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ORIGINAL PAPER

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Spectrum of the axially loaded Timoshenko rod

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Abstract The response to excitation in the axially loaded Timoshenko rod is greatly influenced by the type of axial load, which could be compressive or tensile. The results presented in this article have practical applications, particularly when the axial load is compressive, as non-positive eigenvalues indicate possible buckling. The mathematical model leads to an eigenvalue problem. Abstract theory from a recent article is presented and applied, demonstrating that partial sums of modal solutions converge to a solution of the model problem. This enables one to predict the response of a given structure to excitation.

1 Introduction

The mathematical model for the axially loaded Timoshenko rod is presented in Sect. 2 of this article. The term "rod" is a generic term describing a cable, beam, wire, etc. (The use of the term "rod" is also made in [1–3], for example.) The axially loaded Timoshenko rod model differs from the classical Timoshenko model discussed in [4–6] (where numerous other references are listed). This axial load, which could be compressive or tensile, is not merely a minor variation on the standard Timoshenko model. A compressive force leads to a completely different response to excitation compared to a tensile force. The results presented here are useful for applications. This is especially true for the compressive case where the results have implications for modelling the phenomenon of buckling. If the axial force is zero, the model reduces to the standard Timoshenko model (making the standard model a special case).

It is generally accepted that a structure can be understood and its behaviour predicted when the natural frequencies and modes of vibration are known. The practical side of (experimental) modal analysis is treated in engineering textbooks (see e.g. [7, Chapter 7]). Modal analysis in theory follows the same line of thinking as in practice. The mathematical model leads to an eigenvalue problem, and partial sums of modal solutions are supposed to converge to a solution of the model problem. This enables one to predict the response of a given structure (or body) to excitation. For the vibrating string model, this procedure is not only well known, but the theoretical foundation can be found in textbooks (see, for example, [8]). In [9], an approach to approximate a solution for a general vibration problem by using partial sums of modes is presented. In the process, it becomes clear that the validity of this approximation depends on the completeness of the eigenfunction sequence.

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In computing the eigenvalues and eigenfunctions, we followed the approach in [4], which is completely different to the approach in [5]—both methods are correct. The required modifications for this article were not trivial and quite a challenge. Also, more attention was paid to rigour.

As discussed in [5], the vibration spectrum consists of two vibration mode types. The authors of [5] criticise the practice of some researchers who disregard the eigenfunctions (and, of course, eigenvalues) associated with the so-called "second spectrum". Furthermore, the authors of [4] emphasise that the entire spectrum is needed to present mathematically correct results. One should not ignore any eigenfunctions outright. In fact, ongoing research indicates that the second spectrum is essential when examining solutions of a related nonlinear model. In practice, however, engineers may choose to truncate the series representing the solution and thereby ignore negligible terms. For further details in this regard, see Corollary 5.4.

If the axial force is set equal to zero, the results in the article [4] are obtained as expected and interesting comparisons can be made. A compressive axial load, if large enough, results in non-positive eigenvalues, which is unrealistic for the linear vibration model. This suggests that buckling of the rod is a possibility, but the mathematical model should be extended to include nonlinear effects (see Sect. 8). A large tensile axial force results in large positive eigenvalues. This describes the stiffening effect which occurs. The extent of the stiffening effect may allow for model simplification, where, for example, an Euler–Bernoulli model may be used. In extreme cases, one could even consider the vibrating string model.

The authors of [4] reference [10] with regard to the completeness of the sequence of eigenfunctions, but do not provide any detail. It is not easy to show completeness of eigenfunctions in model problems and [10] considers only simple applications.

The abstract theory of completeness of eigenvector sequences is not easy to apply. In [9], the main strategy is to show the equivalence of the abstract result to another abstract result but this time for a variational formulation instead of one for an abstract linear operator. The theory in [9] then requires conditions on the variational form and bilinear forms for the results to be true. These requirements and associated estimates were established for the rod model in this article (see Sects.3.1 and 5.1). An unexpected benefit of the theory is that it is helpful when eigenvalues and eigenfunctions are calculated.

This article begins by presenting the mathematical model in Sect. 2, as well as the eigenvalue problem to be considered. Sections 3–5 are more theoretical. In Sect. 3, modal analysis and theory from [9] are discussed. Some elementary inequalities are considered in Sect. 4, which are used in Sect. 5 to apply the theory. The general properties of eigenfunctions are described in Sect. 6, where the strategy of finding the solution of the eigenvalue problem is given. In Sects. 7 and 8, the existence and distribution of eigenvectors and eigenvalues of a pinned–pinned rod are discussed. Section 9 is devoted to studying the eigenvectors and eigenvalues of a rod with cantilever and clamped–clamped boundary conditions. Finally, we conclude and mention possible future research in Sect. 10.

2 Model: Timoshenko rod with axial force

In this section, we consider the model for the Timoshenko rod with axial force S (known beforehand). The term "rod" is a generic name which can refer to a range of structures from a sturdy beam to a highly flexible wire. The model includes two partial differential equations, where w represents the relative transverse displacement, ϕ the angle of rotation of a cross section, ℓ the rod length, A the cross-sectional area, and ρ the constant density. The continuous function representing the axial load, S, is not necessarily constant, but for transverse vibration, it is assumed that $\partial_t S = 0$. In the case where $S = 0$, the model is the same as that of the standard Timoshenko rod (discussed in [4–6], for example). Denoting the transverse load by P and the area moment of inertia by I, the equations of motion are

$$\rho A \partial_t^2 w = \partial_x \left(S \partial_x w + V \right) + P, \tag{1}$$

$$\rho I \partial_t^2 \phi = V + \partial_x M. \tag{2}$$

The constitutive equations for the shear force V and bending moment M are

$$V = \kappa^2 A G \left(\partial_x w - \phi \right), \tag{3}$$

$$M = E I \partial_x \phi. \tag{4}$$

Here, κ^2 denotes the shear coefficient, G the shear modulus, and E the Young's modulus.

The standard Timoshenko model was first introduced by S.P. Timoshenko in the articles [11] and [12] in the 1920 s. Numerous other derivations of the standard Timoshenko model can be found in books and articles, for example [13, p337-8], [14, p323-4], [15, p392-5], [7, p337-8].

It has recently come to public attention, as noted in [16], that S.P. Timoshenko collaborated with P. Ehrenfest in deriving what is known as the Timoshenko theory. Some argue that it should be referred to as the Timoshenko–Ehrenfest theory. However, considering Ehrenfest's significant contributions to Physics and to avoid confusion with the well-established terminology, this article will continue to use the term Timoshenko theory.

The introduction of S in Equation (1) appears quite natural and is not new. However, careful explanations are given in [17] and [18], where semi-linear models are compared to the linear one. A continuum mechanics-based derivation can be found in [19], where nonlinear effects are considered. The reader is also referred to a recent book [20] where different beam models are treated with a variety of applications. Article [21] promises to be useful for the calculation of natural frequencies and mode shapes for composite beams. Related articles [22] and [23] deal with nonlinear problems where large displacements and rotations are considered.

In this article, we consider the model in dimensionless form. To derive this, we define scaled variables ξ and τ for distance and time:

$$\xi = \frac{x}{\ell} \quad \text{and} \quad \tau = \frac{t}{t_0}, \quad \text{where} \quad t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}. \tag{5}$$

Next, the dimensionless relative displacement is defined, along with the angle of rotation, which was already dimensionless in the previous notation:

$$\hat{w}(\xi,\tau) = \frac{w(x,t)}{\ell}, \qquad \hat{\phi}(\xi,\tau) = \phi(x,t).$$

Lastly, we scale the moment and forces as follows:

$$\widehat{M}(\xi,\tau) = \frac{M(x,t)}{\ell AG\kappa^2}, \quad \widehat{P}(\xi,\tau) = \frac{\ell P(x,t)}{AG\kappa^2}, \quad \widehat{S}(\xi) = \frac{S(x)}{AG\kappa^2}, \quad \widehat{V}(\xi,\tau) = \frac{V(x,t)}{AG\kappa^2}.$$

The substitutions above are applied to the equations of motion (1)–(2) and the constitutive equations (3)–(4) to obtain the dimensionless form. Since there is no room for confusion, we will use the x and t notation from the start, noting that the dimensionless length of the rod is now 1.

Dynamic Model Problem
The equations of motion are

$$\begin{aligned} \partial_t^2 w &= \partial_x\left(S\partial_x w + V\right) + P, \\ \tfrac{1}{\alpha}\partial_t^2\phi &= V + \partial_x M, \end{aligned}$$

and the constitutive equations are

$$\begin{aligned} M &= \tfrac{1}{\beta}\partial_x\phi, \\ V &= \partial_x w - \phi. \end{aligned}$$

The dimensionless constants are

$$\alpha = \frac{A\ell^2}{I} \quad \text{and} \quad \beta = \frac{AG\kappa^2\ell^2}{EI}, \quad \text{where} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}.$$

Remark 1 The parameters α and β are bounded and piecewise continuous, with a positive minimum, and do not vary greatly. For a realistic model, $\alpha > \beta > 1$ and $\gamma < \frac{1}{2}$. In [5] and [6], the authors consider $\alpha = 4800$, $\beta = 1538.461538$.

An important special case of the model is where S is a constant, which is considered in this article from Sect. 6 onward.

We consider the following boundary conditions (with $t > 0$):

$$w(0,t) = \partial_x \phi(0,t) = w(1,t) = \partial_x \phi(1,t) = 0; \tag{6}$$

$$w(0,t) = \phi(0,t) = S(1)\partial_x w(1,t) + V(1,t) = M(1,t) = 0; \tag{7}$$

$$w(0,t) = \phi(0,t) = w(1,t) = \phi(1,t) = 0. \tag{8}$$

The boundary conditions (6), (7) and (8) describe a pinned–pinned (hinged–hinged) rod, cantilever (clamped-free) rod, and clamped–clamped rod, respectively. Other boundary conditions are possible, but (6), (7) and (8) are sufficient to illustrate the theory.

Modal analysis of the rod leads to an eigenvalue problem.

Problem E

Find a pair $\langle w, \phi\rangle$ *and a number* λ *such that* $\langle w, \phi\rangle \neq 0$ *and*

$$-((1+S)w')' + \phi' - \lambda w = 0, \tag{9}$$

$$-\left(\frac{1}{\beta}\phi'\right)' + \phi - w' - \frac{\lambda}{\alpha}\phi = 0, \tag{10}$$

with one of the following sets of boundary conditions:

Pinned–pinned rod: $w(0) = \phi'(0) = w(1) = \phi'(1) = 0$;
Cantilever rod: $w(0) = \phi(0) = (1+S)w'(1) - \phi(1) = \phi'(1) = 0$;
Clamped–clamped rod: $w(0) = \phi(0) = w(1) = \phi(1) = 0$.

Remark 2 It is possible to decouple the system of equations in the Dynamic Model Problem into a system involving a fourth-order differential equation, which considers the deflection (or angle of rotation), as well as a second-order differential equation, linking the angle of rotation to the deflection. If this is done, then the fourth-order problem cannot be used in isolation. The second-order "link" must be retained to eliminate redundancy in the solutions. This is demonstrated in [5] and [6].

The variational form of Problem E is used for the theory (and can be used for finite element calculation). To derive it, we multiply Eqs.(9) and (10) by arbitrary functions v_1 and v_2 in $C^1[0,1]$ and integrate over $[0,1]$. The result is then added to form one equation. Some additional notation is required.

The generic notation T_P is used throughout to denote the set of test functions which is specific to given boundary conditions. This allows for the variational form of different problems to be defined using one notation. Let

$$T_1[0,1] = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\},$$

$$T_2[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}.$$

Then, for the boundary conditions mentioned in Problem E, there are three variations for T_P:

Case 1 (Pinned–pinned rod): $T_P = T_1[0,1] \times C^1[0,1]$;
Case 2 (Cantilever rod) $T_P = T_2[0,1] \times T_2[0,1]$;
Case 3 (Clamped–clamped rod) $T_P = T_1[0,1] \times T_1[0,1]$.

Using integration by parts, the variational form of Problem E follows. (We use the inner product notation (u, v), where $(u, v) = \int_0^1 uv$.)

Problem EV Find $\langle w, \phi\rangle \in T_P$ such that $\langle w, \phi\rangle \neq 0$ and, for each $\langle v_1, v_2\rangle \in T_P$,

$$\left(\frac{1}{\beta}\phi', v_2'\right) + \left(w' - \phi, v_1' - v_2\right) + (Sw', v_1') = \lambda\left((w, v_1) + \left(\frac{1}{\alpha}\phi, v_2\right)\right). \tag{11}$$

A solution $\langle\langle w, \phi\rangle, \lambda\rangle$ of Problem EV is referred to as an eigensolution, which is also a solution of Problem E.

3 Modal analysis

The reader may browse over Sects. 3, 4 and 5 for a first reading and start with Sect. 6.

3.1 Modal solutions

In this section, we discuss the theoretical foundation for modal analysis given in [9]. The authors state the theory in terms of a general vibration problem and associated eigenvalue problem in variational form using Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$, where $\mathcal{V} \subset \mathcal{W}$. The authors consider the general vibration problem, Problem G, below.

Problem G Find u such that for each $t > 0$, $u(t) \in \mathcal{V}$, $u'(t) \in \mathcal{V}$, $u''(t) \in \mathcal{W}$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = 0 \qquad \text{for each } v \in \mathcal{V},$$

with initial conditions $u(0) = u_0$ and $u'(0) = u_d$.

The bilinear form $a(\cdot, \cdot)$ represents damping, which is not considered in this article. Our dynamic model problem can be written in variational form (see, for example, [24]) and is a special case of Problem G.

To discuss the relevant results of [9], consider the eigenvalue problem, Problem GE, which is obtained by substituting $u(t) = T(t)\tilde{u}$ as a possible solution of Problem G.

Problem GE Find $\tilde{u} \in \mathcal{V}$, such that $\tilde{u} \neq 0$ and

$$b(\tilde{u}, v) = \lambda c(\tilde{u}, v) \text{ for each } v \in \mathcal{V}. \tag{12}$$

Remark 3 A solution $\langle \tilde{u}, \lambda \rangle$ consists of an eigenvector (abstract problem) or eigenfunction (application) $\tilde{u}$ and corresponding eigenvalue λ.

For natural numbers n, if $\langle \tilde{u}_n, \lambda_n \rangle$ is a solution of Problem GE and T_n is a solution of $T_n'' + \lambda_n T_n = 0$, then partial sums $u_m(t) = \sum_{n=1}^{m} T_n(t)\tilde{u}_n$ may be constructed with the idea that these will converge to the solution of Problem G as $m \to \infty$. This can only be considered if the sequence of eigenvectors $(\tilde{u}_n)$ is complete.

Definition 1 [Completeness] In a Hilbert space $\mathcal{H}$, an orthonormal sequence (x_k) is said to be complete if $v = \sum_{k=1}^{\infty} \langle v, x_k \rangle x_k$ for each $v \in \mathcal{H}$.

To prove the existence of a complete sequence of eigenvectors, the following assumptions are made in [9].

Fundamental assumptions

$\mathcal{W}$ is a Hilbert space with inner product c and induced norm $\| \cdot \|_{\mathcal{W}}$; $\mathcal{V}$ is a Hilbert space with inner product b and induced norm $\| \cdot \|_{\mathcal{V}}$; there exists a positive constant C_b such that $\|v\|_{\mathcal{W}} \leq C_b \|v\|_{\mathcal{V}}$ for each $v \in \mathcal{V}$; $\mathcal{V}$ is a dense subset of $\mathcal{W}$ and the embedding of $\mathcal{V}$ into $\mathcal{W}$ is compact.

It is shown that a linear operator T exists such that the eigenvalue problem (12) can be rewritten as

$$T\tilde{u} = \frac{1}{\lambda}\tilde{u}.$$

By the theory of symmetric linear compact operators (see [10] or [25]), the following properties are true for eigenvalues and eigenvectors of T: T has a complete, orthonormal sequence of eigenvectors (x_n) with corresponding positive eigenvalues (μ_n); each eigenspace is finite-dimensional and the sequence of eigenvalues (μ_n) is decreasing, with $\mu_n \to 0$ as $n \to \infty$.

It is not trivial to apply these abstract results to a model problem such as the one in Sect. 2. In [9], the results are applied to Problem GE above and the theorems below are proved.

Theorem 3.1 *The following statements are true:*

The eigenvalue problem (12) has a complete orthonormal sequence of eigenvectors $(\tilde{u}_n) \subset \mathcal{W}$ with corresponding positive eigenvalues (λ_n);
Each eigenspace is finite-dimensional;
The sequence of eigenvalues (λ_n) is increasing and $\lambda_n \to \infty$ as $n \to \infty$.

Remark 4 For each $w \in \mathcal{W}$, it holds that $w = \sum_{n=1}^{\infty} c(w, \tilde{u}_n)\tilde{u}_n$.

Theorem 3.2 *The sequence of eigenvectors $(\tilde{u}_n)$, which is complete in $\mathcal{W}$, is also complete in $\mathcal{V}$.*

In [9], the authors refer also to [26], where the spectral theory for elliptic operators is studied in the form (12). The main result of the present article is to show that Theorems 3.1 and 3.2 can be applied to Problem EV.

3.2 Weak derivatives

Sobolev space theory is required to apply the theory in the previous subsection. Fortunately, we need only the one-dimensional case. In the literature consulted [25,27,28], weak derivatives and other properties of functions are presented for functions defined on subsets of $\mathbb{R}^n$. It is assumed that functions defined on an interval of the real line can be treated as a special case. However, stronger results are sometimes possible with fewer assumptions.

The space $\mathcal{L}^2(0,1)$ is considered to be the set of functions whose square is Lebesgue integrable on the interval $[0,1]$. The inner product for $\mathcal{L}^2(0,1)$ is $(f,g) = \int_0^1 (fg)$, with induced norm $\|f\| = (f,f)^{\frac{1}{2}}$.

A function u is an element of the space $H^1(0,1)$ if it has a weak derivative Du where $(Du,\phi) = (-u,\phi')$ for each $\phi \in C^\infty(0,1)$ with compact support. It is well known that if $u \in H^1(0,1)$, then there exists a sequence $u_n \in C^1[0,1]$ such that $\|u_n - u\| \to 0$ and $\|u_n' - Du\| \to 0$ (see, e.g. [27, Chapter 2]). This result is referred to as the density of $C^1[0,1]$ in $H^1(0,1)$. If a function u has weak derivatives up to order m in $\mathcal{L}^2(0,1)$, it is considered to be in the space denoted by $H^m(0,1)$. $H^m(0,1)$ is a Hilbert space with inner product $(u,v)_m = \sum_{k=0}^m (D^k u, D^k v)$ and induced norm $\|u\|_m$. In particular, if $m = 1$, then $H^1(0,1)$ is a Hilbert space with inner product $(u,v)_1 = (u,v) + (Du,Dv)$ and induced norm $\|u\|_1$.

For the one-dimensional case, it is convenient to think of a function in $H^m(0,1)$ having continuous derivatives up to order $m-1$. That is, if $u \in H^1(0,1)$, then there exists a unique continuous function u^* such that $u = u^*$ $a.e.$, and hence, u may be considered to be continuous. Also, for $x \in [0,1]$, one may define $u(x) = u^*(x)$. These results follow from the density of $C^1[0,1]$ in $H^1(0,1)$.

At the end of this section, Sobolev's embedding theorem is stated. Some preliminary results are presented first.

Suppose $f \in C^1[0,1]$ with maximum and minimum M and m, respectively. Using the fundamental theorem of calculus and the Cauchy–Schwartz inequality, it follows that

$$M - m \leq \|f'\|. \tag{13}$$

Also, $\|f\|_{\sup} \leq \sqrt{2}\|f\|_1$ and, if f has a zero, $\|f\|_{\sup} \leq \|f'\|$. These inequalities can be extended to $H^1(0,1)$ due to the fact that $C^1[0,1]$ is dense in $H^1(0,1)$. Let $u \in H^1(0,1)$ and define $M = \max u^*$ and $m = \min u^*$. If

$$M > 0 \geq m \;\text{ with }\; |m| < M, \tag{14}$$

it follows that $\|u\|_{\sup} \leq \|Du\|$ (which is well known) and that $m \geq M - \|Du\|$. Also, $\|u^*\|_{\sup} \leq \sqrt{2}\|u\|_1$.

Since the interval being considered has length one, it follows that for any $u \in H^1(0,1)$, $m^2 \leq \|u\|^2 \leq M^2$ and hence (using (13))

$$\|u\| - m \leq \|Du\|. \tag{15}$$

Also,

$$\|u\| \leq \|u^*\|_{\sup}. \tag{16}$$

The result below is a Poincaré type inequality which follows from (16). If $u \in H^1(0,1)$ has a zero in $[0,1]$, then

$$\|u\| \leq \|Du\|. \tag{17}$$

Alternatively, if $|m| > M$ in (14), consider the function $v = -u$, where $-m > 0 > -M$. The results (15) - (17) are still obtained since $\|v\| = \|u\|$.

Remark 5 If $\|u\| > \|Du\|$, then $\|u\|_{\sup} > \|Du\|$ (by (16)). This, in turn, implies that u does not have a zero and $\min u = m > 0$.

We conclude this subsection with Sobolev's embedding theorem (presented in [27, Chapter 3], for example). For $u \in C^m[0,1]$, we use the notation $\|u\|_{\sup}^m = \sum_{k=0}^m \|u^{(k)}\|_{\sup}$.

Sobolev's embedding theorem in one dimension If $u \in H^m(0,1)$ then there exists a unique $u^* \in C^{m-1}[0,1]$ such that $u = u^*$ a.e. and

$$\|u^*\|_{\sup}^{m-1} \leq m\sqrt{2}\|u\|_m.$$

4 The energy space and inertia space

To apply the theory of Sect. 3 to the dynamic model problem, the spaces $\mathcal{W}$ and $\mathcal{V}$ need to be constructed. They are referred to as the inertia space and energy space, respectively.

4.1 Estimates

The product space

$$\mathcal{L}^2(0,1)^2 = \mathcal{L}^2(0,1) \times \mathcal{L}^2(0,1)$$

features prominently in the theory. A natural inner product for the space is

$$(f,g)^{(2)} = (f_1, g_1) + (f_2, g_2)\,,$$

and the space is complete since $\mathcal{L}^2(0,1)$ is complete. Let $\|\cdot\|_{\mathcal{L}^2}$ denote the norm induced by $(\cdot,\cdot)^{(2)}$.

Also required is the product space

$$H^k(0,1)^2 = H^k(0,1) \times H^k(0,1),$$

where $k \in \mathbb{N}$, with inner product

$$(f,g)_k^{(2)} = (f_1, g_1)_k + (f_2, g_2)_k\,.$$

The Sobolev space $H^k(0,1)^2$ is complete since $H^k(0,1)$ is complete. For f and g in $H^1(0,1)^2$ (in particular),

$$(f,g)_1^{(2)} = (f_1, g_1) + (Df_1, Dg_1) + (f_2, g_2) + (Df_2, Dg_2)\,.$$

Let $\|\cdot\|_{H^1}$ denote the norm induced by $(\cdot,\cdot)_1^{(2)}$.

Next, recall that α and β are piecewise continuous on the closed interval $[0,1]$ and therefore bounded. Also, recall that $\alpha > \beta > 1$.

Bilinear forms From Equation (11) in Problem EV, we identify two bilinear forms. For f and g in $\mathcal{L}^2(0,1)^2$ and u and v in T_P, define the following symmetric bilinear forms:

$$c_T(f,g) = (f_1, g_1) + \left(\frac{1}{\alpha} f_2, g_2\right),$$

$$b_T(u,v) = \left(\frac{1}{\beta} u_2', v_2'\right) + \left(u_1' - u_2, v_1' - v_2\right).$$

Theorem 4.1 *The bilinear form c_T is an inner product for the space $\mathcal{L}^2(0,1)^2$ and the norm induced by c_T is equivalent to $\|\cdot\|_{\mathcal{L}^2}$.*

Proof The fact that c_T is an inner product for $\mathcal{L}^2(0,1)^2$ follows easily from the fact that $\alpha > 1$ and c_T is composed of inner products on $\mathcal{L}^2(0,1)$. Let $u \in \mathcal{L}^2(0,1)^2$. Then

$$\frac{1}{\alpha_{\max}} \|u\|_{\mathcal{L}^2}^2 \le c_T(u,u) \le \|u\|_{\mathcal{L}^2}^2,$$

where $\alpha_{\max}$ denotes the maximum of α on $[0,1]$. □

The inner product c_T for $\mathcal{L}^2(0,1)^2$ will prove to be useful.

Definition 2 [Inertia Space] The space $\mathcal{L}^2(0,1)^2$ with norm induced by c_T, denoted $\|\cdot\|_{\mathcal{W}}$, is referred to as the inertia space $\mathcal{W}$.

Notation. The closure of T_P in $H^1(0,1)^2$ (with respect to the norm $\|\cdot\|_{H^1}$) is denoted by $\mathcal{V}$.

The energy space for each of the three cases of boundary conditions is now characterised. To do this, the notation $\overline{A}$ is used to denote the closure of the set A. Also, note that for two sets A and B, $\overline{A \times B} = \overline{A} \times \overline{B}$. For Case 1, $\mathcal{V} = \overline{T_1[0,1]} \times H^1(0,1)$, for Case 2, $\mathcal{V} = \overline{T_2[0,1]} \times \overline{T_2[0,1]}$ and for Case 3, $\mathcal{V} = \overline{T_1[0,1]} \times \overline{T_1[0,1]}$.

The bilinear form b_T is now extended to $\mathcal{V}$: for u and v in $\mathcal{V}$,

$$b_T(u,v) = \left(\frac{1}{\beta} Du_2, Dv_2\right) + (Du_1 - u_2, Dv_1 - v_2)\,.$$

The following well-known estimates are stated for easy reference (see [29], for example). Recall that $\beta > 1$. For any $u \in \mathcal{V}$, $u_1(0) = 0$ and

$$\|u_1\|^2 \leq \|Du_1\|^2 \leq 2\left(\|Du_1 - u_2\|^2 + \|u_2\|^2\right) \leq 2\beta_{\max}\left(\|Du_1 - u_2\|^2 + \frac{1}{\beta_{\max}}\|u_2\|^2\right), \tag{18}$$

where $\beta_{\max}$ denotes the maximum of β on $[0,1]$.

Proposition 4.1 *For Cases 2 and 3, if $u \in \mathcal{V}$, then*

$$\|u\|_{\mathcal{W}}^2 \leq (2\beta_{\max} + 1)\, b_T(u,u).$$

Proof Let $u = \langle u_1, u_2\rangle \in \mathcal{V}$. Then, since $u_2(0) = 0$, $\|u_2\| \leq \|Du_2\|$. Therefore, by (18) it follows that for $u \in \mathcal{V}$,

$$\begin{aligned}
\|u\|_{\mathcal{W}}^2 &= \|u_1\|^2 + \left(\frac{1}{\alpha}u_2, u_2\right) \\
&\leq 2\beta_{\max} b_T(u,u) + \left(\frac{1}{\beta}Du_2, Du_2\right) \\
&\leq (2\beta_{\max} + 1) b_T(u,u).
\end{aligned}$$

The result above does not hold for Case 1 (pinned–pinned rod).

Theorem 4.2 *There exists a nonzero real number c^2 such that*

$$c^2\|u\|_{\mathcal{W}}^2 \leq b_T(u,u) \quad \textit{for any } u \in \mathcal{V}. \tag{19}$$

Proof In Cases 2 and 3, $c^2 = (2\beta_{\max} + 1)^{-1}$ by Proposition 4.1.

In Case 1, a direct proof appears to be impossible. For this reason, a proof is given by contradiction. The proof is based on the idea given in appendix of [30]. Let $\mathcal{S}$ denote the unit sphere with centre 0 in $\mathcal{W}$. In order to prove (19), it is required to prove that there exists a nonzero real number c^2 such that

$$b_T(u,u) \geq c^2 \quad \text{for each } u \in \mathcal{V} \cap \mathcal{S}. \tag{20}$$

Suppose not. Then there exists a sequence $(u^n) = (\langle w_n, \phi_n\rangle)$ contained in $\mathcal{V} \cap \mathcal{S}$ such that $b(u^n, u^n) \to 0$ as $n \to \infty$.

In this case, it is true that

$$\|D\phi_n\| \to 0 \;\text{ and }\; \|Dw_n - \phi_n\| \to 0. \tag{21}$$

For each $n \in \mathbb{N}$, w_n and ϕ_n are in $H^1(0,1)$ since (u^n) is contained in $\mathcal{V} \cap \mathcal{S}$. Then, by (21), for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\|Dw_n - \phi_n\| < \varepsilon. \tag{22}$$

In the remainder of the proof, we show that $w_n(1) \neq 0$, which is a contradiction.

Since w_n has a zero in $[0,1]$, it follows from (17) and (22) that for $n > n_0$,

$$\|w_n\| \leq \|Dw_n\| \leq \|Dw_n - \phi_n\| + \|\phi_n\| < \varepsilon + \|\phi_n\|$$

and hence

$$\|w_n\|^2 < 2\varepsilon^2 + 2\|\phi_n\|^2. \tag{23}$$

By (23), since $u^n \in \mathcal{S}$,

$$1 < \|\phi_n\|^2 + 2\varepsilon^2 + 2\|\phi_n\|^2 = 3\|\phi_n\|^2 + 2\varepsilon^2. \tag{24}$$

Choose $\varepsilon = \frac{1}{8}$.

Then, since $u^n \in \mathcal{S}$, (24) implies that there exists $n_1 \in \mathbb{N}$ such that for $n > n_1$

$$\frac{1}{4} < \|\phi_n\|^2 \leq 1.$$

Therefore, since $\|D\phi_n\| \to 0$ (as mentioned in (21)), for $n > n_1$

$$\|D\phi_n\| < \frac{1}{2} < \|\phi_n\|.$$

Note that, for $n > n_1$, (by (15)) $\|\phi_n\| - m_n \leq M_n - m_n \leq \|D\phi_n\| < \frac{1}{8}$. That is, $m_n > \frac{3}{8}$. Consequently,

$$\int_0^1 \phi_n \geq m_n > \frac{3}{8}. \tag{25}$$

Using the Cauchy–Schwartz inequality,

$$\left| \int_0^1 Dw_n - \int_0^1 \phi_n \right| \leq \int_0^1 |Dw_n - \phi_n| \leq \|Dw_n - \phi_n\| < \frac{1}{8} \tag{26}$$

for $n > n_1$. Combining (25) and (26) yields

$$\frac{1}{4} = \frac{3}{8} - \frac{1}{8} < \int_0^1 Dw_n.$$

But this implies that $w_n(1) > w_n(0) = 0$, which contradicts the fact that $u^n \in \mathcal{V}$. Consequently, the estimate (20) holds and hence also (19). □

Corollary 4.1 *The bilinear form b_T is an inner product for the space $\mathcal{V}$.*

Proof By Theorem 4.2, b_T is a positive definite symmetric bilinear form. □

Theorem 4.3 *For any $u \in \mathcal{V}$,*

$$\|Du_1\|^2 \leq 2\left(1 + \frac{\alpha_{\max}}{c^2}\right) b_T(u, u),$$

Proof From (18), for any $u \in \mathcal{V}$,

$$\begin{aligned}
\|Du_1\|^2 &\leq 2\left(\|Du_1 - u_2\|^2 + \|u_2\|^2\right) \\
&\leq 2\left(\|Du_1 - u_2\|^2 + \alpha_{\max}\|u\|_{\mathcal{W}}^2\right) \\
&\leq 2b_T(u, u) + \frac{2\alpha_{\max}}{c^2} b_T(u, u),
\end{aligned}$$

by Theorem 4.2. □

To apply Theorems 3.1 and 3.2, the weak variational form of the problem is considered and the theory applied.

We begin by characterising the required spaces. Recall the Hilbert space $\mathcal{L}^2(0,1)^2$ with the norm $\|\cdot\|_{\mathcal{L}^2}$ (induced by $(\cdot,\cdot)^{(2)}$). In addition, the space $\mathcal{V}$ denotes the closure of the test functions T_P in $H^1(0,1)^2$ with respect to the norm $\|\cdot\|_{H^1}$ (induced by $(\cdot,\cdot)_1^{(2)}$). Also, the space $\mathcal{W}$ denotes the space $\mathcal{L}^2(0,1)^2$ with norm $\|\cdot\|_{\mathcal{W}}$ (induced by c_T).

Since S is continuous, the bilinear form (Sf, g) is bounded in $\mathcal{L}^2(0,1)$. That is, there exists a positive real number μ_S such that

$$|(Sf,g)| \le \mu_S \|f\|\|g\| \tag{27}$$

for each pair $\langle f, g\rangle \in \mathcal{L}^2(0,1)^2$. Since T_P is dense in $\mathcal{V}$, a solution of Problem EV is a solution of Problem EW below.

Problem EW Find λ and $z \in \mathcal{V}$, $z \neq 0$, such that for each $v \in \mathcal{V}$,

$$b_T(z,v) + (SDz_1, Dv_1) = \lambda c_T(z,v).$$

Now, consider Problem EW and let

$$b(u,v) = b_T(u,v) + (SDu_1, Dv_1).$$

If $(SDu_1, Du_1) \ge 0$, then b is an inner product for $\mathcal{V}$ (by Corollary 4.1). Let $\|\cdot\|_{\mathcal{V}}$ denote the norm induced by the inner product b.

Definition 3 [Energy Space] The space $\mathcal{V}$ equipped with the norm $\|\cdot\|_{\mathcal{V}}$ is referred to as the energy space $\mathcal{V}$.

5 Application to the eigenvalue problem

5.1 Existence of a complete sequence of eigenfunctions

To apply Theorem 3.1, we need the next result.

Proposition 5.1 *The norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{V}}$ are equivalent on $\mathcal{V}$.*

Proof Let $u \in \mathcal{V}$. Then, since $\beta > 1$,

$$\begin{aligned}\|u\|_{\mathcal{V}}^2 &\le \|Du_2\|^2 + 2(\|Du_1\|^2 + \|u_2\|^2) + (SDu_1, Du_1)\\ &\le 2\left(\|Du_2\|^2 + \|Du_1\|^2 + \|u_2\|^2\right) + \mu_S\|Du_1\|^2\\ &\le (2+\mu_S)\|u\|_{H^1}^2.\end{aligned}$$

Also, using Theorems 4.2 and 4.3, it follows that

$$\begin{aligned}\|u\|_{H^1}^2 &\le \alpha_{\max}\|u\|_{\mathcal{W}}^2 + \|Du_1\|^2 + \|Du_2\|^2\\ &\le \left(\frac{2\alpha_{\max}}{c^2} + 2\left(1 + \frac{\alpha_{\max}}{c^2}\right) + \beta_{\max}\right)\|u\|_{\mathcal{V}}^2.\end{aligned}$$

Corollary 5.1 *The energy space $\mathcal{V}$ with inner product b is complete.*

The general theory from Sect. 3 can therefore be applied using the spaces $\mathcal{W}$ and $\mathcal{V}$ with inner products c_T and b, respectively. That is, the results of Theorems 3.1 and 3.2 hold for Problem EW in the case where $(SDu_1, Du_1) \ge 0$. Thus, we conclude that the main result has been proved for both the classical Timoshenko problem (in [4], [5] and [6]) and the axially loaded Timoshenko problem where the axial load is tensile.
If $(SDu_1, Du_1) < 0$, then the arguments above need not be true and we consider a slightly different problem. First, an estimate must be derived.

Proposition 5.2 *Suppose $(SDu_1, Du_1) < 0$. For each $u \in \mathcal{V}$,*

$$b_T(u,u) + (SDu_1, Du_1) \ge (1 - 2\mu_S)\, b_T(u,u) - 2\mu_S\|u_2\|^2.$$

Proof By (18) and (27), it follows that for $u \in \mathcal{V}$,

$$\begin{aligned} b_T(u,u) + (SDu_1, Du_1) &\geq b_T(u,u) - \mu_S\|Du_1\|^2 \\ &\geq b_T(u,u) - 2\mu_S\left(\|Du_1 - u_2\|^2 + \|u_2\|^2\right) \\ &= (1-2\mu_S)b_T(u,u) - 2\mu_S\|u_2\|^2 + 2\mu_S\left(\frac{1}{\beta}Du_2, Du_2\right) \\ &\geq (1-2\mu_S)\,b_T(u,u) - 2\mu_S\|u_2\|^2. \end{aligned}$$

For u and v in $\mathcal{V}$, let

$$b^+(u,v) = b_T(u,v) + (SDu_1, Dv_1) + 2\alpha_{\max}c_T(u,v).$$

Proposition 5.3 *If* $(Sf, f) < 0$ *and* $\mu_S \leq \frac{1}{2}$, *then the bilinear form* b^+ *is an inner product for* $\mathcal{V}$.

Proof The symmetry and bilinearity of b^+ is trivial to show. Note that $2\alpha_{\max}c_T(u,u) \geq 2\|u_2\|^2$, and hence, by Proposition 5.2,

$$b^+(u,u) \geq (1-2\mu_S)b_T(u,u) + (2-2\mu_S)\|u_2\|^2 \geq 0. \tag{28}$$

Also, if $b^+(u,u) = 0$, then from (28), $b_T(u,u) = 0$ and therefore $u = 0$ (by Theorem 4.2). □

Let $\|\cdot\|_{\mathcal{V}+}$ denote the norm induced by the inner product b^+.

Proposition 5.4 *The norms* $\|\cdot\|_{H^1}$ *and* $\|\cdot\|_{\mathcal{V}+}$ *are equivalent.*

Proof Let $u \in \mathcal{V}$. Then, from (28) and Proposition 5.1,

$$\begin{aligned} \|u\|^2_{\mathcal{V}+} &\leq \|u\|^2_{\mathcal{V}} + \mu_S\|Du_1\|^2 + 2\alpha_{\max}\|u\|^2_{\mathcal{W}} \\ &\leq 2\|u\|^2_{H^1} + \mu_S\|Du_1\|^2 + 2\alpha_{\max}\|u\|^2_{\mathcal{W}} \\ &\leq (2 + \mu_S + 2\alpha_{\max})\|u\|^2_{H^1}. \end{aligned}$$

Also, by Proposition 5.1, it follows that

$$\|u\|^2_{H^1} \leq K^*\|u\|^2_{\mathcal{V}} \leq K^*\|u\|^2_{\mathcal{V}+},$$

where $K^* = \left(\frac{2\alpha_{\max}}{c^2} + 2\left(1 + \frac{\alpha_{\max}}{c^2}\right) + \beta_{\max}\right)$. □

Corollary 5.2 *The space* $\mathcal{V}$ *with* $\|\cdot\|_{\mathcal{V}+}$ *is complete, since* $H^1(0,1)^2$ *is complete and the norms are equivalent.*

Assumption In the remainder of this article, it is assumed that if $(SDu_1, Du_1) < 0$, then $\mu_S < \frac{1}{2}$.
Theorems 3.1 and 3.2 can now be applied on the spaces $\mathcal{W}$ and $\mathcal{V}$ with inner products c_T and b^+, respectively. Denote the complete orthonormal sequence of eigenvectors by (u_n) and the corresponding positive eigenvalues by λ_n^+.

Then for a given natural number n,

$$b^+(u_n, v) = \lambda_n^+ c_T(u_n, v) \qquad \text{for each} \quad v \in \mathcal{V}$$

if and only if

$$b(u_n, v) = (\lambda_n^+ - 2\alpha_{\max})c_T(u_n, v) \qquad \text{for each} \quad v \in \mathcal{V}.$$

Note that $b(u_n, v) = 0$ if and only if $c_T(u_n, v) = 0$. The main theorem of the article can now be stated.

Theorem 5.1 *Suppose the Fundamental assumptions made in Sect. 3 are met. Then,*

a. *Problem EW has a complete orthonormal sequence of eigenvectors in* $\mathcal{W}$ *with corresponding eigenvalues* $\lambda_k > -2\alpha_{\max}$;

b. *there are at most a finite number of negative eigenvalues and the eigenfunctions can be ordered such that* (λ_k) *is increasing and* $\lambda_k \to \infty$ *as* $k \to \infty$*;*
c. *the sequence of eigenvectors is complete in* $\mathcal{V}$*;*
d. *each eigenspace has finite dimension.*

Corollary 5.3 *Suppose* (u_n) *is a complete sequence of eigenvectors in* $\mathcal{W}$ *(or* $\mathcal{L}^2(0,1)^2$*). Then each* $w \in \mathcal{W}$ *can be written as* $w = \sum_{n=1}^{\infty} c_T(w, u_n)u_n$*.*

Corollary 5.4 *[Parseval] The error in truncating the series is given by* $\|w\|_W^2 - \sum_{n=1}^{N} c_T(w, u_n)^2$*.*

More detail on the error of approximation can be found in [9, Section 5].

Remark 6 The multiplicity of an eigenvalue is equal to the dimension of the eigenspace.

5.2 Regularity

Theorem 5.2 *For a solution* $\langle u, \lambda\rangle$ *of Problem EW,* $(1+S)Du_1$ *and* $\frac{1}{\beta}Du_2$ *are weakly differentiable. Also,* $\langle u, \lambda\rangle$ *is a solution of Problem E (where weak derivatives replace classical derivatives).*

Proof Recall that $\mathcal{V} \subset H^1(0,1)^2$. Let $\langle u, \lambda\rangle$ be a solution of Problem EW. That is, $u \in \mathcal{V}$ is such that for each $v \in \mathcal{V}$,

$$b_T(u, v) + (SDu_1, Dv_1) = \lambda c_T(u, v).$$

Suppose $v \in T_P$ and consider the cases $\langle v_1, 0\rangle$ and $\langle 0, v_2\rangle$. Then, by the definitions of b_T and c_T,

$$\begin{aligned} (Du_1 - u_2, v_1') + (SDu_1, v_1') &= \lambda\,(u_1, v_1) \text{ and} \\ \left(\frac{1}{\beta}Du_2, v_2'\right) - (Du_1 - u_2, v_2) &= \lambda\left(\frac{1}{\alpha}u_2, v_2\right). \end{aligned}$$

That is,

$$\begin{aligned} ((1+S)Du_1 - u_2, v_1') &= -\,(-\lambda u_1, v_1) \text{ and} \\ \left(\frac{1}{\beta}Du_2, v_2'\right) &= -\left(u_2 - Du_1 - \frac{\lambda}{\alpha}u_2, v_2\right). \end{aligned}$$

This implies that the weak derivatives of $(1+S)Du_1 - u_2$ and $\frac{1}{\beta}Du_2$ exist. Since $u \in H^1(0,1)^2$ and $(1+S)Du_1 - u_2 + u_2 = (1+S)Du_1$, the weak derivative of $(1+S)Du_1$ exists. Furthermore,

$$D\left((1+S)Du_1\right) = Du_2 - \lambda u_1, \tag{29}$$

$$D\left(\frac{1}{\beta}Du_2\right) = u_2 - Du_1 - \frac{\lambda}{\alpha}u_2. \tag{30}$$

That is, $\langle u, \lambda\rangle$ is a solution of Problem E using weak derivatives. □

Remark 7 If S, α and β are sufficiently smooth, then the weak derivatives in Equations (29) and (30) may be replaced by classical derivatives and a solution $\langle u, \lambda\rangle$ of Problem EW is a classical solution of Problem E as formulated in Sect. 2.

An important special case of Problem E is where α, β and S are constants. In this case, each solution of Problem EW is a classical solution of Problem E and happens to be infinitely differentiable. Consequently, more results can be achieved using the theory of ordinary differential equations, as shown in the next section. In particular, if $S = S_0$ is constant, then $|(S_0 f, f)| = |S_0|\|f\|^2 \le \mu_S\|f\|^2$ (when $|S_0| \le \mu_S$). If $|S_0| > \mu_S$, then the results obtained thus far may not be true. From a mathematical point of view, one may consider $|S_0| < 1$ instead of $|S_0| < \frac{1}{2}$. From a practical point of view, $\frac{1}{2}$ is very large due to the "scaling" (see Sect. 2), and therefore, $|S_0| \ge \frac{1}{2}$ need not be considered.

6 Properties of eigenfunctions

In the remainder of the article, we assume that all parameters are constant and derive methods to calculate eigenvectors and eigenvalues. In particular, it is assumed that the axial force S is constant ($S = S_0$, where $S_0 > -1$). In the case where $S_0 = 0$, the results presented are valid for the standard Timoshenko rod and may be compared to [4–6].

We take a similar approach to that in [4]. Problem E is a two-dimensional system of linear second-order ordinary differential equations. The idea is to derive a representation of the general solution and then enforce the boundary conditions. The method followed in solving this system is justified by the theory of linear differential equations and systems of differential equations (see [31] or [32], for example).

6.1 General solution of the differential equation

Suppose the function $\langle w(x), \phi(x)\rangle = e^{mx}\bar{u}$ is a solution of the system of differential equations given in Problem E, with λ treated as a known parameter. This is possible if and only if

$$\begin{bmatrix} -(S_0+1)m^2-\lambda & m \\ -\alpha m & -\frac{1}{\gamma}m^2+(\alpha-\lambda) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{31}$$

In order to find a non-trivial solution, it is necessary that the coefficient matrix in (31) is singular. That is,

$$(S_0+1)m^4 + (-S_0\beta + \lambda(1+(S_0+1)\gamma))\, m^2 + \gamma\lambda(\lambda-\alpha) = 0. \tag{32}$$

Equation (32) is of the form $ax^2 + bx + c = 0$, with

$$\begin{aligned} a &= S_0 + 1, \\ b(\lambda) &= -S_0\beta + \lambda(1+(S_0+1)\gamma) \\ &= S_0\beta\left(\frac{\lambda}{\alpha}-1\right) + \lambda\left(1+\frac{\beta}{\alpha}\right), \\ c(\lambda) &= \gamma\lambda(\lambda-\alpha). \end{aligned}$$

It must be emphasised that λ is treated as a given parameter to derive a general solution. It is clear from the theory in Sect. 5 that the case $S_0 < 0$ should be considered carefully, while $S_0 \geq 0$ enables one to adapt the approach in [4].

In the computations that follow, we consider the solution for different values of λ. This is necessary since the general solution has a different form for different values of λ. The results are summarised in Theorem 6.1.

We know (from Sect. 5) that all eigenvalues are positive for $S_0 \geq 0$ and therefore need not consider the possibility that $\lambda \leq 0$ for the general solution of the ODE in this case. The general solution is known when $S_0 = 0$ (see [4] and [5]). For $-1 < S_0 < 0$, all values of λ for the general solution are considered, with special consideration for the case where $\lambda < 0$.

Since $a > 0$, it is not difficult to derive possibilities for solutions by investigating b and c. If λ is in $[0, \alpha]$, then $c \leq 0$, with $c(0) = c(\alpha) = 0$. The graph of b is a straight line with $b(0) = -S_0\beta$ and $b(\alpha) = \alpha + \beta$. This implies that (32) has roots $m^2 = 0$ and $m^2 = \frac{S_0\beta}{S_0+1}$ for $\lambda = 0$ and roots $m^2 = 0$ and $m^2 = -\frac{\alpha+\beta}{S_0+1}$ for $\lambda = \alpha$. Although zero cannot be an eigenvalue if $S_0 > 0$, it is instructive to consider it. If $S_0 > 0$, then the two roots m^2 of (32) are 0 and $\mu^2 = \frac{S_0\beta}{S_0+1} > 0$. In contrast, if $-1 < S_0 < 0$, the two roots m^2 of (32) are 0 and $-\omega^2 = \frac{S_0\beta}{S_0+1} < 0$. The four roots of Equation (32) are 0 (twice) and two others, denoted by $\pm\omega i$. Two linearly independent solutions of Problem E with $\lambda = 0$ can be found by inspection:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

The general solution is therefore

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} x \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C\begin{bmatrix} \sin\omega x \\ -\frac{-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D\begin{bmatrix} \cos\omega x \\ \frac{-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.$$

Next, consider $0 < \lambda < \alpha$. For any $S_0 > -1$, $c < 0$ and the roots m^2 of (32) may be denoted by $\mu^2 > 0$ and $-\omega^2 < 0$. The four roots of Equation (32) are therefore $\pm\mu$ and $\pm\omega i$. In this case, the general solution is

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} \sinh \mu x \end{bmatrix}$$
$$+ \; C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix},$$

where μ and ω are functions of λ which depend on S_0.

Additionally, consider $\lambda = \alpha$. For any $S_0 > -1$, the two roots m^2 of (32) are 0 and $-\omega^2 = -\frac{\alpha+\beta}{S_0+1} < 0$. Therefore, the four roots of Equation (32) are 0 (twice) and two others, denoted by $\pm\omega i$. Two linearly independent solutions of Problem E with $\lambda = \alpha$ can be found by inspection:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha x \end{bmatrix}.$$

The general solution is therefore

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ \alpha x \end{bmatrix} + C \begin{bmatrix} \sin \omega x \\ -\frac{\alpha-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\alpha-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}.$$

Consider now $\lambda > \alpha$. Then, for any $S_0 > -1$, $b > 0$ and $c > 0$ and hence the roots m^2 of (32) are denoted by $-\theta^2 < 0$ and $-\omega^2 < 0$, where θ and ω are functions of λ which depend on S_0. The four roots of Equation (32) are $\pm\theta i$ and $\pm\omega i$. The general solution is therefore

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sin \theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta} \cos \theta x \end{bmatrix} + B \begin{bmatrix} \cos \theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta} \sin \theta x \end{bmatrix}$$
$$+ \; C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}.$$

We now consider $\lambda < 0$. This is only relevant if $S_0 < 0$. As mentioned before, this case requires more careful consideration.

Considering Equation (32) as a quadratic equation of m^2, the discriminant is

$$\begin{aligned} \Delta &= (-S_0\beta + \lambda(1 + (S_0+1)\gamma))^2 - 4(S_0+1)\gamma\lambda(\lambda - \alpha) \\ &= S_0^2\beta^2 + 2S_0\lambda\beta - 2(S_0+1)\beta\lambda\gamma S_0 + \lambda^2 - 2(S_0+1)\lambda^2\gamma + (S_0+1)^2\gamma^2\lambda^2 + 4\lambda\beta \\ &= (S_0\beta + \lambda\,(1 - (S_0+1)\gamma))^2 + 4\lambda\beta. \end{aligned} \tag{33}$$

Note that $\Delta > 0$ for $\lambda \geq 0$. Also, $b(d) = 0$, where $d = \frac{S_0\beta}{1+(S_0+1)\gamma}$ and since $\lambda < 0$, it follows that $c > 0$. Therefore, if $\lambda = d$, then $\Delta < 0$. But Δ is a quadratic function of λ with a positive leading term:

$$\Delta = (1 - (S_0+1)\gamma)^2\,\lambda^2 + 2\beta\,(S_0(1 - (S_0+1)\gamma) + 2)\,\lambda + S_0^2\beta^2.$$

Since $\Delta > 0$ if $\lambda \geq 0$ (as shown in (33)), Δ has two negative real roots. Let the root closest to zero be λ_0. Then, for $\lambda > \lambda_0$, $\Delta > 0$ and the quadratic equation for m^2, Equation (32), has distinct real roots.

Also, b is increasing and therefore since $d < \lambda_0$, $b > 0$ for $\lambda \in (\lambda_0, 0)$. That is, the roots m^2 of Equation (32) are $-\theta^2 < 0$ and $-\omega^2 < 0$, where θ and ω are functions of λ which depend on S_0. The four roots of Equation (32) can be denoted as $\pm\theta i$ and $\pm\omega i$ and the general solution is

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sin \theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta} \cos \theta x \end{bmatrix} + B \begin{bmatrix} \cos \theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta} \sin \theta x \end{bmatrix}$$
$$+ \; C \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} + D \begin{bmatrix} \cos \omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}.$$

It should be noted that ω, μ and θ are uniquely determined by λ:

$$-\omega^2(\lambda) = \frac{1}{2(S_0+1)}\left(S_0\beta - \lambda(1+(1+S_0)\gamma) - \Delta^{\frac{1}{2}}\right), \tag{34}$$

$$\mu^2(\lambda) = \frac{1}{2(S_0+1)}\left(S_0\beta - \lambda(1+(1+S_0)\gamma) + \Delta^{\frac{1}{2}}\right) \text{ for } 0 < \lambda < \alpha, \tag{35}$$

$$-\theta^2(\lambda) = \frac{1}{2(S_0+1)}\left(S_0\beta - \lambda(1+(1+S_0)\gamma) + \Delta^{\frac{1}{2}}\right) \text{ for } \lambda_0 < \lambda \le 0 \text{ or } \lambda > \alpha. \tag{36}$$

The forms of the general solution of the system in Problem E are summarised below. For the three cases where $\lambda > 0$, the results are very similar to [4].

Five intervals for the value of λ have been considered. In each case, four linearly independent solutions have been found. The solution space of Problem E, however, is four-dimensional since it can be written as a system of four linear first-order equations. The following theorem therefore follows.

Theorem 6.1 *In each of the five cases $\lambda_0 < \lambda < 0$, $\lambda = 0$, $\lambda < \alpha$, $\lambda = \alpha$ and $\lambda > \alpha$, respectively, a general solution for the system of differential equations in Problem E exists and has the forms given below, where ω, μ and θ are given by (34) – (36).*

The cases $\lambda_0 < \lambda < 0$ and $\lambda > \alpha$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + B\begin{bmatrix} \cos\theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta}\sin\theta x \end{bmatrix}$$
$$+\ C\begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D\begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.$$

The case $\lambda = 0$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} x \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C\begin{bmatrix} \sin\omega x \\ -\frac{-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D\begin{bmatrix} \cos\omega x \\ \frac{-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.$$

The case $0 < \lambda < \alpha$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} \sinh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\cosh\mu x \end{bmatrix} + B\begin{bmatrix} \cosh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\sinh\mu x \end{bmatrix}$$
$$+\ C\begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D\begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.$$

The case $\lambda = \alpha$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ \alpha x \end{bmatrix} + C\begin{bmatrix} \sin\omega x \\ -\frac{\alpha-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} + D\begin{bmatrix} \cos\omega x \\ \frac{\alpha-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.$$

In [5] and [6], the system of equations describing a Timoshenko rod (where $S_0 = 0$) is decoupled, resulting in a fourth-order differential equation involving the deflection and a second-order differential equation linking the angle of rotation to it. The authors then solve the fourth-order equation but retain the "link" to ensure that there is no redundancy in the solutions.

Remark 8 Although the form of the general solution is different for each of the frequency intervals considered, no general conclusions about the form of the solution can be made. This is because substitution of the boundary conditions may eliminate some of the terms.

6.2 Determining eigenvalues and eigenfunctions

To determine the eigenfunctions and eigenvalues of Problem E, we apply first the boundary conditions at zero. This causes the dimension of the solution space of the system of differential equations to be reduced to two. Any eigenspace, therefore, can have a dimension of at most two. We demonstrate the reduction in the solution space for a rod free at zero in the case where $0 < \lambda < \alpha$. Detail for other boundary conditions is given in the sections that follow. Substitution of boundary conditions at zero yields solutions where two of the four coefficients can be written in terms of the other two. Then, any of the three boundary conditions considered can be applied at $x = 1$ to solve for the remaining coefficients.

For $0 < \lambda < \alpha$, we substitute the boundary conditions of a rod free at zero ($w'(0) - \phi(0) = \phi'(0) = 0$) to reduce the dimension of the solution space to two. That is,

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A \begin{bmatrix} \sinh \mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} \cosh \mu x \end{bmatrix} + B \begin{bmatrix} \cosh \mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} \sinh \mu x \end{bmatrix} \\ &\quad + A \frac{\omega(\lambda+S_0\mu^2)}{\mu(\lambda-S_0\omega^2)} \begin{bmatrix} \sin \omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} - B \frac{\lambda+(S_0+1)\mu^2}{\lambda-(S_0+1)\omega^2} \begin{bmatrix} \cos \omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}.\end{aligned}$$

If the boundary conditions at $x = 1$—clamped ($w(1) = \phi(1) = 0$), free ($w'(1) - \phi(1) = \phi'(1) = 0$) or pinned ($w(1) = \phi'(1) = 0$)—are applied, the following system is obtained and the other two coefficients can be solved for:

$$\begin{bmatrix} d_1(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_2(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \\ d_3(\mu(\lambda), \omega(\lambda), \theta(\lambda)) & d_4(\mu(\lambda), \omega(\lambda), \theta(\lambda)) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{37}$$

If the coefficient matrix of (37) is nonzero, then λ is an eigenvalue if and only if the determinant

$$d_1 d_4 - d_2 d_3 = 0.$$

Note that the solution space, and therefore the eigenspace, would have dimension two only if the coefficient matrix of (37) was a zero matrix. If this were to occur, λ would be referred to as a double eigenvalue.

For a rod free at $x = 0$ and all cases where $\lambda > \lambda_0$, whether the rod is clamped or pinned at $x = 1$, there is at least one nonzero entry in the coefficient matrix in (37). Therefore, all eigenvalues for the rod free at $x = 0$ are simple.

More detail on the computation of eigenvalues and eigenfunctions for a clamped–clamped or cantilever rod is given in Sect. 9, while a pinned–pinned rod is considered in Sects. 7 and 8.

7 Computation of eigenvalues and eigenfunctions for a pinned–pinned rod

We follow the approach in the previous section with a rod that has pinned boundary conditions at $x = 0$ and $x = 1$.

The theorem below shows that, amongst other things, each eigenvalue λ, where $\lambda > \lambda_0$, can be associated with a non-constant eigenfunction $[w\ \phi]^T = [\sin(k\pi x)\ A_k \cos(k\pi x)]^T$, where k is a natural number and A_k is a constant which depends on k and λ.

Theorem 7.1 *The following statements are true for a rod with pinned–pinned boundary conditions.*

a. The value α is an eigenvalue of the pinned–pinned problem with the corresponding eigenfunction $[0\ 1]^T$.

b. If $[w\ \phi]^T$ is a non-constant eigenfunction of the pinned–pinned problem, then given $S_0 > -1$, it follows that a sequence of eigenfunctions is given by

$$\begin{bmatrix} w_k(x) \\ \phi_k(x) \end{bmatrix} = \begin{bmatrix} \sin(k\pi x) \\ A_k \cos(k\pi x) \end{bmatrix}, \tag{38}$$

with k a natural number and A_k a constant depending on k and λ_k.

c. All eigenvalues λ, where $0 \le \lambda < \alpha$, are simple.

Proof a. Simple substitution shows that for $\lambda = \alpha$, $[0\ 1]^T$ is a solution of the pinned–pinned problem.

b. If pinned boundary values at zero ($w(0) = \phi'(0) = 0$) are substituted into the general equations, then the dimension of the solution space is reduced to two. Then, applying the pinned boundary conditions at $x = 1$ ($w(1) = \phi'(1) = 0$), a system of equations is obtained, which may be solved to find the remaining coefficients and thus the eigenfunctions. Details of this will only be shown for the cases $\lambda_0 < \lambda < 0$ (which includes $\lambda > \alpha$, since they are the same) and $0 < \lambda < \alpha$. The other cases ($\lambda = 0$ and $\lambda = \alpha$) are similar, but slightly simpler and are left to the reader. Recall that ω, μ and θ are distinct, nonzero values. *The case* $\lambda_0 < \lambda < 0$ *(and* $\lambda > \alpha$*)*: Substitution of pinned boundary values at zero results in

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix}.$$

Applying the pinned boundary conditions at $x = 1$ results in the following system

$$\begin{bmatrix} \sin\theta & \sin\omega \\ \left(\lambda-(S_0+1)\theta^2\right)\sin\theta & \left(\lambda-(S_0+1)\omega^2\right)\sin\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{39}$$

When (39) is solved, it follows that $C\sin\omega = A\sin\theta = 0$. This is true if $A = 0$ and ω is a multiple of π or $C = 0$ and θ is a multiple of π. It follows that for $\lambda_0 < \lambda < 0$ (and $\lambda > \alpha$), a sequence of eigenfunctions of the given form exists. *The case* $0 < \lambda < \alpha$: Substitution of pinned boundary values at zero results in

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} \sinh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\cosh\mu x \end{bmatrix} + C \begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix}.$$

Applying the pinned boundary conditions at $x = 1$ results in the following system

$$\begin{bmatrix} \sinh\mu & \sin\omega \\ \left(\lambda+(S_0+1)\mu^2\right)\sinh\mu & \left(\lambda-(S_0+1)\omega^2\right)\sin\omega \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{40}$$

When equation (40) is solved, it follows that $A = 0$, and therefore, for a non-constant solution it is required that ω is a multiple of π. That is, for $0 < \lambda < \alpha$, the sequence of eigenfunctions has the given form.

c. From the proof of (b.), it can easily be seen that the solution space is one-dimensional when $0 \le \lambda < \alpha$.

Remark 9 If $\lambda > \alpha$ or $\lambda_0 < \lambda < 0$, then a double eigenvalue is possible if there exist natural numbers j and k such that $j = \frac{\theta}{\pi}$ and $k = \frac{\omega}{\pi}$.

Next, we consider the distribution of λ_k and A_k. Suppose $[w_k\ \phi_k]^T$ is a non-constant eigenfunction of the pinned–pinned problem. Substituting (38) into Problem E, it follows that $[w_k\ \phi_k]^T$ is a solution of the system if and only if the pair $\langle A_k, \lambda_k\rangle$ is a solution of

$$(S_0+1)k^2\pi^2 - A_k k\pi = \lambda_k, \tag{41}$$

$$\frac{k^2\pi^2 A_k}{\gamma} - \alpha k\pi + \alpha A_k = \lambda_k A_k. \tag{42}$$

From Equation (42), it is clear that $A_k \neq 0$. If $\lambda_k = \alpha$, then it follows from Equations (41) and (42) that a necessary condition for α to be a double eigenvalue is

$$S_0 = \frac{\beta+\alpha}{k^2\pi^2} - 1. \tag{43}$$

If there exists a natural number k such that (43) holds, then the eigenvalue $\lambda_k = \alpha$ depends on the elastic constants and the dimensions of the rod.

A necessary condition for 0 to be an eigenvalue is

$$S_0 = \frac{-k^2\pi^2}{k^2\pi^2+\beta}. \tag{44}$$

To determine the distribution of eigenvalues, we substitute Equation (41) into (42). It follows that

$$A_k^2 + \left(k\pi\left(\frac{1}{\gamma} - (S_0+1)\right) + \frac{\alpha}{k\pi}\right)A_k - \alpha = 0, \tag{45}$$

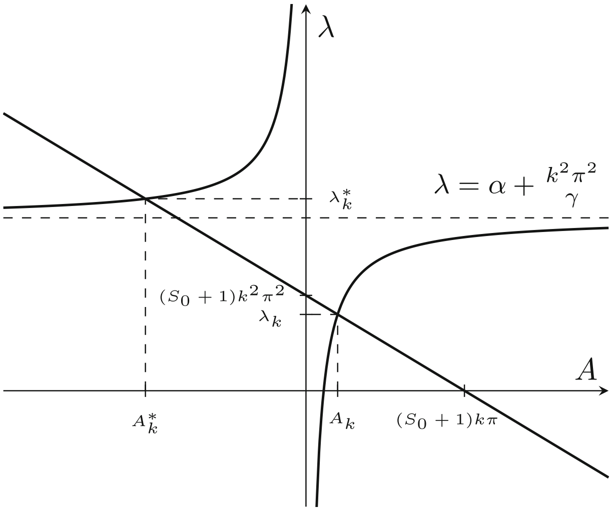

Fig. 1 Graphical representation of λ_k and A_k

$$(S_0+1)k^2\pi^2 - A_k k\pi = \lambda_k. \tag{46}$$

Notice that although λ_k depends on A_k and S_0 for each natural number k, A_k depends only on S_0 by Equation (45). This creates an advantage in further computations. From Equation (45), we obtain two different real values for A_k, resulting in two different values for λ_k.

$$A_k = \frac{k^2\pi^2\left(S_0+1-\gamma^{-1}\right)-\alpha}{2k\pi} \pm \sqrt{\frac{\left(k^2\pi^2\left(S_0+1-\gamma^{-1}\right)-\alpha\right)^2}{4k^2\pi^2}+\alpha}, \tag{47}$$

$$\lambda_k = (S_0+1)k^2\pi^2 - A_k k\pi. \tag{48}$$

It is therefore clear that the system has two solutions $\langle A_k, \lambda_k\rangle$ and $\left\langle A_k^*, \lambda_k^*\right\rangle$, with $A_k^* < 0 < A_k$. Using Equations (41)–(42), graphs may be sketched to visualise the solutions (see Fig. 1). The graphs are similar to those in [4], except that the intercepts of the straight line depend on S_0. Let λ_k^* denote the larger eigenvalue. The authors in [4] refer to the value λ_k as an eigenvalue of Type 1 and to λ_k^* as one of Type 2.

Remark 10 It is interesting to note that the “change" between Type 1 and Type 2 eigenvalues (or the dependence on α) does not depend on S_0, while the value of the eigenvalues themselves do depend on S_0.

Theorem 7.2 *All eigenvalues less than α are Type 1 eigenvalues.*

Proof Equations (45) and (46) imply that

$$A_k\left(\lambda_k - (S_0+1)k^2\pi^2\right) = A_k\left(-A_k k\pi\right) = \frac{A_k k^2\pi^2}{\gamma} - (S_0+1)k^2\pi^2 A_k + \alpha A_k - \alpha k\pi.$$

Therefore,

$$\frac{\alpha k\pi}{A_k} = \frac{k^2\pi^2}{\gamma} + \alpha - \lambda_k.$$

The same is true for $\left\langle A_k^*, \lambda_k^*\right\rangle$. From Equation (47), $A_k^* < 0 < A_k$. Therefore,

$$\frac{k^2\pi^2}{\gamma} + \alpha - \lambda_k^* < 0 < \frac{k^2\pi^2}{\gamma} + \alpha - \lambda_k$$

and hence

$$\lambda_k < \frac{k^2\pi^2}{\gamma} + \alpha < \lambda_k^*.$$

Hence, any eigenvalue less than α is a Type 1 eigenvalue. □

Remark 11 Although all eigenvalues less than α are Type 1, there are Type 1 eigenvalues that are greater than α (when $k > \frac{1}{\pi}\sqrt{\frac{\alpha+\beta}{S_0+1}}$). These eigenvalues intersperse the Type 2 eigenvalues and both sequences tend to infinity, as shown below.

Theorem 7.3 *If* $|S_0| < 1$, *then*

a. $\lim_{k\to\infty} \dfrac{A_k}{k\pi} = 0$, *while* $\lim_{k\to\infty} \dfrac{A_k^*}{k\pi} = -\left(\dfrac{1}{\gamma} - S_0 - 1\right)$.

b. $\lim_{k\to\infty} \dfrac{\lambda_k}{k^2\pi^2} = S_0 + 1$ *and* $\lim_{k\to\infty} \dfrac{\lambda_k^*}{k^2\pi^2} = \dfrac{1}{\gamma}$.

Proof a. Note that since $|S_0| < 1$ and $\gamma < \frac{1}{2}$ (see Sect. 2), it follows that $S_0 + 1 - \frac{1}{\gamma} < 0$. Therefore,

$$\frac{A_k}{k\pi} = -\frac{k^2\pi^2\left(\frac{1}{\gamma} - S_0 - 1\right) + \alpha}{2k^2\pi^2} + \sqrt{\frac{\left(k^2\pi^2\left(\frac{1}{\gamma} - S_0 - 1\right) + \alpha\right)^2}{4k^4\pi^4} + \frac{\alpha}{k^2\pi^2}},$$

$$\frac{A_k^*}{k\pi} = -\frac{k^2\pi^2\left(\frac{1}{\gamma} - S_0 - 1\right) + \alpha}{2k^2\pi^2} - \sqrt{\frac{\left(k^2\pi^2\left(\frac{1}{\gamma} - S_0 - 1\right) + \alpha\right)^2}{4k^4\pi^4} + \frac{\alpha}{k^2\pi^2}}.$$

The limits as $k \to \infty$ may be found by inspection.

b. Equation (48), together with part (a.) of this proof, yields the required result.

Corollary 7.1 *As* $k \to \infty$, $\lambda_k \to \infty$ *and* $\lambda_k^* \to \infty$.

8 Existence and distribution of negative eigenvalues for a pinned–pinned rod

As mentioned in Sect. 1, the existence of non-positive eigenvalues suggests buckling. We therefore investigate the existence and distribution of non-positive eigenvalues for a rod with pinned–pinned boundary conditions. (The phenomenon of buckling is discussed briefly at the end of this section.)

Suppose non-positive eigenvalues exist. One may want to investigate how many there are. To this end, consider λ as a function with domain the positive real line and let $\lambda(n) = 0$ for some non-negative real number n. Then, using Eqs. (47) and (48), it follows that

$$n = 0 \quad \text{or} \quad n = \sqrt{\frac{-S_0\beta}{\pi^2(S_0+1)}}.$$

Thus, for natural numbers k, the first k eigenvalues, where $k < \sqrt{\frac{-S_0\beta}{\pi^2(S_0+1)}}$, are negative.

We now investigate the values of the axial force S_0 that result in non-positive eigenvalues. From Eq. (44), it can be seen that there exists a value for S_0 such that $\lambda_1 = 0$. Denoting this value by S_{crit}, it follows that

$$S_{crit} = \frac{-\pi^2}{\pi^2 + \beta}.$$

Note that $-1 < S_{crit} < 0$.

It is convenient to consider Eqs. (41) and (42) in finding an upper bound for S_0 resulting in negative eigenvalues. Notice that $\lambda_k < 0$ if and only if $A_k > (S_0 + 1)k\pi$ (by Eq. (41)). Also, by Eq. (42), $\lambda_k < 0$ if and only if $k^2\pi^2 - \frac{\beta k\pi}{A_k} + \beta < 0$. That is, $\lambda_k < 0$ if and only if $S_0 < \frac{-k^2\pi^2}{k^2\pi^2+\beta} \leq S_{crit}$.

If negative eigenvalues exist, the sequence (λ_k) is not necessarily increasing while $\lambda_k < 0$. However, since there are finitely many negative eigenvalues, they may be calculated and then re-ordered to have this property. The graphs of $\lambda(k)$ are demonstrated in Fig. 2 for $S_0 < S_{crit}$, $S_0 = S_{crit}$ and $S_0 \geq S_{crit}$.

One should note that if the compressive force exceeds the critical load, it does not necessarily lead to buckling. However, it is possible that the yield stress of the material may be exceeded.

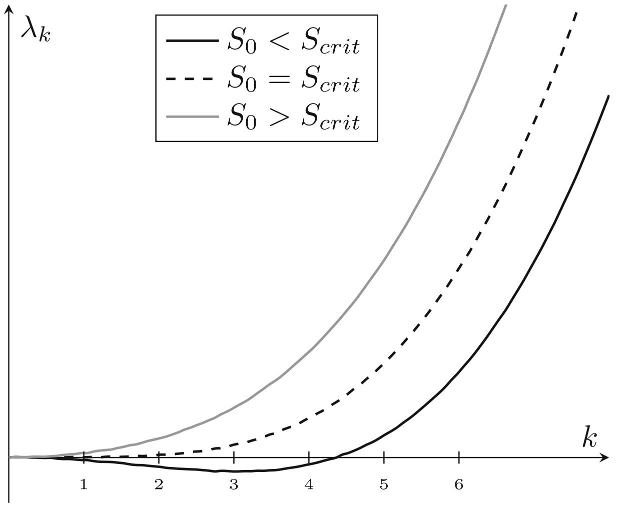

Fig. 2 Graph demonstrating $\lambda(k)$ (where $k \in \mathbb{R}^+$), with differing values for S_0

To conclude this section, we reference a related nonlinear model presented in [17], where the phenomenon of buckling is treated. If the axial force S in the dynamic model problem in Sect. 2 is replaced by the nonlinear force $S(t) = S_0 + \frac{1}{2\gamma}\int_0^1 (\partial_x w(x,t))^2\,dx$, where S_0 is a constant, then the resulting model is qualitatively the same as in [17]. (The authors derived their model from two-dimensional elasticity.) They transformed the problem into a fourth-order system and then calculated possible non-trivial equilibria. This demonstrates the existence of buckled states. Recently, we obtained similar results by considering the aforementioned variation on the dynamic model problem. Our findings formed part of a presentation at the 2024 SANUM conference in Stellenbosch, South Africa, titled "Existence of and properties of solutions of a semi-linear rod model".

9 Computation of eigenvalues and eigenfunctions for a rod with cantilever or clamped–clamped boundary conditions

In Sect. 6, the general solution for the differential equations in Problem E was found. The strategy for finding eigenvectors for specific boundary values was also demonstrated. We now follow this strategy by first substituting the boundary conditions of a rod clamped at zero ($w(0) = \phi(0) = 0$) into the general solution (Theorem 6.1) to reduce the dimension of the solution space from four to two, as follows.

The cases $\lambda_0 < \lambda < 0$ *and* $\lambda > \alpha$:

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A\begin{bmatrix} \sin\theta x \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta}\cos\theta x \end{bmatrix} + B\begin{bmatrix} \cos\theta x \\ \frac{\lambda-(S_0+1)\theta^2}{\theta}\sin\theta x \end{bmatrix} \\ &\quad - A\frac{\omega(\lambda-(S_0+1)\theta^2)}{\theta(\lambda-(S_0+1)\omega^2)}\begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} - B\begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}.\end{aligned}$$

The case $\lambda = 0$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A\begin{bmatrix} x \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{A}{(S_0+1)\omega}\begin{bmatrix} \sin\omega x \\ (S_0+1)\omega\cos\omega x \end{bmatrix} - B\begin{bmatrix} \cos\omega x \\ -(S_0+1)\omega\sin\omega x \end{bmatrix}.$$

The case $0 < \lambda < \alpha$:

$$\begin{aligned}\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} &= A\begin{bmatrix} \sinh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\cosh\mu x \end{bmatrix} + B\begin{bmatrix} \cosh\mu x \\ \frac{\lambda+(S_0+1)\mu^2}{\mu}\sinh\mu x \end{bmatrix} \\ &\quad + A\frac{\omega(\lambda+(S_0+1)\mu^2)}{\mu(\lambda-(S_0+1)\omega^2)}\begin{bmatrix} \sin\omega x \\ -\frac{\lambda-(S_0+1)\omega^2}{\omega}\cos\omega x \end{bmatrix} - B\begin{bmatrix} \cos\omega x \\ \frac{\lambda-(S_0+1)\omega^2}{\omega}\sin\omega x \end{bmatrix}. \qquad (49)\end{aligned}$$

The case $\lambda = \alpha$:

$$\begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ \alpha x \end{bmatrix}$$
$$+ \frac{A\omega}{\alpha - (S_0+1)\omega^2} \begin{bmatrix} \sin \omega x \\ -\frac{\alpha-(S_0+1)\omega^2}{\omega} \cos \omega x \end{bmatrix} - B \begin{bmatrix} \cos \omega x \\ \frac{\alpha-(S_0+1)\omega^2}{\omega} \sin \omega x \end{bmatrix}.$$

For the clamped–clamped rod, the boundary values at $x = 1$ are $w(1) = \phi(1) = 0$. If these boundary values are substituted into the solution, an equation of the form $\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is found, where each d_i is a function of ω, μ and θ. We now consider the frequency equations so that the coefficients in the solution can be solved for. The systems of equations are as follows.

The cases $\lambda_0 < \lambda < 0$ *and* $\lambda > \alpha$:

$$\left[\left\{\begin{matrix} 11 \sin\theta - \frac{\omega(\lambda-(S_0+1)\theta^2)}{\theta(\lambda-(S_0+1)\omega^2)} \sin\omega & \cos\theta - \cos\omega \\ -\frac{\lambda-(S_0+1)\theta^2}{\theta} (\cos\theta - \cos\omega) & \frac{\lambda-(S_0+1)\theta^2}{\theta} \sin\theta - \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin\omega \end{matrix}\right.\right] \left[\left\{\begin{matrix} 11A \\ B \end{matrix}\right.\right] = \left[\left\{\begin{matrix} 110 \\ 0 \end{matrix}\right.\right].$$

For a non-trivial solution,

$$\left(-\frac{\lambda-(S_0+1)\omega^2}{\omega} - \frac{\omega(\lambda-(S_0+1)\theta^2)^2}{\theta^2(\lambda-(S_0+1)\omega^2)}\right) \sin\omega \sin\theta + 2\frac{\lambda-(S_0+1)\theta^2}{\theta}(1 - \cos\theta\cos\omega) = 0.$$

The case $\lambda = 0$:

$$\begin{bmatrix} 1 - \frac{1}{(S_0+1)\omega} \sin\omega & 1 - \cos\omega \\ 1 - \cos\omega & (S_0+1)\omega\sin\omega \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Recall from Sect. 6 that $\omega = \pm\sqrt{-\frac{S_0\beta}{S_0+1}}$. A necessary condition for zero to be an eigenvalue is that S_0 is chosen such that

$$(S_0+1)\omega\sin\omega + 2(\cos\omega - 1) = 0. \tag{50}$$

The case $0 < \lambda < \alpha$:

$$\begin{bmatrix} \sinh\mu + \frac{\omega(\lambda+(S_0+1)\mu^2)}{\mu(\lambda-(S_0+1)\omega^2)} \sin\omega & \cosh\mu - \cos\omega \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} (\cosh\mu - \cos\omega) & \frac{\lambda+(S_0+1)\mu^2}{\mu} \sinh\mu - \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin\omega \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution,

$$\begin{bmatrix} \sinh\mu + \frac{\omega(\lambda+(S_0+1)\mu^2)}{\mu(\lambda-(S_0+1)\omega^2)} \sin\omega & \cosh\mu - \cos\omega \\ \frac{\lambda+(S_0+1)\mu^2}{\mu} (\cosh\mu - \cos\omega) & \frac{\lambda+(S_0+1)\mu^2}{\mu} \sinh\mu - \frac{\lambda-(S_0+1)\omega^2}{\omega} \sin\omega \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again, recall from Sect. 6 that $\omega = \pm\sqrt{\frac{\alpha+\beta}{S_0+1}}$. Therefore, a necessary condition for α to be an eigenvalue is that S_0 is chosen such that

$$\frac{\alpha\omega}{\alpha - (S_0+1)\omega^2} \sin\omega + 2(\cos\omega - 1) = 0.$$

These frequency equations are qualitatively the same as those presented in [6], where $S_0 = 0$.

Theorem 9.1 *For the rod with clamped–clamped boundary conditions, all eigenvalues, where* $0 \le \lambda \le \alpha$, *are simple.*

Proof Eigenvalues of the problem are double if and only if the coefficient matrix $\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ is a zero matrix.

For the cases $\lambda = 0$ and $\lambda = \alpha$, the entries d_1 and d_4 cannot be zero simultaneously. Also, for case where $0 < \lambda < \alpha$, $d_2 \neq 0$ since $\mu \neq 0$. The result therefore follows. □

Existence of negative eigenvalues If we assume that $\lambda = 0$ is an eigenvalue, then we may only consider the form of the solution where $\lambda = 0$. If this is the case, then S_0 must be chosen to satisfy Equation (50) with $-\omega^2 = \frac{S_0\beta}{S_0+1}$. We refer to this choice of S_0 as the critical value. Alternatively, note that for any value of S_0 in the interval $(-1, 0)$, there exists a unique positive value for ω that satisfies Eq. (50). (Existence of such a solution can be seen by sketching graphs.) If S_0 is such that the solution of Eq. (50) is $\omega = \pm\sqrt{-\frac{S_0\beta}{S_0+1}}$, then $\lambda = 0$ is an eigenvalue. Furthermore, if S_0 is less than the critical value, then negative eigenvalues exist depending on β.

For the cantilever rod, the boundary values at $x = 1$ are $w'(1) - \phi(1) = \phi'(1) = 0$. To find the frequency equations, the same approach as the clamped–clamped case may be taken. This is quite a long and tedious process. However, eigenvalues λ can be searched for using the bisection method, where λ is such that the determinant of the coefficient matrix is zero.

It is common practice to approximate eigenvectors and eigenvalues using numerical methods as was done in [6]. These approximations may be substituted into the results found here to determine their accuracy by considering if $|d_1d_4 - d_2d_3|$ is within a desired confidence interval.

Theorem 9.2 *For the rod with cantilever boundary conditions, all eigenvalues are simple.*

Proof As in the proof of Theorem 9.1, we consider whether the matrix $\begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ is a zero matrix.

For the cases where $\lambda_0 < \lambda < 0$ and $\lambda > \alpha$,

$$d_2 = \frac{-\lambda + S_0\theta^2}{\theta}\sin\theta + \frac{\lambda - S_0\omega^2}{\omega}\sin\omega$$

and

$$d_4 = \left(\lambda - (S_0+1)\theta^2\right)\cos\theta - \left(\lambda - (S_0+1)\omega^2\right)\cos\omega.$$

The entries d_2 and d_4 cannot be zero simultaneously since $\omega \neq \theta$. Hence, the coefficient matrix is nonzero.

For the cases where $\lambda = 0$ and $\lambda = \alpha$,

$$d_3 = \omega\sin\omega \text{ and } d_4 = (S_0+1)\omega^2\cos\omega.$$

Since $\sin\omega$ and $\cos\omega$ cannot be zero simultaneously, it follows that the coefficient matrix is nonzero.

Lastly, the case where $0 < \lambda < \alpha$ is trivial, since

$$d_2 = -\frac{\lambda + S_0\mu^2}{\mu}\sinh\mu - \frac{\lambda - (S_0+2)\omega^2}{\omega}\sin\omega \neq 0.$$

We conclude that double eigenvalues are not possible. □

10 Conclusion

In this article, we studied a Timoshenko rod where a known axial load is present. This could be a compressive or tensile force. The model is not merely a variation in the standard Timoshenko model. As mentioned in introduction, the application of a compressive or tensile force leads to completely different responses to excitation. The spectral results confirm this.

The mathematical model leads to an eigenvalue problem, and partial sums of modal solutions are supposed to converge to a solution of the model problem. For the vibrating string model, this procedure is not only well known, but the theoretical foundation can be found in textbooks (see, for example, [8]). In [9], an approach to approximate a solution for a general vibration problem by using partial sums of modes is presented. The authors show that the validity of this method depends on the completeness of the eigenfunction sequence.

It is not easy to show completeness of eigenfunction sequences for complex model problems and, for example, [10] considers only simple applications. In [9], the main strategy is to show the equivalence of the abstract eigenvalue problem to another eigenvalue problem in a variational form. The theory in [9] then requires conditions on the variational form and bilinear forms for the desired results to be true.

These requirements and associated estimates were established for the rod model in this article. To prove the completeness of the orthonormal sequence of eigenfunctions for the Timoshenko model is new. If the axial force is set equal to zero, the results in the articles [4–6] are obtained as expected. In addition, the present article justifies the work done in these papers. Since the sequence of classical eigenfunctions for the rod model is also the sequence of eigenfunctions for the variational problem, it is complete.

Half of the present paper is devoted to developing a strategy to compute the eigenfunctions and eigenvalues for given boundary conditions. It was possible to adapt the methods in [4] but it was by no means trivial. Substitution of pinned–pinned boundary conditions simplified the method by reducing it to solving a system of quadratic equations. Substitution of cantilever and clamped–clamped boundary conditions reduced the strategy to a numerical method to solve so-called frequency equations. An unexpected benefit of the theory is that it is helpful when eigenvalues and eigenfunctions are calculated.

Once boundary conditions have been substituted into the general solution, the solution space (and therefore the eigenspace) is at most two-dimensional. For a rod with cantilever boundary conditions, it was shown that all eigenvalues are simple. That is, all eigenspaces are one-dimensional. For a rod with pinned–pinned or clamped–clamped boundary conditions, conditions for simple eigenvalues were found.

It follows from the theory that all the eigenvalues for the standard Timoshenko rod (zero load) as well as a rod subjected to a tensile load are positive. For a rod subjected to a compressive load, eigenvalues may be positive, zero, or negative. In general, it is believed that negative eigenvalues indicate buckling for a related nonlinear model. If a constant load results in a zero eigenvalue, it is referred to as the critical load. For the pinned–pinned rod, we developed a simple formula for the critical load and for the clamped–clamped rod, a numerical procedure to compute it.

In [17], a model for a pre-stressed Timoshenko rod was introduced with a nonlinear term for the load. One could consider our model in Sect. 2 as a linear approximation in some sense. The zero and negative eigenvalues and associated eigenfunctions for Problem E are important for the analysis of the problem in [17].

The authors of [17] transformed the system of differential equations to a fourth-order system of differential equations. (A remark concerning this approach is made in Sect. 2.) They calculated the eigenvalues and eigenfunctions for an associated linear problem and used the results to prove the existence of buckled states for the nonlinear problem. However, their original semi-linear system can also be investigated using the results of the present paper. Following the alternative approach, we also found buckled states. These results were not published, but included in a conference proceeding as mentioned in Sect. 8. The research is still in progress.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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