

Wishart ratios with dependent structure: new members of the bimatrix beta type IV

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ABSTRACT

In multivariate statistics under normality, the problems of interest are random covariance matrices (known as Wishart matrices) and "ratios" of Wishart matrices that arise in multivariate analysis of variance (MANOVA) (see [24]). The bimatrix variate beta type IV distribution (also known in the literature as bimatrix variate generalised beta; matrix variate generalization of a bivariate beta type I) arises from "ratios" of Wishart matrices. In this paper, we add a further independent Wishart random variate to the "denominator" of one of the ratios; this results in deriving the exact expression for the density function of the *bimatrix variate extended beta type IV* distribution. The latter leads to the proposal of the bimatrix variate extended F distribution. Some interesting characteristics of these newly introduced bimatrix distributions are explored. Lastly, we focus on the *bivariate* variate extended beta type IV distribution (that is an extension of bivariate Jones' beta) with emphasis on $P(X_1 < X_2)$ where X_1 is the random stress variate and X_2 is the random strength variate.

AMS 2000 subject classification: primary 62H10; secondary 62E15

Key words: bimatrix variate beta type IV distribution; bimatrix variate Kummer extended beta type IV distribution; generalized Laguerre polynomial; hypergeometric function of matrix argument; invariant polynomials; Laplace transform; Meijer's G-function; moment generating function; stress-strength

1 Introduction

In multivariate statistics under normality, the problems of interest are random covariance matrices (known as Wishart matrices) and ratios of Wishart matrices that arise in multivariate analysis of variance (see [24]). The Wishart ratio

$$\mathbf{U} = (\mathbf{H} + \mathbf{E})^{-\frac{1}{2}} \mathbf{H} (\mathbf{H} + \mathbf{E})^{-\frac{1}{2}}$$

($\mathbf{H} \sim W_p(n, \mathbf{\Sigma})$ independent of $\mathbf{E} \sim W_p(m, \mathbf{\Sigma})$) is the genesis of the matrix variate beta type I distribution, denoted as $\mathbf{U} \sim B_p^I(n, m)^2$ (see [21]). (Note $\mathbf{C}^{\frac{1}{2}}$ is the unique positive definite square root of \mathbf{C} .) Let $\mathbf{H}_i \sim W_p(n_i, \mathbf{\Sigma})$, $i = 1, 2$, independent of $\mathbf{E} \sim W_p(m, \mathbf{\Sigma})$, where

$$\mathbf{W}_i = (\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_i (\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{E})^{-\frac{1}{2}}, \quad i = 1, 2,$$

then it is evident that $\mathbf{W}_1 \sim B_p^I(n_1, n_2 + m)$ and $\mathbf{W}_2 \sim B_p^I(n_2, n_1 + m)$. However, they are correlated with a common "denominator" and the distribution of $\mathbb{W} = (\mathbf{W}_1 : \mathbf{W}_2)'$ is termed the *bimatrix variate beta type I distribution* (denoted as $\mathbb{W} \sim BB_p^I(n_1, n_2, m)$) (see [11]). The corresponding Dirichlet distribution, that is for $i = 1, \dots, r$, was derived by [28]. The distribution of $\mathbb{U} = (\mathbf{U}_1 : \mathbf{U}_2)'$, where

$$\mathbf{U}_i = (\mathbf{H}_i + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_i (\mathbf{H}_i + \mathbf{E})^{-\frac{1}{2}}, \quad i = 1, 2,$$

($\mathbf{H}_i \sim W_p(n_i, \mathbf{\Sigma})$, $i = 1, 2$, are independent of $\mathbf{E} \sim W_p(m, \mathbf{\Sigma})$) has been independently studied by [2], [7] and [16]. This distribution of \mathbb{U} is referred by [2] as the *bimatrix variate beta type IV distribution*, denoted by $\mathbb{U} \sim BB_p^{IV}(n_1, n_2, m)$. For detailed discussion on bimatrix variate beta distributions with

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²There exist other definitions for the beta matrix \mathbf{U} , see [5].

bounded domain the reader is referred to [11]. The refreshing contributions made by [7], [8] and [9] should also be acknowledged.

In this paper, we propose the Wishart ratios

$$\mathbf{X}_1 = (\mathbf{H}_1 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_1 (\mathbf{H}_1 + \mathbf{E})^{-\frac{1}{2}} \text{ and } \mathbf{X}_2 = (\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_2 (\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}}$$

($\mathbf{H}_i, i = 1, 2, 3$, and \mathbf{E} are independent, where $\mathbf{H}_i \sim W_p(n_i, \boldsymbol{\Sigma}), i = 1, 2, 3$, and $\mathbf{E} \sim W_p(m, \boldsymbol{\Sigma})$). From the construction it is easy to see that $\mathbf{X}_1 \sim B_p^I(n_1, m)$ and $\mathbf{X}_2 \sim B_p^I(n_2, n_3 + m)$ and we refer to the distribution of $\mathbb{X} = (\mathbf{X}_1 : \mathbf{X}_2)'$ as the *bimatrix variate extended beta type IV distribution*, denoted as $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, n_3, m)$. Furthermore, the ratios

$$\mathbf{F}_1 = \mathbf{E}^{-\frac{1}{2}} \mathbf{H}_1 \mathbf{E}^{-\frac{1}{2}} \text{ and } \mathbf{F}_2 = (\mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_2 (\mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}},$$

that is

$$\mathbb{F} = (\mathbf{F}_1, \mathbf{F}_2)' = \left((\mathbf{I}_p - \mathbf{X}_1)^{-\frac{1}{2}} \mathbf{X}_1 (\mathbf{I}_p - \mathbf{X}_1)^{-\frac{1}{2}}, (\mathbf{I}_p - \mathbf{X}_2)^{-\frac{1}{2}} \mathbf{X}_2 (\mathbf{I}_p - \mathbf{X}_2)^{-\frac{1}{2}} \right)'$$

for $\mathbb{X} = (\mathbf{X}_1 : \mathbf{X}_2)' \sim BEB_p^{IV}(n_1, n_2, n_3, m)$ will also be studied; refer to the distribution of \mathbb{F} as the *bimatrix variate extended F distribution*, denoted by $\mathbb{F} \sim BEF_p(n_1, n_2, n_3, m)$.

The rest of the paper is organized as follows: In section 2 we derive the density function of the bimatrix variate extended beta type IV distribution, as well as the bimatrix variate extended F distribution. These two distributions set the platform for section 3.

Section 3 is devoted to the derivation of $E\left(|\mathbf{X}_1|^{h_1} |\mathbf{X}_2|^{h_2}\right)$ where $\mathbb{X} = (\mathbf{X}_1 : \mathbf{X}_2)' \sim BEB_p^{IV}(n_1, n_2, n_3, m)$.

This is used to obtain the exact expression for the density function of $Z = \frac{\Lambda_1}{\Lambda_2}$ where $\Lambda_1 \equiv \left| \frac{\mathbf{H}_1}{\mathbf{H}_1 + \mathbf{E}} \right| = |\mathbf{X}_1|$ and $\Lambda_2 \equiv \left| \frac{\mathbf{H}_2}{\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E}} \right| = |\mathbf{X}_2|$ in terms of Meijer's G-function by the inverse Mellin transform.

Note that Λ_1 is the Wilks' statistic [31] and Λ_2 is a generalized statistic which arises when testing whether two normal populations are identical [1]. (For more generalized statistics, the reader is referred to [11].)

Furthermore, the moment generating function of the bimatrix extended beta type IV distribution is derived, and used to define the *bimatrix variate Kummer extended beta type IV distribution*. For a discussion on Kummer distributions the reader is referred, amongst others, to the work of [26] and [27].

Subsequently, the Laplace transform of \mathbb{F} (where \mathbb{F} has the bimatrix variate extended F distribution) is obtained and this results in the derivation of the density function of $tr(\mathbf{F}_1 + \mathbf{F}_2)$. We conclude the paper by specifically discussing the *bivariate variate extended beta type IV distribution* (an extension of Jones' bivariate beta model). It is well known that the stress-strength model describes the life of a component which has a random strength X_2 and is subjected to random stress X_1 . The component fails if the stress (X_1) applied to it exceeds the strength (X_2) and the component will function satisfactorily whenever $P(X_1 < X_2)$. Therefore the distribution of X_1/X_2 receives attention where (X_1, X_2) has the bivariate extended beta type IV distribution, together with some graphs and percentage points.

2 New members of the bimatrix beta type IV

The key idea of this section is to propose the bimatrix variate extended beta type IV distribution, as well as the bimatrix variate extended F distribution.

Let $\mathbf{H}_i, i = 1, 2, 3$, and \mathbf{E} be independent, where $\mathbf{H}_i \sim W_p(n_i, \boldsymbol{\Sigma}), i = 1, 2, 3$, and $\mathbf{E} \sim W_p(m, \boldsymbol{\Sigma})$ with $\text{Re}(n_i) > \frac{1}{2}(p-1), i = 1, 2, 3$ and $\text{Re}(m) > \frac{1}{2}(p-1)$ and define

$$\mathbf{X}_1 = (\mathbf{H}_1 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_1 (\mathbf{H}_1 + \mathbf{E})^{-\frac{1}{2}} \text{ and } \mathbf{X}_2 = (\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_2 (\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}} \quad (1)$$

then $\mathbb{X} = (\mathbf{X}_1 : \mathbf{X}_2)'$ is said to have an *bimatrix variate extended beta type IV distribution*, denoted as $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, n_3, m)$. Theorem 1 presents the density function of \mathbb{X} , followed by some remarks.

Theorem 1 Assume that $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, n_3, m)$. Then its density function is given by

$$f(\mathbf{X}_1, \mathbf{X}_2) = C \prod_{i=1}^2 |\mathbf{X}_i|^{\frac{(n_i-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{\frac{(n_1+n_3+m-p-1)}{2}} \times {}_2F_1\left(\frac{n_1+m}{2}, \frac{n_1+n_2+n_3+m}{2}; \frac{n_1+n_3+m}{2}; -\mathbf{X}_1(\mathbf{I}_p - \mathbf{X}_2)(\mathbf{I}_p - \mathbf{X}_1)^{-1}\right), \quad (2)$$

where $\mathbf{0} < \mathbf{X}_i < \mathbf{I}_p$, $i = 1, 2$, ${}_2F_1(\cdot)$ is the Gauss hypergeometric function of the matrix argument (see

[25], pp 258, 264), $C = \frac{\beta_p\left(\frac{n_1+m}{2}, \frac{n_3}{2}\right)}{\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}; \frac{m}{2}\right)}$ and $\beta_p(a_1, \dots, a_n) = \frac{\prod_{i=1}^n \Gamma_p(a_i)}{\Gamma_p\left(\sum_{i=1}^n a_i\right)}$ and $\Gamma_p(a)$ is the multivariate gamma function (see [25], pp 62).

Proof:

The joint density function of $\mathbf{H}_i, i = 1, 2, 3$, and \mathbf{E} is

$$f(\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{E}) = K \prod_{i=1}^3 \text{etr}\left(-\frac{1}{2}\Sigma^{-1}\mathbf{H}_i\right) |\mathbf{H}_i|^{\frac{(n_i-p-1)}{2}} \text{etr}\left(-\frac{1}{2}\Sigma^{-1}\mathbf{E}\right) |\mathbf{E}|^{\frac{(m-p-1)}{2}},$$

where

$$K^{-1} = \prod_{i=1}^3 \Gamma_p\left(\frac{n_i}{2}\right) \Gamma_p\left(\frac{m}{2}\right) |2\Sigma|^{\frac{(n_1+n_2+n_3+m)}{2}}. \quad (3)$$

On performing the transformations $\mathbf{Y}_i = \mathbf{H}_i^{-1}$, $i = 1, 2$, $\mathbf{X}_3 = \mathbf{E} + \mathbf{H}_3$, $\mathbf{Z}_1 = \mathbf{E}^{\frac{1}{2}}\mathbf{Y}_1\mathbf{E}^{-\frac{1}{2}}$, $\mathbf{Z}_2 = \mathbf{X}_3^{\frac{1}{2}}\mathbf{Y}_2\mathbf{X}_3^{\frac{1}{2}}$, $\mathbf{X}_i = (\mathbf{I}_p + \mathbf{Z}_i)^{-1}$, $i = 1, 2$, the Jacobian is

$$J(\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{E} \rightarrow \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{E}) = |\mathbf{E}|^{\frac{p+1}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-(p+1)} |\mathbf{X}_3|^{\frac{p+1}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-(p+1)}.$$

Therefore, the joint density of $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ and \mathbf{E} is

$$f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{E}) = K \text{etr}\left\{-\frac{1}{2}\Sigma^{-1}\left[\mathbf{E}^{\frac{1}{2}}(\mathbf{I}_p - \mathbf{X}_1)^{-1}\mathbf{X}_1\mathbf{E}^{\frac{1}{2}} + \mathbf{X}_3^{\frac{1}{2}}(\mathbf{I}_p - \mathbf{X}_2)^{-1}\mathbf{X}_2\mathbf{X}_3^{\frac{1}{2}} + \mathbf{X}_3\right]\right\} \times |\mathbf{E}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} \times |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} |\mathbf{X}_3|^{\frac{n_2}{2}} |\mathbf{X}_3 - \mathbf{E}|^{\frac{(n_3-p-1)}{2}}. \quad (4)$$

We consider the symmetrised density function of $(\mathbf{X}_1, \mathbf{X}_2)$ (see Appendix), that is

$$f_s(\mathbf{X}_1, \mathbf{X}_2) = \int_{\mathbf{X}_3 > \mathbf{0}} \int_{\mathbf{0} < \mathbf{E} < \mathbf{X}_3} \int_{O(p)} f(\mathbf{H}\mathbf{X}_1\mathbf{H}', \mathbf{H}\mathbf{X}_2\mathbf{H}', \mathbf{H}\mathbf{X}_3\mathbf{H}', \mathbf{H}\mathbf{E}\mathbf{H}') d\mathbf{H} d\mathbf{E} d\mathbf{X}_3$$

where \mathbf{H} ($p \times p$) is orthogonal and $d\mathbf{H}$ is the normalised Haar invariant measure on $O(p)$ (see [25], pp 72). Note that $d\mathbf{E} = d\mathbf{H}\mathbf{E}\mathbf{H}'$ and $d\mathbf{X}_3 = d\mathbf{H}\mathbf{X}_3\mathbf{H}'$. (Here $\mathbf{C} < \mathbf{D}$ means that the matrix $\mathbf{D} - \mathbf{C}$ is positive definite.) Therefore

$$f(\mathbf{H}\mathbf{X}_1\mathbf{H}', \mathbf{H}\mathbf{X}_2\mathbf{H}', \mathbf{H}\mathbf{X}_3\mathbf{H}', \mathbf{H}\mathbf{E}\mathbf{H}') = K |\mathbf{E}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{X}_3|^{\frac{n_2+n_3}{2} - \frac{(p+1)}{2}} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} \times |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_3^{-1}\mathbf{E}|^{\frac{(n_3-p-1)}{2}} \times \text{etr}\left\{-\frac{1}{2}\mathbf{H}'\Sigma^{-1}\mathbf{H}\mathbf{E}^{\frac{1}{2}}(\mathbf{I}_p - \mathbf{X}_1)^{-1}\mathbf{X}_1\mathbf{E}^{\frac{1}{2}} - \frac{1}{2}\mathbf{H}'\Sigma^{-1}\mathbf{H}\mathbf{X}_3^{\frac{1}{2}}(\mathbf{I}_p - \mathbf{X}_2)^{-1}\mathbf{X}_2\mathbf{X}_3^{\frac{1}{2}} - \frac{1}{2}\mathbf{H}'\Sigma^{-1}\mathbf{H}\mathbf{X}_3\right\}. \quad (5)$$

On performing the transformation $\mathbf{R} = \mathbf{X}_3^{-\frac{1}{2}} \mathbf{E} \mathbf{X}_3^{-\frac{1}{2}}$, with Jacobian $J(\mathbf{E} \rightarrow \mathbf{R}) = |\mathbf{X}_3|^{\frac{(p+1)}{2}}$, then it follows that

$$\begin{aligned} f_s(\mathbf{X}_1, \mathbf{X}_2) &= K |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} \\ &\quad \times \int_{O(p)} \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \mathbf{H}' \Sigma^{-1} \mathbf{H} \mathbf{X}_3 \right) \\ &\quad \times \text{etr} \left(-\frac{1}{2} \mathbf{H}' \Sigma^{-1} \mathbf{H} \mathbf{X}_3^{\frac{1}{2}} (\mathbf{I}_p - \mathbf{X}_2)^{-1} \mathbf{X}_2 \mathbf{X}_3^{\frac{1}{2}} \right) \times I_1 \, d\mathbf{X}_3 \, d\mathbf{H}. \end{aligned}$$

Applying ${}_0F_0(\mathbf{X}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_{\kappa}(\mathbf{X})}{k!} = \text{etr}(\mathbf{X})$, equation (A5) of [10] and equation (1.6.6) of [15], we obtain that

$$\begin{aligned} I_1 &= \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} \\ &\quad \times \text{etr} \left\{ -\frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \left((\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right)^{\frac{1}{2}} \mathbf{H}' \Sigma^{-1} \mathbf{H} \left((\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right)^{\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \right\} d\mathbf{R} \\ &= \frac{\Gamma_p \left(\frac{(n_1+m)}{2} \right) \Gamma_p \left(\frac{n_3}{2} \right)}{\Gamma_p \left(\frac{(n_1+n_3+m)}{2} \right)} {}_1F_1 \left(\frac{n_1+m}{2}; \frac{(n_1+n_3+m)}{2}; -\frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \left((\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right)^{\frac{1}{2}} \mathbf{H}' \Sigma^{-1} \mathbf{H} \left((\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right)^{\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \right) \end{aligned}$$

where ${}_1F_1(\cdot)$ is the confluent hypergeometric function of matrix argument. To simplify the notation, let us set $\Phi_1 = (\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1$, $\Phi_2 = (\mathbf{I}_p - \mathbf{X}_2)^{-1} \mathbf{X}_2$ and $\Phi_3 = \mathbf{H}' \Sigma^{-1} \mathbf{H}$. On performing the transformation $\mathbf{Q} = \Phi_3^{\frac{1}{2}} \mathbf{X}_3 \Phi_3^{\frac{1}{2}}$ with the Jacobian $J(\mathbf{X}_3 \rightarrow \mathbf{Q}) = |\Phi_3|^{-\frac{(p+1)}{2}}$, using equation (A5) of [10] and equation (1.6.4) of [15], we have (with K defined in (3)):

$$\begin{aligned} f_s(\mathbf{X}_1, \mathbf{X}_2) &= K \frac{\Gamma_p \left(\frac{(n_1+m)}{2} \right) \Gamma_p \left(\frac{n_3}{2} \right)}{\Gamma_p \left(\frac{(n_1+n_3+m)}{2} \right)} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} \\ &\quad \times \int_{O(p)} |\Phi_3|^{-\frac{(n_1+n_2+n_3+m)}{2}} \int_{\mathbf{Q} > \mathbf{0}} |\mathbf{Q}|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \mathbf{Q} (\mathbf{I}_p + \Phi_2) \right) \\ &\quad \times {}_1F_1 \left(\frac{(n_1+m)}{2}; \frac{(n_1+n_3+m)}{2}; -\frac{1}{2} \mathbf{Q} \Phi_1 \right) d\mathbf{Q} \, d\mathbf{H} \\ &= K \frac{\Gamma_p \left(\frac{(n_1+m)}{2} \right) \Gamma_p \left(\frac{n_3}{2} \right) \Gamma_p \left(\frac{(n_1+n_2+n_3+m)}{2} \right) 2^{\frac{1}{2} p (n_1+n_2+n_3+m)}}{\Gamma_p \left(\frac{1}{2} (n_1+n_3+m) \right)} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} \\ &\quad \times |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} \int_{O(p)} |\Phi_3|^{-\frac{(n_1+n_2+n_3+m)}{2}} |\mathbf{I}_p + \Phi_2|^{-\frac{(n_1+n_2+n_3+m)}{2}} \\ &\quad \times {}_2F_1 \left(\frac{(n_1+m)}{2}, \frac{(n_1+n_2+n_3+m)}{2}; \frac{(n_1+n_3+m)}{2}; -\Phi_1 (\mathbf{I}_p + \Phi_2)^{-1} \right) d\mathbf{H} \end{aligned}$$

Note that $\mathbf{I}_p + (\mathbf{I}_p - \mathbf{X}_2)^{-1} \mathbf{X}_2 = (\mathbf{I}_p - \mathbf{X}_2)^{-1}$. Since $f_s(\mathbf{H} \mathbf{X}_1 \mathbf{H}', \mathbf{H} \mathbf{X}_2 \mathbf{H}') = f(\mathbf{X}_1, \mathbf{X}_2)$ (see Appendix), the result (2) follows. \blacksquare

Remark

1. Using equation (4.4.22) pp 199 of [23], and after some simplification, the following alternative representation of the density function of the bimatrix variate extended beta type IV distribution follows:

$$\begin{aligned} f(\mathbf{X}_1, \mathbf{X}_2) &= C \prod_{i=1}^2 |\mathbf{X}_i|^{\frac{(n_i-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{\frac{(n_2+n_3+m-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{\frac{(n_1+n_3+m-p-1)}{2}} \\ &\quad \times |\mathbf{I}_p - \mathbf{X}_1 \mathbf{X}_2|^{-\frac{(n_1+n_2+n_3+m)}{2}} {}_2F_1 \left(\frac{n_3}{2}, \frac{n_1+n_2+n_3+m}{2}; \frac{n_1+n_3+m}{2}; \mathbf{X}_1 (\mathbf{I}_p - \mathbf{X}_2) (\mathbf{I}_p - \mathbf{X}_1 \mathbf{X}_2)^{-1} \right) \end{aligned} \quad (6)$$

where $\mathbf{0} < \mathbf{X}_i < \mathbf{I}_p$, $i = 1, 2$ and $C = \frac{\beta_p \left(\frac{n_1+m}{2}, \frac{n_3}{2} \right)}{\beta_p \left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}; \frac{m}{2} \right)}$.

2. From (6) follows that

$$f(\mathbf{X}_1, \mathbf{X}_2) = \left[f_{BB_p^{IV}}(n_1, n_2, n_3+m)(\mathbf{X}_1, \mathbf{X}_2) \right] \frac{\Gamma_p\left(\frac{(n_1+m)}{2}\right)\Gamma_p\left(\frac{(n_3+m)}{2}\right)}{\Gamma_p\left(\frac{m}{2}\right)\Gamma_p\left(\frac{(n_1+n_3+m)}{2}\right)} \\ \times {}_2F_1\left(\frac{n_3}{2}, \frac{(n_1+n_2+n_3+m)}{2}; \frac{(n_1+n_3+m)}{2}; \mathbf{X}_1(\mathbf{I}_p - \mathbf{X}_2)(\mathbf{I}_p - \mathbf{X}_1\mathbf{X}_2)^{-1}\right)$$

where $f_{BB_p^{IV}}(n_1, n_2, n_3+m)(\mathbf{X}_1, \mathbf{X}_2)$ is the density function of the bimatrix variate beta type IV distribution

(see [2]) with

$$f_{BB_p^{IV}}(a_1, a_2, c)(\mathbf{X}_1, \mathbf{X}_2) \\ = \left\{ \beta_p\left(\frac{a_1}{2}, \frac{a_2}{2}; \frac{c}{2}\right) \right\}^{-1} \prod_{i=1}^2 |\mathbf{X}_i|^{\frac{a_i}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{\frac{(a_2+c)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{\frac{(a_1+c)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{X}_1\mathbf{X}_2|^{-\frac{(a_1+a_2+c)}{2}}$$

$\mathbf{0} < \mathbf{X}_i < \mathbf{I}_p$, $i = 1, 2$. Therefore we refer to (2) as the bimatrix variate extended beta type IV distribution.

3. For $p = 1$, result (6) simplifies to

$$f_{BEB^{IV}}(n_1, n_2, n_3, m)(x_1, x_2) \\ = C^* \prod_{i=1}^2 x_i^{\frac{n_i}{2} - 1} (1 - x_1)^{\frac{(n_2+n_3+m)}{2} - 1} (1 - x_2)^{\frac{(n_1+n_3+m)}{2} - 1} (1 - x_1x_2)^{-\frac{(n_1+n_2+n_3+m)}{2}} \\ \times {}_2F_1\left(\frac{n_3}{2}, \frac{(n_1+n_2+n_3+m)}{2}; \frac{(n_1+n_3+m)}{2}; \frac{x_1(1-x_2)}{1-x_1x_2}\right) \quad (7)$$

for $0 \leq x_1, x_2 \leq 1$, $n_1, n_2, n_3, m > 0$, $C^* = \frac{\Gamma\left(\frac{n_1+n_3}{2}\right)\Gamma\left(\frac{(n_1+n_2+n_3+m)}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_3}{2}\right)\Gamma\left(\frac{(n_1+n_3+m)}{2}\right)}$, and ${}_2F_1(\cdot)$ is the Gauss hypergeometric function with scalar argument. We refer to (7) as the *bivariate extended beta type IV distribution*; denoted as $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$. This distribution (7) was obtained by [12] after applying a transformation componentwise on the extended bivariate F distribution; proposed in their paper.

Subsequently, the bimatrix variate extended F distribution is proposed in the next theorem.

Theorem 2 Let $\mathbf{H}_i, i = 1, 2, 3$, and \mathbf{E} be independent, where $\mathbf{H}_i \sim W_p(n_i, \mathbf{\Sigma})$, $i = 1, 2, 3$, and $\mathbf{E} \sim W_p(m, \mathbf{\Sigma})$ with $\text{Re}(n_i) > \frac{1}{2}(p-1)$, $i = 1, 2, 3$ and $\text{Re}(m) > \frac{1}{2}(p-1)$ and define

$$\mathbf{F}_1 = \mathbf{E}^{-\frac{1}{2}} \mathbf{H}_1 \mathbf{E}^{-\frac{1}{2}} \text{ and } \mathbf{F}_2 = (\mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}} \mathbf{H}_2 (\mathbf{H}_3 + \mathbf{E})^{-\frac{1}{2}}, \quad (8)$$

then $\mathbb{F} = (\mathbf{F}_1 : \mathbf{F}_2)'$ is said to have the *bimatrix variate extended F distribution*, with density function given by

$$f(\mathbf{F}_1, \mathbf{F}_2) = C \prod_{i=1}^2 |\mathbf{F}_i|^{\frac{(n_i-p-1)}{2}} |\mathbf{I}_p + \mathbf{F}_2|^{-\frac{(n_1+n_2+n_3+m)}{2}} {}_2F_1\left(\frac{n_1+m}{2}, \frac{(n_1+n_2+n_3+m)}{2}; \frac{(n_1+n_3+m)}{2}; -\mathbf{F}_1(\mathbf{I}_p + \mathbf{F}_2)^{-1}\right) \quad (9)$$

where $\mathbf{F}_i > \mathbf{0}, i = 1, 2$, and $C = \frac{\beta_p\left(\frac{n_1+m}{2}, \frac{n_3}{2}\right)}{\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}, \frac{m}{2}\right)}$.

Proof:

We know that if $\mathbf{U} \sim B_p^I(n, m)$, then $(\mathbf{I}_p - \mathbf{U})^{-1}\mathbf{U}$ has the matrix-variate beta type II distribution. Now let

$$\mathbb{F} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{I}_p - \mathbf{X}_1)^{-\frac{1}{2}} \mathbf{X}_1 (\mathbf{I}_p - \mathbf{X}_1)^{-\frac{1}{2}} \\ (\mathbf{I}_p - \mathbf{X}_2)^{-\frac{1}{2}} \mathbf{X}_2 (\mathbf{I}_p - \mathbf{X}_2)^{-\frac{1}{2}} \end{pmatrix}$$

with $J(\mathbf{X}_1, \mathbf{X}_2 \rightarrow \mathbf{F}_1, \mathbf{F}_2) = |\mathbf{I}_p + \mathbf{F}_1|^{-(p+1)} |\mathbf{I}_p + \mathbf{F}_2|^{-(p+1)}$, expression (9) follows.

Alternatively: (9) can be proved from (8). Let $\mathbf{X}_3 = \mathbf{E} + \mathbf{H}_3$, then the Jacobian is

$J(\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{E} \rightarrow \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{E}) = |\mathbf{E}|^{\frac{p+1}{2}} |\mathbf{X}_3|^{\frac{p+1}{2}}$ and we have that

$$\begin{aligned} f(\mathbf{F}_1, \mathbf{F}_2) &= K |\mathbf{F}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{F}_2|^{\frac{(n_2-p-1)}{2}} \int_{\mathbf{X}_3 > \mathbf{0}} \int_{\mathbf{0} < \mathbf{E} < \mathbf{X}_3} |\mathbf{X}_3|^{\frac{(n_2+n_3)}{2} - \frac{(p+1)}{2}} |\mathbf{E}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} \\ &\times |\mathbf{I}_p - \mathbf{X}_3^{-1} \mathbf{E}|^{\frac{(n_3-p-1)}{2}} \text{etr} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 - \frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{E}^{\frac{1}{2}} \mathbf{F}_1 \mathbf{E}^{\frac{1}{2}} - \boldsymbol{\Sigma}^{-1} \mathbf{X}_3^{\frac{1}{2}} \mathbf{F}_2 \mathbf{X}_3^{\frac{1}{2}} \right) d\mathbf{E} d\mathbf{X}_3; \end{aligned} \quad (10)$$

proceed now similarly as in the proof of Theorem 1 to obtain (9). \blacksquare

3 Properties

In this section some characteristics of (2) and (9) will be studied. Armed with these results some new density functions will be proposed. Firstly, in this section the product moment for the bimatrix variate extended beta type IV distribution (see (2)) is derived, then the exact expression for the density function of $Z = \frac{\Lambda_1}{\Lambda_2}$ where $\Lambda_1 \equiv \left| \frac{\mathbf{H}_1}{\mathbf{H}_1 + \mathbf{E}} \right| = |\mathbf{X}_1|$ and $\Lambda_2 \equiv \left| \frac{\mathbf{H}_2}{\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E}} \right| = |\mathbf{X}_2|$ is obtained. Secondly, the moment generating function of $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, n_3, m)$ is derived, and as a result the *bimatrix variate Kummer extended beta type IV* distribution is proposed. Lastly, the Laplace transform for $\mathbb{F} \sim BEF_p(n_1, n_2, n_3, m)$, is obtained. Using the expression of the Laplace transform, the density function of $\text{tr}(\mathbf{F}_1 + \mathbf{F}_2)$ is derived.

3.1 Density function of $|\mathbf{X}_1| / |\mathbf{X}_2|$

Lemma 1 Suppose that $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, n_3, m)$, then

$$\begin{aligned} E [|\mathbf{X}_1|^{h_1} |\mathbf{X}_2|^{h_2}] &= \frac{\left[\Gamma_p \left(\frac{p+1}{2} \right) \right]^2}{\Gamma_p \left(\frac{n_1}{2} \right) \Gamma_p \left(\frac{n_2}{2} \right) \Gamma_p \left(\frac{m}{2} \right)} \sum_{\kappa_1, \kappa_2, J_1, J_2; \phi^* \in J_1 \cdot J_2} \binom{\kappa_1}{J_1} \binom{\kappa_2}{J_2} g_{J_1, J_2}^{\phi^*} \frac{\Gamma_p \left(\frac{n_1}{2} + h_1, \kappa_1 \right) \binom{n_1+p+1}{2}_{\kappa_1} C_{\kappa_1}(\mathbf{I}_p)}{k_1! k_2! \Gamma_p \left(\frac{n_1+p+1}{2} + h_1, \kappa_1 \right) \binom{n_1+p+1}{2}_{J_1} C_{J_1}(\mathbf{I}_p)} \\ &\times \frac{\Gamma_p \left(\frac{n_2}{2} + h_2, \kappa_2 \right) \binom{n_2+p+1}{2}_{\kappa_2} C_{\kappa_2}(\mathbf{I}_p)}{\Gamma_p \left(\frac{n_2+p+1}{2} + h_2, \kappa_2 \right) \binom{n_2+p+1}{2}_{J_2} C_{J_2}(\mathbf{I}_p)} \frac{\Gamma_p \left(\frac{n_1+m}{2}, J_1 \right) \Gamma_p \left(\frac{n_1+n_2+n_3+m}{2}, \phi^* \right)}{\Gamma_p \left(\frac{n_1+n_3+m}{2}, J_1 \right)} C_{\phi^*}(-\mathbf{I}_p) \end{aligned} \quad (11)$$

where $\sum_{\kappa_1, \kappa_2, J_1, J_2; \phi^* \in J_1 \cdot J_2} = \sum_{\kappa_1=0}^{\infty} \sum_{\kappa_2=0}^{\infty} \sum_{J_1=0}^{\kappa_1} \sum_{J_2=0}^{\kappa_2} \sum_{\phi^* \in J_1 \cdot J_2}$, $C_{\kappa}(\cdot)$ is the zonal polynomial corresponding to κ (see [18]), $(a)_{\kappa}$ is the generalised Pochhammer symbol of weight κ and $g_{J_1, J_2}^{\phi^*}$ is defined in [4], equation (2.10), pp 467).

Proof:

From (4) and performing the transformation $\mathbf{R} = \mathbf{X}_3^{-\frac{1}{2}} \mathbf{E} \mathbf{X}_3^{-\frac{1}{2}}$, we have that $E [|\mathbf{X}_1|^{h_1} |\mathbf{X}_2|^{h_2}]$ equals

$$K \int_{\mathbf{X}_3 > \mathbf{0}} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} I_2 I_3 d\mathbf{R} d\mathbf{X}_3 \quad (12)$$

where K is defined in (3) (I_2 and I_3 are given below).

Consider the transformation from $\mathbf{X}_1 \rightarrow \mathbf{H} \mathbf{X}_1 \mathbf{H}'$, $\mathbf{H} \in O(p)$, under the normalized invariant measure $d\mathbf{H}$, we have from ${}_0F_0(\mathbf{X}) = \text{etr}(\mathbf{X})$ and equation (A5) of [10] that

$$\begin{aligned} I_2 &= \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \frac{1}{k_1!} \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2} + h_1} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} \\ &\times \int_{O(p)} C_{\kappa_1} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{H} (\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \mathbf{H}' \right) d\mathbf{H} d\mathbf{X}_1 \end{aligned}$$

Using equations (36) pp 243 and (5) pp 259 of [25];

$$I_2 = \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}+h_1} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} {}_0F_0^{(p)} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}}, (\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right) d\mathbf{X}_1$$

Applying ([25] equation (7) pp 283; [4] equation (3.2) pp 469; [25] equation (4) pp 282), I_2 equals

$$\begin{aligned} & \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \frac{L_{\kappa_1}^{\frac{1}{2}n} \left(\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} \right)}{k_1!} \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}+h_1} \frac{C_{\kappa_1}(\mathbf{X}_1)}{C_{\kappa_1}(\mathbf{I}_p)} d\mathbf{X}_1 \\ &= \Gamma_p \left(\frac{p+1}{2} \right) \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \frac{\Gamma_p \left(\frac{n_1}{2} + h_1, \kappa_1 \right) \binom{n_1+p+1}{2}_{\kappa_1}}{k_1! \Gamma_p \left(\frac{n_1+p+1}{2} + h_1, \kappa_1 \right)} C_{\kappa_1}(\mathbf{I}_p) \sum_{j_1=0}^{k_1} \sum_{J_1} \binom{\kappa_1}{J_1} \frac{C_{J_1} \left(-\frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-1} \right)}{\binom{n_1+p+1}{2}_{J_1} C_{J_1}(\mathbf{I}_p)} \end{aligned} \quad (13)$$

where $L_{\kappa}^{\gamma}(\cdot)$ is the generalised Laguerre polynomial (see [25], pp 282).

Similarly,

$$\begin{aligned} I_3 &= \int_{\mathbf{0} < \mathbf{X}_2 < \mathbf{I}_p} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}+h_2} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3^{\frac{1}{2}} \left((\mathbf{I}_p - \mathbf{X}_2)^{-1} \mathbf{X}_2 \right) \mathbf{X}_3^{\frac{1}{2}} \right\} d\mathbf{X}_2 \\ &= \Gamma_p \left(\frac{p+1}{2} \right) \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \frac{\Gamma_p \left(\frac{n_2}{2} + h_2, \kappa_2 \right) \binom{n_2+p+1}{2}_{\kappa_2}}{k_2! \Gamma_p \left(\frac{n_2+p+1}{2} + h_2, \kappa_2 \right)} C_{\kappa_2}(\mathbf{I}_p) \sum_{j_2=0}^{k_2} \sum_{J_2} \binom{\kappa_2}{J_2} \frac{C_{J_2} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right)}{\binom{n_2+p+1}{2}_{J_2} C_{J_2}(\mathbf{I}_p)} \end{aligned} \quad (14)$$

Substituting (13) and (14) in (12), using equation (36) pp 243 of [25]; follows that

$$\begin{aligned} E [|\mathbf{X}_1|^{h_1} |\mathbf{X}_2|^{h_2}] &= K \left[\Gamma_p \left(\frac{p+1}{2} \right) \right]^2 \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{j_1=0}^{k_1} \sum_{J_1} \binom{\kappa_1}{J_1} \frac{\Gamma_p \left(\frac{n_1}{2} + h_1, \kappa_1 \right) \binom{n_1+p+1}{2}_{\kappa_1} C_{\kappa_1}(\mathbf{I}_p)}{k_1! \Gamma_p \left(\frac{n_1+p+1}{2} + h_1, \kappa_1 \right) \binom{n_1+p+1}{2}_{J_1} C_{J_1}(\mathbf{I}_p)} \\ &\quad \times \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \sum_{j_2=0}^{k_2} \sum_{J_2} \binom{\kappa_2}{J_2} \frac{\Gamma_p \left(\frac{n_2}{2} + h_2, \kappa_2 \right) \binom{n_2+p+1}{2}_{\kappa_2} C_{\kappa_2}(\mathbf{I}_p)}{k_2! \Gamma_p \left(\frac{n_2+p+1}{2} + h_2, \kappa_2 \right) \binom{n_2+p+1}{2}_{J_2} C_{J_2}(\mathbf{I}_p)} \\ &\quad \times \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) C_{J_2} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) \\ &\quad \times \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} C_{J_1} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{R} \mathbf{X}_3 \right) d\mathbf{R} d\mathbf{X}_3 \end{aligned} \quad (15)$$

Since (equation (57), pp 254 of [25])

$$\int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} C_{J_1} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \right) d\mathbf{R} = \frac{\Gamma_p \left(\frac{n_1+m}{2}, J_1 \right) \Gamma_p \left(\frac{n_3}{2} \right)}{\Gamma_p \left(\frac{n_1+n_3+m}{2}, J_1 \right)} C_{J_1} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right),$$

it follows from ([4] equation (2.10) pp 467; [25] equation (43) pp. 248) that

$$\begin{aligned} & \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) C_{J_1} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) C_{J_2} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) d\mathbf{X}_3 \\ &= \sum_{\phi^* \in J_1 \cdot J_2} g_{J_1, J_2}^{\phi^*} \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) C_{\phi^*} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3 \right) d\mathbf{X}_3 \\ &= \sum_{\phi^* \in J_1 \cdot J_2} g_{J_1, J_2}^{\phi^*} \Gamma_p \left(\frac{n_1+n_2+n_3+m}{2}, \phi^* \right) |\mathbf{2}\boldsymbol{\Sigma}|^{\frac{n_1+n_2+n_3+m}{2}} C_{\phi^*}(-\mathbf{I}_p). \end{aligned} \quad (16)$$

Substituting (16) in (15); the result (11) follows. ■

Theorem 3 Let $\mathbb{X} \sim BEB_p^{IV}(n_1, n_2, m)$ and $Z = \frac{\Lambda_1}{\Lambda_2}$ where $\Lambda_1 = \left| \frac{\mathbf{H}_1}{\mathbf{H}_1 + \mathbf{E}} \right| = |\mathbf{X}_1|$ and

$\Lambda_2 = \left| \frac{\mathbf{H}_2}{\mathbf{H}_2 + \mathbf{H}_3 + \mathbf{E}} \right| = |\mathbf{X}_2|$, then the density function of Z is given by

$$\begin{aligned} &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right) \right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \sum_{\kappa_1, \kappa_2, J_1, J_2; \phi^* \in J_1 \cdot J_2} \binom{\kappa_1}{J_1} \binom{\kappa_2}{J_2} g_{J_1, J_2}^{\phi^*} \frac{\left(\frac{n_1+p+1}{2}\right)_{\kappa_1} \left(\frac{n_2+p+1}{2}\right)_{\kappa_2} C_{\kappa_1}(\mathbf{I}_p) C_{\kappa_2}(\mathbf{I}_p)}{k_1! k_2! \left(\frac{n_1+p+1}{2}\right)_{J_1} \left(\frac{n_2+p+1}{2}\right)_{J_2} C_{J_1}(\mathbf{I}_p) C_{J_2}(\mathbf{I}_p)} \\ &\times \frac{\Gamma_p\left(\frac{n_1+m}{2}, J_1\right) \Gamma_p\left(\frac{n_1+n_2+n_3+m}{2}, \phi^*\right)}{\Gamma_p\left(\frac{n_1+n_3+m}{2}, J_1\right)} C_{\phi^*}(-\mathbf{I}_p) G_{2p, 2p}^{p, p} \left(z \middle| \begin{matrix} a_1, \dots, a_{2p} \\ b_1, \dots, b_{2p} \end{matrix} \right), \quad z > 0 \end{aligned} \quad (17)$$

where

$$a_j = \begin{cases} -\frac{n_2}{2} - k_{2i} + \frac{(i+1)}{2} & \text{for } i = 1, 2, \dots, p \\ \frac{n_1}{2} + p + k_{1i-p} - \frac{i}{2} & \text{for } i = p+1, p+2, \dots, 2p \end{cases}, \quad b_j = \begin{cases} \frac{n_1}{2} + k_{1j} - \frac{(j+1)}{2} & \text{for } j = 1, 2, \dots, p \\ -\frac{n_2}{2} - k_{2j-p} - p + \frac{j}{2} - 1 & \text{for } j = p+1, p+2, \dots, 2p. \end{cases}$$

where $G(\cdot)$ denotes Meijer's G-function (see [22]).

Proof:

Let $h_1 = h - 1$ and $h_2 = h + 1$. Applying $\Gamma_p(a, \kappa) = \pi^{\frac{p(1-p)}{4}} \prod_{j=1}^p \Gamma\left[a + k_j - \frac{(j-1)}{2}\right]$, ([16] equation (1.5.7)),

Meijer's G-function, the inverse Mellin transform and ([22] equation (2.2.4) pp. 72), the required result (17) follows. \blacksquare

3.2 Bimatrix variate Kummer extended beta type IV distribution

Lemma 2 Suppose that $\mathbb{T} = [\mathbf{T}_1 : \mathbf{T}_2]'$. Then under the assumptions of Theorem 1 the moment generating function of \mathbb{X} is given by

$$\begin{aligned} M(\mathbb{T}) &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right) \right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \sum_{\kappa_1, \Lambda_1, \kappa_2, \Lambda_2; J_1, J_2; \phi \in \kappa_1 \cdot \Lambda_1; \phi^* \in \kappa_2 \cdot \Lambda_2; \Omega \in J_1 \cdot J_2} \left(\theta_{\phi}^{\kappa_1, \Lambda_1} \right)^2 \left(\theta_{\phi^*}^{\kappa_2, \Lambda_2} \right)^2 \binom{\kappa_1}{J_1} \binom{\kappa_2}{J_2} \\ &\times \frac{\Gamma_p\left(\frac{n_1}{2}, \phi\right) \Gamma_p\left(\frac{n_2}{2}, \phi^*\right) \left(\frac{n_1+p+1}{2}\right)_{\kappa_1} \left(\frac{n_1+p+1}{2}\right)_{\kappa_2}}{k_1! \lambda_1! k_2! \lambda_2! \Gamma_p\left(\frac{n_1+p+1}{2}, \phi\right) \Gamma_p\left(\frac{n_2+p+1}{2}, \phi^*\right) \left(\frac{n_1+p+1}{2}\right)_{J_1} \left(\frac{n_2+p+1}{2}\right)_{J_2}} \frac{C_{\phi}(\mathbf{I}_p) C_{\phi}(-\mathbf{T}_1) C_{\phi^*}(\mathbf{I}_p) C_{\phi^*}(-\mathbf{T}_2)}{C_{\Lambda_1}(\mathbf{I}_p) C_{J_1}(\mathbf{I}_p) C_{\Lambda_2}(\mathbf{I}_p) C_{J_2}(\mathbf{I}_p)} \\ &\times g_{J_1 J_2}^{\Omega} \frac{\Gamma_p\left(\frac{n_1+m}{2}, J_1\right) \Gamma_p\left(\frac{n_1+n_2+n_3+m}{2}, \Omega\right) C_{\Omega}(-\mathbf{I}_p)}{\Gamma_p\left(\frac{n_1+n_3+m}{2}, J_1\right)} \end{aligned} \quad (18)$$

$$\text{where} \quad \sum_{\kappa_1, \Lambda_1, \kappa_2, \Lambda_2; J_1, J_2; \phi \in \kappa_1 \cdot \Lambda_1; \phi^* \in \kappa_2 \cdot \Lambda_2; \Omega \in J_1 \cdot J_2} = \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{\lambda_1=0}^{\infty} \sum_{\Lambda_1} \sum_{j_1=0}^{k_1} \sum_{J_1} \sum_{\phi \in \kappa_1 \cdot \Lambda_1} \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \sum_{\lambda_2=0}^{\infty} \sum_{\Lambda_2} \sum_{j_2=0}^{k_2} \sum_{J_2} \sum_{\phi^* \in \kappa_2 \cdot \Lambda_2} \sum_{\Omega \in J_1 \cdot J_2}$$

Proof

Similar as in theorem 3, it follows from (4) and performing the transformation $\mathbf{R} = \mathbf{X}_3^{-\frac{1}{2}} \mathbf{E} \mathbf{X}_3^{-\frac{1}{2}}$, that the moment generating function of \mathbb{X} is given as follows

$$\begin{aligned} M(\mathbb{T}) &= E \text{etr}(\mathbf{T}_1 \mathbf{X}_1 + \mathbf{T}_2 \mathbf{X}_2) \\ &= K \int_{\mathbf{X}_3 > \mathbf{0}} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3\right) \\ &\times |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} I_4 I_5 d\mathbf{R} d\mathbf{X}_3 \end{aligned} \quad (19)$$

where K is defined in (3) (I_4 and I_5 are given below).

Consider the transformation from $\mathbf{X}_1 \rightarrow \mathbf{H} \mathbf{X}_1 \mathbf{H}'$, $\mathbf{H} \in O(p)$, under the normalized invariant measure $d\mathbf{H}$, we have from ([10] equation (A5)) and ([25] equations (36) pp 243 and (5) pp 259) that I_4 equals

$$\begin{aligned} &\int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} (\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 + \mathbf{T}_1 \mathbf{X}_1 \right\} d\mathbf{X}_1 \\ &= \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{(n_1-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{-\frac{(n_1+p+1)}{2}} \text{etr} \{ \mathbf{T}_1 \mathbf{X}_1 \}_0 F_0^{(p)} \left(-\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}}, (\mathbf{I}_p - \mathbf{X}_1)^{-1} \mathbf{X}_1 \right) d\mathbf{X}_1 \end{aligned}$$

Applying ([25] equation (7) pp. 283; [4] equations (4) pp 259 & (2.8) pp. 467; [3] equation (3.32) pp 60; [4] equation (2.2) pp 467; [25] equation (4) pp 282), I_4 equals

$$\begin{aligned} & \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{\lambda_1=0}^{\infty} \sum_{\Lambda_1} \frac{L^{\frac{1}{2}n_1} \left(\frac{1}{2}\Sigma^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \Sigma^{-\frac{1}{2}} \right)}{C_{\kappa_1}(\mathbf{I}_p)^{k_1! \lambda_1!}} \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_1|^{\frac{1}{2}(n_1-p-1)} C_{\kappa_1}(\mathbf{X}_1) C_{\Lambda_1}(\mathbf{T}_1 \mathbf{X}_1) d\mathbf{X}_1 \\ &= \Gamma_p \left(\frac{p+1}{2} \right) \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{\lambda_1=0}^{\infty} \sum_{\Lambda_1} \sum_{j_1=0}^{k_1} \sum_{J_1} \sum_{\phi \in \kappa_1, \Lambda_1} \binom{\kappa_1}{J_1} \left(\theta_{\phi}^{\kappa_1, \Lambda_1} \right)^2 \frac{\Gamma_p \left(\frac{n_1}{2}, \phi \right) \binom{n_1+p+1}{2}_{\kappa_1}}{k_1! \lambda_1! \Gamma_p \left(\frac{n_1+p+1}{2}, \phi \right)} \frac{C_{\phi}(\mathbf{I}_p) C_{\Lambda_1}(-\mathbf{T}_1) C_{J_1} \left(-\frac{1}{2}\Sigma^{-1} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \right)}{\binom{n_1+p+1}{2}_{J_1} C_{\Lambda_1}(\mathbf{I}_p) C_{J_1}(\mathbf{I}_p)} \end{aligned} \quad (20)$$

Similarly, I_5 equals

$$\begin{aligned} & \int_{\mathbf{0} < \mathbf{X}_1 < \mathbf{I}_p} |\mathbf{X}_2|^{\frac{(n_2-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{-\frac{(n_2+p+1)}{2}} \text{etr} \left\{ -\frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \Sigma^{-1} \mathbf{X}_3^{\frac{1}{2}} \left((\mathbf{I}_p - \mathbf{X}_2)^{-1} \mathbf{X}_2 \right) + \mathbf{T}_2 \mathbf{X}_2 \right\} d\mathbf{X}_2 \\ &= \Gamma_p \left(\frac{p+1}{2} \right) \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \sum_{\lambda_2=0}^{\infty} \sum_{\Lambda_2} \sum_{j_2=0}^{k_2} \sum_{J_2} \sum_{\phi^* \in \kappa_2, \Lambda_2} \binom{\kappa_2}{J_2} \left(\theta_{\phi^*}^{\kappa_2, \Lambda_2} \right)^2 \frac{\Gamma_p \left(\frac{n_2}{2}, \phi^* \right) \binom{n_2+p+1}{2}_{\kappa_2}}{k_2! \lambda_2! \Gamma_p \left(\frac{n_2+p+1}{2}, \phi^* \right)} \frac{C_{\phi^*}(\mathbf{I}_p) C_{\Lambda_2}(-\mathbf{T}_2) C_{J_2} \left(-\frac{1}{2}\Sigma^{-1} \mathbf{X}_3 \right)}{\binom{n_2+p+1}{2}_{J_2} C_{\Lambda_2}(\mathbf{I}_p) C_{J_2}(\mathbf{I}_p)} \end{aligned} \quad (21)$$

Substituting (20) and (21) in (19), followed by applying ([25] equation (57) pp 254; [4] equation (2.10) pp 467; [15] equation (1.5.12)), the result (18) follows. \blacksquare

Definition The $p \times p$ symmetric positive definite random matrices \mathbf{X}_1 and \mathbf{X}_2 on the unit p -sphere are said to have the bimatrix variate Kummer extended beta type IV distribution with parameters n_1, n_2, m and Ψ , denoted by $(\mathbf{X}_1, \mathbf{X}_2) \sim BKEB_p^{IV}(n_1, n_2, n_3, m; \Psi)$ if their joint density is given by

$$\begin{aligned} & K_1 \prod_{i=1}^2 |\mathbf{X}_i|^{\frac{(n_i-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1|^{\frac{(n_2+n_3+m-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_2|^{\frac{(n_1+n_3+m-p-1)}{2}} |\mathbf{I}_p - \mathbf{X}_1 \mathbf{X}_2|^{-\frac{(n_1+n_2+n_3+m)}{2}} \\ & \times \text{etr} [-\mathbf{T}(\mathbf{X}_1 + \mathbf{X}_2)] {}_2F_1 \left(\frac{n_3}{2}, \frac{(n_1+n_2+n_3+m)}{2}; \frac{(n_1+n_3+m)}{2}; \mathbf{X}_1(\mathbf{I}_p - \mathbf{X}_2)(\mathbf{I}_p - \mathbf{X}_1 \mathbf{X}_2)^{-1} \right) \end{aligned}$$

where

$$\begin{aligned} K_1^{-1} &= C^{-1} M(-\Psi : -\Psi) \\ &= C^{-1} \frac{\left[\Gamma_p \left(\frac{p+1}{2} \right) \right]^2}{\Gamma_p \left(\frac{n_1}{2} \right) \Gamma_p \left(\frac{n_2}{2} \right) \Gamma_p \left(\frac{m}{2} \right)} \sum_{\kappa_1, \Lambda_1, \kappa_2, \Lambda_2; J_1, J_2; \phi \in \kappa_1, \Lambda_1; \phi^* \in \kappa_2, \Lambda_2; \Omega \in J_1, J_2} \left(\theta_{\phi}^{\kappa_1, \Lambda_1} \right)^2 \left(\theta_{\phi^*}^{\kappa_2, \Lambda_2} \right)^2 \binom{\kappa_1}{J_1} \binom{\kappa_2}{J_2} \\ & \times \frac{\Gamma_p \left(\frac{n_1}{2}, \phi \right) \Gamma_p \left(\frac{n_2}{2}, \phi^* \right) \binom{n_1+p+1}{2}_{\kappa_1} \binom{n_1+p+1}{2}_{\kappa_2}}{k_1! \lambda_1! k_2! \lambda_2! \Gamma_p \left(\frac{n_1+p+1}{2}, \phi \right) \Gamma_p \left(\frac{n_2+p+1}{2}, \phi^* \right) \binom{n_1+p+1}{2}_{J_1} \binom{n_2+p+1}{2}_{J_2}} \frac{C_{\phi}(\mathbf{I}_p) C_{\phi}(\Psi) C_{\phi^*}(\mathbf{I}_p) C_{\phi^*}(\Psi)}{C_{\Lambda_1}(\mathbf{I}_p) C_{J_1}(\mathbf{I}_p) C_{\Lambda_2}(\mathbf{I}_p) C_{J_2}(\mathbf{I}_p)} \\ & \times g_{J_1, J_2}^{\Omega} \frac{\Gamma_p \left(\frac{n_1+m}{2}, J_1 \right) \Gamma_p \left(\frac{n_1+n_2+n_3+m}{2}, \Omega \right) C_{\Omega}(-\mathbf{I}_p)}{\Gamma_p \left(\frac{n_1+n_3+m}{2}, J_1 \right)}. \end{aligned}$$

The expression for K_1 follows directly from (18) and $C = \frac{\beta_p \left(\frac{n_1+m}{2}, \frac{n_3}{2} \right)}{\beta_p \left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{n_3}{2}; \frac{m}{2} \right)}$.

3.3 Density of $\text{tr}(\mathbf{F}_1 + \mathbf{F}_2)$

Lemma 3 Suppose that $\mathbb{S} = [\mathbf{S}_1 : \mathbf{S}_2]'$. Then under the assumptions of Theorem 2 the Laplace transform of \mathbb{F} is given by

$$\begin{aligned} \mathcal{L}(\mathbb{S}) &= \frac{\Gamma_p \left(\frac{n_1+m}{2} \right) \Gamma_p \left(\frac{n_1+n_2+n_3+m}{2} \right)}{\Gamma_p \left(\frac{m}{2} \right) \Gamma_p \left(\frac{n_1+n_3+m}{2} \right)} |\mathbf{S}_1|^{-\frac{n_1}{2}} |\mathbf{S}_2|^{-\frac{n_2}{2}} \sum_{\kappa_1, \kappa_2, \phi \in \kappa_1, \kappa_2} \theta_{\phi}^{\kappa_1, \kappa_2} \\ & \times \frac{\binom{n_1}{2}_{\kappa_1} \binom{n_1+m}{2}_{\kappa_1} \binom{n_2}{2}_{\kappa_2}}{k_1! k_2! \binom{n_1+n_3+m}{2}_{\kappa_1}} C_{\phi}^{\kappa_1, \kappa_2}(-\mathbf{S}_1^{-1}, -\mathbf{S}_2^{-1}) \end{aligned} \quad (22)$$

where $\sum_{\kappa_1, \kappa_2, \phi \in \kappa_1, \kappa_2} = \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \sum_{\phi \in \kappa_1, \kappa_2}$ and $C_{\phi}^{\kappa_1, \kappa_2}(\cdot, \cdot)$ denotes the invariant polynomial with two matrix arguments (see [4]).

Proof

From (10) and performing the transformation $\mathbf{R} = \mathbf{X}_3^{-\frac{1}{2}} \mathbf{E} \mathbf{X}_3^{-\frac{1}{2}}$, we have that the Laplace transform of \mathbb{F} is given by

$$\begin{aligned} L(\mathbb{S}) &= E \text{etr}(-\mathbf{S}_1 \mathbf{F}_1 - \mathbf{S}_2 \mathbf{F}_2) \\ &= K \int_{\mathbf{X}_3 > \mathbf{0}} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_p} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3\right) \\ &\quad \times |\mathbf{R}|^{\frac{(n_1+m)}{2} - \frac{(p+1)}{2}} |\mathbf{I}_p - \mathbf{R}|^{\frac{(n_3-p-1)}{2}} I_6 I_7 d\mathbf{R} d\mathbf{X}_3 \end{aligned} \quad (23)$$

where K is defined in (3) (I_6 and I_7 are given below).

From (equation (1.4.6) pp 19) of [15] follows that I_6 equals

$$\begin{aligned} &\int_{\mathbf{F}_2 > \mathbf{0}} |\mathbf{F}_2|^{\frac{(n_2-p-1)}{2}} \text{etr}\left\{-\left(\mathbf{S}_2 + \frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3^{\frac{1}{2}}\right) \mathbf{F}_2\right\} d\mathbf{F}_2 \\ &= \Gamma_p\left(\frac{n_2}{2}\right) |\mathbf{S}_2|^{-\frac{n_2}{2}} \left|\mathbf{I}_p + \frac{1}{2} \mathbf{S}_2^{-1} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3^{\frac{1}{2}}\right|^{-\frac{n_2}{2}} \end{aligned} \quad (24)$$

and also I_7 equals

$$\begin{aligned} &\int_{\mathbf{F}_1 > \mathbf{0}} |\mathbf{F}_1|^{\frac{(n_1-p-1)}{2}} \text{etr}\left\{-\left(\mathbf{S}_1 + \frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}}\right) \mathbf{F}_1\right\} d\mathbf{F}_1 \\ &= \Gamma_p\left(\frac{n_1}{2}\right) |\mathbf{S}_1|^{-\frac{n_1}{2}} \left|\mathbf{I}_p + \frac{1}{2} \mathbf{S}_1^{-1} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}} \mathbf{R} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}}\right|^{-\frac{n_1}{2}} \end{aligned} \quad (25)$$

Substituting (24) and (25) in (23), using ([25] equation (4) pp 259; [15] equation (1.6.6) pp 36); it follows that

$$\begin{aligned} L(\mathbb{S}) &= K \frac{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{n_3}{2}\right) \Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_3+m}{2}\right)} |\mathbf{S}_1|^{-\frac{n_2}{2}} |\mathbf{S}_2|^{-\frac{n_2}{2}} \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3\right) \\ &\quad \times \left|\mathbf{I}_p + \frac{1}{2} \mathbf{S}_2^{-1} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3 \boldsymbol{\Sigma}^{-\frac{1}{2}}\right|^{-\frac{n_2}{2}} {}_2F_1\left(\frac{n_1}{2}, \frac{n_1+m}{2}, \frac{n_1+n_3+m}{2}, -\frac{1}{2} \mathbf{X}_3^{\frac{1}{2}} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{S}_1^{-1} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3^{\frac{1}{2}}\right) d\mathbf{X}_3 \end{aligned} \quad (26)$$

Applying (equations (1) pp 258 & (4) pp 259) of [25]; it follows that $L(\mathbb{S})$ equals

$$\begin{aligned} &K \frac{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{n_3}{2}\right) \Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_3+m}{2}\right)} |\mathbf{S}_1|^{-\frac{n_2}{2}} |\mathbf{S}_2|^{-\frac{n_2}{2}} \sum_{k_1=0}^{\infty} \sum_{\kappa_1} \sum_{k_2=0}^{\infty} \sum_{\kappa_2} \frac{\binom{n_1}{2}_{\kappa_1} \binom{n_1+m}{2}_{\kappa_1} \binom{n_2}{2}_{\kappa_2}}{k_1! k_2! \binom{n_1+n_3+m}{2}_{\kappa_1}} \\ &\quad \times \int_{\mathbf{X}_3 > \mathbf{0}} |\mathbf{X}_3|^{\frac{(n_1+n_2+n_3+m)}{2} - \frac{(p+1)}{2}} \text{etr}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-1} \mathbf{X}_3\right) C_{\kappa_1}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{S}_1^{-1} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3\right) C_{\kappa_2}\left(-\frac{1}{2} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{S}_2^{-1} \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{X}_3\right) d\mathbf{X}_3 \end{aligned}$$

Using (equation (2.8) pp 467) of [4] and (equation (3.2) pp 58) of [3], we have after simplification (22). ■

Theorem 4 Suppose that $\mathbb{S} = [s\mathbf{I}_p : s\mathbf{I}_p]'$. Then under the assumptions of Lemma 3, the density function of $\text{tr}(\mathbf{F}_1 + \mathbf{F}_2)$ is given by

$$\begin{aligned} f(\text{tr}(\mathbf{F}_1 + \mathbf{F}_2)) &= \frac{\Gamma_p\left(\frac{n_1+m}{2}\right) \Gamma_p\left(\frac{n_1+n_2+n_3+m}{2}\right)}{\Gamma_p\left(\frac{m}{2}\right) \Gamma_p\left(\frac{n_1+n_3+m}{2}\right)} \sum_{\kappa_1, \kappa_2, \phi \in \kappa_1, \kappa_2} \theta_{\phi}^{\kappa_1, \kappa_2} (\text{tr}(\mathbf{F}_1 + \mathbf{F}_2))^{\frac{n_1 p + n_2 p}{2} + k_1 + k_2 - 1} \\ &\quad \times \frac{\binom{n_1}{2}_{\kappa_1} \binom{n_1+m}{2}_{\kappa_1} \binom{n_2}{2}_{\kappa_2}}{k_1! k_2! \binom{n_1+n_3+m}{2}_{\kappa_1} \Gamma\left(\frac{n_1 p + n_2 p}{2} + k_1 + k_2\right)} C_{\phi}^{\kappa_1, \kappa_2}(-\mathbf{I}_p, -\mathbf{I}_p) \end{aligned} \quad (27)$$

Proof

From (22) and making use of the inverse Laplace transformation it follows that

$$\begin{aligned}
 f(\text{tr}(\mathbf{F}_1 + \mathbf{F}_2)) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}(s) \exp(s(\text{tr}(\mathbf{F}_1 + \mathbf{F}_2))) ds \\
 &= \frac{\Gamma_p\left(\frac{n_1+m}{2}\right) \Gamma_p\left(\frac{n_1+n_2+n_3+m}{2}\right)}{\Gamma_p\left(\frac{m}{2}\right) \Gamma_p\left(\frac{n_1+n_3+m}{2}\right)} \sum_{\kappa_1, \kappa_2, \phi \in \kappa_1 \cdot \kappa_2} \theta_{\phi}^{\kappa_1, \kappa_2} \frac{\left(\frac{n_1}{2}\right)_{\kappa_1} \left(\frac{n_1+m}{2}\right)_{\kappa_1} \left(\frac{n_2}{2}\right)_{\kappa_2}}{k_1! k_2! \left(\frac{n_1+n_3+m}{2}\right)_{\kappa_1}} C_{\phi}^{\kappa_1, \kappa_2}(-\mathbf{I}_p, -\mathbf{I}_p) \\
 &\quad \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-(n_1 p + n_2 p + k_1 + k_2)} \exp(s(\text{tr}(\mathbf{F}_1 + \mathbf{F}_2))) ds
 \end{aligned}$$

Therefore from equation (1) pp 238 of [13] follows the result (27). ■

4 Discussion of the bivariate case

In this section we focus on the bivariate case where $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$ with the density function given by (7). Equivalently, (7) can be expressed as an infinite mixture of the popular Jones' bivariate beta distribution (which was independently proposed by [20] and [29]):

$$f_{BEB^{IV}(n_1, n_2, n_3, m)}(x_1, x_2) = \frac{\Gamma\left(\frac{n_1+m}{2}\right) \Gamma\left(\frac{n_3+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_3}{2}\right) \Gamma\left(\frac{m}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{n_1}{2}+k\right) \Gamma\left(\frac{n_3}{2}+k\right)}{k! \Gamma\left(\frac{n_1+n_3+m}{2}+k\right)} f_{BBJONES(n_1+k, n_2, n_3+m)}(x_1, x_2)$$

where for $0 < x_1, x_2 < 1$,

$$f_{BBJONES(a_1, a_2, c)}(x_1, x_2) = \frac{\Gamma\left(\frac{a_1+a_2+c}{2}\right)}{\Gamma\left(\frac{a_1}{2}\right) \Gamma\left(\frac{a_2}{2}\right) \Gamma\left(\frac{c}{2}\right)} x_1^{\frac{a_1}{2}-1} x_2^{\frac{a_2}{2}-1} (1-x_1)^{\frac{(a_2+c)}{2}-1} (1-x_2)^{\frac{(a_1+c)}{2}-1} (1-x_1 x_2)^{-\frac{(a_1+a_2+c)}{2}}, \tag{28}$$

The following will be addressed in this section:

- The effect of the shape parameter n_3 on the density function (7) is illustrated.
- The stress-strength model describes the lifetime of a component with random strength X_2 subjected to a random stress X_1 . The measure $P(X_1 < X_2)$ is of interest and thus the density function of the ratio $Z = \frac{X_1}{X_2}$, where $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$ is studied. Figure 2 illustrates the shape of this density function of Z for different values of the parameter n_3 .
- Tabulation of some percentage points of Z is given, see Table 1.

The programming was done by making use of built-in routines of the package Mathematica. Figure 1 illustrates the effect of the parameter n_3 for $n_3 = 0, 4$ and 10 when $n_1 = m = 4$ and $n_2 = 6$ on the bivariate extended beta type IV density function. This figure also contains the contour plots for easy comparison. As n_3 increased and n_1, n_1, m remain constant, the density function shifts towards the x_1 axes.

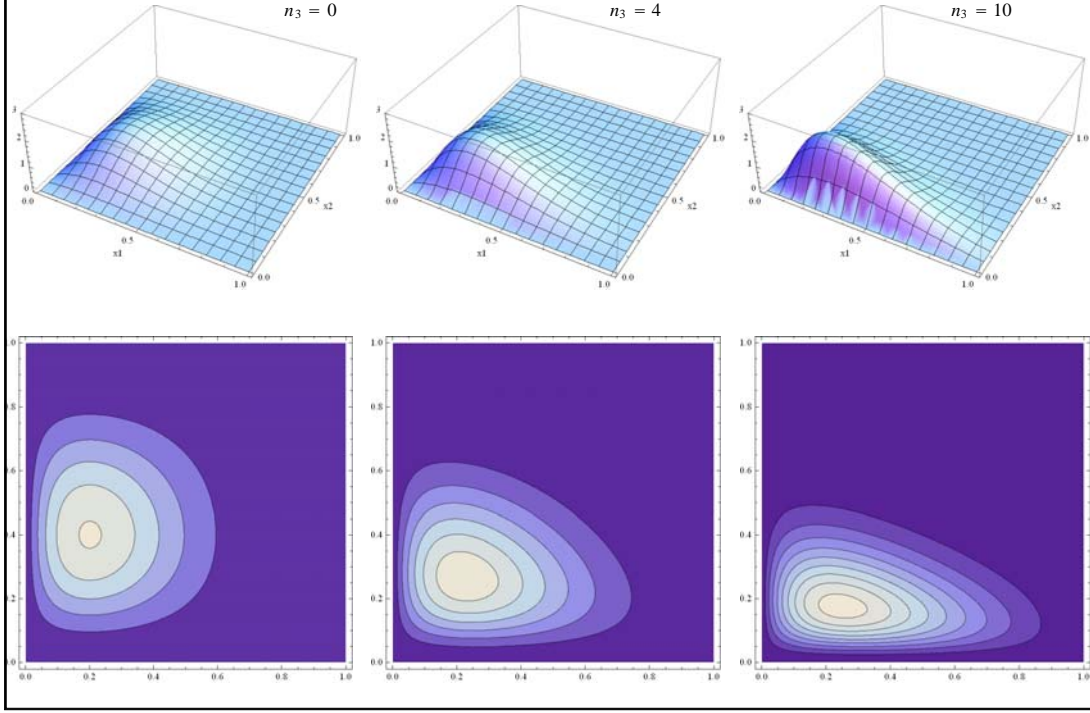


Figure 1: Effect of n_3 on (7).

The density function of $Z = \frac{X_1}{X_2}$ for $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$ (see (7)) is given in the next theorem.

Theorem 5 Suppose that $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$, then the density function of $Z = \frac{X_1}{X_2}$ is given by

$$f(z) = \frac{\Gamma(\frac{n_1+m}{2})\Gamma(\frac{n_2+n_3+m}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\Gamma(\frac{m}{2})\Gamma(\frac{n_3}{2})} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n_3}{2} + j)\Gamma(\frac{n_1+n_2+n_3+m}{2} + j + k)}{j!k!} G_{2,2}^{1,1} \left[z \begin{matrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{matrix} \right] \quad (29)$$

for $z > 0$, where $\alpha_1 = -(\frac{n_2}{2} + k)$, $\alpha_2 = \frac{n_1+n_2+n_3+m}{2} + j + k - 1$, $\beta_1 = \frac{n_1}{2} + j + k - 1$ and $\beta_2 = -(\frac{n_1+n_2+n_3+m}{2} + j + k)$.

Proof

The proof of the density function of Z is based on the product moment of (X_1, X_2) and the inverse Mellin transform. ■

Remark

An alternative expression for the density function of $Z = \frac{X_1}{X_2}$ for $(X_1, X_2) \sim BEB^{IV}(n_1, n_2, n_3, m)$ is: for $0 < z \leq 1$,

$$f(z) = \frac{z^{\frac{n_1}{2}-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\Gamma(\frac{m}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n_1+n_2+n_3+m}{2} + k)\Gamma(\frac{n_1+m}{2} + k)\Gamma(\frac{n_1+n_2}{2} + k)}{\Gamma(n_1 + \frac{n_2+n_3+m}{2} + 2k)} \frac{(-z)^k}{k!} \\ \times {}_2F_1\left(\frac{n_1+n_2}{2} + k, \frac{n_1}{2} + k + 1; n_1 + \frac{n_2+n_3+m}{2} + 2k; z\right)$$

for $z > 1$,

$$f(z) = \frac{z^{-(\frac{n_2}{2}+1)}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{m}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n_1+n_2+n_3+m}{2} + k)\Gamma(\frac{n_1+m}{2} + k)\Gamma(\frac{n_1+n_2}{2} + k)\Gamma(-\frac{n_1}{2} - k)}{\Gamma(\frac{n_1+n_3+m}{2} + k)} \\ \times \frac{(-1)^k}{k!} {}_2F_1\left(\frac{n_1+n_2}{2} + k, 1 - \frac{n_1+n_3+m}{2} - k; \frac{n_2}{2}; \frac{1}{z}\right).$$

Note that since $p = q = 2$ for the $G_{p,q}^{m,n}(\cdot)$ function in (29), it is not necessary to split the domain into $0 < z \leq 1$ and $z > 1$.

The percentage points z_α of Z are obtained numerically by solving the equation $\int_0^{z_\alpha} f(z) dz = \alpha$. Evidently, this involves computation of the Meijer's G-function and routines are widely available. We used the built-in routines of the package Mathematica. Table 1 provides the numerical values of z_α for different values of parameters and α . Similar tabulations can be obtained for other values of the parameters, as well as the upper percentiles. Note the percentage points in Table 1 confirm the shapes for the densities in figure 2.

n_1	n_2	m	n_3	$\alpha = 0.01$	0.025	0.05	0.1
2	2	2	2	0.039786	0.098659	0.194604	0.378128
2	2	2	4	0.049761	0.123500	0.243952	0.475357
2	4	2	2	0.024911	0.061940	0.122754	0.240956
2	2	4	2	0.025015	0.062595	0.125387	0.251612
4	2	2	2	0.362734	0.550152	0.740030	0.976838
2	4	4	2	0.015012	0.037576	0.075310	0.151278
2	2	4	4	0.030032	0.075205	0.150838	0.303511
2	4	2	4	0.029903	0.074395	0.147569	0.290190
4	4	2	2	0.252780	0.388628	0.531727	0.716786
4	2	4	2	0.261564	0.410647	0.575883	0.805010
2	4	4	4	0.017519	0.043872	0.087994	0.177039
4	2	4	4	0.303411	0.477792	0.672716	0.948492
4	4	2	4	0.292869	0.451137	0.618720	0.839218
4	4	4	2	0.175456	0.276560	0.389742	0.548510
4	4	4	4	0.199540	0.315123	0.445135	0.628972
6	6	6	4	0.310616	0.424604	0.539504	0.689438
6	6	4	4	0.373029	0.502772	0.629632	0.790893
6	4	4	4	0.456135	0.612706	0.765111	0.960481
6	4	6	4	0.385280	0.526282	0.668507	0.855642
4	6	6	4	0.125511	0.20039	0.286764	0.413175
4	6	4	4	0.158147	0.250137	0.353973	0.501399
4	4	6	4	0.160546	0.256347	0.366930	0.529095
2	6	2	2	0.019943	0.049642	0.098564	0.194227
2	2	6	2	0.020058	0.050364	0.101484	0.206182

Table 1: The lower percentage points z_α of Z .

Figure 2 illustrates the shape of the density function (29) for different values of the parameters.

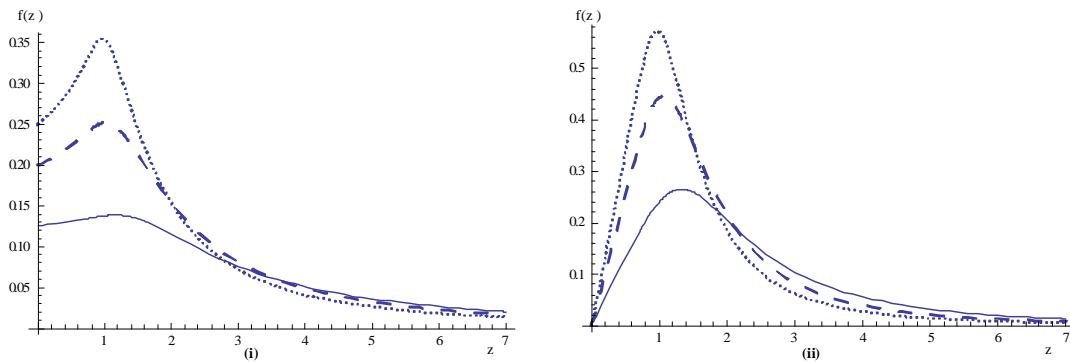


Figure 2: Density (29) for (i) $n_1 = n_2 = m = 1$ (ii) $n_1 = n_2 = m = 4$.

The three curves in each panel are: tiny dotted line $n_3 = 2$, large dotted line $n_3 = 4$, solid line $n_3 = 10$.

5 Conclusion

In this paper, we introduced the bimatrix variate extended beta type IV distribution as well as the bimatrix variate extended F distribution; both originated from "ratios" of Wishart random variates. These are new additions to the bimatrix beta field that currently receives a lot of attention in the literature. Some possible applications of bimatrix variate beta distributions were discussed by [7]. These newly proposed bimatrix distributions (arising from different Wishart ratios) can serve as alternative to the well-known bimatrix variate beta type I distribution.

Some properties of these bimatrix variate distributions in section 2, were explored and here the reader should note the derivation of (i) the density function of the ratio of the determinants of the components of the bimatrix variate extended beta type IV distribution; (ii) density function of the trace of the sum of the components of the bimatrix variate extended F distribution, as well as the defining of the bimatrix variate Kummer extended beta type IV distribution. These exact expressions for the distributions were in terms of complex functions that are suddenly more computable due to dynamic programming and the availability of packages and algorithms. Therefore the theory can be transformed into practice for the user. Applications of the ratio of Wilks' statistic, or a variation of it, for independent components was discussed by [30], thus the potential to extend these applications to dependent components can exist.

Lastly, we shifted attention to the bivariate case (X_1, X_2) with focus on the density of the ratio X_1/X_2 since stress-strength models are important for the practitioner. This study is concluded with a few open problems that need to be addressed in future research: (i) Results for different covariance matrices, Σ_1 and Σ_2 of the Wishart distributions, will be an interesting study; (ii) To explore the use of these expressions in the hypothesis testing context where two samples are present; (iii) In this study the focus was only on bounded domain, but there is always the question about results for other domains. Since matrix theory is now a big subject with applications in many disciplines of science, engineering and finance, it is hoped that the results in this paper will be picked up by many researchers.

6 Appendix A

The expressions in this paper are expressed in terms of Meijer's G-function, zonal polynomials, Laguerre polynomial, hypergeometric functions with matrix argument, or homogeneous invariant polynomials with two matrix arguments. The reader is referred to the papers of ([3], [4], [6], [17], [18], [19]) on these functions; as well as the reference books ([15], [22], [23] and [25]).

[A.1] ([14])

Given a density function $f(\mathbf{X})$, $\mathbf{X} : p \times p$, $\mathbf{X} > 0$, the *symmetrised* density function is defined by [14] as $f_s(\mathbf{X}) \equiv \int_{O(p)} f(\mathbf{H}\mathbf{X}\mathbf{H}') d\mathbf{H}$, $\mathbf{H} \in O(p)$

where $O(p) = \{\mathbf{H} (p \times p) | \mathbf{H}\mathbf{H}' = \mathbf{H}'\mathbf{H} = \mathbf{I}_p\}$ and $d\mathbf{H}$ denotes the normalised Haar invariant measure on the orthogonal group $O(p)$ ([25], pp. 72).

[A.2]

This idea from [14] was applied by [5] in an inverse way, i.e. well-known the explicit expression of the symmetrised density function of X , $f_s(\mathbf{X}) \equiv \int_{O(p)} f(\mathbf{H}\mathbf{X}\mathbf{H}') d\mathbf{H}$. The density function $f(\mathbf{X})$, is then identified from $f(\mathbf{H}\mathbf{X}\mathbf{H}')$. This density obtained by applying the idea underlying, $f_s(\mathbf{X}) \equiv \int_{O(p)} f(\mathbf{H}\mathbf{X}\mathbf{H}') d\mathbf{H}$, was termed by [5] as the nonsymmetrised density function of \mathbf{X} .

[A.3]

Note that $f_s(\mathbf{X}) = f(\mathbf{X})$ if $f(\mathbf{H}\mathbf{X}\mathbf{H}') = f(\mathbf{X})$.

Acknowledgements. Andriette Bekker is grateful to the National Research Foundation, South Africa for support through the grant (Unlocking the future: FA2007043000003). We would like to thank the anonymous reviewer for his/her careful reading and comments.

References

- [1] T.W. Anderson, An introduction to multivariate statistical analysis, 2nd ed., John Wiley & Sons, New York, 1984.

- [2] A. Bekker, J.J.J. Roux, R. Ehlers, M. Arashi, Bimatrix variate beta type IV distribution: relation to Wilks' statistic and bimatrix variate Kummer-beta type IV distribution. *Communications in Statistics- Theory and Methods* (2010) (in production).
- [3] Y. Chikuse, Invariant polynomials with matrix arguments and their applications, in: R P Gupta (ed.), *Multivariate Statistical Analysis*, North-Holland Publishing Company, 1980, pp. 53-68.
- [4] A. W. Davis, Invariant polynomials with two matrix arguments, extending the zonal polynomials: Applications to multivariate distribution theory. *Ann. Inst. Statist. Math. Part A* (1979) 31 465-485.
- [5] J.A. Díaz-García, R. Gutiérrez-Jáimez, Noncentral nonsingular matrix variate beta distribution, *Brazilian Journal of Probability and Statistics* 21 (2007) 175-186.
- [6] J. A. Díaz-García, Special functions: Integral properties of Jack polynomials, hypergeometric functions and invariant polynomials, (2009), *arXiv:0909.1988v1[math.St]*.
- [7] J. A. Díaz-García, R. Gutiérrez-Jáimez, Bimatrix variate generalised beta distributions, *South African Statist. J.* 44 (2010a) 193-208.
- [8] J. A. Díaz-García, R. Gutiérrez-Jáimez, Complex bimatrix variate generalised beta distributions. *Linear Algebra Appl.* 432 (2010b) 571-582.
- [9] J. A. Díaz-García, R. Gutiérrez-Jáimez, Noncentral bimatrix variate generalised beta distributions, *Metrika* 73 (2011) 317-333.
- [10] R. Ehlers, A. Bekker, J.J.J. Roux, The central and non-central matrix variate Dirichlet type III distribution, *South African Statistical J.* 43 (2009) 97-116.
- [11] R. Ehlers, Bimatrix variate distributions of Wishart ratios with application (unpublished thesis, University of Pretoria), 2011.
- [12] A.H. El Bassiouny, M.C. Jones, A bivariate F distribution with marginals on arbitrary numerator and denominator degrees of freedom, and related bivariate beta and F distributions. *Statistical Methods and Appl.* 18 (2009) 465-481.
- [13] A. Erdelyi, et al , *Tables of integral transforms*, vol 2, McGraw-Hill, New:York, 1954.
- [14] M.J. Greenacre, Symmetrized multivariate distributions, *South African Statistical J.* 7 (1973) 95-101.
- [15] A.K. Gupta, D.K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton, 2000.
- [16] A.K. Gupta, D.K. Nagar, Matrix variate generalization of a bivariate beta type I distribution, *Journal of Statistics and Management Systems* 12 (2009) 873-885.
- [17] C.S. Herz, Bessel functions of matrix argument, *Ann of Math.* 61 (1955) 474-523.
- [18] A.T. James, Zonal polynomials of the real positive definite symmetric matrices, *Ann Math.* 33 (1961) 456-469
- [19] A. T. James, Distribution of matrix variate and latent roots derived from normal samples, *Ann Math Statist.* 35 (1964) 475-501.
- [20] M.C. Jones, Multivariate t and the beta distributions associated with the multivariate F distribution, *Metrika* 54 (2001) 215-231.
- [21] C.G. Khatri, On the mutual independence of certain statistics, *The Annals of Mathematical Statistics* 30 (1959) 1258-1262.
- [22] A.M. Mathai, *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Clarendon Press, Oxford, 1993.
- [23] A.M. Mathai, S.B. Provost, T. Hayakawa, *Bilinear forms and zonal polynomials*, Springer-Verlag, New York, 1995.
- [24] D.F. Morrison, *Multivariate Statistical Methods*, Thomson, United Kingdom, 2005.

- [25] R.J. Muirhead, *Aspects of Multivariate Statistical Theory*, John Wiley & Sons, New York, 1982.
- [26] D.K. Nagar, L. Cardeno, Matrix variate Kummer-Gamma distributions, *Random Oper. and Stoch. Equ.* 9 (2001) 207-218.
- [27] D.K. Nagar, A.K. Gupta, Matrix-variate Kummer-Beta distribution, *J. Austral. Math. Soc.* 73 (2002) 11-25.
- [28] I. Olkin, H. Rubin, Multivariate beta distributions and independence properties of the Wishart distribution, *The Annals of Mathematical Statistics* 35 (1964) 261-269. Correction 37 (1966) 297.
- [29] I. Olkin, R. Liu, A Bivariate beta distribution, *Statistics and Probability Letters* 62 (2003) 407-412.
- [30] T. Pham-Gia, Exact distribution of the generalized Wilks's statistic and applications, *Journal of Multivariate Analysis*, 99 (2008) 698-1716.
- [31] S.S. Wilks, Certain generalizations in the analysis of variance, *Biometrika* 24 (1932) 471-494.