Therefore

$$\frac{X''}{X} \quad \text{and} \quad \frac{T''}{T} + 2c\frac{T'}{T}$$

must both be constant. This yields the eigenvalue problem (1.14) again, but this time $T(t)$ must satisfy

$$T'' + 2cT' = \lambda T.$$

Since $\lambda_k = -k^2\pi^2$,

$$T_k'' + 2cT_k' + k^2\pi^2 T_k = 0$$

which has the characteristic equation

$$r^2 + 2cr + k^2\pi^2 = 0. \tag{1.15}$$

Solving for r, we find $r = -c \pm \sqrt{c^2 - k^2\pi^2}$.

Let $\omega_k = \sqrt{k^2\pi^2 - c^2}$ and $\mu_k = \sqrt{c^2 - k^2\pi^2}$.

Now there are three possibilities:

Case I ($k^2\pi^2 > c^2$):

$$r = -c \pm \omega_k i,$$
$$T_k(t) = e^{-ct}[A_k \sin(\omega_k t) + B_k \cos(\omega_k t)]$$

Case II ($k^2\pi^2 = c^2$):

$$r = -c = -k\pi,$$
$$T_k(t) = e^{-ct}(A_k + B_k t)$$

Case III ($k^2\pi^2 < c^2$):

$$r = -c \pm \mu_k,$$
$$T_k(t) = e^{-ct}(A_k e^{-\mu t} + B_k e^{\mu t})$$

We have modal solutions of the form $T_k(t)\sin(k\pi x)$ and a formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} T_k(t)\sin(k\pi x). \tag{1.16}$$

Remarks

1. Some or all the cases above may arise in a given problem: Case I will always occur, Case II will occur only if $k^2\pi^2 = c^2$ for some k and Case I will occur only if $c^2 > \pi^2$.

2. All the modes may be *underdamped* if $c^2 < \pi^2$. Some modes will be *overdamped* if c is large enough that $c^2 > k^2\pi^2$ for some k. Since c is fixed and k is a natural number, there will be infinitely many underdamped modes.

3. Viscous damping is a special case of *weak damping*. A formal definition of weak damping is given in Section 1.7.

4. If we had used the fixed-free boundary conditions (1.9) then $\lambda_k = (k - 1/2)\pi$ and the results above hold with $k\pi$ replaced by $(k - 1/2)\pi$.

Kelvin-Voigt damping

Consider separation of variables for the wave equation with Kelvin-Voigt damping (1.7) and fixed-fixed boundary conditions (1.8). The partial differential equation (1.7) has a solution of the form $u(x,t) = X(x)T(t)$ if and only if

$$T''X = TX'' + 2qT'X''$$

Therefore

$$\frac{X''}{X} \quad \text{and} \quad \frac{T''}{T} - 2q\frac{T'}{T}\frac{X''}{X}$$

must both be constant. This yields the eigenvalue problem (1.14) again, but this time T must satisfy

$$T'' - 2q\lambda T' = \lambda T.$$

Since $\lambda_k = -k^2\pi^2$,

$$T'' + 2qk^2\pi^2 T' + q^2\pi^2 = 0$$

which has the characteristic equation

$$r^2 + 2qk^2\pi^2 r + k^2\pi^2 = 0. \tag{1.17}$$

Solving for r, we find $r = -qk^2\pi^2 \pm k\pi\sqrt{q^2k^2\pi^2 - 1}$.

Let $\omega_k = k\pi\sqrt{1 - q^2k^2\pi^2}$ and $\mu_k = k\pi\sqrt{q^2k^2\pi^2 - 1}$.

Now there are three possibilities:

Case I $\left(k^2\pi^2 < \dfrac{1}{q^2}\right)$:

$$\begin{aligned} r &= -qk^2\pi^2 \pm \omega_k i, \\ T_k(t) &= e^{-qk^2\pi^2 t}[A_k \sin(\omega_k t) + B_k \cos(\omega_k t)] \end{aligned}$$

Case II $\left(k^2\pi^2 = \dfrac{1}{q^2}\right)$:

$$\begin{aligned} r &= -qk^2\pi^2 = \frac{-1}{q}, \\ T_k(t) &= e^{-k\pi t}(A_k + B_k t) \end{aligned}$$

Case III $\left(k^2\pi^2 > \dfrac{1}{q^2}\right)$:

$$\begin{aligned} r &= -qk^2\pi^2 \pm \mu_k, \\ T_k(t) &= e^{-qk^2\pi^2 t}(A_k e^{-\mu t} + B_k e^{\mu t}) \end{aligned}$$

We have modal solutions of the form $T_k(t)\sin(k\pi x)$ and a formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} T_k(t)\sin(k\pi x). \tag{1.18}$$

Remarks

1. Some or all the cases above may arise in a given problem: Case III will always occur, Case II will occur only if $(qk\pi)^2$ for some k and Case I will occur only if $(qk\pi)^2 < 1$.

2. Some modes may be *underdamped* if $q^2 < \dfrac{1}{\pi^2}$. Some modes will be *overdamped* if q is large enough that $q^2 > \dfrac{1}{k^2\pi^2}$ for some k. Since q is fixed and k is a natural number, there will be infinitely many overdamped modes.

3. Kelvin-Voigt damping is a special case of *strong damping*. A formal definition of strong damping is given in Section 1.7.

4. If we had used the fixed-free boundary conditions (1.9) then $\lambda_k = (k - 1/2)\pi$ and the results above hold with $k\pi$ replaced by $(k - 1/2)\pi$.

1.2.3 Validity of the series representation

Suppose initial conditions $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = v_0(x)$ are given. For solutions of the form (1.16) or (1.18) to be valid, it is necessary that the initial conditions can be expressed as

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin k\pi x \quad \text{and} \quad v_0(x) = \sum_{k=1}^{\infty} b_k \sin k\pi x,$$

where a_k and b_k are constants determined by A_k and B_k.

It is well known (see e.g. [We, Section 18, 19] or [Ru]) that a large class of functions can be represented as Fourier series. If one is content with convergence in the mean (L^2-norm), then it is sufficient that the functions are square-integrable.

To write the initial conditions $u_0(x)$ and $v_0(x)$ in terms of the eigenfunctions $\sin k\pi x$ only, one extends the functions $u_0(x)$ and $v_0(x)$ from $[0,1]$ to $[-1,1]$ as odd functions. The procedure for handling this situation is well-known (see e.g. [We, Section 20]).

It is necessary, but not sufficient to be able to express the initial conditions as eigenfunction series. In addition, it must be shown that the series representation of the solution is valid for all $t > 0$.

It is proved in [We, Section 26] (using no more than the theory of Fourier series) that the formal series solution is indeed valid provided that $u_0(x)$ and $v_0(x)$ are sufficiently smooth. The smoothness conditions are used to show that the series representations of u, $\partial_t u$, $\partial_t^2 u$, $\partial_x u$ and $\partial_x^2 u$ converge uniformly. But this is to prove the existence of a classical solution, which is not our concern. Rather, we assume that the existence of a unique solution has been proved using some other method − such as semigroup theory (see [VV02]) or D'Alembert's method for the undamped case (see [We, Section 2]) − and consider the partial sums as an approximation of this solution. This is the approach taken throughout this dissertation.

1.2.4 Approximation in energy

Assume that a classical solution u of the initial value problem consisting of the wave equation with constant viscous damping (1.6), fixed-fixed boundary conditions (1.8) and the initial conditions (1.12) and (1.13) exists. That is, u is a classical solution of

$$\begin{aligned}
\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + 2c\partial_t u(x,t) &= 0, \;\; 0 < x < 1,\, t > 0, \\
u(0,t) = u(1,t) &= 0, \;\; t > 0, \\
u(x,0) &= u_0(x), \\
\partial_t u(x,0) &= v_0(x).
\end{aligned}$$

For the partial sum

$$u_N(x,t) = \sum_{k=1}^{N} T_k(t)\phi_k(x)$$

to be an approximation of the solution u, we must define convergence of this approximation in a meaningful way. We look to physics for a suitable way to measure the difference between the solution u and the approximation u_N.

Definition (Dimensionless energy)

$$E(t) = \frac{1}{2}\int_0^1 (\partial_t u(x,t))^2 \, dx + \frac{1}{2}\int_0^1 (\partial_x u(x,t))^2 \, dx. \tag{1.19}$$

The first term on the right hand side is *dimensionless kinetic energy* and the second term is *dimensionless potential energy*.

We prove that for the initial value problem above, the solution satisfies $E(t) \leq E(0)$ for all $t > 0$. This is to be expected from physics.

Since we assumed the existence of a classical solution,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (\partial_x u)^2 \, dx &= \int_0^1 \partial_x u \partial_t \partial_x u \, dx \\ &= -\int_0^1 \partial_x^2 u \partial_t u \, dx\end{aligned}$$

using integration by parts. Consequently

$$\begin{aligned}E'(t) &= \int_0^1 \partial_t u \partial_t^2 u \, dx - \int_0^1 \partial_x^2 u \partial_t u \, dx \\ &= \int_0^1 \partial_t u (\partial_t^2 u - \partial_x^2 u) \, dx \\ &= -2c \int_0^1 (\partial_t u)^2 \leq 0.\end{aligned}$$

Hence the real-valued function $E(t)$ is decreasing, thus $E(t) \leq E(0)$ for all $t > 0$. This shows that the energy of a solution decays with time, which is to be expected from physics.

Remark The decay of energy of the solution of any of the wave equations defined in Section 1.2.1 may be shown this way.

Suppose that the pair of initial conditions u_0 and v_0 can be approximated in energy by the partial sums. Then as $N \to \infty$:

$$\int_0^1 (u_0'(x) - u_N'(x))^2 \, dx \to 0 \tag{1.20}$$

$$\text{and} \quad \int_0^1 (v_0(x) - v_N(x))^2 \, dx \to 0. \tag{1.21}$$

Note that u_N is a solution of the wave equation with viscous damping (1.6) and fixed-fixed boundary conditions (1.8) and satisfies the following initial conditions

$$u_N(x, 0) =: f_N(x) \quad \text{and} \quad \partial_t u_N(x, 0) =: g_N(x).$$

The wave equations we deal with are linear so the function $u_N^{error} = u - u_N$ satisfies the wave equation with viscous damping (1.6), fixed-fixed boundary conditions (1.8) and initial conditions

$$u_N^{error}(0) = u_0(x) - f_N(x) \quad \text{and} \quad \partial_t u_N^{error}(x, 0) = v_0(x) - g_N(x).$$

Therefore, the energy inequality is valid for u_N^{error}:

$$E^{error}(t) \leq E^{error}(0) = \int_0^1 \left(f'(x) - f_N'(x)\right)^2 dx + \int_0^1 \left(v(x) - v_N(x)\right)^2 dx.$$

We conclude that $E^{error}(t) \to 0$ as $N \to \infty$ if $E^{error}(0) \to 0$ as $N \to \infty$.

Remarks

1. The same procedure can be used to show that $E^{error}(0) \to 0$ as $N \to \infty$ implies $E^{error}(t) \to 0$ as $N \to \infty$ for the wave equation with Kelvin-Voigt damping.
2. Separation of variables does not work for boundary damping.
3. The convergence of (1.21) follows from the theory of Fourier series. A discussion of the convergence of (1.20) is deferred to Appendix C.

1.3 Beam models

1.3.1 The Timoshenko beam

Consider the small transverse vibrations of a beam with density ρ, length ℓ, cross-sectional area A, Young's modulus (elastic modulus) E and area moment of inertia I.

Suppose $w(x,t)$ denotes the transverse displacement of x at time t and $\phi(x,t)$ the rotation of the cross-section at x at time t. Denote the bending moment and shear force for the cross-section x by $M(x,t)$ and $V(x,t)$ respectively.

The **equations of motion** are

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x V \quad \text{and} \\ \rho I \partial_t^2 \phi &= V + \partial_x M, \end{aligned}$$

There are two **constitutive equations** for the model,

$$\begin{aligned} M &= EI\partial_x \phi \\ \text{and} \quad V &= \kappa^2 AG(\partial_x w - \phi), \end{aligned}$$

where κ^2 denotes the shear coefficient and G is an elastic constant (shear modulus). For more detail, see e.g. [In, Section 6.5].

The beam model given by these two equations of motion and two constitutive equations was proposed by Timoshenko in 1921 and is known as the **Timoshenko model** (or theory).

Partial differential equations

If the constitutive equations are substituted into the equations of motion, the following system of partial differential equations is obtained

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x(\kappa^2 AG(\partial_x w - \phi)), \\ \rho I \partial_t^2 \phi &= V + \partial_x(EI\partial_x \phi) \end{aligned}$$

for $0 < x < \ell$.

Boundary conditions

To have a well posed problem, two boundary conditions are required at $x = 0$ and two at $x = \ell$.

Cantilever beam

The left endpoint of the beam is clamped. The right endpoint is free so there is no bending or shear force at $x = \ell$. The boundary conditions are

$$\begin{aligned} w(0,t) = \phi(0,t) &= 0, \\ \partial_x^2 w(\ell,t) &= 0 \quad \text{and} \\ \kappa^2 AG[\partial_x w(\ell,t) - \phi(\ell,t)] &= 0. \end{aligned}$$

Dimensionless form

The Timoshenko beam with cantilever boundary conditions in dimensionless form is

$$\partial_t^2 w = \partial_x\left(\partial_x w - \phi\right), \quad 0 < x < 1, \ t > 0 \tag{1.22}$$

$$\frac{1}{\alpha}\partial_t^2 \phi = (\partial_x w - \phi) + \partial_x\left(\frac{1}{\beta}\partial_x \phi\right), \quad 0 < x < 1, \ t > 0 \tag{1.23}$$

$$\begin{aligned} w(0,t) = \phi(0,t) &= 0, \\ \partial_x \phi(1,t) &= 0, \\ \text{and} \quad \partial_x w(1,t) - \phi(1,t) &= 0, \end{aligned}$$

where

$$\alpha = \frac{A\ell^2}{I}, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E} \tag{1.24}$$

are dimensionless constants.

Remark The validity of the series representation of solutions of the Timoshenko model is studied in [Sh00] and [Sh02].

1.3.2 The Rayleigh and Euler-Bernoulli models

A simplification of the Timoshenko model is obtained if it is assumed that cross-sections remain perpendicular to the neutral surface. The second equation of motion changes to

$$\rho I \partial_t^2 \partial_x w \quad = \quad V + \partial_x M.$$

Only one constitutive equation is now required:

$$M = EI\partial_x^2 w.$$

We obtain the partial differential equation

$$\rho A \partial_t^2 w - \rho I \partial_t^2 \partial_x^2 w = -\partial_x^2 M = -EI\partial_x^4 w \quad \text{for } 0 < x < \ell.$$

This model is sometimes referred to as the **Rayleigh model**. The dimensionless form of the Rayleigh model is

$$\partial_t^2 w - \frac{1}{\alpha}\partial_t^2\partial_x^2 w = -\partial_x^2\left(\frac{1}{\beta}\partial_x^2 w\right) \quad \text{for } 0 < x < 1.$$

In the **Euler-Bernoulli** theory the angular momentum density and shear are ignored. If the dimensionless rotary inertia term $\alpha^{-1}\partial_t^2\partial_x^2 w$ is dropped from the Rayleigh model we obtain the Euler-Bernoulli model (in dimensionless form):

$$\partial_t^2 w = -c^2\partial_x^4 w(x,t), \quad \text{for } 0 < x < 1, \quad \text{where } c^2 = \frac{1}{\beta}. \tag{1.25}$$

Damping due to internal friction in the beam may be taken into account by adding a dimensionless damping term to the constitutive equation as follows,

$$M = \frac{1}{\beta}\partial_x^2 w + k\partial_t\partial_x^2 w.$$

The Euler-Bernoulli model then becomes

$$\partial_t^2 w(x,t) = -c^2 \partial_x^4 w(x,t) + \partial_x^2 (k \partial_t \partial_x^2 w) \quad \text{for } 0 < x < 1. \tag{1.26}$$

This type of damping is referred to as Kelvin-Voigt damping or material damping.

Spectral analysis of the Euler-Bernoulli is discussed in Section 1.4.

Remark The linear theory is only realistic for a beam that executes small transverse vibrations. That is $\partial_x w$, ϕ and $\partial_x \phi$ must be small.

1.4 The Euler-Bernoulli beam

The Euler-Bernoulli model for small transverse vibrations of a beam was derived from the Timoshenko model in Subsection 1.3.2.

1.4.1 Boundary conditions

Two boundary conditions are required at $x = 0$ and two at $x = 1$. The following boundary conditions are used in this dissertation:

Cantilever beam

The left endpoint of the beam is clamped. The right endpoint is free so there is no bending or shear force at $x = 1$. The boundary conditions are

$$w(0,t) = \partial_x w(0,t) = \partial_x^2 w(1,t) = \partial_x^3 w(1,t) = 0. \tag{1.27}$$

The Euler-Bernoulli beam with damping due to internal friction (1.26) and cantilever boundary conditions is considered in [JTW08, Section 6].

Cantilever with linear boundary control

The left endpoint of the beam is clamped. The right endpoint is subject to *linear boundary control.* At $x = 1$, the moment is proportional to the angular velocity of the beam and the force on the end of the beam is proportional to the velocity of the beam. The boundary conditions representing this are

$$\begin{aligned} w(0,t) &= \partial_x w(0,t) = 0, \\ \partial_x^2 w(1,t) &= -k_1 \partial_t \partial_x w(1,t) \\ \text{and} \quad \partial_x^3 w(1,t) &= k_2 \partial_t w(1,t), \end{aligned}$$

respectively, where the damping parameters k_1 and k_2 are positive constants. The Euler-Bernoulli beam with boundary control is considered in [GY01].

Remark The aim of boundary control is to damp out vibrations and is sometimes called boundary damping.

1.4.2 Separation of variables

Consider the boundary value problem for the cantilever Euler-Bernoulli beam in dimensionless form.

$$\begin{aligned} \partial_t^2 w &= -c^2 \partial_x^4 w, \quad 0 < x < 1, \ t > 0, \\ w(0,t) &= \partial_x w(0,t) = \partial_x^2 w(1,t) = \partial_x^3 w(1,t) = 0. \end{aligned}$$

The partial differential equation has a solution $w(x,t) = X(x)T(t)$ if and only if

$$\frac{X^{(4)}}{X} \quad \text{and} \quad \frac{T''}{T}$$

are both constant. This yields the eigenvalue problem

$$X^{(4)} = \lambda X \quad \text{with} \quad X(0) = X'(0) = X''(1) = X'''(1) = 0.$$

If $X(x)$ is a solution of this eigenvalue problem and $T'' = \lambda T$, then the function $w(x,t) = X(x)T(t)$ satisfies the partial differential equation (1.25) and the boundary conditions (1.27).

In Subsection 1.2.2 we saw that the solutions of $T'' - \lambda T = 0$ are given by

$$T_k(t) = [A_k \sin(k\pi t) + B_k \cos(k\pi t)] \quad \text{for } k = 1, 2, \ldots.$$

To solve the eigenvalue problem, we need the general solution of the differential equation $y^{(4)} = ay$. If $a = \tau^4$, then the general solution is

$$y(x) = c_1 \cosh(\tau x) + c_2 \sinh(\tau x) + c_3 \cos(\tau x) + c_4 \sin(\tau x)$$

From the boundary conditions at $x = 0$ we obtain $c_1 = -c_3$ and $c_2 = -c_4$. Let $c_1 = c_k$ and $c_2 = d_k$, then for an eigenvalue τ_k we have the associated eigenfunction

$$y_k(x) = c_k \left[\cosh(\tau_k x) - \cos(\tau_k x)\right] + d_k \left[\sinh(\tau_k x) - \sin(\tau_k x)\right]. \tag{1.28}$$

Note that this means that the algebraic multiplicity of any particular eigenvalue is at most 2 and that the family of of solutions has two parameters (the dimension of the solution space is 2).

The other boundary conditions require that

$$\begin{aligned} c_1[\cosh\tau + \cos\tau] + c_2[\sinh\tau + \sin\tau] &= 0, \\ c_1[\sinh\tau - \sin\tau] + c_2[\cosh\tau + \cos\tau] &= 0. \end{aligned}$$

This system may be written as $Qc = 0$ where

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \cosh\tau + \cos\tau & \sinh\tau + \sin\tau \\ \sinh\tau - \sin\tau & \cosh\tau + \cos\tau \end{bmatrix}$$

The solution of the system above must be nonzero in order to obtain a nonzero solution of the boundary value problem. Therefore we set the determinant of the coefficient matrix Q equal to zero and solve for τ from the resulting equation

$$\cosh\tau\cos\tau = -1.$$

This equation has an infinite sequence of solutions $\{\tau_k\}$. Only the positive solutions are relevant and the sequence can be ordered to be increasing, so

$$\lim_{k\to\infty} \tau_k = \infty.$$

Note that this may also be deduced using the theory in Section 3.1. The eigenvalues are $\lambda_k = \tau_k^4,\ k = 1, 2, \dots$ with corresponding eigenfunctions

$$y_k(x) = \cosh(\tau_k x) - \cos(\tau_k x) + c_k[\sinh(\tau_k x) - \sin(\tau_k x)],$$

where $c_k = 1$ and $d_k = -[\sinh(\tau_k) + \sin(\tau_k)]^{-1}\,[\cosh(\tau_k) + \cos(\tau_k)]$. Note that d_k is well-defined because $\tau_k \neq 0$ and hence $\sinh(\tau_k) \neq 0$. Now we have modal solutions of the boundary value problem of the form:

$$w_k(x,t) = [A_k \sin(k\pi t) + B_k \cos(k\pi t)]\; y_k(x) \text{ for } k = 1, 2, \dots.$$

The series solution of the initial value problem is thus

$$w(x,t) = \sum_{k=1}^{\infty} w_k(x,t),$$

where A_k and B_k are determined by the initial conditions. Suppose the initial conditions

$$w(x,0) = u_0(x) \quad \text{and} \quad \partial_t w(x,0) = v_0(x)$$

are given. Then $u_0(x) = \sum_{k=1}^{\infty} B_k y_k(x)$ and $v_0(x) = \sum_{k=1}^{\infty} k\pi A_k y_k(x)$. Note that this is an eigenfunction series and *not* a Fourier series since the hyperbolic functions are involved. Therefore, the convergence of the series for arbitrary initial conditions is not guaranteed yet. This is will follow from the results in Section 3.1.

Remarks

1. If the beam is free at both ends then $\lambda = 0$ is an eigenvalue with corresponding eigenvector $y(x) = ax + b$. In this case the beam may execute rigid body motion such as rotation and translation.

2. The eigenvalue problem presented here arises in an application in Section 5.3 along with another eigenvalue problem. The solution (1.28) is valid for both problems, with the difference being the choice of coefficients c_k and d_k in (1.25).

1.5 Quadratic eigenvalue problems

The usual separation of variables procedure to derive an eigenvalue problem does not always work. For example, it cannot be used for boundary damping. In this section another approach which may be used in such cases is presented.

First, we consider a situation where separation of variables could be used so that we can compare the eigenvalues and eigenvectors that are obtained from the different approaches.

1.5.1 Viscous damping

It is convenient to use the same example as was used in Subsection 1.2.2 – the wave equation with viscous damping and fixed-fixed boundary conditions.

Consider a solution $\tilde{u}(x,t) = e^{\lambda t}u(x)$. Substitution into the wave equation with viscous damping (1.6) with boundary conditions (1.8) yields the *quadratic eigenvalue problem*:

$$\begin{aligned} u'' &= (2c\lambda + \lambda^2)u, \qquad (1.29) \\ u(0) &= u(1) = 0. \end{aligned}$$

Note that if $c = 0$ we have the undamped case. The choice of boundary conditions above is just an example, this method would have worked for other boundary conditions.

The differential equation (1.29) is of the form $u'' = \alpha u$ with $\alpha = 2c\lambda + \lambda^2$. With homogeneous boundary conditions $u(0) = u(1) = 0$ it has a nontrivial solution if and only if $\alpha = -k^2\pi^2$. The solutions are

$$\phi_k(x) = \sin(k\pi x).$$

Consequently,

$$\alpha = 2c\lambda + \lambda^2 = -k^2\pi^2.$$

The equation above for the eigenvalues is exactly the same as the characteristic equation (1.15) in Subsection 1.2.2 so

$$\lambda_k = -c \pm \sqrt{c^2 - k^2\pi^2}.$$

Let

$$\omega_k = \sqrt{k^2\pi^2 - c^2} \qquad \text{and} \qquad \mu_k = \sqrt{c^2 - k^2\pi^2}.$$

Again we distinguish three cases that depend on the magnitude of c:

Case I ($c^2 > k^2\pi^2$):

$$\lambda_k^+ = -c + \mu_k,$$
$$\lambda_k^- = -c - \mu_k.$$

Case II ($c^2 = k^2\pi^2$):

$$\lambda_k = -c = -k\pi,$$

In this case there is a repeated eigenvalue.

Case III ($c^2 < k^2\pi^2$):

$$\lambda_k^+ = -c + \omega_k i,$$
$$\lambda_k^- = -c - \omega_k i.$$

Remarks

1. Note that the eigenfunctions $\sin k\pi x$ are real-valued.

2. In Cases I and II the eigenvalues are real. In both these cases the function $T_k(t)$ in Subsection 1.2.2 equals $e^{\lambda_k^+ t}$ or $e^{\lambda_k^- t}$.

3. In Case III, the eigenvalues are non-real and $T_k(t)$ is equal to a linear combination of the real and imaginary parts of $e^{\lambda_k t}$.

4. The modal solutions are the same as in Subsection 1.2.2.

5. The results obtained from the quadratic eigenvalue problem are valid for the wave equation without damping if we let $c = 0$.

1.5.2 Kelvin-Voigt damping

Again we use the same example as was used in Subsection 1.2.2 – the wave equation with Kelvin-Voigt damping and fixed-fixed boundary conditions.

Consider a solution $\tilde{u}(x,t) = e^{\lambda t}u(x)$. Substitution of $e^{\lambda t}u(x)$ into the wave equation yields the *quadratic eigenvalue problem*:

$$\begin{aligned}(1 + 2q\lambda)u'' &= \lambda^2 u, \\ u(0) = u(1) &= 0.\end{aligned} \tag{1.30}$$

As in the previous subsection, we find that the eigenfunctions are

$$\phi_k(x) = \sin(k\pi x)$$

and the eigenvalues are

$$\lambda_k^2 = (-k^2\pi^2)(1 + 2q\lambda_k).$$

The equation above for the eigenvalues is exactly the same as the characteristic equation (1.17) in Subsection 1.2.2 so again there are three possibilities:

Case I $\left(k^2\pi^2 < \dfrac{1}{q^2}\right)$:

$$\begin{aligned}\lambda_k^+ &= -qk^2\pi^2 + \omega_k i, \\ \lambda_k^- &= -qk^2\pi^2 - \omega_k i.\end{aligned}$$

Case II $\left(k^2\pi^2 = \dfrac{1}{q^2}\right)$:

$$\lambda_k = -qk^2\pi^2 = \frac{-1}{q}.$$

Case III $\left(k^2\pi^2 > \dfrac{1}{q^2}\right)$:

$$\lambda_k^+ = -qk^2\pi^2 + \mu_k,$$
$$\lambda_k^- = -qk^2\pi^2 - \mu_k.$$

Recall that $\omega_k = k\pi\sqrt{1 - q^2k^2\pi^2}$ and $\mu_k = k\pi\sqrt{q^2k^2\pi^2 - 1}$.

Remarks

1. Note that the eigenfunctions $\sin k\pi x$ are real-valued.

2. In Cases II and III the eigenvalues are real. In both these cases the function $T_k(t)$ in Subsection 1.2.2 equals $e^{\lambda_k^+ t}$ or $e^{\lambda_k^- t}$.

3. In Case I, the eigenvalues are non-real and $T_k(t)$ is equal to a linear combination of the real and imaginary parts of $e^{\lambda_k t}$.

4. The modal solutions are the same as in Subsection 1.2.2.

5. The results obtained from the quadratic eigenvalue problem are valid for the wave equation without damping if we let $q = 0$.

1.5.3 Boundary damping

Separation of variables does not work for boundary damping. Consider a solution of the form $\tilde{u}(x,t) = e^{\lambda t}u(x)$ of the wave equation fixed at $x = 0$ with boundary damping at $x = 1$. The eigenvalue problem in this case is

$$\begin{cases} \qquad u'' = \lambda^2 u, \\ \text{with } \ u(0) = 0 \\ \text{and } \ u'(1) = -k\lambda u(1). \end{cases} \tag{1.31}$$

This problem is considered in [VLV10].

In (1.31), $u'' = \lambda^2 u$ is a second order ordinary differential equation. Hence the general solution of (1.31) is $c_1 e^{\lambda x} + c_2 e^{-\lambda x}$. The boundary condition $u(0) = 0$ implies that $c_2 = -c_1$ and we may let $c_1 = 1$. Therefore, $u(x) = e^{\lambda x} - e^{-\lambda x}$ satisfies the differential equation and the boundary condition at $x = 0$. Consider the other boundary condition:

$$\lambda(e^{\lambda} + e^{-\lambda}) = u'(1) = -k\lambda u(1) = -k\lambda(e^{\lambda} - e^{-\lambda}).$$

Therefore $e^{2\lambda} = \dfrac{k-1}{k+1}$. Let $\lambda = -\alpha + i\omega$. Then

$$e^{2\lambda} = e^{-2\alpha}[\cos(2\omega) + i\sin(2\omega)] = \frac{k-1}{k+1}. \tag{1.32}$$

Remarks

1. Note that *all* the possible solutions of (1.31) are of the form $c_1e^{\lambda x} - c_1e^{-\lambda x}$. If $k = 1$, then the boundary condition at $x = 1$ gives $e^{2\lambda} = 0$, which is impossible. This constitutes a rigorous proof that the eigenvalue problem has no solution for $k = 1$.

2. The existence of eigenvalues and eigenvectors should not be taken for granted. Nonexistence of eigenvalues and eigenvectors is discussed in [CZ] and [CZ95].

Now consider $k \neq 1$, From (1.32):

$$\begin{aligned} e^{-2\alpha}\cos(2\omega) &= \frac{k-1}{k+1}, \\ \sin(2\omega) &= 0. \end{aligned}$$

The second equation has nontrivial solutions $2\omega_n = n\pi$ for $n = 1, 2, \ldots$. There are now two distinct cases for solutions of the first equation :

Case I ($k < 1$):

If $k < 1$, then solutions exist only when $\cos(2\omega_n) < 0$. Consequently,

$$\begin{aligned} \omega_n &= \pi/2,\ 3\pi/2,\ 5\pi/2\ldots \\ \text{and } \alpha &= -\frac{1}{2}\ \ln\frac{1-k}{1+k}. \end{aligned}$$

Case II ($k > 1$):

If $k > 1$, then solutions exist only when $\cos(2\omega_n) > 0$. Consequently

$$\begin{aligned} \omega_n &= n\pi \quad \text{for } n = 1, 2, \ldots \\ \text{and } \alpha &= -\frac{1}{2}\ \ln\frac{k-1}{k+1}. \end{aligned}$$

The eigenvalues are $\lambda_n = -\alpha + \omega_n i$ with eigenfunctions

$$y_n(x) = (e^{-\alpha x} - e^{\alpha x})\cos(\omega_n x) + i(e^{-\alpha x} + e^{\alpha x})(\sin\omega_n x).$$

Now $e^{\lambda_n t} y_n$ is a solution of the wave equation that satisfies the boundary conditions. Thus the formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\lambda_n t} y_n.$$

In [VLV10], this series solution is written in terms of the real and imaginary parts of y_n.

Remarks

1. Boundary damping is also considered in [CZ95].

2. The quadratic eigenvalue problem for the wave equation is presented here to illustrate that when separation of variables is possible, the eigenvalues and eigenvectors obtained from the quadratic eigenvalues are exactly those obtained by separation of variables.

3. The eigenvalues and eigenvectors of the quadratic eigenvalue problem for a beam problem are the same as those obtained by separation of variables (when separation of variables is possible). See Subsection 1.4.2 for the eigenvalue problem for the Euler-Bernoulli beam.

1.6 Variational forms

In this section the independent variables x and t are often suppressed for notational convenience.

1.6.1 The damped wave equation

Notation $C^m[a,b]$ is the space of $m-$times continuously differentiable functions on the interval $[a,b]$ for natural number m.

Viscous damping

Consider the wave equation with viscous damping and fixed-fixed boundary conditions. The damping parameter c may be a bounded real-valued function (as in [CZ94]). Multiply the equation (1.6) by a function $\phi(x) \in C^1[0,1]$ and integrate from 0 to 1. Integration by parts yields

$$\begin{aligned}\int_0^1 \partial_t^2 u(x,t)\phi(x)\,dx &= -\int_0^1 \partial_x u(x,t)\phi'(x)\,dx \\ &\quad -\int_0^1 2c(x)\partial_t u(x,t)\phi(x)\,dx \\ &\quad +[\partial_x u(x,t)\phi(x)]_0^1\end{aligned}$$

Definition (Test functions) Define the space of *test functions* by

$$T[0,1] = \left\{\phi(x) \in C^1[0,1] \mid \phi(0) = \phi(1) = 0\right\}. \tag{1.33}$$

If $\phi \in T[0,1]$, then

$$\int_0^1 \partial_t^2 u\phi = -\int_0^1 \partial_x u\phi'\,dx - \int_0^1 2c\partial_t u\phi\,dx. \tag{1.34}$$

It is now possible to give the variational form of the initial value problem.

Variational problem

Find u such that for each $t > 0$, $u(\cdot,t) \in T[0,1]$ and u satisfies (1.34) with

$$u(x,0) = u_0(x) \quad \text{and} \quad \partial_t u(x,0) = v_0(x).$$

Kelvin-Voigt damping

Consider the wave equation with Kelvin-Voigt damping (1.7) and fixed-fixed boundary conditions (1.8). Repeating the process above, but with

$$T[0,1] = \left\{\phi(x) \in C^1[0,1] \mid \phi(0) = \phi(1) = 0\right\}, \tag{1.35}$$

we find that for $\phi \in T[0,1]$,

$$\int_0^1 \partial_t^2 u(x,t)\phi(x)\,dx = -\int_0^1 \partial_x u(x,t)\phi'(x)\,dx + 2q\int_0^1 \partial_t\partial_x^2 u(x,t)\phi(x)\,dx.$$

Integration by parts on the damping term yields

$$\int_0^1 \partial_t^2 u\phi\,dx = -\int_0^1 \partial_x u\phi'\,dx - 2q\int_0^1 \partial_t\partial_x u\phi'\,dx \tag{1.36}$$

if $\phi \in T[0,1]$.

Then the variational form of the initial value problem is:

Variational problem

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0,1]$ and u satisfies (1.36) with

$$u(x,0) = u_0(x) \quad \text{and} \quad \partial_t u(x,0) = v_0(x).$$

Boundary damping

Lastly, we consider the wave equation with $u(0,t) = 0$ and boundary damping (1.11) at $x = 1$. Again, by integrating by parts, we find that for $\phi \in T[0,1]$ as defined in (1.35),

$$\int_0^1 \partial_t^2 u(x,t)\phi(x)\ dx = -\int_0^1 \partial_x u(x,t)\phi'(x)\ dx + \partial_x u(1,t)\phi(1).$$

Applying the boundary condition (1.11), we have

$$\int_0^1 \partial_t^2 u(x,t)\phi(x)\ dx = -\int_0^1 \partial_x u(x,t)\phi'(x)\ dx - k\partial_t u(1,t)\phi(1). \tag{1.37}$$

if $\phi \in T[0,1]$.

Then the variational form of the initial value problem is:

Variational problem

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0,1]$ and u satisfies (1.37) with

$$u(x,0) = u_0(x) \quad \text{and} \quad \partial_t u(x,0) = v_0(x).$$

Remark If different boundary conditions are used for any of the wave equations above, the space of test functions will change accordingly.

1.6.2 Euler-Bernoulli beam with boundary control

Consider the Euler-Bernoulli beam (1.25) with boundary control in dimensionless form. Multiply (1.25) by a function $v \in C^2[0,1]$ and integrate by parts twice:

$$\begin{aligned}\int_0^1 \partial_t^2 w(x,t)v(x)\ dx &= -c^2\int_0^1 \partial_x^4 w(x,t)v(x)\ dx \\ &= -c^2\int_0^1 \partial_x^2 w(x,t)v''(x)\ dx \\ &\quad - \left[c^2\partial_x^3 w(x,t)v(x)\right]_0^1 + \left[c^2\partial_x^2 w(x,t)v'(x)\right]_0^1.\end{aligned}$$

Test functions

Define the space of test functions by

$$T[0,1] = \left\{ v(x) \in C^2[0,1] \mid v(0) = v'(0) = 0 \right\}. \tag{1.38}$$

If $v \in T[0,1]$, then

$$\int_0^1 \partial_t^2 w \ v \ dx = -c^2 \int_0^1 \partial_x^2 w \ v'' \ dx - c^2 \partial_x^3 w(1,t) v(1) + c^2 \partial_x^2 w(1,t) v'(1).$$

Applying the boundary conditions at $x = 1$, it follows that

$$\begin{aligned} \int_0^1 \partial_t^2 \ v \ dx = -c^2 \int_0^1 \partial_x^2 w \ v'' \ dx \\ - c^2 k_2 \partial_t w(1,t) v(1) - c^2 k_1 \partial_t \partial_x w(1,t) v'(1). \end{aligned} \tag{1.39}$$

It is now possible to give the variational form of the initial value problem.

Variational problem

Find w such that for each $t > 0$, $w(\cdot, t) \in T[0,1]$ and w satisfies (1.39) with

$$w(x,0) = u_0(x) \quad \text{and} \quad \partial_t w(x,0) = v_0(x).$$

1.6.3 Euler-Bernoulli beam with Kelvin-Voigt damping

Consider the Euler-Bernoulli beam (1.26) with Kelvin-Voigt damping in dimensionless form. Multiply (1.26) by a function $v \in C^2[0,1]$ and integrate by parts twice:

$$\begin{aligned} \int_0^1 \partial_t^2 w \ v \ dx &= -c^2 \int_0^1 \partial_x^4 w \ v \ dx + \int_0^1 \partial_x^2 (k \partial_t \partial_x^2 w) v \ dx \\ &= -c^2 \int_0^1 \partial_x^2 w \ v'' \ dx + \int_0^1 k \partial_t \partial_x^2 w v'' \ dx \\ &\quad - \left[(c^2 \partial_x^3 w - \partial_x (k \partial_t \partial_x^2 w)) v \right]_0^1 + \left[(c^2 \partial_x^2 w - k \partial_t \partial_x^2 w) v' \right]_0^1. \end{aligned}$$

Test functions

Define the space of test functions by

$$T[0,1] = \left\{ v(x) \in C^2[0,1] \mid v(0) = v'(0) = 0 \right\}. \tag{1.40}$$

Let $v \in T[0,1]$ and apply the boundary conditions at $x = 1$,

$$\int_0^1 \partial_t^2 \, v \, dx = -c^2 \int_0^1 \partial_x^2 w \; v'' \, dx + \int_0^1 k \partial_t \partial_x^2 w v'' \, dx. \tag{1.41}$$

It is now possible to give the variational form of the initial value problem.

Variational problem

Find w such that for each $t > 0$, $w(\cdot, t) \in T[0,1]$ and w satisfies (1.41) with

$$w(x,0) = u_0(x) \quad \text{and} \quad \partial_t w(x,0) = v_0(x).$$

1.6.4 Timoshenko beam

Consider the Timoshenko beam with cantilever boundary conditions in dimensionless form. Multiply (1.22) by an arbitrary function $v \in C^1[0,1]$ and integrate. Integration by parts yields

$$\begin{aligned} \int_0^1 \partial_t^2 w(x,t) v(x) \, dx &= -\int_0^1 \left(\partial_x w(x,t) - \phi(x,t) \right) v'(x) \, dx \\ &\quad + \left[\left(\partial_x w(x,t) - \phi(x,t) \right) v(x) \right]_0^1. \end{aligned}$$

Multiply (1.23) by an arbitrary function $\psi \in C^1[0,1]$ and integrate. Again, integration by parts yields

$$\begin{aligned} \frac{1}{\alpha} \int_0^1 \partial_t^2 \phi(x,t) \psi(x) \, dx &= \int_0^1 \left(\partial_x w(x,t) - \phi(x,t) \right) \psi(x) \, dx \\ &\quad - \frac{1}{\beta} \int_0^1 \partial_x \phi(x,t) \psi'(x) \, dx \\ &\quad + \frac{1}{\beta} \left[\partial_x \phi(x,t) \psi(x) \right]_0^1 . \end{aligned}$$

Test functions

Define the space of test functions by

$$T[0,1] = \left\{ v(x) \in C^1[0,1] \mid v(0) = 0 \right\}. \tag{1.42}$$

For v and ψ are in $T[0,1]$, we have

$$\begin{aligned}
\int_0^1 \partial_t^2 w(x,t)v(x)\ dx &= -\int_0^1 \left(\partial_x w(x,t) - \phi(x,t)\right) v'(x)\ dx \\
&\quad + \left(\partial_x w(1,t) - \phi(1,t)\right) v(1) \\
\text{and} \quad \frac{1}{\alpha}\int_0^1 \partial_t^2 \phi(x,t)\psi(x)\ dx &= \int_0^1 \left(\partial_x w(x,t) - \phi(x,t)\right) \psi(x)\ dx \\
&\quad -\frac{1}{\beta}\int_0^1 \partial_x \phi(x,t)\psi'(x)\ dx. \\
&\quad +\frac{1}{\beta}\partial_x \phi(1,t)\psi(1).
\end{aligned}$$

Applying the boundary conditions at the right hand side,

$$\begin{aligned}
\int_0^1 \partial_t^2 w\ v &= -\int_0^1 \left(\partial_x w - \phi\right) v' \quad \text{and} \\
\frac{1}{\alpha}\int_0^1 \partial_t^2 \phi\ \psi &= \int_0^1 \left(\partial_x w - \phi\right) \psi - \frac{1}{\beta}\int_0^1 \partial_x \phi \psi'.
\end{aligned}$$

Adding the equations above yields

$$\int_0^1 \partial_t^2 w\ v + \frac{1}{\alpha}\int_0^1 \partial_t^2 \phi\ \psi = -\int_0^1 \left(\partial_x w - \phi\right)\left(v' - \psi\right) - \frac{1}{\beta}\int_0^1 \partial_x \phi\ \psi'. \quad (1.43)$$

It is now possible to present the variational form of the initial value problem.

Variational problem

Find a pair of functions $\langle w, \phi\rangle$ such that for each $t > 0$, $w(\cdot,t)$ and $\phi(\cdot,t)$ are in $T[0,1]$ and the variational problem (1.43) is satisfied for each pair $\langle v, \psi\rangle \in T[0,1] \times T[0,1]$. Further, $\langle w, \phi\rangle$ must satisfy the initial conditions

$$w(x,0) = w_0, \quad \phi(x,0) = \phi_0, \quad \partial_t w(x,0) = w_1 \quad \text{and} \quad \partial_t \phi(x,0) = \phi_1.$$

1.6.5 Bilinear forms

The variational problems for the various beam and wave equations are all of the same general form. We introduce the following notation to make this clear.

Wave equations

For the wave equation (with or without damping), let

$$\begin{aligned} \beta(u,v) &= \int_0^1 u'(x)v'(x)dx, \text{ and} \\ \gamma(u,v) &= \int_0^1 u(x)v(x)dx. \end{aligned}$$

For viscous damping, let

$$\alpha(u,v) = \int_0^1 2c(x)u(x)v(x)dx.$$

For Kelvin-Voigt damping, let

$$\alpha(u,v) = 2k\int_0^1 u'(x)v'(x)dx.$$

For boundary damping, let

$$\alpha(u,v) = ku(1)v(1).$$

Euler-Bernoulli beam

For the Euler-Bernoulli beam, let

$$\begin{aligned} \beta(u,v) &= c^2\int_0^1 u''(x)v''(x)\ dx \\ \text{and} \quad \gamma(u,v) &= \int_0^1 u(x)v(x)\ dx. \end{aligned}$$

For the Euler-Bernoulli beam with boundary control, define

$$\alpha(u,v) = c^2k_2u(1)v(1) + c^2k_1u'(1)v'(1).$$

For the Euler-Bernoulli beam with Kelvin-Voigt damping, define

$$\alpha(u,v) = \int_0^1 k(x)w''(x)v''(x)\ dx.$$

Timoshenko beam

For the Timoshenko beam, let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$ and define

$$\begin{aligned} \beta(u,v) &= \int_0^1 (u_1' - u_2)(v_1' - v_2) + \frac{1}{\beta}\int_0^1 u_2'\ v_2' \\ \text{and} \quad \gamma(u,v) &= \int_0^1 u_1\ v_1 + \frac{1}{\alpha}\int_0^1 u_2\ v_2. \end{aligned}$$

Generalization

Every $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ defined above is linear in both arguments by the linearity of the integral, thus $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are referred to as *bilinear forms.*

The variational equations (1.34), (1.36), (1.37), (1.39) and (1.43) may now be written as

$$\gamma\left(\partial_t^2 u(\cdot,t),\phi\right)+\beta\left(u(\cdot,t),\phi\right)+\alpha\left(\partial_t u(\cdot,t),\phi\right)=0 \text{ for all } \phi\in T[0,1], \quad (1.44)$$

where the appropriate bilinear forms and test functions are used in each case. The existence of solutions of (1.44) is discussed in Subsection 1.7.2.

1.7 Abstract setting

1.7.1 Weak variational form of the general linear vibration problem

To use the weak variational form of the general vibration problem of (1.44), we need to introduce the notion of weak derivatives and the spaces in which weak derivatives are defined.

Notation

The space $L^2(a,b)$ is the Hilbert space consisting of Lebesgue square integrable functions. The inner product for $L^2(a,b)$ is defined by

$$(u,v)_{L^2(a,b)}=\int_a^b uv.$$

Where the integral is the Lebesgue integral over the interval (a,b). The norm on $L^2(a,b)$ is induced in the usual way by

$$\|u\|_{L^2(a,b)}=\sqrt{(u,u)_{L^2(a,b)}}.$$

The space $C_0^m(a,b)$ denotes the space of functions $f\in C^m(a,b)$ that may only take nonzero values on some subinterval (x,y) that is strictly contained in (a,b) so that $f(c)=0$ for $c\in(a,x)\cup(y,b)$. These functions are usually referred to as functions with compact support.

Theorem 1.7.1 ([Ad, Theorem 2.13, p.28]). $C_0^\infty(a,b)$ is dense in $L^2(a,b)$. That is, for any $u \in L^2(a,b)$, there exists a sequence $\{\phi_n\} \subset C_0^\infty(a,b)$ such that

$$\|u - \phi_n\|_{L^2(a,b)} \to 0 \quad \text{as} \quad n \to \infty.$$

Definition (Weak derivative of order m [Ad, p.21]) Suppose $u \in L^2(a,b)$ and there exists a $v \in L^2(a,b)$ such that

$$\left(u, \phi^{(m)}\right)_{L^2(a,b)} = (-1)^m (v, \phi)_{L^2(a,b)} \text{ for all } \phi \in C_0^\infty(a,b),$$

then v is called the *weak derivative* of order m of u and is denoted in the same way as the usual derivative in the literature,

$$v = u^{(m)} = \frac{d^m u}{dx^m} = D^m u.$$

The weak derivative is sometimes referred to as the *generalized derivative.*

Definition (Sobolev space $H^m(a,b)$ [Ad, p.45]) The Sobolev space of order m is defined as the vector space of all functions f whose weak derivatives of up to order m exist and are elements of $L^2(a,b)$. The inner product $(u,v)_m$ for the Sobolev space $H^m(a,b)$ is defined by

$$(u,v)_m = \sum_{k=0}^{m} \left(u^{(k)}, v^{(k)}\right)_{L^2(a,b)}$$

and induces the norm $\|u\|_m = \sqrt{(u,u)_m}$.

Remark If the ordinary derivative of order m exists then the weak derivative is equal to the ordinary derivative. Note that $u^{(0)} = u$ and that $H^m(a,b) \subset L^2(a,b)$.

Theorem 1.7.2 ([Ad, Theorem 3.2, p.45]). The Sobolev space $H^m(a,b)$ is complete. That is, $H^m(a,b)$ is a Hilbert space.

The following theorem is very important and is used often in this dissertation.

Theorem 1.7.3 (Sobolev's Lemma [OR, Theorem 3.10, p.80]). If $u \in H^m(a,b)$ then there exists a $\phi \in C^{m-1}[a,b]$ such that $u = \phi$ almost everywhere and the weak derivative $u^{(m)}$ exists with the property that

$$\left(u^{(m)}, v\right)_{L^2(a,b)} = -\left(u^{(m-1)}, v'\right)_{L^2(a,b)} \text{ for all } v \in C_0^1[a,b]$$

Furthermore, there exists a constant $C > 0$ such that following inequality holds:

$$\left\|u^{(k)}\right\|_{\sup} \leq C \|u\|_m \text{ for all } k = 0, 1, \ldots, m-1. \tag{1.45}$$

Remarks

1. Since the values of functions in Sobolev spaces may be arbitrarily changed on sets of measure zero, we define the boundary values of $u^{(k)} \in H^m(0,1)$ as

$$u^{(k)}(a) = \phi^{(k)}(a) \quad \text{and} \quad u^{(k)}(b) = \phi^{(k)}(b) \text{ for all } 0 \leq k \leq m-1.$$

2. Sobolev's Lemma provides a convenient way to think about Sobolev spaces: A function $u \in H^m(a,b)$ is continuous and has continuous derivatives up to order $m-1$. The m^{th} order derivative of u may only be taken in the weak sense and is square-integrable.

Definition (Derivative) Let w be a function whose values are in a Banach space X. If there exists a $z \in X$ such that

$$\lim_{h \to 0} h^{-1}(w(t+h) - w(t)) = z,$$

then z is the derivative of w at t and is denoted by $w'(t)$. This implies that

$$\lim_{h \to 0} \left\| h^{-1}(w(t+h) - w(t)) - w'(t) \right\|_X = 0.$$

The derivative is denoted $w'(t)$. The derivative (function) w' and the higher order derivatives $w^{(k)}$ are defined in the usual way.

Notation For natural m, $C^m([0,\infty), X)$ is the space of functions that are $m-$times continuously differentiable on the interval $[0,\infty)$ with respect to $\|\cdot\|_X$.

Remark Note that the time derivative defined above depends on the norm used in the definition. It is possible that the derivative exists with respect to one norm but not another.

Energy and inertia spaces

For a linear vibration problem, V denotes the closure of the space of test functions in the relevant Sobolev space and W denotes $L^2(0,1)$, possibly with a different inner product defined by the bilinear form $\gamma(\cdot,\cdot)$.

As example, consider the wave equation with viscous damping. In this case, V is the closure of the test functions with respect to the norm on $H^2(0,1)$ and W is just $L^2(0,1)$. Note that it is necessary to check that the bilinear form $\beta(\cdot,\cdot)$ defines a norm that is equivalent to $(\cdot,\cdot)_2$.

In [VV02], W is called the *inertia space* and V the *energy space*.

Consider the variational equation (1.44) and let $w(t) = u(\cdot,t)$. The weak variational form of the general linear vibration problem is:

Weak variational problem

Find $w(t) \in V$, such that $w'(t) \in V$, $w''(t) \in W$ and

$$\begin{aligned} \gamma(w''(t),v) + \alpha(w'(t),v) + \beta(w(t),v) &= \gamma(f(t),v) \quad \text{for each} \quad v \in V, \\ w(0) &= u_0 \quad \text{and} \\ w'(0) &= v_0. \end{aligned}$$

Remarks

1. A classical solution of the weak variational problem is not (in general) a solution of the original problem, but may be considered to be a *weak solution* of original initial value problem.
2. The forcing function $f = 0$ in the relevant literature, so we restrict our attention to this case.

1.7.2 Existence of solutions of the general linear vibration problem

It is possible to frame the variational forms of linear vibration problems in an abstract setting. Following the approach of [VV02], we make the following assumptions:

Assumptions

(A1) Suppose W is a Hilbert space with inner product $\gamma(\cdot,\cdot)$. The induced norm is denoted by $\|\cdot\|_W$.

(A2) Suppose V is a Hilbert space with inner product $\beta(\cdot,\cdot)$. The induced norm is denoted by $\|\cdot\|_V$.

(A3) Assume that V is a dense subset of W.

(A4) The embedding of V into W is bounded. That is, there exists a constant C such that $\|v\|_W \le C\|v\|_V$ for each $v \in V$.

(A5) A bilinear form $\alpha(\cdot,\cdot)$ is defined on V. It is non-negative, symmetric and bounded on V. That is, there exists a constant C such that

$$\alpha(u,v) \leq C\,\|u\|_V\,\|v\|_V \text{ for all } u,v \in V.$$

(A6) Assume that the embedding of V in W is compact.

Remark Assumption (A6) is not necessary for existence theory. However, it plays a crucial role in spectral analysis.

Problem G: Abstract weak problem

Find $w \in C^1([0,\infty),V)$ such that for each $t > 0$, $w''(t) \in W$ and

$$\begin{aligned} \gamma(w''(t),v) + \alpha(w'(t),v) + \beta(w(t),v) = 0 \quad \text{for each} \quad v \in V, \\ w(0) = u_0 \quad \text{and}, \\ w'(0) = v_0. \end{aligned} \tag{1.46}$$

If assumptions (A1) to (A5) hold then the following theorems from [VV02] give conditions under which Problem G has a unique solution.

Theorem 1.7.4 ([VV02, Theorem 1])**.** If $u_0 \in V$, $v_0 \in V$ and there exists a $y \in W$ such that

$$\beta(u_0,v) + \alpha(v_0,v) = \gamma(y,v) \quad \text{for each} \quad v \in V,$$

then Problem G has a unique solution.

In general, the damping bilinear form $\alpha(\cdot,\cdot)$ need only be defined on V and non-negative. It is possible to identify two special cases where $\alpha(\cdot,\cdot)$ has other additional properties:

Weak damping

Definition (Weak damping [VV02, p.1147]) The damping in Problem G is referred to as *weak damping* if the damping bilinear form $\alpha(\cdot,\cdot)$ is defined on the larger space W and $\alpha(\cdot,\cdot)$ is bounded on W. That is, there exists a constant C such that

$$|\alpha(u,v)| \leq C\,\|u\|_W\,\|v\|_W \text{ for all } u,v \in V.$$

Theorem 1.7.5 (Weak damping [VV02, Theorem 2])**.** If the damping in Problem G is weak, $u_0 \in V$, $v_0 \in V$ and there exists a $y \in W$ such that

$$\beta(u_0,v) = \gamma(y,v) \quad \text{for each} \quad v \in V,$$

then Problem G has a unique solution.

Strong Damping

Definition (Strong damping, [VV02, p.1148]) The damping in Problem G is referred to as *strong damping* if the damping bilinear form $\alpha(\cdot,\cdot)$ is positive definite on V. That is, if

$$\alpha(u,u) \geq C \|u\|_V^2 \text{ for all } u \in V.$$

Theorem 1.7.6 (Strong damping [VV02, Theorem 3])**.** If the damping in Problem G is strong and $u_0 \in V$ and $v_0 \in W$, then Problem G has a unique solution.

Remarks

1. The existence results above are simplified by the assumption that $f = 0$. For the case where f may be nonzero, see [VV02].

2. It is possible that the bilinear form β for a given problem does not satisfy Assumption (A4) (For example, the Euler-Bernoulli beam with free-free boundary conditions). The existence theory is still applicable in this case, see Appendix C.3.

1.7.3 Variational eigenvalue problems

Consider a possible solution of (1.46) of the form $w(t) = e^{\lambda t}u(t)$. Substitution into (1.46) yields the weak variational eigenvalue problem:

$$\lambda^2\gamma(u,v) + \lambda\alpha(u,v) + \beta(u,v) = 0 \text{ for all } v \in V. \tag{1.47}$$

This is an abstract quadratic eigenvalue problem.

If $\alpha = 0$, there is no damping and (1.47) reduces to the ordinary variational eigenvalue problem

$$\beta(u,v) = \mu\gamma(u,v) \text{ for all } v \in V,$$

where $\mu = -\lambda^2$.

We refer to this as the *symmetric case*. There is a well-developed mathematical theory for the series representation of solutions in the symmetric case (see Section 3.1).

When the physical and damping parameters are constant, there are situations where the variational eigenvalue problem for the damped wave equation reduces to the symmetric case.

Special cases of damping

1. If $\alpha(u,v) = 2\kappa\gamma(u,v)$ for some constant κ, (1.47) becomes

$$(\lambda^2 + 2\kappa\lambda)\gamma(u,v) + \beta(u,v) = 0 \text{ for all } v \in V. \tag{1.48}$$

   Thus the quadratic eigenvalue problem reduces to the symmetric case. This is a special case of *weak damping*, see Subsection 1.7.2.

2. If $\alpha(u,v) = 2\nu\beta(u,v)$ for some constant κ, (1.47) becomes

$$\lambda^2\gamma(u,v) + (2\nu\lambda + 1)\,\beta(u,v) = 0 \text{ for all } v \in V. \tag{1.49}$$

   Again, the quadratic eigenvalue problem reduces to the symmetric case. This is a special case of *strong damping*, see Subsection 1.7.2.

The quadratic eigenvalue problem doesn't always reduce to the symmetric case. For instance, (1.47) does not reduce for boundary damping.

Remarks

1. The variational eigenvalue problem may also be derived directly by multiplying the quadratic eigenvalue problem by a test function and integrating by parts. Equation (1.47) is the weak variational form of the quadratic eigenvalue problem.

2. When β does not satisfy Assumption (A4) (positive definiteness), $\lambda = 0$ will be an eigenvalue. In this case, it is possible to define a new bilinear form, which is positive definite, by $\beta_+ = \beta + m\gamma$ for some positive number m. Now,

$$\begin{aligned} &\beta(u,v) = \lambda\gamma(u,v) \quad \text{for all } v \in V \\ \text{if and only if} \quad &\beta_+(u,v) = (\lambda + m)\gamma(u,v) \quad \text{for all } v \in V. \end{aligned}$$

1.7.4 First order systems

In the literature, vibration problems are converted to abstract first order differential equation of the form

$$w'(t) = Aw(t) + F(t). \tag{1.50}$$

in $V \times W$.

Definition (State space) The product space $H = V \times W$ is called the *state space*. Define the inner product on H by

$$(x, y)_H = (\langle x_1, x_2\rangle, \langle y_1, y_2\rangle)_H = (x_1, y_1)_V + (x_2, y_2)_W \text{ for all } x, y \in H.$$

This induces the norm $\|x\|_H^2$, where

$$\|x\|_H^2 = (x_1, x_1)_V + (x_2, x_2)_W = \|x_1\|_V^2 + \|x_2\|_W^2 .$$

It is convenient to have a name for the operator A, so we adopt the terminology suggested in [Sh02].

Definition (Dynamics generator [Sh02]) The linear operator A in (1.50) is referred to as the *dynamics generator*.

The conversion to a first order system allows one to use semigroup theory (See [Pa]).

There is more than one way to get to an abstract first order system. Some authors formulate vibration problems as abstract second order differential equations of the form

$$Au''(t) + Bu'(t) + Cu(t) = f(t),$$

where A, B and C are linear operators on some Hilbert space, see for example [JTW08]. Other authors write the partial differential equations as a first order system and use this to define an abstract first order system of the form (1.50), see for example [CZ94], [GY01] or [Sh02].

The approach of [VV02] is followed here. The approaches taken in [GY01], [JTW08] and [CZ94] are discussed in later chapters. The vibration problem is converted to an abstract first order differential equation using the following linear operator:

Definition (The mapping Λ, [VV02]) The mapping Λ is defined on the state space $H = V \times W$ by

$$\Lambda y = -x,$$

where $-x_2 = y_1 \in V$ and $x_1 \in V$ such that

$$\beta(x_1, v) + \alpha(x_2, v) = \gamma(y_2, v) \text{ for all } v \in V. \tag{1.51}$$

Theorem 1.7.7 (Properties of Λ, [VV02])**.** The mapping Λ is a bounded linear operator that has the following properties:

1. The nullspace of Λ is trivial, so the inverse Λ^{-1} exists.

2. The range $\mathcal{R}(\Lambda)$ is dense in the state space H.

In [VV02], the authors define $A = \Lambda^{-1}$ and prove that A is precisely the operator that takes the vibration problem to an abstract first order differential equation. Consider the following initial value problem:

Problem IVP: First order initial value problem

Find $w = \langle u, v \rangle \in C([0, \infty), H)$ such that for each $t > 0$, $w(t) \in \mathcal{D}(A)$, $w'(t) \in H$ and

$$\begin{aligned} w'(t) &= Aw(t), \\ w(0) &= w_0 = \langle u_0, v_0 \rangle . \end{aligned}$$

Theorem 1.7.8 ([VV02])**.** If w is a solution of Problem IVP with $w(0) = \langle u_0, v_0 \rangle$, then the first component u of w is a solution of Problem G.
On the other hand, if

$$u \in C([0, \infty); V) \cap C^1([0, \infty); W)$$

is a solution of Problem G then $w = \langle u, u' \rangle$ is a solution of Problem IVP with $w_0 = \langle u_0, v_0 \rangle$.

Under the assumptions (A1) to (A5), the dynamics generator has the following important properties. These properties are used often in this dissertation.

Properties of the dynamics generator [VV02]

1. The operator Λ is bounded, so A has a bounded inverse.

2. $\mathcal{R}(\Lambda) = \mathcal{D}(A)$ is dense in H.

3. $\mathcal{D}(\Lambda) = \mathcal{R}(A) = H$.

4. A is a closed operator.

5. A is the infinitesimal generator of a C_0 semigroup.

6. The dynamics generator A is dissipative [VV02, Corollary 3]. That is,

$$(Ax, x)_H \leq 0 \text{ for all } x \in \mathcal{D}(A).$$

7. For any $x \in H$, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$ and
$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \ v \in V.$$

Zero damping

All the results above hold in the important special case that $\alpha(\cdot, \cdot) = 0$. In this case, we denote the dynamics generator by A_0 and its inverse by Λ_0.

Chapter 2

First order systems

2.1 Series representation of solutions

In this section we consider a homogeneous abstract first order system in the state space H of the form of Problem IVPH (see Subsection 1.7.4):

Problem IVPH: Homogeneous case

$$w'(t) = Aw(t), \tag{2.1}$$

$$w(0) = w_0, \tag{2.2}$$

where $w_0 = \langle u_0, v_0 \rangle \in H$.

In order to represent the solution of this initial value problem as a series it is necessary to calculate the eigenvalues and eigenvectors of the dynamics generator A, which may be non-real. Therefore, we must work in a complex state space.

A real Hilbert space X may be embedded in a complex Hilbert space $\widetilde{X}$ as follows (See [Sc, Chapter VI, Section 5] for a rigorous treatment):

Let

$$\widetilde{X} = \{x + iy \mid x \in X, y \in X\}.$$

Note that for the state space H,

$$\widetilde{H} = \widetilde{V} \times \widetilde{W}.$$

Any bilinear form $\theta(\cdot,\cdot)$ on the real space H may be extended to a sesquilinear form $\widetilde{\theta}(\cdot,\cdot)$ on the complex space $\widetilde{H}$ by

$$\widetilde{\theta}(x + iy, u + iv)_{\widetilde{H}} = \theta(x,u)_H + \theta(y,v)_H + i\theta(y,u)_H - i\theta(x,v)_H.$$

The complex form $\widetilde{\theta}\,(\cdot,\cdot)$ is now linear in the first argument and conjugate linear in the second. Such a form is said to be *sesquilinear*.

The inner product on the real space H is a bilinear form, so it may be extended to $\widetilde{H}$ by

$$(x+iy,u+iv)_{\widetilde{H}} = (x,u)_H + (y,v)_H + i\,(y,u)_H - i\,(x,v)_H\,. \tag{2.3}$$

A bilinear form that is symmetric in the real state space will now become *Hermitian* in the complex space. That is

$$\widetilde{\theta}\,(x,y) \quad \text{is the conjugate of} \quad \widetilde{\theta}(y,x) \quad \text{for all } x,y \in \widetilde{H}.$$

A linear operator A defined in H can be extended to $\widetilde{A}$ defined in $\widetilde{H}$ by:

$$\begin{aligned} \mathcal{D}(\widetilde{A}) &= \{x+iy \mid x \in \mathcal{D}(A), y \in \mathcal{D}(A)\}\,, \\ \widetilde{A}(x+iy) &= Ax + iAy. \end{aligned}$$

Proposition 2.1.1. For any $w = x+iy \in \mathcal{D}(\widetilde{A})$

$$\begin{aligned} \mathrm{Re}(\widetilde{A}w,w)_{\widetilde{H}} &= (Ax,x)_H + (Ay,y)_H, \\ \mathrm{Im}(\widetilde{A}w,w)_{\widetilde{H}} &= (Ay,x)_H - (Ax,y)_H. \end{aligned}$$

***Proof*.**

Let $w = x+iy$. By (2.3):

$$(\widetilde{A}(x+iy),x+iy)_{\widetilde{H}} = (Ax,x)_H + (Ay,y)_H + i(Ay,x)_H - i(Ax,y)_H.$$

The result follows since $(\cdot,\cdot)_H$ is real valued. $\square$

Definition (Dissipative operator) A linear operator A in a complex Hilbert space X is dissipative if for every $x \in \mathcal{D}(A)$, $\mathrm{Re}\,(Ax,x)_X \leq 0$.

Properties of $\widetilde{A}$

Recall from Subsection 1.7.4 that the dynamics generator was defined to be $A = \Lambda^{-1}$ and that $\mathcal{R}(\Lambda)$ is dense in H. Therefore A is densely defined in H and consequently $\widetilde{A}$ is densely defined in $\widetilde{H}$.

It follows directly from Proposition 2.1.1 that the complex dynamics generator $\widetilde{A}$ is dissipative since the real dynamics generator A is dissipative (see Subsection 1.7.4).

Remarks

1. We use the original notation A, H, $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ to denote the complex dynamics generator, complex state space and complex bilinear forms respectively.

2. In the finite dimensional case, $\mathbb{R}^n$ is embedded in $\mathbb{C}^n$.

3. In the applications considered in this dissertation, the solutions of Problem IVPH will be real. Since

$$(x+iy)' = x' + iy' \quad \text{and} \quad \widetilde{A}(x+iy) = Ax + iAy,$$

if $w = x + iy$ is a solution of $w' = \widetilde{A}w$, then both x and y are real solutions of $w' = Aw$. If the initial condition w_0 is real, then $w(t)$ will be real for all $t > 0$.

Assuming that the conditions of Theorem 1.7.4 have been fulfilled, the existence of a unique solution of Problem IVPH is guaranteed. Our aim now is to determine conditions under which the series representation of this solution is valid.

2.1.1 Finite dimensional case

Suppose the state space H is $n-$dimensional and let A be a linear operator on H. It is well-known that $e^{\lambda t}v$ is a solution of $w'(t) = Aw(t)$ if and only if $Av = \lambda v$. A has n eigenvalues but A may have less than n linearly independent eigenvectors. Denote the eigenvalues of A by λ_j and the eigenvectors of A by w_j.

Sometimes, the eigenvalue λ is repeated k times. Then there maybe k linearly independent eigenvectors corresponding to λ, or there may be less. In the latter case, it is necessary to find the *generalized eigenvectors* of A corresponding to λ. For completeness, these well-known concepts are defined in Section 2.10.

In the applications considered in this chapter, the simplest case requiring generalized eigenvectors arises. That is, there may be one double eigenvalue λ^* that has only one linearly independent eigenvector v associated with it. Then one generalized eigenvector u is required. This u must satisfy

$$(A - \lambda^* I)u = v.$$

In this case the solutions of $w'(t) = Aw(t)$ corresponding to λ^* are $w_1(t) = e^{\lambda^* t}(vt + u)$ and $w_2(t) = e^{\lambda t}v$.

The general solution of $w'(t) = Aw(t)$ is given by the linear combination of solutions:

$$w(t) = \sum_{j=1}^{n} p_j(t) w_j,$$

where each $p_j(t)$ is a complex function that is uniquely determined by $p_j(0)$ (see Section 2.10). The coefficients $c_j = p_j(0)$ are determined by the initial condition $w(0) = w_0$ since

$$\sum_{j=1}^{n} c_j w_j = w(0) = w_0.$$

Observe that the equation above may be written as

$$Mc = w_0,$$

where $c = [c_1, \ldots, c_n]^T$ and the generalized eigenvectors w_j are the column vectors of M. Since these column vectors are linearly independent, the matrix M is invertible and $Mc = w_0$ has a unique solution. Therefore, the coefficients c_j are uniquely determined and we obtain a unique representation of the solution of Problem IVPH.

2.1.2 Infinite dimensional case

Now let H be an infinite dimensional Hilbert space. It is still true that $e^{\lambda t}v$ is a solution of $w'(t) = Aw(t)$ if and only if $Av = \lambda v$ and generalized eigenvectors are handled in the same way as in the finite dimensional case.

The major difference between the finite dimensional and infinite dimensional cases is that in the infinite dimensional case, there are infinitely many eigenvalues and eigenvectors, so we must consider the convergence of the partial sums

$$w_N(t) = \sum_{j=1}^{N} p_j(t) w_j \tag{2.4}$$

to $w(t)$ as $N \to \infty$, where each w_j is a generalized eigenvector of the dynamics generator and each p_j is a complex function.

For the partial sums (2.4) to converge, it is necessary that the initial condition $w(0) = w_0$ can be represented as

$$w_0 = \sum_{j=1}^{\infty} c_j w_j .$$

If any initial condition $w_0 \in H$ can be represented in this way uniquely, the sequence $\{w_j\}_{j=1}^{\infty}$ is said to form a *basis* for the state space H.

Suppose the generalized eigenvectors of the dynamics generator $\{w_j\}_{j=1}^{\infty}$ form a basis for the state space H. We will prove that this implies that $\|w(t) - w_N(t)\|_H \to 0$ as $N \to 0$ if the dynamics generator is dissipative. We first prove that the energy of a solution of (2.1) decays:

Theorem 2.1.2. If A is a dissipative linear operator and w is a solution of Problem IVPH then $\|w(t)\|_H \leq \|w(0)\|_H$ for all $t > 0$.

Proof.

By Proposition 2.1.1,

$$\begin{aligned}
\frac{d}{dt}\|w(t)\|_H^2 &= \frac{d}{dt}(w(t), w(t))_H \\
&= (w'(t), w(t))_H + (w(t), w'(t))_H \\
&= (Aw(t), w(t))_H + (w(t), Aw(t))_H \\
&= 2(Ax, x)_H + 2(Ay, y)_H \\
&= 2\mathrm{Re}\,(Aw(t), w(t))_H \leq 0.
\end{aligned}$$

The result follows since the derivative of $\|w(t)\|^2$ is non-increasing. □

Remark The condition that $\|w(t)\|_H \leq \|w(0)\|_H$ for all $t > 0$ is referred to as decay of energy. Many authors deal with decay of energy of solutions as it was defined in Subsection 1.2.4, but the norm on the state space H is equivalent to the dimensionless energy $E(t)$ defined there. Therefore, the decay of energy of the solution is equivalent to decay of the norm of the solution.

Theorem 2.1.3. If the dynamics generator A is dissipative and the sequence of generalized eigenvectors $\{w_j\}_1^{\infty}$ of A is a basis of the state space H, then the sequence of partial sums

$$w_N(t) = \sum_{j=1}^{N} p_j(t) w_j$$

converges to the solution of Problem IVPH in the norm of H.

Proof.

Let w be a solution of Problem IVPH with initial condition $w(0) = w_0$. For each N,

$$w_N(t) = \sum_{j=1}^{N} p_j(t)w_j$$

is a solution of $w'(t) = Aw(t)$. Since $\{w_j\}$ is a basis for H, there exists a unique sequence of complex numbers $\{c_j\}$ such that

$$w_0 = \sum_{j=1}^{\infty} c_j w_j.$$

Let $p_j(0) = c_j$, then $w_0^N = \sum_{j=1}^{N} c_j w_j$ and w_N is a solution of the initial value problem

$$w_N' = Aw_N, \quad w_N(0) = w_0^N.$$

Therefore, $w - w_N$ is a solution of Problem IVPH:

$$\begin{aligned} w' - w_N' &= A(w - w_N), \\ (w - w_N)(0) &= w_0 - w_0^N. \end{aligned}$$

Since A is dissipative, $\|w(t) - w_N(t)\|_H \leq \|w(0) - w_N(0)\|_H \overset{N}{\underset{\infty}{\to}} 0$. $\square$

Remark If the partial sums w_N converge to the solution w of Problem IVPH then the series representation of the solution is valid. The theorem above leaves us with the problem of determining when the sequence of generalized eigenvectors of the dynamics generator will form a basis for the state space.

2.1.3 Equivalence of eigenvalue problems

We now show that the eigenvalue problem $Ax = \lambda x$ for the dynamics generator A is equivalent to the weak variational eigenvalue problem (1.47). This is Theorem 2.1.5. To prove the theorem, the following intermediate result is used.

Recall that the dynamics generator A is the inverse of the operator Λ defined in Subsection 1.7.4.

Theorem 2.1.4. For any $x \in H$, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } v \in V.$$

Proof.

The result follows from the characterization of A given at the end of Subsection 1.7.4. $\square$

Remark The theorem above implies that if $x \in \mathcal{D}(A)$, then x_1 and x_2 both lie in V.

Theorem 2.1.5. For the dynamics generator A of Problem IVPH, $Ax = \lambda x$ if and only if

$$x_2 = \lambda x_1 \quad \text{and}$$
$$\beta(x_1, v) + \lambda\alpha(x_1, v) + \lambda^2\gamma(x_1, v) = 0 \text{ for all } \ v \in V.$$

Proof.

By Theorem 2.1.4, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \ v \in V.$$

Let $y = \lambda x$, then $Ax = \lambda x$ if and only if $x_2 = \lambda x_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(\lambda x_2, v) \text{ for all } \ v \in V.$$

So

$$\beta(x_1, v) + \lambda\alpha(x_1, v) + \lambda^2\gamma(x_1, v) = 0 \text{ for all } \ v \in V. \qquad \square$$

In Subsection 2.3.1 and Sections 2.6 to 2.8 the equivalence between the weak variational form of the quadratic eigenvalue problem and the classical form of the quadratic eigenvalue problem will be investigated. The following theorem gives a characterization for $H^m(a,b)$ that will be useful.

Theorem 2.1.6 ([OR, Theorem 2.10, p.53])**.** The space $C^m[a,b]$ is dense $H^m(a,b)$ with respect to the norm of $H^m(a,b)$. Thus for any $u \in H^m(a,b)$, there exists a sequence $\{u_n\} \subset C^m[a,b]$ such that

$$\|u - u_n\|_m \to 0 \quad \text{as} \quad n \to \infty.$$

2.2 The dynamics generator

Many authors define the dynamics generator as a matrix. We show that the dynamics generator defined in Section 2.1 can be expressed this way.

2.2.1 The dynamics generator as a matrix

By Theorem 2.1.4, $A_0x = y$ if and only if $y_1 = x_2$ and

$$\beta(x_1, v) = -\gamma(y_2, v) \quad \text{for all } v \in V.$$

We define the operator $P : W \to V$ by $x_1 = Py_2$ such that

$$\beta(x_1, v) = -\gamma(y_2, v) \text{ for all } v \in V. \tag{2.5}$$

Let $Q = P^{-1}$, then $Q : V \to W$ and

$$\beta(x_1, v) = -\gamma(Qx_1, v) \text{ for all } v \in V. \tag{2.6}$$

The sesquilinear form $\alpha(x, v)$ is bounded on V by Assumption (A5). By the Riesz representation theorem [Kr, Theorem 3.8-4, p.192] there exists a bounded linear operator $\Delta : V \to V$ such that

$$\alpha(x, v) = \beta(\Delta x, v) \quad \text{for all } v \in V. \tag{2.7}$$

For x in $\mathcal{D}(A)$, Theorem 2.1.4 implies that $y_1 = x_2$ and

$$\beta(x_1 + \Delta x_2, v) = -\gamma(y_2, v) \qquad \text{for all } v \in V. \tag{2.8}$$

Therefore (2.8) and (2.6) imply that

$$y_2 = Q(x_1 + \Delta x_2).$$

This allows us to express the dynamics generator A in terms of the operators Δ and Q as follows:

$$Ax = \langle y_1, y_2 \rangle = \langle x_2, Qx_1 + Q\Delta x_2 \rangle. \tag{2.9}$$

Now A may be expressed as the matrix

$$A = \begin{bmatrix} 0 & I \\ Q & Q\Delta \end{bmatrix}.$$

Note that Q and Δ are selfadjoint since:

$$\gamma(Qx, y) = -\beta(x, y) = \gamma(x, Qy) \quad \text{and}$$
$$\beta(\Delta x, y) = \alpha(x, y) = \beta(x, \Delta y) \quad \text{for all } x, v \in V.$$

Zero damping

If $\alpha(\cdot, \cdot) = 0$ then $\Delta = 0$. The dynamics generator $A_0 = \Lambda_0^{-1}$ is simply

$$A_0 \langle u, v \rangle = \langle v, Qu \rangle,$$

which may be written as the matrix

$$A = \begin{bmatrix} 0 & I \\ Q & 0 \end{bmatrix}.$$

2.2.2 Selfadjointness and non-normality

In [CZ95] and [JTW08], the adjoint of the dynamics generator is merely written down for specific vibration problems. This is generalized here by defining the adjoint of the dynamics generator of the general vibration problem and showing that it is not normal. First consider the necessary definitions:

Definition (Hilbert-adjoint operator [Kr, p.527]) Let A be a densely defined linear operator $A : \mathcal{D}(A) \to H$ in a complex Hilbert space H. Then the *Hilbert-adjoint operator* A^* of A (or simply *adjoint*) is the operator $A^* : \mathcal{D}(A^*) \to H$ such that the domain of A^* consists of all $y \in H$ such that there is a $y^* \in H$ satisfying

$$(Ax, y) = (x, y^*) \text{ for all } x \in \mathcal{D}(A).$$

For each such $y \in \mathcal{D}(A^*)$, the adjoint operator is defined in terms of that y^* by

$$y^* = A^* y$$

so that we may write

$$(Ax, y) = (x, A^* y) \text{ for all } x \in \mathcal{D}(A), y \in \mathcal{D}(A^*).$$

Definition (Selfadjoint operator, normal operator [Kr, p.534]) A densely linear operator $A : H \to H$ on a Hilbert space H is said to be *selfadjoint* if $A = A^*$ and *normal* if $AA^* = A^*A$. The operator A is called *skew-symmetric* if $A^* = -A$. Clearly, selfadjoint and skew-symmetric operators are normal. Selfadjoint operators are sometimes referred to as *Hermitian* operators.

There is a well-developed spectral theory of normal operators. See for example [DS, Chapters X, XII]. In applications however, the dynamics generator is often non-normal. Dunford and Schwartz remark that "the problem of extending the spectral theory of selfadjoint operators to non-normal operators is one of the most important unsolved problems in the theory of linear operations [*sic*]" [DS, p.1925].

We show that the adjoint of the dynamics generator A is given by

$$Bx = \langle -x_2, -Qx_1 + Q\Delta x_2 \rangle ,$$

which may be expressed as the matrix

$$B = \begin{bmatrix} 0 & -I \\ -Q & Q\Delta \end{bmatrix}.$$

Let $x, y \in \mathcal{D}(A)$ be arbitrary.

$$\begin{aligned}
(x, By)_H &= \beta\,(x_1, -y_2) + \gamma\,(x_2, -Q(y_1 - \Delta y_2)) \\
&= -\beta\,(x_1, y_2) + \beta\,(x_2, y_1 - \Delta y_2) \\
&= -\beta\,(x_1, y_2) + \beta\,(x_2, y_1) - \beta\,(x_2, \Delta y_2) \\
&= -\beta\,(x_1, y_2) + \beta\,(x_2, y_1) - \beta\,(\Delta x_2, y_2) \\
&= \beta\,(x_2, y_1) - \beta\,(x_1 + \Delta x_2, y_2) \\
&= \beta\,(x_2, y_1) + \gamma\,(Q(x_1 + \Delta x_2), y_2) \\
&= (Ax, y)_H\,.
\end{aligned}$$

Therefore $B = A^*$ indeed.

Theorem 2.2.1. The dynamics generator is not normal.

Proof.

For A to be normal, it must hold that

$$(AA^*x, y)_H = (A^*Ax, y)_H \quad \text{for all } x, y \in \mathcal{D}(A).$$

This is not the case. Let $x, y \in \mathcal{D}(A)$ be arbitrary,

$$\begin{aligned}
(A^*Ax, y)_H &= (Ax, Ay)_H \\
&= \beta\,(x_2, y_2) + \gamma\,(Q(x_1 + \Delta x_2), Q(y_1 + \Delta y_2)) \\
&= \beta\,(x_2, y_2) + \gamma\,(Q(x_1 + \Delta x_2), Qy_1) \\
&\quad +\gamma\,(Q(x_1 + \Delta x_2), Q\Delta y_2)\,.
\end{aligned}$$

And

$$\begin{aligned}
(AA^*x, y)_H &= (A^*x, A^*y)_H \\
&= \beta\,(x_2, y_2) + \gamma\,(-Q(x_1 - \Delta x_2), -Q(y_1 - \Delta y_2)) \\
&= \beta\,(x_2, y_2) + \gamma\,(Q(x_1 - \Delta x_2), Qy_1) \\
&\quad -\gamma\,(Q(x_1 - \Delta x_2), Q\Delta y_2)\,.
\end{aligned}$$

So

$$\begin{aligned}
(A^*Ax, y)_H - (AA^*x, y)_H &= \gamma\,(Q(x_1 + \Delta x_2), Qy_1) \\
&+ \gamma\,(Q(x_1 + \Delta x_2), Q\Delta y_2) \\
&- \gamma\,(Q(x_1 - \Delta x_2), Qy_1) \\
&+ \gamma\,(Q(x_1 - \Delta x_2), Q\Delta y_2) \\
&= 2\gamma\,(Q\Delta x_2, Qy_1) + 2\gamma\,(Qx_1, Q\Delta y_2) \\
&= -2\beta\,(\Delta x_2, Qy_1) - 2\beta\,(Qx_1, \Delta y_2) \\
&= -2\alpha\,(x_2, Qy_1) - 2\alpha\,(Qx_1, y_2)\,,
\end{aligned}$$

which is not identically zero. Therefore A is not a normal operator. □

Remark It is interesting to note that A_0 is skew-symmetric.

2.3 Wave equation with viscous damping

In this section we consider the initial value problem consisting of the wave equation with viscous damping (1.6), fixed-fixed boundary conditions (1.8) and the initial conditions (1.12) and (1.13).

Problem CZ94

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u + 2c\partial_t u &= 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) &= 0, \quad t > 0, \\ u(x,0) &= u_0(x), \\ \partial_t u(x,0) &= v_0(x). \end{aligned}$$

Note that c may be a real-valued function. This problem is considered in [CZ94], where c is assumed to be of bounded variation. Here we assume only that c is bounded.

It is not difficult to see that the bilinear forms defined for this problem in Subsection 1.6.5 satisfy Assumptions (A1) to (A5): (A1) holds since the norm on W is equivalent to the $L^2(0,1)$ norm. (A3) holds since $C_0^\infty[0,1]$ is dense in $L^2(0,1)$. The Poincaré inequality in Appendix C.1 ensures that (A2) and (A4) hold and (A5) holds since the function c is bounded.

2.3.1 Equivalence of problems

We show that the result of Theorem 2.1.4 can be extended for the dynamics generator A of Problem CZ94. Before we do this, we prove the following lemma, which will be used repeatedly in the rest of this chapter.

Lemma 2.3.1. For any $x_1 \in H^2(0,1)$ and $\phi \in C^1[0,1]$,

$$\int_0^1 x_1'' \,\bar{\phi}\, dx = -\int_0^1 x_1' \,\bar{\phi}'\, dx + x_1'(1)\bar{\phi}(1) - x_1'(0)\bar{\phi}(0). \tag{2.10}$$

Proof.

Suppose that $x_1 \in H^2(0,1)$. By Theorem 2.1.6, there exists a sequence $\{u_n\} \subset C^2[0,1]$ such that

$$\|x_1 - u_n\|_2 \to 0 \quad \text{as} \quad n \to \infty,$$
$$u_n'(0) \to x_1'(0) \quad \text{and} \quad u_n'(1) \to x_1'(1).$$

Since $u_n \in C^2[0,1]$, integration by parts yields

$$\int_0^1 u_n'' \, \bar{\phi} \, dx = -\int_0^1 u_n' \, \bar{\phi}' \, dx + u_n'(1)\bar{\phi}(1) - u_n'(0)\bar{\phi}(0).$$

Taking limits, the result follows by the continuity of the inner product on $L^2(0,1)$. □

Theorem 2.3.2. For the dynamics generator A of Problem CZ94, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$, x_1 is in $H^2(0,1)$ and x satisfies

$$\begin{cases} x_1'' = 2cx_2 + y_2, \\ x_1(0) = x_1(1) = 0. \end{cases} \tag{2.11}$$

Proof.

Suppose $x_1 \in H^2(0,1)$ is a solution of (2.11). By Lemma 2.3.1 and the definitions of $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ (See Subsection 1.6.5):

$$\beta(x_1,\phi) + \alpha(x_2,\phi) = -\gamma(y_2,\phi) \text{ for all } \phi \in T[0,1].$$

Since x_1 satisfies the boundary conditions and is in $C^1[0,1]$ (by Sobolev's lemma), x_1 is a test function. The space of test functions is contained in V so $x_1 \in V$.

The energy space V is the closure of $T[0,1]$ in $H^1(0,1)$ so every $v \in V$ is the limit (in $\|\cdot\|_V$) of a sequence of test functions $\{\phi_n\}$. Now for every $v \in V$:

$$\begin{aligned} & \beta(x_1,v) + \alpha(x_2,v) + \gamma(y_2,v) \\ = \; & \beta(x_1, \lim_{n\to\infty} \phi_n) + \alpha(x_2, \lim_{n\to\infty} \phi_n) + \gamma(y_2, \lim_{n\to\infty} \phi_n). \end{aligned}$$

The sesquilinear form $\beta(\cdot,\cdot)$ is an inner product and $\gamma(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$ are bounded on V with respect to $\|\cdot\|_V$ by Assumptions (A4) and (A5) respectively. Therefore all the forms are continuous on V [Kr, Lemma 3.2-2, p.138].

Since $x_2 = y_1$,

$$\begin{aligned}\beta(x_1, v) + \alpha(x_2, v) + \gamma(y_2, v) &= \lim_{n\to\infty} [\beta(x_1, \phi_n) + \alpha(x_2, \phi_n) + \gamma(y_2, \phi_n)] \\ &= 0 \quad \text{for all } v \in V.\end{aligned}$$

So $Ax = y$ by Theorem 2.1.4.

On the other hand, suppose $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.1.4, $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \ v \in V.$$

By definition of the sesquilinear forms,

$$\int_0^1 x_1'\ \bar{v}'\ dx = -\int_0^1 2cx_2\ \bar{v}\ dx - \int_0^1 y_2\ \bar{v}\ dx \text{ for all } \ v \in V.$$

Since $C_0^\infty[0,1] \subset V$,

$$\int_0^1 x_1'\ \bar{v}'\ dx = \int_0^1 (2cx_2 + y_2)\ \bar{v}\ dx \text{ for all } \ v \in C_0^\infty[0,1].$$

By the definition of the weak derivative, the second order weak derivative of x_1 exists ($x_1 \in H^2(0,1)$) and

$$x_1'' = 2cx_2 + y_2.$$

It remains to show that x_1 satisfies the boundary conditions. By Sobolev's lemma, $x_1 \in C^1[0,1]$ and $x \in \mathcal{D}(A)$ implies that $x_1 \in V$. Therefore, x_1 is the limit of some sequence $\{v_n\}$ of test functions. Thus

$$x_1(0) = \lim_{n\to\infty} v_n(0) = 0 \ \text{ and } \ x_1(1) = \lim_{n\to\infty} v_n(1) = 0,$$

since test functions satisfy the forced boundary conditions. □

Remark Note that x_1'' denotes the second order weak derivative of x_1 in (2.11).

Sufficient condition for existence

The existence of solutions of Problem CZ94 was not considered in [CZ94]. The damping in Problem CZ94 is weak since $\alpha(\cdot,\cdot)$ is defined on W and

$$|\alpha(u, v)| = |\gamma(2cu, v)| \le \|c\|_{\sup} \|u\|_W \|v\|_W .$$

It has already been shown that Assumptions (A1) to (A5) are satisfied so Theorem 1.7.5 is applicable if $u_0 \in V$, $v_0 \in V$ and there exists a $y \in W$ such that $\beta(u_0, v) = \gamma(y, v)$ for all $v \in V$. The first part of the proof of Theorem 2.3.2 shows that this condition is satisfied with $y = -y_2 - 2cy_1$ if $u_0 \in H^2(0,1)$. Therefore, a sufficient condition for existence of a unique weak solution of Problem CZ94 is

$$\langle u_0, v_0 \rangle \in (H^2(0,1) \cap V) \times V.$$

2.3.2 Eigenvalue problem for the dynamics generator

It is now of interest to find the eigenvalues and generalized eigenvectors of the dynamics generator of Problem CZ94 and determine whether the sequence of generalized eigenvectors forms a basis for the state space.

Corollary 2.3.3. Let A be the dynamics generator of Problem CZ94.

$$A \langle u, y \rangle = \lambda \langle u, y \rangle$$

if and only if $u \in H^2[0,1]$, $y = \lambda u$ and u satisfies

$$u'' = \left(\lambda 2c + \lambda^2\right) u,$$
$$u(0) = u(1) = 0.$$

Furthermore, if c is continuous, then $u \in C^2[0,1]$.

Proof.

The equivalence of the eigenvalue problems follows directly from Theorem 2.3.2.

Suppose c is continuous, then due to Theorem 2.3.2, $u \in H^2(0,1)$ and

$$u'' = \left(\lambda 2c + \lambda^2\right) u.$$

By Sobolev's Lemma, $u \in C^1[0,1]$. Thus $u'' \in C[0,1]$, which implies $u \in C^2[0,1]$. □

Constant damping

In the rest of this section we deal with constant viscous damping. The case where c is a function is discussed further in Chapter 7.

For constant damping, Corollary 2.3.3 implies that $u \in C^2[0,1]$ and

$$u'' = \left(\lambda 2c + \lambda^2\right) u.$$

Hence $u'' \in C^2[0,1]$, so $u \in C^4[0,1]$. We may continue in this way indefinitely, so the eigenfunctions of (2.11) are in $C^\infty[0,1]$. Recall that the eigenfunctions were found in Section 1.5 to be

$$\phi_k(x) = \sin(k\pi x), \quad \text{for } k = 1, 2, \ldots$$

and that the eigenvalues may be real or non-real. The non-real eigenvalues of A are

$$\lambda_k = -c + i\omega_k \quad \text{and} \quad \bar{\lambda}_k = -c - i\omega_k,$$

where $\omega_k = \sqrt{k^2\pi^2 - c^2}$. In this case, $\langle \phi_k, \lambda_k \phi_k \rangle$ and $\langle \phi_k, \bar{\lambda}_k \phi_k \rangle$ are eigenvectors of A.

There are two possibilities for the eigenvalue λ_k to be real. If $c^2 > k^2\pi^2$ then

$$\lambda_k^+ = -c + \mu_k \quad \text{and} \quad \lambda_k^- = -c - \mu_k$$

are eigenvalues of A, where $\mu_k = \sqrt{c^2 - k^2\pi^2}$. Consequently $\langle \phi_k, \lambda_k^+ \phi_k \rangle$ and $\langle \phi_k, \lambda_k^- \phi_k \rangle$ are both eigenvectors of A.

In the (somewhat unlikely) case that $c^2 = k^2\pi^2$, the eigenvalue $\lambda_k = -c = -k\pi$ is a double eigenvalue. Then A has an eigenvector $\psi_k = \langle \phi_k, -c\phi_k \rangle$ and a generalized eigenvector $\langle \phi_k, (1-c)\phi_k \rangle$ corresponding to λ_k.

It is easy to check that $\psi_{k,2} = \langle \phi_k, (1-c)\phi_k \rangle$ is an generalized eigenvector of A. We must check that

$$A\psi_{k,2} = \lambda\psi_{k,2} + \psi_k.$$

The left hand side is

$$\begin{aligned} A\psi_{k,2} &= \langle (1-c)\phi_k, \phi_k'' - 2c(1-c)\phi_k \rangle \\ &= \left\langle (1-c)\phi_k, -c^2\phi_k - 2c(1-c)\phi_k \right\rangle \\ &= \left\langle (1-c)\phi_k, (c^2 - 2c)\phi_k \right\rangle. \end{aligned}$$

The right hand side is

$$\begin{aligned} \lambda\psi_{k,2} + \psi_k &= \langle -c\phi_k, -c(1-c)\phi_k \rangle + \langle \phi_k, -c\phi_k \rangle \\ &= \left\langle (1-c)\phi_k, (c^2 - 2c)\phi_k \right\rangle. \end{aligned}$$

Therefore $\psi_{k,2} = \langle \phi_k, (1-c)\phi_k \rangle$ is indeed an generalized eigenvector of A.

2.3.3 Approach taken in [CZ94]

The authors of [CZ94] set out to prove a relationship between the spectrum of the dynamics generator and a parameter in the exponential decay formula. To obtain this result they use the fact that the generalized eigenvectors of the dynamics generator to form a basis for the state space.

The authors write Problem CZ94 in the form of Problem IVPH in the state space $H_0^1(0,1) \times L^2(0,1)$ using an apparently simple formal approach:

$$\begin{cases} w'(t) = Aw(t), \\ w(0) = w_0, \end{cases} \tag{2.12}$$

where $w = \langle u, \partial_t u \rangle$, $w_0 = \langle u_0, v_0 \rangle$ and the dynamics generator is defined by

$$A = \begin{bmatrix} 0 & I \\ \partial_x^2 & -2c \end{bmatrix}. \tag{2.13}$$

The domain of A is restricted to pairs of functions that satisfy the boundary conditions:

$$\mathcal{D}(A) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1),$$

where $H_0^1(0,1)$ denotes the subspace of the Sobolev space $H^1(0,1)$ such that the boundary conditions are satisfied. That is, $u \in H^1(0,1)$ such that $u(0) = u(1) = 0$. A reader without sufficient knowledge of Sobolev spaces will not be able to understand this or the equivalence of the operator and classical forms of the eigenvalue problem. This equivalence is made clear in Theorem 2.3.2 and Corollary 2.3.3.

The definition of A given here may seem simpler than the definition of A in Subsection 1.7.4, until one tries to prove that A has the properties required to apply the theory.

The rest of [CZ94] is discussed in Sections 2.5, 2.9 and also in Chapters 3 and 7.

2.4 Basis of generalized eigenvectors of the wave equation with viscous damping

In this section it is shown that the generalized eigenvectors of the dynamics generator A defined in (2.13) form a basis for the state space H. In [CZ94], the authors consider the case where the damping parameter c is constant and proceed to the case where c is a function of bounded variation. The latter case is dealt with in Chapter 7.

In Subsection 2.3.2, the eigenfunctions for the wave equation with constant viscous damping were found to be

$$\phi_k(x) = \sin k\pi x, \text{ for } k = 1, 2, \ldots$$

Consider any $w = \langle u, v\rangle \in H$. We know that $\{\phi_k\}_{k=1}^{\infty}$ is an orthogonal basis for V. Therefore, any $u \in V$ and any $v \in W$ may be expressed uniquely as

$$u = \sum_{k=1}^{\infty} a_k\phi_k \quad \text{and} \quad v = \sum_{k=1}^{\infty} b_k\phi_k.$$

Denote the partial sums

$$u_N = \sum_{k=1}^{N} a_k\phi_k, \quad v_N = \sum_{k=1}^{N} b_k\phi_k \quad \text{and} \quad w_N = \begin{bmatrix} u_N \\ v_N \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} a_k\phi_k \\ b_k\phi_k \end{bmatrix}.$$

It is shown in Appendix C that $\|u - u_N\|_V \to 0$ and $\|v - v_N\|_W \to 0$ as $N \to \infty$.

Therefore $\|w_N - w\|_H^2 = \|u - u_N\|_V^2 + \|v - v_N\|_W^2 \to 0$ as $N \to \infty$.

We now construct an orthogonal basis $\{g_n\}_1^{\infty}$ for H.

Let $g_1 = \langle \phi_1, 0\rangle\,,\ g_2 = \langle 0, \phi_1\rangle\,,\ g_3 = \langle \phi_2, 0\rangle\,, \ldots$. More precisely,

$$\begin{aligned} g_n &= \langle \phi_k, 0\rangle \quad \text{for} \quad n = 2k-1, \\ \text{and } g_n &= \langle 0, \phi_k\rangle \quad \text{for} \quad n = 2k. \end{aligned}$$

Note that $\{g_n\}_1^{\infty}$ will be orthogonal due to the orthogonality of $\{\phi_k\}_1^{\infty}$.

Let $c_n = a_k$ for $n = 2k - 1$ and $c_n = b_k$ for $n = 2k$. Then

$$w_N = \sum_{k=1}^{N} \begin{bmatrix} a_k\phi_k \\ b_k\phi_k \end{bmatrix} = \sum_{n=1}^{2N} c_n g_n.$$

2.4.1 The undamped case

From Subsection 2.3.2, the eigenvalues and eigenfunctions for the undamped case $(c = 0)$ are

$$\lambda_k = \pm k\pi i \quad \text{and} \quad \phi_k(x) = \sin k\pi x, \text{ for } k = 1, 2, \ldots$$

respectively. For $n = 2k$, let

$$\psi_{n-1} = \begin{bmatrix} \phi_k \\ ik\pi\phi_k \end{bmatrix} \quad \text{and} \quad \psi_n = \begin{bmatrix} \phi_k \\ -ik\pi\phi_k \end{bmatrix}.$$

We express each ψ_k in terms of g_n and vice versa to show that $\{\psi_n\}_1^\infty$ forms a basis for the space H.

For $n = 2k$:

$$\begin{aligned} \psi_{n-1} = \begin{bmatrix} \phi_k \\ ik\pi\phi_k \end{bmatrix} &= \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ ik\pi\phi_k \end{bmatrix} = g_{n-1} + ik\pi g_n \\ \text{and} \quad \psi_n = \begin{bmatrix} \phi_k \\ -ik\pi\phi_k \end{bmatrix} &= \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ ik\pi\phi_k \end{bmatrix} = g_{n-1} - ik\pi g_n. \end{aligned}$$

So

$$\begin{aligned} g_{n-1} &= \frac{1}{2}(\psi_{n-1} + \psi_n), \\ g_n &= \frac{1}{2k\pi i}(\psi_{n-1} - \psi_n). \end{aligned}$$

Consequently,

$$\begin{aligned} c_{n-1}g_{n-1} + c_n g_n &= \frac{c_{n-1}}{2}(\psi_{n-1} + \psi_n) + \frac{c_n}{2k\pi i}(\psi_{n-1} - \psi_n), \\ &= \left(\frac{c_{n-1}}{2} + \frac{c_n}{2k\pi i}\right)\psi_{n-1} + \left(\frac{c_{n-1}}{2} - \frac{c_n}{2k\pi i}\right)\psi_n. \end{aligned}$$

Let $d_{n-1} = \left(\frac{c_{n-1}}{2} + \frac{c_n}{2k\pi i}\right)$ and $d_n = \left(\frac{c_{n-1}}{2} - \frac{c_n}{2k\pi i}\right)$, then

$$w_N = \sum_{n=1}^{2N} c_n g_n = \sum_{n=1}^{2N} d_n \psi_n.$$

Recall that $\|w_N - w\|_H \to 0$ as $N \to \infty$. Therefore,

$$w = \lim_{N\to\infty} \sum_{n=1}^{2N} c_n g_n = \lim_{N\to\infty} \sum_{n=1}^{2N} d_n \psi_n.$$

The representation in terms of $\{\psi_n\}_1^\infty$ is unique since $\{g_n\}_1^\infty$ is a basis for H.

2.4.2 Constant viscous damping

Now consider the case where the damping parameter c is a positive constant.

It was shown in Subsection 2.3.2 that the matrix eigenvalue problem $Aw = \lambda w$ is equivalent to the quadratic eigenvalue problem (with the eigenvectors of A being of the form $\langle \phi_k, \lambda_k^{\pm}\phi_k \rangle$) and that the eigenvalues and eigenvectors of the quadratic eigenvalue problem are

$$\lambda_k = -c \pm \sqrt{c^2 - k^2\pi^2} \quad \text{and} \quad \phi_k(x) = \sin k\pi x, \text{ for } k = 1, 2, \ldots$$

respectively. The form of the generalized eigenvectors depends on c.

As in the undamped case, we show that the generalized eigenvectors $\{\psi_k\}_1^\infty$ of A form a basis for the space $H = V \times W$. This is more complicated when there is damping since the spectrum of A is more complicated:

It is necessary to consider three cases as $\{\psi_k\}_1^\infty$ consists of three qualitatively different parts (see Subsection 2.3.2).

Case I ($k^2\pi^2 < c^2$):

Suppose the eigenvalues are real and distinct for $k = 1, ..., k^* - 1$. Let $n = 2k$, then

$$\begin{aligned} \psi_{n-1} &= \begin{bmatrix} \phi_k \\ \lambda_k^+ \phi_k \end{bmatrix} = g_{n-1} + \lambda_k^+ g_n \\ \text{and} \quad \psi_n &= \begin{bmatrix} \phi_k \\ \lambda_k^- \phi_k \end{bmatrix} = g_{n-1} + \lambda_k^- g_n. \end{aligned}$$

So

$$\begin{aligned} \psi_{n-1} - \psi_n &= 2\mu_k g_n \text{ and} \\ \lambda_k^- \psi_{n-1} - \lambda_k^+ \psi_n &= -2\mu_k g_{n-1}, \end{aligned}$$

since $\lambda_k^+ - \lambda_k^- = 2\mu_k$. Consequently,

$$\begin{aligned} c_{n-1}g_{n-1} + c_n g_n &= \frac{-c_{n-1}}{2\mu_k}(\lambda_k^- \psi_{n-1} - \lambda_k^+ \psi_n) + \frac{c_n}{2\mu_k}(\psi_{n-1} - \psi_n), \\ &= \left(\frac{-c_{n-1}\lambda_k^- + c_n}{2\mu_k} \right) \psi_{n-1} + \left(\frac{\lambda_k^+ c_{n-1} - c_n}{2\mu_k} \right) \psi_n. \end{aligned}$$

Let $d_{n-1} = \dfrac{-c_{n-1}\lambda_k^- + c_n}{2\mu_k}$ and $d_n = \dfrac{\lambda_k^+ c_{n-1} - c_n}{2\mu_k}$. Then

$$\sum_{n=1}^{2k^*-2} c_n g_n = \sum_{n=1}^{2k^*-2} d_n \psi_n. \tag{2.14}$$

Case II ($k^2\pi^2 = c^2$):

Suppose $\lambda_{k^*} = -c$ is the double eigenvalue considered in Subsection 2.3.2. For $n = 2k^*$,

$$\begin{aligned} \psi_{n-1} &= \begin{bmatrix} \phi_{k^*} \\ -c\phi_{k^*} \end{bmatrix} = g_{n-1} - cg_n \\ \text{and} \quad \psi_n &= \begin{bmatrix} \phi_{k^*} \\ (1-c)\phi_{k^*} \end{bmatrix} = g_{n-1} + (1-c)g_n. \end{aligned}$$

Then

$$\begin{aligned} \psi_{n-1} - \psi_n &= -g_n \text{ and} \\ (1-c)\psi_{n-1} + c\psi_n &= g_{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} c_{n-1}g_{n-1} + c_n g_n &= c_{n-1}((1-c)\psi_{n-1} + c\psi_n) + c_n(-\psi_{n-1} + \psi_n), \\ &= (c_{n-1}(1-c) - c_n)\psi_{n-1} + (cc_{n-1} + c_n)\psi_n. \end{aligned}$$

Let $d_{n-1} = c_{n-1}(1-c) - c_n$ and $d_n = cc_{n-1} + c_n$.

Then for $n = 2k^*$,

$$c_{n-1}g_{n-1} + c_n g_n = d_n\psi_n + d_{n-1}\psi_{n-1}. \tag{2.15}$$

Case III ($k^2\pi^2 > c^2$):

We have non-real eigenvalues for $k > k^*$. For $k = k^* + 1, k^* + 2, \ldots$ and $n = 2k$,

$$\begin{aligned} \psi_{n-1} &= \begin{bmatrix} \phi_k \\ \lambda_k\phi_k \end{bmatrix} = g_{n-1} + \lambda_k g_n \\ \text{and} \quad \psi_n &= \begin{bmatrix} \phi_k \\ \bar{\lambda}_k\phi_k \end{bmatrix} = g_{n-1} + \bar{\lambda}_k g_n. \end{aligned}$$

Then

$$\begin{aligned}\psi_{n-1} - \psi_n &= (\lambda_k - \bar{\lambda}_k) g_n \text{ and} \\ \bar{\lambda}_k \psi_{n-1} - \lambda_k \psi_n &= -(\lambda_k - \bar{\lambda}_k) g_{n-1}.\end{aligned}$$

But $\lambda_k - \bar{\lambda}_k = 2\omega_k i$, so

$$\begin{aligned}i(\psi_{n-1} - \psi_n) &= -2\omega_k g_n, \\ i(\bar{\lambda}_k \psi_{n-1} - \lambda_k \psi_n) &= 2\omega_k g_{n-1}.\end{aligned}$$

Now

$$\begin{aligned}c_{n-1} g_{n-1} + c_n g_n &= \frac{ic_{n-1}}{2\omega_k} (\bar{\lambda}_k \psi_{n-1} - \lambda_k \psi_n) - \frac{ic_n}{2\omega_k} (\psi_{n-1} - \psi_n), \\ &= \left(\frac{ic_{n-1}\bar{\lambda}_k - ic_n}{2\omega_k} \right) \psi_{n-1} + \left(\frac{-i\lambda_k c_{n-1} + ic_n}{2\omega_k} \right) \psi_n.\end{aligned}$$

Let

$$d_{n-1} = \frac{ic_{n-1}\bar{\lambda}_k - ic_n}{2\omega_k} \quad \text{and} \quad d_n = \frac{-i\lambda_k c_{n-1} + ic_n}{2\omega_k}. \tag{2.16}$$

Then for $2N > 2k^* + 1$,

$$\sum_{n=2k^*+1}^{2N} c_n g_n = \sum_{n=2k^*+1}^{2N} d_n \psi_n. \tag{2.17}$$

Now for $2N > 2k^* + 1$, we combine (2.14), (2.15) and (2.17) to obtain

$$w_N = \sum_{n=1}^{2N} c_n g_n = \sum_{n=1}^{2N} d_n \psi_n.$$

We know that $\|w_N - w\|_H \to 0$ as $N \to \infty$. Therefore,

$$w = \lim_{N\to\infty} \sum_{n=1}^{2N} c_n g_n = \lim_{N\to\infty} \sum_{n=1}^{2N} d_n \psi_n.$$

The representation in terms of $\{\psi_n\}_1^\infty$ is unique since $\{g_n\}_1^\infty$ is a basis for H. Therefore $\{\psi_n\}_1^\infty$ is a basis for H.

Remark The procedure above may be used to show that the generalized eigenvectors of the dynamics generator for the wave equation with Kelvin-Voigt damping form a basis for H.

2.5 Riesz basis with an application

In [CZ94], the authors prove a result concerning the decay of energy of solutions of (2.12). To do so, they require that the series representation of the solution is valid and that there exists a bounded, invertible linear operator T (with a bounded inverse) that maps the sequence of generalized eigenvectors $\{\psi_n\}_1^\infty$ onto an orthogonal basis $\{g_n\}_1^\infty$. Such a sequence is referred to as a *Riesz basis* (see Section 4.3 for a formal definition).

In Section 2.4 we showed that the sequence $\{\psi_n\}_1^\infty$ of generalized eigenvectors of the dynamics generator of Problem CZ94 is a basis for $H = V \times W$. We now show that $\{\psi_n\}_1^\infty$ is a Riesz basis. The following well-known theorem will be used in the proof.

Theorem 2.5.1 (Bounded Inverse Theorem, [Kr, Theorem 4.12-2])**.** A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. If T is a bijection then T^{-1} is bounded.

Theorem 2.5.2. The sequence $\{\psi_n\}_1^\infty$ of generalized eigenvectors of the dynamics generator of Problem CZ94 is a Riesz basis for $H = V \times W$.

Proof.

Define the operator $T : H \to H$ by

$$Tx = \sum_{n=1}^{\infty} d_n g_n, \quad \text{where} \quad x = \sum_{n=1}^{\infty} d_n \psi_n = \sum_{n=1}^{\infty} c_n g_n.$$

Note that $T\psi_n = g_n$. We first show that T is linear. Let

$$x = \sum_{n=1}^{\infty} \alpha_n \psi_n, \quad y = \sum_{n=1}^{\infty} \beta_n \psi_n \in H$$

and a and b be complex numbers. By the linearity properties of limits and summation,

$$T(ax + by) = T\left(\sum_{n=1}^{\infty} (a\alpha_n + b\beta_n)\psi_n \right) = aTx + bTy.$$

We need to show that T is bounded and has a bounded inverse. Since $\{g_n\}_1^\infty$ is orthogonal,

$$\|x\|^2 = \sum_{n=1}^{\infty} |c_n|^2 \|g_n\|_H^2 \quad \text{and} \quad \|Tx\|^2 = \sum_{n=1}^{\infty} |d_n|^2 \|g_n\|_H^2 \quad \text{for all } x \in H.$$

Consider any $x \in H$ such that $\|x\|_H = 1$. The linear operator T is bounded if $\|Tx\|_H$ is finite, but $\sum_{n=1}^{2k^*} |d_n|^2 \|g_n\|_H^2$ must be finite so we only need to show that

$$\sum_{n=2k^*+1}^{\infty} |d_n|^2 \|g_n\|_H^2 < \infty.$$

By the definition of d_n for $n > 2k^* + 1$ (See (2.16)):

$$\begin{aligned} |d_{n-1}| &\leq \frac{1}{2\omega_k} \left(|c_{n-1}||\lambda_k| + |c_n|\right) \quad \text{and} \\ |d_n| &\leq \frac{1}{2\omega_k} \left(|c_{n-1}||\lambda_k| + |c_n|\right). \end{aligned}$$

If $c_n = 0$ then $|d_{n-1}|^2 + |d_n|^2 \leq \dfrac{|c_{n-1}|^2|\lambda_k|^2}{2\omega_k^2}$.

If $c_{n-1} = 0$ then $|d_{n-1}|^2 + |d_n|^2 \leq \dfrac{|c_n|^2}{2\omega_k^2}$.

For nonzero c_n and c_{n-1} the estimate is more complicated:

$$\begin{aligned} |d_{n-1}|^2 + |d_n|^2 &\leq \frac{1}{2\omega_k^2} \left(|c_n|^2 + 2|c_{n-1}||c_n||\lambda_k| + |c_{n-1}|^2|\lambda_k|^2\right) \\ &= \left(\frac{1}{2\omega_k^2} + \frac{|\lambda_k||c_{n-1}|}{\omega_k^2|c_n|}\right)|c_n|^2 + \left(\frac{|\lambda_k|^2}{2\omega_k^2} + \frac{|\lambda_k||c_n|}{\omega_k^2|c_{n-1}|}\right)|c_{n-1}|^2. \end{aligned}$$

But $\dfrac{|\lambda_k|}{\omega_k} \leq 1 + \dfrac{|c|}{\omega_k}$ and $\dfrac{|\lambda_k|}{\omega_k^2} \leq \dfrac{1}{\omega_k} + \dfrac{|c|}{\omega_k^2}$,

So $|d_{n-1}|^2 + |d_n|^2 \leq r|c_n|^2 + s|c_{n-1}|^2$.

where

$$\begin{aligned} r &= \left(\frac{1}{2\omega_k^2} + \left(1 + \frac{|c|}{\omega_k}\right)\frac{|c_{n-1}|}{|c_n|}\right) \quad \text{and} \\ s &= \left(\frac{1}{2}\left(1 + \frac{|c|}{\omega_k}\right)^2 + \left(\frac{1}{\omega_k} + \frac{|c|}{\omega_k^2}\right)\frac{|c_n|}{|c_{n-1}|}\right). \end{aligned}$$

Let $C_\infty^2 = \max\{r, s\}$. Then

$$\sum_{n=2k^*+1}^{\infty} |d_n|^2 \|g_n\|_H^2 \leq C_\infty^2 \sum_{n=2k^*+1}^{\infty} |c_n|^2 \|g_n\|_H^2 < \infty.$$

Therefore, T is a bounded linear operator.

Now we show that T is injective. If $Tx = 0$, then

$$T\sum_{n=1}^{\infty} d_n\psi_n = \sum_{n=1}^{\infty} d_n g_n = 0.$$

But $\{g_n\}_1^\infty$ is a basis for H, so $d_n = 0$ for each n. Consequently, $x = 0$. By [Kr, Theorem 2.6-10, p.88], T^{-1} exists so T must be a bijection and T^{-1} is bounded by the Bounded Inverse Theorem. $\square$

Recall that dimensionless energy was defined in Subsection 1.2.4 as

$$E(t) = \frac{1}{2}\int_0^1 \ (\partial_t u(x,t))^2 \ dx + \frac{1}{2}\int_0^1 \ (\partial_x u(x,t))^2 \ dx$$

and that $2E(t) = \|\langle u(t), \partial_t u(t)\rangle\|_H^2$.

We know that $E(t)$ is decreasing but there is considerable interest in the exponential decay of energy. That is, whether there exists a $C > 0$ such that:

$$E(t) \leq CE(0)e^{2\alpha t} \ \text{ for some } \alpha < 0. \tag{2.18}$$

The authors of [CZ94] use the fact that $\{\psi_n\}_1^\infty$ is a Riesz basis to prove that the decay rate is equal to the spectral abscissa when the damping parameter c is constant [CZ94, Theorem 2.1 p.217]. Before we proceed, note that (2.18) means that there exists a constant K such that $E(t) \leq Ke^{2\alpha(t)}$.

Definition (Decay rate [CZ94, (1.3)]) The decay rate is defined as a function of the damping parameter c by

$$\omega(c) = \inf\{\alpha \text{ such that (2.18) holds}\}.$$

Definition (Spectral abscissa [CZ94, (1.4)]) The *spectral abscissa* of the dynamics generator A (given by 2.13) is defined as a function of the damping parameter c by

$$\mu(c) = \sup \{\text{Re } \lambda \mid \lambda \in \sigma(A)\} .$$

Theorem 2.5.3. If the damping parameter c is a constant then there exists a constant K such that $E(t) \leq Ke^{2\mu t}$.

Proof.

We have shown in Section 2.4 that $\{\psi_n\}_1^\infty$ is a basis for H, so we may express any initial condition $w_0 = \langle u_0, v_0 \rangle \in H$ as

$$w_0 = \sum_{n=1}^{\infty} d_n \psi_n .$$

Let us first consider the simpler case where $c^2 \neq k^2\pi^2$ for any natural number k. Then the generalized eigenvectors of A are all eigenvectors and we may expand the solution of the initial value problem (2.12) as

$$w(t) = \langle u(t), v(t) \rangle = \sum_{n=1}^{\infty} e^{\lambda_n t} d_n \psi_n, \tag{2.19}$$

where the eigenvalues have been arranged in sequence by

$$\{\lambda_n\} = \left\{\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \ldots, \lambda_{2k^*+1}, \bar{\lambda}_{2k^*+1}, \lambda_{2k^*+2}, \bar{\lambda}_{2k^*+2}, \ldots\right\} .$$

Note that for constant viscous damping, either $\mu = \mu(c) = -c$ or $\mu = \mu(c) = -c + \sqrt{c^2 - \pi^2}$ where $c > \pi$. Therefore μ is a maximum in this case.

Now, using the orthogonality of the basis $\{g_n\}_1^\infty$,

$$\begin{aligned}
2E(t) &= \|\langle u(t), v(t)\rangle\|_H^2 \\
&= \left\| \sum_{n=1}^{\infty} e^{\lambda_n t} d_n \psi_n \right\|_H^2 \\
&= \left\| T^{-1} \sum_{n=1}^{\infty} e^{\lambda_n t} d_n g_n \right\|_H^2 \\
&\leq \|T^{-1}\|_H^2 \sum_{n=1}^{\infty} \left|e^{\lambda_n t}\right|^2 |d_n|^2 \, \|g_n\|_H^2 \\
&\leq \|T^{-1}\|_H^2 \; e^{2\mu t} \sum_{n=1}^{\infty} |d_n|^2 \, \|g_n\|_H^2 \\
&= \|T^{-1}\|_H^2 \; e^{2\mu t} \left\| \sum_{n=1}^{\infty} d_n g_n \right\|_H^2 \\
&= \|T^{-1}\|_H^2 \; e^{2\mu t} \left\| T \sum_{n=1}^{\infty} d_n \psi_n \right\|_H^2 \\
&\leq \|T^{-1}\|_H^2 \, \|T\|_H^2 \; e^{2\mu t} \left\| \sum_{n=1}^{\infty} d_n \psi_n \right\|_H^2 \\
&\leq \|T^{-1}\|_H^2 \, \|T\|_H^2 \; e^{2\mu t} \; \|\langle u_0, v_0\rangle\|_H^2 \\
&= 2 \left\|T^{-1}\right\|_H^2 \|T\|_H^2 \, E(0) \; e^{2\mu t}. \qquad (2.20)
\end{aligned}$$

There is a complication if $c^2 = (k^*\pi)^2$ for some natural number k^*. Then there is an eigenvector and a generalized eigenvector corresponding to $\lambda_{k^*} = -c$, see Section 1.5. The expansion 2.19 must be modified to:

$$w(t) = \langle u(t), v(t)\rangle = te^{-ct} d_{2k^*-1}\psi_{2k^*-1} + \sum_{n=1}^{\infty} e^{\lambda_n t} d_n \psi_n, \qquad (2.21)$$

where the eigenvalues have been arranged in sequence by

$$\{\lambda_n\} = \left\{\lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \ldots, \lambda_{2k^*-1}, \lambda_{2k^*}, \lambda_{2k^*+1}, \bar{\lambda}_{2k^*+1}, \ldots\right\}.$$

Again, using the orthogonality of the basis $\{g_n\}_1^\infty$,

$$\begin{aligned}
2E(t) &= \|\langle u(t), v(t)\rangle\|_H^2 \\
&= \left\| te^{-ct} d_{2k^*-1}\psi_{2k^*-1} + \sum_{n=1}^{\infty} e^{\lambda_n t} d_n \psi_n \right\|_H^2 \\
&= \left\| T^{-1}\left(te^{-ct} d_{2k^*-1} g_{2k^*-1} + \sum_{n=1}^{\infty} e^{\lambda_n t} d_n g_n \right) \right\|_H^2 \\
&\leq \|T^{-1}\|_H^2 \left(t^2 e^{-2ct} |d_{2k^*-1}|^2 \, \|g_{2k^*-1}\|_H^2 + \sum_{n=1}^{\infty} |e^{\lambda_n t}|^2 |d_n|^2 \, \|g_n\|_H^2 \right)
\end{aligned}$$

Let $K_0 = \|T^{-1}\|_H^2 \, |d_{2k^*-1}|^2 \, \|g_{2k^*-1}\|_H^2$, then

$$\begin{aligned}
2E(t) &= K_0 t^2 e^{-2ct} + \left\|T^{-1}\right\|_H^2 e^{2\mu t} \sum_{n=1}^{\infty} |d_n|^2 \, \|g_n\|_H^2 \\
&= K_0 t^2 e^{-2ct} + \left\|T^{-1}\right\|_H^2 e^{2\mu t} \left\| \sum_{n=1}^{\infty} d_n g_n \right\|_H^2 \\
&= K_0 t^2 e^{-2ct} + \left\|T^{-1}\right\|_H^2 \; e^{2\mu t} \left\| T \sum_{n=1}^{\infty} d_n \psi_n \right\|_H^2 \\
&\leq K_0 t^2 e^{-2ct} + \left\|T^{-1}\right\|_H^2 \|T\|_H^2 \; e^{2\mu t} \left\| \sum_{n=1}^{\infty} d_n \psi_n \right\|_H^2 \\
&\leq K_0 t^2 e^{-2ct} + \left\|T^{-1}\right\|_H^2 \|T\|_H^2 \; e^{2\mu t} \; \|\langle u_0, v_0\rangle\|_H^2 \\
&= K_0 t^2 e^{-2ct} + 2\left\|T^{-1}\right\|_H^2 \|T\|_H^2 \, E(0) \; e^{2\mu t}.
\end{aligned}$$

If $k^* > 1$ then $\mu = -c + \sqrt{c^2 - \pi^2}$, so $-c < \mu < 0$. Consequently, there exists a constant K_1 such that

$$t^2 e^{-2ct} \leq K_1 e^{2\mu t}. \tag{2.22}$$

Then

$$2E(t) \leq \left(K_1 E(0)^{-1} + 2\left\|T^{-1}\right\|_H^2 \|T\|_H^2 \right) E(0) e^{2\mu t}.$$

If $k^* = 1$ then $\mu = -c$ so this can't be done, but we may choose a real ξ such that $\mu < \xi < 0$. Then there exists a constant K_2 such that

$$t^2 e^{-2ct} \leq K_2 e^{2\xi t}.$$

Then, since $\mu < \xi$,

$$2E(t) \leq \left(K_2 E(0)^{-1} + 2\left\|T^{-1}\right\|_H^2 \|T\|_H^2 \right) E(0) e^{2\xi t}. \qquad \square$$

Remarks

1. If λ_1 is not a repeated eigenvalue then μ belongs to the set such that (2.18) holds. In the exceptional case that λ_1 is a repeated eigenvalue then ξ belongs to the set such that (2.18) holds. But ξ may be chosen as close to μ as one pleases, so μ belongs to the set such that (2.18) holds in this case as well.

2. In [CZ94], the authors claim that "it follows easily that" $\mu(a) \leq \omega(a)$. They do not motivate this or provide a reference.

3. The authors of [CZ94] remark that if $k^* = 1$ then $\omega(c)$ doesn't exist. More clearly, this means that the infimum in the definition of $\omega(c)$ is not attained and is therefore not a minimum.

4. The authors of [CZ94] claim that fact that the generalized eigenvectors $\{\psi_n\}_1^\infty$ form a Riesz basis for the state space follows from the Fredholm Alternative and a theorem from an appendix in a book on inverse spectral theory. Note that we *constructed* a Riesz basis of generalized eigenvectors for H by a direct calculation. This will not work when c is not constant−the abstract theory of Chapter 4 will be needed to prove that the generalized eigenvectors of A form a Riesz basis for H. This is discussed in Chapter 7.

2.6 Wave equation with boundary damping

In this section we consider the wave equation with boundary damping. The problem formulated in Section 1.2 is from [VLV10]. A similar problem was treated in [CZ95].

2.6.1 One end fixed

Consider the wave equation with one end fixed and boundary damping (1.11) at the other end.

Problem VLV10

$$\begin{aligned}
\partial_t^2 u - \partial_x^2 u &= 0, \quad 0 < x < 1,\, t > 0 \\
u(0,t) &= 0, \quad t > 0, \\
\partial_x u(1,t) + k\partial_t u(1,t) &= 0, \quad t > 0, \\
u(x,0) &= u_0(x), \\
\partial_t u(x,0) &= v_0(x).
\end{aligned}$$

It is not difficult to see that the bilinear forms defined in Subsection 1.6.5 satisfy Assumptions (A1) to (A5): (A1) holds since the norm on W is equivalent to the $L^2(0,1)$ norm. (A3) holds since $C_0^\infty[0,1]$ is dense in $L^2(0,1)$. Proposition C.1.1 in Appendix C.1 ensures that (A2), (A4) and (A5) hold.

Equivalence of problems

We show that the result of Theorem 2.1.4 can be extended for the dynamics generator A of Problem VLV10.

Theorem 2.6.1. For the dynamics generator A of Problem VLV10, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$, x_1 is in $H^2(0,1)$ and x satisfies

$$\left\{ \begin{array}{c} x_1'' = y_2, \\ x_1(0) = 0, \\ x_1'(1) + kx_2(1) = 0. \end{array} \right. \tag{2.23}$$

Proof.

Suppose $x_1 \in H^2(0,1)$ is a solution of (2.23). By Lemma 2.3.1 and the definitions of $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ (See Subsection 1.6.5):

$$\beta(x_1,\phi) + \alpha(x_2,\phi) = -\gamma(y_2,\phi) \text{ for all } \phi \in T[0,1].$$

Since x_1 satisfies the boundary conditions and is in $C^1[0,1]$ (by Sobolev's lemma), x_1 is a test function. The space of test functions is contained in V so $x_1 \in V$.

The energy space V is the closure of $T[0,1]$ in $H^1(0,1)$ so every $v \in V$ is the limit (in $\|\cdot\|_V$) of a sequence of test functions $\{\phi_n\}$. Now for every $v \in V$:

$$\begin{aligned} & \beta(x_1,v) + \alpha(x_2,v) + \gamma(y_2,v) \\ = \quad & \beta(x_1, \lim_{n\to\infty} \phi_n) + \alpha(x_2, \lim_{n\to\infty} \phi_n) + \gamma(y_2, \lim_{n\to\infty} \phi_n). \end{aligned}$$

The sesquilinear forms $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are inner products and $\alpha(\cdot,\cdot)$ is bounded on V with respect to $\|\cdot\|_V$ by Assumption (A5). Therefore all the forms are continuous on V [Kr, Lemma 3.2-2, p.138].

Since $x_2 = y_1$,

$$\begin{aligned} \beta(x_1,v) + \alpha(x_2,v) + \gamma(y_2,v) &= \lim_{n\to\infty} [\beta(x_1,\phi_n) + \alpha(x_2,\phi_n) + \gamma(y_2,\phi_n)] \\ &= 0 \quad \text{for all } v \in V. \end{aligned}$$

So $Ax = y$ by Theorem 2.1.4.

On the other hand, suppose $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.1.4, $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \ v \in V.$$

By definition of the sesquilinear forms,

$$\int_0^1 x_1' \, \bar{v}' \, dx + kx_2(1)\bar{v}(1) \, dx + \int_0^1 y_2 \, \bar{v} = 0 \text{ for all } \ v \in V. \tag{2.24}$$

Since $C_0^\infty[0,1] \subset V$,

$$\int_0^1 x_1' \, \bar{v}' \, dx = \int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \ v \in C_0^\infty[0,1].$$

The definition of the weak derivative gives $(x_1')' = y_2$ so the second order weak derivative of x_1 exists ($x_1 \in H^2(0,1)$) and

$$x_1'' = y_2.$$

It remains to show that x_1 satisfies the boundary conditions. By Sobolev's lemma, $x_1 \in C^1[0,1]$ and $x \in \mathcal{D}(A)$ implies that $x_1 \in V$. Therefore, x_1 is the limit of some sequence $\{v_n\}$ of test functions. Thus

$$x_1(0) = \lim_{n\to\infty} v_n(0) = 0,$$

since test functions satisfy the forced boundary conditions.

To show that the other boundary condition is satisfied, we note that

$$\int_0^1 x_1' \, \bar{v}' \, dx = -\int_0^1 x_1' \, \bar{v}' + x_1'(1)\bar{v}(1) \text{ for all } \ v \in V.$$

Since $x_1'' = y_2$,

$$\beta(x_1', \bar{v}') - x_1'(1)\bar{v}(1) + \gamma(y_2, \bar{v}') = 0 \text{ for all } \ v \in V.$$

Comparing this with (2.24) we see that

$$kx_1(1)\bar{v}(1) = -x_1'(1)\bar{v}(1) \text{ for all } \ v \in V.$$

Therefore $x_1'(1) + kx_2(1) = 0.$ $\square$

Remark Note that x_1'' denotes the second order weak derivative of x_1 in (2.11).

Sufficient conditions for existence

The existence of a unique weak solution of Problem VLV10 will follow from Theorem 1.7.4. It has already been shown that Assumptions (A1) to (A5) are satisfied and it was shown in the first part of the proof of Theorem 2.6.1 that if $x_1 \in H^2(0,1)$ and $x_1'(1) + kx_2(1) = 0$ then

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } v \in V.$$

By Theorem 1.7.4, sufficient conditions for the existence of a unique weak solution of Problem VLV10 are

$$\langle u_0, v_0 \rangle \in (H^2(0,1) \cap V) \times V \quad \text{and} \quad x_1'(1) + kx_2(1) = 0.$$

Eigenvalue problem for the dynamics generator

Even more can be done for the eigenvalue problem for the dynamics generator A of Problem VLV10.

Corollary 2.6.2. Let A be the dynamics generator of Problem VLV10.

$$A \langle u, y \rangle = \lambda \langle u, y \rangle$$

if and only if $u \in C^\infty[0,1]$, $y = \lambda u$ and u satisfies

$$\begin{aligned} u'' &= \lambda^2 u, \\ u(0) &= 0, \\ u'(1) + ku(1) &= 0. \end{aligned}$$

Proof.

The result follows directly from Theorem 2.6.1, except that it must be shown that $u \in C^\infty[0,1]$.

Due to Theorem 2.6.1, $u \in H^2(0,1)$ and

$$u'' = \lambda^2 u.$$

By Sobolev's Lemma, $u \in C^1[0,1]$, so $u'' \in C^1[0,1]$, which implies $u \in C^3[0,1]$. We may continue in this way indefinitely, so the eigenvectors of (2.23) are in $C^\infty[0,1]$. □

Corollary 2.6.2 implies that the eigenfunctions are precisely those of the quadratic eigenvalue problem (1.31).

2.6.2 Free string

Consider the wave equation with one end free and boundary damping at the other end.

Problem CZ95

$$
\begin{aligned}
\rho\partial_t^2 u - \partial_x^2 u &= 0, \quad 0 < x < 1,\, t > 0\\
\partial_x u(0,t) &= 0, \quad t > 0,\\
\partial_x u(1,t) + \partial_t u(1,t) &= 0, \quad t > 0,\\
u(x,0) &= u_0(x),\\
\partial_t u(x,0) &= v_0(x).
\end{aligned}
$$

Note that ρ is a function. It is assumed in [CZ95] that ρ is a non-negative function of bounded variation.

Remark It is interesting that all the authors encountered in this study refer to the vibrating string. It is hard to imagine how one can have a tightly stretched string with these boundary conditions. Even if some device can be constructed to make this possible, the two endpoints may have different velocities. This results in an unrealistic model (see Section 1.2). On the other hand, the model is not unrealistic for longitudinal vibrations of a bar. Note that, in general, the bar will undergo translational motion in the direction of the axis as well as vibration.

Equivalence of problems

The sesquilinear form $\beta(\cdot,\cdot)$ is the same as for Problem VLV10. For $\alpha(\cdot,\cdot)$, set $k = 1$ and let

$$\gamma(u,v) = \int_0^1 \rho(x)u(x)v(x)\; dx.$$

These changes have no effect on the proof of the equivalence of problems given for Problem VLV10 and the changes in the boundary conditions necessitate only minor changes.

Existence of solution

Since neither endpoint is fixed, the inequality $\beta(u,u) \geq C\gamma(u,u)$ does not hold. Hence Assumptions (A2) and (A4) are not valid. However, the initial-boundary value problem still has a unique solution, see Appendix C.3.

Invertibility of the dynamics generator

The dynamics generator of Problem CZ95 has $\lambda = 0$ as an eigenvalue and is not invertible. The authors of [CZ95] work around this problem by considering a subspace H_0 of the state space H:

$$H_0 = \left\{ \langle u, v \rangle \in H \mid u(1) + \int_0^1 \rho(x)v(x)\, dx = 0 \right\}.$$

Now suppose $\beta(u, u) + \gamma(v, v) = 0$ for some $\langle u, v\rangle \in H_0$. Then $v = 0$ and consequently $u(1) = 0$. Since both $\beta(u, u)$ and $u(1)$ vanish, $u = 0$. Hence $\|\langle u, v\rangle\|_H^2 = \beta(u, u) + \gamma(v, v)$ defines a norm on H_0.

Remark If $\rho = 1$ then the eigenvalues and eigenvectors of the dynamics generator of Problem CZ95 do not exist. The proof given in [CZ95] is similar to the one given in Subsection 1.5.3.

2.7 Euler Bernoulli beam with boundary control

In [GY01], the initial value problem consisting of the Euler Bernoulli beam fixed at $x = 0$ with boundary control at $x = 1$ and the initial conditions (1.12) and (1.13) is considered.

Problem GY01

$$\begin{aligned}
\partial_t^2 u + \partial_x^4 u &= 0,\ 0 < x < 1, t > 0 \\
u(0,t) = \partial_x u(0,t) &= 0,\ t > 0, \\
\partial_x^2 u(1,t) &= -k_1 \partial_t \partial_x u(1,t),\ t > 0, \\
\partial_x^3 u(1,t) &= k_2 \partial_t u(1,t), t > 0, \\
u(x,0) &= u_0(x), \\
\partial_t u(x,0) &= v_0(x).
\end{aligned}$$

It is not difficult to see that the bilinear forms defined in Subsection 1.6.5 satisfy Assumptions (A1) to (A5):

(A1) holds since the norm on W is equivalent to the $L^2(0,1)$ norm. (A3) holds since $C_0^\infty[0,1]$ is dense in $L^2(0,1)$. (A2) and (A4) hold since the norm on V is equivalent to the $H^2(0,1)$ norm. Proposition C.1.1 and the Poincaré inequality in Appendix C.1 ensure that (A5) holds.

2.7.1 Equivalence of problems

Lemma 2.7.1. For any $x_1 \in H^4(0,1)$ and $\phi \in C^2[0,1]$,

$$\begin{aligned}\int_0^1 x_1^{(4)}\,\bar{\phi}\,dx = \int_0^1 & x_1''\,\bar{\phi}''\,dx \\ & + x_1'''(1)\bar{\phi}(1) - x_1'''(0)\bar{\phi}(0) \\ & - x_1''(1)\bar{\phi}'(1) + x_1''(0)\bar{\phi}'(0)\end{aligned} \tag{2.25}$$

Proof.

Suppose that $x_1 \in H^4(0,1)$. By Theorem 2.1.6, there exists a sequence $\{u_n\} \subset C^4[0,1]$ such that

$$\begin{aligned}&\|x_1 - u_n\|_4 \to 0 \quad \text{as} \quad n \to \infty, \\ &u_n''(0) \to x_1''(0), \quad u_n''(1) \to x_1''(1), \\ &u_n'''(0) \to x_1'''(0) \quad \text{and} \quad u_n'''(1) \to x_1'''(1).\end{aligned}$$

Since $u_n \in C^4[0,1]$, integration by parts yields

$$\begin{aligned}\int_0^1 u_n^{(4)}\,\bar{\phi}\,dx &= \int_0^1 u_n''\,\bar{\phi}''\,dx \\ & \quad + u_n'''(1)\bar{\phi}(1) - u_n'''(0)\bar{\phi}(0) \\ & \quad - u_n''(1)\bar{\phi}'(1) + u_n''(0)\bar{\phi}'(0).\end{aligned}$$

Taking limits, the result follows by the continuity of the inner product on $L^2(0,1)$. □

Theorem 2.7.2. For the dynamics generator A of Problem GY01, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$, x_1 is in $H^4(0,1)$ and x satisfies

$$\begin{cases} x_1^{(4)} = -y_2, \\ x_1(0) = x_1'(0) = 0, \\ x_1''(1) + k_1 x_2'(1) = 0, \\ x_1'''(1) - k_2 x_2(1) = 0. \end{cases} \tag{2.26}$$

Proof.

Suppose $x_1 \in H^4(0,1)$ is a solution of (2.26). By Lemma 2.7.1 and the definitions of $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ (See Subsection 1.6.5):

$$\beta(x_1, \phi) + \alpha(x_2, \phi) = -\gamma(y_2, \phi) \text{ for all } \phi \in T[0,1].$$

Since x_1 satisfies the boundary conditions and is in $C^3[0,1]$ (by Sobolev's lemma), x_1 is a test function. The space of test functions is contained in V so $x_1 \in V$.

The energy space V is the closure of $T[0,1]$ in $H^2(0,1)$ so every $v \in V$ is the limit (in $\|\cdot\|_V$) of a sequence of test functions $\{\phi_n\}$. Now for every $v \in V$:

$$\begin{aligned} & \beta(x_1, v) + \alpha(x_2, v) + \gamma(y_2, v) \\ = \; & \beta(x_1, \lim_{n\to\infty} \phi_n) + \alpha(x_2, \lim_{n\to\infty} \phi_n) + \gamma(y_2, \lim_{n\to\infty} \phi_n). \end{aligned}$$

The sesquilinear forms $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are inner products and $\alpha(\cdot,\cdot)$ is bounded on V with respect to $\|\cdot\|_V$ by Assumption (A5). Therefore all the forms are continuous on V [Kr, Lemma 3.2-2, p.138].

Since $x_2 = y_1$,

$$\begin{aligned} \beta(x_1, v) + \alpha(x_2, v) + \gamma(y_2, v) &= \lim_{n\to\infty} [\beta(x_1, \phi_n) + \alpha(x_2, \phi_n) + \gamma(y_2, \phi_n)] \\ &= 0 \quad \text{for all } v \in V. \end{aligned}$$

So $Ax = y$ by Theorem 2.1.4.

On the other hand, suppose $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.1.4, $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \; v \in V.$$

By definition of the sesquilinear forms,

$$\int_0^1 x_1'' \, \bar{v}'' \, dx + k_2 x_2(1)\bar{v}(1) + k_1 x_2'(1)\bar{v}'(1) = -\int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in V. \tag{2.27}$$

Since $C_0^\infty[0,1] \subset V$,

$$\int_0^1 x_1'' \, \bar{v}'' \, dx = -\int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in C_0^\infty[0,1].$$

The definition of the weak derivative gives

$$\int_0^1 x_1^{(4)} \, \bar{v} \, dx = -\int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in C_0^\infty[0,1].$$

Hence the fourth order weak derivative of x_1 exists ($x_1 \in H^4(0,1)$) and

$$x_1^{(4)} = -y_2.$$

It remains to show that x_1 satisfies the boundary conditions. By Sobolev's lemma, $x_1 \in C^3[0,1]$ and $x \in \mathcal{D}(A)$ implies that $x_1 \in V$. Therefore, x_1 is the limit of some sequence $\{v_n\}$ of test functions. Thus

$$\begin{aligned} x_1(0) &= \lim_{n\to\infty} v_n(0) &= 0, \\ x_1'(0) &= \lim_{n\to\infty} v_n'(0) &= 0, \end{aligned}$$

since test functions satisfy the forced boundary conditions.

To show that x_1 satisfies the boundary conditions at $x = 1$, multiply the differential equation above by an arbitrary test function v and integrate by parts twice:

$$\int_0^1 x_1''\ \bar{v}''\ dx + x_1'''(1)\bar{v}(1) - x_1''(1)\bar{v}'(1) = -\int_0^1 y_2\ \bar{v}\ dx. \tag{2.28}$$

Comparing (2.27) and (2.28), we see that

$$(x_1'''(1) - k_2 x_2(1))\bar{v}(1) - (x_1''(1) + k_1 x_2'(1))\bar{v}'(1) = 0 \quad \text{for all } v \in T[0,1].$$

Choosing $v(x) \in T[0,1]$ such that $v'(1) = 0$ and $v(1) \neq 0$, we see that $x_1'''(1) = k_2 x_2(1)$. Then choosing $v(x) = x$ gives $x_1''(1) = -k_1 x_2'(1)$, so the boundary conditions are satisfied. □

Remark Note that $x_1^{(4)}$ denotes the fourth order weak derivative of x_1 in (2.29).

Sufficient condition for existence

The existence of solutions of Problem GY01 was not considered in [GY01]. Existence of a unique weak solution of Problem GY01 will follow from Theorem 1.7.4. It has already been shown that Assumptions (A1) to (A5) are satisfied and it was shown in the first part of the proof of Theorem 2.7.2 that if $x_1 \in H^4(0,1)$, $x_1''(1) = -k_1 x_2'(1)$ and $x_1'''(1) = k_2 x_2(1)$, then

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } v \in V.$$

By Theorem 1.7.4, sufficient conditions for the existence of a unique weak solution of Problem GY01 are

$$\begin{aligned} \langle u_0, v_0\rangle &\in (H^2(0,1)\cap V) \times V, \\ x_1''(1) &= -k_1 x_2'(1), \\ \text{and } \ x_1'''(1) &= k_2 x_2(1). \end{aligned}$$

2.7.2 Eigenvalue problem for the dynamics generator

Even more can be done for the eigenvalue problem for the dynamics generator A of Problem GY01.

Corollary 2.7.3. Let A be the dynamics generator of Problem GY01.

$$A\langle u, y\rangle = \lambda \langle u, y\rangle$$

if and only if $u \in C^\infty[0,1]$, $y = \lambda u$ and u satisfies

$$\begin{aligned} u^{(4)} &= -\lambda^2 y_2, \\ u(0) &= 0, \\ u''(1) + k_1 u'(1) &= 0, \\ u'''(1) - k_2 u(1) &= 0. \end{aligned}$$

Proof.

The result follows directly from Theorem 2.7.2, except that it must be shown that $u \in C^\infty[0,1]$.

Due to Theorem 2.7.2, $u \in H^4(0,1)$ and

$$u^{(4)} = -\lambda^2 u.$$

By Sobolev's Lemma, $u \in C^1[0,1]$, so $u'' \in C^1[0,1]$, which implies $u \in C^3[0,1]$. We may continue in this way indefinitely, so the eigenvectors of (2.26) are in $C^\infty[0,1]$. $\square$

Remark The quadratic eigenvalue problem for the Euler-Bernoulli beam with cantilever boundary conditions is considered in Section 1.4.2.

2.8 Euler-Bernoulli beam with Kelvin-Voigt damping

In [JTW08], the Euler-Bernoulli beam with Kelvin-Voigt damping and cantilever boundary conditions is considered (see also Section 1.4).

Problem JTW08

$$\begin{aligned}
\partial_t^2 w + c^2\partial_x^4 w - \partial_x^2(k\partial_t\partial_x^2 w) &= 0, \quad 0 < x < 1,\, t > 0\\
w(0,t) = \partial_x w(0,t) &= 0, \quad t > 0,\\
\partial_x^2 w(\ell,t) = \partial_x^3 w(\ell,t) &= 0, \quad t > 0,\\
u(x,0) &= u_0(x),\\
\partial_t u(x,0) &= v_0(x).
\end{aligned}$$

Note that k is a non-negative real-valued function. In [JTW08] it is assumed that k is a strictly positive bounded function. That is, k attains a positive minimum on $[0,1]$.

It is not difficult to see that the bilinear forms defined in Subsection 1.6.5 satisfy Assumptions (A1) to (A5): (A1) holds since the norm on W is equivalent to the $L^2(0,1)$ norm. (A3) holds since $C_0^\infty[0,1]$ is dense in $L^2(0,1)$. (A2), (A4) and (A5) hold since the norm on V is equivalent to the $H^2(0,1)$ norm and k is bounded.

2.8.1 Equivalence of problems

We show that the result of Theorem 2.1.4 can be extended for the dynamics generator A of Problem JTW08 if $k \in H^2(0,1)$.

Theorem 2.8.1. For the dynamics generator A of Problem JTW08, $x \in \mathcal{D}(A)$ and $Ax = y$ if and only if $x_2 = y_1$, x_1 is in $H^4(0,1)$ and x satisfies

$$\left\{\begin{aligned}
x_1^{(4)} &= -(kx_2'')'' - y_2,\\
x_1(0) &= x_1'(0) = 0,\\
x_1''(1) &= x_1'''(1) = 0.
\end{aligned}\right. \tag{2.29}$$

Proof.

Suppose $x_1 \in H^4(0,1)$ is a solution of (2.29). By Lemma 2.3.1, Lemma 2.7.1 and the definitions of $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ (See Subsection 1.6.5):

$$\beta(x_1,\phi) + \alpha(x_2,\phi) = -\gamma(y_2,\phi) \text{ for all } \phi \in T[0,1].$$

Since x_1 satisfies the boundary conditions and is in $C^3[0,1]$ (by Sobolev's lemma), x_1 is a test function. The space of test functions is contained in V so $x_1 \in V$.

The energy space V is the closure of $T[0,1]$ in $H^2(0,1)$ so every $v \in V$ is the limit (in $\|\cdot\|_V$) of a sequence of test functions $\{\phi_n\}$. Now for every $v \in V$:

$$\begin{aligned} & \beta(x_1, v) + \alpha(x_2, v) + \gamma(y_2, v) \\ = \; & \beta(x_1, \lim_{n\to\infty} \phi_n) + \alpha(x_2, \lim_{n\to\infty} \phi_n) + \gamma(y_2, \lim_{n\to\infty} \phi_n). \end{aligned}$$

The sesquilinear forms $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are inner products and $\alpha(\cdot,\cdot)$ is bounded on V with respect to $\|\cdot\|_V$ by Assumption (A5). Therefore all the forms are continuous on V [Kr, Lemma 3.2-2, p.138].

Since $x_2 = y_1$,

$$\begin{aligned} \beta(x_1, v) + \alpha(x_2, v) + \gamma(y_2, v) &= \lim_{n\to\infty} [\beta(x_1, \phi_n) + \alpha(x_2, \phi_n) + \gamma(y_2, \phi_n)] \\ &= 0 \quad \text{for all } v \in V. \end{aligned}$$

So $Ax = y$ by Theorem 2.1.4.

On the other hand, suppose $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.1.4, $x_2 = y_1$ and

$$\beta(x_1, v) + \alpha(x_2, v) = -\gamma(y_2, v) \text{ for all } \; v \in V.$$

By definition of the sesquilinear forms,

$$\int_0^1 x_1'' \, \bar{v}'' \, dx = -\int_0^1 kx_2'' \, \bar{v}'' \, dx - \int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in V. \tag{2.30}$$

Since $C_0^\infty[0,1] \subset V$,

$$\int_0^1 x_1'' \, \bar{v}'' \, dx = -\int_0^1 kx_2'' \, \bar{v}'' \, dx - \int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in C_0^\infty[0,1].$$

The definition of the weak derivative gives

$$\int_0^1 x_1^{(4)} \, \bar{v} \, dx = -\int_0^1 (kx_2'')'' \, \bar{v} \, dx - \int_0^1 y_2 \, \bar{v} \, dx \text{ for all } \; v \in C_0^\infty[0,1].$$

Since $x_2 = y_1$ and y_1 and k are in $H^2(0,1)$, the fourth order weak derivative of x_1 exists ($x_1 \in H^4(0,1)$) and

$$x_1^{(4)} = -(kx_2'')'' - y_2.$$

It remains to show that x_1 satisfies the boundary conditions. By Sobolev's lemma, $x_1 \in C^3[0,1]$ and $x \in \mathcal{D}(A)$ implies that $x_1 \in V$. Therefore, x_1 is the limit of some sequence $\{v_n\}$ of test functions. Thus

$$\begin{aligned} x_1(0) &= \lim_{n\to\infty} v_n(0) = \lim_{n\to\infty} 0 &= 0, \\ x_1'(0) &= \lim_{n\to\infty} v_n'(0) = \lim_{n\to\infty} 0 &= 0, \end{aligned}$$

since test functions satisfy the forced boundary conditions.

To show that x_1 satisfies the natural boundary conditions at $x = 1$, multiply the differential equation above by an arbitrary test function v and integrate by parts twice:

$$\begin{aligned} \int_0^1 x_1'' \, \bar{v}'' \, dx = &- \int_0^1 kx_2'' \, \bar{v}'' \, dx - \int_0^1 y_2 \, \bar{v} \, dx \\ &- x_1'''(1)\bar{v}(1) + x_1''(1)\bar{v}'(1). \end{aligned} \tag{2.31}$$

Comparing (2.30) and (2.31), we see that

$$x_1'''(1)\bar{v}(1) - x_1''(1)\bar{v}'(1) = 0 \quad \text{for all } v \in T[0,1].$$

Choosing $v(x) \in T[0,1]$ such that $v'(1) = 0$ and $v(1) \neq 0$, we see that $x_1'''(1) = 0$. Then choosing $v(x) = x$ gives $x_1''(1) = -k_1 x_2'(1)$, so the boundary conditions are satisfied. □

Remark Note that $x_1^{(4)}$ denotes the fourth order weak derivative of x_1 in (2.29).

Sufficient condition for existence

The existence of solutions of Problem JTW08 was not considered in [JTW08]. The damping parameter in this problem is strictly positive so

$$\alpha(u,u) = \int_0^1 k(x)(u''(x))^2 \, dx \geq \inf_{x\in(0,1)} |k(x)| \, \|u\|_V \, .$$

Therefore, the damping is strong. It has already been shown that Assumptions (A1) to (A5) are satisfied so Theorem 1.7.6 is applicable. Therefore, a sufficient condition for existence of a unique weak solution of Problem JTW08 is simply

$$\langle u_0, v_0 \rangle \in V \times W.$$

Even more can be done for the eigenvalue problem for the dynamics generator A of Problem JTW08.

2.8.2 Eigenvalue problem for the dynamics generator

Corollary 2.8.2. Let A be the dynamics generator of Problem JTW08.

$$A\langle u, y\rangle = \lambda\langle u, y\rangle$$

if and only if $u \in H^4(0,1)$, $y = \lambda u$ and

$$\begin{aligned} u^{(4)} &= -(ku'')'' - \lambda u, \\ u(0) &= u'(0) = 0, \\ u''(1) &= u'''(1) = 0. \end{aligned}$$

Proof.

The result follows directly from Theorem 2.8.1. □

Remark Due to Theorem 2.8.1 and Sobolev's lemma, $u \in C^3[0,1]$. If k is constant, then

$$(1+k)u^{(4)} = -\lambda u.$$

Therefore $u^{(4)} \in C^3[0,1]$, which implies $u \in C^7[0,1]$. We may continue in this way indefinitely, so the eigenvectors of (2.11) are in $C^\infty[0,1]$.

2.9 Compactness

It is very useful to know if the inverse of the dynamics generator is compact as the spectral theory of compact operators is well-developed (See Chapter 3).

Definition (Compact linear operator [Kr, p.405]) Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called a *compact linear operator* if for every bounded subset M of X , the image $T(M)$ is relatively compact. That is, the closure $\overline{T(M)}$ is compact in Y.

The embedding of V in W is said to be compact if a bounded set S in V is pre-compact in W. That is, the closure of S is compact in W.

The following theorems will be used repeatedly to show that the operator Λ is compact.

Theorem 2.9.1 (Rellich's Theorem [Ad, Theorem 6.2, p.144] **).** For any non-negative integer m, the embedding of $H^{m+1}(a,b)$ into $H^m(a,b)$ is compact.

Remark Rellich's Theorem also gives the compactness of the embedding of any Sobolev space $H^m(a,b)$ in $L^2(a,b)$ if we note that $H^0(a,b) = L^2(a,b)$.

Theorem 2.9.2. Let X, Y, U and Z be Hilbert spaces. Suppose X is compactly embedded the space U and Y is compactly embedded in Z. Then the product space $X \times Y$ is compactly embedded in the product space $U \times Z$.

Proof.

Denote the norm of each space by a subscript with its name, i.e. $\|\cdot\|_U$ is the norm on U.

Let $\{\langle x_n, y_n\rangle\}_1^\infty$ be an arbitrary sequence in $X \times Y$. Since the embedding of X in U is compact, there exists a subsequence $\{\langle x_{n_k}, y_{n_k}\rangle\}_1^\infty$ of $\{\langle x_n, y_n\rangle\}_1^\infty$ such that

$$\|x_{n_k} - x_0\|_U \to 0 \quad \text{as } n \to \infty,$$

for some $x_0 \in X$. Re-index this subsequence as $\{\langle x_k, y_k\rangle\}_1^\infty$. Since the embedding of Y in Z is compact, there exists a subsequence $\left\{\left\langle x_{k_j}, y_{k_j}\right\rangle\right\}_1^\infty$ of $\{\langle x_k, y_k\rangle\}_1^\infty$ such that

$$\left\|y_{k_j} - y_0\right\|_Z \to 0 \quad \text{as } n \to \infty,$$

for some $y_0 \in X$. The sequence $\{x_k\}$ converges to x_0, so the subsequence $\left\{x_{k_j}\right\}$ must also converge to x_0. Therefore, the subsequence $\left\{\left\langle x_{k_j}, y_{k_j}\right\rangle\right\}_1^\infty$ of $\{\langle x_n, y_n\rangle\}_1^\infty$ converges to $\langle x_0, y_0\rangle$ in $U \times Z$, since

$$\left\|\left\langle x_{k_j} - x_0, y_{k_j} - y_0\right\rangle\right\|_{U\times Z}^2 = \left\|x_{k_j} - x_0\right\|_U^2 + \left\|x_{k_j} - y_0\right\|_Z^2 \to 0 \quad \text{as } n \to \infty.$$

The sequence $\{\langle x_n, y_n\rangle\}_1^\infty$ was chosen arbitrarily in $X \times Y$, so this implies that $X \times Y$ is compactly embedded in $U \times Z$. $\square$

2.9.1 General case

Suppose that assumption (A6) holds: The embedding of V in W is compact. In all our applications this assumption will be satisfied by the energy space and inertia space due to Rellich's Theorem (Theorem 2.9.1). Consider the dynamics generator A

Lemma 2.9.3. For $x \in \mathcal{D}(A)$, suppose $Ax = y$, then $x_2 = y_1$ and

$$\|x_1\|_V \le C_\alpha \sqrt{\alpha(y_1, y_1)} + \|y_2\|_W . \tag{2.32}$$

***Proof*.**

Let $v = x_1$ and use Theorem 2.1.4:

$$\begin{aligned}
\|x_1\|_V^2 &= \beta(x_1, x_1) \\
&= -\alpha(x_2, x_1) - \gamma(y_2, x_1) \\
&\leq \sqrt{\alpha(x_2, x_2)}\sqrt{\alpha(x_1, x_1)} + \|y_2\|_W \|x_1\|_W \\
&\leq C_{\alpha 0}\sqrt{\alpha(y_1, y_1)}\sqrt{\alpha(x_1, x_1)} + \|y_2\|_W \|x_1\|_V \\
&\leq C_\alpha \sqrt{\alpha(y_1, y_1)}\, \|x_1\|_V + \|y_2\|_W \|x_1\|_V \,,
\end{aligned}$$

where we have used Assumption (A5) and the Cauchy-Schwarz inequality. Note that even though $\alpha(\cdot, \cdot)$ is not an inner product, the Cauchy-Schwarz inequality is still valid. The proof of the Cauchy-Schwarz inequality in [Kr, p.137] holds verbatim for $\alpha(\cdot, \cdot)$ as it does not make use of the positive definiteness of the quadratic form of $\alpha(\cdot, \cdot)$. □

2.9.2 Wave equation with viscous damping

Consider the dynamics generator A of Problem CZ94. Note that we do not assume that c is constant.

Theorem 2.9.4. The inverse Λ of the dynamics generator A is compact.

***Proof*.**

In this case $\alpha(\cdot, \cdot)$ is defined on W and satisfies

$$|\alpha(w, w)| \leq K_\alpha^2 \|w\|_W^2 \text{ for all } w \in W. \tag{2.33}$$

Let $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.3.2, x_1 is in $H^2(0,1)$, $x_2 = y_1$ and x_1 satisfies

$$x_1'' = 2cy_1 + y_2.$$

Hence

$$\|x_2\|_V = \|y_1\|_V \tag{2.34}$$

and

$$\|x_1''\|_W^2 \leq 2\|c\|_{\sup} \|y_1\|_W^2 + \|y_2\|_W^2 \,. \tag{2.35}$$

Note that by the definition of the Sobolev space norms,

$$\|x_1\|_2^2 = \|x_1''\|_0^2 + \|x_1\|_1^2 \tag{2.36}$$

and recall that the norm on the energy space is equivalent to the norm on $H^1(0,1)$ and the norm on the inertia space is equivalent to the norm on $H^0(0,1)$. Combining (2.35) and (2.36), we have

$$\|x_1\|_2^2 \leq 2\,\|c\|_{\sup}\,\|y_1\|_0^2 + \|y_2\|_0^2 + \|x_1\|_1^2\,.$$

Applying Lemma 2.9.3 we have

$$\|x_1\|_2^2 \leq 2\,\|c\|_{\sup}\,\|y_1\|_0^2 + \|y_2\|_0^2 + (C_\alpha\sqrt{\alpha(y_1,y_1)} + \|y_2\|_W)^2.$$

Now applying (2.33)

$$\|x_1\|_2^2 \leq 2\,\|c\|_{\sup}\,\|y_1\|_0^2 + \|y_2\|_0^2 + (C_\alpha K_\alpha\,\|y_1\|_0 + \|y_2\|_W)^2. \tag{2.37}$$

Now suppose $Y \subset H$ is a bounded set such that $\|y\|_H \leq K$ for all $y \in Y$. Let $x = \Lambda y$ for $y \in Y$. Then by (2.34),

$$\|x_2\|_1 = \|y_1\|_1 \leq K$$

and by (2.37)

$$\|x_1\|_2^2 \leq C^2\,\|c\|_{\sup}\,\|y_1\|_0^2 + \|y_2\|_0^2 + (CC_\alpha K_\alpha\,\|y_1\|_V + C\,\|y_2\|_W)^2 \leq K_0.$$

So x_1 is bounded in $H^2(0,1)$ and x_2 is bounded in $H^1(0,1)$. Thus $\Lambda(Y)$ is bounded in $H^2(0,1) \times H^1(0,1)$. By Rellich's Theorem (Theorem 2.9.1), $H^2(0,1)$ is compactly embedded in $H^1(0,1)$ and $H^1(0,1)$ is compactly embedded in $H^0(0,1)$. Therefore, the embedding of $H^2(0,1)\times H^1(0,1)$ in $V\times W$ takes the set $\Lambda(Y)$ which is bounded in $H^2(0,1)\times H^1(0,1)$ onto a pre-compact set in $V \times W$. By the equivalence of norms and Lemma 2.9.2, Λ is compact in $V \times W$. □

Remark In [CZ94, p.219], it is claimed without reference that the compactness of A^{-1} "follows easily" from the Green's operator.

2.9.3 Wave equation with boundary damping

Consider the dynamics generator A of Problem VLV10.

Theorem 2.9.5. The inverse Λ of the dynamics generator A is compact.

Proof.

In this case $\alpha(\cdot,\cdot)$ satisfies

$$\alpha(v,v) = k(y(1))^2 \leq K_\alpha\,\|v\|_V^2 \;\text{ for all } v \in V. \tag{2.38}$$

This follows from Proposition C.1.1 in Appendix C.

Let $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.6.1, x_1 is in $H^2(0,1)$, $x_2 = y_1$ and x_1 satisfies

$$x_1'' = y_2.$$

Hence

$$\|x_2\|_V = \|y_1\|_V \tag{2.39}$$

and

$$\|x_1''\|_W^2 \leq \|y_2\|_W^2 . \tag{2.40}$$

Note that by the definition of the Sobolev space norms,

$$\|x_1\|_2^2 = \|x_1''\|_0^2 + \|x_1\|_1^2 \tag{2.41}$$

and recall that the norm on the energy space is equivalent to the norm on $H^1(0,1)$ and the norm on the inertia space is equivalent to the norm on $H^0(0,1)$. Combining (2.40) and (2.41), we have

$$\|x_1\|_2^2 \leq \|y_2\|_0^2 + \|x_1\|_1^2 .$$

Applying Lemma 2.9.3 we have

$$\|x_1\|_2^2 \leq \|y_2\|_0^2 + (C_\alpha \sqrt{\alpha(y_1,y_1)} + C \|y_2\|_W)^2.$$

Now (2.38) implies

$$\|x_1\|_2^2 \leq \|y_2\|_0^2 + (C_\alpha K_\alpha \|y_1\|_V^2 + C \|y_2\|_W)^2. \tag{2.42}$$

Now suppose $Y \subset H$ is a bounded set such that $\|y\|_H \leq K$ for all $y \in Y$. Let $x = \Lambda y$ for $y \in Y$. Then by (2.34),

$$\|x_2\|_1 = \|y_1\|_1 \leq K$$

and by (2.42)

$$\|x_1\|_2^2 \leq \|y_2\|_0^2 + (C_\alpha K_\alpha \|y_1\|_V^2 + C \|y_2\|_W)^2 \leq K_0.$$

So x_1 is bounded in $H^2(0,1)$ and x_2 is bounded in $H^1(0,1)$. Thus $\Lambda(Y)$ is bounded in $H^2(0,1) \times H^1(0,1)$. By Rellich's Theorem (Theorem 2.9.1), $H^2(0,1)$ is compactly embedded in $H^1(0,1)$ and $H^1(0,1)$ is compactly embedded in $H^0(0,1)$. Therefore, the embedding of $H^2(0,1) \times H^1(0,1)$ in $V \times W$ takes the set $\Lambda(Y)$ which is bounded in $H^2(0,1) \times H^1(0,1)$ onto a pre-compact set in $V \times W$. By the equivalence of norms and Lemma 2.9.2, Λ is compact in $V \times W$. □

Remark The proof of the compactness of A^{-1} for Problem CZ95 in [CZ95, p.551] is incorrect. The authors merely state: "From the boundedness of ρ and the compactness of the imbedding of $H^1(0,1)$ into $L^2(0,1)$ follows the compactness of A^{-1} on V."

2.9.4 Euler-Bernoulli beam with boundary control

Consider the dynamics generator A of Problem GY01.

Theorem 2.9.6. The inverse Λ of the dynamics generator A is compact.

Proof.

In this case $\alpha(\cdot,\cdot)$ satisfies

$$\alpha(v,v) = k_2(v(1))^2 + k_1(v'(1))^2 \leq K_\alpha \|v\|_V^2 \text{ for all } v \in V. \tag{2.43}$$

This follows from Proposition C.1.1 in Appendix C.

Let $x \in \mathcal{D}(A)$ and $Ax = y$. By Theorem 2.7.2, x_1 is in $H^4(0,1)$, $x_2 = y_1$ and x_1 satisfies

$$x_1^{(4)} = -y_2.$$

Hence

$$\|x_2\|_V = \|y_1\|_V \tag{2.44}$$

and

$$\left\|x_1^{(4)}\right\|_W^2 \leq \|y_2\|_W^2. \tag{2.45}$$

Note that by the definition of the Sobolev space norms,

$$\|x_1\|_4^2 = \|x_1\|_2^2 + \|x_1'''\|_0^2 + \left\|x_1^{(4)}\right\|_0^2 \tag{2.46}$$

and recall that the norm on the energy space is equivalent to the norm on $H^2(0,1)$ and the norm on the inertia space is equivalent to the norm on $H^0(0,1)$. Combining (2.45) and (2.46), we have

$$\|x_1\|_4^2 \leq \|x_1\|_2^2 + \|y_2\|_0^2 + \|x_1'''\|_0^2. \tag{2.47}$$

The term $\|x_1'''\|_0^2$ is somewhat troublesome, so we deal with it first. By (C.1),

$$\|x_1'''\|_0 \leq \sup |u'''(x)|$$

and by Proposition C.1.1,

$$|x_1'''(x)| \leq \left\|x_1^{(4)}\right\|_0 + |x_1'''(1)| \text{ for all } x \in [0,1].$$

Applying the boundary condition at $x = 1$, we therefore have

$$\|x_1'''\|_0^2 \leq (\left\|x_1^{(4)}\right\|_0 + |k_2 x_1(1)|)^2.$$

Now applying Proposition C.1.1 again,

$$\|x_1'''\|_0^2 \leq (\left\|x_1^{(4)}\right\|_0 + \|k_2 x_1\|)^2.$$

Combining this with (2.47),

$$\|x_1\|_4^2 \leq \|x_1\|_2^2 + \|y_2\|_0^2 + (\|y_2\|_0 + \|k_2 x_1\|)^2.$$

Applying Lemma 2.9.3 we have

$$\begin{aligned}\|x_1\|_4^2 \quad &\leq \quad \|y_2\|_0^2 + (C_\alpha\sqrt{\alpha(y_1,y_1)} + C\,\|y_2\|_W)^2 \\ &\qquad +(\|y_2\|_0 + k_2 C C_\alpha\sqrt{\alpha(y_1,y_1)})^2.\end{aligned}$$

Now (2.43) implies

$$\begin{aligned}\|x_1\|_4^2 \leq \|y_2\|_0^2 &+ (C_\alpha K_\alpha\,\|y_1\|_V + C\,\|y_2\|_W)^2 \\ &+ (\|y_2\|_0 + k_2 C C_\alpha K_\alpha\,\|y_1\|_V + k_2 C\,\|y_2\|_W)^2. \qquad (2.48)\end{aligned}$$

Now suppose $Y \subset H$ is a bounded set such that $\|y\|_H \leq K$ for all $y \in Y$. Let $x = \Lambda y$ for $y \in Y$. Then by (2.34),

$$\|x_2\|_1 = \|y_1\|_1 \leq K$$

and by (2.48)

$$\|x_1\|_4^2 \leq K_0.$$

So x_1 is bounded in $H^4(0,1)$ and x_2 is bounded in $H^1(0,1)$. Thus $\Lambda(Y)$ is bounded in $H^4(0,1) \times H^1(0,1)$. By Rellich's Theorem (Theorem 2.9.1), $H^4(0,1)$ is compactly embedded in $H^2(0,1)$ and $H^1(0,1)$ is compactly embedded in $H^0(0,1)$. Therefore, the embedding of $H^4(0,1)\times H^1(0,1)$ in $V\times W$ takes the set $\Lambda(Y)$ which is bounded in $H^4(0,1)\times H^1(0,1)$ onto a pre-compact set in $V \times W$. By the equivalence of norms and Lemma 2.9.2, Λ is compact in $V \times W$. □

2.10 Solutions of linear systems

In this section we derive the representation (2.4) for the partial sums of the formal series solution. For completeness, we give the following definitions before we proceed.

Definition (Eigenvalue, eigenvector, generalized eigenvector) Let A be a closed linear operator on a Hilbert space H. The *eigenvalues* of the operator A are the complex numbers λ such that the equation

$$(\lambda I - A)\,x = 0$$

has at least one nonzero solution, $x \in H$, called an *eigenvector*. A nonzero $x \in H$ is called a *generalized eigenvector* of A if there exists a natural number n such that

$$(\lambda I - A)^n x = 0.$$

For any linear operator L on a finite dimensional vector space,

$$e^{tL} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k$$

and $e^{tL}x_0$ is the solution of

$$x' = Lx \quad \text{with} \quad x(0) = x_0.$$

Now suppose that A is a linear operator on the complex Hilbert space H. Consider the eigenvalues $\{\lambda_j\}_1^s$ of A. Denote the (algebraic) multiplicity of each λ_j as m_j and the generalized eigenvectors corresponding to λ_j as v_i^j, for $i = 1, 2, \ldots, m_j$.

Denote the span of all the generalized eigenvectors corresponding to $\{\lambda_j\}_1^s$ by H_1. (Note that the restriction of A to H_1 may be represented by a matrix once a basis is chosen for H_1.) The general solution of $w' = Aw$ is then

$$w(t) = \sum_{j=1}^{s} x_j(t),$$

where each $x_j(t)$ is of the form

$$x(t) = e^{tA} \left[\sum_{i=1}^{m} c_i v_i \right],$$

where $c_i = c_i^j$ are arbitrary complex numbers and the index j is suppressed in the equation above. Note that

$$\begin{aligned} e^{tA} &= e^{\lambda t} e^{t(A - \lambda I)} \\ &= e^{\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} (A - \lambda I)^k . \end{aligned}$$

Since the multiplicity of λ_j is m_j,

$$(A - \lambda_j I)^k v_i^j = 0 \quad \text{for } k \geq m_j.$$

Consequently, each $x_j(t)$ is of the form

$$\begin{aligned}
x(t) &= e^{\lambda t}\sum_{k=0}^{m-1}\frac{t^k}{k!}(A-\lambda I)^k\left[\sum_{i=1}^{m}c_i v_i\right] \\
&= e^{\lambda t}\sum_{i=1}^{m}c_i\sum_{k=0}^{m-1}\frac{t^k}{k!}(A-\lambda I)^k v_i \\
&= e^{\lambda t}\sum_{i=1}^{m}c_i\sum_{k=0}^{m-1}\frac{t^k}{k!}\sum_{r=1}^{m}d_r^k v_r \\
&= e^{\lambda t}\sum_{r=1}^{m}\sum_{k=0}^{m-1}\left[\sum_{i=1}^{m}c_i\frac{t^k}{k!}d_r^k\right]v_r \\
&= e^{\lambda t}\sum_{r=1}^{m}\left[\sum_{k=0}^{m-1}b_r^k\frac{t^k}{k!}\right]v_r \\
&= e^{\lambda t}\sum_{r=1}^{m}q_r(t)v_r,
\end{aligned}$$

where

$$q_r(t)=\sum_{k=0}^{m-1}b_r^k\frac{t^k}{k!}.$$

The general solution of $w' = Aw$ is now

$$w(t)=\sum_{j=1}^{s}e^{\lambda_j t}\sum_{r=1}^{m_j}q_r^j(t)v_r^j.$$

Let $p_r^j(t) = e^{\lambda_j t}q_j(t)$ for $j = 1, 2, \ldots$, then the general solution of the system is

$$w(t)=\sum_{j=1}^{s}\sum_{r=1}^{m_j}p_r^j(t)v_r^j. \tag{2.49}$$

We may change the indices to have

$$p_\nu = p_r^j \quad\text{and}\quad v_\nu = v_r^j \quad\text{with}\quad \nu = 1, 2, \ldots M, \quad\text{where } M=\sum_{j=1}^{s}m_j.$$

It follows that the representation (2.49) is the same as (2.4).

Chapter 3

Completeness

3.1 Spectral results for the symmetric case

The proof of the validity of the formal series solution of the wave equation with constant viscous damping in Subsection 1.2.4 can be generalized to second order abstract differential equations in the undamped case. The general linear vibration problem can be written in the form $u'' = Qu$ (See Section 2.2 for the definition of Q). It is only necessary to show that the eigenvectors of Q form a basis.

Recall from Subsection 1.7.3 that the eigenvalue problem in the symmetric case is to find eigenvectors $u \in V$ and eigenvalues λ such that

$$\beta(u, v) = \lambda\gamma(u, v) \text{ for all } v \in V. \tag{3.1}$$

Many results from the theory of Fourier series can be generalized to orthogonal series for the symmetric case without complication. These generalizations may be found in textbooks on functional analysis such as [Kr]. Theorems 3.1.2 and 3.1.3 summarize the results relevant to our discussion.

A sequence $\{x_n\}$ in an inner product space X is said to be *orthogonal* with respect to the inner product $(\cdot, \cdot)$ if

$$(x_n, x_m) = 0 \quad \text{for } n \neq m.$$

If the sequence $\{x_n\}$ is orthogonal and $\|x_n\| = 1$ for each natural number n then $\{x_n\}$ is called an *orthonormal sequence*.

Separable Hilbert spaces are defined in [Kr]. Due to [Kr, Theorem 3.6-4, p.171], we have the following characterization of a separable Hilbert space:

A Hilbert space H is *separable* if there exists an orthonormal sequence $\{e_n\}_1^\infty$ such that for each $x \in H$ there exists a sequence $\{c_n\}_1^\infty$ of complex numbers such that

$$x = \sum_{n=1}^{\infty} c_n e_n.$$

The sequence $\{e_n\}_1^\infty$ is said to be *complete* in H.

Note that a complete orthogonal sequence in a separable Hilbert space is a basis for that space since each coefficient c_n is uniquely determined by $c_n = (x, e_n)$. The following example shows that the orthogonality of the complete sequence is essential for this to be true.

Example

Consider the sequence $\phi_k(x) = \sin(k\pi x)$ in $L^2(0,1)$. It is well-known that $\{\phi_k\}_1^\infty$ is an orthogonal basis for $L^2(0,1)$. Now define $\phi_0 = 1$. The sequence $\{\phi_k\}_0^\infty$ is complete in $L^2(0,1)$, but for an arbitrary $x \in L^2(0,1)$ we have

$$x = \sum_{k=1}^{\infty} c_k \phi_k \quad \text{and} \quad x - \phi_0 = \sum_{k=1}^{\infty} d_k \phi_k.$$

So x may be expressed as

$$x = \sum_{k=1}^{\infty} c_k \phi_k \quad \text{and} \quad x = \sum_{k=1}^{\infty} d_k \phi_k + \phi_0.$$

Therefore, the representation of x in terms of $\{\phi_k\}_0^\infty$ is not unique so $\{\phi_k\}_0^\infty$ is not a basis. $\triangle$

We only consider separable Hilbert spaces because the Sobolev spaces are separable [Ad, Theorem 3.5, p.47].

For an orthonormal sequence $\{x_n\}_1^\infty$ in a separable Hilbert space H, we have

$$\sum_{n=1}^{\infty} |(x, x_n)| \leq \|x\|^2 \text{ for all } x \in X.$$

This inequality is known as **Bessel's inequality** and the inner products (x, x_n) are the Fourier coefficients of x with respect to $\{x_n\}_1^\infty$ [Kr, p.157].

The following well-known theorem offers characterizations of complete sequences in a separable Hilbert space. The theorem is very useful as it can be extremely difficult to prove the completeness of the sequence of eigenvectors in applications.

Theorem 3.1.1 (Completeness criteria, [Kr, p.170-1])**.** Let $\{\psi_j\}_1^\infty$ be a sequence in a separable Hilbert space H. The following statements are equivalent:

(i) $\{\psi_j\}_1^\infty$ is complete in H.

(ii) If $\{\psi_j\}_1^\infty$ is orthonormal, the Parseval relation,

$$\sum_{j=1}^{\infty} |(x, \psi_j)|^2 = \|x\|^2 \tag{3.2}$$

holds for every $x \in H$.

(iii) If $\{\psi_j\}_1^\infty$ is orthonormal, then $x \perp \{\psi_j\}_1^\infty$ implies that $x = 0$.

We now turn our attention to spectral results for the symmetric case.

Theorem 3.1.2. The eigenvalues and eigenvectors of (3.1) have the following properties:

(i) The eigenvalues are real and positive.

(ii) The eigenvectors are orthogonal in the inertia space W with respect to $\gamma(\cdot, \cdot)$.

Proof.

For λ to be an eigenvalue of (3.1), we must have

$$\beta(u, v) = \lambda \gamma(u, v) \text{ for all } v \in V.$$

The eigenvector $u \in V$, so taking $v = u$ we have

$$\lambda \|u\|_W^2 = \|u\|_V^2 .$$

Therefore the eigenvalues are positive (and real).

Suppose λ and μ are distinct eigenvalues corresponding to eigenvectors u and v respectively. It follows that

$$\lambda \gamma(u, v) = \beta(u, v) = \mu \gamma(u, v).$$

Therefore $(\lambda - \mu)\gamma(u, v) = 0$ and consequently $\gamma(u, v) = 0$. $\square$

Definition (Symmetric linear operator [Kr, p.533]) Let $T : \mathcal{D}(T) \to X$ be a linear operator which is densely defined in a Hilbert space X. Then T is called a *symmetric linear operator* if

$$(Tx, y) = (x, Ty) \text{ for all } x \in H_1, y \in H_2 \text{ for all } x, y \in \mathcal{D}(T).$$

Theorem 3.1.3. If the embedding of V in W is compact then the eigenvalues and eigenvectors of (3.1) have the following properties:

(i) The eigenvalues can be ordered as a sequence $\{\lambda_n\}$ that tends to ∞ as $n \to \infty$.

(ii) The set of eigenvectors can be ordered as a sequence that is complete in W.

Proof.

Consider the linear operator P defined by (2.5). We show that the eigenvalues of P are related to those of (3.1) and that the eigenvectors of P and (3.1) are the same. Suppose u is an eigenvector of (3.1) corresponding to the eigenvalue λ, then

$$\lambda\gamma(u, v) = \beta(u, v) = \gamma(-P^{-1}u, v) \text{ for all } v \in V.$$

This is possible if and only if $Pu = -\lambda^{-1}u$.

The bilinear forms $\beta(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ are symmetric, so for any $f, g \in W$,

$$\gamma(g, Pf) = -\beta(Pg, Pf) = -\beta(Pf, Pg) = \gamma(f, Pg) = \gamma(Pg, f).$$

Therefore, P is symmetric. Now P is a bounded operator from the inertia space W into the energy space V, since

$$\begin{aligned} \|Pf\|_V^2 &= |\beta(Pf, Pf)| \\ &= |\gamma(f, Pf)| \\ &\leq \|f\|_W \|Pf\|_W \\ &\leq C \|f\|_W \|Pf\|_V . \end{aligned}$$

Hence $\|Pf\|_V \leq C \|f\|_W$ for all $f \in W$.

Therefore, if a set A is bounded in the inertia space W, then the image of the set, $P(A)$, is bounded in the energy space V. Consequently, the closure $\overline{P(A)}$ is compact in W due to the compactness of the embedding of V in W.

Therefore the operator P is compact.

The results now follow from the theory of compact symmetric linear operators on a separable Hilbert space, see e.g. [Ze, Theorem 4.A, p.232]: The set of eigenfunctions of P is complete in W and the eigenvalues of P can be ordered as a sequence $\{\lambda_n\}$ that tends to 0 as $n \to \infty$. □

3.2 General spectral theory of compact linear operators

In the symmetric case the completeness of the eigenvectors of (3.1) follows from the compactness of the symmetric operator P. It is natural to investigate the spectral properties of compact operators and consider whether similar results hold in the general case. First, we need the definition of a complete set – in contrast to that of a complete sequence.

Definition (Complete set [Kr, p.168]) A complete set in a normed space X is a subset $M \subset X$ whose span is dense in X. That is $\overline{\text{sp}}M = X$

Definition (Generalized eigenspace [GK, p.5]) Let $sp(A)$ be the subspace spanned by the generalized eigenvectors of A. This space is referred to as the *generalized eigenspace* or *root subspace* of A. The subspace spanned by the generalized eigenvectors of A corresponding to a particular eigenvalue λ of A is called the generalized eigenspace of A corresponding to λ and is denoted $sp_\lambda(A)$. The closure of $sp(A)$ is denoted $\overline{sp}(A)$ and the closure of $sp_\lambda(A)$ is denoted $\overline{sp}_\lambda(A)$.

The spectra of compact linear operators (and linear operators with compact inverses) have many useful properties which are summarized in [Kr, p.420]. These properties are very important and warrant reproduction here.

Definition (Regular value, resolvent operator, resolvent set, spectrum [Kr, p.370]) Let $X \neq \{0\}$ be a normed space. Let $T : \mathcal{D}(T) \to X$ be a linear operator with domain $\mathcal{D}(T)$. A *regular value* λ of T is a complex number such that

(i) $R_\lambda(T) = (\lambda I - T)^{-1}$ exists,

(ii) $R_\lambda(T)$ is bounded,

(iii) $R_\lambda(T)$ is defined on a dense subset of X.

The set of all regular values of T is called the *resolvent set* of T and is denoted by $\rho(T)$. $R_\lambda(T)$ is called the *resolvent operator* of T. The complement of the resolvent is called the *spectrum* of T and is denoted by $\sigma(T) = \mathbb{C} - \rho(T)$. A complex number $\lambda \in \sigma(T)$ is called a *spectral value* of T.

Theorem 3.2.1 ([Kr, Theorem 3.7-2, p.376])**.** The resolvent set of a bounded linear operator T on a Banach space X is open. Hence the spectrum of T is closed.

Definition (Algebraic and geometric multiplicity [GK, p.5]) The *algebraic multiplicity* of an eigenvalue λ of A is the dimension of the space consisting of the zero vector and the generalized eigenvectors corresponding to λ. The *geometric multiplicity* of an eigenvalue λ of A is the dimension of the eigenspace E_λ, which consists of the zero vector and all the eigenvectors associated with λ. An eigenvalue is called *algebraically/geometrically simple* if its algebraic/geometric multiplicity is one.

Remarks

1. In the finite dimensional case, the algebraic multiplicity of an eigenvalue λ of A is its multiplicity as a root of the characteristic equation of A.
2. The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.

Theorem 3.2.2 (Properties of compact linear operators, [Kr, p.420])**.** A compact linear operator $T : X \to X$ on a normed space X has the following properties:

(i) Every spectral value $\lambda \neq 0$ is an eigenvalue. If X is infinite dimensional then the eigenvalues form a sequence that tends to $0 \in \sigma(T)$.

(ii) The set of eigenvalues of T is countable (perhaps even finite or empty).

(iii) $\lambda = 0$ is the only possible point of accumulation of that set.

(iv) The generalized eigenspace corresponding to any eigenvalue of T is finite dimensional.

Theorem 3.2.3 ([DS, p.905])**.** The eigenvectors of a compact normal operator in a Hilbert H space form an orthonormal basis for H.

Definition (Volterra operator [GK, p.16]) An operator A is called a *Volterra operator* if it is compact and has no nonzero eigenvalues.

Lemma 3.2.4 ([GK, Lemma 4.2, p.17])**.** Let A be a compact operator on a Hilbert space H for which $\overline{sp}(A) \neq H$. Let $\mathbb{P}_A$ be the orthogonal projection of H onto $\overline{sp}(A)^{\perp}$. Then $\mathbb{P}_A A \mathbb{P}_A$ and its adjoint $\mathbb{P}_A A^* \mathbb{P}_A$ are Volterra operators.

Remark It is interesting to note that many of the properties of compact symmetric operators on a separable Hilbert space (Theorem 3.1.3) carry through to the more general case of a compact operator on a normed space (Theorem 3.2.2). However, proving the completeness of the set of generalized eigenvectors is a daunting problem.

3.3 Invariant subspaces

Definition (Invariant subspace [Kr, p.374]) A subspace Y of a normed space X is said to be *invariant* under a linear operator $T : X \to X$ if $T(Y) \subset Y$.

Remark Let A be a compact linear operator. The generalized eigenspace of A corresponding to any particular eigenvalue λ (denoted $\overline{sp}_{\lambda}(A)$) is a finite dimensional invariant subspace of the operator A.

Lemma 3.3.1. Let $S \subset H$ be an invariant subspace of H under the operator B. Then $S^{\perp}$ is an invariant subspace under the operator B^*.

Proof.

Let $a \in S$ and $p \in S^{\perp}$ be arbitrary. By definition of an invariant subspace, $Bs \in S$, so $(B^*p, s) = (p, Bs) = 0$. This implies that $B^*p \perp S$. Therefore $B^*p \in S^{\perp}$ and $S^{\perp}$ is invariant under the operator B^*. □

Lemma 3.3.2. Let A be a densely defined bijective linear operator on a Hilbert space H. Suppose A has a bounded inverse and denote $B = A^{-1}$, then B^* is one-to-one.

Proof.

Suppose $B^*x = B^*y$. Then

$$\begin{aligned} (B^*x, z) &= (B^*y, z) \text{ for all } z \in H, \\ \text{so } (x, Bz) &= (y, Bz) \text{ for all } z \in H \\ \text{and } (x - y, Bz) &= 0 \text{ for all } z \in H. \end{aligned}$$

This implies that $x - y \perp \mathcal{R}(B)$. But $R(B) = \mathcal{D}(A)$, which is dense in H, so $x - y = 0$. □

Lemma 3.3.3. Let B be a densely defined linear operator on a Hilbert space H. Then

$$\overline{\mathcal{R}(B)}^{\perp} = \mathcal{N}(B^*).$$

Proof.

First, let us recall that

$$\begin{aligned} \mathcal{N}(B^*) &= \{x \in H \mid B^*x = 0\} \\ \text{and } \mathcal{R}(B) &= \{y \in H \mid y = Bz,\ z \in H\}. \end{aligned}$$

Let $x \in \mathcal{N}(B^*)$ and let $y \in \overline{\mathcal{R}(B)}$ be arbitrary. Then there exists a sequence $\{z_n\} \in \mathcal{R}(B)$ that converges to y. By the continuity of the inner product,

$$\begin{aligned} (x, y) &= \left(x, \lim_{n\to\infty} Bz_n\right) \\ &= \lim_{n\to\infty} (x, Bz_n) \\ &= \lim_{n\to\infty} (B^*x, z_n) \\ &= 0. \end{aligned}$$

Since x and y were chosen arbitrarily, every $x \in \overline{\mathcal{R}(B)}$ and $y \in \mathcal{N}(B^*)$ are orthogonal. □

3.4 Discrete operators

Definition (Discrete operator [DS, p.2291]) An operator T is called *discrete* if there exists a regular value λ in its resolvent set $\rho(T)$ such that $R_\lambda(T)$ is compact.

Lemma 3.4.1 ([DS, Lemma 2, p.2292])**.** Let A be a compact or discrete operator. A has the following properties:

(i) If A is discrete, the resolvent $R_\lambda(A)$ is compact for every $\lambda \in \rho(A)$.

(ii) $\lambda \in \sigma(A)$ if and only if $\lambda^{-1} \in \sigma(A^{-1})$.

(iii) The generalized eigenspaces of A and its inverse are identical. For every nonzero eigenvalue λ of A,

$$\overline{sp}_\lambda(A) = \overline{sp}_{\lambda^{-1}}(A^{-1}).$$

Lemma 3.4.2. If A is a discrete operator and $0 \notin \sigma(A)$ then every eigenspace of A is one dimensional.

Proof.

Suppose f and g are linearly independent eigenfunctions corresponding to the eigenvalue λ of A. Then

$$Af = \lambda f \text{ and } Ag = \lambda g, \text{which implies that } A(f-g) = \lambda(f-g) = 0.$$

But A is a discrete operator and $0 \notin \sigma(A)$ so A is invertible. So $Ax = 0$ if and only if $x = 0$ [Kr, Theorem 2.6-10, p.28]. Therefore $f = g$ and f and g cannot be linearly independent. We conclude that the dimension of the eigenspace corresponding to λ must be one. □

Remark If Assumptions (A1) to (A5) hold then every $\lambda \geq 0$ is in the resolvent $\rho(A)$ of the dynamics generator A. The dynamics generators of all the problems considered in Chapter 2 (with the exception of Problem JTW08) are discrete operators and Lemmas 3.4.1 and 3.4.2 are applicable.

3.5 Completeness results

The following results are due to M.V. Keldysh. They offer fairly general sufficient criteria for the completeness of the generalized eigenvectors of an operator A. These theorems play a crucial role in Chapter 7.

Theorem 3.5.1 ([GK, Theorem 8.1, p.257])**.** Let $A = H(I + S)$ be a linear operator on a Hilbert space X, where S is a compact operator and H is a compact selfadjoint operator such that the sequence of eigenvalues of the operator $(H^*H)^{1/2}$ belongs to ℓ^p. If the operator A is injective (i.e. $\mathcal{N}(A) = \{0\}$), then the system of its generalized eigenvectors is complete.

Definition (Discrete spectrum, [GK, p.276]) The spectrum of an operator is called discrete if it consists entirely of isolated eigenvalues with finite multiplicity with a unique limit point at infinity. An operator with a discrete spectrum must be unbounded.

Remark By Lemma 3.4.1 and Theorem 3.2.2, a discrete operator has a discrete spectrum.

Theorem 3.5.2 ([GK, Theorem 10.1, p.276])**.** Let $A = L + T$, where L is a selfadjoint operator with a discrete spectrum and T is an operator such that $\mathcal{D}(L) = \mathcal{D}(T)$ and $T(L - \rho_0 I)^{-1}$ is compact for some $\rho_0 \in \rho(L)$. If there exists a natural number p for which the sequence of eigenvalues of the operator $(L^{-1}TL^{-1})^{1/p}$ belongs to ℓ^p then the entire spectrum of A consists of isolated eigenvalues of finite multiplicity and the system of its generalized eigenvectors of A is complete.

Remarks

1. Both Keldysh theorems above yield a bonus result - when the conditions of Theorem 3.5.1 or Theorem 3.5.2 are satisfied, we may conclude that the generalized eigenvectors of A^* are also complete in X [GK, Remark 8.1, p.259].

2. The theorems above are extremely difficult to apply, as is evident in [Sh02].

3. Theorem 3.5.2 is quoted in [CZ94, p.220] in the proof of completeness of the generalized eigenvectors of the dynamics generator. This reference is incorrect as this theorem only applies to perturbations of selfadjoint operators and the dynamics generator in [CZ94] is not an operator of this form.

3.6 Krein spaces and operator pencils

In Chapter 6, methods involving operator pencils and Krein spaces used. This section serves as a brief introduction to these topics.

3.6.1 Krein spaces

Loosely speaking, a Krein space is a Hilbert space with an indefinite inner product $[\cdot,\cdot]$ such that the Hilbert space may decomposed into the direct sum of two subspaces which are complete (with respect to $[\cdot,\cdot]$) and orthogonal to each other (with respect to $[\cdot,\cdot]$).

We first consider the definition of an indefinite inner product.

Definition (Hermitian bilinear form) A bilinear form B on some vector space V (over a real or complex scalar field) is *Hermitian* if

$$B\left(x,y\right)=\overline{B\left(y,x\right)}.$$

Definition (Indefinite bilinear form) A definite bilinear form is a bilinear form B over some vector space V (with real or complex scalar field) such that the associated quadratic form

$$Q(x)=B(x,x)$$

is definite. That is, $Q(x)$ has a real value with the same sign (positive or negative) for all non-zero x. According to that sign, B is called positive definite or negative definite. If Q takes both positive and negative values, the bilinear form B is called *indefinite*. If $B(x,x) \geq 0$ for all x, B is said to be positive semidefinite. If $B(x,x) \leq 0$ for all x, B is said to be negative semidefinite.

Definition (Indefinite inner product) An *indefinite inner product* is an indefinite Hermitian bilinear form.

The following is a general rigorous definition of a Krein space.

Definition (General Krein space, Pontryagin space [La81, p.3]) Let X be a Hilbert space and suppose $[\cdot,\cdot]$ is an indefinite inner product on X. The space $(X,[\cdot,\cdot])$ is called a Krein space if it contains two subspaces K_+, K_- with the properties:

1. $X = K_+ \oplus K_-$,
2. $(K_+,[\cdot,\cdot])$ and $(K_-,[\cdot,\cdot])$ are Hilbert spaces,
3. $[K_+,K_-] = \{0\}$.

If, in particular, $k = \min\{\dim K_+, \dim K_-\} < \infty$, the Krein space $(X,[\cdot,\cdot])$ is called a a Pontryagin space of index k, or a π_k-space.

Remark There is another, less general way to define a Krein space which is sufficient for our purposes. See Subsection 6.3.1.

Definition (Krein space [JTW08], [Ha]) Suppose $(H,(\cdot,\cdot))$ is a Hilbert space and $G : H \to H$ is an invertible selfadjoint linear operator such that $G^2 = I$. Then

$$[x,y] = (Gx,y)\,,\ x,y \in H$$

defines an indefinite inner product on H. The space $\mathcal{H} = (H,[\cdot,\cdot])$ is known as a *Krein space*.

Definition (Definitizable operator, [La81, p.11]) A selfadjoint operator in the Krein space $(X,[\cdot,\cdot])$ is called *definitizable* if $\rho(A) \neq 0$ and there exists a polynomial $p(A) = A^k + c_{k-1}A^{k-1} + \ldots + c_1 A + c_0$ of degree k such that $[p(A)x,x] \geq 0$ for $x \in \mathcal{D}(A^k)$.

Theorem 3.6.1 ([La81, Proposition 2.1])**.** Let A be a definitizable operator in the Krein space $\mathcal{H}$. Then the nonreal spectrum of A contains only finitely many conjugate-pairs $\lambda, \overline{\lambda}$. Each isolated spectral point of A is an eigenvalue of finite algebraic multiplicity.

3.6.2 Operator pencils

The theory of operator pencils is widely used in the literature (see for example [Sh02], [KL78(I)] and [KL78(II)]). Some brief highlights of this theory are presented here.

Definition (Operator pencil, [Ma, p.1]) An *operator pencil* is an operator polynomial of the form

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots + \lambda^n A_n,$$

where λ is a spectral parameter and $A_0, \ldots, A_n$ are linear operators acting in a Hilbert space X.

In [JTW08] and [KL78(II), Section 7] an operator pencil of the form

$$L(\lambda) = \lambda^2 I + \lambda B + C \tag{3.3}$$

is considered.

Lemma 3.6.2 ([KL78(I), Section 2.2, p.378])**.** The eigenvectors of the pencil L defined by (3.3) are precisely those of the operator T defined on the Hilbert space X by $T = \begin{bmatrix} 0 & I \\ -C & B \end{bmatrix}$.

Definition (Double completeness, [KL78(I), p.378]) The generalized eigenvectors of the pencil defined by (3.3) are called *double complete* if the generalized eigenvectors $\{\psi_n\}_1^\infty$ of the operator T are such that the sequence $\{\langle \psi_n, \lambda_n \psi_n \rangle\}_1^\infty$ is complete in the product space $X \times X$. Note that λ_n are the eigenvalues corresponding to ψ_n.

Theorem 3.6.3. The double completeness of the generalized eigenvectors of L is equivalent to the completeness of the generalized eigenvectors of T in X.

Proof.

The result follows directly from the definition of double completeness. □

Lemma 3.6.4 ([KL78(I), Lemma 2.2, p.381])**.** The double completeness of the generalized eigenvectors of the pencil L defined by (3.3) is equivalent to the completeness of the generalized eigenvectors of the operator F, defined on the Hilbert space X by

$$F = \begin{bmatrix} 0 & C^{1/2} \\ -C^{1/2} & B \end{bmatrix}.$$

The operator F is said to be the *operator associated with the pencil L*.

Lemma 3.6.5 ([KL78(I), Lemma 2.3, p.381])**.** The spectrum of the pencil L defined by (3.3) coincides with the spectrum of the operator F. That is, $\sigma(L) = \sigma(F)$.

Lemma 3.6.6 ([KL78(I), p.378])**.** The operators T and F are related by the "similarity" transformation $FS = ST$, where

$$S = \begin{bmatrix} C^{1/2} & 0 \\ 0 & I \end{bmatrix}.$$

The inverted commas are used because, by definition, the operator S should be boundedly invertible in a similarity transformation, but here S^{-1} may be an unbounded operator.

The following theorem summarizes Propositions 2.1, 2.2, 2.3 and 2.4 of [KL78(I)].

Theorem 3.6.7. The following results hold for the pencil L defined by (3.3):

1. The spectrum of L is closed.

2. Every non-real $\lambda \in \sigma(L)$ is an eigenvalue of L with finite algebraic multiplicity.

3. The spectrum of L is symmetric with respect to the real axis ($\sigma(L) = \overline{\sigma(L)}$). Moreover, every complex-conjugate pair of eigenvalues λ and $\overline{\lambda}$ of the pencil L have the same geometric and and algebraic multiplicity and the root subspaces λ and $\overline{\lambda}$ have the same structure.

4. If B is non-negative then the spectrum of L is contained in the left half-plane.

Chapter 4

Bases

The results in Chapters 5 to 7 rely on theory that is somewhat esoteric. For example, [DS] only define discrete operators after over 2000 pages. In this chapter, we introduce this theory and examine certain key concepts in more detail.

4.1 Bases and biorthogonal sequences

Bases were defined in Subsection 2.1, but the definition is given again here for convenience.

Definition (Basis [GK, p.306]) A sequence $\{\phi_j\}_1^\infty$ in a Banach space $\mathcal{B}$ is a *basis* of that space if every vector $x \in \mathcal{B}$ can be expanded in a unique way as a series

$$x = \sum_{j=1}^{\infty} c_j \phi_j,$$

which converges in the norm of the space $\mathcal{B}$.

In this expansion, the coefficients c_j are the values of linear functionals Θ_j on $\mathcal{B}$ evaluated at $x \in \mathcal{B}$:

$$c_j = \Theta_j(x) \ , \ \text{for } j = 1, 2, \ldots . \tag{4.1}$$

Definition (Biorthogonal sequences [GK, p.306]) Two sequences $\{\chi_j\}$ and $\{\omega_j\}$ in a Hilbert space H are said to be *biorthogonal* if

$$(\chi_k, \omega_j) = \delta_{jk} \ , \ \text{for } j, k = 1, 2, \ldots .$$

Theorem 4.1.1 ([GK, p.307]). For a basis $\Phi = \{\phi_j\}_1^\infty$ of a Hilbert space H, there exists a sequence $\{\xi_j\}_1^\infty \subset H$ such that $\{\phi_j\}_1^\infty$ and $\{\xi_j\}_1^\infty$ are biorthogonal and every vector $x \in H$ can be expanded in a unique way as a series

$$x = \sum_{j=1}^{\infty} (x, \xi_j)\, \phi_j. \tag{4.2}$$

Proof.

Banach showed in 1932 [Ba, p.68] that for a basis $\Phi = \{\phi_j\}_1^\infty$ of a Hilbert space H, the linear functionals in (4.1) are continuous and there exists a constant C_Φ such that

$$\|\phi_j\|^{-1} \leq \|\Theta_j\| \leq |C_\Phi|\, \|\phi_j\|^{-1} \quad \text{for all } j.$$

The Riesz representation theorem allows us to express the linear functionals Θ_j uniquely as

$$c_j = \Theta_j(x) = (x, \xi_j)\ ,\ (\xi_j \in H; j, k = 1, 2, \ldots).$$

This proves (4.2). Setting $x = \phi_k$, we have $\phi_k = \sum_{j=1}^{\infty} (\phi_k, \xi_j)\, \phi_j$.

By the uniqueness of the coefficients c_j, $(\phi_k, \xi_j) = \delta_{jk}$ for all j and k. $\square$

Proposition 4.1.2. Let $\{\psi_j\}_1^\infty$ be an orthonormal basis of H. Then adding an element to or removing an element from $\{\psi_j\}_1^\infty$ will cause it to cease being a basis for H.

Proof.

Suppose we add the vector $\widetilde{\psi} \in H$ to $\{\psi_j\}_1^\infty$. Then

$$\widetilde{\psi} = \widetilde{\psi} \text{ and } \widetilde{\psi} = \sum_{j=1}^{\infty} c_j \psi_j,$$

so $\{\psi_j\}_1^\infty \cup \{\widetilde{\psi}\}$ is not a basis since the representation of $\widetilde{\psi}$ is not unique.

Suppose we remove an element ψ_p from $\{\psi_j\}_1^\infty$.

Let $\{\gamma_j\}_1^\infty = \{\psi_j\}_1^\infty - \{\psi_p\}$. Then $\psi_p \perp \{\gamma_j\}_1^\infty$ and $\|\psi_p\| = 1 \neq 0$ since ψ_p is a member of the orthonormal set $\{\psi_j\}_1^\infty$.

By Theorem 3.1.1, $\{\gamma_j\}_1^\infty$ cannot be complete in H. Therefore, $\{\gamma_j\}_1^\infty$ is not a basis for H. $\square$

4.2 $\omega-$linear independence

The main result of Chapter 5 (Theorem 5.2.2) pertains to the bases of infinite dimensional Hilbert spaces (with the special property of being equivalent to an orthonormal basis in some sense). In general, intuition gleaned from the finite dimensional case cannot be trusted in the infinite dimensional case, so it is wise to spend some time considering the definition(s) of linear independence in infinite dimensional spaces.

The usual definition of linear independence is given as follows:

Definition (Linear independence [Kr, p.53]) A subset M of a vector space X is *linearly independent* if every nonempty finite subset M_n of M is linearly independent. That is, if $\dim M_n = n$ and $x_1, x_2, \ldots, x_r \in M_n$ $(r \leq n)$, then the only set of scalars $a_1, a_2, \ldots, a_r$ such that

$$\sum_{k=1}^{r} a_k x_k = 0$$

is $a_1 = a_2 = \ldots = a_r = 0$.

The indirect nature of the above definition can make it clumsy to use when the way in which the finite dimensional subspace M_n is chosen may force one to consider many different cases. This is exactly the case in the proof of Theorem 5.2.2.

Gohberg and Krein offer a definition of a different kind of linear independence that deals with infinite dimensional spaces more directly:

Definition (ω-linear independence, [GK, p.316]) A sequence $\{g_j\}_1^\infty$ of vectors in a Hilbert space H is said to be *ω-linearly independent* if the equality

$$\sum_{j=1}^{\infty} c_j g_j = 0$$

is impossible for

$$0 < \sum_{j=1}^{\infty} |c_j|^2 \|g_j\|^2 < \infty,$$

where c_j are scalars.

It is logically equivalent to say that $\{g_j\}_1^\infty$ is ω-linearly independent if

$$0 < \sum_{j=1}^{\infty} |c_j|^2 \|g_j\|^2 < \infty,$$

implies that

$$\sum_{j=1}^{\infty} c_j g_j \neq 0.$$

If a sequence is not $\omega-$linearly independent it is said to be $\omega-$linearly dependent.

Remark The definition of ω-linear independence may seem unnecessarily complicated, but Proposition 4.2.1 shows that in the finite dimensional case, ω-linear independence is equivalent to 'usual' linear independence if we exclude the possibility of zero vectors. The results that follow echo familiar properties of linearly independent sets. Moreover, Corollary 4.2.4 is crucial in the proof of Theorem 5.2.2.

Proposition 4.2.1. Let H_N be a finite dimensional Hilbert space. A subset $\Phi \subset H_N$ is linearly independent if and only if it is ω-linearly independent and $0 \notin \Phi$.

Proof.

Let $\dim H_N = N$ and let $\Phi = \{\phi_k\}_1^N \subset H_N$.

Suppose Φ is linearly independent. Then, by definition, for scalars $c_1, \ldots c_N$:

$$\sum_{k=1}^{N} c_k \phi_k = 0 \quad \text{implies} \quad c_1 = c_2 = \ldots = c_N = 0.$$

Any set containing 0 is linearly dependent, so $\phi_k \neq 0$. Therefore Φ is also ω-linearly independent.

On the other hand, suppose Φ is ω-linearly independent and $0 \notin \Phi$. Then

$$\sum_{k=1}^{N} c_k \phi_k = 0 \quad \text{implies} \quad \sum_{k=1}^{N} |c_k|^2 \|\phi_k\|^2 = 0.$$

Since H_N is finite dimensional and $0 \notin \Phi$, $0 < \|\phi_j\| < \infty$ for each j. It follows immediately that $c_1 = c_2 = \ldots = c_N = 0$ and Φ is linearly independent. $\square$

It is well-known that orthonormal sequences are linearly independent. This remains true for ω-linearly independent sequences, provided that we exclude the possibility of zero vectors.

Proposition 4.2.2. Let $\Phi = \{\phi_k\}_1^\infty$ be a sequence of nonzero vectors in the Hilbert space H. If Φ is orthonormal, then it is also ω-linearly independent.

Proof.

Let $\{c_k\}_1^\infty$ be an arbitrary sequence of scalars. Suppose

$$0 < \sum_{k=1}^{\infty} |c_k|^2 \|\phi_k\|^2 < \infty.$$

Then the orthonormality of Φ implies that

$$0 < \sum_{k=1}^{\infty} |c_k|^2 \|\phi_k\|^2 = \sum_{k=1}^{\infty} |c_k|^2.$$

So there exists a j such that $c_j \neq 0$. But

$$\sum_{k=1}^{\infty} |c_k|^2 = 0 \ \text{ if and only if } \ c_k = 0 \text{ for all } k.$$

Therefore, Φ is ω-linearly independent. □

Proposition 4.2.3. Let $\{\phi_k\}_1^\infty$ be an ω-linearly dependent sequence in a Hilbert space H. Then there exists a $\phi_N \in \{\phi_k\}_1^\infty$ which may be expressed as

$$\phi_N = \sum_{\substack{k=1 \\ k \neq N}}^{\infty} c_k \phi_k.$$

Proof.

Since $\{\phi_k\}_1^\infty$ is an ω-linearly dependent sequence, there must exist scalars $\{c_k\}$ such that

$$\sum_{k=1}^{\infty} c_k \phi_k = 0 \tag{4.3}$$

and

$$0 < \sum_{k=1}^{\infty} |c_k|^2 \|\phi_k\|^2 < \infty.$$

We may choose a $\phi_N \neq 0$ such that $c_N \neq 0$.

Let $\{\gamma_k\} = \{\phi_k\} - \{\phi_N\}$ and denote

$$\begin{aligned} s_n &= \sum_{k=1}^{n} c_k \phi_k \\ \text{and } z_n &= \frac{-1}{c_N} \sum_{k=1}^{n} c_k \gamma_k. \end{aligned}$$

We need to show that $z_n \to \phi_N$ as $n \to \infty$:

By (4.3) we have $s_n \to 0$ as $n \to \infty$. So given any $\varepsilon > 0$, there exists a natural number M such that

$$\|s_J\| = \left\| \sum_{k=1}^{J} c_k \phi_k \right\| < \varepsilon \text{ for all } J > M.$$

Suppose $J > N$ and $J > M$, then

$$\begin{aligned} \|s_J\| &= \left\| c_N \phi_N + \sum_{\substack{k=1 \\ k \neq N}}^{J} c_k \phi_k \right\| \\ &= |c_N| \left\| -\frac{1}{c_N} \sum_{\substack{k=1 \\ k \neq N}}^{J} c_k \phi_k - \phi_N \right\| \\ &= |c_N| \left\| -\frac{1}{c_N} \sum_{\substack{k=1 \\ k \neq N}}^{J} c_k \phi_k - \phi_N \right\| \\ &= |c_N| \, \|z_J - \phi_N\| \\ &< \varepsilon. \end{aligned}$$

Therefore we may write $\phi_N = \sum_{\substack{k=1 \\ k \neq N}}^{\infty} c_k \phi_k$ when $\{\phi_k\}_1^\infty$ is an ω-linearly dependent sequence. $\square$

Corollary 4.2.4. Let $\{\phi_k\}_1^\infty$ be an ω-linearly independent sequence in a Hilbert space H. Then for any $\varphi \neq 0 \in H$, $\{\varphi\} \cup \{\phi_k\}_1^\infty$ is ω-linearly dependent if and only if φ can be expressed as

$$\varphi = \sum_{k=1}^{\infty} c_k \phi_k. \tag{4.4}$$

Proof.

Let $\{\gamma_k\}_1^\infty = \varphi \cup \{\phi_k\}_1^\infty$. Suppose $\{\gamma_k\}_1^\infty$ is ω-linearly dependent. By Proposition 4.2.3, some $\phi_N \in \{\gamma_k\}_1^\infty$ can be written as

$$\phi_N = \sum_{\substack{k=1 \\ k \neq N}}^{\infty} c_k \gamma_k = \sum_{k=1}^{\infty} c_k \phi_k.$$

But no element of $\{\phi_k\}_1^\infty$ can be expressed in this way since this would imply that there exists a sequence of scalars $\{c_k\}$ (with $c_N = -1$) such that

$$\sum_{k=1}^{\infty} c_k \phi_k = 0$$

and

$$0 < \sum_{k=1}^{\infty} |c_k|^2 \|\phi_k\|^2 < \infty,$$

which contradicts the ω-linear independence of $\{\phi_k\}_1^\infty$. Thus we have (4.4).

On the other hand, suppose

$$\varphi = \sum_{k=1}^{\infty} c_k \phi_k.$$

If we index the set $\{\gamma_k\}_1^\infty$ so that $\varphi = \gamma_N$, then for any set of scalars $\{c_k\}$ with $c_N = -1$ such that

$$0 < \sum_{k=1}^{\infty} |c_k|^2 \|\gamma_k\|^2 < \infty.$$

and

$$\sum_{k=1}^{\infty} c_k \gamma_k = c_N \varphi + \sum_{k=1}^{\infty} c_k \phi_k = 0.$$

By definition, $\{\gamma_k\}_1^\infty$ is now ω-linearly dependent. $\square$

4.3 Riesz bases

In practice it can be difficult to construct a Riesz basis directly (as was done in Section 2.4). Fortunately, there are many tools available that make an indirect approach possible. In Sections 5.3, 6.3 and Subsection 7.1.2 the results of this chapter are used to prove that the generalized eigenvectors of an operator A form a Riesz basis for the Hilbert space on which A is defined.

Recall the definition of a Riesz basis:

Definition (Riesz basis [GK, p.309]) Let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis for the Hilbert space H. A basis $\{\psi_j\}_1^\infty$ of H is called a *Riesz basis* of H if there exists an invertible bounded linear operator Q such that Q transforms $\{\phi_j\}_1^\infty$ into $\{\psi_j\}_1^\infty$. That is, $\psi_j = Q\phi_j$ for all $j = 1, 2, \ldots$.

Remark A Riesz basis is sometimes called a *basis equivalent to an orthonormal basis*. Bari's theorem illuminates the aptness of this term.

Theorem 4.3.1 (Bari's Theorem [GK, p.310])**.** Let H be a Hilbert space. The following assertions are equivalent:

(i) The sequence $\{\psi_j\}_1^\infty$ forms a Riesz basis of the space H.

(ii) The sequence $\{\psi_j\}_1^\infty$ becomes an orthonormal basis of H following the appropriate replacement of the inner product $(\cdot, \cdot)$ by a new inner product $(\cdot, \cdot)_\psi$ that is topologically equivalent to the original one. That is, there exist $c_1, c_2 > 0$ such that

$$c_1\,(x, x) \leq (x, x)_\psi \leq c_2\,(x, x) \text{ for all } x \in H.$$

(iii) The sequence $\{\psi_j\}_1^\infty$ is complete in H and there exist positive numbers a_1, a_2 such that for any integer N and complex numbers $\gamma_1, \ldots, \gamma_N$ one has

$$a_1 \sum_{j=1}^{N} |\gamma_j|^2 \leq \left| \sum_{j=1}^{N} \gamma_j \psi_j \right|^2 \leq a_2 \sum_{j=1}^{N} |\gamma_j|^2$$

(iv) The sequence $\{\psi_j\}_1^\infty$ is complete in H, there corresponds to it a complete biorthogonal sequence $\{\chi_j\}_1^\infty$, and for any $f \in H$ one has

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty, \quad \sum_{j=1}^{\infty} |(f, \chi_j)|^2 < \infty.$$

Remark The class of Riesz bases is very large, but not every basis is a Riesz basis. [GK, p.319]. K. I. Babenko showed in his paper *On conjugate functions* (Dokl. Akad. Nauk SSSR 62 (1948) p.157-160) that the sequence

$$\left\{ (\alpha + \frac{1}{2})|x|^\alpha e^{in\pi x} \right\}_{n=-\infty}^{\infty}, \quad -\frac{1}{2} < \alpha < \frac{1}{2};\ \alpha \neq 0$$

is a basis for $L^2(-1, 1)$ but not a Riesz basis.

Definition (Quadratically close sequences [GK, p.316]) The sequences $\{f_j\}_1^\infty$ and $\{g_j\}_1^\infty$ in a Hilbert space H are said to be *quadratically close* if

$$\sum_{j=1}^{\infty} \|f_j - g_j\|^2 < \infty.$$

Remark Note that $\{f_j\}_1^\infty$ and $\{g_j\}_1^\infty$ are quadratically close if there exists a natural number N such that

$$\sum_{j=N}^{\infty} \|f_j - g_j\|^2 < \infty, \quad \text{since} \quad \sum_{j=1}^{N} \|f_j - g_j\|^2 < \infty$$

by definition of a norm.

Theorem 4.3.2 (Bari's 2^{nd} Theorem [GK, p.310])**.** Any ω-linearly independent sequence $\{g_j\}$ which is quadratically close to some Riesz basis $\{\psi_j\}$, is itself a Riesz basis.

Proposition 4.3.3. Let $\{\psi_j\}_1^\infty$ be a Riesz basis of H. Then adding an element to or removing an element from $\{\psi_j\}_1^\infty$ will cause it to cease being a basis for H.

Proof.

By Theorem 4.3.1 there is an inner product $(\cdot,\cdot)_\psi$ equivalent to the usual inner product in H such that $\{\psi_j\}_1^\infty$ is orthonormal with respect to $(\cdot,\cdot)_\psi$. Using the inner product $(\cdot,\cdot)_\psi$, the result follows immediately from Proposition 4.1.2. □

Theorem 4.3.4 ([Rao97])**.** Suppose $\{\phi_n\}_1^\infty$ is a Riesz basis for a Hilbert space H and $\Psi = \{\psi_n\}_{M+1}^\infty$ is an ω-linearly independent sequence in H. If

$$\sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then $\{\psi_n\}_{M+1}^\infty$ is a Riesz basis for the subspace $\overline{sp}(\{\psi_n\}_{M+1}^\infty) \subset H$ that it generates.

Proof.

The proof follows directly from Theorem 4.3.2 since Ψ is ω-linearly independent and quadratically close to $\{\phi_n\}_{M+1}^\infty$, which is an orthonormal basis for $\overline{sp}(\{\psi_n\}_{M+1}^\infty)$. □

Remark Theorem 4.3.4 appeared in [Rao97], which was written in French. The theorem was stated in English in [ZZ03] but not proved.

Definition (Similar operators [Ma, p.168]) An operator A is said to be *similar* to another operator B if there exists an invertible operator C such that $A = C^{-1}BC$.

Lemma 4.3.5 ([Ma, Lemma 30.9, p.168])**.** If a compact operator Z acting on a Hilbert space H is similar to a selfadjoint operator, then there exists a Riesz basis for H which consists of eigenvectors of Z.

Lemma 4.3.6 ([Ma, Lemma 30.10, p.168])**.** If a bounded linear operator Z acting on a Hilbert space H is similar to a selfadjoint operator, then all the generalized eigenvectors of Z are eigenvectors and there exists a Riesz basis for H which consists of eigenvectors of Z.

Proposition 4.3.7 ([GK, p.329])**.** A sequence $\{\psi_j\}$ made up of the bases of the root subspaces of a linear operator is ω-linearly independent. That is, the sequence $\{\psi_j\}$ of generalized eigenvectors of a linear operator is ω-linearly independent.

The following theorem is due to I. S. Iokhvidov.

Theorem 4.3.8 ([Ha, H. Langer, "Krein spaces"])**.** If the spectrum of a definitizable operator is discrete, then the span of the generalized eigenvectors is dense in the Hilbert space H. If A is a selfadjoint compact operator in a Pontryagin space and 0 is not an eigenvalue of A, then there is a Riesz basis of H consisting of generalized eigenvectors of of A.

The following result is due to M. Glazman and was strengthened in [GK]. The stronger form appears here.

Theorem 4.3.9 ([GK, p.328])**.** Let $\{\phi_j\}$ be a system of eigenvectors corresponding to distinct eigenvalues λ_j, $j = 1, 2, \ldots$ of a dissipative operator. If

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} \frac{\mathrm{Im}\lambda_j \mathrm{Im}\lambda_k}{|\lambda_j - \overline{\lambda}_k|} < \infty, \tag{4.5}$$

then the system $\{\phi_j\}$ forms a Riesz basis of $\overline{sp}(\{\phi_j\})$ that is also quadratically close to an orthonormal basis $\{e_j\}$ of $\overline{sp}(\{\phi_j\})$.

Chapter 5

Sequences quadratically close to Riesz bases

In [GY01], Guo and Yu state an abstract result concerning the Riesz basis property of the generalized eigenvectors of a discrete operator (Theorem 5.2.1). Proofs of this theorem are offered in [GY01] and in the subsequent papers [Gu01] and [ZZ03]. For the most part these proofs are almost identical. However, the proof of the fact that *all* the generalized eigenvectors must be used in this basis is glossed over or proved rather cryptically in these papers. This fact is not trivial or trivial to prove. This situation is improved upon in the proof of Theorem 5.2.2.

Section 5.3 deals with the application of Theorem 5.2.2 to the problem that the authors of [GY01] considered – the Euler-Bernoulli beam with boundary control. Many important facts are not proved in [GY01] and the asymptotic expression derived for the eigenvalues there is incorrect. The results of the previous chapters of this dissertation are used to rectify this.

It is unclear how the authors of [GY01] arrived at the conclusion that the dynamics generator for this problem satisfies the conditions of Theorem 5.2.2. Most of their argument is corrected and clarified here, with one exception. This issue is presented and questioned, but not corrected.

5.1 Preliminary Results

Definition (Contraction, [Kr, p.300]) A mapping F on a metric space (X, d) is called a *contraction* if there exists $\theta < 1$ such that

$$d(Fx, Fy) \leq \theta d(x, y) < d(x, y) \text{ for all } x, y \in X.$$

In a normed space, the equation above becomes

$$\|Fx - Fy\| \leq \theta \|x - y\| < \|x - y\| \text{ for all } x, y \in X.$$

Lemma 5.1.1. Let G be a bounded linear operator on a Banach space $\mathcal{B}$ with $\|G\| < 1$. Then the inverse of $(I - G)$ exists and is bounded.

Proof.

Consider, for any $y \in \mathcal{B}$, the equation

$$(I - G)x = y.$$

Rewrite the equation above in fixed point form $x = Gx + y = F(x)$ and consider arbitrary $x, x_0 \in \mathcal{B}$. G is a contraction, so there exists a $\theta < 1$ such that

$$\|F(x) - F(x_0)\| = \|G(x - x_0)\| \leq \theta \|x - x_0\|.$$

By the Banach Fixed Point Theorem [Kr, Theorem 5.1-2 p.300] F has a unique fixed point $x = F(x)$ for any given y. This implies that $(I - G)x = y$ has a unique solution and, in turn, that $(I - G)^{-1}$ exists.

To show that $(I - G)^{-1}$ is bounded we note that

$$\|x\| \leq \|Gx\| + \|y\| \leq \theta \|x\| + \|y\|.$$

So $\|x\| \leq (1 - \theta)^{-1} \|y\|$. Therefore $\|(I - G)^{-1} y\| \leq \|y\|$ for any given $y \in \mathcal{B}$. Consequently $(I - G)^{-1}$ is bounded. $\square$

Remark Lemma 5.1.1 is used in the proof of Lemma 5.1.3 and actually yields the existence of the resolvent of a contraction. The authors of [GY01] use this result without proof or reference.

Lemma 5.1.2 ([GY01])**.** Let $\{\phi_n\}_1^\infty$ be a Riesz basis for a Hilbert space H. Let $\{\psi_n\}_{N+1}^\infty$ be another sequence in H. If there is an $N \geq 0$ such that

$$\sum_{n=N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then there exists an $M \geq N$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is a Riesz basis for H.

Remark This lemma is used to prove Theorem 5.2.2. It may be restated as follows:

Lemma 5.1.3 ([GY01])**.** Let $\{\phi_n\}_1^\infty$ be a Riesz basis for a Hilbert space H. Let $\{\psi_n\}_1^\infty$ be another sequence in H. If

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then there exists a natural number M such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is a Riesz basis for H.

Proof.

We show that there exists a bounded invertible linear operator Q that maps ϕ_n onto ψ_n for each $n > M$.

First, we note that $\{\phi_n\}_1^\infty$ is a Riesz basis, so by definition there exists a bounded invertible linear operator T such that

$$T\phi_n = e_n,$$

where $\{e_n\}_1^\infty$ is an orthonormal basis for H. By the definition of a basis, any $x \in H$ may be expressed uniquely as

$$x = \sum_{n=1}^{\infty} a_n \phi_n. \tag{5.1}$$

Now, since T is a bounded linear operator [Kr, Corollary 2.7-10, p.98],

$$\begin{aligned} Tx &= T\left(\sum_{n=1}^{\infty} a_n \phi_n\right) \\ &= T \lim_{K\to\infty} \sum_{n=1}^{K} a_n \phi_n \\ &= \lim_{K\to\infty} \sum_{n=1}^{K} a_n T\phi_n \\ &= \lim_{K\to\infty} \sum_{n=1}^{K} a_n e_n \\ &= \sum_{n=1}^{\infty} a_n e_n. \end{aligned}$$

Note that $T^{-1} : H \to H$ since $\{e_n\}$ and $\{\phi_n\}$ are bases. Hence T is a bijective bounded linear operator. Consequently, T^{-1} is bounded by the Bounded Inverse Theorem [Kr, p.286]. Since $\{e_n\}_1^\infty$ is an orthonormal basis for H:

$$\|Tx\|^2 = \left\| \sum_{n=1}^{\infty} a_n e_n \right\|^2 = \sum_{n=1}^{\infty} |a_n|^2. \tag{5.2}$$

Let $S_K = \sum_{n=1}^{K} \|\phi_n - \psi_n\|^2$. Then $\{S_K\}$ converges to some real number S since

$$\lim_{K\to\infty} S_K = \sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty.$$

Therefore, given any $\varepsilon > 0$, there exists an M such that

$$S - S_M = \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 < \varepsilon.$$

Let

$$\theta^2 = \|T\|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 < \|T\|^2 \varepsilon. \tag{5.3}$$

If we choose $\varepsilon = 1/\|T\|^2$ we have $\theta^2 < 1$.

We are now in a position to define the operator Q for $x \in H$ using (5.1). Let

$$Qx = \sum_{n=1}^{M} a_n \phi_n + \sum_{n=M+1}^{\infty} a_n \psi_n.$$

Note that $Q\phi_n = \phi_n$ for $1 \leq n \leq M$ and $Q\phi_n = \psi_n$ for $n > M$. The linearity of Q is a direct consequence of the linearity of summation and limits. Now $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ will be a Riesz basis for H if Q is invertible and bounded.

Let $G = I - Q$. For any $x = \sum_{n=1}^{\infty} a_n \phi_n \in H$,

$$\begin{aligned}
\|Gx\|^2 &= \left\| \sum_{n=M+1}^{\infty} a_n(\phi_n - \psi_n) \right\|^2 \\
&\leq \sum_{n=M+1}^{\infty} |a_n|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 \\
&\leq \|T\|^2 \|x\|^2 \sum_{n=M+1}^{\infty} \|\phi_n - \psi_n\|^2 \quad \text{by (5.2)} \\
&= \theta^2 \|x\|^2 \quad \text{by (5.3)}.
\end{aligned}$$

This shows that G is bounded and $\|G\| \leq \theta < 1$. Therefore G is a contraction. By Lemma 5.1.1, $(I - G)^{-1}$ exists and is bounded.

Finally, $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ is a Riesz basis for H because Q is an invertible bounded linear operator such that $Q\phi_n = \phi_n$ for $1 \leq n \leq M$ and $Q\phi_n = \psi_n$ for $n > M$. $\square$

Lemma 5.1.4. Let A be a densely defined discrete operator in a Hilbert space H. Suppose that $0 \in \rho(A)$. Then either $\overline{sp}(A) = H$ or $\overline{sp}(A)^{\perp}$ is infinite dimensional.

Proof.

Suppose $\overline{sp}(A) \neq H$ and $\dim(\overline{sp}(A)^{\perp}) < \infty$. Denote $K = \overline{sp}(A)^{\perp}$. By the orthogonal decomposition theorem, [Kr, p.147]

$$H = \overline{sp}(A) \oplus K, \;\; K \neq \{0\}, \; \dim K < \infty.$$

A is a discrete operator and $0 \in \rho(A)$ so $B = A^{-1}$ exists and is a bounded linear operator defined on H. In fact, since A is discrete, B is actually compact.

By Lemma 3.4.1, $\lambda \in \sigma(A)$ implies $\lambda^{-1} \in \sigma(B)$ and the root subspace of A corresponding to λ is the same as the the root subspace of B corresponding to λ^{-1} (The generalized eigenvectors of B and A are the same).

Let $\mathbb{P}$ be the orthogonal projection from H onto K. By Lemma 3.2.4, the operator $\mathbb{P}B\mathbb{P}$ and its adjoint $(\mathbb{P}B\mathbb{P})^* = \mathbb{P}B^*\mathbb{P}$ are Volterra operators.

Because $\overline{sp}(A) = \overline{sp}(B)$ is an invariant subspace of B, Lemma 3.3.1 guarantees that K is an invariant subspace of B^*. That is, $B^*(K) \subset K$. Note that B^* cannot be the 0 operator since it is one-to-one by Lemma 3.3.2. Now, since $\mathbb{P}|_K = I$ and $B^*(K) \subset K$,

$$\mathbb{P}B^*\mathbb{P}K = \mathbb{P}(B^*(K)) = B^*(K).$$

Therefore, $\mathbb{P}B^*\mathbb{P}K|_K = B^*|_K$ is a Volterra operator and has no nonzero eigenvalue.

But K is finite dimensional so $B^*|_K$ must have at least one eigenvalue and eigenvector (This result is well-known from linear algebra, see for example [Kr, Theorem 7.1-4, p.368]). It follows that this eigenvalue must be 0. Thus there is an $x_0 \in K, x_0 \neq 0$ such that $B^*x_0 = 0$.

We now arrive at a contradiction:

By Lemma 3.3.3, $\overline{\mathcal{R}(B)}^{\perp} = \mathcal{N}(B^*)$.

Further, we have that $H = \overline{\mathcal{R}(B)} \oplus \mathcal{N}(B^*)$ by the projection theorem [Kr, Theorem 3.3-4, p.146].

But A is densely defined in H so $\overline{\mathcal{R}(B)} = \overline{\mathcal{D}(A)} = H$.

Therefore $\mathcal{N}(B^*) = \{0\}$, but we deduced that there exists an $x_0 \neq 0 \in \mathcal{N}(B^*)$. A contradiction arises, so the assumption that $\overline{sp}(A) \neq H$ and $\dim \overline{sp}(A) < \infty$ is false. Therefore, either $\overline{sp}(A) = H$ or $\overline{sp}(A)^{\perp}$ is infinite dimensional. $\square$

5.2 Guo & Yu's Theorem

We arrive at the main result of this chapter. The theorem is stated as follows in [GY01]:

Theorem 5.2.1 ([GY01])**.** Let A be a densely defined discrete operator in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for a Hilbert space H. If there is an $N \geq 0$ with a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of A such that

$$\sum_{n=N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty.$$

Then

(i) there is a constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of A such that $\{\psi_{n0}\}_1^M \cup \{\psi_{n0}\}_{M+1}^\infty$ forms a Riesz basis for H.

(ii) Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to eigenvalues $\{\lambda_n\}_1^\infty$ of A. Then $\sigma(A) = \{\lambda_n\}_1^\infty$, where λ_n is counted according to its algebraic multiplicity.

(iii) If there is an $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that λ_n is algebraically simple for all $n > N_0$.

Remarks

1. The summation from $M + 1$ to ∞ may be replaced by summation from 1 to ∞ in the statement of the theorem, since the sum of the norms of finitely many vectors must be finite. This is consistent with the definition of quadratically close sequences. In fact, this relates the theorem above even more closely to Bari's Theorem (Theorem 4.3.2). These two theorems are so intimately related that the authors of [AY09] refer to Theorem 5.2.1 as the Bari-Guo theorem and Guo refers to this theorem as a corollary to Bari's theorem in [Gu02].

2. The first line in the statement of the theorem and the subsequent re-indexing of the vectors is misleading – we want conditions under which the generalized eigenvectors of A form a Riesz basis for H.

We restate the theorem with these comments in mind and offer a proof which is based on that given in [GY01], but which is clearer and easier to read.

Theorem 5.2.2 ([GY01])**.** Let A be a densely defined discrete operator with generalized eigenvectors $\{\psi_n\}_1^\infty$ in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. If

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty.$$

Then

(i) the generalized eigenvectors $\{\psi_n\}_1^\infty$ of A form a Riesz basis for H.

(ii) Let $\{\psi_n\}_1^\infty$ correspond to eigenvalues $\{\lambda_n\}_1^\infty$ of A. Then $\sigma(A) = \{\lambda_n\}_1^\infty$, where λ_n is counted according to its algebraic multiplicity.

(iii) If there is an $M_0 > 0$ such that $\lambda_n \neq \lambda_m$ for all $m, n > M_0$, then there is an $N_0 > M_0$ such that λ_n is algebraically simple for all $n > N_0$.

Proof.

Guo and Yu claim that assertions (ii) and (iii) are consequences of (i) and do not prove them [GY01]. We prove that (i) does in fact imply (ii). Looking at the results in Chapter 3, it can be seen that assertion (iii) cannot be trivial to prove.

(i) We first show the completeness of the generalized eigenvectors in H and then proceed to show that they form a Riesz basis for H.

By Lemma 5.1.3, there exists an M such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H. By Bari's Theorem (Theorem 4.3.1), there exists an inner product $(\cdot,\cdot)_M$ on H (that is topologically equivalent to the original inner product) such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is an orthonormal basis for H with respect to $(\cdot,\cdot)_M$.

Denote $V_M = sp(\{\phi_n\}_1^M)$. It follows from Rao's Theorem (Theorem 4.3.4) that $\{\psi_n\}_{M+1}^\infty$ is a Riesz basis for the subspace $\overline{sp}(\{\psi_n\}_{M+1}^\infty) \subset H$ that it generates. Thus,

$$H = V_M \oplus \overline{sp}(\{\psi_n\}_{M+1}^\infty). \tag{5.4}$$

By definition, $\overline{sp}(A)$ is a closed subspace of H. By [Kr, Theorem 1.4-7, p.30], $\overline{sp}(A)$ is a Hilbert space and moreover,

$$H = \overline{sp}(A) \oplus \overline{sp}(A)^\perp. \tag{5.5}$$

Note that the inner product $(\cdot,\cdot)_M$ is used in (5.4) and (5.5) so that V_M and $\overline{sp}(\{\psi_n\}_{M+1}^\infty)$ are orthogonal.

Consider (5.4) and (5.5). Since $\overline{sp}(\{\psi_n\}_{M+1}^{\infty}) \subset \overline{sp}(A)$, $\overline{sp}(A)^{\perp} \subset V_M$. But V_M is finite dimensional, so $\overline{sp}(A)^{\perp}$ must be finite dimensional. Recalling that we deal only with operators with the property that $0 \in \rho(A)$, we may apply Lemma 5.1.4 to conclude that $\overline{sp}(A) = H$. This proves the completeness of the generalized eigenvectors of A in H.

Our aim now is to replace the Riesz basis vectors $\{\phi_n\}_1^M$ with generalized eigenvectors of A, so that we have a Riesz basis of H that is comprised solely of generalized eigenvectors of A.

Since $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ is a basis for H, $\{\psi_n\}_{M+1}^{\infty}$ must be ω-linearly independent. Consider, for $k \leq M$, the generalized eigenvectors ψ_k of A which do not belong to $\overline{sp}(\{\psi_n\}_{M+1}^{\infty})$. Suppose we can add up to $L \leq M$ such ψ_k to $\{\psi_n\}_{M+1}^{\infty}$ such that the union $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ remains ω-linearly independent. Then for any other generalized eigenvector ψ of A the set $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty} \cup \{\psi\}$ must be ω-linearly dependent. Therefore, $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ is a basis for $\overline{sp}(A) = H$ by Theorem 4.3.4.

By hypothesis, $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ is quadratically close to the Riesz basis $\{\phi_n\}_1^{\infty}$. Theorem 4.3.2 now guarantees that $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ is a Riesz basis for $\overline{sp}(A) = H$.

Finally, we show that $L = M$. Suppose $L < M$. By Theorem 4.3.1, there is an inner product $(\cdot,\cdot)_M$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ is an orthonormal basis for H with respect to $(\cdot,\cdot)_M$. By definition of a Riesz basis, there must exist an isomorphism Q between this orthonormal basis and the Riesz basis $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ such that

$$Q\psi_j = \begin{cases} \phi_j \text{ for } 1 \leq j \leq L \\ \psi_j \text{ for } j \geq M+1 \end{cases} \tag{5.6}$$

Again, by Theorem 4.3.1, there is an inner product $(\cdot,\cdot)_L$ such that $\{\psi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ is an orthonormal basis for H with respect to $(\cdot,\cdot)_L$. Now (5.6) implies that $\{\phi_n\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ is a Riesz basis for H. Since $L < M$, there must now exist a k such that $L < k \leq M$ and

$$\phi_k = \sum_{j=1}^{L} c_j\phi_j + \sum_{j=M+1}^{\infty} c_j\psi_j.$$

But this contradicts the ω-linear independence of $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ by Corollary 4.2.4. Therefore $L = M$ and $\{\psi_n\}_1^{\infty}$ is a Riesz basis for H.

(ii) By Theorem 3.4.1, every $\lambda^{-1} \in \sigma(A^{-1})$ is an eigenvalue of A^{-1}.

By Lemma 3.4.1, $\lambda^{-1} \in \sigma(A^{-1})$ implies that $\lambda \in \sigma(A)$. Further, the root vectors associated with $\overline{sp}_{\lambda^{-1}}(A^{-1}) = \overline{sp}_{\lambda}(A)$. But each spectral value of A^{-1} is an eigenvalue, so there exists a $\psi \in \{\psi_n\}_1^\infty$ such that

$$A^{-1}\psi = \lambda^{-1}\psi \text{ which implies that } \lambda\psi = A\psi,$$

whence every $\lambda \in \sigma(A)$ is an eigenvalue of A. □

Remark In [ZZ03] the authors prove that Theorem 5.2.2 holds if the operator A is compact (instead of being discrete, as in [GY01]). The proof given above holds for compact A with only minor adjustments:

1. Lemma 5.1.4 holds for compact A. The proof of this lemma is simpler: since A itself is compact it is not necessary to consider A^{-1}.

2. Part (ii) of Theorem 5.2.2 is immediate from Theorem 3.2.2.

5.3 Application to a beam problem

We apply Theorem 5.2.2 to the beam problem that the authors of [GY01] originally applied it to, the Euler-Bernoulli beam equation with one end fixed and linear boundary feedback control at the other end (see Section 1.4).

Problem GY01

$$\begin{aligned}
\partial_t^2 u + \partial_x^4 u &= 0,\ 0 < x < 1, t > 0 \\
u(0,t) = \partial_x u(0,t) &= 0,\ t > 0, \\
\partial_x^2 u(1,t) &= -k_1 \partial_t \partial_x u(1,t),\ t > 0, \\
\partial_x^3 u(1,t) &= k_2 \partial_t u(1,t), t > 0,
\end{aligned}$$

where k_1 and k_2 are positive real numbers.

Definition Let $H_E^2(0,1) = \{f(x) \in H^2(0,1) \mid f(0) = f'(0) = 0\}$. Define the *state Hilbert space* by $H = H_E^2(0,1) \times L^2(0,1)$ with the inner product induced by the norm

$$\|\langle f, g\rangle\|^2 = \int_0^1 |f''(x)|^2 + |g(x)|^2 dx.$$

Then Problem GY01 can be written as a first order equation in the state space H as

$$\frac{d}{dt}Y(t) = AY(t),$$

where $Y(t) = \langle y, y_t \rangle$ and the operator A is defined in [GY01] by

$$A \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} -g(x) \\ f^{(4)}(x) \end{bmatrix}. \tag{5.7}$$

The domain of A is

$$\begin{aligned} \mathcal{D}(A) \quad = \quad & \left\{ \langle f, g \rangle \in (H^4(0,1) \cap H^2_E(0,1)) \times H^2_E(0,1) \right. \\ & \quad \left. \text{such that} \quad f''(1) = -k_1 g'(1),\ f'''(1) = k_2 g(1) \right\}. \end{aligned}$$

We may write the dynamics generator as $A = \begin{bmatrix} 0 & -I \\ \partial_x^4 & 0 \end{bmatrix}$.

Remarks

1. The definitions of the energy, inertia and state spaces for Problem GY01 were given in Section 2.7. Those that appear in [GY01] are included here for completeness.

2. To define the dynamics generator the way it is defined in [GY01], one needs to understand the boundary conditions. Since $\mathcal{D}(A) \subset H^4(0,1)$, the derivatives given in the definition are weak. Therefore, function values may be changed arbitrarily on a set of measure zero. The boundary conditions can be defined in such a way that this is not a problem, but it is not a simple matter at all. Further, it must be proved that A is densely defined on the state space, which is not done in [GY01].

3. These problems may be circumvented by defining A only for functions in $C^4[0,1]$ that satisfy the boundary conditions, then A can be extended to the dynamics generator given in Chapter 2. If this is done, then the relevant results of Chapters 1 and 2 apply. The results in Section 2.7 are especially pertinent.

We will follow the proof that the generalized eigenvectors of the dynamics generator constitute a Riesz basis for the state space H given in [GY01] in which the authors attempt to show that A satisfies the hypotheses of Theorem 5.2.2. The discussion of this in [GY01] is very difficult to follow and occasionally incorrect.

The discussion in [GY01] is broken into three very technical lemmas and a theorem. The lemmas proved here differ from their counterparts in [GY01]. They have been altered to clarify their meaning and Lemma 5.3.4 corrects the errors made in [GY01, Lemma 3].

Lemma 5.3.1 ([GY01, Lemma 2])**.** For any real k_1, k_2, the following hold:

(i) The dynamics generator A of Problem GY01 is a densely defined discrete operator in H.

(ii) The spectrum $\sigma(A)$ consists entirely of geometrically simple eigenvalues.

(iii) The eigenvalues of the dynamics generator A are of the form $\lambda = \pm i\tau^2$, where τ satisfies the following characteristic equation:

$$\begin{aligned}
&k_1 i\tau^2[\sinh\tau\cos\tau + \cosh\tau\sin\tau]\\
&+\tau[1+\cosh\tau\cos\tau - k_1k_2(1-\cosh\tau\cos\tau)]\\
&+k_2 i[\cosh\tau\sin\tau - \sinh\tau\cos\tau] = 0. \qquad (5.8)
\end{aligned}$$

The eigenvector of A corresponding to the eigenvalue $\lambda = i\tau$ is of the form $\langle f(x), \lambda f(x)\rangle$, where

$$\begin{aligned}
f(x) \;=\; &\tau[\cosh\tau(1-x) - \cos\tau(1-x) + \sinh\tau\sin\tau x\\
&+\sinh\tau x\sin\tau - \cosh\tau\cos\tau x + \cosh\tau x\cos\tau]\\
&+k_2 i[-\sinh\tau(1-x) - \sin\tau(1-x) + \sinh\tau\cos\tau x\\
&+\cosh\tau x\sin\tau - \cosh\tau\sin\tau x - \sinh\tau x\cos\tau]. \qquad (5.9)
\end{aligned}$$

Proof.

(i) The authors of [GY01] claim that this is trivial and omit the proof. We proved that A has a compact inverse in Theorem 2.9.6, so A is indeed discrete.

(ii) The spectrum consists entirely of eigenvalues by part (iii) of Theorem 3.2.2. Lemmas 3.4.1 and 3.4.2 give the geometric simplicity of the eigenvalues.

(iii) For nonzero $\langle f(x), g(x)\rangle \in \mathcal{D}(A)$,

$$\begin{aligned}
A\langle f(x), g(x)\rangle &= \lambda\langle f(x), g(x)\rangle\\
\text{if and only if} \begin{bmatrix} g(x) \\ -f^{(4)}(x)\end{bmatrix} &= \begin{bmatrix}\lambda f(x)\\ \lambda g(x)\end{bmatrix}.
\end{aligned}$$

Hence $g(x) = \lambda f(x)$ and by Corollary 2.7.3, $f(x)$ must satisfy

$$\begin{aligned} f^{(4)}(x) - \lambda^2 f(x) &= 0, \\ f(0) = f'(0) &= 0, \\ f''(1) = k_1 g'(1) &= -k_1 \lambda f'(1), \\ f'''(1) = k_2 g(1) &= k_2 \lambda f(1). \end{aligned}$$

Using the results in Subsection 1.4.2 it follows that the eigenvalues of this problem are $\lambda = \pm i\tau^2$ and that the general solution of $f^{(4)}(x) - \tau^4 f(x) = 0$ subject to $f(0) = f'(0) = 0$ is

$$f(x) = c_1(\cosh \tau x - \cos \tau x) + c_2(\sinh \tau x - \sin \tau x), \tag{5.10}$$

where c_1 and c_2 are constants.

Using the boundary condition $f'''(1) = k_2 i\tau^2 f(1)$, these constants are found to be

$$\begin{aligned} c_1 &= \tau(\cosh \tau + \cos \tau) - k_2 i(\sinh \tau - \sin \tau) \text{ and} \\ c_2 &= -\tau(\sinh \tau - \sin \tau) + k_2 i(\cosh \tau - \cos \tau). \end{aligned}$$

We substitute these into (5.10) and use the sum and difference formulae for trigonometric and hyperbolic functions (See for example [SS]) to obtain (5.9).

The boundary condition $f''(1) = -k_1 i\tau^2 f'(1)$ gives (5.8) after multiplying out, grouping terms and dividing by 2. □

A reference Riesz basis is required to apply Theorem 5.2.2 to Problem GY01. To obtain such a basis, the authors of [GY01] suppose that $k_2 = 0$ and consider two special cases of Problem GY01, $y_{xt}(1,t) = 0$ and $y_{xx}(1,t) = 0$.

Special case I:

Define the operator A_0 by (5.7) but with the domain given by

$$\mathcal{D}(A_0) = \{\langle f, g\rangle \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) \text{ such that } f'''(1) = g'(1) = 0\}. \tag{5.11}$$

Special case II:

Let the operator A_0 be the restriction of A (defined by (5.7)) such that

$$\mathcal{D}(A_0) = \{\langle f, g\rangle \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) \text{ such that } f''(1) = f'''(1) = 0\}. \tag{5.12}$$

Remark The authors of Guo and Yu represent the special cases by letting $k_1^{-1} = 0$, $(k_1 \neq 0)$ for $y_{xt}(1,t) = 0$ and $k_1 = 0$ for $y_{xx}(1,t) = 0$ – this procedure is very dangerous and not used here.

Lemma 5.3.2 ([GY01, Lemma 4])**.** Let A_0 be the operator defined for Special case I or Special case II. Then

(i) A_0 is skew-adjoint and discrete. Therefore, the eigenvalues of A_0 are algebraically simple and the sequence of generalized eigenvectors of A_0 forms a Riesz basis for H.

(ii) The eigenvalues of A_0 are of the form $\{\mu_n, \bar{\mu}_n\}$, $\mu_n = i\omega_n^2$ and ω_n may be expressed asymptotically as

$$\omega_n = m\pi + \mathcal{O}(e^{-n}), \tag{5.13}$$

for large n, where

$$m = \begin{cases} (n - 1/4)\pi & \text{for Special case I} \\ (n - 1/2)\pi & \text{for Special case II.} \end{cases}$$

(iii) The eigenvectors $\langle \phi_n, \mu\phi_n \rangle$ of A_0 corresponding to $\mu_n = i\omega_n^2$ may be expressed asymptotically as

$$\begin{bmatrix} \phi_n''(x) \\ \mu_n\phi_n(x) \end{bmatrix} = \frac{1}{2}\omega_n^3 e^{\omega_n} G_n(x),$$

where

$$G_n(x) = \begin{bmatrix} e^{-m\pi x} + e^{m\pi(x-1)}(\sin m\pi + \cos m\pi) - (\sin m\pi x - \cos m\pi x) \\ ie^{-m\pi x} + ie^{m\pi(x-1)}(\sin m\pi + \cos m\pi) + i(\sin m\pi x - \cos m\pi x) \end{bmatrix} + \mathcal{O}(n^{-1}). \tag{5.14}$$

Proof.

(i) It is clear from Section 2.2 that A_0 is skew-adjoint in general. The inverse of A_0 is compact by Theorem 2.9.6 so A_0 is discrete. The fact that the eigenvalues are imaginary and the eigenvectors form a Riesz basis for H may be shown by the process given for the wave equation with constant viscous damping in Chapter 2.

(ii) Consider Special case II first. It was shown in Subsection 1.4.2 that the eigenvalues are $\mu_n = \pm i\omega_n^2$ where ω_n is a positive solution of

$$1 + \cosh\omega\cos\omega = 0. \tag{5.15}$$

This equation is equivalent to

$$-\cos\omega = \frac{1}{\cosh\omega}.$$

For $\omega > 0$, $\dfrac{1}{\cosh\omega} < 2e^{-\omega}$.

The positive solutions of $-\cos\alpha = 0$ are $\alpha_n = (n-1/2)\pi$ for $n = 1, 2, \ldots$. By the Mean Value Theorem,

$$|\alpha - \omega| = \frac{1}{|\sin c|}\frac{1}{\cosh\omega} < 2e^{-\omega}$$

for some $c \in (\min\{\omega, \alpha\}, \max\{\omega, \alpha\})$. Therefore, ω_n tends to α_n rapidly as n increases. Hence the positive solutions ω_n of (5.16) are of the form

$$\omega_n = (n-1/2)\pi + \mathcal{O}(e^{-n})$$

for large positive integer n.

Similarly for Special case I, the eigenvalues are $\mu_n = \pm i\omega_n^2$ where ω_n is a positive solution of

$$\sinh\omega\cos\omega + \cosh\omega\sin\omega = 0 \tag{5.16}$$

and the positive solutions of (5.16) have the form

$$\omega_n = (n-1/4)\pi + \mathcal{O}(e^{-n})$$

for large positive integer n. Hence ω_n and may be expressed asymptotically as

$$\omega_n = m\pi + \mathcal{O}(e^{-n}),$$

for large n, where

$$m = \begin{cases} (n-1/4)\pi & \text{for Special case I} \\ (n-1/2)\pi & \text{for Special case II.} \end{cases}$$

(iii) This is proved for the generalized eigenvectors of A in the proof of Lemma 5.3.4. The result will hold for the eigenvectors of A_0 since A_0 is a special case of A. □

We'll need Rouché's Theorem from complex analysis to obtain our next result.

Definition (Analytic function [SS, p.70]) A complex-valued function $f(z)$ is said to be *analytic* on an open set G if it has a derivative at every point of G.

Theorem 5.3.3 (Rouché's Theorem [SS, Theorem 4, p.361])**.** Suppose f and h are analytic functions on some closed ball $\bar{B}(\xi, r)$ with boundary $C(\xi, r)$ in the complex plane. If the strict inequality

$$|h(z)| < |f(z)|$$

holds for each point z *on the boundary* $C(\xi, r)$, then f and $f + h$ have the same number of zeros (counting multiplicities) *inside* $C(\xi, r)$.

Lemma 5.3.4 ([GY01, Lemma 3])**.** Consider the dynamics generator A defined by (5.7).

(i) The eigenvalues of the dynamics generator A of Problem GY01 may be expressed asymptotically as

$$\tau_n = m\pi + \mathcal{O}(n^{-1}). \tag{5.17}$$

That is, there exists a natural number N such that (5.17) holds for all $n > N$, where

$$m = \begin{cases} (n-1/4)\pi & \text{if } k_1 \neq 0, \\ (n-1/2)\pi & \text{if } k_1 = 0. \end{cases}$$

Moreover,

$$\lambda_n = \begin{cases} -2(k_2 + \frac{1}{k_1})\cos m\pi + i(m\pi)^2 + \mathcal{O}(n^{-1}) & \text{if } k_1 \neq 0, \\ 2k_2 \sin m\pi + i(m\pi)^2 + \mathcal{O}(n^{-1}) & \text{if } k_1 = 0. \end{cases} \tag{5.18}$$

(ii) The eigenvectors $\langle f_n, \lambda_n f_n \rangle$ of A corresponding to $\lambda_n = i\tau_n^2$ may be expressed asymptotically as

$$\begin{bmatrix} f_n''(x) \\ \lambda_n f_n(x) \end{bmatrix} = F_n(x) = \frac{1}{2}\tau_n^3 e^{\tau_n}$$

where

$$F_n(x) = \begin{bmatrix} e^{-m\pi x} + e^{m\pi(x-1)}(\sin m\pi + \cos m\pi) - (\sin m\pi x - \cos m\pi x) \\ ie^{-m\pi x} + ie^{m\pi(x-1)}(\sin m\pi + \cos m\pi) + i(\sin m\pi x - \cos m\pi x) \end{bmatrix} + \mathcal{O}(n^{-1}). \tag{5.19}$$

Remark The authors of [GY01] claim that the asymptotic expression (5.17) is only valid for a 'family' of eigenvalues of A when it is in fact valid for the eigenvalues λ_n of A for n larger than some N. All remarks regarding 'families' in [GY01] may be disregarded.

Proof.

By Lemma 5.3.1, the eigenvalues of A are $\{\lambda_n, \bar{\lambda}_n\}$, where $\lambda_n = i\tau_n^2$ and τ is a solution of (5.8).

We show that the solutions τ_n of (5.8) may be expressed asymptotically by (5.17) and then use this to show that the eigenvalues of A admit the asymptotic representation (5.18). We consider two different cases.

Special case I: $k_1 \neq 0$.

Note that we are only interested in nonzero τ. In this case the characteristic equation (5.8) simplifies to

$$\begin{aligned} \sinh\tau\cos\tau + \cosh\tau\sin\tau &= \frac{i}{k_1\tau}[1 + \cosh\tau\cos\tau - k_1k_2(1 - \cosh\tau\cos\tau)] \\ &- \frac{k_2}{k_1\tau^2}[\cosh\tau\sin\tau - \sinh\tau\cos\tau]. \end{aligned}$$

Dividing the equation above by the function $\cosh\tau$ (which never takes the value 0), we find

$$\begin{aligned} \tanh\tau\cos\tau + \sin\tau &= \frac{i}{\tau}\left[\frac{1}{k_1}\frac{1}{\cosh\tau} + \frac{1}{k_1}\cos\tau - k_2\frac{1}{\cosh\tau} + k_2\cos\tau\right] \\ &- \frac{k_2}{k_1\tau^2}[\sin\tau - \tanh\tau\cos\tau]. \end{aligned}$$

By expressing $\tanh\tau$ and $\cosh\tau$ in terms of the exponential function, it can be seen that

$$\begin{aligned} |1-\tanh\tau| &\le 2e^{-\operatorname{Re}\tau} \\ \text{and } \frac{1}{|\cosh\tau|} &\le 2e^{-\operatorname{Re}\tau}. \end{aligned}$$

Therefore, $\tanh\tau\cos\tau = \cos\tau + \mathcal{O}(e^{-\operatorname{Re}\tau})$ and consequently

$$\cos\tau + \sin\tau = \frac{i}{\tau}\left(\frac{1}{k_1}+k_2\right)\cos\tau + \mathcal{O}(|\tau|^{-1}e^{-\operatorname{Re}\tau}) + \mathcal{O}(|\tau|^{-2}) \tag{5.20}$$
$$\text{or } \cos\tau + \sin\tau = \mathcal{O}(|\tau|^{-1}).$$

Squaring this last equation and applying well-known trigonometric identities, we have

$$(\cos\tau+\sin\tau)^2 = \sin 2\tau + 1 = \mathcal{O}(|\tau|^{-2}).$$

The positive solutions of $\sin 2\alpha = -1$ are of the form $\alpha_n = (n-1/4)\pi$, for $n = 1, 2, \ldots$.

For τ and n large enough that $|\tau|^{-2} < |\sin 2\tau - 1|$ on a circle of radius $1/n$ around $(n-1/4)\pi$, we may apply Rouché's Theorem (Theorem 5.3.3) to conclude that $\sin 2\tau + 1 = O(|\tau|^{-2})$ has a single solution lying inside this circle. That is, $\tau_n = (n-1/4)\pi + \mathcal{O}(n^{-1})$.

By the continuity of $\exp x$, $\sin x$ and $\cos x$, (5.17) implies that for any $y > 0$ and $0 \le x \le 1$:

$$e^{-\tau_n y} = e^{-m\pi y} + \mathcal{O}(n^{-1}), \tag{5.21}$$
$$\sin\tau_n x = \sin m\pi x + \mathcal{O}(n^{-1}) \text{ and } \cos\tau_n x = \cos m\pi x + \mathcal{O}(n^{-1}). \tag{5.22}$$

We derive the estimate (5.18) by substituting (5.17) into (5.20) and using (5.22):

$$\begin{aligned} \sin\tau_n + \cos\tau_n &= \sin m\pi + \cos m\pi + C_1 n^{-1} \\ &= \frac{i\cos m\pi}{m\pi}\left(k_2 + \frac{1}{k_1}\right) + C_2 n^{-1}. \end{aligned}$$

Now $\sin m\pi + \cos m\pi = 0$, so $\dfrac{i}{m\pi}\left(k_2+\dfrac{1}{k_1}\right)\cos m\pi = (C_1 - C_2)n^{-1}$.

Since we already know that $\tau_n = m\pi + \mathcal{O}(n^{-1})$, we can write

$$\tau_n = m\pi + \frac{i}{m\pi}\left(k_2+\frac{1}{k_1}\right)\cos m\pi + \mathcal{O}(n^{-1}).$$

Whence $\lambda_n = i\tau_n^2 = -2\left(k_2 + \dfrac{1}{k_1}\right)\cos m\pi + i(m\pi)^2 + \mathcal{O}(n^{-1})$.

The authors of [GY01] drop the $\cos m\pi$ in their asymptotic expression for λ_n, but this is incorrect as the contribution of this term cannot be ignored – it takes the values $\pm 1/\sqrt{2}$ and alternates the sign of $2(k_2 + k_1^{-1})$ depending on whether n is odd or even.

Special case II: $k_1 = 0$.

We are only interested in nonzero τ so the characteristic equation simplifies to

$$1 + \cosh\tau\cos\tau + \frac{ik_2}{\tau}(\cosh\tau\sin\tau - \sinh\tau\cos\tau) = 0.$$

Dividing the equation above by the nonzero function $\cosh\tau$, we find

$$\cos\tau = -\frac{1}{\cosh\tau} - \frac{ik_2}{\tau}(\sin\tau - \tanh\tau\cos\tau).$$

Following the same procedure as in the previous case, we find the asymptotic expression

$$\cos\tau = -\frac{ik_2}{\tau}(\sin\tau - \cos\tau) + \mathcal{O}(e^{-\operatorname{Re}\tau}) \qquad (5.23)$$
$$\text{or } \cos\tau = O(|\tau|^{-1}).$$

The positive solutions of $\cos\alpha = 0$ are of the form $\alpha_n = (n - 1/2)\pi$, for $n = 1, 2, \ldots$.

For τ and n large enough that $|\tau|^{-1} < |\cos\tau|$ on a circle of radius $1/n$ around $(n - 1/2)\pi$, we may apply Rouché's Theorem (Theorem 5.3.3) to conclude that $\cos\tau = O(|\tau|^{-1})$ has a single solution lying inside this circle. That is, $\tau_n = (n - 1/2)\pi + \mathcal{O}(n^{-1})$.

Thus (5.17) is established. To establish the estimate (5.18), substitute (5.17) into (5.23) and use (5.22) to obtain:

$$\begin{aligned}\cos\tau_n &= \cos m\pi + C_1 n^{-1}\\ &= \frac{i}{m\pi}k_2(\cos m\pi - \sin m\pi) + C_2 n^{-1}.\end{aligned}$$

But $\cos m\pi = 0$, so $-\dfrac{i}{m\pi}k_2\sin m\pi = (C_1 - C_2)n^{-1}$.

We already have $\tau_n = m\pi + \mathcal{O}(n^{-1})$, so we can write

$$\tau_n = m\pi - \frac{i}{m\pi}k_2 \sin m\pi + C_3 n^{-1}.$$

Therefore, $\lambda_n = i\tau_n^2 = 2k_2 \sin m\pi + i(m\pi)^2 + \mathcal{O}(n^{-1})$.

The authors of [GY01] drop the $\sin m\pi$ in their asymptotic expression for λ_n, but this is incorrect as this term alternates the sign of $2k_2$ depending on whether n is odd or even.

We have established the corrected asymptotic expression for the eigenvalues $\lambda_n = i\tau_n^2$ of A. All that remains is to prove that the eigenfunctions associated with these eigenvalues may be expressed by (5.19).

Let $\langle f_n(x), \lambda_n f_n(x)\rangle$ be the eigenfunction of A corresponding to λ_n where $f_n(x)$ is defined by (5.9) with $\tau = \tau_n$. Then

$$\begin{aligned}\tau^{-2} f''(x) = \tau[&\cosh \tau(1-x) + \cos \tau(1-x) - \sinh \tau \sin \tau x \\ &+ \sinh \tau x \sin \tau + \cosh \tau \cos \tau x + \cosh \tau x \cos \tau] \\ &+ k_2 i[-\sinh \tau(1-x) + \sin \tau(1-x) + \sinh \tau \cos \tau x \\ &- \cosh \tau x \sin \tau + \cosh \tau \sin \tau x - \sinh \tau x \cos \tau].\end{aligned}$$

Now letting $\tau = \tau_n$ and expressing the trigonometric and hyperbolic functions in terms of the exponential function, we find

$$2e^{-\tau}\tau^{-3} f''(x) = e^{-\tau x} - (\cos \tau x - \sin \tau x) + e^{-\tau(1-x)}(\cos \tau - \sin \tau) + \mathcal{O}(e^{-\operatorname{Re}\tau}).$$

Applying (5.21) and (5.22) to this, we have the first component of (5.19). The estimate for the second component follows in exactly the same way. $\square$

Remarks

1. The authors of [GY01] take a very dubious step to obtain the asymptotic expression of the eigenvalues. They substitute the expression obtained for τ into the the simplified characteristic equation and use an expression of the form $2\mathcal{O}(n^{-1}) = x$ to conclude that $\mathcal{O}(n^{-1}) = x/2$. This is a completely unjustified abuse of "Big O" notation as the "=" sign here does not have the usual symmetric properties of equality. In fact, $2\mathcal{O}(n^{-1}) = \mathcal{O}(n^{-1})$. Unsurprisingly, the asymptotic expressions arrived at in [GY01] are incorrect.

2. Note that by (5.19) and (5.14),

$$F_n(x) - G_n(x) = \mathcal{O}(n^{-1}). \tag{5.24}$$

This will be used in the proof of the next theorem.

Theorem 5.3.5 ([GY01, Theorem 2]). Let A be defined as in (5.7). Then

(i) The sequence of generalized eigenvectors of A forms a Riesz basis for the state space H.

(ii) The eigenvalues of A have an asymptotic expression (5.18).

(iii) There is an $M > 0$ such that every λ_n is algebraically simple for $n > M$.

Proof.

By Lemma 5.3.1, A is a densely defined discrete operator on H. By Lemma 5.3.2, the eigenvectors $\langle \phi_n(x), \mu\phi_n(x)\rangle$ of A_0 and their conjugates form a Riesz basis for H. We need only show that the eigenvectors of A are quadratically close to those of A_0. Then the hypotheses of Theorem 5.2.2 are satisfied and parts (i) and (iii) follow directly.

By Lemmas 5.3.4 and 5.3.2, the generalized eigenvectors $\langle f_n(x), \lambda f_n(x)\rangle$ of A and $\langle \phi_n(x), \mu\phi_n(x)\rangle$ of A_0, may be expressed asymptotically in such a way that (5.19) and (5.14) hold respectively:

$$F_n(x) = \frac{2\tau_n^{-3}}{e^{\tau_n}} \begin{bmatrix} f_n''(x) \\ \lambda_n f_n(x) \end{bmatrix}$$

and

$$G_n(x) = \frac{2\omega_n^{-3}}{e^{\omega_n}} \begin{bmatrix} \phi_n''(x) \\ \mu_n \phi_n(x) \end{bmatrix},$$

The authors now show that $2\tau_n^{-3}e^{-\tau_n}\langle f_n(x), \lambda_n f_n(x)\rangle$ is quadratically close to $2\omega_n^{-3}e^{-\omega_n}\langle \phi_n(x), \mu_n\phi_n(x)\rangle$ using $F_n(x)$ and $G_n(x)$.

$$\begin{aligned}
&\sum_{n>N}^{\infty} \left\| \frac{2\tau_n^{-3}}{e^{\tau_n}} \begin{bmatrix} f_n(x) \\ \lambda_n f_n(x) \end{bmatrix} - \frac{2\omega_n^{-3}}{e^{\omega_n}} \begin{bmatrix} \phi_n(x) \\ \mu_n \phi_n(x) \end{bmatrix} \right\|_H^2 \\
=\ &\sum_{n>N}^{\infty} \left\| \frac{2\tau_n^{-3}}{e^{\tau_n}} \begin{bmatrix} f_n''(x) \\ \lambda_n f_n(x) \end{bmatrix} - \frac{2\omega_n^{-3}}{e^{\omega_n}} \begin{bmatrix} \phi_n''(x) \\ \mu_n \phi_n(x) \end{bmatrix} \right\|_{L^2(0,1)}^2 \\
=\ &\sum_{n>N}^{\infty} \|F_n(x) - G_n(x)\|_{L^2(0,1)}^2 \\
\leq\ &C \sum_{n>N}^{\infty} \frac{1}{n^2} < \infty.
\end{aligned}$$

The authors of [GY01] assume that this implies that $\{\langle f_n(x), \lambda_n f_n(x)\rangle\}_1^{\infty}$ is quadratically close to $\{\langle g_n(x), \mu_n g_n(x)\rangle\}_1^{\infty}$. Further, it is unclear as to whether it is legitimate to conclude from this that $\{\langle f_n(x), \lambda_n f_n(x)\rangle\}_1^{\infty}$ is a Riesz basis. $\square$

Chapter 6

Operator pencil approach

In this chapter, the approach and results of [JTW08] are discussed. The authors of [JTW08] use the theory of *polynomial operator pencils* and *Krein spaces* to arrive at their results. Brief introductions to these topics (which include the results used here) are given in Section 3.6.

In order to compare the approach taken in this dissertation to that taken in [JTW08] it is necessary to take a closer look at the dynamics generator A.

6.1 Properties of operators

Theorem 6.1.1 (Properties of operators)**.** The dynamics generator A has the following properties:

1. A has a bounded inverse ($A = \Lambda^{-1}$, See Subsection 1.7.4).

2. The domain $\mathcal{D}(A)$ is dense in the state space H ($A = \Lambda^{-1}$, See Subsection 1.7.4).

3. The range of A is the entire state space H, ($A = \Lambda^{-1}$, See Subsection 1.7.4).

4. A is injective ($A = \Lambda^{-1}$, See Subsection 1.7.4).

5. A is dissipative (See Section 2.1).

6. The eigenvalues of A have negative real part [VV02, Corollary 1].

7. A is the infinitesimal generator of a C_0 semigroup, (See [VV02, Proof of Theorem 1]).

8. A is a closed operator (See [VV02, Lemma 3]).

The operator Q has the following properties:

1. $-Q$ is selfadjoint and uniformly positive on W:

   It was shown in Section 2.2 that Q is selfadjoint. $-Q$ is uniformly positive since

   $$\gamma(-Qw, w) = \beta(w, w) \geq C\gamma(w, w) \geq 0.$$

2. Q has a bounded inverse P that is selfadjoint (See Section 3.1).

3. $P = Q^{-1}$ is compact if the embedding of V in W is compact (See Proof of Theorem 3.1.3).

The damping operator Δ has the following properties:

1. The linear operator Δ is bounded (By definition of Δ using the Riesz Representation Theorem).

2. The operators Δ and $P^{1/2}DP^{1/2} : \mathcal{D}(P^{1/2}DP^{1/2}) \subset V \to W$ are selfadjoint:

   It was shown in Section 2.2 that Δ is selfadjoint. Now since P is selfadjoint,[Kr, Theorem 9.4-2, p.476] implies that there exists a unique bounded selfadjoint linear operator $P^{1/2} : \mathcal{D}(P^{1/2}) \subset V \to W$ such that $P = P^{1/2}P^{1/2}$.

   $$\begin{aligned} \beta(P^{1/2}\Delta P^{1/2}x, y) &= \beta(\Delta P^{1/2}x, P^{1/2}y) \\ &= \beta(P^{1/2}x, \Delta P^{1/2}y) \\ &= \beta(x, P^{1/2}\Delta P^{1/2}y). \end{aligned}$$

3. The damping operator Δ is non-negative:

   For $x \neq 0 \in V$, $\beta(\Delta x, x) = \alpha(x, x) \geq 0$.

6.2 Operator theoretic approach

In [JTW08], the authors consider vibration problems of the same type as considered in Chapters 1 and 2. Their approach is based on abstract operator theory, which leads them to write the vibration problem as a second order differential equation in an abstract Hilbert space:

$$z''(t) + Dz'(t) + A_0 z(t) = 0, \tag{6.1}$$

where A_0 is the stiffness operator and D is the damping operator. These operators contain the spatial derivatives.

The authors of [JTW08] convert this equation a first order system and prove that under certain conditions on the operators involved, the generalized eigenvectors of the dynamics generator A form a Riesz basis for H and that A generates a holomorphic C_0 semigroup in H.

It will be shown in Section 6.4 that the work of [JTW08] can be framed in the setting of the rest of this dissertation.

Before examining the results of [JTW08], it is necessary to consider the setting the authors work in.

6.2.1 Assumptions and definitions

Unfortunately, it is impossible to change all the notation in [JTW08] to be consistent with the rest of this dissertation. **The reader is thus advised to bear in mind that the symbols in this section may have different meanings than in the rest of this dissertation.**

The following table summarizes the major differences in notation.

Dissertation	Q	Δ	$\alpha(\cdot,\cdot)$	$\beta(\cdot,\cdot)$	$\gamma(\cdot,\cdot)$
[JTW08]	A_0	D	$(D\cdot,\cdot)_W$	$(A_0\cdot,\cdot)_W$	$(\cdot,\cdot)_W$

Table 6.1: Notation

The authors of [JTW08] make the following assumptions and definitions:

Assumption on V and W:

Assume that V and W are infinite dimensional. Denote the norm and inner product on W by $\|\cdot\|_W$ and $(\cdot,\cdot)_W$ respectively. The norm and inner product on V are denoted similarly with the subscript V.

Definition (Non-negative operator, positive operator [KL78(I), p. 365]) A selfadjoint linear operator on a Hilbert space X is *non-negative* if

$$(Tx,x)_X \geq 0 \text{ for all } x \in X.$$

T is a *positive operator* if $(Tx,x)_X > 0$ for all $x \neq 0 \in X$.

Definition (Uniformly positive operator [KL78(II), p.365]) A selfadjoint linear operator on a Hilbert space X is *uniformly positive* if

$$\inf_{\substack{x\in X\\ x\neq 0}} \frac{(Tx,x)_X}{(x,x)_X} > 0.$$

Assumptions on A_0:

(JA1) The linear operator $A_0 : \mathcal{D}(A_0) \subset W \to W$ is selfadjoint and uniformly positive operator on W.

(JA2) A_0 has a bounded inverse.

(JA3) A_0^{-1} is compact. The authors of [KL78(I)] remark that this assumption is fundamental.

By [Kr, Theorem 9.4-2, p.476] there exists a unique bounded selfadjoint linear operator $A_0^{1/2} : \mathcal{D}(A_0^{1/2}) \subset V \to W$ such that $A_0 = A_0^{1/2}A_0^{1/2}$.

Definition Let $V = \mathcal{D}(A_0^{1/2})$ be a Hilbert space with the inner product

$$(x,y)_V = \left(A_0^{1/2}x, A_0^{1/2}y\right)_W = (A_0x,y)_W\,.$$

This inner product induces the norm $\|x\|_V = \left\|A_0^{1/2}x\right\|_W$.

Assumptions on D:

(JD1) The linear operator $D : V \subset W \to W$ is bounded.

(JD2) The operator $A_0^{-1/2}DA_0^{-1/2} : W \to W$ is bounded and selfadjoint.

(JD3) The damping operator D is non-negative. That is,

$$(Dz,z)_W \geq 0 \text{ for all } z \in V \tag{6.2}$$

Definition Define the state space H as before as $H = V \times W$. Define the inner product on H by

$$(x, y)_H = (\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle) = (x_1, y_1)_V + (x_2, y_2)_W \text{ for all } x, y \in H.$$

This induces the norm

$$\|x\|_H^2 = (x_1, x_1)_V + (x_2, x_2)_W = \|x_1\|_V^2 + \|x_2\|_W^2 .$$

Note that the inner product and state space are complex.

Remarks

1. The definition of V in [JTW08] is counter-intuitive for applications, where one is not faced with an operator, but rather a partial differential equation (possibly in variational form). It is more natural to consider the relevant spaces first and then define operators in those spaces (as is done in the rest of this dissertation).
2. The assumption that A_0^{-1} is compact forces the relationship between these spaces which would arise naturally if V and W were defined as in the rest of this dissertation.
3. Starting with operators and working back to problems may lead one to make assumptions that make the results obtained inapplicable, especially when assumptions are made ad hoc (as they are in [JTW08]).

6.2.2 Conversion to first order system

The authors of [JTW08] convert (6.1) to a first order system as follows:

Let $w(t) = z'(t)$. Then $w'(t) = -A_0 z(t) - Dw(t)$.

This system of equations can be expressed as a first order ordinary differential equation in the product space $H = V \times W$ by letting $x(t) = \langle z(t), w(t) \rangle$. Then

$$x'(t) = Ax(t)$$

where $A : \mathcal{D}(A) \subset V \times W \to V \times W$ is the dynamics generator defined on

$$\mathcal{D}(A) = \{\langle z, w \rangle \in V \times V \ : \ A_0 z + Dw \in W\}$$

by

$$A \langle x_1, x_2 \rangle = \langle x_2, -A_0 x_1 - Dx_2 \rangle$$

which the authors of [JTW08] write as the matrix

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}.$$

According to [TW03], this dynamics generator is dissipative and has a bounded inverse given by

$$A^{-1}\langle x_1, x_2\rangle = \left\langle -A_0^{-1}(Dx_1 + x_2), x_1 \right\rangle, \tag{6.3}$$

where $A_0^{-1}D : V \to V$. The inverse may be denoted as the matrix

$$A = \begin{bmatrix} -A_0^{-1}D & -A_0^{-1} \\ I & 0 \end{bmatrix}.$$

6.3 Results

In the results that follow, it is implicit that all the assumptions in Subsection 6.2.1 hold.

Proposition 6.3.1 ([JTW08, Proposition 3.1])**.** The dynamics generator A has a bounded inverse and is the generator of a strongly continuous semigroup of contractions on H. The spectrum of A is contained in the closed left half plane and is symmetric with respect to the real axis.

Remark The authors of [JTW08] claim that it is well-known that A is the generator of a strongly continuous semigroup of contractions on H (This is proved in [VV02]). By the Hille-Yosida theorem [Pa, Theorem 3.1, p.8], $\mathcal{D}(A)$ is dense in H and the resolvent of A contains the right half-plane $\{x \in \mathbb{C} \mid \operatorname{Re} x > 0\}$.

6.3.1 Selfadjointness of the dynamics generator in a Krein space

It was shown in Chapter 2 that the dynamics generator A is in general not selfadjoint in the Hilbert space $H = V \times W$. However, according to [JTW08], the operator

$$JA, \text{ where } J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is selfadjoint in H. This suggests that by changing the inner product on the state space H, we may treat A as a selfadjoint operator in the new inner product space. This motivates the use of so-called Krein spaces, where the new inner product may be indefinite.

Definition (Krein space [JTW08], [Ha]) Suppose $(H, (\cdot, \cdot))$ is a Hilbert space and $G : H \to H$ is an invertible selfadjoint linear operator such that $G^2 = I$. Then

$$[x, y] = (Gx, y)\,, \;\; x, y \in H$$

defines an indefinite inner product on H. The space $\mathcal{H} = (H, [\cdot, \cdot])$ is known as a *Krein space*.

Remark This is not the most general way to define a Krein space, but it is natural and sufficiently general for our purposes. It also spares us from various topological nightmares. A more general definition may be found in Subsection 3.6.1.

Definition In [JTW08], the authors define an indefinite inner product on H by

$$\begin{aligned}
[\langle x_1, x_2\rangle\,, \langle y_1, y_2\rangle] &= (J\,\langle x_1, x_2\rangle, \langle y_1, y_2\rangle)_H \\
&= (\langle x_1, -x_2\rangle, \langle y_1, y_2\rangle)_H \\
&= (x_1, y_1)_V - (x_2, y_2)_W\,.
\end{aligned}$$

Note that J is selfadjoint, invertible, $\|J\|_H = 1$ and $J^2 = I$. We denote the Krein space $(H, [\cdot, \cdot])$ by $\mathcal{H}$.

Remark All topological notions may be understood with respect to the Hilbert space norm $\|\cdot\|_H$ since $[\cdot, \cdot]$ is $\|\cdot\|_H$ −continuous:

For any $x, y \in H$: $|[x, y]|^2 = |(Jx, y)|^2 \leq \|J\|_H \, \|x\|_H \, \|y\|_H$.

Lemma 6.3.2. The damping operator D is selfadjoint with respect to $(\cdot, \cdot)_W$

Proof.

Let $x = A_0^{-1/2}x_0, y = A_0^{-1/2}y_0 \in V$ be arbitrary, then

$$\begin{aligned}
(Dx, y)_W &= \left(DA_0^{-1/2}x_0, A_0^{-1/2}y_0\right)_W \\
&= \left(A_0^{-1/2}DA_0^{-1/2}x_0, y_0\right)_W \\
&= \left(x_0, A_0^{-1/2}DA_0^{-1/2}y_0\right)_W \\
&= \left(x_0, A_0^{-1/2}Dy\right)_W \\
&= \left(A_0^{-1/2}x_0, Dy\right)_W \\
&= (x, Dy)_W\,,
\end{aligned}$$

since $A_0^{-1/2}DA_0^{-1/2}$ is selfadjoint by assumption (JD2). □

Theorem 6.3.3. The dynamics generator A is selfadjoint in the Krein space $(H, [\cdot,\cdot])$.

Proof.

Let x and $y \in H = V \times W$ be arbitrary:

$$\begin{aligned}
[Ax, y] &= (x_2, y_1)_V - (-A_0 x_1 - D x_2, y_2)_W \\
&= (x_2, y_1)_V + (A_0 x_1, y_2)_W + (D x_2, y_2)_W \\
&= (x_2, A_0 y_1)_W + (x_1, y_2)_V + (D x_2, y_2)_W \\
\text{and } [x, Ay] &= (x_1, y_2)_V - (x_2, -A_0 y_1 - D y_2)_W \\
&= (x_1, y_2)_V + (x_2, A_0 y_1)_W + (x_2, D y_2)_W \\
&= [Ax, y],
\end{aligned}$$

Since D is self adjoint by Lemma 6.3.2 and A_0 is selfadjoint by assumption (JA1). □

6.3.2 Main results

The authors of [JTW08] denote the spectral points of an operator T that are not eigenvalues of finite multiplicity by $\sigma_{ess}(T)$. The main results of [JTW08] are as follows:

Theorem 6.3.4. [JTW08, Theorem 4.1] Assume that A_0^{-1} is a compact operator in H and that $0 \notin \sigma_{ess}(A_0^{-1}D)$. Then A generates an analytic semigroup on H.

Theorem 6.3.5 ([JTW08, Theorem 5.1])**.** Assume that A_0^{-1} is a compact operator in H and that $0 \notin \sigma_{ess}(A_0^{-1}D)$. Assume that the set $\sigma_{ess}(A_0^{-1}D)$ is countable and has at most countably many accumulation points. Moreover, let at least one of the following conditions be satisfied:

(a) (Strong Damping) There exists a $\delta > 0$ such that for all $f \in V$ with $\|f\|_V = 1$ we have

$$\left(A_0^{-1} D f, f\right)_V^2 > \delta + 4\left(A_0^{-1} f, f\right)_V . \tag{6.4}$$

(b) For all $\mu \in \sigma_{ess}(A_0^{-1}D)$ we have either $\frac{1}{\mu} \notin \sigma_p(A)$ or, if $\frac{1}{\mu} \in \sigma_p(A)$, there exists no non-zero $\begin{bmatrix} y \\ \mu^{-1} y \end{bmatrix} \in \mathcal{N}(A - \mu^{-1} I)$ such that

$$\mu^2 (y, w)_V = (y, w)_W \quad \text{for all } \begin{bmatrix} w \\ \mu^{-1} w \end{bmatrix} \in \mathcal{N}(A - \mu^{-1} I). \tag{6.5}$$

(c) $\left\| A_0^{-1/2} \right\| < \inf \left\{ \lambda > 0 \mid \lambda \in \sigma_{ess}(A_0^{-1}D) \right\}$.

Then

(a) There exists a subspace of H of at most finite codimension which has a Riesz basis consisting of eigenvectors of A.

(b) The generalized eigenvectors of A form a Riesz basis of H.

(c) If condition (a) holds, then all the generalized eigenvectors of A are eigenvectors of A and the spectrum of A is real.

The result for the case where condition (a) holds was proved in [KL78(II), Section 7] (see Section 6.6 for more details). The approach taken in [JTW08] is to prove that the pencil is *strongly hyperbolic* (see [Ma, Lemma 31.23]) and that this yields the result. This is frustrating, but the result is true and can be obtained using [Ma, Lemma 31.1], [Ma, Lemma 31.16], [Ma, Lemma 31.18] and [Ma, Lemma 30.10].

The proof of Theorem 6.3.5 in [JTW08] for the case where condition (b) or (c) holds is cryptic at best. The proof hinges on a method the authors developed in [JT07] that involves decomposing the spectrum into parts that exhibit various properties. After this decomposition of the spectrum, the authors of [JTW08] simply state that the generalized eigenvectors of A form a Riesz basis for H, without motivation or citing a reference.

After correspondence with Prof. Trunk about the proof of Theorem 6.3.5, he was kind enough to send his more recent work [Tr10]. Using this paper and the general results collected in Subsection 3.6.1 it is possible to prove the result of Theorem 6.3.5 under more general conditions than those of Theorem 6.3.5. This is done for the general dynamics generator A once we have returned to the framework used elsewhere in this dissertation.

6.4 Return to general framework

We now show that the work of [JTW08] can be framed in the setting of [VV02]. Once this is done we can return to the notation used elsewhere in this dissertation. We make the following definitions:

Definition Let $\alpha(\cdot,\cdot) = (D\cdot,\cdot)_W$, $\beta(\cdot,\cdot) = (A_0\cdot,\cdot)_W$ and $\gamma(\cdot,\cdot) = (\cdot,\cdot)_W$.

We need to show that the bilinear forms defined in [JTW08] satisfy the assumptions on the bilinear forms in [VV02].

Theorem 6.4.1. The bilinear forms defined above have the following properties:

1. $V = \mathcal{D}(A_0^{1/2})$ is dense in W,
2. There exists a constant C such that $\|v\|_W \leq C \|v\|_V$ for each $v \in V$,
3. The bilinear form $\alpha(\cdot,\cdot)$ is non-negative, symmetric (Hermitian in the complex case) and bounded on V.

Proof.

We deal with each point in turn.

1. Since A is the infinitesimal generator of a C_0 semigroup, $\mathcal{D}(A)$ is dense in $H = V \times W$, but $\mathcal{D}(A) \subset V \times V$. Therefore V must be dense in W.
2. For any $v \in V$:
$$\|v\|_W^2 = \left|\left(A_0^{-1}v, v\right)_V\right| \leq \left\|A_0^{-1}v\right\|_V \|v\|_V \leq \left\|A_0^{-1}\right\|_V \|v\|_V^2 .$$
So we can merely let $C = \sqrt{\left\|A_0^{-1}\right\|_V}$.
3. By assumption (JD3), the bilinear form $\alpha(\cdot,\cdot)$ is non-negative, see (6.2). It is also assumed that D is bounded, so for any $x, y \in V$:
$$|\alpha\,(x,y)| = |(Dx,y)_W| \leq C^2 \|D\|_V \|x\|_V \|y\|_V .$$
Therefore, $\alpha(\cdot,\cdot)$ is bounded. Finally, we show that $\alpha(\cdot,\cdot)$ is Hermitian (symmetric in the real case).

   Let $x = A_0^{-1/2}x_0, y = A_0^{-1/2}y_0 \in V$ be arbitrary, then
$$\alpha\,(x,y) = (Dx,y)_W = \overline{(Dy,x)_W} = \overline{\alpha\,(y,x)},$$
since D is selfadjoint by Lemma 6.3.2. □

This shows if we make the same assumptions as [JTW08], we find ourselves in the framework of [VV02]. It remains to show that the dynamics generators are equal.

Theorem 6.4.2. The dynamics generator A defined in [JTW08] is equal to the dynamics generator defined for Problem JTW08 in Chapter 2.

Proof.

We show that the inverse $A^{-1} : \mathcal{R}(A) \to \mathcal{D}(A) \subset V \times V$ of the dynamics generator defined in [JTW08] corresponds to Λ defined in Chapter 2:

Let $A^{-1}y = -x$. That is

$$\begin{bmatrix} -A_0^{-1}D & -A_0^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}.$$

Therefore $y_1 = -x_2$ and

$$A_0^{-1}Dy_1 + A_0^{-1}y_2 = x_1.$$

So

$$-A_0^{-1}Dx_2 + A_0^{-1}y_2 = x_1.$$

Note that x and y are in $V \times V$ so we can take the inner product $(\cdot,\cdot)_V$ with an arbitrary element of V:

$$\begin{aligned} \left(-A_0^{-1}Dx_2, v\right)_V + \left(A_0^{-1}y_2, v\right)_V &= (x_1, v)_V \text{ for all } v \in V, \\ \text{so} \quad (x_1, v)_V + (Dx_2, v)_W &= (y_2, v)_W \text{ for all } v \in V \\ \text{and} \quad \beta(x_1, v) + \alpha(x_2, v) &= \gamma(y_2, v) \text{ for all } v \in V. \end{aligned}$$

Therefore $A^{-1}y = \Lambda y = -x$.

On the other hand, let $\Lambda y = -x$. Then $y_1 = -x_2$ and

$$\beta(x_1, v) + \alpha(x_2, v) = \gamma(y_2, v) \text{ for all } v \in V.$$

Since $\alpha(\cdot,\cdot) = (D\cdot,\cdot)_W$, $\beta(\cdot,\cdot) = (A_0\cdot,\cdot)_W$ and $\gamma(\cdot,\cdot) = (\cdot,\cdot)_W$,

$$\begin{aligned} (x_1, v)_V + (Dx_2, v)_W - (y_2, v)_W &= 0 \text{ for all } v \in V, \\ \text{so} \quad -\left(A_0^{-1}Dy_1, v\right)_V - \left(A_0^{-1}y_2, v\right)_V + (x_1, v)_V &= 0 \text{ for all } v \in V, \\ \text{hence} \quad \left(-A_0^{-1}Dy_1 - A_0^{-1}y_2, v\right)_V + (x_1, v)_V &= 0 \text{ for all } v \in V. \end{aligned}$$

Since $y_1 = -x_2$, $(y_1 + x_2, w)_W = 0$ for every w in W. Combining this with the equation above and letting $z = \langle v, w\rangle$ denote an arbitrary element of H, we have,

$$\left(-A_0^{-1}Dy_1 - A_0^{-1}y_2 + x_1, v\right)_V + (y_1 + x_2, w)_W = 0 \text{ for all } z \in H.$$

Hence

$$\begin{aligned}
&\left(\begin{bmatrix} -A_0^{-1}Dy_1 - A_0^{-1}y_2 + x_1 \\ y_1 + x_2 \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix}\right)_H \\
= &\left(\begin{bmatrix} -A_0^{-1}Dy_1 - A_0^{-1}y_2 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix}\right)_H \\
= &\left(\begin{bmatrix} -A_0^{-1}D & -A_0^{-1} \\ I & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} v \\ w \end{bmatrix}\right)_H \\
= &\ 0 \text{ for all } z \in H.
\end{aligned}$$

Therefore

$$\left(A^{-1}y + x, z\right) = 0 \text{ for all } z \in H.$$

This implies that $A^{-1}y+x$ is orthogonal to every element of H, so $A^{-1}y+x = 0$. Therefore $\Lambda y = A^{-1}y = -x$. Taking the inverses completes the proof. □

We may now compare the constituent operators of the two dynamics generators:

Chapter 2

$$Ax = \langle y_1, y_2\rangle = \langle x_2, Qx_1 + Q\Delta x_2\rangle\,, \quad A = \begin{bmatrix} 0 & I \\ Q & Q\Delta \end{bmatrix}.$$

[JTW08]

$$A\,\langle x_1, x_2\rangle = \langle x_2, -A_0x_1 - Dx_2\rangle\,, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}.$$

So

$$Q = -A_0 \quad \text{and} \quad Q\Delta = -D.$$

We now dispose of the framework of [JTW08] and return to the setting and notation of the rest of the dissertation.

6.5 A theorem

In the proof of Theorem 6.3.5, it is shown that under condition (b) or (c) the dynamics generator A has only finitely many associate eigenvectors. In the following theorem these (very technical) conditions are replaced by the single (very clear) condition that A must have only finitely many associate eigenvectors.

Theorem 6.5.1. Assume that the assumptions of Subsection 6.2.1 hold, Q^{-1} is a compact operator in H and that Δ is boundedly invertible in V. If the dynamics generator A has only finitely many associate eigenvectors then:

(i) There exists a subspace E of H with finite dimensional complement which has a Riesz basis consisting of eigenvectors of A.

(ii) The generalized eigenvectors of A form a Riesz basis of H.

Proof.

The proof is divided into three steps:

(i) The sequence of generalized eigenvectors of A are complete in H.

(ii) The eigenvectors of A forms a Riesz basis for the closure of their span E in H.

(iii) The complement of E is finite dimensional and generalized eigenvectors of A form a Riesz basis of H.

We recall that A is selfadjoint in the Krein space $\mathcal{H}$ by Theorem 6.3.3 and dissipative (See Subsection 2.1). Proceeding stepwise:

(i) By Theorem 4.3.8, the sequence of generalized eigenvectors of A is complete in H if its spectrum is discrete. By [Tr10, Proposition 3.5], $\sigma(A)$ consists entirely of isolated eigenvalues of finite multiplicity, except for possibly the point 0, if it is in the spectrum of A. We assume that $0 \notin \sigma(A)$ for all our results, so the spectrum of A is discrete and Theorem 4.3.8 yields the completeness of the generalized eigenvectors of A in H.

(ii) Denote the eigenvalues of A by $\{\lambda_n\}_1^\infty$ and the closure of the span of their corresponding eigenvectors by E. By Theorem [JTW08, Theorem 4.2], A has only finitely many non-real eigenvalues. Say there are N such eigenvalues and order the sequence of eigenvalues so that the non-real eigenvalues are $\{\lambda_n\}_1^N$. Then $\mathrm{Im}\lambda_n = 0$ for $n > N$. Now it is clear that

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{\infty} \frac{\mathrm{Im}\lambda_j \mathrm{Im}\lambda_k}{|\lambda_j - \overline{\lambda}_k|} = \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{\mathrm{Im}\lambda_j \mathrm{Im}\lambda_k}{|\lambda_j - \overline{\lambda}_k|} < \infty.$$

Hence the eigenvectors of A form a basis quadratically close to an orthonormal basis in E by Theorem 4.3.9. The eigenvectors of A are $\omega-$linearly independent by Proposition 4.3.7. Now, by Theorem 4.3.2, the eigenvectors of A form a Riesz basis for E.

(iii) The generalized eigenvectors of A are complete in H and are $\omega-$linearly independent by Proposition 4.3.7, so the complement of E is spanned by the associate eigenvectors of A. We assumed that A has only finitely many associate eigenvectors so $E^{\perp}$ is finite dimensional.

The span of the generalized eigenvectors of A is complete and $\omega-$linearly independent in H and the sequence of eigenvectors is quadratically close to an orthonormal basis of E. There are only finitely many associate eigenvectors, so the sequence of generalized eigenvectors is quadratically close to an orthonormal basis for H. The sequence of generalized eigenvectors thus forms a Riesz basis of H by Theorem 4.3.2. □

Remark The assumption that the dynamics generator A has only finitely many associated eigenvectors is realistic – if the damping in the general linear vibration problem is modal, it will be satisfied.

6.6 Strong Damping

In [JTW08], the authors refer to strong damping as it was defined for pencils by Krein and Langer in [KL78(II), Section 7].

Definition (Strongly damped operator pencil, [KL78(II), Section 7]) An operator pencil $L(\lambda) = \lambda^2 I + \lambda B + C$ in a Hilbert space H is said to be *strongly damped* if

$$(Bx, x)_H^2 > 4\,(Cx, x)_H\,(x, x)_H \text{ for all } x \neq 0.$$

Remark In terms of the bilinear forms of [VV02], L is a strongly damped pencil if

$$\alpha\,(x, x)^2 > 4\beta\,(x, x)\,\gamma\,(x, x) \text{ for all } x \neq 0 \in H = V \times W.$$

The following results are given for strongly damped pencils in [KL78(II), Section 7]:

Proposition 6.6.1. [KL78(II), Proposition 7.1, p.553] Every eigenvalue of a strongly damped pencil L is real and negative.

Proposition 6.6.2. [KL78(II), Theorem 7.1, p.557] The eigenvectors of a strongly damped pencil L form a Riesz basis for H.

Remark In [KL78(II)], the details of this are not fleshed out. The authors of that paper prove that the eigenvectors of the operator root Z of the pencil L form a Riesz basis for H. By [Ma, Lemma 22.10, p115], the eigenvectors of the pencil L form a Riesz basis for H.

Corollary 6.6.3. By definition of the eigenvectors of a pencil, the eigenvectors of a A form a Riesz basis for H.

Proposition 6.6.4. [KL78(II), Proposition 7.3, p.555] If L is a strongly damped pencil, then B is uniformly positive.

Remark This means that the definition of strong damping in [KL78(II)] is a special case of strong damping as defined in [VV02]. Langer himself referred to this type of damping as "über stark" damping, which translates to super strong or overly strong damping. We will follow his example and refer to the situation where L is a strongly damped pencil as *very strong damping*.

Definition (Very strong damping, [KL78(II), Section 7]) The damping in the general vibration problem is referred to as *very strong damping* if

$$\alpha(x,x)^2 > 4\beta(x,x)\gamma(x,x) \text{ for all } x \neq 0 \in H = V \times W.$$

In [VV02], the following (weaker) definition of strong damping is used.

Definition (Strong damping, [VV02, p.1148]) The damping in the general vibration problem is referred to as *strong damping* if the damping bilinear form $\alpha(\cdot,\cdot)$ is positive definite on V. That is, if $\alpha(u,u) \geq C\|u\|_V^2$ for any $u \in V$.

Remark The authors of [KL78(II)] remark, but do not prove, that (with some complications) their results for very strong damping hold for strong damping. One such complication is that the pencil L has at most finitely many non-real eigenvalues and that in this case, it is the generalized eigenvectors of L which form a Riesz basis for H.

Let us now turn our attention to the situation at hand. The pencil considered in [JTW08] is

$$\begin{aligned} L(\lambda) &= \lambda^2 I + \lambda A_0^{-1} D + A_0^{-1} \\ &= \lambda^2 I + \lambda\Delta + (-Q). \end{aligned}$$

The condition given in (6.4) is that there exists a $\delta > 0$ such that for all $f \in V$ with $\|f\|_V = 1$ we have

$$(\Delta f, f)_V^2 > \delta + 4((-Q)f, f)_V .$$

recall that $-Q$ is uniformly positive on W. So if we let $B = \Delta$ and $C = -Q$ we have

$$(Bf, f)_V^2 > \delta + 4\,(Cf, f)_V\,(f, f)_V \text{ for all } f \text{ such that } \|f\|_V^2 = 1.$$

Further, the damping must satisfy the weaker condition, since $\alpha\,(f, f) = (\Delta f, f)_V$ by definition. Therefore

$$\begin{aligned}
\alpha\,(f, f)^2 &= (\Delta f, f)_V^2 \\
&= (Bf, f)_V^2 \\
&> (Cf, f)_V\,(f, f)_V \\
&= ((-Q)f, f)_V\,(f, f)_V \\
&= (f, f)_W\,(f, f)_V \\
&= \|f\|_W^2\,\|f\|_V^2\,.
\end{aligned}$$

Therefore, the results of [KL78(II)] apply when condition (a) of Theorem 6.3.5 is satisfied. In this case, Corollary 6.6.3 gives the result of Theorem 6.3.5, obviating the proof in [JTW08].

Chapter 7

Conclusion

7.1 Biorthogonality

In this section we present the approach taken in [CZ94, Section 6] to prove that the generalized eigenvectors of the dynamics generator of Problem CZ94 form a Riesz basis for the Hilbert space H. This approach is also taken in [Sh02]. It relies on Bari's theorem (Theorem 4.3.1), specifically the equivalence of conditions (ii) and (iv). We restate this result here for convenience.

Theorem ([GK, p.311]) The sequence $\{\psi_j\}_1^\infty$ forms a Riesz basis of the space H if and only if $\{\psi_j\}_1^\infty$ is complete in H and there corresponds to it a complete biorthogonal sequence $\{\chi_j\}_1^\infty$ such that

$$\sum_{j=1}^{\infty} |(f, \psi_j)|^2 < \infty, \quad \sum_{j=1}^{\infty} |(f, \chi_j)|^2 < \infty \quad \text{for all} \quad f \in H. \tag{7.1}$$

Recall the definition of biorthogonality:

Definition (Biorthogonal sequences [GK, p.306]) Two sequences $\{\chi_j\}$ and $\{\omega_j\}$ in a Hilbert space H are said to be *biorthogonal* if

$$(\chi_k, \omega_j) = \delta_{jk} \ , \ (j, k = 1, 2, \ldots).$$

The strategy here is to prove that given the completeness of the generalized eigenvectors of A, the generalized eigenvectors of the adjoint A^* are complete, biorthogonal to those of A and that (7.1) holds.

This approach is less constructive than that of [GY01], but has the strength of not requiring that the generalized eigenvectors of A be quadratically close to some orthonormal basis of H. We present the approach for the specific case given in [CZ94].

7.1.1 Setting and preliminary results

Recall the problem considered in [CZ94]:

Problem CZ94

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u + 2c\partial_t u &= 0, \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) &= 0, \quad t > 0, \\ u(x,0) &= u_0(x), \\ \partial_t u(x,0) &= v_0(x). \end{aligned}$$

It is assumed in [CZ94] that c is a non-negative function of bounded variation and that c is strictly positive on some subinterval (a,b) of $(0,1)$.

Recall that the state space for this problem is $H = H_0^1(0,1) \times L^2(0,1)$, with the inner product

$$(\langle f,g\rangle, \langle u,v\rangle) = \int_0^1 f'\bar{u}' + g\bar{v}dx = \beta(f,u) + \gamma(g,v)$$

and that the dynamics generator for Problem CZ94 is

$$A = \begin{bmatrix} 0 & I \\ \partial_x^2 & -2c \end{bmatrix}, \tag{7.2}$$

See Subsection 2.3.3 for the details. We suppose that the following holds:

Theorem 7.1.1 ([CZ94, Theorem 3.1]). For the dynamics generator A of Problem CZ94:

(i) A possesses a compact inverse and so a discrete spectrum $\sigma(A)$ of eigenvalues of finite algebraic multiplicity.

(ii) The eigenvalues are the roots of $\lambda \mapsto y_2(1,\lambda)$, where where y_2 is a solution of

$$z'' - \lambda^2 z + 2\lambda c z = 0,\ z(0,\lambda) = 0,\ z'(0,\lambda) = 1.$$

If λ_n is such a root then $y_2(x,\lambda_n)\,\langle 1,\lambda_n\rangle$ spans the corresponding eigenspace and its algebraic multiplicity is the order to which $\lambda \mapsto y_2(1,\lambda)$ vanishes.

(iii) $\sigma(A)$ is symmetric about the real axis.

(iv) The generalized eigenvectors of A are complete in H.

Remarks

1. We proved that A has a compact inverse in Theorem 2.9.4 so A is discrete and consequently has a discrete spectrum.
2. To prove part (iv) of the Theorem, the authors of [CZ94] claim that A is a bounded perturbation of a skew symmetric (undamped) operator and that the completeness of its generalized eigenvectors "follows directly from [GK, Theorem 10.1, Section 5]." (Theorem 3.5.2 in this text). However, this theorem only applies to perturbations of selfadjoint operators.
3. It is not at all clear how the authors of [CZ94] applied Theorem 3.5.2 obtain their result. In [Sh02], the author uses two full pages of very technical work to justify the use of this theorem for a skew-symmetric operator.

We'll need the following estimates from [CZ94] to prove convergence of (7.1).

Theorem 7.1.2 ([CZ94, Theorem 5.1])**.** If c is of bounded variation then there exist constants C_0 and C_1 such that

$$\left| y_2(x,\lambda) - \frac{1}{\lambda}\sinh(\lambda x + \int_0^x c dt) \right| \leq \frac{C_0}{|\lambda|^2} \tag{7.3}$$

and

$$\left| y_2'(x,\lambda) - \cosh(\lambda x + \int_0^x c dt) \right| \leq \frac{C_1}{|\lambda|} \tag{7.4}$$

uniformly for $0 < x < 1$.

Theorem 7.1.3 ([CZ94, Theorem 5.5])**.** If c is of bounded variation then there exist constants C_2 and C_3 and C_4 such that

$$|\lambda_n + c_0 - in\pi| \leq \frac{C_2}{|n|\pi} \text{ for } |n| > N \tag{7.5}$$

where $c_0 = x\int_0^1 c dx$. Furthermore,

$$\left| y_2(x,\lambda) - \frac{\sinh(\xi(x) + in\pi x)}{\lambda} \right| \leq \frac{C_3}{|\lambda|^2} \tag{7.6}$$

and

$$|y_2'(x,\lambda) - \cosh(\xi(x) + in\pi x)| \leq \frac{C_4}{|\lambda|} \tag{7.7}$$

where $\xi(x) = \int_0^x c dt - c_0$ measures the deviation of c from a constant. Since c is bounded, $\xi(x)$ is bounded. It is simple to check that $\xi(x) = 0$ when c is constant.

Remark Note that (7.5) implies that if there exists a constant K_1 such that $|a-b| \leq \frac{K_1}{|\lambda_n|}$, then there also exists a constant K_2 such that $|a-b| \leq \frac{K_2}{|n|}$.

Definition Denote the algebraic multiplicity of λ_n by m_n. The generalized eigenvectors corresponding to λ_n are denoted $\{V_{n,j}\}_{j=0}^{m_n-1}$,

$$\begin{aligned} V_{n,0} &= y_2(x,\lambda_n)\langle 1,\lambda_n\rangle , \\ AV_{n,j} &= \lambda_n V_{n,j} + V_{n,j-1},\ (V_{n,j},V_{n,0}) = 0,\ j=1,\ldots,m_n-1. \end{aligned}$$

$V_{n,0}$ is an eigenvector and $\{V_{n,j}\}_{j=0}^{m_n-1}$ is a basis for the root subspace

$$L_n = \{V \mid (A-\lambda_n)^{m_n} V = 0\}.$$

7.1.2 Riesz basis

We now show that the generalized eigenvectors of the dynamics generator A defined in (7.2) form a Riesz basis for H.

By [CZ94, Theorem 5.3], there exists an N such that $m_n = 1$ for all $|n| > N$.

Since $\sigma(A)$ is symmetric with respect to the real axis, $\lambda_n \in \sigma(A)$ if and only if $\overline{\lambda}_n \in \sigma(A)$. Also, A is a closed, densely defined operator (see Subsection 1.7.4), so $\lambda_n \in \sigma(A)$ if and only if $\overline{\lambda}_n \in \sigma(A^*)$. Therefore $\sigma(A) = \sigma(A^*)$, including multiplicities.

The generalized eigenvectors corresponding to $\overline{\lambda}_n$ are denoted $\{W_{n,j}\}_{j=0}^{m_n-1}$,

$$\begin{aligned} W_{n,0} &= y_2(x,\overline{\lambda}_n)\left\langle 1,-\overline{\lambda}_n\right\rangle , \\ A^*W_{n,j} &= \overline{\lambda}_n W_{n,j} + W_{n,j-1},\ (W_{n,j},V_{n,m_n-1}) = 0,\ j=1,\ldots,m_n-1. \end{aligned}$$

$W_{n,0}$ is an eigenvector of A^* and $\{V_{n,j}\}_{j=0}^{m_n-1}$ is a basis for the root subspace

$$L_n^* = \left\{W \mid (A^*-\overline{\lambda}_n)^{m_n} W = 0\right\}.$$

The $W_{n,j}$ are uniquely determined as long as $(W_{n,0},V_{n,m_n-1}) \neq 0$. We need precisely one element from each of L_n and L_n^* to not be orthogonal for each n. We make this precise in the following lemma.

Lemma 7.1.4 ([CZ94, Lemma 6.2])**.** There exists a $c > 0$ such that

$$\begin{aligned} (V_{n,p},W_{j,k}) &= (V_{n,p},W_{n,m_n-1-p})\,\delta_{n,j}\delta_{m_n-1-p,k} \\ &\geq c\delta_{n,j}\delta_{m_n-1-p,k}. \end{aligned}$$

That is, the generalized eigenvectors of A and A^* can be biorthonormalized.

Proof.

The statement above actually means two things:

1. The spaces L_j and L_k^* are orthogonal when $j \neq k$ (This comes from the $\delta_{n,j}$ factor).
2. The spaces L_n, L_n^* are such that each element of L_n is orthogonal to every element of L_n^*, except one, to which it cannot be orthogonal. This relationship is very precise, for each $p = 0, \ldots, m_n - 1$:

$$(V_{n,p}, W_{n,k}) = \begin{cases} c > 0 & \text{if } k = m_n - 1 - p, \\ 0 & \text{otherwise.} \end{cases}$$

We prove each part separately.

1. Suppose $j \neq k$. We show that $L_j \perp L_k^*$:

$$\begin{aligned} (AV_{j,0}, W_{k,0}) &= \lambda_j (V_{j,0}, W_{k,0}), \text{ and} \\ (AV_{j,0}, W_{k,0}) &= (V_{j,0}, A^* W_{k,0}) = \lambda_k (V_{j,0}, W_{k,0}). \end{aligned}$$

So $(\lambda_j - \lambda_k)(V_{j,0}, W_{k,0}) = 0$, which is only possible if $(V_{j,0}, W_{k,0}) = 0$. For the generalized eigenvectors,

$$\begin{aligned} (AV_{j,}, W_{k,1}) &= \lambda_j (V_{j,0}, W_{k,1}), \text{ and} \\ (AV_{j,0}, W_{k,1}) &= (V_{j,0}, A^* W_{k,1}) = \lambda_k (V_{j,0}, W_{k,1}) + (V_{j,0}, W_{k,0}) \end{aligned}$$

So $(\lambda_j - \lambda_k)(V_{j,0}, W_{k,1}) = 0$, which is only possible if $(V_{j,0}, W_{k,1}) = 0$. Proceeding in this way up to $j = m_n - 1$, we see that $L_j \perp L_k^*$.

2. The relationship between L_n and L_n^* is more complicated.
For $j = 1, \ldots, m_n - 1$:

$$\begin{aligned} (V_{n,j}, A^* W_{n,0}) &= (AV_{n,j}, W_{n,0}) \\ &= (\lambda_n V_{n,j}, W_{n,0}) + (V_{n,j-1}, W_{n,0}) \\ &= \left(V_{n,j}, \overline{\lambda}_n W_{n,0}\right) + (V_{n,j-1}, W_{n,0}) \\ &= (V_{n,j}, A^* W_{n,0}) + (V_{n,j-1}, W_{n,0}). \end{aligned}$$

So $(V_{n,j-1}, W_{n,0}) = 0$ for $j = 1, \ldots, m_n - 1$. Or more explicitly,

$$(V_{n,j}, W_{n,0}) = 0 \text{ for } j = 0, \ldots, m_n - 2.$$

Since the generalized eigenvectors of A are complete in H, there must be at least one generalized eigenvector of A that isn't orthogonal to $W_{n,0}$

(otherwise $W_{n,0} = 0$). The only possibility is that $(V_{n,m_n-1}, W_{n,0}) \neq 0$.

The same argument shows that $(V_{n,0}, W_{n,j}) = 0$ for $j = 0, \ldots, m_n - 2$ and $(V_{n,0}, W_{n,m_n-1}) \neq 0$. Note that the completeness of the generalized eigenvectors of A^* follows from Theorem 7.1.1, See [GK, Remark 8.1, p.259].

We now show that each associate eigenvector in L_n is orthogonal to each associate eigenvector in L_n^*, except one. Compare

$$\begin{aligned}(AV_{n,1}, W_{n,m_n-k}) &= \lambda_n (V_{n,1}, W_{n,m_n-k}) + (V_{n,0}, W_{n,m_n-k}), \text{ and} \\ (V_{n,1}, A^* W_{n,m_n-k}) &= \left(V_{n,1}, \overline{\lambda}_n W_{n,m_n-k}\right) + (V_{n,1}, W_{n,m_n-k-1}).\end{aligned}$$

The equations above imply that $(V_{n,1}, W_{n,m_n-k-1}) = (V_{n,0}, W_{n,m_n-k})$. Letting $j = m_n - k - 1$ we see that that

$$(V_{n,1}, W_{n,j}) = 0 \text{ unless } j = m_n - 2, \ (j + 1 = m_n - 1),$$

in which case this inner product cannot be zero.

We repeat the argument for each $V_{n,j}$ and note that in doing so we exhaust all the generalized eigenvectors in L_n^*. We give the argument for $V_{n,2}$ to illustrate the pattern clearly. Compare

$$\begin{aligned}(AV_{n,2}, W_{n,m_n-k}) &= \lambda_n (V_{n,2}, W_{n,m_n-k}) + (V_{n,1}, W_{n,m_n-k}), \text{ and} \\ (V_{n,2}, A^* W_{n,m_n-k}) &= \left(V_{n,2}, \overline{\lambda}_n W_{n,m_n-k}\right) + (V_{n,2}, W_{n,m_n-k-1}).\end{aligned}$$

The equations above imply that

$$(V_{n,2}, W_{n,m_n-k-1}) = (V_{n,1}, W_{n,m_n-k}) = (V_{n,0}, W_{n,m_n-k+1}).$$

Letting $j = m_n - k - 1$, we see from the previous equation that

$$(V_{n,2}, W_{n,j}) = 0 \text{ unless } j = m_n - 3, \ (j + 2 = m_n - 1),$$

in which case this inner product cannot be zero.

This settles the question of orthogonality. We turn our attention to the task of binormalizing the systems. We deal with the eigenvectors first. We need the asymptotic estimates for the high frequencies $n > N$, so we use Theorem 7.1.2:

$$\begin{aligned}\lambda_n y_2(x, \lambda_n) &= \sinh(\lambda_n x + \int_0^x c \; dt) + \mathcal{O}\left(\frac{1}{|\lambda_n|}\right) \text{ and} \\ y_2'(x, \lambda_n) &= \cosh(\lambda_n x + \int_0^x c \; dt) + \mathcal{O}\left(\frac{1}{|\lambda_n|}\right).\end{aligned}$$

Now, for all $n > N$ we have

$$\begin{aligned}
&(V_{n,0}, W_{n,0}) = \left(y_2(x,\lambda_n) \begin{bmatrix} 1 \\ \lambda_n \end{bmatrix}, y_2(x,\overline{\lambda}_n) \begin{bmatrix} 1 \\ -\overline{\lambda}_n \end{bmatrix} \right) \\
&= \int_0^1 (y_2'(x,\lambda_n))^2 - \lambda_n^2 y_2^2(x,\lambda_n) dx \\
&= \int_0^1 \cosh^2(\lambda_n x + \int_0^x c dt) - \sinh^2(\lambda_n x + \int_0^x c dt) dx + \mathcal{O}\left(\frac{1}{|\lambda_n|}\right) \\
&= 1 + \mathcal{O}\left(\frac{1}{|n|}\right).
\end{aligned}$$

For the high frequencies we introduce the normalized eigenvectors

$$\begin{aligned}
\tilde{V}_{n,0}(x) &= (V_{n,0}, W_{n,0})^{-1/2} V_{n,0}(x) = V_{n,0}(x) + \mathcal{O}\left(1/|n|\right), \text{ and} \\
\tilde{W}_{n,0}(x) &= (V_{n,0}, W_{n,0})^{-1/2} W_{n,0}(x) = W_{n,0}(x) + \mathcal{O}\left(1/|n|\right).
\end{aligned}$$

Now consider the low frequencies $\lambda_n, n \leq N$. Here the algebraic multiplicity m_n of an eigenvalue λ_n may exceed 1. We let

$$\begin{aligned}
\tilde{V}_{n,j} &= (V_{n,j}, W_{n,m_n-1-j})^{-1/2} V_{n,j} \text{ and} \\
\tilde{W}_{n,j} &= (V_{n,m_n-1-j}, W_{n,j})^{-1/2} V_{n,j} \text{ for } j = 0, 2, \ldots, m_n - 1.
\end{aligned}$$

Since there are only finitely many eigenvalues with multiplicity larger than 1, they do not affect the convergence of (7.1) so we don't need estimates for them. □

Remark Note that using the equality sign in a relation such as $a = b + \mathcal{O}(n^k)$ is an abuse of notation. What this statement really means is that there exists a constant C such that $a - b \leq Cn^k$.

Theorem 7.1.5 ([CZ94, Theorem 6.3]). The normalized generalized eigenvectors

$$\left\{ \tilde{V}_{n,j} \;:\; n = \pm 1, \pm 2, \ldots;\; j = 0, 1, \ldots, m_n - 1 \right\}$$

form a Riesz basis for H.

Proof.

All that remains is to show that (7.1) holds for every $\langle f, g \rangle \in H$. To do this, we use the estimates from Theorems 7.1.2 and 7.1.3 for $V_{n,0}$, $n > N$ and show that this implies convergence for $\tilde{V}_{n,0}$.

$$
\begin{aligned}
& \sum_{n>N} |(V_{n,0}, \langle f, g\rangle)|^2 \\
= \quad & \sum_{n>N} \left| \int_0^1 y_2'(x,\lambda_n)\overline{f}'(x) + \lambda_n y_2(x,\lambda_n)\overline{g}'(x) \right|^2 \\
\leq \quad & \sum_{n>N} \left| \int_0^1 \cosh(\lambda_n x + \int_0^x c dt)\overline{f}'(x) + \sinh(\lambda_n x + \int_0^x c dt)\overline{g}'(x) \right|^2 \\
& + C \sum_{n>N} \frac{1}{n^2} \\
\leq \quad & \sum_{n>N} \left| \int_0^1 (\cosh \xi(x)\overline{f}'(x) + \sinh \xi(x)\overline{g}'(x)) \cos n\pi x dx \right|^2 \\
& + \sum_{n>N} \left| \int_0^1 (\sinh \xi(x)\overline{f}'(x) + \cosh \xi(x)\overline{g}'(x)) \sin n\pi x dx \right|^2 + K \sum_{n>N} \frac{1}{n^2} \\
\leq \quad & \left\| \cosh \xi(x)\overline{f}'(x) + \sinh \xi(x)\overline{g}'(x) \right\|^2_{L^2(0,1)} \\
& + \left\| \sinh \xi(x)\overline{f}'(x) + \cosh \xi(x)\overline{g}'(x) \right\|^2_{L^2(0,1)} + K_\ell < \infty.
\end{aligned}
$$

Since ξ is bounded and $\overline{f}'$ and $\overline{g}$ are in $L^2(0,1)$, the coefficients of $\cos n\pi x$ and $\sin n\pi x$ belong to $L^2(0,1)$. It is well-known that $\{1, \cos n\pi x, \sin n\pi x\}$ is an orthogonal basis for $L^2(0,1)$ (See for example [Kr] or [Ru]). The series above are Fourier series representations of these coefficients, so the Parseval relation [Kr, p.170] gives the final equality above.

Now we show that this implies the convergence of (7.1) for the binormalized eigenvector $\tilde{V}_{n,0}$.

$$
\begin{aligned}
\sum_{n>N} \left| \left(\tilde{V}_{n,0}, \langle f, g\rangle \right) \right|^2 &= \sum_{n>N} \left| \left(V_{n,0} + \frac{C}{|n|}, \langle f, g\rangle \right) \right|^2 \\
&= \sum_{n>N} \left| (V_{n,0}, \langle f, g\rangle) + \left(\frac{C}{|n|}, \langle f, g\rangle \right) \right|^2 \\
&\leq \sum_{n>N} |(V_{n,0}, \langle f, g\rangle)|^2 + \sum_{n>N} \left| \left(\frac{C}{|n|}, \langle f, g\rangle \right) \right|^2 \\
&\leq \sum_{n>N} |(V_{n,0}, \langle f, g\rangle)|^2 + \|\langle f, g\rangle\|^2 \sum_{n>N} \frac{C}{n^2} < \infty.
\end{aligned}
$$

The sum over negative n is handled identically, as is the convergence of

$$\sum_{n>N} \left| \left(\tilde{W}_{n,0}, \langle f, g \rangle \right) \right|^2 .$$

By Bari's theorem (Theorem 4.3.1), the normalized generalized eigenvectors of A constitute a Riesz basis for H. $\square$

7.2 Overview

The spectral theory of nonselfadjoint operators is broad and deep, but by no means complete. Rather, it is a vast collection of approaches and results which may be applied to specific situations.

This dissertation focused on vibration problems. The dynamics generators of vibration problems are in general non-normal so special cases are considered in the literature. Furthermore, completeness of a sequence does not imply that the sequence is a basis and not all bases are Riesz bases. Therefore, one is often forced to prove completeness and the Riesz basis property separately.

Three different approaches were encountered in the literature. All three make use of Keldysh's theorems on completeness (see Section 3.5) and Bari's theorems on Riesz bases (see Section 4.3).

The theories of operator pencils and Krein spaces are widely used, see for example [KL78(I)], [KL78(II)], [Sh96A], [Sh96B] [Sh00], [Sh02] or [JTW08]. Most of the results require that the spectrum of the dynamics generator satisfies certain properties, which quickly leads into deep functional analysis (see Section 3.6 and Chapter 6).

Bari's theorem may be applied directly, but this requires showing that the sequence of generalized eigenvectors is complete first and then showing that this sequence is biorthogonal to some other complete sequence. This biorthogonality approach may be powerful when used in conjunction with Keldysh's theorems (which give the completeness of the generalized eigenvectors of the adjoint of the dynamics generator as a bonus result). This approach is taken in [CZ94] (see Section 7.1).

Another approach based on Bari's theorems is to show that the sequence of generalized eigenvectors is quadratically close to a reference Riesz basis. This is attempted in [GY01] (see Chapter 5).

The general setting (due to [VV02]) outlined in Chapters 1 and 2 can be used to frame all these approaches.

Almost all the literature deals exclusively with one dimensional problems. This is because one often needs an asymptotic expression for the eigenvalues and eigenvectors of the dynamics generator. It may be exceptionally difficult to obtain an asymptotic expression for the eigenvectors in higher dimensions. The biorthogonality approach may be of use for problems in higher dimensions as it may be used without an explicit asymptotic expression of eigenvectors.

M. A. Shubov considered a three dimensional model in *Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustic waves by a spherically symmetric inhomogeneity of the density* (Differential Integral Equations 8 (1995), no. 5, 1073 – 1115), but here spherical symmetry is assumed.

Fortunately, for one dimensional linear vibration problems the situation is not too dire, the dynamics generator is always dissipative and this is linked to the dynamics generator being a discrete operator or at least having a discrete spectrum. In this case, there are fairly general results which may be used to prove that the generalized eigenvectors of the dynamics generator form a Riesz basis – provided that the necessary asymptotic estimates for the eigenvalues and eigenvectors can be obtained.

It must be noted that the papers [CZ94], [CZ95], [GY01] and [JTW08] were found to contain grave errors and omissions. Almost all of these were corrected and clarified in this dissertation. In some cases (such as Theorem 6.5.1) the results of these papers were generalized and made more readily applicable in the process.

Judging from recent publications, what Shubov stated in [Sh97] still holds, "at the present time there is no general spectral theory of nonselfadjoint operators".

Appendix A

Definitions

The formal definitions of concepts dealt with in this dissertation are collected here in alphabetical order for the reader's convenience.

Definition (Algebraic and geometric multiplicity [GK, p.5]) The *algebraic multiplicity* of an eigenvalue λ of A is the dimension of the space consisting of the zero vector and the generalized eigenvectors corresponding to λ. In the finite dimensional case, the algebraic multiplicity of an eigenvalue λ of A is its multiplicity as a root of the characteristic equation of A. An eigenvalue is called *algebraically simple* if its algebraic multiplicity is one.

The *geometric multiplicity* of an eigenvalue λ of A is the dimension of the eigenspace E_λ, which consists of the zero vector and all the eigenvectors associated with λ. The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity. An eigenvalue is called *geometrically simple* if its geometric multiplicity is one.

Geometric multiplicity is called *proper multiplicity* by some authors.

Definition (Analytic function [SS, p.70]) A complex-valued function $f(z)$ is said to be *analytic* on an open set G if it has a derivative at every point of G.

Definition (Basis [GK, p.306]) A sequence $\{\phi_j\}_1^\infty$ in a Banach space $\mathcal{B}$ is a *basis* of that space if every vector $x \in \mathcal{B}$ can be expanded in a unique way as a series

$$x = \sum_{j=1}^{\infty} c_j \phi_j,$$

which converges in the norm of the space $\mathcal{B}$.

Definition (Biorthogonal sequences [GK, p.306]) Two sequences $\{\chi_j\}$ and $\{\omega_j\}$ in a Hilbert space H are said to be *biorthogonal* if

$$(\chi_k, \omega_j) = \delta_{jk} \ , \ \text{for } j = 1, 2, \ldots .$$

Definition (Bounded bilinear form) A bilinear form $\phi(\cdot, \cdot)$ is *bounded* on an inner product space Y if there exists a constant C such that

$$\phi(u, v) \leq C \|u\| \|v\| \ \text{for all } u, v \in Y.$$

Definition (Closed linear operator [Kr, p.536]) Let X be a Hilbert space and suppose $T : \mathcal{D}(T) \subset X \to X$. Then T is called a *closed linear operator* if it's *graph*

$$\mathcal{G}(T) = \{\langle x, Tx \rangle \ | \ x \in \mathcal{D}(()T)\}$$

is closed in $X \times X$, where the norm on $X \times X$ is defined by

$$\|\langle x, y \rangle\|^2 = \|x\|^2 + \|y\|^2 .$$

Theorem [Kr, Theorem 10.3-2] gives a is a convenient characterization of closed operators: T is closed if and only if, for every sequence $\{x_n\} \in \mathcal{D}(T)$,

$$x_n \to x \quad \text{and} \quad Tx_n \to y$$

together imply that $x \in \mathcal{D}(T)$ and $Tx = y$.

Definition (Compact linear operator [Kr, p.405]) Let X and Y be normed spaces. A linear operator $T : X \to Y$ is called a *compact linear operator* if for every bounded subset M of X , the image $T(M)$ is relatively compact. That is, the closure $\overline{T(M)}$ is compact in Y.

Definition (Complete set [Kr, p.168]) A complete set in a normed space X is a subset $M \subset X$ whose span is dense in X.

Definition (Contraction [Kr, p.300]) A mapping F on a metric space (X, d) is called a *contraction* if there exists $\theta < 1$ such that

$$d(Fx, Fy) \leq \theta d(x, y) < d(x, y) \ \text{for all } x, y \in X.$$

In a normed space, the equation above becomes

$$\|Fx - Fy\| \leq \theta \|x - y\| < \|x - y\| \ \text{for all } x, y \in X.$$

Definition (Decay rate [CZ94, (1.3)]) The decay rate is defined as a function of the damping parameter c by

$$\omega(c) = \inf \left\{ \alpha \text{ such that there exists } C > 0 \; s.t. \; E(t) \leq CE(0)e^{2\alpha t} \right\}.$$

Definition (Definitizable operator [La81, p.11]) A selfadjoint operator in the Krein space $(X, [\cdot, \cdot])$ is called *definitizable* if $\rho(A) \neq 0$ and there exists a polynomial $p(A) = A^k + c_{k-1}A^{k-1} + \ldots + c_1 A + c_0$ of degree k such that $[Ax, x] \geq 0$ for $x \in \mathcal{D}(A^k)$.

Definition (Densely defined operator [Kr, p.527]) An operator T on a Hilbert space H is said to be *densely defined* if its domain $\mathcal{D}(T)$ is dense in H.

Definition (Dimensionless energy)

$$E(t) = \frac{1}{2} \int_0^1 \left(\partial_t u(x,t)\right)^2 \, dx + \frac{1}{2} \int_0^1 \left(\partial_x u(x,t)\right)^2 \, dx. \tag{A.1}$$

The first term on the right hand side is *dimensionless kinetic energy* and the second term is *dimensionless potential energy.*

Definition (Discrete operator [DS, p.2291]) An operator T is called *discrete* if there exists a regular value λ in its resolvent set $\rho(T)$ such that $R_\lambda(T)$ is compact.

Definition (Discrete spectrum [GK, p.276]) The spectrum of an operator is called discrete if it consists entirely of isolated eigenvalues with finite multiplicity with a unique limit point at infinity. An operator with a discrete spectrum must be unbounded.

Definition (Dissipative operator) A linear operator A in a complex Hilbert space X is dissipative if for every $x \in \mathcal{D}(A)$, $\mathrm{Re}\,(Ax, x)_X \leq 0$.

Definition (Double completeness [KL78(I), p.378]) The generalized eigenvectors of the pencil defined by (3.3) are called *double complete* if the generalized eigenvectors $\{\psi_n\}_1^\infty$ of the operator T are such that the sequence $\left\{[\psi_n, \lambda_n \psi_n]^T\right\}_1^\infty$ is complete in the product space $X \times X$. Note that λ_n are the eigenvalues corresponding to ψ_n.

Definition (Dynamics generator [Sh02]) The linear operator A in (1.50) is referred to as the *dynamics generator.*

Definition (Eigenvalue, eigenvector, generalized eigenvector) Let A be a closed linear operator on a Hilbert space H. The *eigenvalues* of the operator A are the complex numbers λ such that the equation

$$(\lambda I - A)\, x = 0$$

has at least one nonzero solution, $x \in H$, called an *eigenvector*. The spectrum $\sigma(A)$ contains all the eigenvalues of A.

$(0 \neq) x \in H$ is called a *generalized eigenvector* of A if there exists a natural number n such that

$$(\lambda I - A)^n\, x = 0.$$

Generalized eigenvectors are sometimes referred to as *root vectors*.

Generalized eigenvectors of A that are not eigenvectors of A are called *associated eigenvectors*.

Definition (Energy space, inertia space, state space) In [VV02] W is called the *inertia space* and V the *energy space*. See Subsection 1.7.2 for the definitions of V and W.

Definition (Generalized eigenspace [GK, p.5]) Let $\overline{sp}(A)$ be the closed subspace spanned by the generalized eigenvectors of A. This space is referred to as the *generalized eigenspace* or *root subspace* of A. The closed subspace spanned by the generalized eigenvectors of A corresponding to a particular eigenvalue λ of A is called the generalized eigenspace of A corresponding to λ and is denoted $\overline{sp}_\lambda(A)$.

Definition (Hermitian bilinear form) A bilinear form B on some vector space V (over a real or complex scalar field) is *Hermitian* if

$$B\,(x, y) = \overline{B\,(y, x)}.$$

Definition (Hilbert-adjoint operator [Kr, p.527]) Let A be a densely defined linear operator $A : \mathcal{D}(A) \to H$ in a Hilbert space H. Then the *Hilbert-adjoint operator* A^* of A (or simply *adjoint*) is the operator $A^* : \mathcal{D}(A^*) \to H$ such that the domain of A^* consists of all $y \in H$ such that there is a $y^* \in H$ satisfying

$$(Ax, y) = (x, y^*) \text{ for all } x \in \mathcal{D}(A).$$

For each such $y \in \mathcal{D}(A^*)$, the adjoint operator is defined in terms of that y^* by

$$y^* = A^* y$$

so that we may write

$$(Ax, y) = (x, A^*y) \ \text{ for all } x \in \mathcal{D}(A), y \in \mathcal{D}(A^*).$$

Definition (Indefinite bilinear form) A definite bilinear form is a bilinear form B over some vector space V (with real or complex scalar field) such that the associated quadratic form

$$Q(x) = B(x, x)$$

is definite. That is, $Q(x)$ has a real value with the same sign (positive or negative) for all non-zero x. According to that sign, B is called positive definite or negative definite. If Q takes both positive and negative values, the bilinear form B is called *indefinite*. If $B(x, x) \geq 0$ for all x, B is said to be positive semidefinite. If $B(x, x) \leq 0$ for all x, B is said to be negative semidefinite.

Definition (Indefinite inner product) An *indefinite inner product* is an indefinite Hermitian bilinear form.

Definition (Invariant subspace [Kr, p.374]) A subspace Y of a normed space X is said to be *invariant* under a linear operator $T : X \to X$ if $T(Y) \subset Y$.

Definition (Krein space [JTW08], [Ha]) Suppose $(H, (\cdot, \cdot))$ is a Hilbert space and $G : H \to H$ is an invertible selfadjoint linear operator such that $G^2 = I$. Then

$$[x, y] = (Gx, y)\,, \ x, y \in H$$

defines an indefinite inner product on H. The space $\mathcal{H} = (H, [\cdot, \cdot])$ is known as a *Krein space*. This is not the most general way to define a Krein space, but it is natural and sufficiently general for our purposes. It also spares us from various topological nightmares. For a more general definition, see [Ma, p.206] or [Ha].

Definition (General Krein space, Pontryagin space [La81, p.3]) Let X be a Hilbert space and suppose $[\cdot, \cdot]$ is an indefinite inner product on X. The space $(X, [\cdot, \cdot])$ is called a Krein space if it contains two subspaces K_+, K_- with the properties:

1. $X = K_+ \oplus K_-$,
2. $(K_+, [\cdot, \cdot])$ and $(K_-, [\cdot, \cdot])$ are Hilbert spaces,
3. $[K_+, K_-] = \{0\}$.

If, in particular, $k = \min\{\dim K_+, \dim K_-\} < \infty$, the Krein space $(X, [\cdot, \cdot])$ is called a a Pontryagin space of index k, or a π_k-space.

Definition (Linear independence [Kr, p.53]) A subset M of a vector space X is *linearly independent* if every nonempty finite subset M_n of M is linearly independent. That is, if $\dim M_n = n$ and $x_1, x_2, \ldots, x_r \in M_n$ $(r \leq n)$, then the only set of scalars $a_1, a_2, \ldots, a_r$ such that

$$\sum_{k=1}^{r} a_k x_k = 0$$

is $a_1 = a_2 = \ldots = a_r = 0$.

Definition (ω-linear independence [GK, p.316]) A sequence $\{g_j\}_1^\infty$ of vectors in a Hilbert space H is said to be ω*-linearly independent* if the equality

$$\sum_{j=1}^{\infty} c_j g_j = 0$$

is impossible for

$$0 < \sum_{j=1}^{\infty} |c_j|^2 \|g_j\|^2 < \infty,$$

where c_j are scalars.

It is logically equivalent to say that $\{g_j\}_1^\infty$ is ω-linearly independent if

$$0 < \sum_{j=1}^{\infty} |c_j|^2 \|g_j\|^2 < \infty,$$

implies that

$$\sum_{j=1}^{\infty} c_j g_j \neq 0.$$

If a sequence is not $\omega-$linearly independent it is said to be $\omega-$linearly dependent.

Definition (Linear operator [Kr, p.82]) A *linear operator* T is a mapping such that

(i) The domain $\mathcal{D}(T)$ is a vector space and the range $\mathcal{R}(T)$ is in a vector space over the same field,

(ii) for all x, $y \in \mathcal{D}(T)$ and scalars α and β,

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

Definition (The space ℓ^p [Kr, p.21,61]) For a real number p, $1 \leq p < \infty$, the space ℓ^p consists of sequences $x = \{\xi_j\}_1^\infty$ such that

$$\sum_{j=1}^{\infty} |\xi_j|^p$$

converges. The norm on ℓ^p is given by

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p}.$$

Definition (Non-negative operator, positive operator [KL78(I), p. 365]) A selfadjoint linear operator on a Hilbert space X is *non-negative* if

$$(Tx, x)_X \geq 0 \text{ for all } x \in X.$$

T is a *positive operator* if $(Tx, x)_X > 0$ for all $x \neq 0 \in X$.

Definition (Operator pencil [Ma, p.1]) An *operator pencil* is an operator polynomial of the form

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots + \lambda^n A_n,$$

where λ is a spectral parameter and $A_0, \ldots, A_n$ are linear operators acting in a Hilbert space X.

Definition (Quadratically close sequences [GK, p.316]) The sequences $\{f_j\}_1^\infty$ and $\{g_j\}_1^\infty$ in a Hilbert space H are said to be *quadratically close* if

$$\sum_{j=1}^{\infty} \|f_j - g_j\|^2 < \infty.$$

Definition (Regular value, resolvent operator, resolvent set, spectrum [Kr, p.370]) Let $X \neq \{0\}$ be a normed space. Let $T : \mathcal{D}(T) \to X$ be a linear operator with domain $\mathcal{D}(T)$. A *regular value* λ of T is a complex number such that

(i) $R_\lambda(T) = (\lambda I - T)^{-1}$ exists,

(ii) $R_\lambda(T)$ is bounded,

(iii) $R_\lambda(T)$ is defined on a dense subset of X.

The set of all regular values of T is called the *resolvent set* of T and is denoted by $\rho(T)$. $R_\lambda(T)$ is called the *resolvent operator* of T. The complement of the resolvent is called the *spectrum* of T and is denoted by $\sigma(T) = \mathbb{C} - \rho(T)$. A complex number $\lambda \in \sigma(T)$ is called a *spectral value* of T.

Definition (Riesz basis [GK, p.309]) Let $\{\phi_j\}_1^\infty$ be an arbitrary orthonormal basis for the Hilbert space H. A basis $\{\psi_j\}_1^\infty$ of H is called a *Riesz basis* of H if there exists an invertible bounded linear operator Q such that Q transforms $\{\phi_j\}_1^\infty$ into $\{\psi_j\}_1^\infty$. That is, $\psi_j = Q\phi_j$ for all $j = 1, 2, \ldots$. A Riesz basis is sometimes called a *basis equivalent to an orthonormal basis*.

Definition (Selfadjoint operator, normal operator [Kr, p.534]) A densely linear operator $A : H \to H$ on a Hilbert space H is said to be *selfadjoint* if $A = A^*$ and *normal* if $AA^* = A^*A$. The operator A is called *skew-symmetric* if $A^* = -A$. Clearly, selfadjoint and skew-symmetric operators are normal. Selfadjoint operators are sometimes referred to as *Hermitian* operators.

Definition (Separable space [Kr, p.171]) A Hilbert space H is separable if it contains an orthonormal sequence $\{\phi_n\}_1^\infty$ such that the span of $\{\phi_n\}_1^\infty$ is dense in H.

Definition (Similar operators [Ma, p.168]) An operator A is said to be *similar* to another operator B if there exists an invertible operator C such that

$$A = C^{-1}BC.$$

Definition (Sobolev space $H^m(a,b)$ [Ad, p.45]) The Sobolev space of order m is defined as the vector space of all functions f whose weak derivatives of up to order m exist and are elements of $L^2(a,b)$. The inner product $(u,v)_m$ for the Sobolev space $H^m(a,b)$ is defined by

$$(u,v)_m = \sum_{k=0}^{m} \left(u^{(k)}, v^{(k)}\right)_{L^2(a,b)}$$

and induces the norm $\|u\|_m = \sqrt{(u,u)_m}$.

Definition (Spectral abscissa [CZ94, (1.4)]) The *spectral abscissa* of the dynamics generator A (given by 2.13) is defined as a function of the damping parameter c by

$$\mu(c) = \sup \{\text{Re } \lambda \mid \lambda \in \sigma(A)\}.$$

Definition (State space) In [CZ94] and [GY01], the product space $H = V \times W$ is called the *state space*. Define the inner product on H by

$$(x, y)_H = (\langle x_1, x_2\rangle, \langle y_1, y_2\rangle)_H = (x_1, y_1)_V + (x_2, y_2)_W \text{ for all } x, y \in H.$$

This induces the norm $\|x\|_H^2$, where

$$\|x\|_H^2 = (x_1, x_1)_V + (x_2, x_2)_W = \|x_1\|_V^2 + \|x_2\|_W^2.$$

Definition (Strong damping [VV02, p.1148]) The damping in the general vibration problem is referred to as *strong damping* if the damping bilinear form $\alpha(\cdot, \cdot)$ is positive definite on V. That is, if $\alpha(u, u) \geq C \|u\|_V^2$ for any $u \in V$.

Definition (Strongly damped operator pencil, [KL78(II), Section 7]) An operator pencil $L(\lambda) = \lambda^2 I + \lambda B + C$ in a Hilbert space H is said to be *strongly damped* if

$$(Bx, x)_H^2 > 4 (Cx, x)_H (x, x)_H \text{ for all } x \neq 0.$$

In terms of the bilinear forms of [VV02], L is a strongly damped pencil if

$$\alpha(x, x)^2 > 4\beta(x, x)\gamma(x, x) \text{ for all } x \neq 0 \in H = V \times W.$$

Definition (Symmetric linear operator [Kr, p.533]) Let $T : \mathcal{D}(T) \to X$ be a linear operator which is densely defined in a Hilbert space X. Then T is called a *symmetric linear operator* if

$$(Tx, y) = (x, Ty) \text{ for all } x, y \in \mathcal{D}(T).$$

Definition (Uniformly positive operator [KL78(II), p.365]) A selfadjoint linear operator on a Hilbert space X is *uniformly positive* if

$$\inf_{\substack{x \in X \\ x \neq 0}} \frac{(Tx, x)_X}{(x, x)_X} > 0.$$

Definition (Very strong damping, [KL78(II), Section 7]) The damping in the general vibration problem is referred to as *very strong damping* if

$$\alpha(x, x)^2 > 4\beta(x, x)\gamma(x, x) \text{ for all } x \neq 0 \in H = V \times W.$$

Definition (Volterra operator [GK, p.16]) An operator A is called a *Volterra operator* if it is compact and has no nonzero eigenvalues.

Definition (Weak damping) The damping in the general vibration problem is referred to as *weak damping* if the damping bilinear form $\alpha(\cdot,\cdot)$ is defined on the larger space W and $\alpha(\cdot,\cdot)$ is bounded on W. That is, there exists a constant C such that

$$|\alpha(u,v)| \leq C\,\|u\|_W\,\|v\|_W \text{ for all } u,v\in V.$$

Definition (Weak derivative of order m [Ad, p.21]) Suppose $u \in L^2(a,b)$ and there exists a $v \in L^2(a,b)$ such that

$$\left(u,\phi^{(m)}\right)_{L^2(a,b)} = (-1)^m\,(v,\phi)_{L^2(a,b)} \text{ for all } \phi\in C_0^\infty(a,b),$$

then v is called the *weak derivative* of order m of u and is denoted in the same way as the usual derivative in the literature,

$$v = u^{(m)} = \frac{d^m u}{dx^m} = D^m u.$$

The weak derivative is sometimes referred to as the *generalized derivative*.

Appendix B

Sobolev spaces

The definitions and results concerning Sobolev spaces are collected here for convenient reference.

Notation

The space $L^2(a,b)$ is the Hilbert space consisting of the Lebesgue square-integrable functions. The inner product for $L^2(a,b)$ is defined by

$$(u,v)_{L^2(a,b)} = \int_a^b uv.$$

Where the integral is the Lebesgue integral over the interval (a,b). The norm on $L^2(a,b)$ is induced in the usual way by

$$\|u\|_{L^2(a,b)} = \sqrt{(u,u)}_{L^2(a,b)}.$$

The space $C_0^m(a,b)$ denotes the space of functions $f \in C^m(a,b)$ that may only take nonzero values on some subinterval (x,y) that is strictly contained in (a,b) so that $f(c) = 0$ for $c \in (a,x) \cup (y,b)$.

In a higher dimensional space X, the *support* of a function $f \in C^m(X)$ is defined by

$$\text{supp } f = \{x \in X \mid f(x) \neq 0\}.$$

The function f is said to have compact support if supp f is compact.

Theorem B.1.1 ([Ad, Theorem 2.13, p.28])**.** $C_0^\infty(a,b)$ is dense in $L^2(a,b)$. That is, for any $u \in L^2(a,b)$, there exists a sequence $\{\phi_n\} \subset C_0^\infty(a,b)$ such that

$$\|u - \phi_n\|_{L^2(a,b)} \to 0 \quad \text{as} \quad n \to \infty.$$

Definition (Weak derivative of order m [Ad, p.21]) Suppose $u \in L^2(a,b)$ and there exists a $v \in L^2(a,b)$ such that

$$\left(u, \phi^{(m)}\right)_{L^2(a,b)} = (-1)^m \left(v, \phi\right)_{L^2(a,b)} \text{ for all } \phi \in C_0^\infty(a,b),$$

then v is called the *weak derivative* of order m of u and is denoted in the same way as the usual derivative in the literature, $v = \partial^m u$.

The weak derivative is sometimes referred to as the *generalized derivative.*

Definition (Sobolev space $H^m(a,b)$ [Ad, p.45]) The Sobolev space of order m is defined as the vector space of all functions f whose weak derivatives of up to order m exist. The inner product $(u,v)_m$ for the Sobolev space $H^m(a,b)$ is defined by

$$(u,v)_m = \sum_{k=0}^{m} \left(u^{(k)}, v^{(k)}\right)_{L^2(a,b)}$$

and induces the norm $\|u\|_m = \sqrt{(u,u)_m}$.

Remark If the ordinary derivative of order m exists then the weak derivative is equal to the ordinary derivative.
Note that $u^{(0)} = u$ and that $H^m(a,b) \subset L^2(a,b)$.

Theorem B.1.2 ([Ad, Theorem 3.2, p.45])**.** The Sobolev space $H^m(a,b)$ is complete. That is, $H^m(a,b)$ is a Hilbert space.

Definition (Separable space [Kr, p.21, p.171]) A metric space X is *separable* if it has a countable subset which is dense in X. A Hilbert space H is separable if it contains an orthonormal sequence $\{\phi_n\}_1^\infty$ such that $sp(\{\phi_n\}_1^\infty)$ is dense in H.

Theorem B.1.3 ([Ad, Theorem 3.5, p.47])**.** The Sobolev space $H^m(a,b)$ is separable.

Theorem B.1.4 (Sobolev's Lemma [OR, Theorem 3.10, p.80])**.** If $u \in H^m(a,b)$ then there exists a $\phi \in C^{m-1}[a,b]$ such that $u = \phi$ almost everywhere and the weak derivative $u^{(m)}$ exists with the property that

$$\left(u^{(m)}, v\right)_{L^2(a,b)} = -\left(u^{(m-1)}, v'\right)_{L^2(a,b)} \text{ for all } v \in C_0^1[a,b]$$

Furthermore, there exists a constant $C > 0$ such that following inequality holds:

$$\left\|u^{(k)}\right\|_{\sup} \leq C \|u\|_m \text{ for all } k = 0,1,\ldots,m-1. \tag{B.1}$$

Remark Since the values of functions in Sobolev spaces may be arbitrarily changed on sets of measure zero, we define the boundary values of $u^{(k)} \in H^m(0,1)$ as

$$u^{(k)}(a) = \phi^{(k)}(b) \text{ and } u^{(k)}(b) = \phi^{(k)}(b) \text{ for all } 0 \leq k \leq m-1.$$

The following theorem gives another useful characterization for $H^m(a,b)$.

Theorem B.1.5 ([OR, Theorem 2.10, p.53])**.** The space $C^m[a,b]$ is dense $H^m(a,b)$ with respect to the norm of $H^m(a,b)$. Thus for any $u \in H^m(a,b)$, there exists a sequence $\{u_n\} \subset C^m[a,b]$ such that

$$\|u - u_n\|_m \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem B.1.6 (Rellich's Theorem [Ad, Theorem 6.2, p.144])**.** For any non-negative integer m, the embedding of $H^{m+1}(a,b)$ into $H^m(a,b)$ is compact.

Remark Rellich's Theorem also gives the compactness of the embedding of any Sobolev space $H^m(a,b)$ in $L^2(a,b)$ if we note that $H^0(a,b) = L^2(a,b)$.

Definition (Derivative) Let w be a function whose values are in a Banach space X. If there exists a $z \in X$ such that

$$\lim_{h \to 0} h^{-1}(w(t+h) - w(t)) = z,$$

then z is the derivative of w at t and is denoted by $w'(t)$. This implies that

$$\lim_{h \to 0} \left\| h^{-1}(w(t+h) - w(t)) - w'(t) \right\|_X = 0.$$

The derivative is denoted $w'(t)$. The derivative (function) w' and the higher order derivatives $w^{(k)}$ are defined in the usual way.

Appendix C

Convergence

The results in this Appendix may be found in [De] and [Va10]. They are reproduced here for ease of reference.

Note that the subscript $L^2(0,1)$ is suppressed for inner products and norms. The inner product for $L^2(0,1)$ is $(f,g) = \int_0^1 fg$ and the induced norm is denoted by $\|f\|$.

C.1 Inequalities

Poincaré type inequalities can be found in the theory of partial differential equations and the finite element method. The simple results in this section are from [De].

Proposition C.1.1. Consider any $u \in C^1[0,1]$. For any x and y in $[0,1]$,

$$|u(x)| \leq \|u'\| + |u(y)|.$$

Proof.

We may assume without loss of generality that $x > y$. By the fundamental theorem of calculus,

$$u(x) = \int_y^x u' + u(y).$$

For any $f, g \in L^2(0,1)$ we have the Cauchy-Schwarz inequality

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right).$$

Choosing $g = 1$,

$$\left(\int_y^x f\right)^2 \leq \int_y^x \left(f^2\right)(x-y) \leq \|f\|^2.$$

Hence $\left|\int_y^x f\right| \leq \|f\|$ for each $f \in L^2(0,1)$. This inequality also holds for u' since u' is integrable by assumption. For $u \in C^1[0,1]$ it follows that

$$|u(x)| \leq \left|\int_y^x u'\right| + |u(y)| \leq \|u'\| + |u(y)|. \qquad \square$$

Proposition C.1.2. For any $u \in C^1[0,1]$ with a zero in $[0,1]$ we have

$$\|u\|_{\sup} \leq \|u'\|.$$

Proof.

Suppose $u(y) = 0$, then $|u(x)| \leq \|u'\|$ by Proposition C.1.1.

Since x is arbitrary, $\|u'\|$ is an upper bound for for $|u|$, but $\|u\|_{\sup}$ is the least upper bound for $|u|$. Therefore $\|u\|_{\sup} \leq \|u'\|$. $\square$

Proposition C.1.3. For any $u \in C^1[0,1]$ with a zero in $[0,1]$,

$$\|u\| \leq \|u'\|.$$

Proof.

$$\|u\|^2 = \int_0^1 (u(x))^2\,dx \leq \|u\|_{\sup}^2 \int_0^1 dx \leq \|u\|_{\sup}^2. \tag{C.1}$$

Proposition C.1.2 implies that $\|u\| \leq \|u\|_{\sup} \leq \|u\|'$. $\square$

Lemma C.1.4 (Poincaré inequality). Let $T[0,1]$ be any space of test functions such that $u(0) = 0$ or $u(1) = 0$ for every $u \in T$. Denote the closure of $T[0,1]$ in $H^1(0,1)$ by V. Then

$$\|u\| \leq \|u'\| \quad \text{for all } u \in V. \tag{C.2}$$

Proof.

For any $u \in V$, there exists a sequence $\{u_n\} \in T[0,1]$ such that

$$\|u_n - u\|_1 \to 0 \ \text{ as } n \to \infty.$$

This implies that $\|u_n\| \to \|u\|$ as $n \to \infty$, since

$$|\|u_n\| - \|u\|| \leq \|u_n - u\| \leq \|u_n - u\|_1 .$$

By the same argument, $\|u_n'\| \to \|u'\|$ as $n \to \infty$.

Every $u_n \in T[0,1]$ so by Proposition C.1.2, $\|u_n\| \leq \|u_n'\|$ for all n. Taking limits we find that $\|u\| \leq \|u'\|$. $\square$

C.2 Convergence

The result of this section is from [Va10].

In Subsections 1.2 and 2.4 we used the fact that

$$\int_0^1 \left(u_0'(x) - u_N'(x)\right)^2 \, dx \to 0,$$

where $u_N(x) = \displaystyle\sum_{n=1}^{N} d_n u_n(x)$.

The space of test functions is given in (1.33) as

$$T[0,1] = \left\{\phi(x) \in C^1[0,1] \mid \phi(0) = \phi(1) = 0\right\}.$$

and V is the closure of T in $H^1(0,1)$. Recall that in Subsection 1.6.5 the bilinear form β was defined as

$$\beta(u,v) = \int_0^1 u'v'.$$

By the Poincaré inequality (C.2),

$$\beta(u,u) \geq K \|u\|^2 \quad \text{for all } u \in V. \tag{C.3}$$

Therefore β is an inner product for V. Let $\|u\|_V = \sqrt{\beta(u,u)}$.

Proposition C.2.1. Let $\phi_n = \sin n\pi x$. The completeness of $\{\phi_n\}$ in $L^2(0,1)$ implies that $\{\phi_n\}$ is complete in $H^2(0,1) \cap V$ with respect to the norm of V.

Proof.

Let $u \in H^2(0,1) \cap V$ and suppose $u'' = f$. Then

$$\beta(u,v) = (f,v) \quad \text{for all } v \in V.$$

Let $\sum_{n=1}^{N} c_n\phi_n = f_N$ where $\{c_n\}$ is the sequence of Fourier coefficients. Then

$$\|f - f_N\| \to 0 \quad \text{as} \quad N \to \infty.$$

Since $\{\phi_n\}$ is the sequence of eigenvectors, $\beta(\phi_n, v) = \lambda_n(\phi_n, v)$ for each $v \in V$. Therefore:

$$\begin{aligned}
(f_N, v) &= \sum_{n=1}^{N} c_n(\phi_n, v) \\
&= \sum_{n=1}^{N} c_n\lambda_n^{-1}\beta(\phi_n, v) \\
&= \beta\left(\sum_{n=1}^{N} c_n\lambda_n^{-1}\phi_n, v\right) \quad \text{for all } v \in V.
\end{aligned}$$

Let $u_N = \sum_{n=1}^{N} c_n\lambda_n^{-1}\phi_n$, then

$$\beta(u_N, v) = (f_N, v) \quad \text{for all } v \in V.$$

Consequently

$$\beta(u - u_N, v) = (f - f_N, v) \quad \text{for all } v \in V.$$

But

$$\begin{aligned}
\|u - u_N\|_V^2 &= \beta(u - u_N, u - u_N) \\
&= (f - f_N, u - u_N) \\
&\leq C\,\|f - f_N\|\,\|u - u_N\|_V\,,
\end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality and (C.3). Finally,

$$\|u - u_N\|_V \leq K\,\|f - f_N\|\,,$$

so $\|u - u_N\|_V \to 0$ as $N \to \infty$. $\square$

C.3 Existence

The following result is from [Va10].

Suppose β doesn't satisfy Assumption (A4), but rather

$$\beta(u,u) \geq c_1(u,u)_V - c_0\gamma(u,u).$$

Let

$$\begin{aligned} u &= e^{mt}w. \\ \text{Then } u' &= me^{mt}w + e^{mt}w' \\ \text{and } u'' &= m^2e^{mt}w + 2me^{mt}w' + e^{mt}w''. \end{aligned}$$

Now

$$\begin{aligned} & \gamma(u''(t),v) + \alpha(u'(t),v) + \beta(u(t),v) \\ = \; & e^{mt}[\gamma(w''(t),v) + 2m\gamma(w'(t),v) + m^2\gamma(w(t),v) \\ + \; & \alpha(w'(t),v) + m\alpha(w(t),v) + \beta(w(t),v)] \\ = \; & e^{mt}[\gamma(w''(t),v) + 2m\gamma(w'(t),v) + \alpha(w'(t),v) \\ + \; & m^2\gamma(w(t),v) + m\alpha(w(t),v) + \beta(w(t),v)] \\ = \; & e^{mt}[\gamma(w''(t),v) + \tilde{\alpha}(w'(t),v) + \tilde{\beta}(w(t),v)], \end{aligned}$$

where

$$\begin{aligned} \tilde{\beta}(u,v) &= m^2\gamma(u,v) + m\alpha(u,v) + \beta(u,v) \\ \text{and } \; \tilde{\alpha}(u,v) &= m\gamma(u,v) + \alpha(u,v). \end{aligned}$$

Note that

$$\begin{aligned} \tilde{\beta}(u,u) &= m^2\gamma(u,u) + m\alpha(u,u) + \beta(u,u) \\ &\geq \beta(u,u) + m^2\gamma(u,u) \geq c_1\|u\|_V^2 \\ \tilde{\alpha}(u,u) &= m\gamma(u,u) + \alpha(u,u) \geq 0 \\ \tilde{\alpha}(u,u) &= m\gamma(u,u) + \alpha(u,u) \leq K\|u\|_V^2. \end{aligned}$$

So $\tilde{\alpha}(u,u$ and $\tilde{\beta}$ satisfy Assumptions (A1) to (A5) and

$$\gamma(u''(t),v) + \alpha(u'(t),v) + \beta(u(t),v) = f(t)$$

if and only if

$$\gamma(w''(t),v) + \tilde{\alpha}(w'(t),v) + \tilde{\beta}(w(t),v) = e^{-mt}f(t) \quad \text{for all} \quad v \in V.$$

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