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Characterizing Brownian motion by martingale properties

by

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Declaration

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Summary

Title	Characterizing Brownian motion by martingale properties
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We prove Paul Lévy's celebrated *martingale characterization of Brownian motion.* Let us recall here that if $(X_t)_{t \geq 0}$ is a Brownian motion in $\mathbb{R}$, then both $(X_t)_{t \geq 0}$ and $(X_t^2 - t)_{t \geq 0}$ are martingales. Next, assume that a real-valued stochastic process $(X)_{t \geq 0}$ with continuous sample paths satisfy:

- $(X_t)_{t \geq 0}$ is a martingale; and
- $(X_t^2 - t)_{t \geq 0}$ is a martingale.

It turns out that this property characterizes Brownian motion.

The main source for much of the discussion is Stromberg [4]. Firstly, a few concepts underlying Lévy's theorem are given, which includes reference to some of Doob's inequalities. Lastly, the proof of Lévy's theorem is given.

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Contents

1 Introduction

The main task of this dissertation is to prove Paul Lévy's characterization of Brownian motion. It is a fact that every real - valued Brownian motion $(X_t)_{t\geq 0}$ satisfies

(i). $(X_t)_{t\geq 0}$ is a martingale;

(ii). $(X_t^2 - t)_{t\geq 0}$ is a martingale.

Lévy shows in his theorem that whenever a real - valued stochastic process $(X_t)_{t\geq 0}$ with continuous sample paths satisfies (i) and (ii) above, then $(X_t)_{t\geq 0}$ is a Brownian motion. The proof given by Stromberg [4] of Lévy's theorem was used as the building block of the proof that appears in this dissertation.

Section 2 states the basic definition of a Brownian motion. In section 3 introductory theorems are proved which form the basis of the proof of the objective of the dissertation: the proof of Lévy's theorem in section 4.

2 Basic definitions

Definition 1. We fix a probability space $(\Omega, \mathfrak{F}, P)$. A *stochastic process* is a measurable function on $\Omega \times [0, \infty)$ into $\mathbb{R}$. The value of a stochastic process X at time t is the random variable $X(\cdot, t) : \Omega \mapsto \mathbb{R}$.

Definition 2. A *standard Brownian motion* is a stochastic process X defined by the properties:

BM 1: $X_0(\omega) = 0 \ \forall \omega \in \Omega$;

BM 2: *stationary increments: for any times* t *and* $s > t$, $X_s - X_t$ *is normally distributed with mean zero and variance* $s - t$;

BM 3: *independent increments: for any times* $t_0, \ldots, t_n$ *such that* $0 \leq t_0 < t_1 <, \ldots < t_n < \infty$, *the random variables* $X(\omega, t_0), X(\omega, t_1) - X(\omega, t_0), \ldots, X(\omega, t_n) - X(\omega, t_{n-1})$ *are independently distributed; and*

BM 4: *continuity: for each* $\omega \in \Omega$, *the sample path* $t \mapsto X(\omega, t)$ *is continuous.*

3 Introductory theorems

Theorem 3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ a σ-subalgebra of $\mathcal{F}$. Suppose $f, g : \Omega \mapsto [0, \infty]$ are $\mathcal{F}$-measurable. Then $1 < p < \infty, q = \frac{p}{p-1}$ implies*

(i). $\mathbb{E}(fg|\mathcal{G}) \leq \mathbb{E}[(f^p|\mathcal{G})]^{\frac{1}{p}}\mathbb{E}[(g^q|\mathcal{G})]^{\frac{1}{q}}$ a.s., and

(ii). $[\mathbb{E}((f+g)^p|\mathcal{G})]^{\frac{1}{p}} \leq [\mathbb{E}(f^p|\mathcal{G})]^{\frac{1}{p}} + [\mathbb{E}(g^p|\mathcal{G})]^{\frac{1}{p}}$ a.s.

Proof. First suppose that f and g are simple functions. Without loss of generality we may assume there exist disjoint $A_1, A_2, \cdots, A_n \in \mathcal{F}$ and nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ such that $f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ and $g = \sum_{j=1}^{n} \beta_j \chi_{A_j}$. Note that some α_j and β_j, $j = 1, \ldots, n$ may be zero. Let $h_j = \mathbb{E}\left(\chi_{A_j}|\mathcal{G}\right)$ for $j = 1, 2, \ldots, n$. By definition $h_j : \Omega \mapsto [0, 1]$ is $\mathcal{G}-$measurable. Then

$$\mathbb{E}(fg|\mathcal{G}) = \mathbb{E}\left(\sum_{j=1}^{n} \alpha_j\beta_j\chi_{A_j}\middle|\mathcal{G}\right)$$

By linearity it follows that

$$\begin{aligned}\mathbb{E}(fg|\mathcal{G}) &= \sum_{j=1}^{n} \alpha_j\beta_j\mathbb{E}(\chi_{A_j}|\mathcal{G})\\ &= \sum_{j=1}^{n} \alpha_j\beta_j h_j\\ &= \sum_{j=1}^{n} (\alpha_j h_j^{\frac{1}{p}})(\beta_j h_j^{\frac{1}{q}})\end{aligned}$$

Hölder's inequality gives

$$\begin{aligned}\mathbb{E}(fg|\mathcal{G}) &\leq [\sum_{j=1}^{n} \alpha_j^p h_j]^{\frac{1}{p}}[\sum_{j=1}^{n} \beta_j^q h_j]^{\frac{1}{q}}\\ &= [\mathbb{E}(f^p|\mathcal{G})]^{\frac{1}{p}}[\mathbb{E}(g^q|\mathcal{G})]^{\frac{1}{q}} \qquad (1)\end{aligned}$$

which completes the proof of the conditional expectation form of Hölder's inequality. To prove the conditional expectation form of the Minkowski inequality we see that

$$\begin{aligned}[\mathbb{E}((f+g)^p)|\mathcal{G})]^{\frac{1}{p}} &= \left[\sum_{j=1}^{n}(\alpha_j+\beta_j)^p h_j\right]^{\frac{1}{p}} \\ &= \left[\sum_{j=1}^{n}\left(\alpha_j h_j^{\frac{1}{p}}+\beta_j h_j^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}\end{aligned}$$

by the classical Minkowski inequality it follows that

$$\begin{aligned}[\mathbb{E}((f+g)^p)|\mathcal{G})]^{\frac{1}{p}} &\leq \left[\sum_{j=1}^{n}\alpha_j^p h_j\right]^{\frac{1}{p}} + \left[\sum_{j=1}^{n}\beta_j^p h_j\right]^{\frac{1}{p}} \\ &= [\mathbb{E}\left(f^p|\mathcal{G}\right)]^{\frac{1}{p}} + [\mathbb{E}\left(g^p|\mathcal{G}\right)]^{\frac{1}{p}} \end{aligned} \tag{2}$$

We have now proved that (i) and (ii) hold if f and g are simple functions.

Assume now that f and g [0,∞]-valued $\mathcal{F}$-measurable functions on Ω. Then there exist non-decreasing sequences $\{f_n\}$ and $\{g_n\}$ of [0,∞)-valued simple measurable functions on Ω such that

$$f(\omega) = \lim_n f_n(\omega) \text{ and } g(\omega) = \lim_n g_n(\omega) \tag{3}$$

The sequences $\{\phi_n\}$ and $\{\psi_n\}$ defined by $\phi_n = \mathbb{E}\left(f_n g_n|\mathcal{G}\right)$ and $\psi_n = \mathbb{E}\left((f_n+g_n)^p\,|\mathcal{G}\right)$ are non-decreasing, so their respective pointwise limits ϕ and ψ exist and are $\mathcal{G}$-measurable. For all $A \in \mathcal{G}$ we have by Lebesgue's Monotone Convergence Theorem

$$\int_A \phi\,\mathrm{d}P = \lim_{n\to\infty}\int_A \phi_n\,\mathrm{d}P = \lim_{n\to\infty}\int_A f_n g_n\,\mathrm{d}P = \int_A fg\,\mathrm{d}P$$

and

$$\int_A \psi\,\mathrm{d}P = \lim_{n\to\infty}\int_A \psi_n\,\mathrm{d}P = \lim_{n\to\infty}(f_n+g_n)^p\,\mathrm{d}P = \int_A (f+g)^p\mathrm{d}P.$$

This proves that $\phi = \mathbb{E}\left(fg|\mathcal{G}\right)$ and $\psi = \mathbb{E}((f+g)^p|\mathcal{G})$. By the preceding paragraph we have

$$\phi_n \leq [\mathbb{E}\left(f_n^p|\mathcal{G}\right)]^{\frac{1}{p}}\,[\mathbb{E}\left(g_n^q|\mathcal{G}\right)]^{\frac{1}{q}} \leq [\mathbb{E}\left(f^p|\mathcal{G}\right)]^{\frac{1}{p}}\,[\mathbb{E}\left(g^q|\mathcal{G}\right)]^{\frac{1}{q}}$$

where the first inequality follows from the deduction of (1) and the second inequality from the fact that $f_n \leq f$ and $g_n \leq g$. We also have

$$\psi_n \leq [\mathbb{E}(f_n^p|\mathcal{G})]^{\frac{1}{p}} + [\mathbb{E}(g_n^q|\mathcal{G})]^{\frac{1}{q}} \leq [\mathbb{E}(f^p|\mathcal{G})]^{\frac{1}{p}} + [\mathbb{E}(g^q|\mathcal{G})]^{\frac{1}{q}}$$

where the first inequality follows from (2) and the second from the fact that $f_n \leq f$ and $g_n \leq g$.

If we now let $n \to \infty$, we obtain (i) and (ii).

□

Theorem 4 (Doob's inequalities). *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_1 \subset \mathcal{F}_2 \cdots \subset \mathcal{F}_n$ be σ-subalgebras of $\mathcal{F}$. Suppose $f_1, f_2, \ldots, f_n \in L^1$ satisfy*

- *f_k is real-valued and $\mathcal{F}_k-$measurable for $k = 1, 2, \ldots, n$ and*
- *$\mathbb{E}(f_{k+1}|\mathcal{F}_k) = 0$ for $k = 1, 2, \ldots, n-1$.*

Write

$$S_0 := 0, \qquad S_k := f_1 + f_2 + \cdots + f_k,\ 0 \leq k \leq n$$
$$M_n := \max_{1\leq k\leq n}\{S_k\}, \qquad M_n^* := \max_{1\leq k\leq n}\{|S_k|\}.$$

Then $1 < p < \infty$, $q = \frac{p}{p-1}$ implies

(i). $\|M_n^+\|_p \leq q\|S_n^+\|_p$ and

(ii). $\|M_n^*\|_p \leq q\|S_n\|_p$

Proof. Note that we use the notation $f^+ = \sup\{f, 0\}$.

We now have

$$\mathbb{E}(S_n|\mathcal{F}_k) = \mathbb{E}(f_1 + f_2 \cdots + f_k + f_{k+1} + \cdots + f_n|\mathcal{F}_k)$$

By linearity of conditional expectation we get

$$\mathbb{E}(S_n|\mathcal{F}_k) = \mathbb{E}(f_1 + f_2 \cdots + f_k|\mathcal{F}_k) + 0$$

Since f_k is $\mathcal{F}_k$-measurable we have

$$\begin{aligned}\mathbb{E}\left(S_n|\mathcal{F}_k\right) &= f_1 + f_2 + \cdots + f_k \\ &= S_k. \end{aligned} \tag{4}$$

Fix $t > 0$ and set $A_1 = \{\omega \in \Omega : S_1(\omega) \geq t\}$ and

$$\begin{aligned}A_k = \{\omega \in \Omega : S_1(\omega) < t\} \cap \{\omega \in \Omega : S_2(\omega) < t\} \cap \cdots \cap \\ \{\omega \in \Omega : S_{k-1}(\omega) < t\} \cap \{\omega \in \Omega : S_k(\omega) \geq t\},\ 2 \leq k \leq n.\end{aligned}$$

It is clear that $A_k \in \mathcal{F}_k$. This follows from the measurability of S_k. The sets A_k are also disjoint and $\{\omega \in \Omega : M_n(\omega) \geq t\} = \bigcup_{k=1}^n A_k$. Therefore

$$\begin{aligned}\int_{\{\omega\in\Omega:M_n(\omega)\geq t\}} S_n^+(\omega)P(\mathrm{d}\omega) &\geq \int_{\{\omega\in\Omega:M_n(\omega)\geq t\}} S_n(\omega)P(\mathrm{d}\omega) \\ &= \int_{\bigcup_{k=1}^n A_k} \mathbb{E}\left(S_n|\mathcal{F}_k\right) P(\mathrm{d}\omega)\end{aligned}$$

Since the A_k's are disjoint we can use summation

$$\begin{aligned} &= \sum_{k=1}^n \int_{A_k} S_k(\omega)P(\mathrm{d}\omega) \\ &\geq \sum_{k=1}^n tP(A_k) \end{aligned} \tag{5}$$

To explain (5) we see that the relation $0 \leq t\chi_{A_k} \leq S_k(\omega)\chi_{A_k} \leq S_k(\omega)$ implies that

$$\int t\chi_{A_k}\, P(\mathrm{d}\omega) \leq \int_{A_k} S_k(\omega)\, P(\mathrm{d}\omega) \leq \int S_k(\omega)\, P(\mathrm{d}\omega).$$

Since $\int t\chi_{A_k}\, P(\mathrm{d}\omega) = tP(A_k)$, we conclude therefore that

$$\int_{\{\omega\in\Omega:M_n(\omega)\geq t\}} S_n^+(\omega)P(\mathrm{d}\omega) \geq tP\left(\{\omega \in \Omega : M_n(\omega) \geq t\}\right)$$

From the above, the following inequality is obvious

$$P\left(\{\omega \in \Omega : M_n(\omega) \geq t\}\right) \leq \frac{1}{t}\int_{\{\omega\in\Omega:M_n(\omega)\geq t\}} S_n^+(\omega)P(\mathrm{d}\omega). \tag{6}$$

Note that

$$\boxed{\mathbb{E}((M_n^+)^p) = \int_0^\infty pt^{p-1}\mathbb{P}(M_n^+ \geq t)\mathrm{d}t.}$$

To show this, wee see that

$$\int_0^\infty pt^{p-1}P(M_n^+ \geq t)\mathrm{d}t = \int_0^\infty pt^{p-1}\int_\Omega \chi_{\{\omega\in\Omega:M^+(\omega)\geq t\}}P(\mathrm{d}\omega)\mathrm{d}t$$

By Fubini's theorem we get

$$\begin{aligned}
\int_0^\infty pt^{p-1}P(M^+ \geq t)\mathrm{d}t &= \int_\Omega\int_0^\infty pt^{p-1}\chi_{\{\omega\in\Omega:M^+(\omega)\geq t\}}\mathrm{d}tP(\mathrm{d}\omega) \\
&= \int_\Omega\int_0^{M_n^+} pt^{p-1}\mathrm{d}tP(\mathrm{d}\omega) \\
&= \int_\Omega (M_n^+)^pP(\mathrm{d}\omega) \\
&= \mathbb{E}((M_n^+)^p)
\end{aligned}$$

From the above it follows that

$$\|M_n^+\|_p^p = \int_0^\infty pt^{p-1}P(M_n^+ \geq t)\mathrm{d}t.$$

It is clear from (6) that $P\left(\{\omega\in\Omega : M_n^+(\omega)\geq t\}\right) \leq P\left(\{\omega\in\Omega : M_n(\omega)\geq t\}\right)$. This allows us to write

$$\begin{aligned}
\|M_n^+\|_p^p &\leq \int_0^\infty pt^{p-2}\int_{\{\omega\in\Omega:M_n^+(\omega)\geq t\}} S_n^+P(\mathrm{d}\omega)\mathrm{d}t \\
&= p\int_\Omega S_n^+\int_0^{M_n^+} t^{p-2}\mathrm{d}tP(\mathrm{d}\omega)
\end{aligned}$$

(by Fubini's theorem)

$$= p\int_\Omega S_n^+\cdot\frac{(M_n^+)^{p-1}}{p-1}P(\mathrm{d}\omega)$$

By Hölder's inequality we get

$$\begin{aligned}
\|M_n^+\|_p^p &\leq \frac{p}{p-1}\|S_n^+\|_p\cdot\left(\int_\Omega |M_n^+|^{(p-1)q}P(\mathrm{d}\omega)\right)^{\frac{1}{q}} \\
&= q\|S_n^+\|_p\cdot\|M_n^+\|_p^{p-1}.
\end{aligned}$$

This yields (i) as desired.

We will now prove (ii). Write $E_1 = \{\omega \in \Omega : S_1(\omega) \geq t\}$ and $F_1 = \{\omega \in \Omega : S_1(\omega) \leq -t\}$. These sets are obviously measurable with respect to $\mathcal{F}_1$. Further to this, we will write

$$\begin{aligned} E_k = \{\omega \in \Omega : |S_1(\omega)| \leq t\} \cap \{\omega \in \Omega : |S_2(\omega)| \leq t\} \cap \cdots \cap \\ \{\omega \in \Omega : |S_{k-1}(\omega)| \leq t\} \cap \{\omega \in \Omega : S_k(\omega) > t\},\ 2 \leq k \leq n \end{aligned}$$

and

$$\begin{aligned} F_k = \{\omega \in \Omega : |S_1(\omega)| \leq t\} \cap \{\omega \in \Omega : |S_2(\omega)| \leq t\} \cap \cdots \cap \\ \{\omega \in \Omega : |S_{k-1}(\omega)| \leq t\} \cap \{\omega \in \Omega : S_k(\omega) < -t\},\ 2 \leq k \leq n. \end{aligned}$$

Since the sum of random variables is a random variable, it follows that S_i, $i = 1, 2, \ldots, n$ is measurable and it follows that E_k and F_k as defined above are measurable with respect to $\mathcal{F}_k$. Also, defining E_k and F_k as we did above, it is obvious that these $2n$ events are also pairwise disjoint, and we can write

$$\{\omega \in \Omega : M_n^*(\omega) \geq t\} = \left(\bigcup_{k=1}^{n} E_k\right) \cup \left(\bigcup_{k=1}^{n} F_k\right).$$

$$\begin{aligned} \int_{\{\omega\in\Omega : M_n^*(\omega)\geq t\}} |S_n| P(\mathrm{d}\omega) &= \int_{\{(\bigcup_{k=1}^{n} E_k)\cup(\bigcup_{k=1}^{n} F_k)\}} |S_n|\, P(\mathrm{d}\omega) \\ &\geq \int_{\{(\bigcup_{k=1}^{n} E_k)\cup(\bigcup_{k=1}^{n} F_k)\}} S_n\, P(\mathrm{d}\omega) \end{aligned}$$

As in the proof of (i),

$$\int_{\{(\bigcup_{k=1}^{n} E_k)\cup(\bigcup_{k=1}^{n} F_k)\}} S_n\, P(\mathrm{d}\omega) = \sum_{k=1}^{n} \left(\int_{E_k} S_k\, P(\mathrm{d}\omega) + \int_{F_k} S_k\, P(\mathrm{d}\omega) \right)$$

As we are only integrating negative S_k as specfied in the definition of F_k we can write

$$\begin{aligned} \int_{\{(\bigcup_{k=1}^{n} E_k)\cup(\bigcup_{k=1}^{n} F_k)\}} S_n\, P(\mathrm{d}\omega) &= \sum_{k=1}^{n} \left(\int_{E_k} S_k\, P(\mathrm{d}\omega) + \int_{F_k} -S_k\, P(\mathrm{d}\omega) \right) \\ &\geq t\mathbb{P}(M_n^* \geq t). \end{aligned}$$

Then we can write

$$\|M_n^*\|_p^p \leq \int_0^\infty pt^{p-2} \int_{\{\omega\in\Omega: M_n^*(\omega)\geq t\}} S_n P(\mathrm{d}\omega)\mathrm{d}t$$

By Fubini's theorem we get

$$\begin{aligned}\|M_n^*\|_p^p &\leq p\int_\Omega S_n \int_0^{M_n^*} t^{p-2}\mathrm{d}tP(\mathrm{d}\omega)\\ &= p\int_\Omega S_n \cdot \frac{(M_n^*)^{p-1}}{p-1}\mathbb{P}(\mathrm{d}\omega)\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}\|M_n^*\|_p^p &\leq \frac{p}{p-1}\|S_n\|_p \cdot \left(\int_\Omega |M_n^*|^{(p-1)q}\mathbb{P}(\mathrm{d}\omega)\right)^{\frac{1}{q}}\\ &= q\|S_n\|_p \cdot \|M_n^*\|_p^{p-1}.\end{aligned}$$

This yields (ii) as desired.

□

Remark 5. Please note that Theorem 4 holds for any σ-finite positive measure.

Theorem 6. *Let $(\Omega, \mathcal{F}, P)$ be a measure space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_n$ be σ-subalgebras of $\mathcal{F}$. Suppose $f_0, f_1, f_2, \ldots, f_n \in L^1$ satisfy*

- *f_k is real-valued and $\mathcal{F}_k$–measurable for $k = 0, 1, \ldots, n$ and*
- *$\mathbb{E}\left(f_{k+1}|\mathcal{F}_k\right) = 0$ for $k = 0, 1, \ldots, n-1$.*

Write

$$\begin{aligned}S_0 &:= 0, & S_k &:= f_0 + f_1 + \cdots + f_k,\ 0 \leq k \leq n\\ M_n &:= \max_{0\leq k\leq n}\{S_k\}, & M_n^* &:= \max_{0\leq k\leq n}\{|S_k|\}.\end{aligned}$$

Then $1 < p < \infty$, $q = \frac{p}{p-1}$ implies

(i). $[\mathbb{E}((M_n^+)^p|\mathcal{F}_0)]^{\frac{1}{p}} \leq q\,[\mathbb{E}((S_n^+)^p|\mathcal{F}_0)]^{\frac{1}{p}}$ *a.s. and*

(ii). $[\mathbb{E}((M_n^*)^p|\mathcal{F}_0)]^{\frac{1}{p}} \leq q\,[\mathbb{E}(|S_n|^p|\mathcal{F}_0)]^{\frac{1}{p}}$ *a.s.*

Proof. Let $A \in \mathcal{F}_0$ and define P_A on $\mathcal{F}$ by $P_A(E) := P(E \cap A)$. Then $P_A(\emptyset) = 0$. Furthermore, if $\{E_i\}$ is a sequence of disjoint sets in Ω that belongs to $\mathcal{F}$ with $\bigcup_{i=1}^{\infty} E_i = E$ then

$$\begin{aligned} P_A(E) &= P(A \cap E) \\ &= P(A \cap (E_1 \cup E_2 \cup \cdots)) \\ &= P((A \cap E_1) \cup (A \cap E_2) \cup \cdots) \end{aligned}$$

Then by the additivity of P we get

$$\begin{aligned} P_A(E) &= P(A \cap E_1) + P(A \cap E_2) + \cdots \\ &= \sum_{i=1}^{\infty} P(A \cap E_i) \\ &= \sum_{i=1}^{\infty} P_A(E_i) \end{aligned}$$

So, P_A is a measure on $(\Omega, \mathcal{F})$. Note that we are not proving, at this stage, that P_A is a probability measure. Also, $\int_A h\mathrm{d}P(\mathrm{d}\omega) = \int_\Omega h\mathrm{d}P_A(\mathrm{d}\omega)$ for any nonnegative $\mathcal{F}_0-$measurable h on Ω. We can see this by assuming first that h is a simple function,

$$h = \sum_{i=1}^{n} h_i \chi_{E_i}$$

where $h_1, \ldots, h_n$ are real numbers and $E_1, \ldots, E_n$ are as defined above, then

$$\begin{aligned} \int_A h\mathrm{d}P(d\omega) &= \sum_{i=1}^{n} \int_A h_i \chi_{E_i} P(\mathrm{d}\omega) \\ &= \sum_{i=1}^{n} \int_\Omega h_i \chi_{E_i \cap A} P(\mathrm{d}\omega) \\ &= \sum_{i=1}^{n} h_i P(E_i \cap A) \end{aligned}$$

and

$$\begin{aligned}\int_\Omega hP_A(\mathrm{d}\omega) &= \sum_{i=1}^n \int_\Omega h_i \chi_{E_i} P_A(\mathrm{d}\omega)\\ &= \sum_{i=1}^n h_i P_A(E_i)\\ &= \sum_{i=1}^n h_i P(E_i \cap A).\end{aligned}$$

We conclude therefore that for simple functions that the relation below holds:

$$\int_A h\, P(\mathrm{d}\omega) = \int_\Omega h\, P_A(\mathrm{d}\omega). \tag{7}$$

Next, any nonnegative Borel function h can be approximated by a non-decreasing sequence of simple functions. For such an h the result follows by Lebesgue's Monotone Convergence Theorem of integrals. Finally, this implies the desired equality for all Borel functions h, since each can be split into its positive and negative parts, $h = h^+ - h^-$, where $h^+, h^- \geq 0$.
We have from Theorem 4 that

$$\left(\overbrace{\int_\Omega (M_n^+)^p\, P(\mathrm{d}\omega)}^{C}\right)^{\frac{1}{p}} \leq q \left(\overbrace{\int_\Omega (S_n^+)^p\, P(\mathrm{d}\omega)}^{D}\right)^{\frac{1}{p}}.$$

If $A \in \mathcal{F}_0$ and using (7) we can write

$$C = \int_\Omega (M_n^+)^p\, P_A(\mathrm{d}\omega) = \int_A (M_n^+)^p\, P(\mathrm{d}\omega)$$

and

$$D = \int_\Omega (S_n^+)^p\, P_A(\mathrm{d}\omega) = \int_A (S_n^+)^p\, P(\mathrm{d}\omega).$$

It follows that

$$\left(\int_A (M_n^+)^p\, P(\mathrm{d}\omega)\right)^{\frac{1}{p}} \leq q \left(\int_A (S_n^+)^p\, P(\mathrm{d}\omega)\right)^{\frac{1}{p}}.$$

That is,

$$\begin{aligned}\int_A \mathbb{E}((M_n^+)|\mathcal{F}_0)\, P(\mathrm{d}\omega) &= \int_A (M_n^+)^p\, P(\mathrm{d}\omega)\\ &\leq q^p \int_A (S_n^+)^p\, P(\mathrm{d}\omega)\\ &= q^p \int_A \mathbb{E}((S_n^+)|\mathcal{F}_0)\, P(\mathrm{d}\omega)\ \ \forall A \in \mathcal{F}_0.\end{aligned}$$

If we apply the above equation to the $\mathcal{F}_0-$sets

$$A_k := \{k \geq \mathbb{E}((M_n^+)|\mathcal{F}_0) > q^p\mathbb{E}((S_n^+)|\mathcal{F}_0)\} \tag{8}$$

and get $P(A_k) = 0$, valid for every $k \in \mathbb{N}$. Thus $P\left(\bigcup_{k\in\mathbb{N}} A_k\right) = 0$ and that is what is said by (i). To prove (ii), we see that

$$\left[\int_A (M_n^*)^p \mathrm{d}P\right]^{\frac{1}{p}} \leq \left[\int_A (S_n)^p \mathrm{d}P\right]^{\frac{1}{p}}$$

that is

$$\begin{aligned}\int_A \mathbb{E}((M_n^*)^p|\mathcal{F}_0)\mathrm{d}P &= \int_A (M_n^*)^p \mathrm{d}P\\ &\leq q^p \int_A (S_n)^p \mathrm{d}P\\ &= q^p \int_A \mathbb{E}((S_n)^p|\mathcal{F}_0)\mathrm{d}P\ \forall\, A \in \mathcal{F}_0.\end{aligned}$$

Apply this to the $\mathcal{F}_0-$sets

$$A_k := \{k \geq \mathbb{E}((M_n^*)|\mathcal{F}_0) > q^p\mathbb{E}((S_n)|\mathcal{F}_0)\} \tag{9}$$

and get $P(A_k) = 0$, valid for every $k \in \mathbb{N}$. Thus $P\left(\bigcup_{k\in\mathbb{N}} A_k\right) = 0$.

□

Theorem 7. *Let* $m \in \mathbb{N}$, *and* $(\mathcal{F})_{k=0}^m$ *be a filtration of* $(\Omega, \mathcal{F}, \mathbb{P})$, *and let* $(M_k)_{k=0}^m \subset L^\infty$ *be a real-valued martingale adapted to this filtration with* $M_0 = 0$. *For* $i \in \mathbb{N}$, $1 \leq k \leq m$ *define*

$$f_k := M_k - M_{k-1}, \quad M_k^* := \max_{1\leq j\leq k} |M_j|, \quad \textit{and } \mathbb{E}_{i,m} := \sum_{k=0}^{m-1} \mathbb{E}(|f_{k+1}|^i|\mathcal{F}_k).$$

Then there exists a finite constant K independent of m and $p \in \mathbb{N}$ such that for every even $p \in \mathbb{N}$ we have

$$[\mathbb{E}((M_m^*)^p|\mathcal{F}_0)]^{\frac{1}{p}} \leq Kp \max_{2\leq i\leq p} \left[\mathbb{E}(\mathbb{E}_{i,m}^{\frac{p}{i}}|\mathcal{F}_0)\right]^{\frac{1}{p}} \; a.s. \tag{10}$$

Proof. Fix $p \in \mathbb{N}$, p even. Let m be given, $1 \leq k \leq m$. Then

$$\mathbb{E}(M_k^p|\mathcal{F}_0) = \mathbb{E}((M_{k-1} + f_k)^p|\mathcal{F}_0), \quad \text{from the definition of } f_k$$

By linearity and the fact that $(a+b)^n = a^n + \sum_{i=1}^n \binom{n}{i} a^{n-i} b^i$ we get

$$\begin{aligned}
\mathbb{E}(M_k^p|\mathcal{F}_0) &= \mathbb{E}(M_{k-1}^p|\mathcal{F}_0) + \sum_{i=1}^{p} \binom{p}{i} \mathbb{E}(M_{k-1}^{p-i} f_k^i|\mathcal{F}_0), \\
&= \mathbb{E}(M_{k-1}^p|\mathcal{F}_0) + \sum_{i=1}^{p} \binom{p}{i} \mathbb{E}(\mathbb{E}(M_{k-1}^{p-i} f_k^i|\mathcal{F}_{k-1})|\mathcal{F}_0)
\end{aligned}$$

(by the Tower Property)

$$= \mathbb{E}(M_{k-1}^p|\mathcal{F}_0) + \sum_{i=1}^{p} \binom{p}{i} \mathbb{E}(M_{k-1}^{p-i} \mathbb{E}(f_k^i|\mathcal{F}_{k-1})|\mathcal{F}_0)$$

(since M_{k-1} is $\mathcal{F}_{k-1}-$measurable)

$$= \mathbb{E}(M_{k-1}^p|\mathcal{F}_0) + \sum_{i=2}^{p}, \binom{p}{i} \mathbb{E}(M_{k-1}^{p-i} \mathbb{E}(f_k^i|\mathcal{F}_{k-1})|\mathcal{F}_0)$$

(since $\mathbb{E}(f_k^1|\mathcal{F}_{k-1}) = 0$)

$$\begin{aligned}
&\leq \mathbb{E}(M_{k-1}^p|\mathcal{F}_0) \\
&\quad + \sum_{i=2}^{p} \binom{p}{i} \mathbb{E}((M_m^*)^{p-i} \mathbb{E}(|f_k|^i|\mathcal{F}_{k-1})|\mathcal{F}_0), \; 1 \leq k \leq m.
\end{aligned} \tag{11}$$

If we assume that $k = m$ in (11) we get:

$$\mathbb{E}(M_m^p|\mathcal{F}_0) \leq \mathbb{E}(M_{m-1}^p|\mathcal{F}_0) + \sum_{i=2}^{p} \binom{p}{i} \mathbb{E}((M_m^*)^{p-i} \mathbb{E}(|f_m|^i|\mathcal{F}_{m-1})|\mathcal{F}_0) \tag{12}$$

And if $k = m-1$ in (11) we have

$$\mathbb{E}(M^p_{m-1}|\mathfrak{F}_0) \leq \mathbb{E}(M^p_{m-2}|\mathfrak{F}_0) + \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}((M^*_m)^{p-i}\mathbb{E}(|f_{m-1}|^i|\mathfrak{F}_{m-2})|\mathfrak{F}_0)$$

Substituting $\mathbb{E}(M^p_{m-1}|\mathfrak{F}_0)$ into (12):

$$\mathbb{E}(M^p_m|\mathfrak{F}_0) \leq \mathbb{E}(M^p_{m-2}|\mathfrak{F}_0) + \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}((M^*_m)^{p-i}\mathbb{E}(|f_m|^i|\mathfrak{F}_{m-1})|\mathfrak{F}_0)$$
$$+ \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}((M^*_m)^{p-i}\mathbb{E}(|f_{m-1}|^i|\mathfrak{F}_{m-2})|\mathfrak{F}_0)$$

After simplifying we get

$$\mathbb{E}(M^p_m|\mathfrak{F}_0) \leq \mathbb{E}(M^p_{m-2}|\mathfrak{F}_0) + \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}\left((M^*_m)^{p-i}\sum_{k=m-1}^{m}\mathbb{E}(|f_k|^i|\mathfrak{F}_{k-1})|\mathfrak{F}_0\right)$$

Continuing in this way we are able to deduce the following

$$\mathbb{E}(M^p_m|\mathfrak{F}_0) \leq \mathbb{E}(M^p_{m-1}|\mathfrak{F}_0) + \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}((M^*_m)^{p-i}\mathbb{E}(|f_m|^i|\mathfrak{F}_{m-1})|\mathfrak{F}_0)$$

(The equation above is the same as (12))

$$\leq \cdots \leq \mathbb{E}(M^p_0|\mathfrak{F}_0) + \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}\left(\left((M^*_m)^{p-i}\sum_{k=1}^{m}\mathbb{E}(|f_k|^i|\mathfrak{F}_{k-1})\right)|\mathfrak{F}_0\right)$$
$$= \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}\left(\left((M^*_m)^{p-i}\sum_{k=0}^{m-1}\mathbb{E}(|f_{k+1}|^i|\mathfrak{F}_{k-1})\right)|\mathfrak{F}_0\right), \text{ since } M_0 = 0$$

Writing $\mathbb{E}_{i,m}$ for $\sum_{k=0}^{m-1}\mathbb{E}(|f_{k+1}|^i\,|\mathfrak{F}_{k-1})$ in this last expression is:

$$\mathbb{E}(M^p_m|\mathfrak{F}_0) \leq \sum_{i=2}^{p}\binom{p}{i}\mathbb{E}((M^*_m)^{p-i}\mathbb{E}_{i,m}|\mathfrak{F}_0).$$

Using the conditional expectation form of Hölder's inequality as deduced in Theorem 3 we get

$$\mathbb{E}(M^p_m|\mathfrak{F}_0) \leq \sum_{i=2}^{p}\binom{p}{i}\left(\mathbb{E}((M^*_m)^p|\mathfrak{F}_0)\right)^{\frac{p-i}{p}}\left(\mathbb{E}(\mathbb{E}^{\frac{p}{i}}_{i,m}|\mathfrak{F}_0\right)^{\frac{1}{p}}. \tag{13}$$

Note: If we set $p' = \frac{p}{p-i}$ and $q' = \frac{p}{i}$ in (13) then $\frac{1}{p'} + \frac{1}{q'} = 1$.

Define the following variables

$$x_p := (\mathbb{E}((M_m^*)^p|\mathcal{F}_0))^{\frac{1}{p}}, \qquad y_p := \max_{2\le i\le p}\left(\mathbb{E}(\mathbb{E}_{i,m}^{\frac{p}{i}}|\mathcal{F}_0)\right)^{\frac{1}{p}}.$$

Then (13) can also be written as

$$\begin{aligned}\mathbb{E}(M_m^p) &\le \sum_{i=2}^{p}\binom{p}{i}x_p^{p-i}y_p^i \\ &\le (x_p+y_p)^p - x_p^p \end{aligned} \tag{14}$$

The last inequality stems from the fact that i starts counting from 2 upwards in the first inequality. There is therefore $p-1$ terms in the first inequality and p terms in (14) .

Next, if we set $f_k = M_k - M_{k-1}$ we see that:

$$\begin{aligned}\mathbb{E}(f_{k+1}|\mathcal{F}_k) &= \mathbb{E}(M_{k+1}-M_k|\mathcal{F}_k) \\ &= \mathbb{E}(M_{k+1}|\mathcal{F}_k) - \mathbb{E}(M_k|\mathcal{F}_k) \\ &= M_k - M_k \\ &= 0 \end{aligned} \tag{15}$$

and

$$0 < \mathbb{E}(|f_k|) = \mathbb{E}(|M_k - M_{k-1}|) < \infty. \tag{16}$$

The last equation follows because $L^\infty \subseteq L^1$, and L^1 is a linear space.

Thus, $f_k = M_k - M_{k-1}$ satisfies the hypotheses of Theorem 6 on page 8 as shown by (15) and (16) together with the fact that f_k is $\mathcal{F}_k$-measurable. By using the notation of Theorem 6 we get the following:

$$\begin{aligned}S_m &= f_1 + f_2 + \cdots + f_m \\ &= (M_1 - M_0) + (M_2 - M_1) + \cdots + (M_m - M_{m-1}) \\ &= M_m - M_0 \\ &= M_m, \text{ since } M_0 = 0\end{aligned}$$

By (ii) of Theorem 6 we have the following inequality:

$$(\mathbb{E}((M_m^*)^p)|\mathcal{F}_0)^{\frac{1}{p}} \leq \frac{p}{p-1}(\mathbb{E}(|M_m|^p)|\mathcal{F}_0)^{\frac{1}{p}}$$

Write $d_p := \frac{p^p}{(p-1)^p}$ and $c_p := (1+d_p)^{\frac{1}{p}} - d_p^{\frac{1}{p}}$. Then the inequality above helps us to deduce the following:

$$\begin{aligned} x_p^p &\leq d_p \mathbb{E}(|M_m|^p|\mathcal{F}_0) \\ &= d_p \mathbb{E}(M_m^p|\mathcal{F}_0), \text{ since } p \text{ is even} \\ &\leq d_p[(x_p + y_p)^p - x_p^p], \text{ by (14)} \end{aligned}$$

This gives

$$x_p \leq \frac{d_p^{\frac{1}{p}}}{c_p} \cdot y_p \tag{17}$$

One sees that $\lim_{n\to\infty} d_n = e$ and $n[a^{\frac{1}{n}} - 1] \downarrow \ln a$ whenever $a > 1$, so

$$\lim_{n\to\infty} \frac{d_n^{\frac{1}{n}}}{c_p} = \frac{1}{\ln(1+e^{-1})}.$$

Therefore, $K = \sup_{p\in 2\mathbb{N}} \frac{d_p^{\frac{1}{p}}}{pc_p}$ is finite.

□

Theorem 8. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and let $(X_t)_{t\geq 0}$ be a real-valued martingale adapted to it. Suppose that all sample paths are continuous, $\mathbb{E}(|X_t|^2) < \infty \, \forall \, t \geq 0$, and*

$$E((X_t - X_s)^2|\mathcal{F}_s) = t - s \text{ a.s. } \forall \, 0 \leq s < t. \tag{18}$$

Then for each $p \in \mathbb{N}$, there exists a finite constant K_p such that

$$[\mathbb{E}\left(|X_t - X_s|^p \, |\mathcal{F}_s\right)]^{\frac{1}{p}} \leq K_p(t-s)^{\frac{1}{2}} \text{ a.s. } \forall \, 0 \leq s < t. \tag{19}$$

Note that under the hypothesis that $(X_t)_{t\geq 0}$ is a martingale, (18) *is equivalent to*

$$\{X_t^2 - t\}_{t\geq 0} \text{ is a martingale.} \tag{20}$$

Proof. We use mathematical induction to prove the theorem. It is given that $\mathbb{E}((X_t - X_s)^2|\mathcal{F}_s) = t - s$. If we let $f = |X_t - X_s|$ and $g = 1$ in (i) of

Theorem 3 (the conditional expectation form of Hölder's inequality) then we see that for $p = 1$

$$\begin{aligned}
\mathbb{E}(|X_t - X_s| \cdot 1|\mathfrak{F}_s) &\leq \left[\mathbb{E}(|X_t - X_s|^2|\mathfrak{F}_s)\right]^{\frac{1}{2}} \left[\mathbb{E}(1^2|\mathfrak{F}_s)\right]^{\frac{1}{2}} \\
&= \left[\mathbb{E}(|X_t - X_s|^2|\mathfrak{F}_s)\right]^{\frac{1}{2}} \\
&= 1 \times (t-s)^{\frac{1}{2}} \text{ by (18),}
\end{aligned}$$

and therefore may assume that $K_1 = 1$ in (19).
Let $p = 2$. Then

$$\mathbb{E}(|X_t - X_s|^2|\mathfrak{F}_s)^{\frac{1}{2}} = (t-s)^{\frac{1}{2}} \text{ by (18)}$$

Thus,

$$\left[\mathbb{E}(|X_t - X_s|^2|\mathfrak{F}_s)\right]^{\frac{1}{2}} = 1 \times (t-s)^{\frac{1}{2}}$$

and we can therefore assume $K_1 = K_2 = 1$ in (19).

Continuing in this way, assume therefore that for some even $p \geq 4$ that constants $K_1, K_2, \ldots, K_{p-2}$ have been found that satisfy

$$\left(\mathbb{E}(|X_t - X_s|^i|\mathfrak{F}_s)\right)^{\frac{1}{i}} \leq K_i(t-s)^{\frac{1}{2}} \; a.s. \forall\, 0 \leq s \leq t,\; 1 \leq i \leq p-2 \tag{21}$$

We will now attempt to find K_{p-1} and K_p. So, fix $0 \leq s < t$ and write $\alpha = (t-s)^{\frac{1}{2}}$. For $m \in \mathbb{N}$ define

$$A_m^{\mathbb{R}} := \{\omega \in \Omega : |X_u(\omega) - X_v(\omega)| \leq \alpha \,\forall u, v \in [s,t] \text{ with } |u - v| \leq \frac{\alpha^2}{m}\}$$

and

$$A_m^{\mathbb{Q}} := \{\omega \in \Omega : |X_u(\omega) - X_v(\omega)| \leq \alpha \,\forall u, v \in [s,t] \cap \mathbb{Q} \text{ with } |u - v| \leq \frac{\alpha^2}{m}\}$$

We first want to show that if $\omega \in A_m^{\mathbb{Q}}$ then $\omega \in A_m^{\mathbb{R}}$. Therefore, let $\omega \in A_m^{\mathbb{Q}}$ and $[u, v] \in [s, t]$ such that $|u - v| < \frac{\alpha^2}{m}$. Then there exist $u_n, v_n \in \mathbb{Q}$ such that $|u_n - v_n| < \frac{\alpha^2}{m}$. This implies that

$$|X_{u_n}(\omega) - X_{v_n}(\omega)| \leq \alpha, \text{ since } \omega \in A_m^{\mathbb{Q}}.$$

Also, there exist sequences $\{u_n\}$ and $\{v_n\}$ in $\mathbb{Q}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. This follows from the denseness of $\mathbb{Q}$ in $\mathbb{R}$. This, together with the fact that every sample path is continuous (this is given) allows us to write

$$\lim_{n\to\infty} |X_{u_n}(\omega) - X_{v_n}(\omega)| = |X_u(\omega) - X_v(\omega)| \leq \alpha.$$

We conclude therefore that whenever $\omega \in A_m^{\mathbb{Q}}$ then $\omega \in A_m^{\mathbb{R}}$. It is also clear that $A_m^{\mathbb{Q}} \in \mathfrak{F}_t$.

Next we want to show that $\bigcup_{m=1}^{\infty} A_m^{\mathbb{Q}} = \Omega$. To do this, we let $\omega \in \Omega$. Since $u \mapsto X_u(\omega)$ is continuous; on a closed interval it is also uniformly continuous, it follows that there exists for each $\epsilon > 0$ a $\delta > 0$ such that

$$|X_u(\omega) - X_v(\omega)| < \epsilon \; \forall \; u, v \text{ such that } |u - v| < \delta.$$

For $\epsilon = \alpha$, choose m such that $\frac{\alpha^2}{m} < \delta$. Hence, $\omega \in A_m^{\mathbb{Q}}$ and it follows that $\bigcup_{m=1}^{\infty} A_m^{\mathbb{Q}} = \Omega$. It is therefore enought to find $K_{p-1}, K_p < \infty$ independent of m such that for each $m \in \mathbb{N}$

$$[\mathbb{E}(|X_t - X_s|^i \chi_{A_m^{\mathbb{Q}}} |\mathfrak{F}_s)]^{\frac{1}{i}} \leq K_i \cdot \alpha \;\; \text{for } i = p-1 \text{ and } i = p. \tag{22}$$

We note in (22) that $|X_t - X_s|^i \chi_{A_1^{\mathbb{Q}}} \leq |X_t - X_s|^i \chi_{A_2^{\mathbb{Q}}} \leq \cdots$. By using the Monotone Convergence Theorem for conditional expectations (22) reduces to (21). Now, fix $m \in \mathbb{N}$ and define

$$t_k := s + \frac{k(t-s)}{m}, \; 0 \leq k \leq m \tag{23}$$
$$\Delta_k := X_{t_{k+1}} - X_{t_k} \tag{24}$$
$$g_k := \Delta_k \chi_{|\Delta_k| \leq \alpha} \tag{25}$$
$$G_k := g_k - \mathbb{E}(g_k | \mathfrak{F}_{t_k}) \tag{26}$$

Also see that $\mathfrak{F}_s = \mathfrak{F}_{t_0} \subset \mathfrak{F}_{t_1} \subset \cdots \subset \mathfrak{F}_{t_{m-1}} \subset \mathfrak{F}_{t_m} = \mathfrak{F}_t$. When $\omega \in A_m^{\mathbb{Q}}$, then $|\Delta_k(\omega)| \leq \alpha$ for $0 \leq k \leq m-1$, because $t_{k+1} - t_k = \frac{t-s}{m} = \frac{\alpha^2}{m}$. So, on $A_m^{\mathbb{Q}}$ we have the following

$$X_t(\omega) - X_s(\omega) = \sum_{k=0}^{m-1} \Delta_k(\omega) = \sum_{k=0}^{m-1} \Delta_k \chi_{\{\omega \in \Omega : |\Delta_k(\omega)| \leq \alpha\}} = \sum_{k=0}^{m-1} g_k. \tag{27}$$

So,

$$\begin{aligned}
\left[\mathbb{E}(|X_t - X_s|^p \chi_{A_m^{\mathbb{Q}}} |\mathcal{F}_s)\right]^{\frac{1}{p}} &= \left[\mathbb{E}\left(\left|\sum_{k=0}^{m-1} g_k\right|^p \chi_{A_m^{\mathbb{Q}}} |\mathcal{F}_s\right)\right]^{\frac{1}{p}}, \text{ follows from (27)} \\
&\leq \left[\mathbb{E}\left(\left|\sum_{k=0}^{m-1} g_k\right|^p |\mathcal{F}_s\right)\right]^{\frac{1}{p}} \\
&= \left[\mathbb{E}\left(\left|\sum_{k=0}^{m-1}(G_k + \mathbb{E}(g_k|\mathcal{F}_{t_k}))\right|^p |\mathcal{F}_s\right)\right]^{\frac{1}{p}}, \text{ from (25)} \\
&\leq \left[\mathbb{E}\left(\left(\left|\sum_{k=0}^{m-1} G_k\right| + \left|\sum_{k=0}^{m-1} \mathbb{E}(g_k|\mathcal{F}_{t_k})\right|\right)^p \middle| \mathcal{F}_s\right)\right]^{\frac{1}{p}}
\end{aligned}$$

With the help of the conditional expectation form of Minkowski's inequality, i.e. (ii) of Theorem 3, we get

$$\begin{aligned}
\left[\mathbb{E}(|X_t - X_s|^p \chi_{A_m^{\mathbb{Q}}} |\mathcal{F}_s)\right]^{\frac{1}{p}} \leq & \left[\mathbb{E}\left(\left(\left|\sum_{k=0}^{m-1} G_k\right|\right)^p \middle| \mathcal{F}_s\right)\right]^{\frac{1}{p}} \\
& + \left[\mathbb{E}\left(\left(\left|\sum_{k=0}^{m-1} \mathbb{E}(g_k|\mathcal{F}_{t_k})\right|\right)^p \middle| \mathcal{F}_s\right)\right]^{\frac{1}{p}}
\end{aligned} \tag{28}$$

Next, by martingale properties we know that

$$\begin{aligned}
\mathbb{E}(\Delta_k|\mathcal{F}_{t_k}) &= \mathbb{E}(X_{t_{k+1}} - X_{t_k}|\mathcal{F}_{t_k}) \\
&= X_{t_k} - X_{t_k} \\
&= 0.
\end{aligned}$$

Thus, it follows that

$$|\mathbb{E}(\Delta_k \chi_{\{\omega\in\Omega:|\Delta_k(\omega)|\leq\alpha\}}|\mathcal{F}_{t_k})| = |\mathbb{E}(\omega \in \Omega : \Delta_k \chi_{\{|\Delta_k(\omega)|>\alpha\}}|\mathcal{F}_{t_k})|.$$

And we have the following inequalities

$$|\mathbb{E}(g_k|\mathcal{F}_{t_k})| = |\mathbb{E}(\Delta_k \chi_{\{|\Delta_k|>\alpha\}}|\mathcal{F}_{t_k})|$$

Then by the properties of the integral we have

$$|\mathbb{E}(g_k|\mathcal{F}_{t_k})| \le \mathbb{E}(|\Delta_k|\chi_{\{|\Delta_k|>\alpha\}}|\mathcal{F}_{t_k})$$

Note that $\chi_{\{|\Delta_k|>\alpha\}} \le \frac{1}{\alpha}\Delta_k$. It follows then that

$$\begin{aligned}
|\mathbb{E}(g_k|\mathcal{F}_{t_k})| &= |\mathbb{E}(\Delta_k\chi_{\{|\Delta_k|>\alpha\}}|\mathcal{F}_{t_k})| \\
&\le \frac{1}{\alpha}\mathbb{E}(\Delta_k^2|\mathcal{F}_{t_k}) \\
&= \frac{1}{\alpha}(t_{k+1}-t_k), \quad \text{as per (18)} \\
&= \frac{\alpha^2}{\alpha}\cdot m \\
&= \frac{\alpha}{m}.
\end{aligned}$$

So,

$$\begin{aligned}
\left[\mathbb{E}\left(\left|\sum_{k=0}^{m-1}\mathbb{E}(g_k|\mathcal{F}_{t_k})\right|^p \middle|\mathcal{F}_s\right)\right]^{\frac{1}{p}} &\le \mathbb{E}(|\sum_{k=0}^{m-1}\frac{\alpha}{m}|^p\mathcal{F}_s)^{\frac{1}{p}} \\
&= m\cdot\frac{\alpha}{m} \\
&= \alpha \\
&= (t-s)^{\frac{1}{2}}
\end{aligned} \tag{29}$$

Define:

$$M_0 := 0,\ 1 \le k \le m \qquad\qquad M_k := \sum_{j=0}^{k-1} G_j.$$

Then

$$\begin{aligned}
\mathbb{E}(M_{k+1}|\mathcal{F}_{t_k}) &= \mathbb{E}(G_0+G_1+\cdots+G_k|\mathcal{F}_{t_k}) \\
&= \mathbb{E}(g_0-\mathbb{E}(g_0|\mathcal{F}_{t_0})+\cdots+g_k-\mathbb{E}(g_k|\mathcal{F}_{t_k})|\mathcal{F}_{t_k}),\ \text{by (26)} \\
&= \mathbb{E}(g_0-\mathbb{E}(g_0|\mathcal{F}_{t_0})+\cdots+g_{k-1}-\mathbb{E}(g_{k-1}|\mathcal{F}_{t_{k-1}})|\mathcal{F}_{t_k}) \\
&\quad+\mathbb{E}(g_k|\mathcal{F}_{t_k})-\mathbb{E}\left(\mathbb{E}(g_k|\mathcal{F}_{t_k}|)\,\mathcal{F}_{t_k}\right)
\end{aligned}$$

$$\mathbb{E}(M_{k+1}|\mathcal{F}_{t_k}) = \mathbb{E}(G_0 + G_1 + \cdots + G_{k-1}|\mathcal{F}_{t_k})$$

Since $g_v,\ 0 \leq v \leq k-1$ is $\mathcal{F}_{t_k}-$measurable, we have

$$\begin{aligned} &= g_0 - \mathbb{E}(g_0|\mathcal{F}_{t_0}) + \cdots + g_{k-1} - \mathbb{E}(g_{k-1}|\mathcal{F}_{t_{k-1}}) \\ &= M_k \end{aligned}$$

We have now established the foundations to use Theorem 7: The f_k there is G_{k-1} and the $\mathcal{F}_0$ there is $\mathcal{F}_s$. The following inequalities are now deduced:

$$\begin{aligned} \left[\mathbb{E}\left(\left|\sum_{j=0}^{m-1} G_j\right|^p |\mathcal{F}_s\right)\right]^{\frac{1}{p}} &= \left[\mathbb{E}\left(|M_m|^p|\mathcal{F}_s\right)\right]^{\frac{1}{p}}, \text{from the definition of } M_m \\ &\leq \left[\mathbb{E}\left(\max_{1\leq j\leq m}|M_j|^p|\mathcal{F}_s\right)\right]^{\frac{1}{p}} \\ &\leq Kp \max_{2\leq i\leq p}\left[\mathbb{E}\left(\left[\sum_{k=0}^{m-1}\mathbb{E}(|G_k|^i|\mathcal{F}_{t_k})\right]^{\frac{p}{i}}\middle|\mathcal{F}_s\right)\right]^{\frac{1}{p}}, \text{ by (10).} \end{aligned}$$

Define

$$S_{m,i} := \left[\mathbb{E}\left(\left[\sum_{k=0}^{m-1}\mathbb{E}(|G_k|^i|\mathcal{F}_{t_k})\right]^{\frac{p}{i}}\middle|\mathcal{F}_s\right)\right]^{\frac{1}{p}} \tag{30}$$

We are able to conclude that:

$$\left[\mathbb{E}\left(\left|\sum_{j=0}^{m-1} G_j\right|^p\middle|\mathcal{F}_s\right)\right]^{\frac{1}{p}} \leq Kp\max_{2\leq i\leq p} S_{m,i} \tag{31}$$

We will now attempt to estimate each $S_{m,i}$. Recall in (26) that we defined $G_k := g_k - \mathbb{E}(g_k|\mathcal{F}_{t_k})$. By taking expectations, for $2 \leq i \leq p$ we deduce the following:

$$\mathbb{E}(|G_k|^i|\mathcal{F}_{t_k}) \leq \mathbb{E}\left(2^i \max\left(|g_k|^i,\ |\mathbb{E}(|g_k|\ |\mathcal{F}_{t_k})|^i\right)|\mathcal{F}_{t_k}\right)$$

Using Theorem 3 and setting $g = 1$ in (i), we have

$$\begin{aligned} \mathbb{E}(|G_k|^i|\mathcal{F}_{t_k}) &\leq \mathbb{E}\left(2^i\max\left(|g_k|^i, \mathbb{E}(|g_k|^i|\mathcal{F}_{t_k})^i\right)|\mathcal{F}_{t_k}\right) \\ &\leq 2^i\mathbb{E}\left(\left(|g_k|^i + \mathbb{E}(|g_k|^i|\mathcal{F}_{t_k})^i\right)|\mathcal{F}_{t_k}\right) \\ &= 2^{i+1}\mathbb{E}(|g_k|^i|\mathcal{F}_{t_k}) \end{aligned} \tag{32}$$

Then for $2 \leq i \leq p-2$ we have the following inequalities:

$$\begin{aligned}
\mathbb{E}(|G_k|^i|\mathcal{F}_{t_k}) &\leq 2^{i+1}\mathbb{E}(|\Delta_k|^i|\mathcal{F}_{t_k}) && (33)\\
&\leq 2^{i+1}K_i(t_{k+1}-t_k)^{\frac{i}{2}}, \text{ by the induction hypothesis} \\
&= 2^{i+1}K_i\frac{(t-s)^{\frac{i}{2}}}{m^{\frac{i}{2}}} && (34)\\
&\leq 2^{i+1}K_i\frac{(t-s)^{\frac{i}{2}}}{m}
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{k=0}^{m-1}\mathbb{E}(|G_k|^i|\mathcal{F}_{t_k}) &\leq 2^{i+1}K_i\frac{(t-s)^{\frac{i}{2}}}{m}\cdot m && (35)\\
&< 2^{2i}K_i(t-s)^{\frac{i}{2}} && (36)
\end{aligned}$$

Now, by substituting (36) into $S_{m,i}$ as defined above we get:

$$\begin{aligned}
S_{m,i} &< \left[\mathbb{E}\left(\left[2^{2i}K_i(t-s)^{\frac{i}{2}}\right]^{\frac{p}{i}}\middle|\mathcal{F}_s\right)\right]^{\frac{1}{p}} \\
&= \left(\left(2^{2i}K_i(t-s)^{\frac{i}{2}}\right)^{\frac{p}{i}}\right)^{\frac{1}{p}} \\
&= 4K_i^{\frac{1}{i}}(t-s)^{\frac{1}{2}}, \text{ if } 2 \leq i \leq p-2. && (*)
\end{aligned}$$

We will now consider $S_{m,p-1}$. By (32) with $i = p-1$ the following inequalities are deduced:

$$\begin{aligned}
\mathbb{E}(|G_k|^{p-1}|\mathcal{F}_{t_k}) &\leq 2^{(p-1)+1}\mathbb{E}(|g_k|^{p-1}|\mathcal{F}_{t_k}) \\
&\leq 2^p\alpha\mathbb{E}(|g_k|^{p-2}|\mathcal{F}_{t_k}), \text{ since } |g_k| \leq \alpha \\
&\leq 2^p\alpha K_{p-2}\frac{(t-s)^{\frac{p-2}{2}}}{m^{\frac{p-2}{2}}}
\end{aligned}$$

(similar to the way in which (34) was deduced from (32)).

$$\mathbb{E}(|G_k|^{p-1}|\mathcal{F}_{t_k}) \leq 2^pK_{p-2}\frac{(t-s)^{\frac{p-1}{2}}}{m}, \text{ since } \alpha = (t-s)^{\frac{1}{2}}$$

Thus

$$\sum_{k=0}^{m-1} \mathbb{E}(|G_k|^{p-1}|\mathfrak{F}_{t_k}) \leq 2^p K_{p-2} \frac{(t-s)^{\frac{p-1}{2}}}{m} \cdot m = 2^p K_{p-2}(t-s)^{\frac{p-1}{2}}$$

We now have the following equation (using (30))

$$\begin{aligned} S_{m,p-1} &\leq 2^{\frac{p}{p-1}} K_{p-2}^{\frac{1}{p-1}} (t-s)^{\frac{1}{2}} \\ &\leq 4K_{p-2}^{\frac{1}{p-1}} (t-s)^{\frac{1}{2}} \end{aligned} \tag{**}$$

We will now consider the last case, namely $S_{m,p}$. If we set $i = p$ in (32) we get

$$\begin{aligned} \mathbb{E}(|G_k|^p|\mathfrak{F}_{t_k}) &\leq 2^{p+1}\mathbb{E}(|g_k|^p|\mathfrak{F}_{t_k}) \\ &\leq 2^{p+1}\alpha^2\mathbb{E}(|g_k|^{p-2}|\mathfrak{F}_{t_k}) \\ &\leq 2^{p+1}\alpha^2 K_{p-2} \frac{(t-s)^{\frac{p-2}{2}}}{m^{\frac{p-2}{2}}} \\ &\leq 2^{p+1} K_{p-2} \frac{(t-s)^{\frac{p}{2}}}{m}. \end{aligned}$$

Thus

$$\sum_{k=0}^{m-1} \mathbb{E}(|G_k|^p|\mathfrak{F}_{t_k}) \leq 2^{p+1} K_{p-2}(t-s)^{\frac{p}{2}} \tag{37}$$

Substituting (37) in (30) we get the following equation:

$$\begin{aligned} S_{m,p} &\leq 2^{\frac{p+1}{p}} K_{p-2}^{\frac{1}{p}} \\ &\leq 4K_{p-2}^{\frac{1}{p}}(t-s)^{\frac{1}{2}} \end{aligned} \tag{***}$$

Combining (*), (**) and (***) gives

$$\begin{aligned} \max_{2\leq i\leq p} S_{m,i} &\leq \max_{2\leq i\leq p-2} \max\{4K_i^{\frac{1}{i}}(t-s)^{\frac{1}{2}}, 4K_{p-2}^{\frac{1}{p-1}}(t-s)^{\frac{1}{2}}, 4K_{p-2}^{\frac{1}{p}}(t-s)^{\frac{1}{2}}\} \\ &= L_p(t-s)^{\frac{1}{2}} \end{aligned}$$

where L_p is dependent on p. Using it in (31) we deduce the following:

$$\left[\mathbb{E}\left(\left|\sum_{j=0}^{m-1} G_j\right|^p \middle| \mathcal{F}_s\right)\right]^{\frac{1}{p}} \leq KpL_p(t-s)^{\frac{1}{2}} \text{ a.s.} \tag{38}$$

Using (29) and (38) in (28) we get

$$\begin{aligned}[\mathbb{E}(|X_t - X_s|^p \chi_{A_m}|\mathcal{F}_s)]^{\frac{1}{p}} &\leq (t-s)^{\frac{1}{2}} + KpL_p(t-s)^{\frac{1}{2}} \text{ a.s.} \\ &= \overbrace{(1+KpL_p)}^{K_p}(t-s)^{\frac{1}{2}} \text{ a.s.}\end{aligned} \tag{39}$$

To find K_{p-1} we will use Theorem 3:

$$\begin{aligned}\mathbb{E}(|X_t - X_s|^{p-1}\chi_{A_m}|\mathcal{F}_s) &\leq \left[\mathbb{E}\left(\left(|X_t - X_s|^{p-1}\chi_{A_m}\right)^q |\mathcal{F}_s\right)\right]^{\frac{1}{q}} \\ &= [\mathbb{E}(|X_t - X_s|^p \chi_{A_m}|\mathcal{F}_s)]^{\frac{p}{p-1}}\end{aligned}$$

So

$$\left[\mathbb{E}(|X_t - X_s|^{p-1}\chi_{A_m}|\mathcal{F}_s)\right]^{\frac{1}{p-1}} \leq [\mathbb{E}(|X_t - X_s|^p \chi_{A_m}|\mathcal{F}_s)]^{\frac{1}{p}} \text{ a.s.}$$

And (39) shows that we can take $K_{p-1} := K_P$. This completes the proof. □

Theorem 9. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and let $(X_t)_{t\geq 0}$ be a real-valued martingale adapted to it. Suppose that all sample paths are continuous, $\mathbb{E}(|X_t|^2) < \infty \,\forall\, t \geq 0$, and*

$$E((X_t - X_s)^2|\mathcal{F}_s) = t - s \text{ a.s. } \forall\, 0 \leq s < t. \tag{40}$$

Then for every $p \in \mathbb{Z}^+$ and every $t > s \geq 0$ we have

$$\mathbb{E}((X_t - X_s)^p|\mathcal{F}_s) = \mathfrak{m}_p(t-s)^{\frac{p}{2}} \text{ a.s.,} \tag{41}$$

where

$$\mathfrak{m}_p := \begin{cases} 0 & \text{if } p \text{ is odd} \\ \frac{p!}{2^{\frac{p}{2}}(\frac{p}{2})!} & \text{if } p \text{ is even.} \end{cases} \tag{42}$$

Proof. We will prove this theorem by using mathematical induction on p in (41). Therefore, fix $s, t \in \mathbb{R}$ such that $t > s \geq 0$.

Let $p = 0$, then

$$\mathbb{E}((X_t - X_s)^0 | \mathcal{F}_s) = 1 = \mathfrak{m}_0.$$

and (41) holds.

If we let $p = 1$ then

$$\begin{aligned}\mathbb{E}((X_t - X_s)^1 | \mathcal{F}_s) &= X_s - X_s, \text{ because } (X_t)_{t \geq 0} \text{ is a martingale} \\ &= 0.\end{aligned}$$

and (41) holds for $p = 1$ with $\mathfrak{m}_1 = 0$.

For $p = 2$ we get

$$\mathbb{E}((X_t - X_s)^2 | \mathcal{F}_s) = t - s, \text{ by (40)}$$

and we may therefore take $\mathfrak{m}_2 = 1$. Continuing in this way, assume then that for some $p > 2$ and all $q = 0, 1, \ldots, p-1$ we deduced the following

$$\mathbb{E}((X_v - X_u)^q | \mathcal{F}_u) = \mathfrak{m}_q (v - u)^{\frac{q}{2}} \text{ a.s. } \forall u, v \in [s, t] \text{ and } u < v, \tag{43}$$

where

$$\mathfrak{m}_q := \begin{cases} 0 & \text{if } q \text{ is odd} \\ \frac{q!}{2^{\frac{q}{2}} (\frac{q}{2})!} & \text{if } q \text{ is even.} \end{cases} \tag{44}$$

Fix $m \in \mathbb{N}$. Let t_k and Δ_k be as defined in (23) and (24). Then

$$\begin{aligned}(X_t - X_s)^p &= \left(\sum_{k=0}^{m-1} \Delta_k \right)^p \\ &= \sum_{\substack{1 \leq n \leq p \\ 0 \leq i_1 < i_2 < \cdots < i_n \leq m-1 \\ p_1 + p_2 + \cdots + p_n = p}} \overbrace{\left(\prod_{j=1}^{n} \Delta_{i_j}^{p_j} \right)}^{\mho} \frac{p!}{p_1! \cdots p_n!} \, ,\end{aligned} \tag{45}$$

We can split (45) into two parts: the first part containing *odd* exponents (i.e. where at least one of $(p_1, p_2, \ldots, p_n)$ is odd) and the second part containing only *even* exponents (i.e. where all of $(p_1, p_2, \ldots, p_n)$ is even). We can therefore write (45) as follows:

$$(X_t - X_s)^p = A_m + \sum_{n=1}^{\frac{p}{2}} B(m,n), \tag{46}$$

where A_m consist of all the terms in which at least one p_j is odd and

$$B(m,n) := \sum_{(b_1,b_2,\ldots,b_n)} \sum_{(i_1,\ldots,i_n)} \left(\prod_{j=1}^{n} \Delta_{i_j}^{2b_j}\right) \frac{p!}{(2b_1)!\cdots(2b_n)!}, \tag{47}$$

the outer sum being over those $(b_1, b_2, \ldots, b_n)$ with $2b_1 + \cdots + 2b_n = p$. Note that $B(m,n) = 0$ whenever p is an odd number.

Before we continue with any remarks on A_m and $B(m,n)$ above, we will first consider $\mho$ in (45). Let $2 \leq n \leq p$. Then $p_j < p$, $j = 1, 2, \ldots, n$, and we have the following

$$\begin{aligned}
\mathbb{E}\left(\prod_{j=1}^{n} \Delta_{i_j}^{p_j} \middle| \mathcal{F}_s\right) &= \mathbb{E}\left(\Delta_{i_n}^{p_n} \prod_{j=1}^{n-1} \Delta_{i_j}^{p_j} \middle| \mathcal{F}_s\right) \text{ a.s.} \\
&= \mathbb{E}\left(\mathbb{E}\left(\Delta_{i_n}^{p_n} \prod_{j=1}^{n-1} \Delta_{i_j}^{p_j} \middle| \mathcal{F}_{t_{i_n}}\right) \middle| \mathcal{F}_s\right) \text{ a.s.}
\end{aligned}$$

(by the tower property of conditional expectations)

$$= \mathbb{E}\left(\mathbb{E}\left(\Delta_{i_n}^{p_n} | \mathcal{F}_{t_{i_n}}\right) \prod_{j=1}^{n-1} \Delta_{i_j}^{p_j} \middle| \mathcal{F}_s\right) \text{ a.s.}$$

(since $\prod_{j=1}^{n-1} \Delta_{i_j}^{p_j}$ is $\mathcal{F}_{t_{i_n}}$ – measurable)

$$= \mathbb{E}\left(\mathfrak{m}_{p_n} (t_{i_n+1} - t_{i_n})^{\frac{p_n}{2}} \prod_{j=1}^{n-1} \Delta_{i_j}^{p_j} \Big| \mathcal{F}_s \right), \text{ a.s. by (43)}$$

$$= \mathbb{E}\left(\Delta_{i_{n-1}}^{p_{n-1}} \mathfrak{m}_{p_n} \left(t_{i_n+1} - t_{i_n}\right)^{\frac{p_n}{2}} \prod_{j=1}^{n-2} \Delta_{i_j}^{p_j} \Big| \mathcal{F}_s \right) \text{ a.s.}$$

$$= \mathbb{E}\left(\mathfrak{m}_{p_n} \cdot \mathfrak{m}_{p_{n-1}} \left(t_{i_n+1} - t_n\right)^{\frac{p_{n-1}+p_n}{2}} \prod_{j=1}^{n-2} \Delta_{i_j}^{p_j} \Big| \mathcal{F}_s \right) \text{ a.s.}$$

(since $(t_{i_n+1} - t_{i_n}) = (t_{i_n} - t_{i_n-1})$)

$$= \cdots = \left(\frac{t-s}{m} \right)^{\frac{p_1+p_2+\cdots+p_n}{2}} \prod_{j=1}^{n} \mathfrak{m}_{p_j} \text{ a.s.}$$

(since $t_{i_n+1} - t_{i_n} = \frac{t-s}{m}$)

$$= \left(\frac{t-s}{m} \right)^{\frac{p}{2}} \prod_{j=1}^{n} \mathfrak{m}_{p_j} \text{ a.s.} \tag{48}$$

It now follows from our induction assumptions, the definition of $\mathfrak{m}_{p_j}$ and (48) that, if $2 \leq n \leq p$ with at least one p_j odd that

$$\mathbb{E}\left(\prod_{j=1}^{n} \Delta_{i_j}^{p_j} \Big| \mathcal{F}_s \right) = 0 \text{ a.s.} \tag{49}$$

If we let $p_1 = p_2 = \cdots = p_n = 2$, then $p = 2n$ is even and all $\mathfrak{m}_{p_j} = \mathfrak{m}_2 = 1$. Also, $p > 2$ and $p = 2n$ means that $n \geq 2$, so from (48)

$$\mathbb{E}\left(\prod_{j=1}^{n} \Delta_{i_j}^{2} \Big| \mathcal{F}_s \right) = \left(\frac{t-s}{m} \right)^{\frac{p}{2}} \text{ a.s., if } p_1 = p_2 = \cdots = p_n = 2. \tag{50}$$

We deduce the following (in a similar manner in which (48) was deduced and by using (19)):

$$\mathbb{E}\left(\prod_{j=1}^{n} |\Delta_{i_j}^{p_j}| \Big| \mathcal{F}_s \right) = \mathbb{E}\left(|\Delta_{i_n}^{p_n}| \prod_{j=1}^{n-1} |\Delta_{i_j}^{p_j}| \Big| \mathcal{F}_s \right) \text{ a.s.}$$

$$= \mathbb{E}\left(\mathbb{E}\left(|\Delta_{i_n}^{p_n}| \prod_{j=1}^{n-1} |\Delta_{i_j}^{p_j}| \Big| \mathcal{F}_{t_n} \right) \Big| \mathcal{F}_s \right) \text{ a.s.}$$

(by the tower property of conditional expectations)

$$= \mathbb{E}\left(\mathbb{E}\left(|\Delta_{i_n}^{p_n}|\big|\mathcal{F}_{t_n}\right)\prod_{j=1}^{n-1}|\Delta_{i_j}^{p_j}|\Big|\mathcal{F}_s\right) \text{ a.s.}$$

(since $\prod_{j=1}^{n-1}|\Delta_{i_j}^{p_j}|$ is $\mathcal{F}_{t_n}-$ measurable)

$$\begin{aligned}
&\leq \mathbb{E}\left(K_{p_n}(t_{i_n+1}-t_{i_n})^{\frac{1}{2}}\prod_{j=1}^{n-1}|\Delta_{i_j}^{p_j}|\Big|\mathcal{F}_s\right) \text{ a.s., by (19)}\\
&= \mathbb{E}\left(|\Delta_{i_{n-1}}^{p_{n-1}}|K_{p_n}(t_{i_n+1}-t_{i_n}))^{\frac{1}{2}}\prod_{j=1}^{n-2}|\Delta_{i_j}^{p_j}|\Big|\mathcal{F}_s\right) \text{ a.s.}\\
&\leq \left(\frac{t-s}{m}\right)^{\frac{p}{2}}\prod_{j=1}^{n}K_{p_j}\\
&\leq L_p\cdot\left(\frac{t-s}{m}\right)^{\frac{p}{2}} \qquad (51)
\end{aligned}$$

where

$$L_p := \max\{\prod_{j=1}^{n}K_{p_j} : (p_1,\ldots,p_n)\in\mathbb{N}^n, p_1+\cdots+p_n=p,\ 1\leq n\leq p\}$$

depends on p. This concludes our remarks on $\mho$ in (45) and will be using it for future reference. Our attention will now shift to A_m and $B(m,n)$.

We will now estimate $|\mathbb{E}(A_m|\mathcal{F}_s)|$. We have already seen in (48) that for $n\geq 2$ that (49) applies, that is $\mathbb{E}\left(\prod_{j=1}^{n}\Delta_{i_j}^{p_j}\Big|\mathcal{F}_s\right)=0$ if any p_j is odd. The only case that remains is $n=1, p_1=p$. Hence,

$$
\begin{aligned}
|\mathbb{E}(A_m|\mathcal{F}_s)| &= \left|\mathbb{E}\left(\sum \text{ products with } n=1, p_1=p\,|\mathcal{F}_s\right)\right| \\
&= \left|\mathbb{E}\left(\sum_{i_1=0}^{m-1} \Delta_{i_1}^{p_1}\,|\mathcal{F}_s\right)\right| \\
&\leq \mathbb{E}\left(\sum_{i_1=0}^{m-1} |\Delta_{i_1}^{p_1}|\,|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(|\Delta_0^{p_1}|\big|\,\mathcal{F}_s\right) + \cdots + \mathbb{E}\left(|\Delta_{m-1}^{p_1}|\big|\,\mathcal{F}_s\right), \text{ by linearity} \\
&\leq \overbrace{L_p\cdot\left(\frac{t-s}{m}\right)^{\frac{p}{2}} + \cdots + L_p\cdot\left(\frac{t-s}{m}\right)^{\frac{p}{2}}}^{m \text{ terms}}, \text{ by (51)} \\
&= m\cdot L_p\cdot\left(\frac{t-s}{m}\right)^{\frac{p}{2}} \\
&= m^{1-\frac{p}{2}} L_p\cdot\left(\frac{t-s}{1}\right)^{\frac{p}{2}}, \text{ a.s.}
\end{aligned}
\tag{52}
$$

Thus, if p is odd, then by taking expectations in (46) we get

$$
\begin{aligned}
\mathbb{E}\left((X_t-X_s)^p\,|\mathcal{F}_s\right) &= \mathbb{E}\left(A_m + \sum_{n=1}^{\frac{p}{2}} B(m,n)\,|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(A_m\,|\mathcal{F}_s\right) \\
&\leq m^{1-\frac{p}{2}} L_p\cdot\left(\frac{t-s}{1}\right)^{\frac{p}{2}}
\end{aligned}
$$

Since $p>2, m^{1-\frac{1}{2}} \leq m^{-\frac{1}{2}}$ and therefore by letting $m\to\infty$ we get $\mathbb{E}((X_t - X_s)^p|\mathcal{F}_s) = 0$ almost surely, which is what (41) claims for odd p.

It remains to settle the case when p is even. In order to do this, we will inspect (47) in more detail. We take note of some important aspects concerning (47):

(i). Let $p \leq q$. Let $K(p,q)$ be the number of $p-$tuples $(a_1,\ldots,a_p)$ such that $a_1+\cdots a_p = q$. Then $K(p,q) < q^p$. This is obvious, since each $a_i \leq q$, $i=1,\ldots,p$ and hence $a_i \in \{1,2,\ldots,q\}$

(ii). Let $L(p,q)$ be the number of $p-$tuples $(a_1,\ldots,a_p)$ such that $0 \leq a_1 < a_2 < \cdots < a_p \leq q$. Then $L(p,q) < q^p$

Consider $B(m,n)$: At most m^n (from (ii.) above) different $n-$tuples $(i_1,\ldots,i_n)$ are involved in $B(m,n)$ and for each of these at most $\left(\frac{p}{2}\right)^n$ different $n-$tuples $(b_1,\ldots,b_n)$ are available (from (i.) above), so $B(m,n)$ has at most $m^n\left(\frac{p}{2}\right)^n \leq m^n\left(\frac{p}{2}\right)^{\frac{p}{2}}$ terms.

Let $n<\frac{p}{2}$. The following is now deduced:

$$\mathbb{E}(B(m,n)\,|\mathcal{F}_s) \leq \overbrace{m^n\left(\frac{p}{2}\right)^{\frac{p}{2}}}^{\text{number of terms}} \times L_p \times \overbrace{\left(\frac{t-s}{m}\right)^{\frac{p}{2}}}^{\text{from (51)}} \times \overbrace{B'}^{\text{biggest factor}}$$

where

$$B' = \max_{\substack{(b_1,\ldots,b_n)\\ 2\leq n\leq\frac{p}{2}-1}}\left\{\frac{p!}{(2b_1)!\cdots(2b_n)!} : b_1+\cdots+b_n=\frac{p}{2}\right\}.$$

If we set $L'_p := L_p\cdot B'$, then

$$\mathbb{E}(B(m,n)\,|\mathcal{F}_s) \leq m^n\left(\frac{p}{2}\right)^{\frac{p}{2}} L'_p\left(\frac{t-s}{m}\right)^{\frac{p}{2}}$$

We note that $m^n \leq m^{\frac{p}{2}-1}$ for the n under consideration. We get from the above inequality the following:

$$\begin{aligned}\mathbb{E}(B(m,n)|\mathcal{F}_s) &\leq L'_p\left(\frac{p}{2}\right)^{\frac{p}{2}} m^{\frac{p}{2}-1}\left(\frac{t-s}{m}\right)^{\frac{p}{2}}\\ &= \frac{L'_p\left(\frac{p}{2}\right)^{\frac{p}{2}}}{m}\cdot(t-s)^{\frac{p}{2}}.\end{aligned}$$

As before, we let $m\to\infty$ to get

$$\lim_{m\to\infty}\mathbb{E}(B(m,n)|\mathcal{F}_s) = 0 \text{ a.s., if } n<\frac{p}{2}. \tag{53}$$

If we let $n=\frac{p}{2}$, then $\frac{p}{2}=b_1+\cdots+b_{\frac{p}{2}}$. Given the fact that $b_j\geq 1, j=1,\ldots\frac{p}{2}$ helps us to conclude that each b_j is exactly 1. Consequently, in $B(m,\frac{p}{2})$ there is only one $n-$tuple of $(b_1,\ldots,b_n)$ and exactly $\binom{m}{\frac{p}{2}}$ possible $n-$tuples $(i_1,\ldots i_n)$. In this scenario, $B(m,\frac{p}{2})$ has $\binom{m}{\frac{p}{2}}$ terms in total, each of the form (50). So (for $n=\frac{p}{2}$) we get,

$$\mathbb{E}(B(m,n)|\mathcal{F}_s) = \binom{m}{\frac{p}{2}} \left(\frac{t-s}{m}\right)^{\frac{p}{2}} \frac{p!}{2^{\frac{p}{2}}}$$
$$= \mathfrak{m}_p(t-s)^{\frac{p}{2}} \frac{m!}{(m-\frac{p}{2})!m^{\frac{p}{2}}}. \tag{54}$$

If we let $m \to \infty$ we see that

$$\lim_{m\to\infty} \mathbb{E}(B(m,n)|\mathcal{F}_s) = \mathfrak{m}_p(t-s)^{\frac{p}{2}} \text{ a.s.,} \tag{55}$$

because for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{m!}{(m-k)!m^k} = \lim_{m\to\infty} \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{k-1}{m}\right)$$
$$= 1.$$

where $k = \frac{p}{2}$ in (54). This completes the proof.

$\square$

Definition 10. The *Gaussian kernel* on $\mathbb{R}^d$ is the function G given by $G(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{(x-y)^2}{2t}}$, $x, t \in \mathbb{R}^d$ and $t > 0$.

Definition 11. Let $\mu : \mathcal{B}^d \mapsto \mathbb{C}$ be a measure where $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d)$ is the $\sigma-$algebra of all Borel subsets of $\mathbb{R}^d$. Then we define the *Fourier transform* on $\mathbb{R}^d$ of μ by

$$\hat{\mu}(t) := \int_{\mathbb{R}^d} e^{-itx} \mathrm{d}\mu(x).$$

Also, if f is an integrable function, then its Fourier transform $\hat{f}(t)$ on $\mathbb{R}^d$ is defined by

$$\hat{f}(t) := \int_{\mathbb{R}^d} e^{-itx} f(x)dx,$$

where i is the usual imaginary number, often denoted $\sqrt{-1}$.

4 The proof of Lévy's theorem

Theorem 12 (Levy). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and let $(X_t)_{t\geq 0}$ be a real-valued martingale adapted to it. Suppose that all sample paths are continuous, $\mathbb{E}(|X_t|^2) < \infty \,\forall\, t \geq 0$, and*

$$E((X_t - X_s)^2|\mathcal{F}_s) = t - s \text{ a.s. } \forall\, 0 \leq s < t.$$

Then $(X_t)_{t\geq 0}$ is a Brownian motion (except possibly for the condition that $X_0 = 0$).

Proof. To prove this theorem we need to verify (BM 1) to (BM 4) of Definition 2. (BM 1) is not asserted and continuity of sample paths are true by hypothesis, so (BM 4) is not required to prove. Therefore, we only need to verify (BM 2) and (BM 3).

We will start of with the proof of (BM 2). So, let $s \geq 0$ and $t > 0$. Let μ denote the distribution of $X_{s+t} - X_s$. We must show that μ has a Normal (or Gaussian) distribution on $\mathbb{R}$ with mean zero and variance t. Our objective is to show that the measures μ and ν_t are the same where ν_t is the normal distribution on $\mathbb{R}$ with mean 0 and variance t. Recall that μ, the distribution function of $Y = X_{s+t} - X_s$, defines a measure on $\mathcal{B}(\mathbb{R})$ via the assignment $\mu((-\infty, a]) = \text{Prob}(Y \leq a)$. Indeed, $\mu(A) = \text{Prob}(Y \in A)$ for any $A \in \mathcal{B}(\mathbb{R})$. To show that μ and ν_t are the same, it is enough to show that the two measures just specified transmit the same Fourier transforms. It is also a general result that the Fourier transform of the measure ν_t is

$$\hat{\nu}_t(u) = e^{-\frac{tu^2}{2}}\ \forall u \in \mathbb{R}.$$

We will now show that $\hat{u}(u) = \hat{\nu}_t(u)$. Then

$$\begin{aligned}\int_{\mathbb{R}} x^p \mathrm{d}\mu(x) &= \int_{\Omega} Y^p \mathrm{d}P \\ &= \mathfrak{m}_p t^{\frac{p}{2}},\ \ \forall\, p \in \mathbb{Z}^+,\ \text{by (41).}\end{aligned} \tag{56}$$

For all $u \in \mathbb{R}$ we have

$$\begin{aligned}\hat{\mu}(u) &= \int_{\mathbb{R}} e^{-iux} d\mu(x),\ \text{ by Definition 11} \\ &= \int_{\mathbb{R}} \lim_{N\to\infty} \sum_{p=0}^{N} \frac{(-iux)^p}{p!} d\mu(x),\ \text{ power series expansion of } e^{-iux} \\ &= \lim_{N\to\infty} \sum_{p=0}^{N} \frac{(-iu)^p}{p!} \int_{\mathbb{R}} x^p d\mu(x), \text{ by LDCT}\end{aligned} \tag{57}$$

Note that in order to apply LDCT in the last step, we need to show that the functions $\left|\sum_{p=0}^{N}\frac{(-iux)^p}{p!}\right|$ are indeed dominated by an $\mu-$integrable function. We will show this later on. For now we continue with the proof of the theorem. From (56) and (57) we have

$$\begin{aligned}
\hat{\mu}(u) &= \sum_{p=0}^{\infty}\frac{(-iu)^p}{p!}\mathfrak{m}_p t^{\frac{p}{2}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{2n!}\cdot\frac{(2n)!}{2^n n!}\cdot t^n, \text{ by letting } p=2n \text{ and the use of (42)}\\
&= \sum_{n=0}^{\infty}\frac{(-tu^2)^n}{2^n n!}\\
&= e^{\frac{-tu^2}{2}}\\
&= \hat{\nu}_t(u).
\end{aligned}$$

We will now show that the use of LDCT in (57) is indeed justified. Let $u, x \in \mathbb{R}$ and $N \in \mathbb{Z}^+$. We note that

$$\begin{aligned}
\left|\sum_{p=0}^{N}\frac{(-iu)p}{p!}x^p\right| &\le \sum_{p=0}^{N}\frac{|ux|^p}{p!}\\
&\le e^{|ux|}\\
&\le e^{ux}+e^{-ux}\\
&= 2\cosh ux
\end{aligned} \tag{58}$$

We now get

$$\begin{aligned}
\int_{\mathbb{R}}\cosh ux\, d\mu(x) &= \int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{(ux)^{2n}}{(2n)!}d\mu(x), \text{ power expansion of } \cosh ux\\
&= \sum_{n=0}^{\infty}\frac{u^{2n}}{(2n)!}\int_{\mathbb{R}}x^{2n}d\mu(x), \text{ by Monotone Convergence Theorem}\\
&= \sum_{n=0}^{\infty}\frac{u^{2n}}{(2n)!}\cdot\frac{(2n)!}{2^n n!}t^n, \text{ by (56)}\\
&= \sum_{n=0}^{\infty}\frac{(\frac{tu^2}{2})^n}{n!}\\
&= e^{\frac{tu^2}{2}}\\
&< \infty,
\end{aligned}$$

which concludes the proof of (BM 2).

We still have to prove (BM 3). Let $s \geq 0, t > 0$, and μ be the distribution of $X_{s+t} - X_s$ and let $A \in \mathfrak{F}_s$ be arbitrary. Let the measure P_A on $\mathfrak{F}_s$ be defined by $P_A(E) = P(E \cap A)$. We will now define Borel measures μ_1 and μ_2 on $\mathbb{R}$. For any Borel set $B \subseteq \mathbb{R}$, let

$$\mu_1(B) = P_A[(X_{s+t} - X_s)^{-1}(B)],$$

hence μ_1 is the P_A- distribution of $X_{s+t} - X_s$. Also, let

$$\mu_2(B) = P(A)\mu(B).$$

Now,

$$\begin{aligned}
\int_{\mathbb{R}} x^p \mathrm{d}\mu_1(x) &= \int_{\Omega} (X_{s+t} - X_s)^p \, \mathrm{d}P_A \\
&= \int_A (X_{s+t} - X_s)^p \, \mathrm{d}P \\
&= \int_A \mathbb{E}\left((X_{s+t} - X_s)^p \, |\mathfrak{F}_s\right) \, \mathrm{d}P \\
&= P(A)\mathfrak{m}_p t^{\frac{p}{2}}, \text{by (41)} \\
&= P(A) \int_{\mathbb{R}} x^p \, \mathrm{d}\mu(x) \\
&= \int_{\mathbb{R}} x^p \, \mathrm{d}\mu_2(x).
\end{aligned}$$

It still remains to prove that the measures μ_1 and μ_2 are equal. We again use the fact that two measures are equal if they transmit the same Fourier transforms. Let $\hat{\mu}_1$ and $\hat{\mu}_2$ denote the Fourier transforms of μ_1 and μ_2 respectively. Recalling our earlier deduction of the Fourier transform of $\hat{\mu}$, then, for all $u \in \mathbb{R}$ we have

$$\begin{aligned}
\hat{\mu}_1(u) &= \int_{\mathbb{R}} e^{-iux} \mathrm{d}\mu_1(x) \\
&= \lim_{N\to\infty} \sum_{p=0}^{N} \frac{(-iu)^p}{p!} \int_{\mathbb{R}} x^p \mathrm{d}\mu_1(x) \\
&= \lim_{N\to\infty} \sum_{p=0}^{N} \frac{(-iu)^p}{p!} P(A) \int_{\mathbb{R}} x^p \mathrm{d}\mu(x) \\
&= \lim_{N\to\infty} \sum_{p=0}^{N} \frac{(-iu)^p}{p!} \int_{\mathbb{R}} x^p \mathrm{d}\mu_2(x) \\
&= \int_{\mathbb{R}} e^{-iux} \, \mathrm{d}\mu_2(x) \\
&= \hat{\mu}_2(u).
\end{aligned}$$

This concludes therefore that $\mu_1 = \mu_2$. So, if B is any Borel set in $\mathbb{R}$, then

$$\begin{aligned}
P(A)P\left(\{(X_{s+t} - X_s) \in B\}\right) &= P(A)\mu(B) \\
&= \mu_2(B) \\
&= \mu_1(B) \\
&= P_A\left((X_{s+t} - X_s)^{-1}(B)\right) \\
&= P\left(A \cap \{(X_{s+t} - X_s) \in B\}\right).
\end{aligned}$$

Recall that the set $A \in \mathcal{F}_s$ was arbitrary. The statements above shows that $X_{s+t} - X_s$ is independent of $\mathcal{F}_s$. We have thus shown that

$$0 \le s < u \implies X_u - X_s \text{ is independent of } \mathcal{F}_s. \tag{59}$$

However, we require the form (BM 3) when $X_0 \neq 0$:

$$\begin{cases} n > 1, 0 \le t_1 < t_2 < \cdots < t_n \implies \\ \{X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}\} \text{ is independent.} \end{cases} \tag{60}$$

Given $0 \le t_1 < t_2 < \cdots < t_n$ and Borel subsets $B_1, B_2, \ldots, B_n$ of $\mathbb{R}$, the event

$$A := \{X_{t_1} \in B_1, X_{t_j} - X_{t_{j-1}} \in B_j \text{ for } 2 \le j < n\}$$

belongs to $\mathcal{F}_{t_{n-1}}$, so (59) yields

$$P\left(A \cap \{(X_{t_n} - X_{t_{n-1}}) \in B_n\}\right) = P(A)P\left(\{(X_{t_n} - X_{t_{n-1}}) \in B_n\}\right).$$

The proof follows immediately: (60) follows from (59) by induction on n.

□

Corollary 13. *Suppose $X := (X_t)_{t\geq 0}$ is a real-valued process with continuous sample paths, independent increments, and $X_0 = 0$. That is, it satisfies (BM 1), (BM 3), and (BM 4) of Definition 2. Suppose also that*

$$\mathbb{E}(X_t) = 0 \;\; \forall t \geq 0, \text{ and} \tag{i}$$

$$\mathbb{E}(X_t^2) = t \;\; \forall t \geq 0. \tag{ii}$$

Then X is a Brownian motion.

Proof. Let $0 \leq s < t$ in $\mathbb{R}$. Choose $r \leq s$, then $X_r = X_r - X_0$ is independent of $X_t - X_s$. But $\mathcal{F}_s$ is generated by such X_r. Thus

$$X_t - X_s \text{ is independent of } \mathcal{F}_s. \tag{61}$$

The following almost sure equalities follow:

$$\begin{aligned}\mathbb{E}(X_t - X_s\,|\mathcal{F}_s) &= \mathbb{E}(X_t - X_s)\\ &= 0, \text{ by (i).}\end{aligned}$$

and therefore $\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(X_s|\mathcal{F}_s) = X_s$ a.s. We have then that $(X_t)_{t\geq 0}$ is a martingale. By (61) we have the following:

$$\begin{aligned}\mathbb{E}((X_t - X_s)^2|\mathcal{F}_s) &= \mathbb{E}((X_t - X_s)^2)\\ &= \mathbb{E}((X_t - X_s)X_t) - \mathbb{E}((X_t - X_s)X_s)\\ &= \mathbb{E}((X_t - X_s)X_t) + \mathbb{E}((X_t - X_s)X_s)\end{aligned}$$

(since by (BM 3) and (i) we have $\mathbb{E}((X_t - X_s)(X_s)) = \mathbb{E}(X_t - X_s)\mathbb{E}(X_s) = 0$)

$$\begin{aligned}&= \mathbb{E}((X_t - X_s)(X_t + X_s))\\ &= \mathbb{E}(X_t^2 - X_s^2)\\ &= t - s \text{ by (ii),}\end{aligned}$$

which is the hypothesis (18). We have now fulfilled all the requirements for X to be a Brownian motion and the objective of the dissertation has now been achieved.

□

Properties of Conditional Expectation and Other Definitions

All $X's$ satisfy $\mathbb{E}(|X|) < \infty$ in this list of properties. Of course, $\mathcal{G}$ and $\mathcal{H}$ denote sub-σ algebras of $\mathcal{F}$.

(i). *Linearity:* $\mathbb{E}(a_1X_1 + a_2X_2|\mathcal{G}) = a_1\mathbb{E}(X_1|\mathcal{G}) + a_2\mathbb{E}(X_2|\mathcal{G})$

(ii). *Tower Property:* If $\mathcal{H}$ is a sub-σ algebra of $\mathcal{G}$, then

$$\mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{H}] = \mathbb{E}(X|\mathcal{H}), \text{ a.s.}$$

(iii). *Taking out what is known:* If Z is $\mathcal{G}-$measurable and bounded then

$$\mathbb{E}\left(ZX|\mathcal{G}\right) = Z\mathbb{E}\left(X|\mathcal{G}\right), \text{ a.s.}$$

(iv). *Role of independence:* If X is independent of $\mathcal{H}$, then $\mathbb{E}(X|\mathcal{H}) = \mathbb{E}(X), \text{ a.s.}$

(v). *LDCT* will refer to Lebesgue's Dominated Convergence Theorem.

References

[1] J.L. Doob. *Stochastic Processes*, Wiley Classics Library Edition, Wiley, 1990

[2] L.D.G. Rogers, D. Williams *Diffusions, Markov Processes and Martingales, Volume 1, Foundations*, 2nd Edition, Cambridge University Press, 2000

[3] D. Williams *Probability with Martingales*, 1st Edition, Cambridge University Press, 2003

[4] K.R. Stromberg *Probability for Analysts*, 1st Edition, Chapman & Hall, 1994

[5] D.L. Cohn *Measure Theory*, Birkhäuser, 1997

[6] J. Protter, P. Protter *Probability Essentials*, 2nd Edition, Springer, 2003

[7] M. Adams, V.G. Guillemin *Measure Theory and Probability*, Wadsworth & Brooks, 1986

[8] D. Duffie *Dynamic Asset Pricing Theory*, 3rd Edition, Princeton University Press, 2001

[9] I. Karatzas, S.E. Shreve *Brownian Motion and Stochastic Calculus*, 2nd Edition, Springer, 1991

[10] E.B. Saff, A.D. Snider *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 2nd Edition, Prentice Hall, 1993

[11] N. Dunford, J.T. Schwartz *Linear Operators, Part I, General Theory*, Wiley Clasics Library Edition, Wiley, 1988

[12] J.W. Brown, R.V. Churchill *Fourier Series and Boundary Value Problems*, 5th Edition, McGraw-Hill , 1993

[13] L. Breiman *Probability*, 2nd Edition, Society for Industrial and Applied Mathematics, 1992

[14] W. Rudin *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill, 1976

[15] R. Ellis, D. Gulick *Calculus with Analytic Geometry*, 4th Edition, Harcourt Brace Jovanovich College Publishers, 1990

[16] C. Clark *Elementary Mathematical Analysis*, 2nd Edition, Brooks/Cole Publishing Company, 1982