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The performance of the preliminary test estimator under different loss functions

by

JUDITH KLEYN

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South Africa

Supervisor: Professor M. Arashi
Co-supervisor: Professor A. Bekker

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Declaration of originality

I, Judith Kleyn, declare that the thesis, which I hereby submit for the degree Philosophiae Doctor (Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Summary

In this thesis different situations are considered in which the preliminary test estimator is applied and the performance of the preliminary test estimator under different proposed loss functions, namely the reflected normal , linear exponential (LINEX) and bounded LINEX (BLINEX) loss functions is evaluated. In order to motivate the use of the BLINEX loss function rather than the reflected normal loss or the LINEX loss function, the risk for the preliminary test estimator and its component estimators derived under BLINEX loss is compared to the risk of the preliminary test estimator and its components estimators derived under both reflected normal loss and LINEX loss analytically (in some sections) and computationally. It is shown that both the risk under reflected normal loss and the risk under LINEX loss is higher than the risk under BLINEX loss. The key focus point under consideration is the estimation of the regression coefficients of a multiple regression model under two conditions, namely the presence of multicollinearity and linear restrictions imposed on the regression coefficients. In order to address the multicollinearity problem, the regression coefficients were adjusted by making use of Hoerl and Kennard's (1970) approach in ridge regression. Furthermore, in situations where under- or overestimation exist, symmetric loss functions will not give optimal results and it was necessary to consider asymmetric loss functions. In the economic application, it was shown that a loss function which is both asymmetric and bounded to ensure a maximum upper bound for the loss, is the most appropriate function to use. In order to evaluate the effect that different ridge parameters have on the estimation, the risk values were calculated for all three ridge regression estimators under different conditions, namely an increase in variance, an increase in the level of multicollinearity, an increase in the number of parameters to be estimated in the regression model and an increase in the sample size. These results were compared to each other and summarised for all the proposed estimators and proposed loss functions. The comparison of the three proposed ridge regression estimators under all the proposed loss functions was also summarised for an increase in the sample size and an increase in variance.

Keywords: Block bootstrapping; Bounded LINEX loss function; Feasible Bayes estimator; LINEX loss function; Preliminary test estimator; Reflected normal loss function; Restricted estimator; Ridge regression;Risk function; Unrestricted estimator.

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Preface

In problems of statistical inference, one may use the sample information available in a random sample for the estimation of some or all the population parameters of interest. Sometimes some non-sample information is also available about the unknown population parameters. This information can be used and interpreted as restrictions or constraints imposed on the model and can therefore be used to develop better estimators (Tabatabaey, 1995).

This source of prior information usually comes from previous studies or expert opinions and beliefs (prior information via introspection). Applications for these proposed restrictions can be found in Don (1982), Sengupta and Jammalamadaka (2003) and Xu and Yang (2012). In a number of econometric applications, preliminary tests are used to firstly evaluate the accuracy of the restrictions placed on the population parameters and secondly to justify the relevance of the restrictions.

In 1944, Bancroft proposed the preliminary test estimator as a suitable combination of sample and non-sample information which he used to obtain an estimator with better performance, which implies that this proposed estimator has lower risk than other standard estimators.

The estimates obtained in any estimation problem will improve if it can be ensured that the proposed estimator exhibits good performance, which implies lower risk. The choice of the specific loss function will play a deterministic role in improving the performance of the estimators. In order to choose the appropriate loss function it is necessary to take practical and reasonable losses into account.

In situations where under- or overestimation exist, symmetric loss functions do not give optimal results and it is necessitates consideration of asymmetric loss functions. In some econometric problems it is necessary to use loss functions which are both asymmetric and bounded to ensure a maximum upper bound for the loss.

The research problem in essence is to consider different situations in which the preliminary test estimator is applicable and to evaluate the performance of the preliminary test estimator under different loss functions, namely the reflected normal loss function, linear exponential (LINEX) and bounded LINEX (BLINEX) loss functions.

A huge amount of research is available on the idea of testing a hypothesis of specific parameters and pre-testing is used to make the correct decision on the hypothesis. Therefore the results of the preliminary test procedure will give an indication of what the correct decision or action should be regarding the parameter. For an extensive overview and detailed discussion on the preliminary test estimator (PTE) see Saleh (2006) and his references.

In this thesis, situations will be considered for which there is known prior information available about an unknown population parameter that could be used in the form of a constraint.

Furthermore, situations where a model is specified under uncertainty will be considered where specific stochastic model assumptions must be tested before the appropriate statistical model can be specified. The outcome of the test will have an impact on the model specification and the performance of the preliminary test estimator.

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The main objectives addressed in this study are

- To compare the performance of the preliminary test estimator to other known and commonly used estimators in terms of the risk functions associated with each estimator for different choices of loss functions.
- To examine the performance of the estimators using both the classical approach as well as the Bayesian approach.
- To compare the specific loss functions (reflected normal loss, LINEX loss and BLINEX loss) and to propose the preferred loss function based on some analytical and computational results.

One of the proposed problems under investigation is the estimation of a location parameter for a normal model. In order to add value to an existing problem, different possible estimators are defined or derived as possible estimators for the location parameter, namely the restricted maximum likelihood estimator (RMLE), the unrestricted maximum likelihood estimator (UMLE), the preliminary test estimator (PTE) and Bayes estimators.

In the case of the Bayes estimator(s) it was not possible to derive estimators under specific asymmetric loss functions (due to the complexity of the functions) and new methods were developed to obtain feasible Bayes estimators. The performance of the estimators was evaluated by determining and comparing the risk functions of the estimators under LINEX and BLINEX loss functions. In situations where there is no known method available to determine the exact risk functions, new methods were developed to obtain approximated risk functions or conditional risk functions.

The performance of the different estimators was compared in various ways, namely

- The risk functions of the different classical estimators (UMLE, RMLE and PTE) calculated under the same loss function were compared to each other both analytically and computationally.
- The risk function of a specific classical estimator (e.g. PTE) calculated under different loss functions (e.g. PTE under LINEX loss versus PTE under BLINEX loss) was computationally compared to each other.
- The risk function of the classical PTE calculated under BLINEX loss was computationally compared to the risk function of the Bayes PTE calculated under BLINEX loss.
- The previous result was extended to examining the performance of classical estimators versus Bayes estimators under different loss functions.

New and improved preliminary test estimators were considered under different loss functions. These estimators were used in a relevant practical application in order to show that the proposed estimators exhibit superior characteristics under specific conditions. The practical application which was considered is the estimation of the regression coefficients of a multiple regression model under two conditions. Firstly the estimation of the coefficients will take place under the assumption that multicollinearity is present in the data and secondly linear restrictions will be imposed on the

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regression coefficients. In order to correct for the multicollinearity the coefficients will be adjusted by making use of the ridge regression approach of Hoerl and Kennard (1970). This problem will be illustrated in a practical context where the Cobb-Douglas production function will be considered. The two explanatory variables in the analysis, namely labour input and capital input are closely related to each other in practice and therefore multicollinearity is a problem which should be addressed. The presence of multicollinearity between the two explanatory variables in the Cobb-Douglas production function has been one of the criticisms of the production function by Fraser (2002). The interested reader may refer to Akdeniz and Erol (2003), Akdeniz and Kaciranlar (1995), Akdeniz and Tabakan (2009), Akdeniz and Akdeniz (2012), Kibria (1996, 2003, 2012), Kibria and Saleh (2012), Kristofer, Kibria and Shukur (2012), Saleh and Kibria (2011) for more related contributions in the field of multicollinearity. Furthermore a linear restriction will be imposed on the model coefficients where it is assumed that constant return to scale is present. The different estimators considered under these two conditions were the unrestricted ridge regression estimator (URRE), the restricted ridge regression estimator (RRRE) and the preliminary test ridge regression estimator (PTRRE).

The use of asymmetric and bounded loss function for econometric modelling has been discussed by Granger (1999) where he gives different examples of the advantage of penalising over- or underestimation of estimators, depending on which one of the two is the most serious. In the example of the Cobb-Douglas production function it can be deemed that the loss/cost of overestimating production response to capital and labour input is more serious than underestimating production response and therefore overestimation should be penalised more heavily. The three loss functions which were considered in this study were therefore the reflected normal loss function (bounded, not necessarily asymmetric), the LINEX loss function (asymmetric but not bounded) and the BLINEX loss function (both asymmetric and bounded). Therefore the performance of the different estimators in the context of the practical example will be compared in various ways, namely

- The risk functions of the different classical estimators (URRE, RRRE and PTRRE) calculated under the same loss function will be derived and then compared to each other computationally.
- The risk function of a specific classical estimator (e.g. PTRRE) calculated under different loss functions (e.g. PTRRE under reflected normal loss, PTRRE under LINEX loss versus PTRRE under BLINEX loss) will be computationally compared to each other.

In the context of the above-mentioned scenario, it is apparent that in any inference regarding the Cobb-Douglas production function, the following issues should be addressed:

1. Hyperspace restrictions
2. Bounded asymmetric loss functions
3. Multicollinearity

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There is therefore a need to consider a situation that fulfills all the above conditions. The main goal of this study is to accommodate all the above-mentioned issues in one model and to consider the performance of the related estimators. Further research could still be done in the evaluation of the risk performance of Stein-type and its positive part shrinkage estimators which falls outside the scope of this study.

In section 1.1 of Chapter 1 a brief literature review is given, in section 1.2 all the univariate and multivariate distributions of importance in this thesis are discussed together with their properties. Section 1.3 covers all the models and proposed estimators to be considered in the thesis and in section 1.4 all the important theorems which are extensively used in this thesis, together with important results are given. In Chapter 2 the discussion of the reflected normal loss function, its properties and the derivation of the risk functions for each of the proposed ridge regression estimators are given, therefore the risk of the preliminary ridge regression estimator and its two component estimators (URRE and RRRE) are derived under reflected normal loss. In section 2.5 the simulation study procedure is discussed. As part of the simulation study, different ridge parameters are defined and used in order to determine the effect of the specific choice of ridge parameter on the performance of the estimator. For all three ridge regression estimators (URRE, RRRE and PTRRE) the risk values are calculated under different conditions, namely an increase in variance, an increase in the level of multicollinearity, an increase in the number of parameters to be estimated in the regression model and an increase in the sample size. The last part of the simulation study focussed on the comparison of the three proposed ridge regression estimators under the reflected normal loss function for an increase in the sample size and an increase in the variance.

In Chapter 3 the linear exponential (LINEX) loss function is discussed, together with its properties. The risk function of the three proposed ridge regression estimators (preliminary test ridge regression estimator and its two component estimators) is derived in sections 3.2 - 3.4 and in section 3.5 a similar simulation study was conducted as in Chapter 2 in order to evaluate the effect of the different proposed ridge parameters on the estimators under the four conditions mentioned above. The comparison of the performance of the three proposed ridge regression estimators to each other under LINEX loss for an increase in the sample size and an increase in variance is also discussed at the end of this section.

In Chapter 4 the focus is on the bounded linear exponential (BLINEX) loss function and its properties. In section 4.2 the focus is on an application where the classical non-ridge preliminary test estimator is defined, together with its two component estimators (UMLE and RMLE). The risk functions of the three proposed estimators are derived under BLINEX loss and at the end of the section are compared to each other analytically and computationally. In section 4.3 the classical ridge preliminary test estimator (PTRRE) is considered under BLINEX loss and the risk functions of the preliminary test estimator and its two component estimators (URRE and RRRE) are derived. The same simulation study as in Chapters 2 and 3 was conducted at the end of this section under BLINEX loss in order to establish the effect of the different ridge parameters on the performance of the different estimators and also to compare the performance of the three estimators under BLINEX loss for an increase in the sample size and an increase in the variance. Section 4.4 the Bayes preliminary test estimator is discussed, where two feasible Bayes estimators are proposed and the feasible Bayes preliminary test estimator is defined. At the end of this section the results based on a simulation study are discussed, where the performance of the classical preliminary test estimator is compared to performance of the feasible Bayes preliminary test estimator.

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In Chapter 5 an economic application for the classical ridge preliminary test estimator is considered. The Cobb-Douglas production function is considered under the economic restriction of constant return to scale. In this application it will become clear that the two explanatory variables in the model are highly negatively correlated with each other. Therefore the problem of multicollinearity also has to be addressed in the estimation of the parameters of the model. Making use of a South African data set of the manufacturing sector for the sample period 1985 to 2012, the Cobb-Douglas production function is estimated for all three proposed ridge regression estimators and the results are given and discussed in this chapter. Another simulation study is also conducted, by making use of the block bootstrapping technique due to the fact that the data is time dependent. The empirical distributions of the three proposed ridge regression estimators are determined under all the proposed ridge parameters and are graphically represented and discussed at the end of the chapter. The sampling properties of these estimators are not well known and some of these properties were studied. The practical application of preliminary test estimators for economic indices were studied and comparisons will be drawn between the relevant proposed estimators.

In Chapter 6 conclusions and summaries of results obtained in the work as discussed in Chapters 2 to 4, are provided. An alternative method to derive the risk functions for the proposed ridge regression estimators under reflected normal loss, LINEX loss and BLINEX loss is given and future directions for possible research in this field are also addressed. All the simulation programs, data sets, a glossary and a complete list of references are provided in Appendices A to E at the back of the thesis. The data set used in Chapter 5 is obtained from the South African Reserve bank's website (http://www.resbank.co.za/pages/default.aspx), which is an open data source for public use.

A list of research outputs, papers and conferences attended related to this thesis is given below.

- Coetsee, J., Bekker, A. and Millard, S. (2012). Preliminary-test and Bayes Estimation of a Location Parameter Under BLINEX loss, Accepted in 2012 for publication in Communications in Statistics - Theory and Methods.
- Coetsee, J., Bekker, A. and Millard, S. (2012). "Comparison of test estimators for location parameter of normal model under BLINEX loss with focus on preliminary test estimation". 8th World Congress on Probability and Statistics, Istanbul, Turkey.
- Coetsee, J., Bekker, A. and Millard, S. (2012). "Estimation of a location parameter for a normal model with special reference to preliminary test estimators under BLINEX and LINEX loss". Annual conference of the South African Statistical Association, Port Elizabeth, South Africa.
- Kleyn, J., Arashi, M., Bekker, A. and Millard, S. (2013). Preliminary testing of the Cobb-Douglas production function and related inferential issues, Submitted in Communications in Statistics - Simulation and Computations.

Note: My maiden name is Coetsee and my current surname is Kleyn.

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Acronyms

Throughout this thesis all matrices are denoted by capital bold letters, vectors are denoted by small bold letters and scalars are denoted by small letters. When random variables are defined, the random variable is denoted by an uppercase letter and its value by the corresponding lowercase letter. The following notation will be used in this thesis:

$\mathbb{R}^p$	p- dimensional real space
$\mathbb{R}^+$	Set of positive real numbers
$I_A(F)$	Indicator function of set A, that is $I_A(F)=1$ if $F \in A$ and $I_A(F)=0$ if $F \notin A$
$\mathbf{A}_{p\times n}$	$p \times n$ matrix
$\mathbf{A}'$	Transpose of $\mathbf{A}$
$\lvert\mathbf{A}\rvert$	Determinant of $\mathbf{A}$
$tr\mathbf{A}$	Trace of a square matrix $\mathbf{A}$
$\lVert\mathbf{A}\rVert$	Norm of a square matrix $\mathbf{A}_{p\times p}$, defined as $\sqrt{a_{11}^2+a_{22}^2+....+a_{pp}^2}$
$\mathbf{A}^{\frac{1}{2}}$	Root of a semi positive definite $\mathbf{A}$ such that $\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}=\mathbf{A}$
$\mathbf{I}_p$	$p \times p$ dimensional identity matrix
$\varphi(X)$	Borel measurable function of the random variable X
$N(\mu,\sigma^2)$	Normal distribution with mean μ and variance σ^2
χ^2_ν	Chi-square distribution with ν degrees of freedom
t_k	Student t distribution with k degrees of freedom
F_{k_1,k_2}	F distribution with k_1 and k_2 degrees of freedom
$\Gamma(a)$	Gamma function at point a
$B(a,b)$	Beta function at points a and b
$\widetilde{\boldsymbol{\beta}}_n$	Unrestricted regression estimator
$\widetilde{\boldsymbol{\beta}}_n(k)$	Unrestricted ridge regression estimator
$\widehat{\boldsymbol{\beta}}_n$	Restricted regression estimator
$\widehat{\boldsymbol{\beta}}_n(k)$	Restricted ridge regression estimator
$\widehat{\boldsymbol{\beta}}_n^{PT}$	Preliminary test regression estimator
$\widehat{\boldsymbol{\beta}}_n^{PT}(k)$	Preliminary test ridge regression estimator
$\widehat{\mu}_B$	Bayes estimator of the location parameter
$\widehat{\mu}_{FB}^{(1)}$	First feasible Bayes estimator of the location parameter
$\widehat{\mu}_{FB}^{(2)}$	Second feasible Bayes estimator of the location parameter
μ_{FBP}	Feasible Bayes Preliminary test estimator of the location parameter
$\arg\min f(x)$	Argument of the minimum, i.e. the set of points of the given argument for which the given function attains its minimum value.
$\arg\max f(x)$	Argument of the maximum (see above)

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BLINEX	Bounded linear exponential
EL	Expected loss
FB	Feasible Bayes
FBPTE	Feasible Bayes preliminary test estimator
LINEX	Linear exponential
PTE	Preliminary test estimator
PTRRE	Preliminary test ridge regression estimator
RMLE	Restricted maximum likelihood estimator
RN	Reflected normal
RRE	Restricted regression estimator
RRRE	Restricted ridge regression estimator
UMLE	Unrestricted maximum likelihood estimator
URE	Unrestricted regression estimator
URRE	Unrestricted ridge regression estimator

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Chapter 1

Preliminaries

1.1 Introduction and literature review

The first preliminary test procedures were studied by Bancroft in 1944, Mosteller (1948), Kitawaga (1963), Huntsberger (1955), Bancroft (1964), Larson and Bancroft (1963a, 1963b) and Ashar (1970). This work mainly focussed on the properties of sample statistics in terms of their means and mean square errors.

More recently a number of papers have been published on the preliminary test estimators using different proposed loss structures, namely Giles and Giles (1996), Giles (2002), Ohtani et al. (1997), Kibria (2003), Kibria and Saleh (2006), Saleh (2006), Arashi et al (2008) to name a few. In the papers by Parsian and Kirmani (2002), Porosiński and Kamińska (2009), Arashi (2010) and Arashi and Tabatabaey (2010) the properties of the LINEX loss function, which were first proposed by Varian (1975), are presented together with results of estimation under LINEX loss for a number of probability distributions.

In a paper by Wen and Levy (2001) a new parametric family of bounded and asymmetric loss functions, called the BLINEX loss function was developed and the mathematical properties of the BLINEX loss function were discussed in the paper. The BLINEX function is the bounded alternative to the LINEX loss function and therefore it doesn't have the same limitations as the LINEX loss function.

One of the limitations of the LINEX loss function is that it exhibits a huge increase in expected loss which limits the application of this loss function in practice. The BLINEX function is therefore both bounded and asymmetric, which allows the same flexibility as exhibited by the LINEX loss function but it also has the added advantage of being bounded. Wen and Levy (2004) developed algorithms for fitting BLINEX loss parameters, which can be applied under certain conditions. Wen et al (2009) considered an application of the Bayes estimator under BLINEX loss and Kamińska (2010) showed that a unique Bayes estimator exists under BLINEX loss, but no explicit formula was proposed.

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In the field of econometrics it is assumed that when an econometric model is correctly specified, known statistical tests can be used to estimate unknown parameters. Researchers in this field are mainly restricted to non-experimental data, therefore it is never clear which theoretical constraints should be imposed on the model and it is also unclear which statistical model would be the most appropriate to use.

The choice of both the economic and the statistical model can be influenced by a number of factors ranging from

- the number of variables to be included in the economic model
- the stochastic assumptions underlying the statistical model
- the algebraic form of the economic model
- and many more.

Researchers start building econometric models based on a set of specifications and have to adapt their models by making use of hypothesis testing procedures. The final model specification therefore depends on the outcome of these tests and this process of using hypothesis testing to determine the model specification is known as preliminary testing and in the process preliminary test estimators are obtained.

An application in which some work has been done is a situation commonly found in econometric modelling. One of the stochastic statistical assumptions underlying a statistical model is that the stochastic error terms are independently distributed. In econometrics the researcher usually works with non-experimental data and therefore this assumption is easily violated and should be tested.

There are a number of procedures available for testing for serial correlation in the error terms, namely the Durbin-Watson procedure, the method of generalised least squares, the Cochrane-Orcutt two-stage procedure etc. Studies have been conducted by Rao and Griliches (1969), Berenblutt and Webb (1974), L'Esperance and Taylor (1975) and Judge and Bock (1978) on the relative power properties for serial correlation and preliminary test risk. The choice of testing procedure applied, level of significance chosen and the outcome of these tests will have an impact on the performance of the preliminary test estimators and has already been considered under squared error loss. In the work done by Nakale (2013) this work was revisited and re-evaluated under different (asymmetric) loss functions, namely LINEX and BLINEX loss functions. In his work he extended the work of Simons (1988), King and Giles (1984) and Judge and Bock (1978) on serial correlation preliminary test estimation of first order autoregressive error models. These studies investigated the finite sample risk performances of the various feasible generalised least square estimators used in models with serially correlated error terms and their corresponding preliminary test estimators, relative to the squared error loss function. The literature was expanded by investigating the impact of the asymmetric loss functions, namely LINEX and BLINEX loss functions, on the established results based on squared error loss as provided in the literature. Furthermore the followed work done by Simons (1988) in order to compare the risk performance of the classical serial correlation preliminary test estimator and the Bayesian preliminary test estimator to one another. By making use of bootstrap techniques Nakale (2013) was able to establish some aspects of the sampling distributions

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of the classical serial correlation preliminary test estimator and Bayesian preliminary test estimator in order to approximate probability distributions.

In a paper by Ruge-Murcia (2002), which is an extension of the monetary policy model of Barro and Gordon (1983), it is shown that in a situation of unexpected inflation, unnatural low levels of unemployment can be produced if a government or central bank has a preference for either inflation or unemployment, it will not be able to formulate or decide on an optimal monetary policy.

The approach followed by Ruge-Murcia is to allow the central banker's preferences to be asymmetric around the optimal inflation rate. Therefore different weights are allocated to positive or negative deviations from the inflation target. When inflation is below the optimal rate it could be an indication for deflation bias and if inflation is above the optimal rate it could be an indication of inflation bias.

If inflation bias is present the central banker has asymmetric unemployment preferences, which will aim to reduce unemployment below its natural rate. Positive unemployment deviations from its natural rate are therefore weighted more severely than negative deviations in the loss function.

In the work of Ruge-Murcia the LINEX loss function was used, which is an asymmetric loss function, but this work can be extended to the BLINEX loss function, where a maximum loss can be specified. One of the first papers to use LINEX, first proposed by Varian (1975), in monetary policy specification, was a paper by Nobay and Peel (1998).

1.2 Some results from distribution theory

The following results can be found in any text book on inferential statistics. We refer to Muirhead (2005) and Johnson, Kotz and Balakrishnan (1994, 1995) for more detail.

1.2.1 Normal distribution

The probability density function (pdf) of the normal distribution, denoted by $N(\mu, \sigma^2)$, is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \text{ with } x, \mu \in \mathbb{R} \text{ and } \sigma \in \mathbb{R}^+$$

where μ is the mean and σ^2 is the variance of the random variable X.

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Properties:

1. If X is $N(\mu, \sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma}$ is a standard normal variable, denoted by $N(0,1)$ and the probability density function is given by

$$f(z;0,1) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} \text{ with } z \in \mathbb{R}\ ,\ \mu = 0 \text{ and } \sigma = 1$$

2. If $X_1, X_2, ..., X_n$ are n independent $N(\mu, \sigma^2)$ random variables, then the sample mean, $\overline{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n)$ is $N\left(\mu, \dfrac{\sigma^2}{n}\right)$

The p-dimensional normal pdf for the random vector $\mathbf{X}' = (X_1, X_2, ..., X_p)$ has the form

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu})' \mathbf{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})/2\right\}$$

where $\mathbf{x}' = (x_1, x_2, ..., x_p)$, $-\infty < x_i < \infty, i = 1, 2, ...p$, $\mathbf{\Sigma}$ is a symmetric positive definite variance-covariance matrix, $\boldsymbol{\mu}$ the mean vector. The p-dimensional normal pdf will be denoted by $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$.

1.2.2 Chi-squared distribution

If X is $N(0,1)$, then X^2 has the central chi-square distribution with one degree of freedom, denoted by $\chi^2_{(1,0)}$. The pdf of the central chi-square distribution with ν degrees of freedom is given by:

$$\chi^2_{(\nu,0)} = \frac{\left(\chi^2_{(1,0)}\right)^{(\nu/2)-1} e^{-\chi^2_{(1,0)}/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}$$

when $\frac{\nu}{2}$ is a positive integer, $\Gamma\left(\frac{\nu}{2}\right) = \left(\frac{\nu}{2}-1\right)!$, when $\frac{\nu}{2}$ is a complex number with a real positive part, $\Gamma\left(\frac{\nu}{2}\right)$ is defined via a convergent improper integral

$$\Gamma\left(\frac{\nu}{2}\right) = \int_0^\infty x^{\frac{\nu}{2}-1} e^{-x} dx.$$

where ν is the mean and 2ν is the variance of the random variable X^2 with ν degrees of freedom.

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When X is $N(\mu, \sigma^2)$ then X^2 has a noncentral chi-square distribution, denoted by $\chi^2_{\left(1,\frac{1}{2}\Delta^2\right)}$, with one degree of freedom and noncentrality parameter $\frac{\mu^2}{2\sigma^2} = \frac{1}{2}\Delta^2$. The pdf of a noncentral chi-square variable with ν degrees of freedom and noncentrality parameter $\frac{1}{2}\Delta^2$ is given by

$$h\left[\chi^2_{\left(\nu,\frac{1}{2}\Delta^2\right)}\right] = e^{-\frac{1}{2}\Delta^2}{}_0F_1\left(\frac{1}{2} : \frac{1}{4}\Delta^2\chi^2_{\left(1,\frac{1}{2}\Delta^2\right)}\right) h\left[\chi^2_{(1,0)}\right]$$

where $h\left[\chi^2_{(1,0)}\right]$ is the pdf of the central chi-square distribution with 1 degrees of freedom and ${}_0F_1\left(\frac{1}{2} : \frac{1}{4}\Delta^2\chi^2_{\left(1,\frac{1}{2}\Delta^2\right)}\right)$ is the hypergeometric function of the argument $\frac{1}{4}\Delta^2\chi^2_{\left(1,\frac{1}{2}\Delta^2\right)}$.

Further the hypergeometric function of matrix argument is defined by

$${}_mF_n\left(a_1,....,a_m;b_1,....,b_n;S\right) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(a_1\right)_\kappa....\left(a_m\right)_\kappa}{\left(b_1\right)_\kappa....\left(b_n\right)_\kappa}\frac{C_\kappa\left(S\right)}{k!},$$

where $a_i, i = 1, ..., m; b_j, j = 1, ..., n$ are arbitrary complex numbers, $S_{p\times p}$ is a complex symmetric matrix, $\sum_{\kappa}$ denotes summation over all partitions κ and $C_\kappa\left(S\right)$ is a zonal polynomial. Refer to Gupta and Nagar (2000).

1.2.3 t-distribution

If Z_1 is a standard normal random variable i.e. $Z_1 \sim N(0,1)$ and Z_2 follows the chi-square distribution with k degrees of freedom and is independent of Z_1, then the variable defined as

$$t = \frac{Z_1}{\sqrt{Z_2/k}} \sim t_k$$

follows the Student's t distribution with k degrees of freedom. The pdf of the univariate t distribution with k degrees of freedom is given by

$$f\left(x;k\right) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{x^2}{k}\right)^{-\frac{k+1}{2}}, x \in \mathbb{R}$$

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1.2.4 F-distribution

If Z_1 and Z_2 are independently distributed chi-square random variables with k_1 and k_2 degrees of freedom respectively, then the variable

$$F = \frac{Z_1/k_1}{Z_2/k_2} \sim F_{k_1,k_2}$$

has the F distribution with k_1 and k_2 degrees of freedom, where k_1 is known as the numerator degrees of freedom and k_2 the denominator degrees of freedom.

The square of the Student's t variable with k degrees of freedom, has an F distribution with $k_1 = 1$ degrees of freedom and $k_2 = k$ degrees of freedom, i.e.

$$t_k^2 = F_{1,k}$$

The pdf of the univariate F distribution with k_1 and k_2 degrees of freedom is given by

$$f\left(x; k_1, k_2\right) = \frac{1}{\beta\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} \left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}} x^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} x\right)^{-\left(\frac{k_1+k_2}{2}\right)}, x \in \mathbb{R}^+$$

with $k_1, k_2 > 0$ and $B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)$ is the Beta function given by

$$B\left(\frac{k_1}{2}, \frac{k_2}{2}\right) = \int_0^1 t^{\frac{k_1}{2}-1} \left(1-t\right)^{\frac{k_2}{2}-1} dt = \frac{\Gamma\left(\frac{k_1}{2}\right)\Gamma\left(\frac{k_2}{2}\right)}{\Gamma\left(\frac{k_1+k_2}{2}\right)}$$

If Y_1 is the noncentral χ^2 random variable with noncentrality parameter λ and ν_1 degrees of freedom; Y_2 is the central χ^2 random variable with ν_2 degrees of freedom and statistically independent of Y_1, then $F^* = \frac{Y_1/\nu_1}{Y_2/\nu_2}$ is the noncentral F distributed random variable. The pdf of the noncentral F distribution is given by

$$f\left(x; \nu_1, \nu_2, \lambda\right) = \sum_{k=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^k}{B\left(\frac{\nu_2}{2}, \frac{\nu_1}{2}+k\right) k!} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}+k} \left(\frac{\nu_2}{\nu_2+\nu_1 x}\right)^{\frac{\nu_1+\nu_2}{2}+k} x^{\frac{\nu_1}{2}-1+k} \qquad x \in \mathbb{R}^+$$

with $\nu_1, \nu_2 > 0$, $\lambda \geq 0$.

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1.3 Estimators under study

1.3.1 Model

In this thesis the well-known multiple regression model was studied under both conditions of well and ill conditioned design matrices.

Multiple regression model

Consider the multiple regression model (MRM)

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where $\mathbf{Y}_{n\times 1}$ is the vector of response variables, $\mathbf{X}_{n\times p}$ is regarded as the non-stochastic design matrix of full column rank $(p < n)$ and $\boldsymbol{\beta} : p \times 1$ is the vector of unknown regression parameters. Also $\boldsymbol{\epsilon}$ is the vector of random error terms with the assumption that $\boldsymbol{\epsilon} \sim N_n\left(\mathbf{0}, \sigma^2\mathbf{I}_p\right)$, where $\sigma \in \mathbb{R}^+$ is unknown.

1.3.2 Proposed estimators

In this part some well-known estimators are proposed.

Ordinary least squares estimators

Unrestricted regression estimator (URE) For the MRM the URE of $\boldsymbol{\beta}$ under least squares and maximum likelihood estimation, is defined as:

$$\widetilde{\boldsymbol{\beta}}_n = (\mathbf{C})^{-1}\mathbf{X}'\mathbf{Y}, \quad \mathbf{C} = \mathbf{X}'\mathbf{X} \tag{1.1}$$

with

$$\widetilde{\boldsymbol{\beta}}_n \sim N_p\left(\boldsymbol{\beta}, \sigma^2\mathbf{C}^{-1}\right) \tag{1.2}$$

and the unrestricted unbiased estimator of σ^2 is given by

$$s_\varepsilon^2 = \frac{1}{m}\left(\mathbf{Y} - \mathbf{X}\widetilde{\boldsymbol{\beta}}_n\right)'\left(\mathbf{Y} - \mathbf{X}\widetilde{\boldsymbol{\beta}}_n\right)$$

with $m = n - p$ and $n = n_1 + n_2 + ... + n_p$.

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Restricted regression estimator (RRE) For the MRM the RRE of $\boldsymbol{\beta}$ is defined as:

$$\widehat{\boldsymbol{\beta}}_n = \widetilde{\boldsymbol{\beta}}_n - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right) \tag{1.3}$$

with $\mathbf{H}$ and $\mathbf{h}$, defined in the following linear restriction, which will be placed on the regression parameters in the form of the following hypotheses:

$$H_0 : \mathbf{H}\boldsymbol{\beta} = \mathbf{h} \tag{1.4}$$

for a given known matrix $\mathbf{H}_{q\times p}$ of rank q $(q < p)$ and a pre-specified $\mathbf{h}_{q\times 1}$, will be tested against

$$H_A : \mathbf{H}\boldsymbol{\beta} \neq \mathbf{h}$$

where q represents the number of linear restrictions placed on the model.

Preliminary test regression estimator (PTRE) For the MRM, the PTRE of $\boldsymbol{\beta}$ is defined as:

$$\widehat{\boldsymbol{\beta}}_n^{PT} = \widetilde{\boldsymbol{\beta}}_n - \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) I_{NR}\left(F\right) \tag{1.5}$$

The indicator function of PTRE of the regression coefficients is defined below:

$$\begin{aligned} I_{NR}\left(F\right) = I\left(\pounds_n < F_{q,m}\left(\alpha\right)\right) &= 1 \quad \text{when the null hypothesis is not rejected} \\ &= 0 \quad \text{otherwise} \\ I_{R}\left(F\right) = I\left(\pounds_n \geq F_{q,m}\left(\alpha\right)\right) &= 1 \quad \text{when the null hypothesis is rejected} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

with $\pounds_n$ the likelihood ratio test statistic for testing H_0 against H_A given by

$$\pounds_n = \frac{\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right)\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right)}{qs_{\varepsilon}^2} \tag{1.6}$$

where $F_{q,m}\left(\alpha\right)$ is the critical value of the central F-distribution with q and m degrees of freedom and α is the level of significance.

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Under H_A the likelihood ratio test statistic has a noncentral F-distribution with q and m degrees of freedom and noncentrality parameter $\frac{\Delta^2}{2}$

where

$$\Delta^2 = \frac{(\mathbf{H}\boldsymbol{\beta} - \mathbf{h})' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} (\mathbf{H}\boldsymbol{\beta} - \mathbf{h})}{\sigma^2} \tag{1.7}$$

The distribution of URE and RRE are independent of the distribution of

$$\frac{ms_\varepsilon^2}{\sigma^2}$$

Ridge regression The original interpretation of multicollinearity was the existence of a perfect linear relationship between some or all of the explanatory variables in the model which are illustrated below for a p-variable regression model:

$$c_1X_1 + c_2X_2 + ... + c_pX_p = 0$$

where $c_1, c_2, ..., c_p$ are constants which are not all simultaneously equal to zero.

Today's interpretation of multicollinearity is viewed in a less restrictive sense which include perfect multicollinearity as well as the case where variables are intercorrelated, but not necessarily perfectly correlated:

$$c_1X_1 + c_2X_2 + ... + c_pX_p + \nu_i = 0$$

where ν_i is a stochastic error term.

In the presence of perfect multicollinearity $\mathbf{C} = \mathbf{X}'\mathbf{X}$ is not of full rank and therefore cannot be inverted which implies that the parameter estimates cannot be calculated. In the case of nearly perfect multicollinearity it is possible to calculate the inverse of $\mathbf{X}'\mathbf{X}$. The inverse will be ill-conditioned and therefore the inverse will be very sensitive to variation in the data and can provide very inaccurate results, which will lead to very large sampling variances for this estimator. Hoerl and Kennard (1970) suggested the following correction:

$$\mathbf{C}_{(k)} = \mathbf{X}'\mathbf{X} + k\mathbf{I}_p$$

where $k > 0$ is known as the ridge parameter.

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Unrestricted ridge regression estimator (URRE) For the MRM, the URRE of $\boldsymbol{\beta}$ is defined as:

$$\begin{aligned}\widetilde{\boldsymbol{\beta}}_n(k) &= (\mathbf{X}'\mathbf{X}+k\mathbf{I}_p)^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{C}_{(k)}^{-1}\mathbf{X}'\mathbf{Y} \\ &= \left(\mathbf{I}_p + k(\mathbf{X}'\mathbf{X})^{-1}\right)^{-1}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ &= \mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n\end{aligned} \tag{1.8}$$

with

$$\mathbf{R}(k) = \left(\mathbf{I}_p + k(\mathbf{X}'\mathbf{X})^{-1}\right)^{-1} \tag{1.9}$$

The distribution of URRE is given by

$$\widetilde{\boldsymbol{\beta}}_n(k) \sim N_p\left(\boldsymbol{\beta}, \sigma^2\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)\right) \equiv N_p\left(\mathbf{R}(k)\,\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right) \tag{1.10}$$

Restricted ridge regression estimator (RRRE) For the MRM, the RRRE of $\boldsymbol{\beta}$ is defined as:

$$\widehat{\boldsymbol{\beta}}_n(k) = \mathbf{R}(k)\,\widehat{\boldsymbol{\beta}}_n \tag{1.11}$$

The distribution of RRRE is

$$\widehat{\boldsymbol{\beta}}_n(k) \sim N_p\left(\mathbf{R}(k)\,\boldsymbol{\beta}, \sigma^2\mathbf{R}(k)\,\mathbf{A}\mathbf{R}'(k)\right) \equiv N_p\left(\mathbf{R}(k)\,\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\right) \tag{1.12}$$

where

$$\mathbf{A} = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}$$

Preliminary test ridge regression estimators (PTRRE) For the MRM, the PTRRE is defined as:

$$\widehat{\boldsymbol{\beta}}_n^{PT}(k) = \mathbf{R}(k)\,\widehat{\boldsymbol{\beta}}_n^{PT} \tag{1.13}$$

The derivation and details of these results can be found in Saleh (2006).

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1.4 Theorems

The following results and theorems were repeatedly used throughout this thesis.

Results based on the proposed estimators

Consider a matrix of the form $\mathbf{F} = \mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}$ which is a symmetric idempotent matrix of rank $q \leq p$ and therefore there always exists an orthogonal matrix $\mathbf{Q}$ such that

$$\begin{aligned}\mathbf{Q}\mathbf{F}\mathbf{Q}' &= \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}' \\ &= \begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\end{aligned} \tag{1.14}$$

where $\mathbf{C} = \mathbf{X}'\mathbf{X}$ and $\mathbf{H}$ is defined as in (1.4) and

$$\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}' = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix} = \mathbf{D} \tag{1.15}$$

with $\mathbf{R}(k)$ is defined as in (1.9).

Result 1

$$\begin{aligned}&\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}' \\ &= \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}' \\ &= (\mathbf{Q}\mathbf{F}\mathbf{Q}')\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right) \\ &= \begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \text{or} \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\end{aligned} \tag{1.16}$$

and in a similar way as above it was found that

$$\begin{aligned}&\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}' \\ &= \begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\end{aligned} \tag{1.17}$$

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The matrices $\mathbf{D}_{11}$ and $\mathbf{D}_{22}$ are of order q and $q-p$ respectively.

Define the random variable $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\widetilde{\boldsymbol{\beta}}_n - \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{h} \tag{1.18}$$

Using (1.2), the distribution of $\boldsymbol{\omega}$ is given as $\boldsymbol{\omega} \sim N_p\left(\boldsymbol{\eta}, \sigma^2\mathbf{I}_p\right)$

where

$$\boldsymbol{\eta} = E\left(\boldsymbol{\omega}\right) = \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\boldsymbol{\beta} - \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{h} \tag{1.19}$$

therefore

$$\boldsymbol{\omega} - \boldsymbol{\eta} = \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) \tag{1.20}$$

and the distribution of $\boldsymbol{\omega} - \boldsymbol{\eta}$ is given as $\left(\boldsymbol{\omega} - \boldsymbol{\eta}\right) \sim N_p\left(\mathbf{0}, \sigma^2\mathbf{I}_p\right)$.

By partitioning the vector $\boldsymbol{\omega} = \left(\boldsymbol{\omega}_1', \boldsymbol{\omega}_2'\right)'$ and $\boldsymbol{\eta} = \left(\boldsymbol{\eta}_1', \boldsymbol{\eta}_2'\right)'$ the independent sub-vectors of order q and $p-q$ is obtained, namely $\boldsymbol{\omega}_1 \sim N_q\left(\boldsymbol{\eta}_1, \sigma^2\mathbf{I}_q\right)$ and $\boldsymbol{\omega}_2 \sim N_{p-q}\left(\boldsymbol{\eta}_2, \sigma^2\mathbf{I}_{p-q}\right)$.

In the last part of this section necessary relations between the estimators according to the above given definitions is proposed. In this section extensive use is made of the results of Saleh (2006). For specific consideration detailed referencing is provided.

Lemma 1 *Let $\widetilde{\boldsymbol{\beta}}_n$ be the estimator of $\boldsymbol{\beta}$ according to (1.1). Then*

$$\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) = \mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega} - \boldsymbol{\eta}\right) \tag{1.21}$$

where $\boldsymbol{\omega} = \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\widetilde{\boldsymbol{\beta}}_n - \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{h}, \boldsymbol{\eta} = \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\boldsymbol{\beta} - \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{h}$, $\mathbf{Q}$ is an orthogonal matrix obtained from the factorisation $\mathbf{QFQ}'$ and $\mathbf{F} = \mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}$ is a symmetric idempotent matrix of rank $q \leq p$.

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Lemma 2 *Let $\widetilde{\boldsymbol{\beta}}_n$ be an estimator of $\boldsymbol{\beta}$ according to (1.1) and $\widehat{\boldsymbol{\beta}}_n$ be an estimator of $\boldsymbol{\beta}$ according to (1.3), then*

$$\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) = \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right) \tag{1.22}$$

by premultiplying (1.18) by $\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'$ the following expression was obtained

$$\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} = \left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right) \tag{1.23}$$

Therefore

$$\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) = \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} \tag{1.24}$$

Lemma 3 *Let $\widetilde{\boldsymbol{\beta}}_n$ be one estimator of $\boldsymbol{\beta}$ according to (1.1) and $\widehat{\boldsymbol{\beta}}_n$ be another estimator of $\boldsymbol{\beta}$ according to (1.3), then*

$$\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta} = \mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega} - \boldsymbol{\eta}\right) - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} \tag{1.25}$$

Proof. Subtracting $\boldsymbol{\beta}$ on both sides of (1.3) the following expression was obtained

$$\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta} = \left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\left(\mathbf{H}\widetilde{\boldsymbol{\beta}}_n - \mathbf{h}\right)$$

Substituting (1.21) and (1.23) into this expression completes the proof. ■

Theorem 1 *Let $\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}$ be given by (1.21) and $\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n$ be given by (1.24) then*

$$\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}^2\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) = \left(\boldsymbol{\omega}_1 - \boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1 + \left(\boldsymbol{\omega}_2 - \boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 \tag{1.26}$$

Proof. By substituting (1.21) and (1.24) into the left hand side of (1.26) the following expression was obtained

$$\begin{aligned}
&\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}^2\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) \\
= &\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega} - \boldsymbol{\eta}\right)\right)'\mathbf{R}^2\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} \\
= &\left(\boldsymbol{\omega} - \boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}
\end{aligned}$$

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by using (1.14) and (1.15) the following expression was obtained

$$\begin{aligned}
& \begin{pmatrix} (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)' \end{pmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \mathbf{QFQ}' \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} \\
= & \begin{pmatrix} (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)' \end{pmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} \\
= & \begin{pmatrix} (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)' \end{pmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} \\
= & (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' \mathbf{D}_{11} \boldsymbol{\omega}_1 + (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)' \mathbf{D}_{21} \boldsymbol{\omega}_2
\end{aligned}$$

which completes the proof. ■

Theorem 2 *Let $\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n$ be given by (1.24) then*

$$\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right)' \mathbf{R}^2(k) \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) = \boldsymbol{\omega}_1' \mathbf{D}_{11} \boldsymbol{\omega}_1 \tag{1.27}$$

Proof. By substituting (1.24) into the left hand side of (1.27) the following expression was obtained

$$\begin{aligned}
& \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right)' \mathbf{R}^2(k) \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) \\
= & \left(\mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}} \mathbf{Q}'\boldsymbol{\omega}\right)' \mathbf{R}^2(k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}} \mathbf{Q}'\boldsymbol{\omega} \\
= & \boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-1}\mathbf{R}^2(k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}} \mathbf{Q}'\boldsymbol{\omega} \\
= & \boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}} \mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k) \mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}} \mathbf{Q}'\boldsymbol{\omega}
\end{aligned}$$

by using (1.14) and (1.15) the following expression was obtained

$$\begin{aligned}
\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right)' \mathbf{R}^2(k) \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) &= \boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega} \\
&= \begin{pmatrix} \boldsymbol{\omega}_1' & \boldsymbol{\omega}_2' \end{pmatrix} \begin{bmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\omega}_1' & \boldsymbol{\omega}_2' \end{pmatrix} \begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} \\
&= \boldsymbol{\omega}_1' \mathbf{D}_{11} \boldsymbol{\omega}_1
\end{aligned}$$

which completes the proof. ■

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Result 2 Given $\mathbf{R}(k) - \mathbf{I}_p = -k\mathbf{C}_{(k)}^{-1}$ where $\mathbf{C}_{(k)} = \mathbf{C} + k\mathbf{I}_p$.

Proof Since $\mathbf{R}(k) = (\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}$ and therefore $\mathbf{R}^{-1}(k) = \mathbf{I}_p + k\mathbf{C}^{-1}$ by multiplying both sided with $\mathbf{R}(k)$ we obtain $\mathbf{I}_p = \mathbf{R}(k) + k\mathbf{R}(k)\mathbf{C}^{-1}$ and by rearranging terms

$$\mathbf{R}(k) - \mathbf{I}_p = -k\mathbf{R}(k)\mathbf{C}^{-1} = -k(\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}\mathbf{C}^{-1} = -k(\mathbf{C} + k\mathbf{I}_p)^{-1} = -k\mathbf{C}_{(k)}^{-1}$$

(1.28)

Lemma 4 *(Judge and Bock, 1978, p320). Let w be a non-central chi-square random variable with g degrees of freedom and non-centrality parameter θ, let $\varphi(\cdot)$ be a Borel measurable function and let n be any real value such that $n > \frac{-g}{2}$, then*

$$E(w^n\varphi(w)) = 2^n\sum_{m=0}^{\infty}\frac{e^{-\theta}\theta^m}{m!}\frac{\left[\Gamma\frac{1}{2}(g+2n+2m)\right]}{\Gamma\left[\frac{1}{2}(g+2m)\right]}E\left[\varphi\left(\chi^2_{(g+2n+2m)}\right)\right]$$

Theorem 3 *(Judge and Bock, 1978, p321) If $\boldsymbol{\omega}$ is a $(p\times 1)$ normally distributed random vector with mean $\boldsymbol{\eta}$ vector and covariance $\sigma^2\mathbf{I}_p$, $\varphi(.)$ be any Borel measurable function, then*

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}}{\sigma}\right] = \frac{\boldsymbol{\eta}}{\sigma}E\left[\varphi\left(\chi^2_{p+2,\,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}}\right)\right]$$

Result 3.1 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ then

$$E\left[\frac{\boldsymbol{\omega}}{\sigma}\right] = \frac{\boldsymbol{\eta}}{\sigma} \tag{1.29}$$

Result 3.2 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ then

$$E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0} \tag{1.30}$$

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Result 3.3 (Saleh, 2006, p353) For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$ the following expectations are defined

$$E\left[\varphi\left(\chi^2_{(p+2,\xi)}\right)\right] = E\left(\varphi\left(\frac{n-p}{q}\frac{\chi^2_{q+2,\xi}}{\chi^2_{n-p}}\right)\right)$$

$$= E\left[I\left(\pounds_n < F_{q,m}(\alpha)\right)\right] = E\left\{I\left(F_{q+2,n-p,\xi} < \tfrac{q}{q+2}F_{q,m}(\alpha)\right)\right\}$$

with $\xi = \frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$ the noncentrality parameter and $F_{q,m}(\alpha)$ the critical value of the F-distribution with q and m degrees of freedom and α is the level of significance, then

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}}{\sigma}\right] = \frac{\boldsymbol{\eta}}{\sigma}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] \tag{1.31}$$

Result 3.4 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$ since $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$,then

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\right] = \mathbf{0} \tag{1.32}$$

Theorem 4 *(Judge and Bock, 1978, p322). If $\boldsymbol{\omega}$ is a $(p \times 1)$ normally distributed random vector with mean $\boldsymbol{\eta}$ vector and covariance $\sigma^2\mathbf{I}_p$,$\mathbf{D}$ is any positive definite symmetric matrix and $\varphi(.)$ be any Borel measurable function, then*

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]tr\mathbf{D} + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2}$$

Result 4.1 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ then

$$E\left[\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right] = tr\mathbf{D} + \frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2} \tag{1.33}$$

Result 4.2 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ and $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$ then

$$E\left[\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\right] = tr\mathbf{D} \tag{1.34}$$

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Result 4.3 (Saleh, 2006, p353) For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$ the following expectations are defined

$$E\left[\varphi\left(\chi^2_{(p+2,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+2,\xi}}{\chi^2_{n-p}}\right)\right)$$

$$= E\left[I\left(\pounds_n < F_{q,m}(\alpha)\right)\right] = E\left\{I\left(F_{q+2,n-p,\xi} < \tfrac{q}{q+2}F_{q,m}(\alpha)\right)\right\} \text{ and}$$

$$E\left[\varphi\left(\chi^2_{(p+4,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+4,\xi}}{\chi^2_{n-p}}\right)\right)$$

$$= E\left[I\left(\pounds_n < F_{q,m}(\alpha)\right)\right] = E\left\{I\left(F_{q+4,n-p,\xi} < \tfrac{q}{q+4}F_{q,m}(\alpha)\right)\right\}$$

with $\xi = \frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$ the noncentrality parameter and $F_{q,m}(\alpha)$ the critical value of the F-distribution with q and m degrees of freedom and α is the level of significance, therefore

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]tr\mathbf{D} + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2} \tag{1.35}$$

Result 4.4 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$, since $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$, then

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]tr\mathbf{D} \tag{1.36}$$

Theorem 5 *(Judge and Bock, 1978, p323). If the $(p \times 1)$ vector $\boldsymbol{\omega}$ is distributed normally with mean vector $\boldsymbol{\eta}$ and covariance $\sigma^2\mathbf{I}_p$ and $\varphi(.)$ be any Borel measurable function, then*

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}}{\sigma}\frac{\boldsymbol{\omega}'}{\sigma}\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\frac{\boldsymbol{\eta}\boldsymbol{\eta}'}{\sigma^2}$$

Result 5.1 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ then

$$E\left[\frac{\boldsymbol{\omega}}{\sigma}\frac{\boldsymbol{\omega}'}{\sigma}\right] = \mathbf{I}_p + \frac{\boldsymbol{\eta}\boldsymbol{\eta}'}{\sigma^2} \tag{1.37}$$

Result 5.2 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$, since $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$ then

$$E\left[\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)'\right] = \mathbf{I}_p \tag{1.38}$$

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Result 5.3 (Saleh, 2006, p353) For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$ the following expectations are defined

$$E\left[\varphi\left(\chi^2_{(p+2,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+2,\xi}}{\chi^2_{n-p}}\right)\right)$$

$$= E\left[I\left(\pounds_n < F_{q,m}(\alpha)\right)\right] = E\left\{I\left(F_{q+2,n-p,\xi} < \tfrac{q}{q+2}F_{q,m}(\alpha)\right)\right\} \text{and}$$

$$E\left[\varphi\left(\chi^2_{(p+4,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+4,\xi}}{\chi^2_{n-p}}\right)\right)$$

$$= E\left[I\left(\pounds_n < F_{q,m}(\alpha)\right)\right] = E\left\{I\left(F_{q+4,n-p,\xi} < \tfrac{q}{q+4}F_{q,m}(\alpha)\right)\right\}$$

with $\xi = \frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$ the noncentrality parameter and $F_{q,m}(\alpha)$ the critical value of the F-distribution with q and m degrees of freedom and α is the level of significance, therefore

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\frac{\boldsymbol{\omega}}{\sigma}\frac{\boldsymbol{\omega}'}{\sigma}\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\frac{\boldsymbol{\eta}\boldsymbol{\eta}'}{\sigma^2} \tag{1.39}$$

Result 5.4 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$, since $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$, then

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)'\right] = E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p \tag{1.40}$$

Lemma 5 *(Arashi, 2008) Assume that the random variable $X_{(p)}$ has a $\chi^2_{(p)}$ distribution. Then for every Borel measurable function φ , then*

$$E\left[\varphi\left(X_{(p)}\right)X^k_{(p)}\right] = \frac{2^k\Gamma\left(k+\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}E\left[\varphi\left(X_{(2k+p)}\right)\right]$$

Proof. Using the probability density function of χ^2_p the following result was obtained

$$E\left[\varphi\left(X_{(p)}\right)X^k_{(p)}\right] = \int_0^\infty \frac{x^k\varphi(x)\,e^{-\frac{x}{2}}\left(\frac{x}{2}\right)^{\frac{p}{2}-1}}{2\Gamma\left(\frac{p}{2}\right)}dx = \int_0^\infty \frac{2^{\frac{2k+p}{2}}\Gamma\left(\frac{2k+p}{2}\right)\varphi(x)\,x^{\frac{2k+p-2}{2}}e^{-\frac{x}{2}}}{2^{\frac{2k+p}{2}}\Gamma\left(\frac{2k+p}{2}\right)\Gamma\left(\frac{p}{2}\right)}dx$$

$$= \frac{2^k\Gamma\left(k+\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\int_0^\infty \frac{\varphi(x)\,x^{\frac{2k+p}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{2k+p}{2}}\Gamma\left(\frac{2k+p}{2}\right)}dx = \frac{2^k\Gamma\left(k+\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}E\left[\varphi\left(X_{(2k+p)}\right)\right]$$

which completes the proof. ■

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Lemma 6 *(Arashi, 2008). Assume that the random variable W is distributed as $N(\mu, 1)$. Then for every Borel measurable function φ, then*

$$E\left[\varphi(W^2)(W^2)^2\right] = 3E\left[\varphi\left(X_{(5,\mu^2)}\right)\right] + 6\mu^2 E\left[\varphi\left(X_{(7,\mu^2)}\right)\right] + \mu^4 E\left[\varphi\left(X_{(9,\mu^2)}\right)\right]$$

where $X_{(k,\mu^2)} \sim \chi^2_{(k,\mu^2)}$.

Proof. From $W^2 \sim \chi^2_{(1,\mu^2)}$ and the definition of the non-central chi-square the following result was obtained

$$E\left[\varphi(W^2)(W^2)^2\right] = E\left[\varphi\left(X_{(1,\mu^2)}\right)X^2_{(1,\mu^2)}\right] = \sum_{j=0}^{\infty}\frac{\left(\frac{\mu 2}{2}\right)^j e^{-\frac{\mu^2}{2}}}{\Gamma(j+1)}E\left[\varphi\left(X_{(1+2j)}\right)X^2_{(1+2j)}\right]$$

Using Lemma 5 for $k = 2$ and $p = 1 + 2j$, then

$$E\left[\varphi\left(X_{(1+2j)}\right)X^2_{(1+2j)}\right] = (1+2j)(3+2j)\,E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

Therefore the following result was obtained $E\left[\varphi(W^2)(W^2)^2\right]$

$$= \sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}(1+2j)(3+2j)}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

$$= 3\sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right] + 8\sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}j}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

$$+4\sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}j^2}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

$$= 3E\left[\varphi\left(X_{(5,\mu^2)}\right)\right] + 8A + 4B$$

where $A = \sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}j}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right] = \left(\frac{\mu^2}{2}\right)\sum_{j=1}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^{j-1} e^{-\frac{\mu^2}{2}}}{\Gamma(j)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$

and

$$B = \sum_{j=0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^j e^{-\frac{\mu^2}{2}}j^2}{\Gamma(j+1)}E\left[\varphi\left(X_{(5+2j)}\right)\right] = \left(\frac{\mu^2}{2}\right)\sum_{j=1}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^{j-1} e^{-\frac{\mu^2}{2}}j}{\Gamma(j)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

Substituting $u = j - 1$ and $t = u - 1$, then

$$A = \left(\frac{\mu^2}{2}\right)\sum_{u-0}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^u e^{-\frac{\mu^2}{2}}}{\Gamma(u+1)}E\left[\varphi\left(X_{(7+2u)}\right)\right] = \left(\frac{\mu^2}{2}\right)\sum_{j=1}^{\infty}\frac{\left(\frac{\mu^2}{2}\right)^{j-1} e^{-\frac{\mu^2}{2}}j}{\Gamma(j)}E\left[\varphi\left(X_{(5+2j)}\right)\right]$$

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$$= \left(\frac{\mu^2}{2}\right) E\left[\varphi\left(X_{(7,\mu^2)}\right)\right]$$

$$B = \left(\frac{\mu^2}{2}\right) \sum_{u=0}^{\infty} \frac{\left(\frac{\mu^2}{2}\right)^u e^{-\frac{\mu^2}{2}} (u+1)}{\Gamma(u+1)} E\left[\varphi\left(X_{(7+2u)}\right)\right]$$

$$= \left(\frac{\mu^2}{2}\right) \left[\left(\frac{\mu^2}{2}\right) \sum_{t=0}^{\infty} \frac{\left(\frac{\mu^2}{2}\right)^t e^{-\frac{\mu^2}{2}}}{\Gamma(t+1)} E\left[\varphi\left(X_{(9+2t)}\right)\right]\right] + \sum_{u=0}^{\infty} \frac{\left(\frac{\mu^2}{2}\right)^u e^{-\frac{\mu^2}{2}}}{\Gamma(u+1)} E\left[\varphi\left(X_{(7+2u)}\right)\right]$$

$$= \left(\frac{\mu^2}{2}\right)^2 E\left[\varphi\left(X_{(9+2t)}\right)\right] + A$$

Putting all these terms together completes the proof. ■

Theorem 6 *(Arashi, 2008). If $\boldsymbol{\omega}$ is a $(p \times 1)$ normally distributed random vector with mean $\boldsymbol{\eta}$ vector and covariance $\sigma^2 \mathbf{I}_p$ and $\mathbf{D}$ is any positive definite symmetric matrix, and $\varphi(.)$ be any Borel measurable function, then*

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right)^2\right] = 3tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] + 6\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2}E\left[\varphi\left(\chi^2_{\left(p+6,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$
$$+\left(\frac{\boldsymbol{\eta}'\mathbf{D}^{\frac{1}{2}}\boldsymbol{\eta}}{\sigma^2}\right)^2 E\left[\varphi\left(\chi^2_{\left(p+8,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

Proof. Suppose $\mathbf{P}$ is an orthogonal matrix such that $\mathbf{PDP} = \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_p \end{pmatrix}$

where $\boldsymbol{\Lambda}$ is a diagonal matrix and $\lambda_i > 0, i = 1, 2, ..., p$ is the eigenvalues of $\mathbf{D}$. Define the random variable $\mathbf{Z} = \mathbf{P}\boldsymbol{\omega}$ with $\mathbf{Z} \sim N_p(\mathbf{P}\boldsymbol{\eta}, \mathbf{I}_p)$.

Using Lemma 6 the following result was obtained

$$E\left[\phi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right)^2\right] = E\left[\varphi\left(\mathbf{Z}'\mathbf{Z}\right)\left(\mathbf{Z}'\boldsymbol{\Lambda}\mathbf{Z}\right)^2\right]$$

$$= \sum_{i=1}^{p} \lambda_i E\left\{E\left[\varphi\left(Z_i^2 + \sum_{j\neq i} Z_j^2\right) Z_i^2 \middle| Z_j, j \neq i\right]\right\}$$

$$= \sum_{i=1}^{p} \lambda_i \left\{3E\left[\varphi\left(\chi^2_{\left(5,(\mathbf{P}_i'\boldsymbol{\eta})^2\right)}\right) + \sum_{j\neq i} Z_j^2\right] + 6\left(\mathbf{P}_i'\boldsymbol{\eta}\right)^2 E\left[\varphi\left(\chi^2_{\left(7,(\mathbf{P}_i'\boldsymbol{\eta})^2\right)}\right) + \sum_{j\neq i} Z_j^2\right]\right.$$

$$\left. + \left(\mathbf{P}_i'\boldsymbol{\eta}\right)^4 E\left[\varphi\left(\chi^2_{\left(9,(\mathbf{P}_i'\boldsymbol{\eta})^2\right)}\right) + \sum_{j\neq i} Z_j^2\right]\right\}$$ where $\mathbf{P}_i'$ is the i^{th} row of $\mathbf{P}$.

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Furthermore

$$\sum_{i=1}^{p}\lambda_i = tr\left(\mathbf{D}\right),\ \sum_{i=1}^{p}\lambda_i\left(\mathbf{P}_i'\boldsymbol{\eta}\right)^2 = \boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta} \text{ and } \sum_{i=1}^{p}\lambda_i\left(\mathbf{P}_i'\boldsymbol{\eta}\right)^4 = \sum_{i=1}^{p}\left(\lambda_i^{\frac{1}{2}}\left(\mathbf{P}_i'\boldsymbol{\eta}\right)^2\right)^2 = \left(\boldsymbol{\eta}\mathbf{D}^{\frac{1}{2}}\boldsymbol{\eta}\right)^2$$

Therefore we conclude that

$$E\left[\phi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right)^2\right] = 3tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] + 6\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2}E\left[\varphi\left(\chi^2_{\left(p+6,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$
$$+\left(\frac{\boldsymbol{\eta}'\mathbf{D}^{\frac{1}{2}}\boldsymbol{\eta}}{\sigma^2}\right)^2 E\left[\varphi\left(\chi^2_{\left(p+8,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] \blacksquare$$

Result 6.1 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$, then

$$E\left[\left(\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right)^2\right] = 3tr\mathbf{D}+6\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2} + \left(\frac{\boldsymbol{\eta}'\mathbf{D}^{\frac{1}{2}}\boldsymbol{\eta}}{\sigma^2}\right)^2 \tag{1.41}$$

Result 6.2 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = 1$ and $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$, then

$$E\left[\left(\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)' D\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\right)^2\right] = 3tr\mathbf{D} \tag{1.42}$$

Result 6.3 (Saleh, 2006, p353) For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}\left(F\right) = I\left(\pounds_n < F_{q,m}\left(\alpha\right)\right)$ the following expectations are defined

$$E\left[\varphi\left(\chi^2_{(p+4,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+4,\xi}}{\chi^2_{n-p}}\right)\right)$$
$$= E\left[I\left(\pounds_n < F_{q,m}\left(\alpha\right)\right)\right] = E\left\{I\left(F_{q+4,n-p,\xi} < \tfrac{q}{q+4}F_{q,m}\left(\alpha\right)\right)\right\};$$
$$E\left[\varphi\left(\chi^2_{(p+6,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+6,\xi}}{\chi^2_{n-p}}\right)\right)$$
$$= E\left[I\left(\pounds_n < F_{q,m}\left(\alpha\right)\right)\right] = E\left\{I\left(F_{q+6,n-p,\xi} < \tfrac{q}{q+6}F_{q,m}\left(\alpha\right)\right)\right\} \text{ and}$$
$$E\left[\varphi\left(\chi^2_{(p+8,\xi)}\right)\right] = E\left(\varphi\left(\tfrac{n-p}{q}\tfrac{\chi^2_{q+8,\xi}}{\chi^2_{n-p}}\right)\right)$$
$$= E\left[I\left(\pounds_n < F_{q,m}\left(\alpha\right)\right)\right] = E\left\{I\left(F_{q+8,n-p,\xi} < \tfrac{q}{q+8}F_{q,m}\left(\alpha\right)\right)\right\}$$

with $\xi = \frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}$ the noncentrality parameter and $F_{q,m}\left(\alpha\right)$ the critical value of the F-distribution with q and m degrees of freedom and α is the level of significance

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therefore

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}'}{\sigma}\mathbf{D}\frac{\boldsymbol{\omega}}{\sigma}\right)^2\right] = 3tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] + 6\frac{\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}}{\sigma^2}E\left[\varphi\left(\chi^2_{\left(p+6,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] + \left(\frac{\boldsymbol{\eta}'\mathbf{D}^{\frac{1}{2}}\boldsymbol{\eta}}{\sigma^2}\right)^2 E\left[\varphi\left(\chi^2_{\left(p+8,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] \tag{1.43}$$

Result 6.4 For $\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right) = I_{NR}(F) = I\left(\pounds_n < F_{q,m}(\alpha)\right)$, since $E\left[\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right] = \mathbf{0}$, then

$$E\left[\varphi\left(\frac{\boldsymbol{\omega}'}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}\right)\left(\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma} - \frac{\boldsymbol{\eta}}{\sigma}\right)\right)^2\right] = 3tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right] \tag{1.44}$$

Theorem 7 *(Stein's identity) If* $\boldsymbol{\omega}$ *is* $N_p(\boldsymbol{\eta}, \sigma^2\mathbf{I}_p)$ *and if a function* $\phi(\boldsymbol{\omega})$ *is partially differentiable, that is*

$$\left|\frac{\partial\phi_i(\boldsymbol{\omega})}{\partial\omega_i}\right| < \infty, \quad i = 1, 2, ..., p$$

and $\phi(\boldsymbol{\omega})$ *is a continuous function of* ω_i *for all vectors* $\boldsymbol{\omega}_{[i]} = (\omega_1, \omega_2, ..., \omega_{i-1}, \omega_{i+1}, ...\omega_p)'$ *then*

$$E\left\{[\phi(\boldsymbol{\omega})]'(\boldsymbol{\omega} - \boldsymbol{\eta})\right\} = \sigma^2 E\left[\sum_{i=1}^{p}\frac{\partial\phi_i(\boldsymbol{\omega})}{\partial\omega_i}\right]$$

provided that $E\|\phi(\boldsymbol{\omega})\|^2 < \infty$ *where* $\|.\|^2$ *is the squared norm of the function.*

Result 7.1 $E\left[(\boldsymbol{\omega} - \boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega} - \boldsymbol{\eta})(\boldsymbol{\omega} - \boldsymbol{\eta})'\right] = E\left[(\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} - 2\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} + \boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta})(\boldsymbol{\omega} - \boldsymbol{\eta})'\right]$

$= E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}(\boldsymbol{\omega} - \boldsymbol{\eta})'\right]$

(1.45)

$-2E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta}(\boldsymbol{\omega} - \boldsymbol{\eta})'\right]$

(1.46)

$+\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}E(\boldsymbol{\omega} - \boldsymbol{\eta})'$

(1.47)

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Expectation (1.45): Using theorem 7 and taking $\phi(\boldsymbol{\omega}') = \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}$ the following expression was obtained

$$E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right] = \sigma^2 E\left[\frac{\partial \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}}{\partial \boldsymbol{\omega}}\right] = 2\sigma^2 E\left[\boldsymbol{\omega}'\mathbf{D}\right] = 2\sigma^2 E\left[\boldsymbol{\omega}'\right]\mathbf{D} = 2\sigma^2\boldsymbol{\eta}'\mathbf{D}.$$

Expectation (1.46): Using theorem 7 and taking $\varphi(\boldsymbol{\omega}') = \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta}$

$$-2E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right] = -2\sigma^2 E\left[\frac{\partial \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta}}{\partial \boldsymbol{\omega}}\right] = \sigma^2\boldsymbol{\eta}'\mathbf{D}.$$

Expectation (1.47): Since $E\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)' = \mathbf{0}'$

$$\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}E\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)' = \mathbf{0}'$$

Combining the simplified expressions of (1.45) - (1.47) the following expression was obtained:

$$E\left[\left(\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} - \mathbf{2}\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} + \boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right] = 2\sigma^2\boldsymbol{\eta}'\mathbf{D} - 2\sigma^2\boldsymbol{\eta}'\mathbf{D} = \mathbf{0}'.$$

(1.48)

Result 7.2 $E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)I_{NR}\left(F\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]$

$$= E\left[\left(\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} - \mathbf{2}\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} + \boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)' I_{NR}\left(F\right)\right]$$

$$= E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} I_{NR}\left(F\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]$$

(1.49)

$$-2E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} I_{NR}\left(F\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]$$

(1.50)

$$+\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}E\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)I_{NR}\left(F\right)\right)'$$

(1.51)

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Expectation (1.49): Using theorem 7 and taking $\phi(\boldsymbol{\omega}') = \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}$ the following expression was obtained

$$E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right] = \sigma^2 E\left[\frac{\partial \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} I_{NR}(F)}{\partial \boldsymbol{\omega}}\right] = 2\sigma^2 E\left[\boldsymbol{\omega}' I_{NR}(F)\mathbf{D}\right]$$

$$= 2\sigma^2 E\left[\boldsymbol{\omega}' I_{NR}(F)\right]\mathbf{D} = 2\sigma^2\boldsymbol{\eta}'\mathbf{D}.E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (1.50): Using theorem 7 and taking $\varphi(\boldsymbol{\omega}') = \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} I_{NR}(F)$

$$-2E\left[\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} I_{NR}(F)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right] = -2\sigma^2 E\left[\frac{\partial \boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} I_{NR}(F)}{\partial \boldsymbol{\omega}}\right] = -2\sigma^2\boldsymbol{\eta}'\mathbf{D}E\left[I_{NR}(F)\right].$$

Expectation (1.51): Since $E\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)' = \mathbf{0}'$

$$\boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta}E\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)I_{NR}(F)\right)' = \mathbf{0}'$$

Combining the simplified expressions of (1.49) - (1.51) the following expression was obtained:

$$E\left[\left(\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\omega} I_{NR}(F) - 2\boldsymbol{\omega}'\mathbf{D}\boldsymbol{\eta} I_{NR}(F) + \boldsymbol{\eta}'\mathbf{D}\boldsymbol{\eta} I_{NR}(F)\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]$$

$$= 2\sigma^2\boldsymbol{\eta}'\mathbf{D}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] - 2\sigma^2\boldsymbol{\eta}'\mathbf{D}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] = \mathbf{0}'. \tag{1.52}$$

Cauchy-Schwarz inequality Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be nonnegative numbers, then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \tag{1.53}$$

Furthermore, the equality holds if and only if $[x_1.....x_n]'$ and $[y_1.....y_n]'$ are linearly dependent.

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1.5 Algorithm

The following algorithm is used in the simulation exercises in section 2.5, section 3.5 and section 4.3.4:

1. $p+1$ standardised normal random variables are generated, namely

$$z_{i\kappa} \sim N(0,1) \qquad i = 1, 2, ..., n \text{ and } \kappa = 1, 2, ...p+1$$

2. p multicollinear explanatory variables are computed by using the technique described by McDonald and Galarneau (1975), namely

$$x_{ij} = \left(1-\gamma^2\right)^{\frac{1}{2}} z_{ij} + \gamma z_{i(p+1)} \qquad i = 1, 2, ..., n \text{ and } j = 1, 2, ..., p$$

where γ is chosen in such a way that the correlation between any two explanatory variables are given by γ^2.

3. The explanatory variables generated in the previous step are standardised in order to ensure that $\mathbf{X}'\mathbf{X}$ is in correlation form.

4. The true $\boldsymbol{\beta}$ vector is calculated as the normalised eigenvector corresponding to the largest eigenvalue of the $\mathbf{X}'\mathbf{X}$ matrix, subject to the constraint that $\boldsymbol{\beta}'\boldsymbol{\beta} = 1$.

5. Error terms, e_i, are generated from a $N(0, \sigma^2)$ distribution.

6. The dependent variable,y_i, is generated using $\boldsymbol{\beta}$ and e_i :

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + ... + e_i \qquad i = 1, 2, ..., n$$

7. Seven different ridge parameters, k, are calculated based on the definitions on pages 110 and 111 of the thesis.

8. The three proposed estimators, namely URRE, RRRE and PTRRE are each calculated seven times, using the different ridge parameters in the previous step.

9. The three loss functions, namely reflected normal loss, LINEX loss and BLINEX loss (as defined on pages 28, 118 and 134) are calculated for each proposed estimator calculated in the previous step.

10. This process is repeated 2000 times for every proposed estimator under each for the three loss functions and the risk values (expected or average loss) for each proposed estimator are calculated and tabulated.

11. This process is also repeated by changing specific parameters, namely the variance, σ^2, the level of multicollinearity present, γ, the number of parameters to be estimated,p, and the sample size, n.

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Chapter 2

Performance of the preliminary test ridge regression estimator using reflected normal loss

2.1 Reflected normal loss function

In order to evaluate the performance of the preliminary test estimator under the reflected normal loss the risk functions of the three estimators, namely the unrestricted ridge regression estimator, the restricted ridge regression estimator and the preliminary test ridge regression estimator were derived. The reflected normal loss function, which is based on the reflection of the normal density function, was first proposed by Taguchi (1986) where he motivated the use of this loss function in practical applications in quality engineering. The function is of practical use since it is a bounded loss function and can be customised in order to allow the practitioner to set the maximum allowable loss depending on his/her requirements. Ghari et al. (2009) considered the estimation of the mean vector of a multivariate normal model under the reflected normal loss function. Assume $\boldsymbol{\beta}^*$ is a $p\times 1$ vector containing regression coefficient estimators and $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown regression coefficient parameters. The multivariate reflected normal loss function is given by

$$\mathcal{L}_{RN}\left(\boldsymbol{\beta}^*, \boldsymbol{\beta}\right) = c\left[1 - \exp\left(-\frac{\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right)' \mathbf{W}\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right)}{2\gamma^2}\right)\right]$$

where c is the maximum loss for the function, γ is a shape parameter and $\mathbf{W}$ is a matrix containing weights. The loss function can also be transformed into an asymmetric loss function, by specifying different maximum loss values and different scaling parameters to the left and right of $\boldsymbol{\beta}$ which is referred to the extended reflected normal loss function by Spiring (1993). The shape parameter $\gamma = \frac{\Delta^*}{4}$ can be used to change the general form of the loss function, where Δ^* represents the distance from $\boldsymbol{\beta}$ to the point where the maximum loss c occurs for the first time, therefore at $\boldsymbol{\beta}+\Delta^*$.

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The following panel of sketches illustrate the effect of different values of c and γ on the reflected normal loss function for $p = 2$:

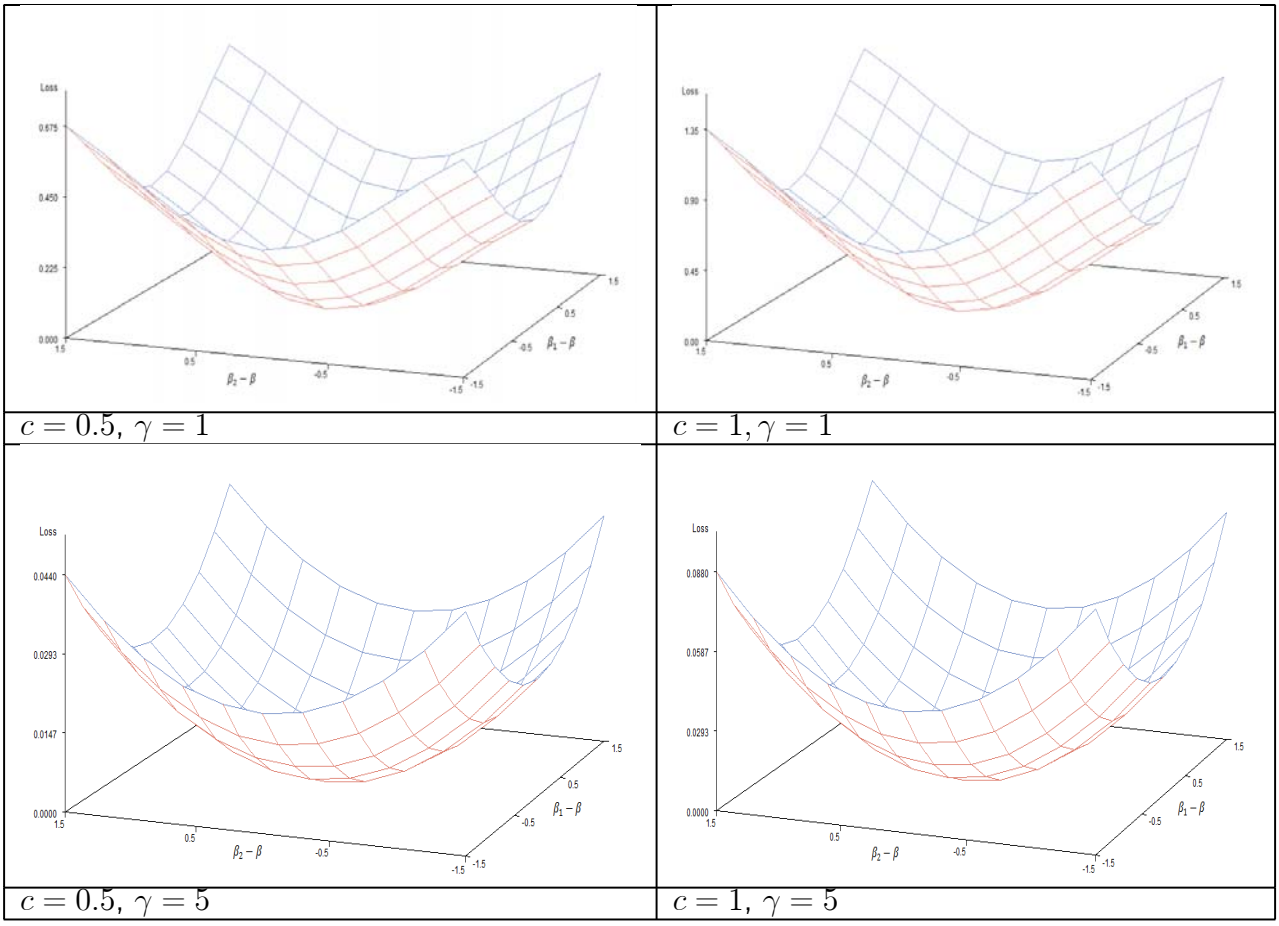

Figure 2.1:The effect of c and γ on the reflected normal loss function.

The increase in c, the bounding parameter of the reflected normal loss function will result in an increase in allowable loss. The scale parameter γ influences the shape of the loss function. The sketches above illustrate that as γ decreases the curve is flatter compared to larger γ values. The scale parameter can be used to transform the reflected normal loss function from a symmetric to asymmetric loss function, known as the extended reflected normal loss function.

The risk function under reflected normal loss is defined as:

$$\Re_{RN}\left(\boldsymbol{\beta}^{*}, \boldsymbol{\beta}\right) = cE\left[1 - \exp\left(-\frac{\left(\boldsymbol{\beta}^{*} - \boldsymbol{\beta}\right)' \mathbf{W}\left(\boldsymbol{\beta}^{*} - \boldsymbol{\beta}\right)}{2\gamma^{2}}\right)\right]$$

In order to evaluate all the estimators on the same scale, the weight matrix $\mathbf{W}$ was consistently chosen as the identity matrix of order p, namely $\mathbf{I}_p$.

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2.2 Risk function of the URRE

Theorem 8 *Ignoring the terms of order 3, the approximate risk function of the URRE under reflected normal loss is given by:*

$$E\left[\mathcal{L}_{RN}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] \approx -cE\left[\sum_{r=1}^{2}\frac{\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^r}{(-1)^r(2\gamma^2)^r r!}\right]$$

$$= -c\left[\frac{(\sigma^2 tr\mathbf{D}++k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta})}{-2\gamma^2}+\right.$$

$$\left.\frac{\left(3\sigma^4 tr\mathbf{D}+2k^2\sigma^2 tr\mathbf{D}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}+4k^2\sigma^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}+k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2\right)}{8\gamma^4}\right]$$

Proof By choosing $\mathbf{W}=\mathbf{I}_p$ it follows that the risk function for $\widetilde{\boldsymbol{\beta}}_n(k)$ under reflected normal loss function is:

$$\Re_{RN}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)=E\left[\mathcal{L}_{RN}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]=cE\left[1-\exp\frac{\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right]$$

and from the Taylor expansion

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+O\left(x^3\right)$$

the risk function reduces to

$$-cE\left[\sum_{r=1}^{2}\frac{\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^r}{(-1)^r(2\gamma^2)^r r!}+\ldots\right] \tag{2.1}$$

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The first term of the expansion (2.1) namely

$$-c\frac{E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]}{-2\gamma^2}$$

will now be considered.

The expectation in the numerator namely

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right] \tag{2.2}$$

can be simplified by adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ within each bracket:

$$= E\left[\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$= \left[E\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right.$$

$$\left.+2\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right] \tag{2.3}$$

$$+2E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right] \tag{2.4}$$

$$+E\left[\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right] \tag{2.5}$$

Expression (2.3): Using (1.8):

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right] = E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)'\mathbf{R}^2(k)\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)\right]$$

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Using the results (1.21) and (1.15) the following result was obtained

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)'\mathbf{R}^2(k)\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)\right]=E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right]$$

$$=E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right]$$

$$=\sigma^2E\left[\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)\right]$$

By using (1.34) it was found that

$$\sigma^2E\left[\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)\right]=\sigma^2tr\mathbf{D}$$

Expression (2.4): From (1.8) and from the fact that $E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)\right]=\mathbf{0}$

$$2E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]=2E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\right](\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$=2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)\right](\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$=2\mathbf{R}(k)E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)\right](\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})=0$$

Expression (2.5): Using (1.28) the following is obtained

$$E\left[(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]=(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$=\boldsymbol{\beta}'(\mathbf{R}(k)-\mathbf{I}p)'(\mathbf{R}(k)-\mathbf{I}p)\boldsymbol{\beta}$$

$$=k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

(2.6)

By combining the simplified expressions of (2.3), (2.4) and (2.5) an expression for the numerator (2.2) was obtained:

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]=\sigma^2tr\mathbf{D}+k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

(2.7)

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The second term of the expansion (2.1) namely

$$-c\frac{E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^2}{(-1)^2(2\gamma^2)^2\,2!}$$

will now be considered.

The expectation in the numerator

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^2 \tag{2.8}$$

can be simplified by adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ in both brackets:

$$= E\left[\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]^2$$

$$= \Big[E\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)$$

$$+2\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\Big]^2$$

Using the definition of (1.8) and (1.21) and simplifying the expectation it was found that:

$$= E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})+2\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\mathbf{R}'(k)\left(\mathbf{R}(k)-\mathbf{I}_p\right)\boldsymbol{\beta}+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]^2$$

Multiplying the brackets out if was found that:

$$= E\Big[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)$$

$$+4\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\mathbf{R}'(k)\left(\mathbf{R}(k)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)$$

$$+2\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$+4\left(\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\mathbf{R}'(k)\left(\mathbf{R}(k)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)'\left(\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\mathbf{R}'(k)\left(\mathbf{R}(k)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)$$

$$+4\left(\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\mathbf{R}'(k)\left(\mathbf{R}(k)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$+\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\Big]$$

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Taking the expectation of each term in the bracket resulted in

$$= E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)^2\right] \tag{2.9}$$

$$+4E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta}\right) \tag{2.10}$$

$$+2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right) \tag{2.11}$$

$$+4\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)'\mathbf{R}\left(k\right)E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta} \tag{2.12}$$

$$+4E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\left(\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right) \tag{2.13}$$

$$+E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right] \tag{2.14}$$

Expression (2.9): Using (1.42) resulted in

$$E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)^2\right] = \sigma^4 E\left[\left(\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)\right)^2\right] = 3\sigma^4 tr\mathbf{D}$$

Expression (2.10): Using (1.48) it was found that

$$4E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta}\right) = 0$$

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Expression (2.11): Using (1.34) and (1.28) it was found that

$$2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$=2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}p\right)'\left(\mathbf{R}\left(k\right)-\mathbf{I}p\right)\boldsymbol{\beta}=2k^{2}\sigma^{2}tr\mathbf{D}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

Expression (2.12):

$$4\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

This expression can be simplified by noting that $E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]=\sigma^{2}E\left[\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)'\right]$

Using (1.38) resulted in

$$E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]=\sigma^{2}\mathbf{I}_{p}$$

Therefore expression (2.12) is equal to

$$4\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{I}_{p}\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

since $\mathbf{Q}$ is an orthogonal matrix it follows that $\mathbf{Q}'\mathbf{Q}=\mathbf{I}$, therefore

$$4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

Finally using (1.28) resulted in

$$4\sigma^{2}\boldsymbol{\beta}'\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_{p}\right)\boldsymbol{\beta}$$

$$=4k^{2}\sigma^{2}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

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Expression (2.13): Using (1.30) it was found that

$$4E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\left(\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$=4E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}'\mathbf{C}^{-\frac{1}{2}}\left(\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)-\mathbf{I}_p\right)\boldsymbol{\beta}\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$=0$$

Expression (2.14): Using (1.28) this expression reduces to

$$E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$=k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2$$

Combining all the simplified expression of (2.9) to (2.14) it follows that$E\left[\left(\widetilde{\boldsymbol{\beta}}_n\left(k\right)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n\left(k\right)-\boldsymbol{\beta}\right)\right]^2$

$$=3\sigma^4 tr\mathbf{D}+2k^2\sigma^2 tr\mathbf{D}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}+4k^2\sigma^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}+k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2 \tag{2.15}$$

Substituting (2.7) and (2.15) into (2.1) completes the proof.■

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2.3 Risk function of the RRRE

Theorem 9 *Ignoring the terms of order 3, the approximate risk function of the RRRE under reflected normal loss is given by:*

$$E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] \approx \frac{c}{2\gamma^2}\left[\sigma^2 tr\mathbf{D}-\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right.$$

$$\left.-2k\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}+k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right]$$

$$-\frac{c}{8\gamma^4}\left[3\sigma^4 tr\mathbf{D}-12\sigma^4 tr\mathbf{D}_{11}-8\sigma^4 tr\mathbf{D}_{11}^*-4\sigma^4 tr\mathbf{D}_{22}tr\mathbf{D}_{11}+6\sigma^4 tr\mathbf{D}_{11}+2\sigma^2 tr\mathbf{D}_{11}\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right.$$

$$+2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]+4k\sigma^2 tr\mathbf{D}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$+2k^2\sigma^2 tr\mathbf{D}\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)+12\sigma^4 tr\mathbf{D}_{11}+24\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+4\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$$

$$+4\boldsymbol{\eta}_1'\mathbf{D_{11}}\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D_{11}}\boldsymbol{\eta}_1-16\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D}_{11}\boldsymbol{\eta}_1-8\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right)$$

$$+4\sigma^2 tr\mathbf{D}^*-12\sigma^4 tr\mathbf{D}_{11}-24\sigma^2\boldsymbol{\eta}_1\mathbf{D}_{11}\boldsymbol{\eta}_1-4\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D_{11}}\boldsymbol{\eta}_1$$

$$+4\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1+8k\sigma^2\boldsymbol{\eta}_1'\left[\mathbf{D}_{11}\mathbf{j}_1+\mathbf{D}_{12}\mathbf{j}_2\right]$$

$$-8k\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{j}_1^*-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^*-8k\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

$$-8k\boldsymbol{\eta}_1'\mathbf{D}_{12}\left[\sigma^2\mathbf{I}_{p-q}+\boldsymbol{\eta}_2\boldsymbol{\eta}_2'\right]\mathbf{j}_2^*+8k\left(\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*-4k^2\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$+3\sigma^4 tr\mathbf{D}_{11}+6\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1+8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^*$$

$$-4k\sigma^2\left[tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{j}_1^*\right)+4k\left[\sigma^2 trD_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

$$+2k^2\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)+4k^2\sigma^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$-8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2\mathbf{I}_p+\boldsymbol{\eta}\boldsymbol{\eta}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$+8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$+\left[4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2\mathbf{I}_p+\boldsymbol{\eta}\boldsymbol{\eta}'\right]\right.$$

$$\left.\times\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right]$$

$$\left.+4k^3\left(\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)+k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2\right]$$

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with $\mathbf{j}_{p\times 1} = \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \end{pmatrix}$ where $\mathbf{j}_1 : q\times 1$ and $\mathbf{j}_2 : (p-q)\times 1$,

$\mathbf{j}^*_{p\times 1} = \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} = \begin{pmatrix} \mathbf{j}^*_1 \\ \mathbf{j}^*_2 \end{pmatrix}$ where $\mathbf{j}^*_1 : q\times 1$ and $\mathbf{j}^*_2 : (p-q)\times 1$,

$\mathbf{D}^* = \mathbf{D}'_{12}\left[\sigma^2\mathbf{I}_q + \boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D}_{12}$ and $\mathbf{D}^*_{11} = \mathbf{D}'_{21}\mathbf{D}'_{12}$

Proof. By choosing $\mathbf{W} = \mathbf{I}_p$ it follows that the risk function for $\widehat{\boldsymbol{\beta}}_n(k)$ under reflected normal loss function is:

$$E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] = cE\left[1-\exp\frac{\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right]$$

and using Taylor's expansion the risk function reduced to

$$-cE\left[\sum_{r=1}^{2}\frac{\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^r}{(-1)^r(2\gamma^2)^r\,r!}+O\left(x^3\right)\right] \tag{2.16}$$

The first term in the expansion (2.16) above namely

$$-c\frac{E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]}{-2\gamma^2}$$

will now be considered.

The expectation in the numerator namely

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]$$

can be simplified by adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ within each bracket.

$$= E\left[\left(\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$= E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right.$$

$$\left.+2\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

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$$= E\left[\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)\right] \tag{2.17}$$

$$+2E\left[\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right] \tag{2.18}$$

$$+E\left[\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right] \tag{2.19}$$

Expression (2.17): Using (1.11) and (1.25) it was found that

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)\right] = E\left[\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta}) - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)'\mathbf{R}'(k)\right.$$

$$\left.\times\mathbf{R}(k)\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta}) - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)\right]$$

$$= E\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)\right]$$

$$-2E\left[\left(\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)\right]$$

$$+E\left[\left(\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)\right]$$

$$= E\left[(\boldsymbol{\omega} - \boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right]$$

$$-2E\left[\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right]$$

$$+E\left[\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right]$$

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Using (1.15), (1.16) and (1.17) the above three expectations reduces to

$$= E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\,\mathbf{D}\,(\boldsymbol{\omega}-\boldsymbol{\eta})\right] \tag{2.20}$$

$$-2E\left[\boldsymbol{\omega}'\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}(\boldsymbol{\omega}-\boldsymbol{\eta})\right] \tag{2.21}$$

$$+E\left[\boldsymbol{\omega}'\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\boldsymbol{\omega}\right] \tag{2.22}$$

Expectation (2.20): Using (1.34) the following was obtained

$$E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\,\mathbf{D}\,(\boldsymbol{\omega}-\boldsymbol{\eta})\right] = \sigma^2 tr\mathbf{D}$$

Expectation (2.21)

$$-2E\left[\boldsymbol{\omega}'\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}(\boldsymbol{\omega}-\boldsymbol{\eta})\right]$$

$$= -2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\,(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\,(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]$$

$$= -2E\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)+2E\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\right)\boldsymbol{\eta}_1-2E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2\right)+2E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2\right)$$

since $E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2\right) = E\left\{E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2|\boldsymbol{\omega}_2\right)\right\}$ by using double expectations, it was found that

$$-2E\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)+2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1-2E\left\{E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2|\boldsymbol{\omega}_2\right)\right\}+2\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2$$

using (1.33) and $E\left\{E\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2|\boldsymbol{\omega}_2\right)\right\} = \boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2$ resulted in

$$= -2\sigma^2 tr\mathbf{D}_{11}-2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1-2E\left(\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\omega}_2\right)+2\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2$$

$$= -2\sigma^2 tr\mathbf{D}_{11}$$

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Expectation (2.22): Using (1.33) the result was

$$E\left[\boldsymbol{\omega}'\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_q & 0\\ 0 & 0\end{bmatrix}\boldsymbol{\omega}\right] = E\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]$$

$$= \sigma^2 tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Combining the results of (2.20) - (2.22) a simplified expression for (2.17) was obtained

$$E\left[\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)\right]$$

$$= \sigma^2 tr\mathbf{D} - \sigma^2 tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expression (2.18): Using (1.11), (1.25) and (1.28) this expression simplifies to

$$-2E\left[\left(\widehat{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right] = -2E\left[\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right]$$

$$= -2E\left[\left(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}'(k)\left(\mathbf{R}(k) - \mathbf{I}_p\right)\boldsymbol{\beta}\right]$$

$$= 2kE\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta}) - \mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)'\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\right]\boldsymbol{\beta}$$

$$= 2kE\left[\left(\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)'\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\right]\boldsymbol{\beta} - 2kE\left[\left(\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\right)'\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\right]\boldsymbol{\beta}$$

$$= 2k\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}E\left[(\boldsymbol{\omega} - \boldsymbol{\eta})'\right]\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} - 2kE(\boldsymbol{\omega}')\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

using (1.29) and (1.30) it was found that

$$-2k\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expression (2.19): Using (2.6) resulted in

$$E\left[(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta})\right] = (\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta})$$

$$= k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

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By combining the simplified expressions of (2.17), (2.18) and (2.19) the following expression was obtained

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]$$
$$=\sigma^2 tr\mathbf{D}-\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1-2k\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$
$$+k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta} \tag{2.23}$$

The second term of the expansion (2.16) namely

$$-c\frac{E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^2}{(2\gamma^2)^2\,2!} \tag{2.24}$$

will now be considered.

The expectation in the numerator

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]^2$$

can be simplified by adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ in each bracket

$$=E\left[\left(\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]^2$$
$$=E\left[\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right.$$
$$\left.+2\left(\widehat{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]^2$$

using (1.11) it was found that

$$=E\left[\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)\right.$$
$$\left.+2\left(\mathbf{R}(k)\widehat{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]^2$$

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can be extended to the following form by using (1.25)

$$= E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right.\right.$$

$$-2\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})$$

$$+\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}$$

$$+2(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$\left.\left.+(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)^2\right]$$

Using (1.14), (1.15), (1.16) and (1.17) the expectation reduces to

$$= E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})-2\boldsymbol{\omega}'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(\boldsymbol{\omega}-\boldsymbol{\eta})+\boldsymbol{\omega}'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}\right.$$

$$+2(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})-2\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$\left.+(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)^2\Big]$$

By multiplying the brackets out resulted in

$$= E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)^2\right]$$

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)\right]$$

$$+2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)\right]$$

$$+4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)\right]$$

$$-4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)\right]$$

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$$-8E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$+8E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$-4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+E\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)$$

$$+4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$-4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$+2E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$-8E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$+4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+4E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right.$$

$$\left.\times\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$-4E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$+E\left[\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

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Using (1.28) and (1.30) this expression reduces to

$$= E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)^{2}\right]$$

(2.25)

$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)\right]$$

(2.26)

$$+2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)\right]$$

(2.27)

$$-4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.28)

$$+4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\boldsymbol{\omega}'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.29)

$$+2k^{2}E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

(2.30)

$$+4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)\right]$$

(2.31)

$$-4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

(2.32)

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$$+8kE\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} \tag{2.33}$$

$$-8kE\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} \tag{2.34}$$

$$-4k^2E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta} \tag{2.35}$$

$$+E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right] \tag{2.36}$$

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right)\right] \tag{2.37}$$

$$+4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'(\boldsymbol{\omega}')\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right) \tag{2.38}$$

$$+2k^2E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\right]\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right) \tag{2.39}$$

$$+4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\mathbf{C}^{-\frac{1}{2}}(k)\,\mathbf{Q}'E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta} \tag{2.40}$$

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$$-8k^2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right] \tag{2.41}$$

$$+4k^2E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right] \tag{2.42}$$

$$+4k^3\left(\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right) \tag{2.43}$$

$$+k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2 \tag{2.44}$$

In order to simplify these expressions, each one of the above expressions has to be considered separately:

Expression (2.25): Using (1.42) it was found that

$$E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)^2\right]=\sigma^4E\left[\left(\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)'\mathbf{D}\left(\frac{\boldsymbol{\omega}}{\sigma}-\frac{\boldsymbol{\eta}}{\sigma}\right)\right)^2\right]$$

$$=3\sigma^4tr\mathbf{D}$$

Expression (2.26):

$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left[\begin{array}{cc}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{array}\right]\left[\begin{array}{cc}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{array}\right]\right)\right]$$

$$=-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left[\begin{array}{cc}\boldsymbol{\omega}_1'\mathbf{D}_{11} & \boldsymbol{\omega}_1'\mathbf{D}_{12}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{array}\right]\right)\right]$$

$$=-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\right]$$

$$=-4E\left[\left(\begin{array}{cc}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' & \left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\end{array}\right)\left[\begin{array}{cc}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{array}\right]\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\right]$$

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$$= -4E\left[\begin{pmatrix} (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21} & (\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega}_1-\boldsymbol{\eta}_1 \\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2 \end{pmatrix}\right.$$

$$\left.\times\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right)\right]$$

$$= -4E\left[\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right.\right.$$

$$\left.+(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right.$$

$$\left.\left.\times\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$$

(2.45)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

(2.46)

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$$

(2.47)

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

(2.48)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$$

(2.49)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

(2.50)

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$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right] \tag{2.51}$$

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right] \tag{2.52}$$

Expectation (2.45): Using (1.42) and (1.48) it was found that

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\right]\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)\right)\right]$$

$$=-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2\right]-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$=-12\sigma^4 tr\mathbf{D}_{11}$$

Expectation (2.46):

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]\right]=0 \text{ since } E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]=\mathbf{0}$$

Expectation (2.47):

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\middle|\,\boldsymbol{\omega}_1\right]=0 \text{ since } E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$$

Expectation (2.48):

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{12}'\boldsymbol{\omega}_1\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{12}'\boldsymbol{\omega}_1\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{12}'\boldsymbol{\omega}_1\right]$$

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using (1.38) resulted in

$E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\sigma^2\mathbf{I}_{p-q}$ therefore

$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}'_{21}E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}'_{12}\boldsymbol{\omega}_1\right]$

$=-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}'_{21}\mathbf{D}'_{12}\boldsymbol{\omega}_1\right]$

$=-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}'_{21}\mathbf{D}'_{12}((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)\right]$

$=-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}'_{21}\mathbf{D}'_{12}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}'_{21}\mathbf{D}'_{12}\boldsymbol{\eta}_1\right]$

with $\mathbf{D}'_{21}\mathbf{D}'_{12}=\mathbf{D}^*_{11}$ it was found that

$=-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}^*_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]-4\sigma^2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$

using (1.30) and (1.34) it was found that

$=-4\sigma^4tr\mathbf{D}^*_{11}$

Expectation (2.49):

$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]=0$

(see simplification of (2.46))

Expectation (2.50):

$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}'_1\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\right]=-4\sigma^4tr\mathbf{D}^*_{11}$

(see simplification of (2.48))

Expectation (2.51): Using (1.34) resulted in

$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$

$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\middle|\,\boldsymbol{\omega}_1\right]\right\}$

$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]$

$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\right]$

$=-4\sigma^2tr\mathbf{D}_{22}E\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]$

$=-4\sigma^2tr\mathbf{D}_{22}\left[E\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}\boldsymbol{\omega}_1\right]-E\left[\boldsymbol{\omega}'_1\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\right]$

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using (1.29) and (1.33) resulted in

$$-4\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1-\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]=-4\sigma^4 tr\mathbf{D}_{22}tr\mathbf{D}_{11}$$

Expectation (2.52):

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right]$$

$$=-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{12}\boldsymbol{\omega}_1\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{12}\boldsymbol{\omega}_1\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{12}\boldsymbol{\omega}_1\right]$$

using (1.48) it was found that $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$

Therefore

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{12}\boldsymbol{\omega}_1\right]=0$$

By combining (2.45) - (2.52) an expression for (2.26) was obtained

$$-12\sigma^4 tr\mathbf{D}_{11}-8\sigma^4 tr\mathbf{D}_{11}^*-4\sigma^4 tr\mathbf{D}_{22}tr\mathbf{D}_{11}$$

Expression (2.27):

$$2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{bmatrix}\right)\right]$$

$$=2E\left[\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right.\right.$$

$$\left.\left.+(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right](\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]$$

$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right] \tag{2.53}$$

$$+2E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right] \tag{2.54}$$

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$$+2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right] \tag{2.55}$$

$$+2E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right] \tag{2.56}$$

Expectation *(2.53)*:

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]$$

$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left(((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)'\mathbf{D}_{11}((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)\right)\right]$$

$$=2E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2\right]+4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

$$+2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

since $E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]=0$ as shown in (1.48) it was found that

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]$$

$$=2E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2\right]+2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

using (1.34) and (1.42) it was found that

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]=6\sigma^4 tr\mathbf{D}_{11}+2\sigma^2 tr\mathbf{D}_{11}(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1)$$

Expectation *(2.54)*:

$$2E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]$$

$$=E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]=0 \text{ since } E(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'=\mathbf{0}'$$

Expectation (2.55):

In a similar way to (2.54) it can be shown that

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1)\right]=0 \text{ since } E(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)=\mathbf{0}$$

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Expectation (2.56): Using (1.33) and ((1.34) it was found that

$$2E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

$$=2E\left\{E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=2E\left[E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

$$=2\sigma^2 tr\mathbf{D}_{22}E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

$$=2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

Combining (2.53) - (2.56) an expression for (2.27) was obtained:

$$=6\sigma^4 tr\mathbf{D}_{11}+2\sigma^2 tr\mathbf{D}_{11}\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)+2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

Expression (2.28):

$$-4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}=0$$

as shown in (1.48).

Expression (2.29):

$$4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\boldsymbol{\omega}'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'+\boldsymbol{\eta}'\right]\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.57)

$$4kE\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\boldsymbol{\eta}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.58)

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Expectation (2.57): Using (1.48) it was found that

$$4kE\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}=0$$

Expectation (2.58): Using result (1.34) it was found that

$$4kE\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right]\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=4k\sigma^2 tr\mathbf{D}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

By combining (2.57) and (2.58) an expression for (2.29) was obtained:

$$4k\sigma^2 tr\mathbf{D}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expression (2.30): Using (1.34) resulted in

$$2k^2E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right]\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

$$=2k^2\sigma^2 tr\mathbf{D}\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

Expression (2.31):

$$4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2\right]$$

(2.59)

$$+4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right)^2\right]$$

(2.60)

$$+8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right)\right]$$

(2.61)

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Expectation (2.59):

$$4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2\right]=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1-\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)^2\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]+4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)^2\right]-8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right] \tag{2.62}$$

$$+4\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left(\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\right)\mathbf{D}_{11}\boldsymbol{\eta}_1 \tag{2.63}$$

$$-8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right] \tag{2.64}$$

Expectation (2.62): Using (1.41) it was found that

$$4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]=12\sigma^4 tr\mathbf{D}_{11}+24\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+4\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$$

Expectation (2.63): Using (1.37) it was found that

$$4\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left(\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\right)\mathbf{D}_{11}\boldsymbol{\eta}_1=4\boldsymbol{\eta}_1'\mathbf{D}_{11}\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.64): Using (1.33) and the final result of (1.45) it was found that

$$-8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right]$$

$$=-8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]-8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

$$=-16\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D}_{11}\boldsymbol{\eta}_1-8\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)$$ and

Combining (2.62) - (2.64) an expression for (2.59) was obtained:

$$12\sigma^4 tr\mathbf{D}_{11}+24\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+4\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2+4\boldsymbol{\eta}_1'\mathbf{D}_{11}\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$-16\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D}_{11}\boldsymbol{\eta}_1-8\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)$$

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Expectation (2.60): Using result (1.37) it was found that

$$4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)'\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\right]=4E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{12}'\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]$$

$$=4E\left[E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{12}'\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)|\boldsymbol{\omega}_2\right]\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{12}'E\left(\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\right)\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{12}'\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]$$

Assuming that $\mathbf{D}^*=\mathbf{D}_{12}'\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D}_{12}$ is a symmetric positive definite matrix, thus using (1.34) resulted in

$$=4E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}^*\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]$$

$$=4\sigma^2 tr\mathbf{D}^*$$

Expectation (2.61):

$$8E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\right]=8E\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)|\boldsymbol{\omega}_1\right]\right]$$

$$=E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\right)\right]=0 \text{ since } E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]=\mathbf{0}$$

By combining (2.59) - (2.61) an expression for (2.31) was obtained:

$$12\sigma^4 tr\mathbf{D}_{11}+24\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+4\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2+4\boldsymbol{\eta}_1'\mathbf{D_{11}}\left[\sigma^2\mathbf{I}_q+\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{D_{11}}\boldsymbol{\eta}_1$$

$$-16\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D}_{11}\boldsymbol{\eta}_1-8\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right)+4\sigma^2 tr\mathbf{D}^*$$

Expression (2.32):

$$-4E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

$$=-4E\left[\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right]\right]$$

$$=-4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right] \tag{2.65}$$

$$-4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right] \tag{2.66}$$

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Expectation (2.65):

$$-4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right]=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1-\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]+4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right]$$

where $E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]=3\sigma^4 tr\mathbf{D}_{11}+6\sigma^2\boldsymbol{\eta}_1\mathbf{D}_{11}\boldsymbol{\eta}_1+\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$ using (1.41)

$$E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right]=E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D_{11}}\boldsymbol{\eta}_1\right)\right]$$

$$=E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\right]\mathbf{D_{11}}\boldsymbol{\eta}_1+E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1$$

using (1.33) and (1.45) it was found that

$$2\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D_{11}}\boldsymbol{\eta}_1+\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1$$

Combining these results an expression for (2.65) was obtained:

$$\text{-}4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]+4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right]$$

$$=-12\sigma^4 tr\mathbf{D}_{11}-24\sigma^2\boldsymbol{\eta}_1\mathbf{D}_{11}\boldsymbol{\eta}_1-4\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$$

$$+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D_{11}}\boldsymbol{\eta}_1+4\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1$$

Expectation (2.66):

$$-4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right]=-4E\left[E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1|\boldsymbol{\omega}_1\right]\right]$$

$$=-4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right]=0 \text{ since } E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)=\mathbf{0}.$$

Combining (2.65) and (2.66) an expression for (2.32) was obtained:

$$-12\sigma^4 tr\mathbf{D}_{11}-24\sigma^2\boldsymbol{\eta}_1\mathbf{D}_{11}\boldsymbol{\eta}_1-4\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$$

$$+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{D_{11}}\boldsymbol{\eta}_1+4\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{D_{11}}\boldsymbol{\eta}_1$$

Expression (2.33):

$$8kE\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Defining $\mathbf{j}_{p\times 1}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}=\begin{pmatrix}\mathbf{j}_1\\ \vdots\\ \mathbf{j}_2\end{pmatrix}$ where $\mathbf{j}_1: q\times 1$ and $\mathbf{j}_2: (p-q)\times 1$.

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Therefore this expression reduces to

$$= 8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2))'\begin{pmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1 & \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{pmatrix}'\begin{pmatrix}\mathbf{j}_1\\ \mathbf{j}_2\end{pmatrix}\right]$$

$$= 8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2))'\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{j}_2\right)\right]$$

$$= 8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1))\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right)\right]$$

(2.67)

$$+8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1))\left((\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{j}_2\right)\right]$$

(2.68)

$$+8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2))\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right)\right]$$

(2.69)

$$+8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2))\left((\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{j}_2\right)\right]$$

(2.70)

Expectation (2.67)

$$8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1))\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right)\right]$$

$$= 8kE\left[\left(((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right)\right]$$

$$= 8kE\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right]+8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right]$$

Since $E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right]=0$ as shown in (1.48)

and

$E\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right]=\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\right]\mathbf{j}_1$ using (1.38) it was found that

$$= \boldsymbol{\eta}_1'\mathbf{D}_{11}\sigma^2\mathbf{I}_q\mathbf{j}_1$$

Combining these results an expression for (2.67) was obtained

$$8kE\left[(\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1))\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{j}_1\right)\right]=8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1$$

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Expectation (2.68)

$$8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right)\right]$$

$$=8kE\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right)|\boldsymbol{\omega}_1\right]\right]$$

$$=8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\mathbf{j}_2\right]=0 \text{ since } E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)=\mathbf{0}'$$

Expectation (2.69)

$$8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1\right)\right]$$

$$=8kE\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1\right)|\boldsymbol{\omega}_1\right]\right]$$

$$=8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1\right]=0 \text{ since } E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'=\mathbf{0}'$$

Expectation (2.70)

$$8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right)\right]$$

$$=8kE\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right)|\boldsymbol{\omega}_1\right]\right]$$

$$=8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{j}_2\right]$$

using (1.38) it was found that

$$8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left[\sigma^2\mathbf{I}_{p-q}\right]\mathbf{j}_2\right]$$

$$=8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{12}\mathbf{j}_2$$

Combining (2.67) - (2.70) an expression for (2.33) was obtained:

$$8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1+8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{12}\mathbf{j}_2=8k\sigma^2\boldsymbol{\eta}_1'\left[\mathbf{D}_{11}\mathbf{j}_1+\mathbf{D}_{12}\mathbf{j}_2\right]$$

Expression (2.34):

$$-8kE\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Define $\mathbf{j}_{p\times1}^*=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}=\begin{pmatrix}\mathbf{j}_1^*\\ \mathbf{j}_2^*\end{pmatrix}$

where $\mathbf{j}_1^*: q\times1$ and $\mathbf{j}_2^*: (p-q)\times1$.

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This expression reduces to

$$-8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{pmatrix}\mathbf{j}_1^* \\ \mathbf{j}_2^*\end{pmatrix}\right]$$

$$=-8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right)\left(\boldsymbol{\omega}_1'\mathbf{j}_1^*+\boldsymbol{\omega}_2'\mathbf{j}_2^*\right)\right]$$

$$=-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*\right]$$

(2.71)

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]$$

(2.72)

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*\right]$$

(2.73)

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]$$

(2.74)

Expectation (2.71):

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*\right]=8kE\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*\right]$$

$$=-8kE\left[\left(\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*+\boldsymbol{\eta}_1'\mathbf{j}_1^*\right)\right]$$

$$=-8kE\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*+\boldsymbol{\eta}_1'\mathbf{j}_1^*\right)\right]$$

$$=-8kE\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*\right]$$

(2.75)

$$-8kE\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{j}_1^*\right]$$

(2.76)

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$$-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*\right] \tag{2.77}$$

$$-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{j}_1^*\right] \tag{2.78}$$

Expectation (2.75):

$-8kE\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*\right]=0$ as shown in (1.48)

Expectation (2.76):

$-8kE\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{j}_1^*\right]=-8k\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{j}_1^*$, using (1.34)

Expectation (2.77):

$-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1^*\right]=-8k\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\right]\mathbf{j}_1^*$

$=-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_q\mathbf{j}_1^*=-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^*$ using (1.38)

Expectation (2.78):

$-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{j}_1^*\right]=0$ using (1.30)

Combining (2.75) - (2.78) an expression for (2.71) was obtained

$-8k\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{j}_1^*-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^*$

Expectation (2.72): Using (1.29) and (1.33) it was found that

$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]=-8kE\left[E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*|\boldsymbol{\omega}_1\right]\right]$

$=-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*\right]=-8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1-\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*\right]$

$=-8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*\right]+8kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*\right]$

$=-8k\left[\sigma^2 tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_2'\mathbf{j}_2^*+8k\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*$

$=-8k\sigma^2 tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{j}_2^*$

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Expectation (2.73):

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*\right]=-8kE\left[E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*|\boldsymbol{\omega}_1\right]\right]$$

$$=-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{j}_1^*\right]=0 \text{ since } E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)=\mathbf{0}.$$

Expectation (2.74): Using (1.29) and (1.37) resulted in

$$-8kE\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]=-8kE\left[E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*|\boldsymbol{\omega}_2\right]\right]$$

$$=-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]$$

$$=-8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2\boldsymbol{\omega}_2'\mathbf{j}_2^*\right)\right]+8kE\left[\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2\boldsymbol{\omega}_2'\mathbf{j}_2^*\right]$$

$$=-8k\boldsymbol{\eta}_1'\mathbf{D}_{12}E\left(\boldsymbol{\omega}_2\boldsymbol{\omega}_2'\right)\mathbf{j}_2^*+8k\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2E\left(\boldsymbol{\omega}_2'\right)\mathbf{j}_2^*$$

$$=-8k\boldsymbol{\eta}_1'\mathbf{D}_{12}\left[\sigma^2\mathbf{I}_{p-q}+\boldsymbol{\eta}_2\boldsymbol{\eta}_2'\right]\mathbf{j}_2^*+8k\left(\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

Combining (2.71) - (2.74) an expression for (2.34) was obtained:

$$-8k\sigma^2tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{j}_1^*-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^*-8k\sigma^2tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{j}_2^*-8k\boldsymbol{\eta}_1'\mathbf{D}_{12}\left[\sigma^2\mathbf{I}_{p-q}+\boldsymbol{\eta}_2\boldsymbol{\eta}_2'\right]\mathbf{j}_2^*+8k\left(\boldsymbol{\eta}_1'\mathbf{D}_{12}\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

Expression (2.35):

$$-4k^2E\left[\left(\begin{bmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{bmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{bmatrix}\right)'\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1-\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

(2.79)

$$-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

(2.80)

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Expectation (2.79): Using (1.29) and (1.33) resulted in

$$-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1-\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}+4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2\left[\sigma^2tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}+4k^2\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2\sigma^2tr\mathbf{D}_{11}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

Expectation (2.80):

$$-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}=-4k^2E\left[E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)|\boldsymbol{\omega}_1\right]\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

$$=-4k^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{12}E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}=0 \text{ since } E\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)=\mathbf{0}$$

Combining (2.79) - (2.80) an expression for (2.35) was obtained

$$-4k^2\sigma^2tr\mathbf{D}_{11}\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

Expression (2.36): Using (1.41) it was found that

$$E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2\right]$$

$$=3\sigma^4tr\mathbf{D}_{11}+6\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1+\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2$$

Expression (2.37):

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right)\right]$$

Define $\mathbf{j}_{p\times1}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)=\begin{pmatrix}\mathbf{j}_1\\ \mathbf{j}_2\end{pmatrix}$ where $\mathbf{j}_1: q\times1$ and $\mathbf{j}_2:(p-q)\times1$.

Therefore expression (2.37) reduces to

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\left(\begin{pmatrix}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' & \left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\end{pmatrix}\begin{pmatrix}\mathbf{j}_1\\ \mathbf{j}_2\end{pmatrix}\right)\right]$$

$$=-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1+\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right)\right]$$

$$=-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1\right] \tag{2.81}$$

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right] \tag{2.82}$$

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Expectation (2.81): Using (1.45) the following was obtained

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{j}_1\right]$$

$$=-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1$$

Expectation (2.82)

$$-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2\right]=-4kE\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{j}_2|\boldsymbol{\omega}_1\right]\right]$$

$$=-4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{j}_2\right]=0 \text{ since } E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]=\mathbf{0}'$$

Combining (2.81) - (2.82) an expression for (2.37) was obtained:

$$-8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1$$

Expression (2.38):

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\left(\boldsymbol{\omega}'\right)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)$$

Define $\mathbf{j}_{p\times 1}^{*}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)=\begin{pmatrix}\mathbf{j}_1^{*}\\ \mathbf{j}_2^{*}\end{pmatrix}$ where $\mathbf{j}_1^{*}: q\times 1$

and $\mathbf{j}_2^{*}: (p-q)\times 1$.

Therefore expression (2.38) reduces to

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\right]\begin{pmatrix}\mathbf{j}_1^{*}\\ \mathbf{j}_2^{*}\end{pmatrix}$$

$$=4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{j}_1^{*}+\boldsymbol{\omega}_2'\mathbf{j}_2^{*}\right)\right]$$

$$=4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_1'\mathbf{j}_1^{*}\right]$$

(2.83)

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^{*}\right]$$

(2.84)

Expectation (2.83): Using (1.33) and (1.45) it was found that

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_1'\mathbf{j}_1^{*}\right]$$

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_1'\right]\mathbf{j}_1^{*}=4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\right]\mathbf{j}_1^{*}$$

$$=4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\right]\mathbf{j}_1^{*}+4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\right]\boldsymbol{\eta}_1'\mathbf{j}_1^{*}$$

$$=8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^{*}-+4k\sigma^2\left[tr\mathbf{D}_{11}+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{j}_1^{*}\right)$$

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Expectation (2.84): Using (1.33) resulted in

$$4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*\right] = 4kE\left[E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_2'\mathbf{j}_2^*|\boldsymbol{\omega}_1\right]\right]$$

$$= 4kE\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\eta}_2'\mathbf{j}_2^*\right]$$

$$= 4k\left[\sigma^2 tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

Combining (2.83) - (2.84) an expression for (2.38) was obtained:

$$8k\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{j}_1^* + 4k\sigma^2\left[tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\eta}_1'\mathbf{j}_1^*\right) + 4k\left[\sigma^2 tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_2'\mathbf{j}_2^*$$

Expression (2.39): Using (1.33) resulted in

$$2k^2E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)'\right]\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

$$= 2k^2\left[\sigma^2 tr\mathbf{D}_{11} + \boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

Expression (2.40): Using (1.38) it was found that

$$4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\mathbf{C}^{-\frac{1}{2}}\left(k\right)\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$= 4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\sigma^2\mathbf{I}_p\right)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$= 4k^2\sigma^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$= 4k^2\sigma^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expression (2.41):

$$-8k^2E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right]$$

$$= -8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$= -8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'-\boldsymbol{\eta}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$= -8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.85)

$$+8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\eta}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

(2.86)

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Expectation (2.85): Using (1.37) it was found that

$$-8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=-8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2\mathbf{I}_p+\boldsymbol{\eta}\boldsymbol{\eta}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expectation (2.86): Using (1.29) it was found that

$$8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\eta}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Combining (2.85) - (2.86) an expression for (2.41) was obtained

$$-8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2\mathbf{I}_p+\boldsymbol{\eta}\boldsymbol{\eta}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$+8k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expression (2.42): Using result (1.37) the following was obtained

$$4k^2E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\right.$$

$$\left.\times\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)\right]$$

$$=4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

$$=4k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-1}\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2\mathbf{I}_p+\boldsymbol{\eta}\boldsymbol{\eta}'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}$$

Expression (2.43):

$$4k^3\left(\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\mathbf{C}_{(k)}^{-1}\boldsymbol{\beta}\right)'\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

Expression (2.44):

$$k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2$$

Combining the final expressions of (2.25) to (2.44) a final expression for (2.8) was obtained

(2.87)

Substituting (2.23) and (2.87) into (2.16) completes the proof.■

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2.4 Risk function of the PTRRE

Theorem 10 *Ignoring the terms of order 3, the approximate risk function of the PTRRE under reflected normal loss is given by:*

$$E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right] \approx \frac{c}{2\gamma^2}\left[\sigma^2 tr\mathbf{D}+2\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\right.$$

$$+2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\left[E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\right]+k^2\boldsymbol{\beta}'\mathbf{C}^{-2}_{(k)}\boldsymbol{\beta}$$

$$+2\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}(k)\left[\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right]\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

$$+\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\Big]$$

$$-\frac{c}{8\gamma^4}\left[3\sigma^4 tr\mathbf{D}-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*\right.$$

$$-4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*-4\sigma^4 tr\mathbf{D}_{22}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}$$

$$+2\sigma^2 tr\mathbf{D}\left(\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$-4\sigma^2 tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$+6\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+2\sigma^2 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$+2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

$$+4\sigma^2\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$-8\sigma^2\mathbf{v}_1'^{*}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1-8\sigma^2\mathbf{v}_2'^{*}\mathbf{D}_{21}\boldsymbol{\eta}_1E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

$$-8\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{s}+8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^*$$

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$$+12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+4\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$+4\sigma^2\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}^*\boldsymbol{\eta}_1\right]$$

$$-4\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$+8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{v}_1^{**}+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_qE\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{v}_1^{**}$$

$$+8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{v}_2^{**}+8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{21}'\mathbf{v}_2^{**}$$

$$-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-4\sigma^2 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$-8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1+k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2$$

$$-4\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

$$+2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

$$+\mathbf{v}_1^{*\prime}\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{I}_q+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{v}_1^*-8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^*$$

$$-4\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{v}_1^*$$

$$+3\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+6\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1E\left[\varphi\left(\chi^2_{\left(q+6,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2E\left[\varphi\left(\chi^2_{\left(q+8,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

with $\mathbf{v}_{p\times1}^{*\prime}=\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'=\begin{pmatrix}\mathbf{v}_1^{*\prime} & \mathbf{v}_2^{*\prime}\end{pmatrix}$ *where* $\mathbf{v}_1^{*\prime}:1\times q$ *and* $\mathbf{v}_2^{*\prime}:1\times(p-q)$,

$\mathbf{v}_{p\times1}^{**}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)=\begin{pmatrix}\mathbf{v}_1^{**}\\ \mathbf{v}_2^{**}\end{pmatrix}$ *where* $\mathbf{v}_1^{**}:q\times1$ *and* $\mathbf{v}_2^{**}:(p-q)\times1$

$\mathbf{D}^*=\sigma^2\mathbf{D}_{12}\mathbf{D}_{21}$ *and* $\mathbf{s}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$ *with* $\mathbf{s}$ *a* $p\times1$ *vector*

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Proof. The risk function for $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$ under reflected normal loss and choosing $\mathbf{W} = \mathbf{I}_p$ can be written as

$$E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)\right] = cE\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)}{-2\gamma^2}\right)\right]$$

using

$$\widehat{\boldsymbol{\beta}}_n^{PT}(k) = \widetilde{\boldsymbol{\beta}}_n(k) - \left(\widetilde{\boldsymbol{\beta}}_n(k) - \widehat{\boldsymbol{\beta}}_n(k)\right) I_{NR}(F) \tag{2.88}$$

and the Taylor's expansion the risk function can be written as:

$$\begin{aligned} E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)\right] &= c - cE\left[\sum_{r=0}^{2} \frac{\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)/-2\gamma^2\right]^r}{r!} + O\left(x^3\right)\right] \\ &= -cE\left[\sum_{r=1}^{2} \frac{\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)\right]^r}{(-1)^r (2\gamma^2)^r r!} + O\left(x^3\right)\right] \end{aligned} \tag{2.89}$$

The first term of the expansion (2.89)

$$-c\frac{E\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)\right]}{-2\gamma^2} \tag{2.90}$$

will now be considered.

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The expectation in the numerator of the first term can be simplified by using (1.5) and (1.13)

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]$$

$$=E\left[\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)'\right.$$

$$\left.\times\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

By adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ in both brackets resulted in

$$=E\left[\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)'\right.$$

$$\left.\times\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

$$=E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)\right]$$

(2.91)

$$+2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

(2.92)

$$-2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

(2.93)

$$+E\left[\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

(2.94)

$$-2E\left[\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

(2.95)

$$+E\left[\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)'\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

(2.96)

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Expression (2.91): Using (1.21), (1.15) and (1.34) it was found that

$$E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}'(k)\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right]$$

$$= E\left[\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}^2(k)\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)\right]$$

$$= E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right]$$

$$= \sigma^2 tr\mathbf{D}$$

Expression (2.92):

$$2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$= 2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\right](\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

$$= 2E\left[\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\right]\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}) = 0 \text{ since } E\left[\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\right] = \mathbf{0}'$$

Expression (2.93): Using result (1.26) resulted in

$$2E\left[\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n - \mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]$$

$$= 2E\left[\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right)'\mathbf{R}^2(k)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right]$$

$$= 2E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 + (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\right)I_{NR}(F)\right]$$

$$= E\left[2\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F) - 2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F) + 2\boldsymbol{\omega}_2'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F) - 2\boldsymbol{\eta}_2'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= 2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] - \left(2\boldsymbol{\eta}_1'\mathbf{D}_{11} + 2\boldsymbol{\eta}_2'\mathbf{D}_{21}\right)E\left[\boldsymbol{\omega}_1 I_{NR}(F)\right] + 2E\left[\boldsymbol{\omega}_2'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= 2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.97}$$

$$-2\left(\boldsymbol{\eta}_1'\mathbf{D}_{11} + \boldsymbol{\eta}_2'\mathbf{D}_{21}\right)E\left[\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.98}$$

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$$+2E\left[\boldsymbol{\omega}_2'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

(2.99)

Expectation (2.97): Using (1.35) it was found that

$$2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11} + 2E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.98): Using (1.31) resulted in

$$-2\left(\boldsymbol{\eta}_1'\mathbf{D}_{11} + \boldsymbol{\eta}_2'\mathbf{D}_{21}\right)E\left[\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= -2\left(\boldsymbol{\eta}_1'\mathbf{D}_{11} + \boldsymbol{\eta}_2'\mathbf{D}_{21}\right)\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (2.99): Using (1.29) and (1.31) the following was obtained

$$2E\left[\boldsymbol{\omega}_2'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right] = 2E\left\{\boldsymbol{\omega}_2'\mathbf{D}_{21}E\left[\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]|\boldsymbol{\omega}_2\right\}$$

$$= 2E\left[\boldsymbol{\omega}_2'\mathbf{D}_{21}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\right]$$

$$= 2\boldsymbol{\eta}_2'\mathbf{D}_{21}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Combining (2.97) - (2.99) an expression for (2.93) was obtained:

$$= 2\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11} + 2E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$-2\left(\boldsymbol{\eta}_1'\mathbf{D}_{11} + \boldsymbol{\eta}_2'\mathbf{D}_{21}\right)\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] + 2\boldsymbol{\eta}_2'\mathbf{D}_{21}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

$$= 2\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11} + 2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\left[E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] - E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\right]$$

Expression (2.94): Using (1.28) it was found that

$$E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\right] = \left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)$$

$$= \boldsymbol{\beta}'\left(\mathbf{R}\left(k\right) - \mathbf{I}_p\right)'\left(\mathbf{R}\left(k\right) - \mathbf{I}_p\right)\boldsymbol{\beta}$$

$$= k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}$$

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Expression (2.95): Using (1.24) and (1.31) resulted in

$$2E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right)\right]$$

$$=2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\left[\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right]E\left[\boldsymbol{\omega}I_{NR}\left(F\right)\right]$$

$$=2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\left[\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right]\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right].$$

Expression (2.96): Since $I_{NR}\left(F\right)\times I_{NR}\left(F\right)=I_{NR}\left(F\right)$ it was found that

$$E\left[\left(\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right)'\left(\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right)\right]$$

$$=E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)'\mathbf{R}'\left(k\right)\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right]$$

$$=E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)'\mathbf{R}^2\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right]$$

Using result (1.27) and (1.35) the following was obtained

$$E\left[\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)'\mathbf{R}^2\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right]=E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1I_{NR}\left(F\right)\right]$$

$$=\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

By combining (2.91) - (2.96) an expression for the numerator of (2.90) was obtained:

$$\sigma^2tr\mathbf{D}+2\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\left[E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\right]$$

$$+k^2\boldsymbol{\beta}'\mathbf{C}^{-2}_{(k)}\boldsymbol{\beta}+2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\left[\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right]\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

$$+\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 \tag{2.100}$$

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The second term of the expansion (2.89) namely

$$-c\frac{E\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]^2}{(2\gamma^2)^2\,2!} \tag{2.101}$$

will be now be considered.

The expected value in the numerator of the previous term

$$E\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]^2$$

can be simplified as follows:

$$\begin{aligned}&E\left[\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]^2\\&=E\left[\left(\left(\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)'\right.\\&\left.\times\left(\left(\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]^2\end{aligned} \tag{2.102}$$

In order to simplify this term, firstly the focus will be on the simplification of

$$\left(\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)$$

Adding and subtracting $\mathbf{R}(k)\,\boldsymbol{\beta}$ then

$$\left(\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\,\boldsymbol{\beta}+\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)$$

$$=\left(\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)$$

Using (1.21) and (1.24) resulted in

$$\left(\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)$$

$$=\left(\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})+\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)-\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)$$

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The simplification continues and therefore

$$\left(\left(\mathbf{R}\left(k\right)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right)'\left(\left(\mathbf{R}\left(k\right)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)-\mathbf{R}\left(k\right)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}\left(F\right)\right)$$

$$=\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)+\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'$$

$$\times\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)+\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$=\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)+2\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$-2\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$+\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(\mathbf{k}\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$+\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$=\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)+2\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$-2\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)$$

$$+\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$+\boldsymbol{\omega}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

since in the last term of the previous expression it is known that $I_{NR}\left(F\right)$ is a constant and $I_{NR}\left(F\right)\times I_{NR}\left(F\right)=I_{NR}\left(F\right)$.

$$=\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)+2\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$-2\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)$$

$$+\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-2\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)$$

$$+\left[\boldsymbol{\omega}'\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right)\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right)\right.$$

$$\left.\times\left(\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\right)\boldsymbol{\omega}I_{NR}\left(F\right)\right]$$

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Using (1.15) and (1.14) it was found that

$$= (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta}) + 2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2 (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F) + (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' \left(\mathbf{R} (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right) + \boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)$$

Finally,

$$\left[\left(\left(\mathbf{R} (k) \widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) + \mathbf{R} (k) \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) I_{NR} (F)\right)' \left(\left(\mathbf{R} (k) \widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) + \mathbf{R} (k) \left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) I_{NR} (F)\right)\right]^2$$

$$= \Big[(\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta}) + 2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2 (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F) + (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' \left(\mathbf{R} (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right) + \boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\Big]'$$

$$\times \Big[(\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta}) + 2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2 (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F) + (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})$$

$$-2 (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' \left(\mathbf{R} (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right) + \boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\Big]$$

$$= \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)$$

$$+4 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$-4 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2 (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right)$$

$$+2 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' \left((\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$-4 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' \left((\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})' \left(\mathbf{R} (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right)\right)$$

$$+2 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{D} (\boldsymbol{\omega}-\boldsymbol{\eta})\right)' (\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}) I_{NR} (F)$$

$$+4 \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})\right)' \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$-8 \Big[\left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}' (k) (\mathbf{R} (k) \boldsymbol{\beta}-\boldsymbol{\beta})\right)'$$

$$\times \left((\boldsymbol{\omega}-\boldsymbol{\eta})' \mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2 (k) \mathbf{C}^{-1}\mathbf{H}' \left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1} \mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR} (F)\right)\Big]$$

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$$
\begin{aligned}
&+4\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\\
&-8\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right.\\
&\left.\times\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right]\\
&+4\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\\
&+4\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\right.\\
&\left.\times\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]\\
&-4\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\\
&+8\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\right.\\
&\left.\times\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right]\\
&-4\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\\
&+\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\\
&-4\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\\
&+2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\\
&+4\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\right.\\
&\left.\times\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right]\\
&-4\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\\
&+\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)
\end{aligned}
$$

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Which finally leads to solving the following expression in order to obtain the numerator of (2.101):

$$E\left[\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)+\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)'\left(\left(\mathbf{R}(k)\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right)+\mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n-\widehat{\boldsymbol{\beta}}_n\right)I_{NR}(F)\right)\right]^2$$

$$=E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)\right]$$

(2.103)

$$+4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

(2.104)

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

(2.105)

$$+2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

(2.106)

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right)\right]$$

(2.107)

$$+2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega})I_{NR}(F)\right]$$

(2.108)

$$+4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

(2.109)

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$$-8E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right.$$

$$\left.\times\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right] \tag{2.110}$$

$$+4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right] \tag{2.111}$$

$$-8E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right.$$

$$\left.\times\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right] \tag{2.112}$$

$$+4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.113}$$

$$+4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\right.$$

$$\left.\times\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right] \tag{2.114}$$

$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right] \tag{2.115}$$

$$+8E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\right.$$

$$\left.\times\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.116}$$

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$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right] \tag{2.117}$$

$$+E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right] \tag{2.118}$$

$$-4E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right] \tag{2.119}$$

$$+2E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.120}$$

$$+4E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\right.$$
$$\left.\times\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.121}$$

$$-4E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.122}$$

$$+E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right] \tag{2.123}$$

Expression (2.103): Using (1.42) resulted in

$$E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)\right]$$
$$=E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{D}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)^{2}\right]=3\sigma^{4}tr\mathbf{D}$$

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Expression (2.104): Since we know that $E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'(\boldsymbol{\omega}-\boldsymbol{\eta})'\right]=\mathbf{0}'$ from (1.48) it was found that

$$4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'(\boldsymbol{\omega}-\boldsymbol{\eta})'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$=0$$

Expression (2.105): Using (1.14), (1.15) and (1.16) the following was obtained

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^{2}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\right)\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}(F)\right]$$

$$=-4E\left[\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\right]$$

By multiplying the matrices out one obtains

$$=-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}(F)\right]$$

(2.124)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1I_{NR}(F)\right]$$

(2.125)

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1I_{NR}(F)\right]$$

(2.126)

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1I_{NR}(F)\right]$$

(2.127)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1I_{NR}(F)\right]$$

(2.128)

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$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.129}$$

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.130}$$

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.131}$$

Expectation (2.124):

$$\begin{aligned}
&-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\right) I_{NR}(F)\right] \\
&= -4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1+\boldsymbol{\eta}_1)\right) I_{NR}(F)\right] \\
&= -4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right) I_{NR}(F)\right] \\
&-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right) I_{NR}(F)\right] \\
&= -4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2 I_{NR}(F)\right]
\end{aligned} \tag{2.132}$$

$$-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right) I_{NR}(F)\right] \tag{2.133}$$

Expectation (2.132): Using (1.44) it was found that

$$\begin{aligned}
&-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2 I_{NR}(F)\right] \\
&= -12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]
\end{aligned}$$

Expectation (2.133): Using (1.52) resulted in

$$\begin{aligned}
&-4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right] \\
&= -4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1 \\
&= \mathbf{0}'\mathbf{D}_{11}\boldsymbol{\eta}_1 = 0
\end{aligned}$$

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By combining (2.132) and (2.133) an expression for (2.124) was obtained:

$$-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (2.125):

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]=0$$

Expectation (2.126):

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]=0$$

Expectation (2.127): Using (1.38) it was found that

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{21}'\sigma^2\mathbf{I}_{p-q}\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

assuming that $\mathbf{D}^*=\sigma^2\mathbf{D}_{12}\mathbf{D}_{21}$ is a positive definite symmetric matrix, it was found that

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}^*\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}^*(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1+\boldsymbol{\eta}_1)I_{NR}(F)\right]$$

$$=-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}^*(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)I_{NR}(F)\right]-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'I_{NR}(F)\right]\mathbf{D}^*\boldsymbol{\eta}_1$$

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using (1.32) and (1.36) it was found that

$$= -4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}^*$$

Expectation (2.128): Similar to the simplification of (2.125) and (2.126) it was found

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= 0$$

Expectation (2.129): Similar to the simplification of (2.127) it was found that

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= -4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}^*$$

assuming that $\mathbf{D}^* = \sigma^2\mathbf{D}_{12}\mathbf{D}_{21}$ is a positive definite symmetric matrix.

Expectation (2.130): Using (1.34) resulted in

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= -4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$= -4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\right](\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= -4E\left[\sigma^2 tr\mathbf{D}_{22}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= -4E\left[\sigma^2 tr\mathbf{D}_{22}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1+\boldsymbol{\eta}_1) I_{NR}(F)\right]$$

$$= -4E\left[\sigma^2 tr\mathbf{D}_{22}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)\right] - 4E\left[\sigma^2 tr\mathbf{D}_{22}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' I_{NR}(F)\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

$$= -4\sigma^2 tr\mathbf{D}_{22}E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)\right] - 4\sigma^2 tr\mathbf{D}_{22}E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' I_{NR}(F)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

using (1.32) and (1.36) the following was obtained

$$-4\sigma^4 tr\mathbf{D}_{22}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11}$$

Expectation (2.131):

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$= -4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$= -4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

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using (1.48) it was found that

$$E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$$

and therefore

$$-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]=0$$

Combining (2.124) - (2.131) an expression for (2.105) was obtained

$$-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-4\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*$$

$$-4\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*-4\sigma^4 tr\mathbf{D}_{22}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}$$

Expression (2.106): Using (1.34) resulted in

$$2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$=2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$=2\sigma^2 tr\mathbf{D}\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

Expression (2.107):

$$-4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right)\right]$$

$$=-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\boldsymbol{\omega}'I_{NR}(F)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$=-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta}+\boldsymbol{\eta})'I_{NR}(F)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$=-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})'I_{NR}(F)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

(2.134)

$$-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

(2.135)

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Expectation (2.134):

$$-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})'I_{NR}(F)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$=-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

using (1.52) it was found that

$$-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$=\mathbf{0}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})=0$$

Expectation (2.135)

$$-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

using (1.36) the following was found

$$-4E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

$$=-4\sigma^2 tr\mathbf{D}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})$$

Expression (2.108): Using (1.17) it was found that

$$2E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{D}(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega})I_{NR}(F)\right]$$

$$=2E\left[\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\\ \boldsymbol{\omega}_2-\boldsymbol{\eta}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\right]$$

by multiplying the matrices out it was found that

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.136}$$

$$+2E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.137}$$

$$+2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{12}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.138}$$

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$$+2E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{22}(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.139}$$

Expectation (2.136):

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$
$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1+\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1+\boldsymbol{\eta}_1)I_{NR}(F)\right]$$
$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+2(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]I_{NR}(F)\right]$$

$$=2E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2 I_{NR}(F)\right] \tag{2.140}$$

$$+4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right] \tag{2.141}$$

$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right] \tag{2.142}$$

Expectation (2.140): Using (1.44) resulted in

$$2E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)^2 I_{NR}(F)\right]$$
$$=6\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (2.141): Using (1.52) it was found that

$$4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right]$$
$$4E\left[\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)_1 I_{NR}(F)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\right]\mathbf{D}_{11}\boldsymbol{\eta}\right]=\mathbf{0}'\mathbf{D}_{11}\boldsymbol{\eta}=0.$$

Expectation (2.142): Using (1.36) the following was obtained

$$2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right]$$
$$=2E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)I_{NR}(F)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

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$$= 2\sigma^2 tr\mathbf{D}_{11} E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

By combining (2.140) - (2.142) an expression for (2.136) was obtained:

$$6\sigma^4 tr\mathbf{D}_{11} E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] + 2\sigma^2 tr\mathbf{D}_{11} E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.137):

$$2E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2E\left\{E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$= 2E\left[E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

since $E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right] = \mathbf{0}'$ it was found that

$$2E\left[E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right] = 0$$

Expectation (2.138): In a similar way to the simplification of (2.137) it was found that

$$2E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{12}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right] = 0$$

Expectation (2.139): Using (1.34) and (1.35) resulted in

$$2E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2E\left\{E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$= 2E\left[E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{22}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2E\left[\sigma^2 tr\mathbf{D}_{22}\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2\sigma^2 tr\mathbf{D}_{22} E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$= 2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11} + E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

By combining (2.136) - (2.139) an expression for (2.108) was obtained:

$$6\sigma^4 tr\mathbf{D}_{11} E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] + 2\sigma^2 tr\mathbf{D}_{11} E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$+2\sigma^2 tr\mathbf{D}_{22}\left[\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11} + E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

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Expression (2.109): Using (1.38) it was found that

$$4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=4E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

$$=4\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$=4\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\sigma^{2}\mathbf{I}_{p}\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$=4\sigma^{2}\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$=4\sigma^{2}\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

Expression (2.110): Using (1.16) the following was found

$$-8E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=-8E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}\left(F\right)\right]$$

Define $\mathbf{v}_{p\times 1}'^{*}=\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'=\begin{pmatrix}\mathbf{v}_{1}'^{*} & \mathbf{v}_{2}'^{*}\end{pmatrix}$ where $\mathbf{v}_{1}'^{*}:1\times q$ and $\mathbf{v}_{2}'^{*}:1\times\left(p-q\right)$ it was found that

$$-8E\left[\begin{pmatrix}\mathbf{v}_{1}'^{*} & \mathbf{v}_{2}'^{*}\end{pmatrix}\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}\left(F\right)\right]$$

$$=-8E\left[\begin{pmatrix}\mathbf{v}_{1}'^{*} & \mathbf{v}_{2}'^{*}\end{pmatrix}\begin{pmatrix}\boldsymbol{\omega}_{1}-\boldsymbol{\eta}_{1}\\ \boldsymbol{\omega}_{2}-\boldsymbol{\eta}_{2}\end{pmatrix}\begin{pmatrix}\left(\boldsymbol{\omega}_{1}-\boldsymbol{\eta}_{1}\right)' & \left(\boldsymbol{\omega}_{2}-\boldsymbol{\eta}_{2}\right)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_{1}\\ \boldsymbol{\omega}_{2}\end{pmatrix}I_{NR}\left(F\right)\right]$$

By multiplying the matrices out resulted in

$$-8E\left[\mathbf{v}_{1}'^{*}\left(\boldsymbol{\omega}_{1}-\boldsymbol{\eta}_{1}\right)\left(\boldsymbol{\omega}_{1}-\boldsymbol{\eta}_{1}\right)'\mathbf{D}_{11}\boldsymbol{\omega}_{1}I_{NR}\left(F\right)\right] \tag{2.143}$$

$$-8E\left[\mathbf{v}_{1}'^{*}\left(\boldsymbol{\omega}_{1}-\boldsymbol{\eta}_{1}\right)\left(\boldsymbol{\omega}_{2}-\boldsymbol{\eta}_{2}\right)'\mathbf{D}_{21}\boldsymbol{\omega}_{1}I_{NR}\left(F\right)\right] \tag{2.144}$$

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$$-8E\left[\mathbf{v}_2'^{*}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right] \tag{2.145}$$

$$-8E\left[\mathbf{v}_2'^{*}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right] \tag{2.146}$$

Expectation (2.143):

$$-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$
$$=-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right) I_{NR}\left(F\right)\right]$$
$$=-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right) I_{NR}\left(F\right)\right] \tag{2.147}$$

$$-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right] \tag{2.148}$$

Expectation (2.147): Using (1.52) it was found that

$$-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right) I_{NR}\left(F\right)\right]$$
$$=-8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right) I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^{*}\right]$$
$$=\mathbf{0}'\mathbf{v}_1^{*}=0$$

Expectation (2.148): Using (1.40) the following was obtained

$$-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]$$
$$=-8\left[\mathbf{v}_1'^{*}E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' I_{NR}\left(F\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$
$$=-8\sigma^2\mathbf{v}_1'^{*}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{I}_q\mathbf{D}_{11}\boldsymbol{\eta}_1=-8\sigma^2\mathbf{v}_1'^{*}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

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Combining (2.147) and (2.148) an expression for (2.143) was obtained:

$$-8\sigma^2\mathbf{v}_1'^{*}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.144): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=-8E\left\{E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=-8E\left[\mathbf{v}_1'^{*}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]=0$$

Expectation (2.145): This expectation can be simplified in a similar way to (2.144)

$$-8E\left[\mathbf{v}_2'^{*}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]=0$$

Expectation (2.146): Using (1.31) and (1.38) resulted in

$$-8E\left[\mathbf{v}_2'^{*}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=-8E\left\{E\left[\mathbf{v}_2'^{*}\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=-8E\left[\mathbf{v}_2'^{*}E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=-8\sigma^2E\left[\mathbf{v}_2'^{*}\mathbf{I}_{p-q}\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=-8\sigma^2\mathbf{v}_2'^{*}\mathbf{D}_{21}E\left[\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=-8\sigma^2\mathbf{v}_2'^{*}\mathbf{D}_{21}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Combining (2.143) - (2.146) an expression for (2.110) was obtained:

$$-8\sigma^2\mathbf{v}_1'^{*}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1-8\sigma^2\mathbf{v}_2'^{*}\mathbf{D}_{21}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expression (2.111): Since $E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})\right]=\mathbf{0}$ it was found that

$$4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=4\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right]\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)=0$$

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Expression (2.112):

$$-8E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right.$$

$$\left.\times\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=-8E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right.$$

$$\left.\times\boldsymbol{\omega}'I_{NR}\left(F\right)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

$$=-8E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right.$$

$$\left.\times\left(\boldsymbol{\omega}-\boldsymbol{\eta}+\boldsymbol{\eta}\right)'I_{NR}\left(F\right)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

Let $\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)=\mathbf{s}$ where $\mathbf{s}$ is a $p\times 1$ vector

$$=-8\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'I_{NR}\left(F\right)\right]\mathbf{s}$$

(2.149)

$$-8\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)I_{NR}\left(F\right)\right]\boldsymbol{\eta}'\mathbf{s}$$

(2.150)

Expectation (2.149): Using using (1.40) it was found that

$$-8\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'I_{NR}\left(F\right)\right]\mathbf{s}$$

$$=-8\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{s}$$

Expectation (2.150): Using (1.32) resulted in

$$-8\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)I_{NR}\left(F\right)\right]\boldsymbol{\eta}'\mathbf{s}=0$$

Combining (2.149) and (2.150) an expression for (2.112) was obtained:

$$-8\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{s}$$

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Expression (2.113): Using (1.27) it was found that

$$4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right)\right]$$

$$=4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

Define $\mathbf{v}^*_{p\times 1}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)=\begin{pmatrix}\mathbf{v}_1^*\\ \mathbf{v}_2^*\end{pmatrix}$ where $\mathbf{v}_1^*: q\times 1$ and $\mathbf{v}_2^*:(p-q)\times 1$ resulted in

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\begin{pmatrix}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' & \left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\end{pmatrix}\begin{pmatrix}\mathbf{v}_1^*\\ \mathbf{v}_2^*\end{pmatrix}\right]$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^*+\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^*\right)\right]$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^*\right]$$

(2.151)

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^*\right]$$

(2.152)

Expectation (2.151): Using (1.49) and (1.31) it was found that

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^*\right]$$

$$=8\sigma^2E\left[\boldsymbol{\omega}_1'I_{NR}\left(F\right)\mathbf{D}_{11}\right]\mathbf{v}_1^*$$

$$=8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^*$$

Expectation (2.152):

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^*\right]$$

$$=4E\left\{E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^*\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{v}_2^*\right]$$

since $E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]=\mathbf{0}'$ we obtain

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{v}_2^*\right]=0$$

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By combining (2.151) and (2.152) an expression for (2.113) was obtained:

$$8\sigma^2\boldsymbol{\eta}_1' E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^*$$

Expression (2.114): Using (1.16) it was found that

$$4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)'\right.$$

$$\left.\times(\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right]$$

$$=4E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}(F)\right)'\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=4E\left[\left(\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\right)'\right.$$

$$\left.\times\left(\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\right)\right]$$

Multiplying the brackets out it resulted in

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.153}$$

$$+4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.154}$$

$$+4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.155}$$

$$+4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right] \tag{2.156}$$

Expectation (2.153):

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]$$

$$=4E\left[((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1)I_{NR}(F)\right]$$

$$=4E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\right)I_{NR}(F)\right]$$

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$$= 4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2 I_{NR}\left(F\right)\right] \tag{2.157}$$

$$+8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)I_{NR}\left(F\right)\right] \tag{2.158}$$

$$+4E\left[\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2 I_{NR}\left(F\right)\right] \tag{2.159}$$

Expectation (2.157): Using (1.44) it was found that

$$4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2 I_{NR}\left(F\right)\right]$$

$$= 12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (2.158): Using (1.52) resulted in

$$8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)I_{NR}\left(F\right)\right]$$

$$= 8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\right]\mathbf{D}_{11}\boldsymbol{\eta}_1 = 0$$

Expectation (2.159): Using (1.40) the following was obtained

$$4E\left[\left(\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2 I_{NR}\left(F\right)\right]$$

$$= 4E\left[\boldsymbol{\eta}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]$$

$$= 4\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'I_{NR}\left(F\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$= 4\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

By combining (2.157) - (2.159) an expression for (2.153) was obtained:

$$12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] + 4\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

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Expectation (2.154): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=4E\left\{E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]=0$$

Expectation (2.155): In a similar way to the simplification of (2.154) resulted in

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]=0$$

Expectation (2.156): Using (1.38) the following was obtained

$$4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=4E\left\{E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\sigma^2\mathbf{I}_{p-q}\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=4E\left[\boldsymbol{\omega}_1'\sigma^2\mathbf{D}_{21}'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

Assuming that $\mathbf{D}^*=\sigma^2\mathbf{D}_{21}'\mathbf{D}_{21}$ is a positive definite symmetric matrix and making use of (1.35) resulted in

$$=4E\left[\boldsymbol{\omega}_1'\mathbf{D}^*\boldsymbol{\omega}_1 I_{NR}\left(F\right)\right]$$

$$=4\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}^*\boldsymbol{\eta}_1\right]$$

Combining (2.153) - (2.156) an expression for (2.114) was obtained:

$$12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+4\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$+4\sigma^2\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}^*+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}^*\boldsymbol{\eta}_1\right]$$

Expression (2.115): Using (1.16) it was found that

$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\boldsymbol{\omega}I_{NR}\left(F\right)\right]\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

$$=-4E\left[\begin{pmatrix}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' & \left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}\left(F\right)\right]\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)$$

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$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F) + (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

(2.160)

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

(2.161)

Expectation (2.160):

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1\right) I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

(2.162)

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

(2.163)

Expectation (2.162): Using (1.36) it was found that

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1) I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$= -4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11}\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

Expectation (2.163): Using (1.32) resulted in

$$-4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$= -4E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' I_{NR}(F)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right) = 0$$

Combining (2.162) and (2.163) an expression for (2.160) was obtained:

$$-4\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right] tr\mathbf{D}_{11}\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

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Expectation (2.161): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right]=0$$

Combining (2.160) and (2.161) an expression for (2.115) was obtained:

$$-4\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

Expression (2.116): Using (1.16) it was found that

$$8E\left[\left((\boldsymbol{\omega}-\boldsymbol{\eta})'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)'\right.$$

$$\left.\times(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=8E\left[\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\right)$$

$$\left.\times\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right]$$

Define $\mathbf{v}^{**}_{p\times1}=\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})=\begin{pmatrix}\mathbf{v}_1^{**}\\ \mathbf{v}_2^{**}\end{pmatrix}$ where $\mathbf{v}_1^{**}: q\times1$ and $\mathbf{v}_2^{**}:(p-q)\times1$ resulted in

$$8E\left[\begin{pmatrix}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)' & (\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}(F)\left(\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{pmatrix}\mathbf{v}_1^{**}\\ \mathbf{v}_2^{**}\end{pmatrix}\right)\right]$$

$$=8E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)+(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1 I_{NR}(F)\right)(\boldsymbol{\omega}_1'\mathbf{v}_1^{**}+\boldsymbol{\omega}_2'\mathbf{v}_2^{**})\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.164)

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\omega}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

(2.165)

$$+8E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.166)

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$$+8E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

(2.167)

Expectation (2.164)

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$
$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1\right)\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1\right)'\mathbf{v}_1^{**}I_{NR}(F)\right]$$
$$=8E\left[\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{v}_1^{**}I_{NR}(F)+\boldsymbol{\eta}_1'\mathbf{v}_1^{**}I_{NR}(F)\right)\right]$$
$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.168)

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.169)

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.170)

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

(2.171)

Expectation (2.168): Using (1.52) it was found that

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{v}_1^{**}I_{NR}(F)\right]$$
$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)I_{NR}(F)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\right]\mathbf{v}_1^{**}=0$$

Expectation (2.169): Using (1.36) resulted in

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$
$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)I_{NR}(F)\right]\boldsymbol{\eta}_1'\mathbf{v}_1^{**}$$
$$=8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{v}_1^{**}$$

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Expectation (2.170): Using (1.40) the following was obtained

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

$$=8\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'I_{NR}(F)\right]\mathbf{v}_1^{**}$$

$$=8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_qE\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{v}_1^{**}$$

Expectation (2.171): Using (1.32) it was found that

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'I_{NR}(F)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{v}_1^{**}=0$$

Combining (2.168) - (2.171) an expression for (2.164) was obtained:

$$8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{v}_1^{**}+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_qE\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{v}_1^{**}$$

Expectation (2.165)

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\omega}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\left((\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)+\boldsymbol{\eta}_2\right)'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{v}_2^{**}I_{NR}(F)\right] \tag{2.172}$$

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right] \tag{2.173}$$

Expectation (2.172): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left\{E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{v}_2^{**}I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{v}_2^{**}I_{NR}(F)\right]=0$$

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Expectation (2.173):

$$8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\omega}_1\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\left((\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)+\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'\mathbf{D}_{11}(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)I_{NR}(F)\right]\boldsymbol{\eta}_2'\mathbf{v}_2^{**}$$

$$+8E\left[(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1)'I_{NR}(F)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_2'\mathbf{v}_2^{***}$$

Using (1.32)

$$=8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{v}_2^{**}$$

Combining (2.172) and (2.173) an expression for (2.165) was obtained:

$$8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{v}_2^{**}$$

Expectation (2.166): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$8E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]$$

$$=8E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=8E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^{**}I_{NR}(F)\right]=0$$

Expectation (2.167):

$$8E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\boldsymbol{\omega}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\omega}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\left((\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)+\boldsymbol{\eta}_2\right)'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

$$=8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

(2.174)

$$+8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}(F)\right]$$

(2.175)

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Expectation (2.174): Using (1.31) and (1.38) the following was obtained

$$8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^{**}I_{NR}\left(F\right)\right]$$

$$=8E\left\{E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{v}_2^{**}I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\right]\mathbf{v}_2^{**}I_{NR}\left(F\right)\right]$$

$$=8\sigma^2E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\mathbf{I}_{p-q}\mathbf{v}_2^{**}I_{NR}\left(F\right)\right]$$

$$=8\sigma^2E\left[\boldsymbol{\omega}_1'I_{NR}\left(F\right)\right]\mathbf{D}_{21}'\mathbf{v}_2^{**}$$

$$=8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{21}'\mathbf{v}_2^{**}$$

Expectation (2.175): Since $E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]=\mathbf{0}$ it was found that

$$8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}\left(F\right)\right]$$

$$=8E\left\{E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}\left(F\right)\middle|\,\boldsymbol{\omega}_1\right]\right\}$$

$$=8E\left[\boldsymbol{\omega}_1'\mathbf{D}_{21}'E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)\right]\boldsymbol{\eta}_2'\mathbf{v}_2^{**}I_{NR}\left(F\right)\right]=0$$

Combining (2.174) and (2.175) an expression for (2.167) was obtained:

$$8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{21}'\mathbf{v}_2^{**}$$

Combining (2.164) - (2.167) an expression for (2.116) was obtained:

$$8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_1'\mathbf{v}_1^{**}+8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_qE\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{v}_1^{**}$$

$$+8\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}\boldsymbol{\eta}_2'\mathbf{v}_2^{**}+8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{21}'\mathbf{v}_2^{**}$$

Expression (2.117):

$$-4E\left[\left(\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}^2\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right]$$

using (1.16) and (1.17) it was found that

$$=-4E\left[\begin{pmatrix}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)' & \left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{D}_{21} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right]$$

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Multiplying the brackets out resulted in

$$-4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1+\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1\right)\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right] \tag{2.176}$$

$$-4E\left[\left(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2\right)'\mathbf{D}_{21}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right] \tag{2.177}$$

Expectation (2.176):

$$-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)\left(\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)\right)I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)\right.$$

$$\left.\times\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+2\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1+\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right)I_{NR}\left(F\right)\right]$$

Multiplying the brackets out resulted in

$$=-4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2I_{NR}\left(F\right)\right] \tag{2.178}$$

$$-12E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1I_{NR}\left(F\right)\right] \tag{2.179}$$

$$-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1I_{NR}\left(F\right)\right] \tag{2.180}$$

$$-8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1I_{NR}\left(F\right)\right] \tag{2.181}$$

$$-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1I_{NR}\left(F\right)\right] \tag{2.182}$$

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Expectation (2.178): using (1.44) it was found that

$$-4E\left[\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\right)^2 I_{NR}\left(F\right)\right]$$

$$=-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

Expectation (2.179): Using (1.52) the following was obtained

$$-12E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]=0$$

Expectation (2.180): Using (1.36) it was found that

$$-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)I_{NR}\left(F\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$=-4\sigma^2 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.181): Using (1.40) resulted in

$$-8E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]$$

$$=-8\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'I_{NR}\left(F\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$=-8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\mathbf{I}_q E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$=-8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expectation (2.182): Using (1.32) it was found that

$$-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'I_{NR}\left(F\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1=0$$

Combining (2.178) - (2.182) we obtain an expression for (2.176)

$$-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-4\sigma^2 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$-8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

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Expectation (2.177): Since $E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]=\mathbf{0}'$ it was found that

$$-4E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}(F)\right]$$

$$=-4E\left\{E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\mathbf{D}_{21}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}(F)\middle|\boldsymbol{\omega}_1\right]\right\}$$

$$=-4E\left[E\left[(\boldsymbol{\omega}_2-\boldsymbol{\eta}_2)'\right]\mathbf{D}_{21}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}(F)\right]=0$$

Combining (2.176) and (2.177) an expression for (2.117) was obtained:

$$-12\sigma^4 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]-4\sigma^2 tr\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1$$

$$-8\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\boldsymbol{\eta}_1$$

Expression (2.118): Using (1.28) resulted in

$$E\left[\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$=\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)$$

$$=\left(\boldsymbol{\beta}'(\mathbf{R}(k)-\mathbf{I}_p)'(\mathbf{R}(k)-\mathbf{I}_p)\boldsymbol{\beta}\right)\left(\boldsymbol{\beta}'(\mathbf{R}(k)-\mathbf{I}_p)'(\mathbf{R}(k)-\mathbf{I}_p)\boldsymbol{\beta}\right)$$

$$=\left(k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)\left(k^2\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)$$

$$=k^4\left(\boldsymbol{\beta}'\mathbf{C}_{(k)}^{-2}\boldsymbol{\beta}\right)^2$$

Expression (2.119): Using (1.31) the following was obtained

$$-4E\left[\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)'\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\left(\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right)\right]$$

$$=-4\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}I_{NR}(F)\right]$$

$$=-4\left[\left((\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})\right)'(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta})'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\right.$$

$$\left.\times\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\right]$$

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Expression (2.120): Using (1.17) and (1.35) it was found that

$$2E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\boldsymbol{\omega}'\mathbf{QFQ}'\mathbf{DQFQ}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'E\left[\left(\boldsymbol{\omega}'\mathbf{QFQ}'\mathbf{DQFQ}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'E\left[\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}\left(F\right)\right]$$

$$=2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'E\left[\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1I_{NR}\left(F\right)\right]$$

$$=2\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]$$

Expression (2.121):

$$4E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{HC}^{-1}\mathbf{H}'\right)^{-1}\mathbf{HC}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\right.$$

$$\left.\times\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{HC}^{-1}\mathbf{H}'\right)^{-1}\mathbf{HC}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)\right]$$

$$=E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{QC}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{HC}^{-1}\mathbf{H}'\right)^{-1}\mathbf{HC}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right.$$

$$\left.\times\boldsymbol{\omega}'I_{NR}\left(F\right)\mathbf{QC}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{HC}^{-1}\mathbf{H}'\right)^{-1}\mathbf{HC}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{QC}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

$$=E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}I_{NR}\left(F\right)\mathbf{QC}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]$$

Define $\mathbf{v}^*_{p\times1}=\mathbf{QC}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\left(\mathbf{R}\left(\mathbf{k}\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)=\begin{pmatrix}\mathbf{v}_1^*\\ \mathbf{v}_2^*\end{pmatrix}$ where $\mathbf{v}_1^*:q\times1$ and $\mathbf{v}_2^*:(p-q)\times1$

$$=E\left[\mathbf{v}_1'^*\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\mathbf{v}_1^*I_{NR}\left(F\right)\right]$$

$$=\mathbf{v}_1'^*E\left[\boldsymbol{\omega}_1\boldsymbol{\omega}_1'I_{NR}\left(F\right)\right]\mathbf{v}_1^*$$

using (1.39) resulted in

$$E\left[\boldsymbol{\omega}_1\boldsymbol{\omega}_1'I_{NR}\left(F\right)\right]=\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{I}_q+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1\boldsymbol{\eta}_1'$$

Therefore

$$\mathbf{v}_1'^*E\left[\boldsymbol{\omega}_1\boldsymbol{\omega}_1'I_{NR}\left(F\right)\right]\mathbf{v}_1^*=\mathbf{v}_1'^*\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{I}_q+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1\boldsymbol{\eta}_1'\right]\mathbf{v}_1^*$$

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Expression (2.122): Using (1.17) resulted in

$$-4E\left[\left(\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=-4E\left[\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\left(\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

$$=-4E\left[\left(\mathbf{R}\left(\mathbf{k}\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\begin{bmatrix}\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}\right.$$

$$\left.\times\left(\begin{pmatrix}\boldsymbol{\omega}_1' & \boldsymbol{\omega}_2'\end{pmatrix}\begin{bmatrix}\mathbf{D}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{pmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}I_{NR}\left(F\right)\right)\right]$$

$$=-4E\left[\mathbf{v}_1'^{*}\boldsymbol{\omega}_1\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)\boldsymbol{\omega}_1'\mathbf{v}_1^{*}I_{NR}\left(F\right)\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\left(\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)+\boldsymbol{\eta}_1\right)'\mathbf{v}_1^{*}\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^{*}\right] \tag{2.183}$$

$$-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\boldsymbol{\eta}_1'\mathbf{v}_1^{*}\right] \tag{2.184}$$

Expectation (2.183): Using (1.49) and (1.31) the following was obtained

$$-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\mathbf{v}_1^{*}\right]$$

$$=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\left(\boldsymbol{\omega}_1-\boldsymbol{\eta}_1\right)'\right]\mathbf{v}_1^{*}$$

$$=-4\sigma^2E\left[\frac{\partial\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1I_{NR}\left(F\right)}{\partial\boldsymbol{\omega}_1}\right]\mathbf{v}_1^{*}=-8\sigma^2E\left[\boldsymbol{\omega}_1'I_{NR}\left(F\right)\mathbf{D}_{11}\right]\mathbf{v}_1^{*}$$

$$=-8\sigma^2\boldsymbol{\eta}_1'E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^{*}$$

Expectation (2.184): Using (1.35) it was found that

$$-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\boldsymbol{\eta}_1'\mathbf{v}_1^{*}\right]=-4E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}\left(F\right)\right]\boldsymbol{\eta}_1'\mathbf{v}_1^{*}$$

$$=-4\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{v}_1^{*}$$

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Combining (2.183) and (2.184) an expression for (2.122) was obtained:

$$-8\sigma^2\boldsymbol{\eta}_1' E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\mathbf{D}_{11}\mathbf{v}_1^*$$

$$-4\left[\sigma^2 E\left[\varphi\left(\chi^2_{\left(q+2,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]tr\mathbf{D}_{11}+E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1\right]\boldsymbol{\eta}_1'\mathbf{v}_1^*$$

Expression (2.123): Using (1.27) and (1.43) it was found that

$$E\left[\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)'\left(\boldsymbol{\omega}'\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{D}\mathbf{Q}\mathbf{F}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1 I_{NR}(F)\right)'\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)I_{NR}(F)\right]=E\left[\left(\boldsymbol{\omega}_1'\mathbf{D}_{11}\boldsymbol{\omega}_1\right)^2 I_{NR}(F)\right]$$

$$=3\sigma^4 tr\left(\mathbf{D}_{11}\right)E\left[\varphi\left(\chi^2_{\left(q+4,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]+6\sigma^2\boldsymbol{\eta}_1'\mathbf{D}_{11}\boldsymbol{\eta}_1 E\left[\varphi\left(\chi^2_{\left(q+6,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

$$+\left(\boldsymbol{\eta}_1\mathbf{D}_{11}^{\frac{1}{2}}\boldsymbol{\eta}_1\right)^2 E\left[\varphi\left(\chi^2_{\left(q+8,\frac{\boldsymbol{\eta}_1'\boldsymbol{\eta}_1}{2\sigma^2}\right)}\right)\right]$$

By combining the final expressions for (2.103) - (2.123) an expression for the numerator of expression (2.101) was obtained:

(2.185)

Substituting (2.100) into (2.90) and (2.185) into (2.101) and then adding (2.90) to (2.101) completes the proof.■

It is clear that the risk functions under reflected normal loss are extremely complicated and cannot be compared to each other analytically. Therefore a simulation study will be conducted in the next section in order to compare the performance of the different ridge estimators under reflected normal loss with each other computationally.

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2.5 Simulations

Kibria(2003), Gisela and Kibria (2009) and Najarian et al. (2013) extensively discussed the performance of ridge estimators for different ridge parameters. They only considered the squared error loss function as a measure of the performance of the proposed estimators. In this section a numerical study of ridge estimators are provided by using different ridge parameters under reflected normal loss. In Chapters 3 and 4 similar results will be discussed under LINEX and BLINEX loss functions respectively. The simulation technique adopted is similar to the technique described in McDonald and Galarneau (1975) and Gibbons (1981). For the detailed description of the simulation technique see section 1.5 page 28 of the thesis. For completeness the a short version of the simulation technique is given again.

The $p = 4$ multicollinear explanatory variables are computed by

$$x_{ij} = \left(1 - \gamma^2\right)^{\frac{1}{2}} z_{ij} + \gamma z_{i5} \qquad i = 1, 2, ..., n \text{ and } j = 1, 2, ...4$$

with $z_{ij} \sim N(0, 1)$ random variables and γ chosen in such a way that the correlation between any two explanatory variables are given by γ^2. The explanatory variables are standardised in order to ensure that $\mathbf{X}'\mathbf{X}$ is in correlation form.

In order to generate the dependent variable y_i, the true $\boldsymbol{\beta}$ vector is needed when using the following equation:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + e_i \qquad i = 1, 2, ...n \tag{2.186}$$

with $e_i \sim N(0, \sigma^2)$ and $\beta_0 = 0$. The $\boldsymbol{\beta}$ vector is determined as the normalised eigenvector corresponding to the largest eigenvalue of the $\mathbf{X}'\mathbf{X}$ matrix, which is subject to the constraint that $\boldsymbol{\beta}'\boldsymbol{\beta} = 1$ as suggested by Newhouse and Oman (1971).

These regression coefficients were introduced into (2.186) in order to generate the dependent variable in the analysis. These simulated dependent and explanatory variables were then used to estimate the values of $\boldsymbol{\beta}$, using the three proposed estimators in the thesis, namely URRE, RRRE and PTRRE. The performance of each estimator is then determined by evaluating the performance under the reflected normal loss function. The performance is evaluated by calculating the risk (average loss) values for each estimator for all the different parameter values. In order to calculate the risk values the process was repeated 2000 times (2000 iterations).

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The different ridge regression estimators were further evaluated by introducing different ridge regression estimators (k) into the estimators. The different ridge regression estimators used as briefly discussed below:

Hoerl and Kennard (1970) ($\widehat{k}_{HK}$ or HK):

$$\widehat{k}_{HK} = \frac{\widehat{\sigma}^2}{\alpha_i^2}$$

where $\widehat{\sigma}^2 = \dfrac{\sum \widehat{e}_i^2}{n-p}$ (the residual mean square error) , $\boldsymbol{\alpha} = \boldsymbol{\Gamma\beta} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\Gamma}$ represents the eigenvectors of $\mathbf{X'X}$. α_i^2 will be replaced by α_{Max}^2, the maximum squared element of $\boldsymbol{\alpha}$.

Hoerl, Kennard and Baldwin (1975) ($\widehat{k}_{HKB}$ or HKB):

$$\widehat{k}_{HKB} = \frac{p\widehat{\sigma}^2}{\widehat{\boldsymbol{\alpha}}'\widehat{\boldsymbol{\alpha}}}$$

Lawless and Wang (1976) ($\widehat{k}_{LW}$ or LW):

$$\widehat{k}_{LW} = \frac{p\widehat{\sigma}^2}{\widehat{\boldsymbol{\alpha}}'\mathbf{X'X}\widehat{\boldsymbol{\alpha}}}$$

Hocking,Speed and Lynn(1976) ($\widehat{k}_{LW}$ or LW):

$$\widehat{k}_{HSL} = \widehat{\sigma}^2 \frac{\sum_{i=1}^{p} (\lambda_i \widehat{\alpha}_i)^2}{\sum_{i=1}^{p} \left(\lambda_i \widehat{\alpha}_i^2\right)^2}$$

Kibria (2003) ($\widehat{k}_{AM}$ or AM): This represents the arithmetic mean of of k_{HK}.

$$\widehat{k}_{AM} = \frac{1}{p}\sum_{i=1}^{p}\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}$$

Kibria (2003) ($\widehat{k}_{GM}$ or GM): This represents the geometric mean of k_{HK}.

$$\widehat{k}_{GM} = \frac{\widehat{\sigma}^2}{\left(\prod_{i=1}^{p} \widehat{\alpha}_i^2\right)^{\frac{1}{p}}}$$

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Kibria (2003) ($\widehat{k}_{MED}$ or MED) for $p \geq 3$: This represents the median of k_{HK}.

$$\widehat{k}_{MED} = Median\left(\frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}\right), \quad i = 1, 2, ...p$$

The following simulation results evaluate the performance of the different proposed ridge parameters. By keeping specific parameters fixed in the simulation process and varying only one parameter at a time, the behaviour of the risk values can be evaluated. The ridge parameters were calculated as proposed at the beginning of this section and all the estimators (URRE, RRRE and PTRRE) were repeatedly calculated using all the proposed ridge parameters. All three proposed estimators (URRE, RRRE and PTRRE) were considered under the reflected normal loss function. In order to calculate the risk values, the average loss was calculated for each estimator calculated with the different ridge parameter. In the first table in each section only the error variance varied while the second table represents the risk values over different levels of multicollinearity. The third table represents the effect of an increase in the number of parameters that needs to be estimated and the fourth table represents the effect of an increase in the sample size. The SAS program for these simulations is available in Appendix A.

Unrestricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the variance(σ^2).

Table 2.1: Effect of an increase in the variance on the risk of the URRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	0.0000169	0.0000169	0.0000169	0.0000169	0.0004497	0.0000169	0.0000169
0.1	0.0016183	0.0016099	0.001623	0.0016165	0.0066346	0.0015784	0.0015922
0.25	0.0102039	0.0099029	0.0103973	0.0101278	0.0251271	0.0089392	0.0089621
0.5	0.0374972	0.0341275	0.0401476	0.0365041	0.0597485	0.0266543	0.0270407
1	0.1159532	0.0972911	0.1452599	0.1081491	0.1049873	0.0570376	0.0587568
4	0.2659404	0.4535828	0.7620413	0.2332226	0.2814325	0.219202	0.2179398
9	0.2802593	0.7565972	0.9525279	0.2749721	0.3589985	0.3352665	0.3343223
20	0.3371624	0.908707	0.9946903	0.3445348	0.3860661	0.3802397	0.3800933

(see Figure E1 panel (a) in Appendix E for all results based on the URRE)

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Unrestricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 2.2: Effect of an increase in the level of multicollinearity on the risk of the URRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0919835	0.0791547	0.1092779	0.0916944	0.1177358	0.0586821	0.0617445
0.8	0.1144414	0.0961581	0.1438745	0.1072252	0.109294	0.0578618	0.0584121
0.9	0.1585982	0.1334387	0.2299965	0.1142642	0.0952602	0.0522485	0.0549788
0.95	0.2031011	0.1931234	0.3601067	0.1061584	0.0965546	0.0482935	0.0518413
0.99	0.1976727	0.4260104	0.7057415	0.0700818	0.0806708	0.0268449	0.0291966

Unrestricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the number of parameters (p)

Table 2.3: Effect of an increase in the number of parameters that needs to be estimated on the risk of the URRE under reflected normal loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.1144414	0.0961581	0.1438745	0.1072252	0.109294	0.0578618	0.0584121
10	0.2657388	0.2203197	0.4324877	0.2381448	0.1622041	0.0493141	0.0646302
20	0.3900201	0.4806742	0.848574	0.3278158	0.193024	0.0425016	0.0600911
25	0.4576327	0.7140402	0.963327	0.3838384	0.206854	0.05795	0.0838016

Unrestricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the sample size (n)

Table 2.4: Effect of an increase in the sample size on the risk of the URRE under reflected normal loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.180749	0.1604742	0.2622493	0.1601093	0.1389418	0.084708	0.0879856
30	0.1144414	0.0961581	0.1438745	0.1072252	0.109294	0.0578618	0.0584121
50	0.0772754	0.0668003	0.0896149	0.0737845	0.0815214	0.0449302	0.046489
100	0.0400969	0.0363409	0.0433226	0.0391661	0.0605349	0.0281894	0.0284955

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Restricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the variance(σ^2).

Table 2.5: Effect of an increase in the variance on the risk of the RRRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	0.000017	0.0000169	0.000017	0.000017	0.0004488	0.0000169	0.0000169
0.1	0.0015539	0.0015452	0.0015587	0.0015521	0.0065587	0.0015123	0.0015249
0.25	0.0097425	0.0094416	0.0099338	0.009668	0.0247232	0.008493	0.0022096
0.5	0.0359275	0.0325296	0.0385642	0.0349448	0.0587512	0.0252099	0.0256165
1	0.1106823	0.0915612	0.1399216	0.1029762	0.1016887	0.0524852	0.0542029
4	0.2234999	0.4115349	0.7368753	0.1948099	0.2771398	0.2104379	0.208772
9	0.2357076	0.7048577	0.937174	0.2422777	0.3582475	0.3338246	0.333387
20	0.3292381	0.8602704	0.9907287	0.339335	0.3855637	0.3798181	0.3797286

(see Figure E2 panel (a) in Appendix E for all results based on the RRRE)

Restricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 2.6: Effect on an increase in the level of multicollinearity on the risk of RRRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0853514	0.0720441	0.1023917	0.0851229	0.1148721	0.0542194	0.0727785
0.8	0.1089415	0.0902539	0.1382052	0.1018291	0.1062525	0.0536277	0.0542765
0.9	0.1539642	0.1284592	0.2254633	0.1097708	0.0919092	0.0483655	0.0510171
0.95	0.1994402	0.1892551	0.3569338	0.1021805	0.0936924	0.044836	0.0482893
0.99	0.1943711	0.4235415	0.7043764	0.0664903	0.0778187	0.0233205	0.025528

Restricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the number of parameters (p)

Table 2.7: Effect on an increase in the number of parameters to be estimated on the risk of the RRRE under reflected normal loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.1089415	0.0902539	0.1382052	0.1018291	0.1062525	0.0536277	0.0542765
10	0.3416348	0.2911386	0.5175468	0.3115058	0.2265265	0.1289673	0.1406642
20	0.6185015	0.7045859	0.9813425	0.5303648	0.2691035	0.1616054	0.1810673
25	0.6739893	0.9027447	0.9997857	0.5751968	0.2619219	0.1511962	0.182771

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Restricted ridge regression estimator under reflected normal loss, comparing the effect of a change in the sample size (n)

Table 2.8: Effect on an increase in the sample size on the risk of the RRRE under reflected normal loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.1687555	0.1471631	0.2484981	0.1484689	0.1344379	0.0771563	0.0807604
30	0.1089415	0.0902539	0.1382052	0.1018291	0.1062525	0.0536277	0.0542765
50	0.074138	0.0634703	0.0864373	0.0706513	0.0798865	0.0421587	0.0439048
100	0.0383618	0.0345535	0.0415664	0.0374354	0.0592498	0.0265544	0.026925

Preliminary test ridge regression estimator under reflected normal loss, comparing the effect of a change in the variance(σ^2)**.**

Table 2.9: Effect of an increase in the variance on the risk of the PTRRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	0.0000166	0.0000165	0.0000166	0.0000166	0.0001784	0.0000165	0.0000165
0.1	0.0016471	0.0016381	0.0016521	0.0016451	0.0062737	0.0016031	0.0016003
0.25	0.0100036	0.0096968	0.0101984	0.0099354	0.0239456	0.008751	0.0087633
0.5	0.0365188	0.0330048	0.0392042	0.035486	0.0543462	0.0252808	0.0254319
1	0.1112168	0.0922945	0.1409591	0.1033731	0.1109183	0.05524	0.0555483
4	0.236488	0.4258012	0.7412309	0.2028267	0.2731629	0.208132	0.2093742
9	0.2451612	0.6871811	0.9364953	0.2488641	0.3597922	0.3350945	0.3348939
20	0.3307011	0.8714112	0.984512	0.3392821	0.3856343	0.3794501	0.3793543

(see Figure E3 panel (a) in Appendix E for all results based on the PTRRE)

Preliminary test ridge regression estimator under reflected normal loss, comparing the effect of a change in the level of multicollinearity(γ)**.**

Table 2.10: Effect of an increase in the level of multicollinearity on the risk of the PTRRE under reflected normal loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0893581	0.0762784	0.1069666	0.0888034	0.1149501	0.0543776	0.0593886
0.8	0.11061	0.0924855	0.1391345	0.1029873	0.1089375	0.056623	0.0586928
0.9	0.1615028	0.1368722	0.2369163	0.1164266	0.100637	0.0525923	0.0562409
0.95	0.1978958	0.1894625	0.3563428	0.1011977	0.0880028	0.0445128	0.0480834
0.99	0.1900256	0.3979498	0.679225	0.0621754	0.0757749	0.0224531	0.0251799

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Preliminary test ridge regression estimator under reflected normal loss, comparing the effect of a change in the number of parameters (p)

Table 2.11: Effect of an increase in the number of parameters to be estimated on the risk of the PTRRE under reflected normal loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.11061	0.0924855	0.1391345	0.1029873	0.1089375	0.056623	0.0586928
10	0.2607062	0.2138464	0.427464	0.231376	0.1719465	0.0471389	0.0617497
20	0.3898779	0.4835283	0.8590115	0.3209366	0.1951895	0.0423395	0.0583444
25	0.4711574	0.7300629	0.9663871	0.3864171	0.1997036	0.0581011	0.0839177

Preliminary test ridge regression estimator under reflected normal loss, comparing the effect of a change in the sample size (n)

Table 2.12: Effect of an increase in the sample size on the risk of the PTRRE under reflected normal loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.1871091	0.1688937	0.2767531	0.1605017	0.1475324	0.0826207	0.0835359
30	0.11061	0.0924855	0.1391345	0.1029873	0.1089375	0.056623	0.0586928
50	0.0724991	0.0618931	0.0844816	0.0691327	0.0901583	0.0409087	0.0422614
100	0.0412222	0.0371169	0.044615	0.0403039	0.0624968	0.0274219	0.0278773

In order to summarise the results in tables 2.1 to 2.12, Figures E1 to E3 in Appendix E, panel (a) was used. For all three estimators, namely URRE, RRRE and PTRRE it was found that for relative small values of σ i.e. $0.01 \leq \sigma \leq 4$, the GM and MED ridge parameters lead to the lowest risk, whereas for $\sigma > 4$ the risk for all estimators are relatively small when using the HK and HSL ridge parameter. In all cases the LW ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE it was found that over all the tested levels of multicollinearity, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE it was found that for an increasing number of parameters to be estimated, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE it was found that when the sample size increases, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst, except for cases where the sample size is above 50, it seems as if the AM ridge parameter performs the worst.

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2.5.1 Evaluating the performance of the estimators relative to each other under reflected normal loss

The risk of RRRE, URRE and PTRRE calculated under reflected normal loss are computationally compared to one another in order to determine the performance of the estimators relative to each other. In Figure 2.2 the dashed (blue) line represents the risk of the RRRE, the solid (red) line represents the risk of the URRE and the dashed-dot (green) line represents the risk of the PTRRE. The panel of sketches in this section will be discussed in two ways, firstly by keeping σ constant and allowing the sample size, n to increase (across the rows of the panel sketches) and secondly by keeping n constant and allowing σ to increase (across the columns of the panel sketches). All the risk functions are represented as a function of $\boldsymbol{\delta} = \mathbf{H}\boldsymbol{\beta} - \mathbf{h}$. In the practical application in Chapter 5 only one linear restriction is placed on the estimators of the multiple regression model and therefore δ was regarded as a scalar in this simulation exercise. In these graphs the range of δ considered were $-1 \leq \delta \leq 1$, on the horizontal axis. For other ranges of δ the results did not differ substantially. For cases where $\boldsymbol{\delta}$ is not a scalar, the norm of $\boldsymbol{\delta}$ can be considered on the horizontal axis. The SAS program used for these simulations is in Appendix B.

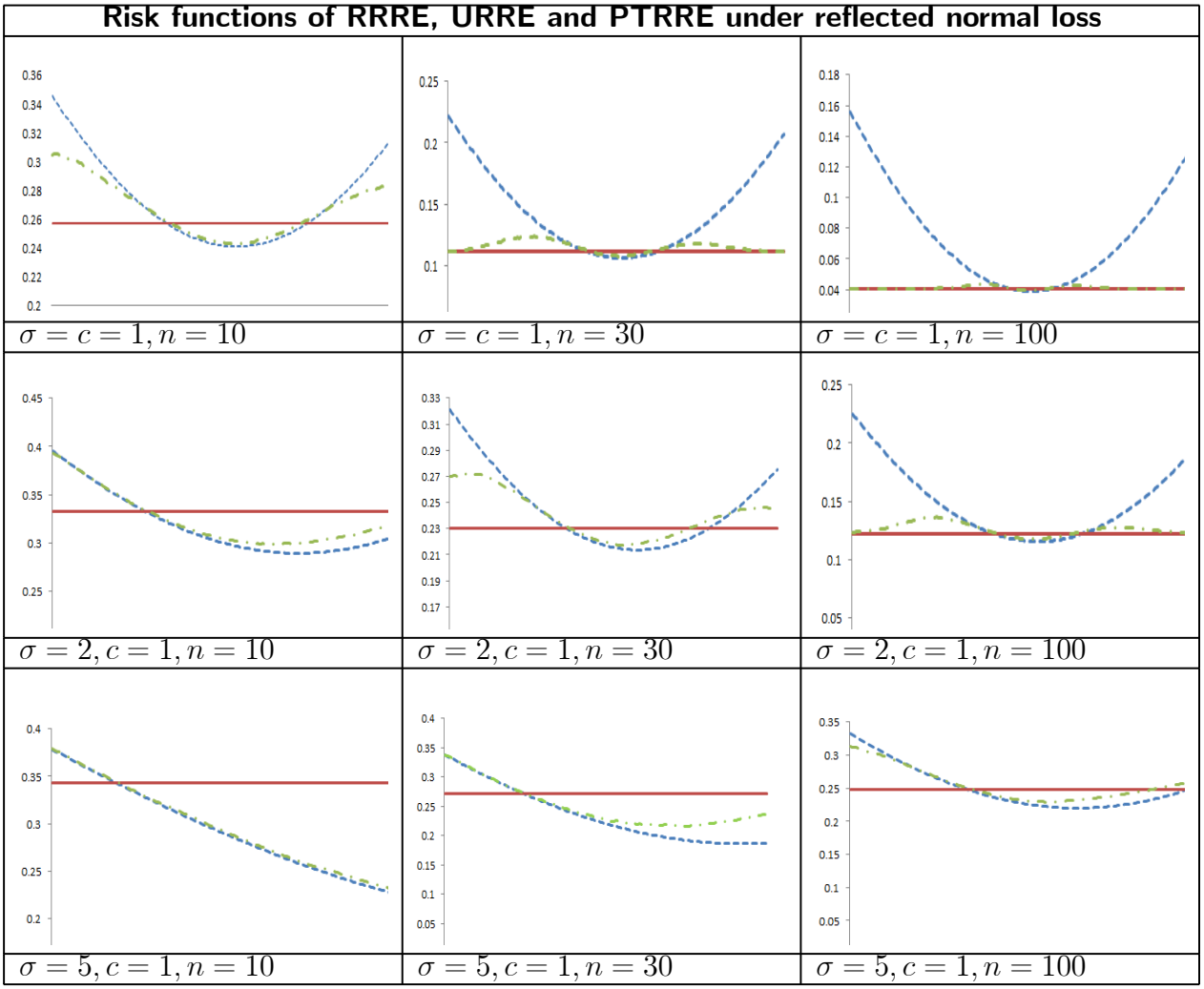

Figure 2.2 : Comparison of URRE, RRRE and PTRRE across different values of σ and n, under reflected normal loss.

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For the case where $\sigma = 1$, it is clear that as the sample size increases for this choice of σ, the risk of PTRRE tends towards the risk of URRE as $\delta \to \pm\infty$. It is well-known that URRE is the most efficient estimator as the sample size increases, since the variance of URRE, i.e. $\widetilde{\boldsymbol{\beta}}_n(k)$ is $\sigma^2 \mathbf{R}(k) \mathbf{C}^{-1} \mathbf{R}'(k)$ and $\sigma^2 = \dfrac{\sum e_i^2}{n-p}$ It is clear that as the sample size increases the variance of URRE will tend to zero. Therefore as the sample size increases, PTRRE becomes as efficient as the URRE. It can also be seen that as the sample size increases the risk associated with all the estimators decreases. Similar results are obtained for different values of σ.

When σ increases for a specific sample size, it can be seen that in the interval near the origin, the interval in which the PTRRE has lower risk relative to the URRE, becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTRRE is a more efficient estimator compared to the URRE.

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Chapter 3

Performance of the preliminary test ridge regression estimator using LINEX loss

3.1 Linear exponential loss function (LINEX loss function)

In this chapter the performance of the preliminary test ridge regression estimator is considered under LINEX loss, but it is again necessary to compare the performance of the preliminary test ridge regression estimator to that of its component estimators, namely the unrestricted ridge regression estimator and the restricted ridge regression estimator. It is therefore necessary to derive the risk functions of all three estimators under LINEX loss. In the papers by Parsian and Kirmani (2002), Porosiński and Kamińska (2009), Arashi (2010) and Arashi and Tabatabaey (2010) the properties of the LINEX function, which was first proposed by Varian (1975) are presented together with results of estimation under LINEX loss for a number of probability distributions. One of the disadvantages of the LINEX loss function is that it exhibits a huge increase in expected loss as the estimator error namely $\boldsymbol{\beta}^* - \boldsymbol{\beta}$ increases, which limits the application of the loss function in practice. Assume $\boldsymbol{\beta}^*$ is a vector $p \times 1$ containing regression coefficient estimators and $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown regression coefficient parameters. The multivariate LINEX loss function is given by:

$$\mathcal{L}_{LINEX}\left(\boldsymbol{\beta}^*, \boldsymbol{\beta}\right) = d\left[\exp\left(\mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right)\right) - \mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right) - 1\right]$$

where d is referred to as the scale parameter and $\mathbf{a}$ is a vector containing shape parameters. When the values of $\mathbf{a}$ is close to zero, the LINEX loss function tends to be more symmetric compared to larger positive or negative values of $\mathbf{a}$. As mentioned before, estimation error vector in the loss function is given by $\boldsymbol{\beta}^* - \boldsymbol{\beta}$, therefore if $\mathbf{a}$ contains negative values it suggests that negative errors (underestimation) are penalised and when $\mathbf{a}$ contains positive values it suggests that positive errors (overestimation) are penalised.

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The following panel of sketches illustrate the effect of $\mathbf{a}$ on the LINEX loss function for $p = 2$:

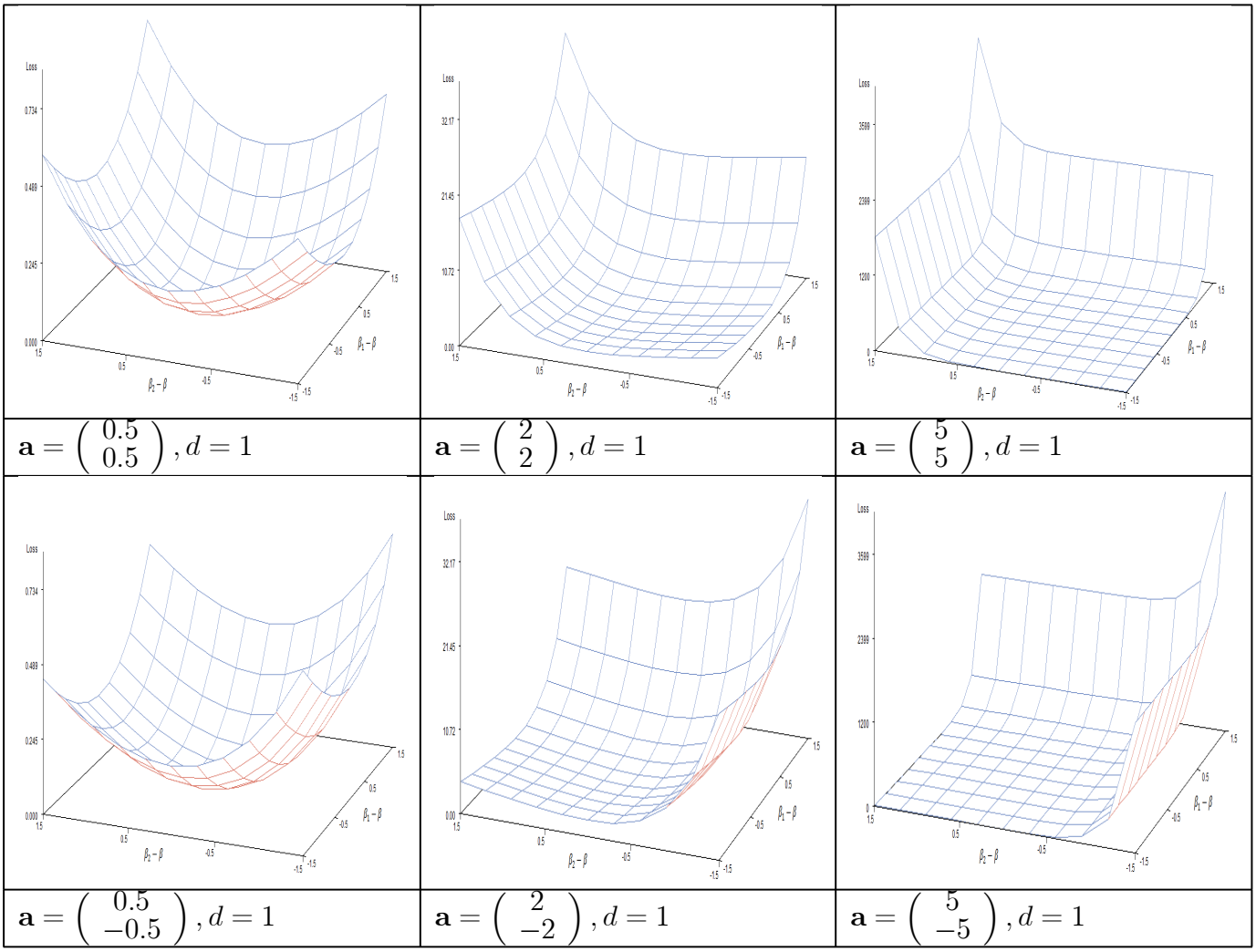

Figure 3.1 : The effect of $\mathbf{a}$ *on the LINEX loss function.*

It is clear that as the a_i values increase in absolute value, the higher the level of asymmetry of the LINEX loss function. The vector norm, namely $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + ... + a_p^2}$ can also be considered to evaluate the degree of asymmetry. As the value of $\|\mathbf{a}\|$ increases it indicates a higher degree of asymmetry as indicated in the sketches above. The only aspect to consider when using this measure is that the measure doesn't reflect the direction of the asymmetry. The flexibility of the LINEX loss function is also easily seen by penalising positive and negative errors in the same model.

The risk function under LINEX loss is therefore given by:

$$\Re_{LINEX}\left(\boldsymbol{\beta}^*, \boldsymbol{\beta}\right) = dE\left[\exp\left(\mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right)\right) - \mathbf{a}'\left(\boldsymbol{\beta}^* - \boldsymbol{\beta}\right) - 1\right]$$

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3.2 Risk function of the URRE

Theorem 11 *The risk function of the URRE under LINEX loss is given by:*

$$E\left[\mathcal{L}_{LINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]=d\left[\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-1\right]$$

When $\mathbf{R}(k)=\mathbf{I}_p$ *the risk function reduces to the risk function of the URE under LINEX loss.*

$$E\left[\mathcal{L}_{LINEX}\left(\widetilde{\boldsymbol{\beta}}_n,\boldsymbol{\beta}\right)\right]=d\left[\exp\left(\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-1\right]$$

Proof. The risk function for $\widetilde{\boldsymbol{\beta}}_n(k)$ under the LINEX loss function is defined as:

$$E\left[\mathcal{L}_{LINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$$

$$=dE\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]$$

by adding and subtracting $\mathbf{R}(k)\,\boldsymbol{\beta}$ the expression reduces to

$$=dE\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}+\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)-1\right]$$

$$=dE\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]-d\mathbf{a}'E\left[\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}\right]-d\mathbf{a}'E\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-d$$

$$=d\exp\left[-\boldsymbol{\beta}'\mathbf{a}\right]E\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\,\mathbf{a}\right)\right]$$

(3.1)

$$-d\mathbf{a}'E\left[\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}\right]$$

(3.2)

$$-d\mathbf{a}'E\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-d$$

(3.3)

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Expression (3.1):

Since the moment generating function of the multivariate normal distribution (see 1.10) is defined as

$$MGF_{\widetilde{\boldsymbol{\beta}}'_n(k)}(\mathbf{a}) = E\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\,\mathbf{a}\right)\right]$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\,\mathbf{a}\right)\right.$$

$$\left.\times\frac{1}{(2\pi)^{\frac{p}{2}}\left|\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right|^{\frac{1}{2}}}\exp\left(-\tfrac{1}{2}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}\right)'\boldsymbol{\Sigma}^{-1}_{\widetilde{\boldsymbol{\beta}}_n(k)}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}\right)\right)d\widetilde{\boldsymbol{\beta}}_n(k)\right]$$

$$= \exp\left(\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) \tag{3.4}$$

therefore,

$$d\exp\left[-\boldsymbol{\beta}'\mathbf{a}\right]E\left[\exp\left(\widetilde{\boldsymbol{\beta}}'_n(k)\,\mathbf{a}\right)\right] = d\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

Expression (3.2): Since $E\left[\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right]=\mathbf{0}$

$$-d\mathbf{a}'E\left[\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\,\boldsymbol{\beta}\right]$$

$$= -d\mathbf{a}'E\left[\mathbf{R}(k)\,\widetilde{\boldsymbol{\beta}}_n-\mathbf{R}(k)\,\boldsymbol{\beta}\right]$$

$$= -d\mathbf{a}'\mathbf{R}(k)\,E\left[\widetilde{\boldsymbol{\beta}}_n-\boldsymbol{\beta}\right]$$

$$= -d\mathbf{a}'\mathbf{0} = 0$$

Expression (3.3):

$$-d\mathbf{a}'E\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-d = -d\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)-d$$

Combining the final expressions of (3.1) - (3.3) completes the proof.■

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3.3 Risk function of the RRRE

Theorem 12 *The risk function of the RRRE under LINEX loss is given by:*

$$E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]=d\left[\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\left[\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right]-1\right]$$

When $\mathbf{R}(k)=\mathbf{I}_p$ the risk function reduces to the risk function of the URE under LINEX loss.

Proof. The proof is similar to the proof of Theorem 11. ■

3.4 Risk function of the PTRRE

Theorem 13 *Ignoring the terms of order 3, the approximate risk function of the PTRRE under LINEX loss is given by:*

$$E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right]\approx\frac{1}{2}\left[d\sigma^2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}\right.$$

$$-2d\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

$$+d\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$+2d\mathbf{a}'\left(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

$$+d\left[\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p+E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'\right]\right.$$

$$\left.\left.\times\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}\right]\right]$$

When $\mathbf{R}(k)=\mathbf{I}_p$ the risk function reduces to the risk function of the URE under LINEX loss.

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Proof. In order to derive the risk function of the preliminary test ridge regression estimator under LINEX loss, $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$, it is necessary to write the estimator in terms of the expressions $(1.21 \text{ and } 1.24)$. From (1.5) and (1.13) it is known that

$$\widehat{\boldsymbol{\beta}}_n^{PT}(k) = \widetilde{\boldsymbol{\beta}}_n(k) - \left(\widetilde{\boldsymbol{\beta}}_n(k) - \widehat{\boldsymbol{\beta}}_n(k)\right) I_{NR}(F)$$

and by subtracting $\boldsymbol{\beta}$ on both sides of the equation $\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)$ resulted in

$$= \left(\widetilde{\boldsymbol{\beta}}_n(k) - \boldsymbol{\beta}\right) - \left(\widetilde{\boldsymbol{\beta}}_n(k) - \widehat{\boldsymbol{\beta}}_n(k)\right) I_{NR}(F)$$

$$= \left(\widetilde{\boldsymbol{\beta}}_n(k) - \mathbf{R}(k)\boldsymbol{\beta} + \mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}\right) - \mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) I_{NR}(F)$$

$$= \mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right) + (\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}) - \mathbf{R}(k)\left(\widetilde{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\beta}}_n\right) I_{NR}(F)$$

using results (1.21) and (1.24) it was found that

$$= \mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta}) + (\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}) - \mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR}(F) \tag{3.5}$$

The risk function for $\widehat{\boldsymbol{\beta}}^{PT}(k)$ under LINEX loss is defined as:

$$\Re_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right) = dE\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)\right] - \mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right) - 1\right]$$

Using the Taylor expansion, then

$$\Re_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right) = \frac{dE\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)\right]^2}{2!} + O\left(x^3\right) \tag{3.6}$$

The expectation in the numerator

$$E\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k) - \boldsymbol{\beta}\right)\right]^2$$

can be simplified by substituting (3.5) into the expectation:

$$E\left[\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta}) + \mathbf{a}'(\mathbf{R}(k)\boldsymbol{\beta} - \boldsymbol{\beta}) - \mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega} I_{NR}(F)\right]^2$$

$$= E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)'\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega} - \boldsymbol{\eta})\right)\right]$$

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(3.7)

$$+2E\left[\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

(3.8)

$$-2E\left[\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

(3.9)

$$+E\left[\left(\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

(3.10)

$$-2E\left[\left(\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

(3.11)

$$+E\left[\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)'\right.$$
$$\left.\times\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}\left(F\right)\right)\right]$$

(3.12)

Expression (3.7):

$$E\left[\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)'\left(\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right)\right]$$
$$=E\left[\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\right]$$
$$=E\left[\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{a}\right]$$
$$=\mathbf{a}'\mathbf{R}\left(k\right)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)\left(\boldsymbol{\omega}-\boldsymbol{\eta}\right)'\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'\left(k\right)\mathbf{a}$$

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using (1.38) it was found that

$$\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\sigma^2\mathbf{I}_p\right)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{a}$$

$$=\sigma^2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{R}'(k)\,\mathbf{a}$$

$$=\sigma^2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

Expression (3.8): Since $E[(\boldsymbol{\omega}-\boldsymbol{\eta})]=\mathbf{0}$ the following was obtained

$$2E\left[\left(\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\mathbf{a}'(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$=2E\left[\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\left(\mathbf{a}'(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta})\right)\right]$$

$$=2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E[(\boldsymbol{\omega}-\boldsymbol{\eta})]\left(\mathbf{a}'(\mathbf{R}(k)\,\boldsymbol{\beta}-\boldsymbol{\beta})\right)=0$$

Expression (3.9):

$$-2E\left[\left(\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\right)'\left(\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=-2E\left[\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'(\boldsymbol{\omega}-\boldsymbol{\eta})\,\boldsymbol{\omega}'I_{NR}(F)\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}\right]$$

$$=-2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E[(\boldsymbol{\omega}-\boldsymbol{\eta})\,\boldsymbol{\omega}'I_{NR}(F)]\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

$$=-2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})\left((\boldsymbol{\omega}-\boldsymbol{\eta})+\boldsymbol{\eta}\right)'I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

$$=-2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})'I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

(3.13)

$$-2\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E[(\boldsymbol{\omega}-\boldsymbol{\eta})\,\boldsymbol{\eta}'I_{NR}(F)]\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

(3.14)

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Expectation (3.13): Using (1.40) it was found that

$$-2\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})(\boldsymbol{\omega}-\boldsymbol{\eta})' I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

$$=-2\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left(\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p\right)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

$$=-2\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

$$=-2\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

Expectation (3.14): Since $E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})\boldsymbol{\eta}' I_{NR}(F)\right]=E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})I_{NR}(F)\right]\boldsymbol{\eta}'$ and using (1.32) resulted in

$$-2\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[(\boldsymbol{\omega}-\boldsymbol{\eta})\boldsymbol{\eta}' I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}=0$$

Combining (3.13) and (3.14) an expression for (3.9) was obtained:

$$-2\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

Expression (3.10):

$$E\left[\left(\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\right]$$

$$=\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

Expression (3.11): Using (1.31) it was found that

$$2E\left[\left(\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=2E\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right]$$

$$=2\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}I_{NR}(F)\right]$$

$$=2\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

Expression (3.12): Using (1.39) the following was obtained

$$E\left[\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)'\right.$$

$$\left.\times\left(\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}I_{NR}(F)\right)\right]$$

$$=E\left[\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\omega}\boldsymbol{\omega}'I_{NR}(F)\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\right]$$

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$$= \mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

with $E\left[\boldsymbol{\omega}\boldsymbol{\omega}'I_{NR}(F)\right] = \sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'$ therefore,

$$\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'E\left[\boldsymbol{\omega}\boldsymbol{\omega}'I_{NR}(F)\right]\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}$$

$$= \Bigg[\mathbf{a}'\mathbf{R}(k)\,\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2 E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p + E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'\right]$$

$$\times\,\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\,\mathbf{a}\Bigg]$$

By combining (3.7) - (3.12) an expression for the numerator of (3.6) was obtained.

(3.15)

Substituting (3.15) into (3.6) completes the proof.■

3.5 Simulations

In this section the same simulation exercise was conducted as in Chapter 2 (for more detail see section 1.5 page 28) and the results are given in a similar way. All three proposed estimators (URRE, RRRE and PTRRE) are considered under the LINEX loss function $(\mathbf{a} = \mathbf{1}, d = 1.)$ In the first table in each section only the error variance varied, the second table represents the risk values over different levels of multicollinearity. The third table represents the effect of an increase in the number of parameters that needs to be estimated and the fourth table represents the effect of an increase in the sample size. The SAS program for these simulations is available in Appendix A.

Unrestricted ridge regression estimator under LINEX loss, comparing the effect of a change in the variance(σ^2).

Table 3.1: Effect of an increase in the variance on the risk of the URRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.49E-06	2.49E-06	2.49E-06	2.49E-06	0.0013919	2.49E-06	2.51E-06
0.1	0.000222	0.0002229	0.0002218	0.0002221	0.0164399	0.0002523	0.0003237
0.25	0.0015779	0.0015911	0.0015855	0.0015775	0.0573444	0.0019879	0.0022404
0.5	0.0060267	0.0062432	0.0061068	0.0060221	0.1367345	0.0130565	0.0137515
1	0.0228799	0.0256124	0.0237123	0.0227638	0.2548971	0.0619075	0.0668131
4	0.2687104	0.3491463	0.4444086	0.2718964	0.8038869	0.6118955	0.6066501
9	0.6184285	2.3418732	4.1699215	0.6542804	1.0337162	0.9641137	0.9614041
20	0.9470142	331.14739	1639.8916	0.9783371	1.1125986	1.095838	1.0954383

(see Figure E1, panel (b) in Appendix E for all the results of URRE)

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Unrestricted ridge regression estimator under LINEX loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 3.2: Effect of an increase in the level of multicollinearity on the risk of the URRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0272922	0.0307117	0.0287744	0.0272555	0.2889368	0.075849	0.0911682
0.8	0.0234975	0.0260209	0.0245722	0.0234705	0.263243	0.0625855	0.0618597
0.9	0.0195846	0.0212434	0.0200445	0.0202895	0.2263896	0.0472795	0.0506391
0.95	0.0180375	0.018881	0.0183726	0.0219692	0.2360793	0.0471423	0.0481491
0.99	0.0161885	0.0164978	0.0165378	0.0412554	0.2232969	0.0427467	0.0407586

Unrestricted ridge regression estimator under LINEX loss, comparing the effect of a change in the number of parameters (p)

Table 3.3: Effect of an increase in the number of parameters to be estimated on the risk of the URRE under LINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.0234975	0.0260209	0.0245722	0.0234705	0.263243	0.0625855	0.0618597
10	0.0257605	0.0272743	0.0271941	0.0258191	0.9396982	0.1236577	0.0933821
20	0.0281054	0.0289691	0.033652	0.0289419	1.855324	0.192965	0.1178431
25	0.0280919	0.0306888	0.0446579	0.029549	2.2721672	0.2069142	0.1172139

Unrestricted ridge regression estimator under LINEX loss, comparing the effect of a change in the sample size (n)

Table 3.4: Effect of an increase in the sample size on the risk of the URRE under LINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0503534	0.0571268	0.0546896	0.0510983	0.3484839	0.1371877	0.1518859
30	0.0234975	0.0260209	0.0245722	0.0234705	0.263243	0.0625855	0.0618597
50	0.0133175	0.0143882	0.0136717	0.0133472	0.1851107	0.0328133	0.0340946
100	0.0069222	0.0072964	0.0070231	0.0069269	0.1379556	0.0143201	0.0146389

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Restricted ridge regression estimator under LINEX loss, comparing the effect of a change in the variance(σ^2).

Table 3.5: Effect of an increase in the variance on the risk of the RRRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.29E-06	2.28E-06	2.29E-06	2.29E-06	0.0013886	2.24E-06	2.28E-06
0.1	1.76E-06	1.21E-06	2.18E-06	1.65E-06	0.0161678	0.0000241	0.0000929
0.25	1.43E-06	9.19E-06	2.04E-06	2.43E-06	0.0559977	0.0004596	0.0007365
0.5	0.0000271	0.0001957	1.58E-06	0.000057	0.1332403	0.0077864	0.0086111
1	0.0005445	0.0025758	4.75E-06	0.0010233	0.2441713	0.0457966	0.0508376
4	0.0820385	0.0565302	0.0005747	0.1195946	0.7930312	0.5896867	0.5832996
9	0.4632639	0.0864715	0.0017016	0.5479301	1.0308931	0.9587859	0.9574256
20	0.9077195	0.1002413	0.0018098	0.9497898	1.1107399	1.0937663	1.0935007

(see Figure E2, panel (b) in Appendix E for all the results of RRRE)

Restricted ridge regression estimator under LINEX loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 3.6: Effect on an increase in the level of multicollinearity on the risk of the RRRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.000641	0.0036064	0.0000252	0.0006262	0.2792505	0.0605898	0.0764599
0.8	0.0005526	0.0025592	4.62E-06	0.0010242	0.2534618	0.0478898	0.047272
0.9	0.000452	0.0015232	9.99E-06	0.002338	0.2152412	0.0334304	0.0367768
0.95	0.0004049	0.0008783	0.0000193	0.0051408	0.2261691	0.0343994	0.0347464
0.99	0.0003628	0.0001872	0.0000452	0.0275562	0.2139202	0.0301837	0.0275512

Restricted ridge regression estimator under LINEX loss, comparing the effect of a change in the number of parameters (p)

Table 3.7: Effect of an increase in the number of parameters to be estimated on the risk of the RRRE under LINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.0005526	0.0025592	4.62E-06	0.0010242	0.2534618	0.0478898	0.047272
10	0.4768643	0.484579	0.4720913	0.4805061	1.3172652	0.6584262	0.6107194
20	1.2659497	1.290517	1.4905098	1.2573528	2.5258614	1.4263526	1.3456467
25	1.1648705	1.3128668	1.8121345	1.1487571	2.8368007	1.3076337	1.2088602

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Restricted ridge regression estimator under LINEX loss, comparing the effect of a change in the sample size (n)

Table 3.8: Effect of an increase in the sample size on the risk of the RRRE under LINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0022472	0.0076932	0.0001033	0.0045034	0.3342832	0.1123389	0.1284607
30	0.0005526	0.0025592	4.62E-06	0.0010242	0.2534618	0.0478898	0.047272
50	0.0002117	0.0010701	4.94E-07	0.0003791	0.1796453	0.0229607	0.0247722
100	0.0000536	0.0003123	7.93E-08	0.0000906	0.1334878	0.0082621	0.0087817

Preliminary test ridge regression estimator under LINEX loss, comparing the effect of a change in the variance(σ^2)**.**

Table 3.9: Effect of an increase in the variance on the risk of the PTRRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.17E-06	2.16E-06	2.17E-06	2.17E-06	0.0028249	2.14E-06	2.14E-06
0.1	0.0000739	0.0000738	0.0000741	0.0000739	0.0189713	0.0000813	0.0000859
0.25	0.0004638	0.0004687	0.0004687	0.0004645	0.0698066	0.0010349	0.0013247
0.5	0.0016369	0.001814	0.0016404	0.0016554	0.1312955	0.0083023	0.0096624
1	0.0067331	0.0088424	0.0066409	0.0070137	0.261523	0.0529673	0.049579
4	0.1412467	0.1620255	0.1440023	0.1690607	0.7814624	0.5861453	0.5924365
9	0.5523396	2.4555379	4.2110122	0.6085024	1.0278617	0.9549981	0.9510714
20	0.9251724	238.50027	908.27665	0.9632687	1.1112948	1.0945344	1.0953171

(see Figure E3, panel (b) in Appendix E for all the results of PTRRE)

Preliminary test ridge regression estimator under LINEX loss, comparing the effect of a change in the level of multicollinearity(γ)**.**

Table 3.10: Effect of an increase in the level of multicollinearity on the risk of the PTRRE under LINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0098944	0.0129995	0.0102398	0.00988	0.2687538	0.0626524	0.0850684
0.8	0.0069031	0.0089518	0.0068472	0.0072279	0.2429004	0.0517412	0.0545445
0.9	0.0051672	0.0062399	0.0049448	0.0067154	0.241459	0.0395513	0.0385507
0.95	0.0057597	0.0063717	0.0055642	0.010398	0.2317894	0.0378813	0.0376763
0.99	0.0043706	0.0043111	0.004217	0.0304182	0.2345355	0.0352696	0.0362339

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Preliminary test ridge regression estimator under LINEX loss, comparing the effect of a change in the number of parameters (p)

Table 3.11: Effect of an increase in the number of parameters to be estimated on the risk of the PTRRE under LINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.0069031	0.0089518	0.0068472	0.0072279	0.2429004	0.0517412	0.0545445
10	0.0242353	0.0256523	0.0253732	0.024257	0.9554551	0.1265353	0.0920304
20	0.0273602	0.0278644	0.0319506	0.0283394	1.9363356	0.1954489	0.1140029
25	0.0278778	0.0310244	0.0459858	0.0294221	2.2549517	0.213404	0.1240177

Preliminary test ridge regression estimator under LINEX loss, comparing the effect of a change in the sample size (n)

Table 3.12: Effect of an increase in the sample size on the risk of the PTRRE under LINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0161828	0.0218173	0.0154884	0.0182105	0.3658404	0.1196617	0.1238406
30	0.0069031	0.0089518	0.0068472	0.0072279	0.2429004	0.0517412	0.0545445
50	0.0042419	0.0052618	0.0041853	0.0043618	0.1863505	0.0263496	0.0284935
100	0.0020705	0.0023623	0.0020529	0.0020967	0.1345911	0.0097424	0.0097547

In order to summarise the results in tables 3.1 to 3.12, Figures E1 to E3 in Appendix E, panel (b) was used. For URRE and PTRRE is was found that the risk is the lowest when using the HK and HSL ridge parameters, whereas for RRRE the LW ridge parameter leads to the lowest risk. In all cases the AM ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, LW and HKB ridge parameters lead to the lowest risk over all the tested levels of multicollinearity. For the HSL ridge parameter the risks are also low for $\gamma < 0.95$ the AM ridge parameter performed the worst overall. For all three estimators, namely URRE, RRRE and PTRRE is was found that the HK and HSL ridge parameters lead to the lowest risks over an increasing number of parameters to be estimated. The HKB ridge parameter also performs well, except for RRRE when $p > 20$ the AM ridge parameter performed the worst overall. For all the estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB, LW and HSL ridge parameters perform the best over different sample sizes and the AM ridge parameter performs the worst overall.

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3.5.1 Evaluation the performance of the estimators relative to each other under LINEX loss

The risk of RRRE, URRE and PTRRE calculated under LINEX loss are computationally compared to each other in order to determine the performance of the estimators relative to each other. In Figure 3.2 the dashed (blue) line represents the risk of the RRRE, the solid (red) line represents the risk of the URRE and the dashed-dot (green) line represents the risk of the PTRRE. The panel of sketches in this section will be discussed in two ways, firstly by keeping σ constant and allowing the sample size, n to increase (across the rows of the panel sketches) and secondly by keeping n constant and allowing σ to increase (across the columns of the panel sketches). All the risk functions are represented as a function of $\boldsymbol{\delta} = \mathbf{H}\boldsymbol{\beta} - \mathbf{h}$ (in this application δ was a scalar as explained in section 2.5.1), where $-1 \leq \delta \leq 1$ on the horizontal axis. The SAS program used for these simulations is in Appendix B.

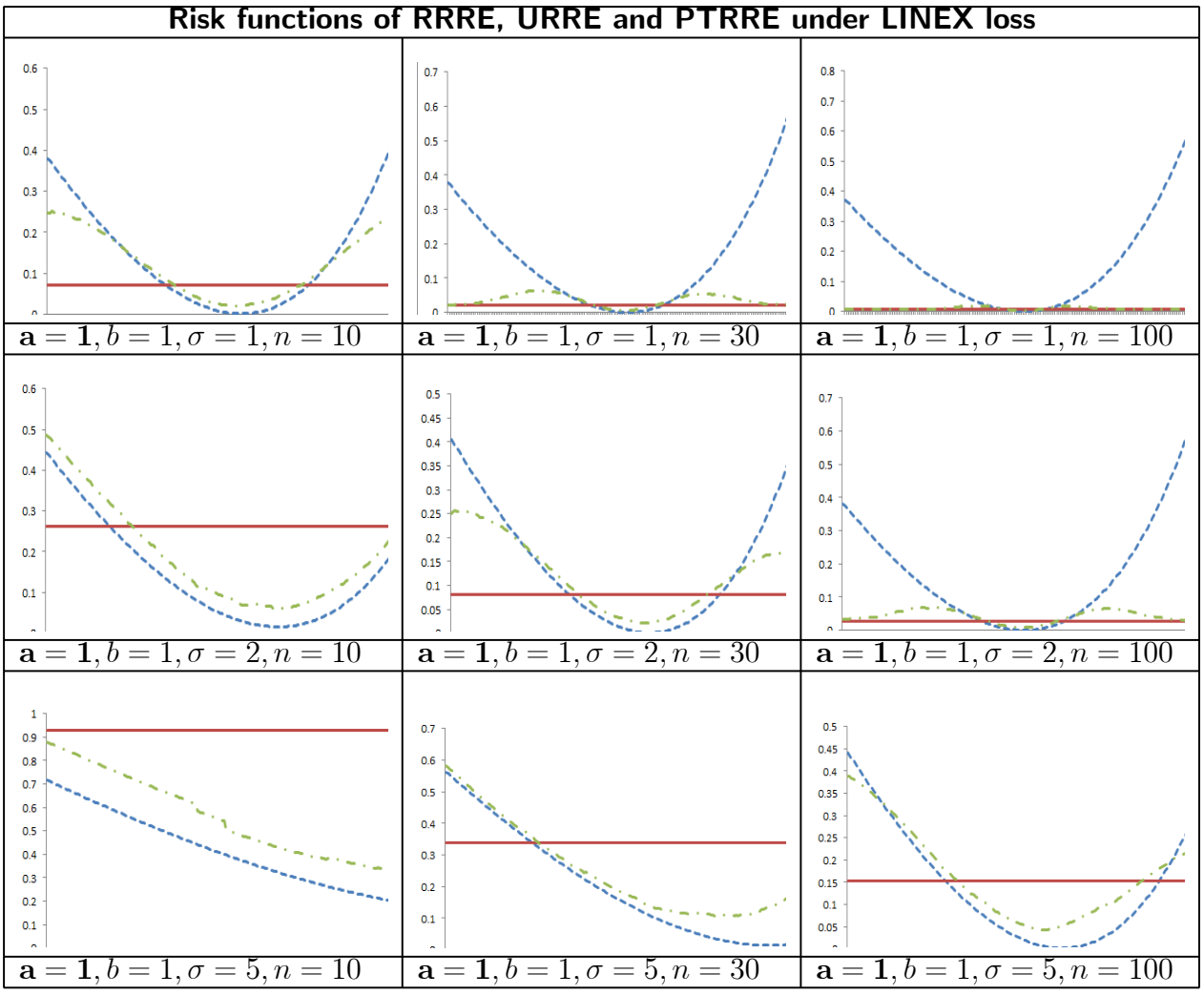

Figure 3.2 : Comparison of URRE, RRRE and PTRRE across different values of σ and n under LINEX loss.

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For the case where $\sigma = 1$, it is again clear that as the sample size increases for this choice of σ, the risk of PTRRE tends towards the risk of URRE as $\delta \rightarrow \pm\infty$. It is well-known that URRE is the most efficient estimator as the sample size increases, since the variance of URRE, i.e. $\widetilde{\boldsymbol{\beta}}_n(k)$ is $\sigma^2 \mathbf{R}(k) \mathbf{C}^{-1} \mathbf{R}'(k)$ and $\sigma^2 = \dfrac{\sum e_i^2}{n-p}$ It is clear that as the sample size increases the variance of URRE will tend to zero. Therefore as the sample size increases, PTRRE becomes equally efficient as the URRE. It is clear that as the sample size increases the risk associated with all the estimators decreases as shown in the previous section. Similar results are obtained for different values of σ.

When σ increases for a specific sample size, it can be seen that in the interval near the origin, the interval in which the PTRRE has lower risk relative to the URRE, becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTRRE is a more efficient estimator compared to the URRE.

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Chapter 4

Performance of the preliminary test estimator using BLINEX loss

4.1 Bounded linear exponential loss function (BLINEX loss function)

In this chapter the performance of the preliminary test estimator under BLINEX loss is considered and in all the sections in this chapter the performance of the preliminary test estimator is compared to the performance of its component estimators. Therefore the risk functions of all three the estimators are derived and compared to each other, either numerically or computationally. In a paper by Wen and Levy (2001) a new parametric family of bounded and asymmetric loss functions, called the BLINEX loss function, was developed and the mathematical properties of the BLINEX loss function were discussed in the paper. The BLINEX function is the bounded alternative to the LINEX loss function, therefore it doesn't have the same limitations as the LINEX loss function as mentioned in the previous chapter. The BLINEX function is therefore both bounded and asymmetric, which allowed the same flexibility as exhibited by the LINEX loss function but it also has the added advantage of being bounded. The relationship between the BLINEX and LINEX loss functions will be considered next.

Assume that β^* is an estimator of the unknown parameter β where the LINEX loss function is then defined as follows:

$$\mathcal{L}_{LINEX}\left(\beta^*, \beta\right) = d\left(e^{a(\beta^* - \beta)} - a\left(\beta^* - \beta\right) - 1\right)$$

where a is the shape parameter and d is the scale parameter.

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The BLINEX loss function is derived from the LINEX loss function, let

$$\begin{aligned}\mathcal{L}_{BLINEX}(\tau,\theta) &= \frac{d\left(e^{a(\beta^*-\beta)}-a(\beta^*-\beta)-1\right)}{1+\lambda d\left(e^{a(\beta^*-\beta)}-a(\beta^*-\beta)-1\right)} \\ &= \frac{\mathcal{L}_{LINEX}(\beta^*,\beta)}{1+\lambda\mathcal{L}_{LINEX}(\beta^*,\beta)}\end{aligned} \tag{4.1}$$

with $b=\lambda d$.

There is a need to investigate the performance of the preliminary test estimator when a linear restriction is placed on the regression model, working under a bounded loss function and also taking into account the presence of multicollinearity. Assume $\boldsymbol{\beta}^*$ is a $p\times 1$ vector containing regression coefficient estimators and $\boldsymbol{\beta}$ is a $p\times 1$ vector of unknown regression coefficient parameters.

The multivariate BLINEX loss function is given by:

$$\mathcal{L}_{BLINEX}(\boldsymbol{\beta}^*,\boldsymbol{\beta})=\frac{1}{\lambda}\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})\right)-\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})-1\right]}\right]$$

with $b=\lambda d$ where $d=$the scale parameter of the LINEX loss function.

The BLINEX loss function depends on the selected values of three parameters, namely, the constant vector $\mathbf{a}$, the constants b and λ, where each of these values plays a significant role in the shape of the loss function. The bounding parameter λ, specifies the range of the loss which is bounded between 0 and $\frac{1}{\lambda}$. The sign of the values in the constant vector $\mathbf{a}$ will determine the direction of error penalisation, where negative values will penalise negative errors and a positive value will penalise positive errors. This constant also influences the flatness of the curve. For small positive values of the constant vector $\mathbf{a}$, the curve will be flatter and large positive values will result in a steeper curve. The constant b is an asymmetry parameter and it can be seen that for smaller values of b the loss function is more asymmetric and for larger values of b the function is more symmetric.

The risk function under BLINEX loss is therefore given by:

$$\Re_{BLINEX}(\boldsymbol{\beta}^*,\boldsymbol{\beta})=\frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})\right)-\mathbf{a}'(\boldsymbol{\beta}^*-\boldsymbol{\beta})-1\right]}\right]$$

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The following panel of sketches shows the effect of different values of $\mathbf{a}$ and b on the BLINEX loss function for $p = 2$:

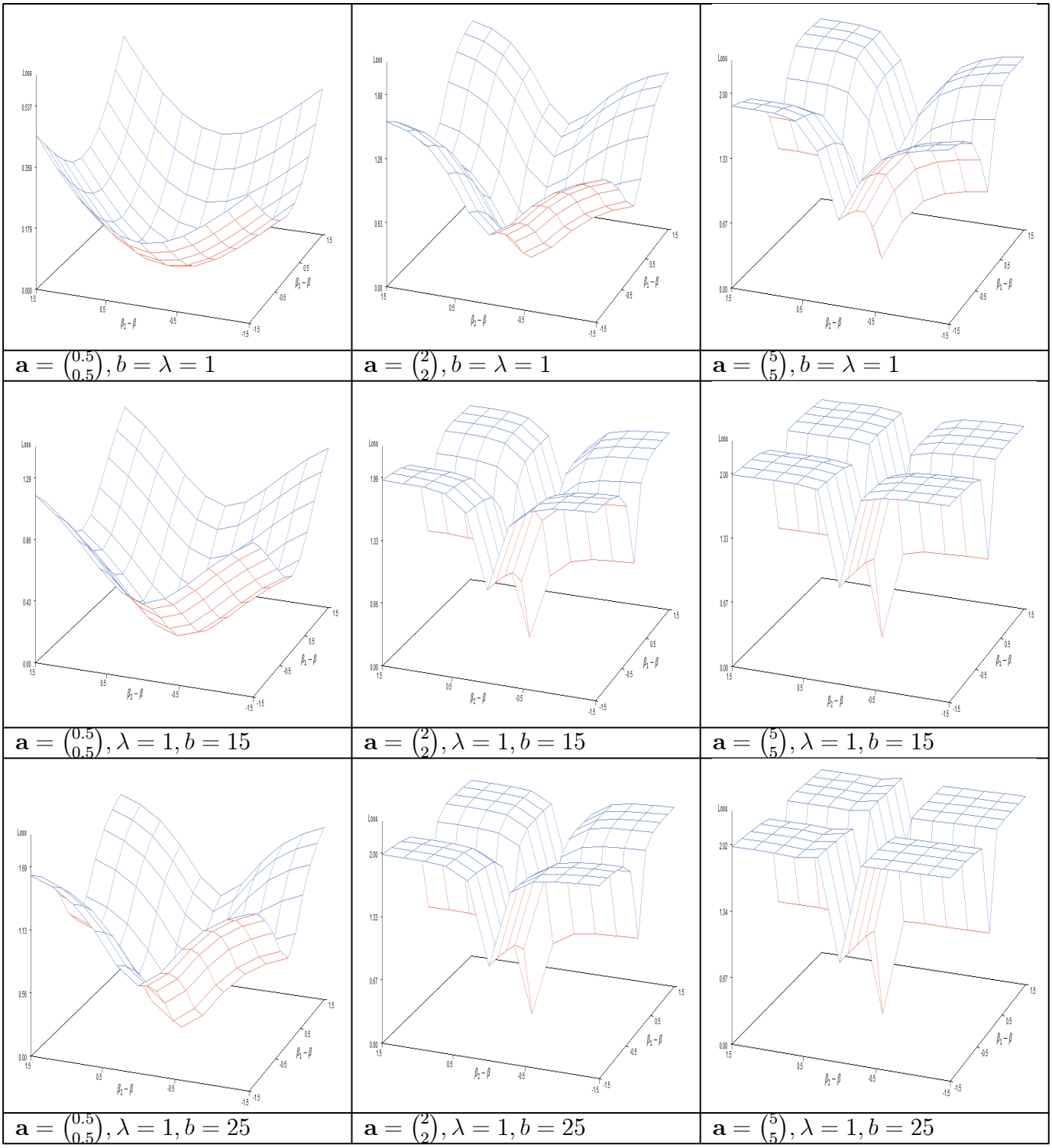

Figure 4.2 : The effect of $\mathbf{a}$ *and* b *on the BLINEX loss function.*

The role of $\mathbf{a}$ is exactly the same as for the LINEX loss function. It can be seen that as the absolute values of a_i increases, the level of positive or negative error penalisation increases, where as the value of b increases the loss function shows higher levels of symmetry in the loss functions.

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4.2 Classical non-ridge preliminary test estimator

In this section the performance the preliminary test estimator (PTE), restricted maximum likelihood estimator (RMLE) and unrestricted maximum likelihood estimator (UMLE) are all considered under the BLINEX loss function and the risk functions are compared to each other in order to determine which estimator performs better compared to the rest. The multicollinearity assumption is not taken into account to analyse the risk performance under both classical as well as the Bayes paradigms. The proposed results are general and can be considered in ridge regression as well. A number of papers have already been published on the preliminary test estimators using different proposed loss structures by Bancroft (1944,1964), Giles and Giles (1996), Giles (2002), Ohtani et al. (1997) Kibria and Saleh (1993, 2006), Saleh (2006), Arashi et al. (2008) to name a few.

The problem under study is to consider the estimation of a location parameter μ from a normal population with unknown variance in order to evaluate the finite-sample properties of the preliminary test estimator in terms of its risk. Suppose that $X_{1,}X_2, ..., X_n$ is a random sample from $N(\mu, \sigma^2)$ where both $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are unknown.

The outcome of a preliminary test on μ will determine which of the estimators will be used to estimate the population mean μ. The unrestricted maximum likelihood estimator, $\overline{X} = \frac{\Sigma X_i}{n}$, will be regarded as the estimator of μ when nothing is a priori known about the parameter μ, whereas the restricted maximum likelihood estimator will be chosen when some constraints are imposed to the model as a null hypothesis. The standard $t - test$ for the population mean will be the appropriate test since the scale parameter σ is unknown.

Now consider the case where the following hypotheses is dealt with:

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_A : \mu \neq \mu_0 \end{cases}$$

Then it is well known that the test statistic is given as $t = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t(n-1)$ with $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ (see section 1.2.3)

Alternatively the following statistic $t^2 = \frac{\left(\overline{X} - \mu_0\right)^2}{\frac{s^2}{n}} = F \sim F(1, n-1)$ (see section 1.2.4) can be used.

In more general terms, the sampling distribution of the test statistic will be the noncentral F variate, $F \sim F(1, n-1, \lambda^{**})$ with $\lambda^{**} = \frac{n\delta^2}{2\sigma^2}$ known as the noncentrality parameter and the estimation error is defined as $\delta = \mu_0 - \mu$. It is clear that when $\delta = 0$, the sampling distribution under the null hypothesis will be the central $F(1, n-1, 0)$ distribution. Based on this proposal the unrestricted maximum likelihood estimator(UMLE) is given by $\mu_1 = \overline{X}$ and the restricted maximum likelihood estimator (RMLE) is equal to μ_0.

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Then, the PTE of μ is defined as

$$\mu_p = [I_R(F) \times \mu_1] + [I_{NR}(F) \times \mu_0]$$

The null hypothesis will not be rejected when $F \leq c_\alpha$, where c_α is a critical value defined as $F_{\alpha,1,n-1,\lambda^{**}=0}$, where α is the level of significance.

4.2.1 Risk function of RMLE

The risk function of RMLE is defined and followed by an illustration of how the risk function behaves for different choices of the parameters in the function.

The risk function of the RMLE is defined as follows:

$$\Re_{BLINEX}(\mu_0, \mu) = E\left[\frac{1}{\lambda}\left[1 - \frac{1}{1+b\left(e^{a\delta}-a\delta-1\right)}\right]\right] = \frac{1}{\lambda}\left[1 - \frac{1}{1+b\left(e^{a\delta}-a\delta-1\right)}\right]$$

with $\delta = \mu_0 - \mu$.

In the case where $\delta = 0, \Re_{BLINEX}(\mu_0, \mu) = 0$.

By examining the risk function it can be seen that when both a and δ are negative values or when both a and δ are positive, with δ either tending to $-\infty$ or ∞, the risk of RMLE ($\Re_{BLINEX}(\mu_0, \mu)$) tends to $\frac{1}{\lambda}$. When δ tends to zero, the term $b\left(e^{a\delta} - a\delta - 1\right)$ tends to zero and therefore $\Re_{BLINEX}(\mu_0, \mu)$ tends to 0.

4.2.2 Risk function of UMLE

By definition the risk function of UMLE can be calculated as below:

$$\Re_{BLINEX}(\mu_1, \mu) = \int_{-\infty}^{\infty} \frac{1}{\lambda}\left[1 - \frac{1}{1 + b\left[e^{a(\mu_1-\mu)} - a(\mu_1 - \mu) - 1\right]}\right] p(\mu_1)\, d\mu_1 \tag{4.2}$$

where $p(\mu_1)$ is the density of $N\left(\mu, \frac{\sigma^2}{n}\right)$.

Therefore by applying the binomial expansion to $(1-Z)^{-1}$ where $Z = b\left[a(\mu_1 - \mu) - e^{a(\mu_1-\mu)} + 1\right]$

$$\Re_{BLINEX}(\mu_1, \mu) = \frac{1}{\lambda} - \frac{1}{\lambda}\left(\frac{n}{2\pi\sigma^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-(\mu_1-\mu)^2\left(\frac{n}{2\sigma^2}\right)}}{1-z} d\mu_1$$

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The risk function of the UMLE can be expanded to:

$$
\begin{aligned}
&= \frac{b}{\lambda}\left[e^{\frac{a^2\sigma^2}{2n}} - 1\right] \\
&+ \frac{a^2b^2\sigma^2}{\lambda n}\left[2e^{\frac{a^2\sigma^2}{2n}} - 1\right] + \frac{b^2}{\lambda}\left[2e^{\frac{a^2\sigma^2}{2n}} - e^{\frac{2a^2\sigma^2}{n}} - 1\right] \\
&+ \frac{3a^2b^3\sigma^2}{\lambda n}\left[e^{\frac{a^2\sigma^2}{2n}} + e^{\frac{a^2\sigma^2}{2n}}\left(\frac{a^2\sigma^2}{n}\right) - 1\right] \\
&+ \frac{6a^2b^3\sigma^2}{\lambda n}\left[e^{\frac{a^2\sigma^2}{2n}} - e^{\frac{2a^2\sigma^2}{n}}\right] + \frac{3b^3}{\lambda}\left[e^{\frac{a^2\sigma^2}{2n}} - e^{\frac{2a^2\sigma^2}{n}}\right] + \frac{b^3}{\lambda}\left[e^{\frac{9a^2\sigma^2}{2n}} - 1\right] \\
&+ ....
\end{aligned}
\tag{4.3}
$$

The risk function of the UMLE is not a function of the estimation error, δ and is therefore also constant with regard to the noncentrality parameter λ^{**}.

Monte Carlo simulations of risk of UMLE

The risk function of UMLE was also evaluated as a function of λ and it was found that the risk function is a decreasing function of both n and λ. The following table gives the simulation results where the function was evaluated for different values of a, b, σ and n and using $p = 1000$ iterations. The SAS program used for these simulations is in Appendix C.

Table 4.1: Risk of UMLE for specific choices of a, b, σ, n and $\lambda = 1$.

	$\mathbf{a=1, b=0.1, \lambda=1}$			$\mathbf{a=0.1, b=1, \lambda=1}$			$\mathbf{a=0.1, b=0.1, \lambda=1}$		
Sample size	$\boldsymbol{\sigma=1}$	$\boldsymbol{\sigma=2}$	$\boldsymbol{\sigma=5}$	$\boldsymbol{\sigma=1}$	$\boldsymbol{\sigma=2}$	$\boldsymbol{\sigma=5}$	$\boldsymbol{\sigma=1}$	$\boldsymbol{\sigma=2}$	$\boldsymbol{\sigma=5}$
$\mathbf{n=10}$	0.0047	0.0190	0.1077	0.0005	0.0019	0.0113	0.0001	0.0002	0.0012
$\mathbf{n=30}$	0.0016	0.0065	0.0401	0.0002	0.0007	0.0040	0.0000	0.0001	0.0004
$\mathbf{n=50}$	0.0011	0.0042	0.0263	0.0001	0.0004	0.0026	0.0000	0.0000	0.0003
$\mathbf{n=100}$	0.0005	0.0020	0.0128	0.0001	0.0002	0.0013	0.00000	0.0000	0.0001
$\mathbf{n=1000}$	0.0001	0.0002	0.0015	0.0000	0.0000	0.0001	0.00000	0.00000	0.0000

The results in table 4.1 confirm some of the expected characteristics of the risk function, namely that the risk of UMLE is a decreasing function of both λ and n, but the risk is an increasing function of the variance for a specific choice of parameter.

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4.2.3 Risk of PTE

In this section, the null risk (risk under the null hypothesis) of the PTE is derived under BLINEX loss.

Theorem 14 *Ignoring the terms of order 3, the risk function of the preliminary test estimator using the BLINEX loss function under H_0 is given by*

$$\Re_{BLINEX}\left(\mu_p,\mu\right) \approx \frac{2ab}{\lambda}\left(1+e^{a\delta}\right)\left(\frac{2\sigma^2}{n}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**m}}{m!}\frac{\Gamma\left[\frac{1}{2}(2+2m)\right]}{\Gamma\left[\frac{1}{2}(1+2m)\right]}P\left[F_1<c_\alpha^{*}\right]$$

$$+\frac{3b}{\lambda}\left[\sum_{r=1}^{\infty}\frac{a^r}{r!}\left(\frac{\left(\frac{2\sigma^2}{n}\right)^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)}{2\pi}+\delta^r P\left[F_2<c_\alpha^{**}\right]\right.\right.$$

$$\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\frac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\frac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3<c_\alpha^{***}\right]\right)\right]$$

$$-\frac{a^2b}{\lambda}\left[\frac{\left(\frac{2\sigma^2}{n}\right)\Gamma\left(\frac{3}{2}\right)}{2\pi}+\delta P\left[F_2<c_\alpha^{**}\right]-\sum_{j=0}^{1}\delta^j\left[\frac{2\sigma^2}{n}\right]^{\frac{(1-j)}{2}}\sum_{k=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**k}}{k!}\frac{\Gamma\left[\frac{1}{2}(2-j+2k)\right]}{\Gamma\left[\frac{1}{2}(1+2k)\right]}P\left[F_4<c_\alpha^{****}\right]\right]$$

$$-\frac{b}{\lambda}\left[\sum_{r=1}^{\infty}\frac{(2a)^r}{r!}\left(\frac{\left(\frac{2\sigma^2}{n}\right)^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)}{2\pi}+\delta^r P\left[F_2<c_\alpha^{**}\right]\right.\right.$$

$$\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\frac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\frac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3<c_\alpha^{***}\right]\right)\right]$$

with $F_1=\frac{(n-1)\chi^2_{(2+2m,\lambda^{**})}}{(2+2m)\chi^2_{(n-1)}}$, $F_2=\frac{(n-1)\chi^2_{(1,,\lambda^{**})}}{\chi^2_{(n-1)}}$, $F_3=\frac{(n-1)\chi^2_{(1+r-j+2i,\lambda^{**})}}{(1+r-j+2i)\chi^2_{(n-1)}}$

and $F_4=\frac{(n-1)\chi^2_{(2-j+2k,\lambda^{**})}}{(2-j+2k)\chi^2_{(n-1)}}$

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Proof. By applying the binomial expansion for $Z = b\left[a\left(\mu_p - \mu\right) - e^{a\left(\mu_p - \mu\right)} + 1\right]$ the risk function is given by

$$\begin{aligned} \Re_{BLINEX}\left(\mu_p, \mu\right) &= E\left[\mathcal{L}_{BLINEX}\left(\mu_p, \mu\right)\right] \\ &\approx -\frac{1}{\lambda}\left[E\left(Z\right) + E\left(Z^2\right) + O\left(Z^3\right)\right] \end{aligned} \tag{4.4}$$

where

$$E\left(Z\right) = -b\left[\sum_{r=2}^{\infty} \frac{a^r E\left(\mu_p - \mu\right)^r}{r!}\right]$$

Using the fact that

$$\begin{aligned} \left(\mu_p - \mu\right)^r &= \left(\mu_1 - \mu\right)^r I_R\left(F\right) - \delta^r I_{NR}\left(F\right) \\ &= \left(\mu_1 - \mu\right)^r + \left[\delta^r - \left(\mu_1 - \mu\right)^r\right] I_{NR}\left(F\right) \quad for \quad r = 1, 2, 3, ... \\ &= \left(\mu_1 - \mu\right)^r + \left[\delta^r - \sum_{j=0}^{r} \left(\delta^j\right) \binom{r}{j} \left(\mu_1 - \mu_0\right)^{r-j}\right] I_{NR}\left(F\right) \\ &= \left(\mu_1 - \mu\right)^r + \left[\delta^r - \sum_{j=0}^{r} \left(\delta^j\right) \binom{r}{j} \left[\frac{\sigma^2 \chi^2_{(1,\lambda^{**})}}{n}\right]^{\frac{(r-j)}{2}}\right] I_{NR}\left(F\right) \end{aligned}$$

where $\chi^2_{(1,\lambda^{**})}$ is a noncentral Chi-square random variable with 1 degree of freedom and a noncentrality parameter $\lambda^{**} = \frac{n\delta^2}{2\sigma^2}$ (see section 1.2.2).

Therefore,

$$E\left(\mu_p - \mu\right)^r = E\left[\left(\mu_1 - \mu\right)^r\right] + \delta^r E\left[I_{NR}\left(F\right)\right] - \sum_{j=0}^{r} \left(\delta^j\right) \binom{r}{j} \left[\frac{\sigma^2}{n}\right]^{\frac{(r-j)}{2}} E\left[I_{NR}\left(F\right)\left(\chi^2_{(1,\lambda^{**})}\right)^{\frac{(r-j)}{2}}\right]$$

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Using Lemma 4 it was found that :

$$E\left[I_{NR}(F)\left(\chi^2_{(1,\lambda^{**})}\right)^{\frac{(r-j)}{2}}\right] = \sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\frac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\frac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3 < c_\alpha^{***}\right]$$

$$E(Z) = -b\left[\sum_{r=2}^{\infty}\frac{a^r}{r!}\left(E\left[(\mu_1-\mu)^r\right] + \delta^r P\left[F_2 < c_\alpha^{**}\right]\right.\right.$$

$$\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\frac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\frac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3 < c_\alpha^{***}\right]\right)\right]$$

Consider the following expressions (see Zellner, 1971, p364-365)

$$\begin{aligned} E(\mu_1-\mu)^{2r} &= \frac{\left(\frac{2\sigma^2}{n}\right)^r\Gamma\left(r+\frac{1}{2}\right)}{2\pi} \quad for \quad r=1,2,3,... \\ E(\mu_1-\mu)^{2r-1} &= 0 \quad for \quad r=1,2,3,.... \end{aligned}$$

an expression for $E(Z)$ was obtained:

$$\begin{aligned} &E(Z) \\ &= -b\left[\sum_{r=2}^{\infty}\frac{a^r}{r!}\left(\frac{\left(\frac{2\sigma^2}{n}\right)^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)}{2\pi} + \delta^r P\left[F_2 < c_\alpha^{**}\right]\right.\right. \\ &\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\frac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\frac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\frac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3 < c_\alpha^{***}\right]\right)\right] \end{aligned} \tag{4.5}$$

Furthermore,

$$E(Z^2) = a^2bE\left(\mu_p-\mu\right)^2 + 2abE\left(\mu_p-\mu\right) - 2abE\left[\left(\mu_p-\mu\right)e^{a\left(\mu_p-\mu\right)}\right] - 2E\left[e^{a\left(\mu_p-\mu\right)}\right]$$

$$+bE\left[e^{2a\left(\mu_p-\mu\right)}\right] + b$$

Since

$$\begin{aligned} \left(\mu_p-\mu\right) &= (\mu_1-\mu)I_R(F) + (\mu_0-\mu)I_{NR}(F) \\ &= (\mu_1-\mu) + [\delta-(\mu_1-\mu)]I_{NR}(F) \\ &= (\mu_1-\mu) + (\mu_1-\mu_0)I_{NR}(F) \\ &= (\mu_1-\mu) - \left(\frac{\sigma^2}{n}\right)^{\frac{1}{2}}\left(\chi^2_{1,\lambda^{**}}\right)^{\frac{1}{2}}I_{NR}(F) \end{aligned}$$

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and under H_0 it was found that

$$\begin{aligned} E\left[(\mu_p-\mu)\,e^{a(\mu_p-\mu)}\right] &= E(\mu_1-\mu)-\left(\tfrac{\sigma^2}{n}\right)^{\frac{1}{2}}E\left[\left(\chi^2_{1,\lambda^{**}}\right)^{\frac{1}{2}}I_{NR}(F)\,e^{a(\mu_p-\mu)}\right] \\ &= -\left(\tfrac{\sigma^2}{n}\right)^{\frac{1}{2}}E\left[\left(\chi^2_{1,\lambda^{**}}\right)^{\frac{1}{2}}I_{NR}(F)\,e^{a(\mu_1-\mu)I_R(F)+a(\mu_0-\mu)I_{NR}(F)}\right] \\ &= -\left(\tfrac{\sigma^2}{n}\right)^{\frac{1}{2}}e^{a(\mu_0-\mu)}E\left[\left(\chi^2_{1,\lambda^{**}}\right)^{\frac{1}{2}}I_{NR}(F)\right] \\ &= -\left(\tfrac{2\sigma^2}{n}\right)^{\frac{1}{2}}e^{a(\delta)}\sum_{m=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**m}}{m!}\tfrac{\Gamma\left[\frac{1}{2}(2+2m)\right]}{\Gamma\left[\frac{1}{2}(1+2m)\right]}P\left[F_1<c^*_\alpha\right] \end{aligned}$$

otherwise $E\left[(\mu_p-\mu)\,e^{a(\mu_p-\mu)}\right]=0$

Therefore $E(Z^2)$ under H_0:

$$E(Z^2)=a^2b\left[\tfrac{\left(\frac{2\sigma^2}{n}\right)\Gamma\left(\frac{3}{2}\right)}{2\pi}+\delta P\left(F_2<c^{**}_\alpha\right)-\sum_{j=0}^{1}\delta^j\left(\tfrac{2\sigma^2}{n}\right)^{\frac{1-j}{2}}\sum_{k=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**k}}{k!}\tfrac{\Gamma\left[\frac{1}{2}(2-j+2k)\right]}{\Gamma\left[\frac{1}{2}(1+2k)\right]}P\left[F_4<c^{****}_\alpha\right]\right]$$

$$-2ab\left(\tfrac{2\sigma^2}{n}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**m}}{m!}\tfrac{\Gamma\left[\frac{1}{2}(2+2m)\right]}{\Gamma\left[\frac{1}{2}(1+2m)\right]}P\left[F_1<c^*_\alpha\right]+2ab\left(e^{a\delta}\right)\left(\tfrac{2\sigma^2}{n}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**m}}{m!}\tfrac{\Gamma\left[\frac{1}{2}(2+2m)\right]}{\Gamma\left[\frac{1}{2}(1+2m)\right]}P\left[F_1<c^*_\alpha\right]$$

$$-2b\left[\sum_{r=1}^{\infty}\tfrac{a^r}{r!}\left(\tfrac{\left(\frac{2\sigma^2}{n}\right)^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)}{2\pi}+\delta^r P\left[F_2<c^{**}_\alpha\right]\right.\right.$$

$$\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\tfrac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\tfrac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3<c^{***}_\alpha\right]\right)\right]$$

$$+b\left[\sum_{r=1}^{\infty}\tfrac{(2a)^r}{r!}\left(\tfrac{\left(\frac{2\sigma^2}{n}\right)^{\frac{r}{2}}\Gamma\left(\frac{r+1}{2}\right)}{2\pi}+\delta^r P\left[F_2<c^{**}_\alpha\right]\right.\right.$$

$$\left.\left.-\sum_{j=0}^{r}\left(\delta^j\right)\binom{r}{j}\left[\tfrac{2\sigma^2}{n}\right]^{\frac{(r-j)}{2}}\sum_{i=0}^{\infty}\tfrac{e^{-\lambda^{**}}\lambda^{**i}}{i!}\tfrac{\Gamma\left[\frac{1}{2}(1+r-j+2i)\right]}{\Gamma\left[\frac{1}{2}(1+2i)\right]}P\left[F_3<c^{***}_\alpha\right]\right)\right] \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) the result follows. ■

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4.2.4 Comparison results

The risk of UMLE and PTE are analytically compared to each other and the result is given in the following theorem.

Theorem 15 *The PTE performs better than the UMLE in the sense of having smaller risk if the following inequality holds:*

$$\delta \leq \frac{1}{a} \ln \left[\left(\frac{\frac{1}{b} E\left[1 - \frac{1}{x_{\mu_1}}\right] + aE\left(\mu_p - \mu\right) + 1}{E\left[e^{a(\mu_1 - \mu) I_R[F]}\right]} - P\left(F > c\right) \right) / P\left(F < c\right) \right]$$

Proof. It is enough to show that $\Re_{BLINEX}\left(\mu_p, \mu\right) \leq \Re_{BLINEX}\left(\mu_1, \mu\right)$. Therefore it is known that $E\left[1 - \frac{1}{x_{\mu_p}}\right] \leq E\left[1 - \frac{1}{x_{\mu_1}}\right]$, where $x_{\mu_p} = 1 + b\left[e^{a\left(\mu_p - \mu\right)} - a\left(\mu_p - \mu\right) - 1\right]$ and $x_{\mu_1} = 1 + b\left[e^{a(\mu_1 - \mu)} - a\left(\mu_1 - \mu\right) - 1\right]$, knowing that $E\left[1 - \frac{1}{x_{\mu_1}}\right]$ is not a function of δ.

Applying the inequality, $1 - \frac{1}{x} \leq x - 1, \forall\, x > 0$ to the expression above it is sufficient to prove that $E\left[x_{\mu_p} - 1\right] \leq E\left[1 - \frac{1}{x_{\mu_1}}\right]$.

Simplifying the relevant term it yields

$$E\left[e^{a\left(\mu_p - \mu\right)}\right] \leq \frac{1}{b} E\left[1 - \frac{1}{x_{\mu_1}}\right] + aE\left(\mu_p - \mu\right) + 1$$

Setting $\frac{1}{b} E\left[1 - \frac{1}{x_{\mu_1}}\right] + aE\left(\mu_p - \mu\right) + 1 = Q$ it was found that

$$E\left[e^{a\left(\mu_p - \mu\right)}\right] \leq Q \tag{4.7}$$

The LHS of the above inequality can be written as

$$E\left[e^{a\left(\mu_p - \mu\right)}\right] = E\left[e^{a(\mu_1 - \mu) I_R[F]} e^{a\delta I_{NR}[F]}\right] \tag{4.8}$$

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By applying the (1.53) and using (4.7) and (4.8) it was found that

$$E\left[e^{a(\mu_1-\mu)I_R[F]}e^{a\delta I_{NR}[F]}\right] \leq E\left[e^{a(\mu_1-\mu)I_R[F]}\right] E\left[e^{a\delta I_{NR}[F]}\right]$$

Then it is enough to show that $E\left[e^{a(\mu_1-\mu)I_R[F]}\right] E\left[e^{a\delta I_{NR}[F]}\right] \leq Q$

Therefore, from the fact that $E\left[e^{a(\mu_1-\mu)I_R[F]}\right] > 0$,

$$E\left[e^{a\delta I_{NR}[F]}\right] \leq \frac{Q}{E\left[e^{a(\mu_1-\mu)I_R[F]}\right]} \tag{4.9}$$

But the LHS of (4.9) can be written as

$$\begin{aligned} E\left[e^{a\delta I_{NR}[F]}\right] &= E\left[\sum_{k=0}^{\infty}\frac{(a\delta)^k}{k!}I_{NR}[F]\right] \\ &= P(F>c) + P(F<c)\left[\sum_{k=0}^{\infty}\frac{(a\delta)^k}{k!}\right] \\ &= P(F>c) + P(F<c)\,e^{a\delta} \end{aligned} \tag{4.10}$$

By making use of the expression in (4.9) and (4.10)

$$P(F>c) + P(F<c)\,e^{a\delta} \leq \frac{Q}{E\left[e^{a(\mu_1-\mu)I_R[F]}\right]}$$

By simplifying and taking logs on both sides of the inequality the result follows. ■

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The risk of RMLE, UMLE and PTE calculated under BLINEX loss are computationally compared to each other in order to determine the performance of the estimators relative to one another. The panel of sketches (see Figure 4.3) in this section will be discussed in two ways, firstly by keeping σ constant and allowing the sample size, n, to increase and secondly by keeping n constant and allowing σ to increase. The SAS program used for these simulations is in Appendix C.

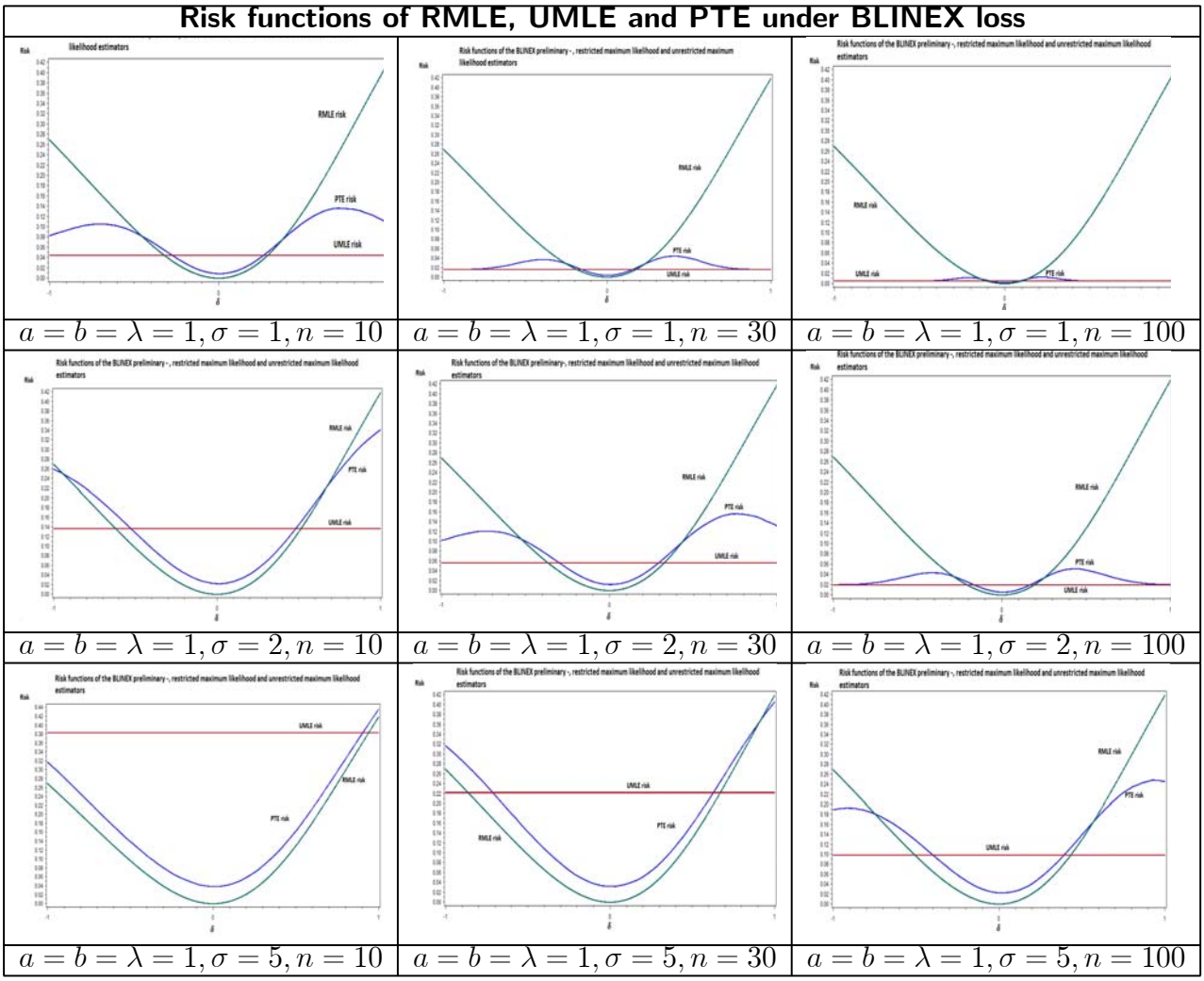

Figure 4.3 : Comparison of UMLE, RMLE and PTE across different values of σ and n under BLINEX loss.

For the case where $\sigma = 1$, as the sample size increases for this choice of σ, the risk of PTE tends towards the risk of UMLE as $\delta \to \pm\infty$. It is well-known that $\overline{X}$ (UMLE) is the most efficient estimator as the sample size increases, since $Var\left(\overline{X}\right) = \frac{\sigma^2}{n}$. It is clear that as the sample size increases the $Var\left(\overline{X}\right)$ will tend to zero. Therefore as the sample size increases, PTE becomes just as efficient as the UMLE. Similar results are obtained for different values of σ.

When σ increases for a specific sample size, in the interval near the origin, the interval where the risk of the of PTE is lower relative to the risk of the UMLE becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTE is a more efficient estimator compared to the UMLE.

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4.3 Classical preliminary test ridge regression estimator

In this section, the previous results are extended to the ridge regression model.

4.3.1 Risk function of the URRE

Theorem 16 *Ignoring the terms of order 3, the approximated risk function for URRE under BLINEX loss is given as*

$$E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] \approx -\tfrac{1}{\lambda}\left[E[M]+E[M^2]\right]$$

$$= -\tfrac{1}{\lambda}\left[b\left[1+\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]-\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right]\right.$$

$$\left.+b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\right]\mathbf{a}\right.$$

$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\boldsymbol{\beta}-1\right]$$

$$\left.+2b^2\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b^2\exp\left(-2\boldsymbol{\beta}'\mathbf{a}+2\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+4\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)+b^2\right]$$

Proof. The risk function for $\widetilde{\boldsymbol{\beta}}_n(k)$ under BLINEX loss is defined as:

$$\begin{aligned}\Re_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right) &= E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]\\ &= \frac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]}\right]\end{aligned}$$

By applying the binomial expansion $(1-M)^{-1}=1+M+M^2+O(M^3)$

where

$$M=b\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)+1\right]$$

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The risk function reduces to

$$\begin{aligned} E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] &= \frac{1}{\lambda}E\left[1-\left(1+M+M^2+...\right)\right] \\ &= -\frac{1}{\lambda}E\left[M+M^2+...\right] \\ &= -\frac{1}{\lambda}\left[E[M]+E\left[M^2\right]\right]+... \end{aligned}$$

When calculating $E[M]$ we make use of the final expressions of (3.1) - (3.3) and therefore:

$$E[M] = bE\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}+\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)-\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)+1\right]$$

$$= bE\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}+\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]-bE\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]+b$$

$$= -b\exp\left(-\boldsymbol{\beta}'\mathbf{a}\right)E\left[\exp\left(\widetilde{\boldsymbol{\beta}}_n'(k)\,\mathbf{a}\right)\right]+b\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b$$

$$= -b\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)+b\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b$$

$$= b\left[1+\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]-\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\,\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right]$$

For the next expectation, namely $E[M^2]$, it is necessary to define M^2 :

$$= b^2\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\mathbf{a}\right]-2b^2\mathbf{a}'\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]$$

$$+2b^2a'\left[\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right]-2b^2\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]+b^2\left[\exp\left(2\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]+b^2$$

Therefore $E[M^2]$ can be written as

$$b^2E\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\mathbf{a}\right] \tag{4.11}$$

$$-2b^2\mathbf{a}'E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right] \tag{4.12}$$

$$+2b^2a'E\left[\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right] \tag{4.13}$$

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$$-2b^2E\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right] \tag{4.14}$$

$$+b^2E\left[\exp\left(2\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]+b^2 \tag{4.15}$$

Expression (4.11): By adding and subtracting $\mathbf{R}(k)\boldsymbol{\beta}$ in both brackets it follows that

$$b^2E\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\mathbf{a}\right]$$

$$=b^2\mathbf{a}'E\left[\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)\left(\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right)'\right]\mathbf{a}'$$

by multiplying the brackets out and knowing that the $E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\right]=0$ it follows that

$$b^2E\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\mathbf{a}\right]$$

$$=b^2E\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\mathbf{a}\right]+b^2E\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{a}\right]$$

$$=b^2\mathbf{a}'E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\right]\mathbf{a}+b^2E\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{a}\right]$$

since $E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\right]=\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}$ the expression reduces to

$$=b^2\left[\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}+\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\mathbf{a}\right]$$

$$=b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\right]\mathbf{a}$$

Expression (4.12):

$$-2b^2\mathbf{a}'E\left[\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]$$

$$=-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}\right)\mathbf{a}'E\left[\widetilde{\boldsymbol{\beta}}_n(k)\exp\left(\mathbf{a}'\widetilde{\boldsymbol{\beta}}_n(k)\right)\right] \tag{4.16}$$

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$$+2b^2\mathbf{a}'\boldsymbol{\beta}E\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]$$

(4.17)

Expectation (4.16):

$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}\right)\mathbf{a}'E\left[\widetilde{\boldsymbol{\beta}}_n(k)\exp\left(\mathbf{a}'\widetilde{\boldsymbol{\beta}}_n(k)\right)\right]$$

therefore from (1.10)

$$E\left[\widetilde{\boldsymbol{\beta}}_n(k)\exp\left(\mathbf{a}'\widetilde{\boldsymbol{\beta}}_n(k)\right)\right]$$

$$=\int_{-\infty}^{\infty}\ldots\int_{\infty}^{\infty}\left(\widetilde{\boldsymbol{\beta}}_n(k)\exp\left(\widetilde{\boldsymbol{\beta}}_n'(k)\mathbf{a}\right)\frac{1}{(2\pi)^{\frac{p}{2}}\left|\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right|^{\frac{1}{2}}}\right.$$

$$\left.\times\exp\left(-\tfrac{1}{2}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)'\boldsymbol{\Sigma}^{-1}_{\widetilde{\boldsymbol{\beta}}_n(k)}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\mathbf{R}(k)\boldsymbol{\beta}\right)\right)d\widetilde{\boldsymbol{\beta}}_n(k)\right)$$

$$=\exp\left(\boldsymbol{\beta}'\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\int_{-\infty}^{\infty}\ldots\int_{\infty}^{\infty}\widetilde{\boldsymbol{\beta}}_n(k)f\left(\widetilde{\boldsymbol{\beta}}_n(k)\right)d\widetilde{\boldsymbol{\beta}}_n(k)$$

where the probability density function of $f\left(\widetilde{\boldsymbol{\beta}}_n(k)\right)$

$$=\left(\frac{1}{(2\pi)^{\frac{p}{2}}\left|\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\right|^{\frac{1}{2}}}\right.$$

$$\left.\times\exp\left(-\tfrac{1}{2}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\left(\mathbf{R}(k)\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right)'\boldsymbol{\Sigma}^{-1}_{\widetilde{\boldsymbol{\beta}}_n(k)}\left(\widetilde{\boldsymbol{\beta}}_n(k)-\left(\mathbf{R}(k)\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right)\right)d\widetilde{\boldsymbol{\beta}}_n(k)\right)$$

Therefore

$$E\left[\widetilde{\boldsymbol{\beta}}_n(k)\exp\left(\mathbf{a}'\widetilde{\boldsymbol{\beta}}_n(k)\right)\right]=\exp\left(\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)E\left[\widetilde{\boldsymbol{\beta}}_n(k)\right]$$

$$=\exp\left(\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

Expectation (4.17): Using (3.4) it was found that

$$2b^2\mathbf{a}'\boldsymbol{\beta}E\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)\right]$$

$$=2b^2\mathbf{a}'\boldsymbol{\beta}\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

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Combining (4.16) and (4.17) an expression for (4.12) was obtained:

$$= -2b^2 \exp\left(-\boldsymbol{\beta}'\mathbf{a}\right) \mathbf{a}' E\left[\widetilde{\boldsymbol{\beta}}_n\left(k\right) \exp\left(\mathbf{a}'\widetilde{\boldsymbol{\beta}}_n\left(k\right)\right)\right] + 2b^2\mathbf{a}'\boldsymbol{\beta} \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

$$= -2b^2 \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) \mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} + \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

$$+2b^2\mathbf{a}'\boldsymbol{\beta} \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

$$= -2b^2 \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} + \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) - \mathbf{a}'\boldsymbol{\beta}\right]$$

Expression (4.13):

$$2b^2 a' E\left[\widetilde{\boldsymbol{\beta}}_n\left(k\right) - \boldsymbol{\beta}\right] = 2b^2\mathbf{a}'\left[\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right]$$

Expression (4.14): Using (3.4) resulted in

$$-2b^2 E\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n\left(k\right) - \boldsymbol{\beta}\right)\right)\right]$$

$$= 2b^2 \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)$$

Expression (4.15): Using (3.4) the following was obtained

$$b^2 E\left[\exp\left(2\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n\left(k\right) - \boldsymbol{\beta}\right)\right)\right] + b^2$$

$$= b^2 \exp\left(-2\boldsymbol{\beta}'\mathbf{a} + 2\boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + 4\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) + b^2$$

Combining the final expressions of (4.11) - (4.15) an expression for $E\left[M^2\right]$was obtained:

$$= b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)} + \left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right)'\right]\mathbf{a}$$

$$-2b^2 \exp\left(-\boldsymbol{\beta}'\mathbf{a} + \boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + \tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}\left(k\right)\boldsymbol{\beta} + \boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) - \mathbf{a}'\boldsymbol{\beta} - 1\right]$$

$$+2b^2\mathbf{a}'\left[\mathbf{R}\left(k\right)\boldsymbol{\beta} - \boldsymbol{\beta}\right] + b^2 \exp\left(-2\boldsymbol{\beta}'\mathbf{a} + 2\boldsymbol{\beta}'\mathbf{R}'\left(k\right)\mathbf{a} + 4\mathbf{a}'\boldsymbol{\Sigma}_{\widetilde{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right) + b^2$$

Combining all the results completes the proof. ■

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4.3.2 Risk function of the RRRE

Theorem 17 *Ignoring the terms of order 3, the approximated risk function for RRRE under BLINEX loss is given as*

$$E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right] \approx -\tfrac{1}{\lambda}\left[E[M]+E[M^2]\right]$$

$$= -\tfrac{1}{\lambda}\left[b\left[1+\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]-\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\right]\right.$$

$$+b^2\mathbf{a}'\left[\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}+\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)'\right]\mathbf{a}$$

$$-2b^2\exp\left(-\boldsymbol{\beta}'\mathbf{a}+\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+\tfrac{1}{2}\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)\left[\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}+\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)-\mathbf{a}'\boldsymbol{\beta}-1\right]$$

$$\left.+2b^2\mathbf{a}'\left[\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right]+b^2\exp\left(-2\boldsymbol{\beta}'\mathbf{a}+2\boldsymbol{\beta}'\mathbf{R}'(k)\mathbf{a}+4\mathbf{a}'\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}_n(k)}\mathbf{a}\right)+b^2\right]$$

Proof. The proof is similar to the proof of Theorem 16. ■

4.3.3 Risk function of the PTRRE

Theorem 18 *Ignoring the terms of order 2, the approximated risk function for RRRE under BLINEX loss is given as*

$$E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right] \approx \frac{-b}{2\lambda}\left[\sigma^2\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\right.$$

$$-2\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}$$

$$+\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)$$

$$+2\mathbf{a}'\left(\mathbf{R}(k)\boldsymbol{\beta}-\boldsymbol{\beta}\right)\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\boldsymbol{\eta}E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]$$

$$+\left[\mathbf{a}'\mathbf{R}(k)\mathbf{C}^{-1}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-\frac{1}{2}}\mathbf{Q}'\left[\sigma^2E\left[\varphi\left(\chi^2_{\left(p+2,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\mathbf{I}_p+E\left[\varphi\left(\chi^2_{\left(p+4,\frac{\boldsymbol{\eta}'\boldsymbol{\eta}}{2\sigma^2}\right)}\right)\right]\boldsymbol{\eta}\boldsymbol{\eta}'\right]\right.$$

$$\left.\left.\times\mathbf{Q}\mathbf{C}^{-\frac{1}{2}}\mathbf{H}'\left(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'\right)^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{R}'(k)\mathbf{a}\right]\right]$$

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Proof. The risk function for $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$ under BLINEX loss can be written as

$$\begin{aligned} E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)\right] &= \frac{1}{\lambda}E\left[1-\left(1+M+M^2+O\left(M^3\right)\right)\right] \\ &= -\frac{1}{\lambda}E\left[M+M^2+...\right] \\ &= -\frac{1}{\lambda}\left[E\left[M\right]+E\left[M^2\right]\right]+... \end{aligned}$$

with

$$M = b\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)+1\right]$$

or

$$M = -b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]$$

Due to the complexity of the BLINEX loss function, the risk function will only be extended to

$$= -\frac{1}{\lambda}\left[E\left[M\right]+...\right]$$

$$E\left[M\right] = -bE\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]$$

therefore

$$E\left[M\right] = -b\left[\frac{E\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]^2}{2!}\right] \tag{4.18}$$

and the expectation

$$E\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]^2$$

This expectation was calculated in section 3.4 under LINEX loss as seen in (3.15). By substituting this result into (4.18) an approximated risk function for the preliminary test ridge regression estimator under BLINEX loss is obtained which completes the proof.■

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4.3.4 Simulations

In this section the exact same simulation exercise was conducted as in Chapters 2 and 3 and the results are presented in a similar way (for more detail see section 1.5, page 28). All three proposed estimators (URRE, RRRE and PTRRE) were considered under the BLINEX loss function. In all the simulations the parameters of the BLINEX function was fixed at $\lambda = b = 1$ and $\mathbf{a} = \mathbf{1}$ respectively. In the first table in each section only the error variance varied, the second table represents the risk values over different levels of multicollinearity. The third table represents the effect of an increase in the number of parameters that needs to be estimated and the fourth table represents the effect of an increase in the sample size. The SAS program used for these simulations is in Appendix A.

Unrestricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the variance(σ^2).

Table 4.2: Effect of an increase in the variance on the risk of the URRE under BLINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.49E-06	2.49E-06	2.49E-06	2.49E-06	0.0010707	2.49E-06	2.51E-06
0.1	0.0002218	0.0002228	0.0002217	0.000222	0.0109315	0.000252	0.0003204
0.25	0.0015712	0.0015843	0.0015788	0.0015709	0.0363489	0.0019744	0.0022096
0.5	0.0059318	0.0061442	0.0060081	0.005928	0.0856562	0.0120154	0.0125617
1	0.0215563	0.0240307	0.0222496	0.0214666	0.1615253	0.0528128	0.0560122
4	0.1751268	0.1994822	0.212875	0.1799516	0.4270138	0.3577132	0.3556461
9	0.3178539	0.4066363	0.4564228	0.3439198	0.5042928	0.4841858	0.4834418
20	0.4523026	0.5976154	0.6560864	0.4700296	0.5259865	0.5215678	0.5214459

(see Figure E1, panel (c) in Appendix E for all the results of URRE)

Unrestricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 4.3: Effect of an increase in the level of multicollinearity on the risk of the URRE under BLINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0254274	0.0284393	0.0266898	0.0254013	0.1777523	0.063313	0.0727785
0.8	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
0.9	0.0186432	0.0201741	0.0190362	0.0192973	0.1454451	0.0420804	0.0438371
0.95	0.0171629	0.0179473	0.0174455	0.0208226	0.1480076	0.0413856	0.0413403
0.99	0.0154786	0.0157537	0.0157828	0.0370948	0.1403219	0.0374711	0.0355819

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Unrestricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the number of parameters (p).

Table 4.4: Effect of an increase in the number of parameters to be estimated on the risk of the URRE under BLINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
10	0.0241123	0.0255239	0.0252804	0.0242387	0.4066395	0.0998795	0.0757541
20	0.0261123	0.0268236	0.0306112	0.0268788	0.5758074	0.1459182	0.0949289
25	0.0261692	0.0283196	0.0396431	0.0274805	0.6171153	0.1501062	0.0937188

Unrestricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the sample size (n).

Table 4.5: Effect on an increase in the sample size on the risk of the URRE under BLINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0444846	0.0501069	0.0476796	0.0452737	0.2148647	0.1061582	0.1155797
30	0.0220932	0.0243859	0.0229981	0.0220907	0.164138	0.0530384	0.0524669
50	0.0128204	0.0138229	0.0131457	0.0128557	0.1171487	0.0287464	0.0294292
100	0.0067695	0.0071331	0.0068621	0.0067742	0.08703	0.0132359	0.0134058

Restricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the variance(σ^2)**.**

Table 4.6: Effect on an increase in the variance on the risk of the RRRE under BLINEX loss and across different ridge parameters:

	$n = 30\ p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.29E-06	2.28E-06	2.29E-06	2.29E-06	0.0010676	2.24E-06	2.28E-06
0.1	1.76E-06	1.21E-06	2.18E-06	1.65E-06	0.0106626	0.000024	0.0000899
0.25	1.43E-06	9.19E-06	2.04E-06	2.43E-06	0.0351159	0.0004564	0.0007198
0.5	0.0000271	0.0001957	1.58E-06	0.000057	0.0824737	0.0069221	0.0075724
1	0.000544	0.0025669	4.75E-06	0.0010216	0.154075	0.0393545	0.0426766
4	0.0747552	0.0490267	0.0005563	0.1042537	0.42935	0.3605744	0.3577684
9	0.3130289	0.0688724	0.0015839	0.349236	0.5064233	0.4883548	0.4879562
20	0.4750527	0.0810547	0.0017288	0.4863195	0.5261745	0.5223268	0.5222596

(see Figure E2, panel (c) in Appendix E for all the results of RRRE)

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Restricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 4.7: Effect of an increase in the level of multicollinearity on the risk of the RRRE under BLINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0006403	0.0035879	0.0000252	0.0006256	0.1703246	0.0509422	0.0608166
0.8	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
0.9	0.0004517	0.0015199	9.99E-06	0.0023279	0.1367048	0.02977	0.0317504
0.95	0.0004046	0.0008769	0.0000193	0.0050786	0.1397372	0.0299106	0.0290612
0.99	0.0003626	0.0001871	0.0000451	0.0250294	0.1331218	0.0263292	0.0238209

Restricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the number of parameters (p).

Table 4.8: Effect of an increase in the number of parameters to be estimated on the risk of the RRRE under BLINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
10	0.3227751	0.3262358	0.3206902	0.3244085	0.5450151	0.3939551	0.3758208
20	0.5544572	0.559757	0.5979607	0.5525225	0.7042255	0.5831416	0.5688081
25	0.5146786	0.5477095	0.6388293	0.5112429	0.7194949	0.5481884	0.5271954

Restricted ridge regression estimator under BLINEX loss, comparing the effect of a change in the sample size (n).

Table 4.9: Effect of an increase in the sample size on the risk of the RRRE under BLINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0022388	0.0075825	0.0001032	0.0044595	0.2058656	0.0881903	0.0990456
30	0.0005522	0.0025503	4.62E-06	0.0010226	0.1570133	0.0406039	0.0402203
50	0.0002117	0.0010687	4.94E-07	0.0003789	0.1125626	0.0198642	0.020909
100	0.0000536	0.0003122	7.93E-08	0.0000906	0.0831591	0.0074833	0.0078169

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Preliminary test ridge regression estimator under BLINEX loss, comparing the effect of a change in the variance(σ^2).

Table 4.10: Effect of an increase in the variance on the risk of the PTRRE under BLINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\gamma = 0.8$						
σ^2	HK	HKB	LW	HSL	AM	GM	MED
0.01	2.22E-06	2.22E-06	2.22E-06	2.22E-06	0.0017563	2.20E-06	2.20E-06
0.1	0.0000653	0.000065	0.0000657	0.0000653	0.0122171	0.000092	0.0001423
0.25	0.0004213	0.0004277	0.0004252	0.0004212	0.0389342	0.0009127	0.0008031
0.5	0.0009996	0.0011918	0.0009794	0.0010285	0.0808159	0.0075407	0.0077568
1	0.0068635	0.0088707	0.0066511	0.0072474	0.1540832	0.0432259	0.0452642
4	0.0950263	0.0746971	0.032596	0.1232195	0.4236086	0.3551171	0.3550905
9	0.3285672	0.1142532	0.0519684	0.3542052	0.5069341	0.4882761	0.4875254
20	0.4704927	0.1365946	0.0561257	0.4820313	0.5257225	0.521726	0.5216334

(see Figure E3, panel (c) in Appendix E for all the results of PTRRE)

Preliminary test ridge regression estimator under BLINEX loss, comparing the effect of a change in the level of multicollinearity(γ).

Table 4.11: Effect of an increase in the level of multicollinearity on the risk of the PTRRE under BLINEX loss and across different ridge parameters:

	$n = 30$ $p = 4$ and $\sigma = 1$						
γ	HK	HKB	LW	HSL	AM	GM	MED
0.7	0.0083307	0.0114513	0.0082199	0.008318	0.1686848	0.0566555	0.0690406
0.8	0.0060882	0.0080786	0.0060507	0.0063681	0.152002	0.0424793	0.0453755
0.9	0.0044254	0.0054715	0.0041551	0.0061405	0.1435205	0.0334427	0.0340364
0.95	0.004267	0.0047627	0.0040002	0.0084174	0.1452784	0.0324955	0.0312116
0.99	0.0039417	0.0038505	0.0037822	0.0272357	0.1411934	0.0306878	0.0282437

Preliminary test ridge regression estimator under BLINEX loss, comparing the effect of a change in the number of parameters (p).

Table 4.12: Effect of an increase in the number of parameters to be estimated on the risk of the PTRRE under BLINEX loss and across different ridge parameters:

	$n = 30, \gamma = 0.8$ and $\sigma = 1$						
p	HK	HKB	LW	HSL	AM	GM	MED
4	0.006565	0.0085628	0.0063321	0.0069623	0.1472063	0.0404024	0.0418997
10	0.0234876	0.0250226	0.0242882	0.0241134	0.4177921	0.1029818	0.0717673
20	0.0253094	0.0256043	0.0289253	0.0261733	0.6029361	0.1550401	0.0977558
25	0.0266363	0.029402	0.0401802	0.0277912	0.6099069	0.1510385	0.0957948

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Preliminary test ridge regression estimator under BLINEX loss, comparing the effect of a change in the sample size (n) .

Table 4.13: Effect of an increase in the sample size on the risk of the PTRRE under BLINEX loss and across different ridge parameters:

	$\gamma = 0.8, p = 4$ and $\sigma = 1$						
n	HK	HKB	LW	HSL	AM	GM	MED
15	0.0150589	0.0203113	0.0136204	0.0174938	0.221718	0.0957336	0.1030331
30	0.006565	0.0085628	0.0063321	0.0069623	0.1472063	0.0404024	0.0418997
50	0.0045046	0.0055789	0.0043295	0.0046473	0.1373385	0.0276937	0.026425
100	0.0021742	0.0024503	0.0021468	0.0021968	0.0804973	0.0076744	0.0080379

In order to summarise the results in tables 4.2 to 4.13, Figures E1 to E3 in Appendix E, panel (c) was used. For RRRE and PTRRE it was found that the LW ridge parameter performs the best, whereas for URRE the HK and HSL ridge parameters performed the best over different values of σ. In all cases the AM ridge parameter performed the worst. For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, LW and HKB ridge parameters lead to the lowest risk over different levels of multicollinearity. For the HSL ridge parameter the risks are also low for $\gamma < 0.95$. the AM ridge parameter performed the worst overall. For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB and HSL perform the best over an increasing number of parameters to be estimated. For the LW ridge parameter also performs well, except for RRRE when $p > 20$ the AM ridge parameter performed the worst overall. For all the estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB, LW and HSL ridge parameters performs the best over different sample sizes and the AM ridge parameter perform the worst overall.

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4.3.5 Evaluation the performance of the estimators relative to each other under BLINEX loss

The risk of RRRE, URRE and PTRRE calculated under BLINEX loss are computationally compared to each other in order to determine the performance of the estimators relative to each other. The panel of sketches (see Figure 4.4) in this section will be discussed in two ways, firstly by keeping σ constant and allowing the sample size, n, to increase and secondly by keeping n constant and allowing σ to increase. The SAS program used for these simulations is in Appendix B.

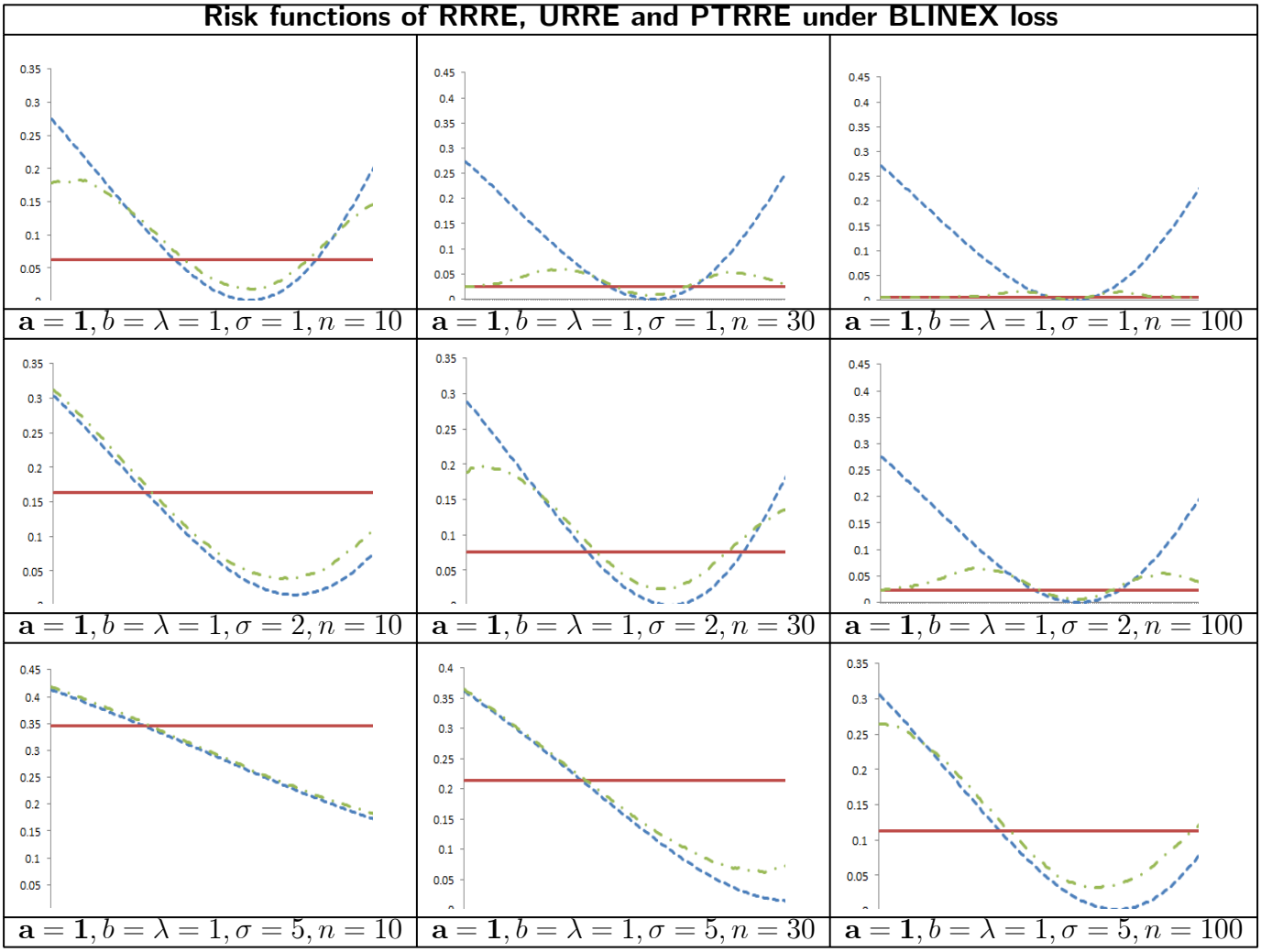

Figure 4.4 : Comparison of URRE, RRRE and PTRRE across different values of σ and n under BLINEX loss.

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Similar to the other two loss functions for the case where $\sigma = 1$, it is clear that as the sample size increases for this choice of σ, the risk of PTRRE tends towards the risk of URRE as $\delta \longrightarrow \pm\infty$. It is well-known that URRE is the most efficient estimator as the sample size increases, since the variance of URRE, i.e. $\widetilde{\boldsymbol{\beta}}_n(k)$ is $\sigma^2 \mathbf{R}(k) \mathbf{C}^{-1} \mathbf{R}'(k)$ and $\sigma^2 = \dfrac{\sum e_i^2}{n-p}$ It is clear that as the sample size increases the variance of URRE will tend to zero. Therefore as the sample size increases, PTRRE becomes equally as efficient as the URRE. As the sample size increases the risk associated with all the estimators decreases as shown in the previous section. Similar results are obtained for different values of σ.

When σ increases for a specific sample size, it is clear that in the interval near the origin, the interval in which the PTRRE has lower risk relative to the URRE, becomes larger. Therefore when working with a specific sample size and if the variance of the sample is relatively high, the PTRRE is a more efficient estimator compared to the URRE.

4.4 Bayes preliminary test estimator

In this section, only the Bayes PTE is considered in the context of the normal model, rather than a regression model as these results can be extended to the regression model as in the previous sections.

4.4.1 Feasible Bayes Estimator

The Bayes estimator under BLINEX loss is derived in this section and due to the complex nature of BLINEX loss function it was necessary to make use of alternative methods in order to propose feasible estimators.

Suppose that $X_1, X_2, ..., X_n$ is a sample from $N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ is assumed to be known. For the case of σ unknown, one may look at the work of Arashi (2010). The results obtained in his paper are very complex. It is well-known that the conjugate prior for the unknown parameter μ is $N(\mu', \sigma'^2)$ and the posterior of $\mu|\mathbf{x}$ is $N(\mu'', \sigma''^2)$ where $\mu'' = \frac{\left(n\mu_1\sigma'^2 + \mu'\left(\sigma^2\right)\right)}{n\sigma'^2 + \sigma^2}$ and $\sigma''^2 = \frac{\sigma'^2\sigma^2}{n\sigma'^2 + \sigma^2}$.

To derive the Bayes estimator, the expected loss is equal to

$$EL = \int_{-\infty}^{\infty} \mathcal{L}_{BLINEX}(\mu_B, \mu)\, p(\mu|\mathbf{x})\, d\mu$$

where $p(\mu|\mathbf{x})$ is the posterior distribution of $\mu|\mathbf{x}$ and $\mathbf{x} = (x_1, x_2, ..., x_n)$

Then under the pre-specified assumptions it was found that

$$EL(\mu_B) = \frac{1}{\lambda} - \frac{1}{\lambda} E\left[\frac{1}{1 + b\left(e^{a(\mu_B - \mu)} - a(\mu_B - \mu) - 1\right)}\right]$$

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(4.19)

Then the Bayes estimator of μ denoted by $\widehat{\mu}_B$ is the one that minimises the posterior expectation of the loss function, i.e.

$$\widehat{\mu}_B = \arg\min_{\mu_B} \; EL(\mu_B).$$

Note that

$$\frac{\partial EL(\mu_B)}{\partial \mu_B} = \frac{b}{\lambda} \, E\left[\frac{ae^{a(\mu_B-\mu)} - a}{\left[1 + b\left(e^{a(\mu_B-\mu)} - a(\mu_B-\mu) - 1\right)\right]^2}\right] = 0$$

Take $f_a\left(\mu_B - \mu\right) = ae^{a(\mu_B-\mu)} - a$; it can be seen that $f_a\left(\mu_B - \mu\right) \geq 0$ when $\mu_B \geq \mu$ and $f_a\left(\mu_B - \mu\right) < 0$ when $\mu_B < \mu$. Also note that for a random variable $X \geq 0$, $E\left(X\right) = 0$ implies $X = 0 \; a.e.$ Therefore subject to the constraint $\mu_B \geq \mu$ the Bayes estimator is equal to

$$\widehat{\mu}_B = \mu \tag{4.20}$$

Now under the restriction $\mu_B < \mu$, consider the following optimisation problem

$$\arg\max_{\mu_B} EL(\mu_B)$$

$$\arg\max_{\mu_B} E\left[\frac{1}{1 + b\left(e^{a(\mu_B-\mu)} - a(\mu_B-\mu) - 1\right)}\right] \tag{4.21}$$

Consider that $g\left(\mu_B - \mu\right) = \left[\frac{1}{1+b\left(e^{a(\mu_B-\mu)} - a(\mu_B-\mu) - 1\right)}\right]$ is an increasing function since $g'\left(\mu_B - \mu\right) > 0$ when $\mu_B < \mu$. Also from the fact that $e^{a(\mu_B-\mu)} > a(\mu_B - \mu) + 1$ when $\mu_B < \mu$ it can be seen that $g\left(\mu_B - \mu\right) > 0$. There is no unique solution for (4.21), therefore (4.20) is a valid result.

The problem with the above solution (4.20) is that $\widehat{\mu}_B$ is not a valid estimator, because μ is unknown. In this case a feasible estimator, denoted by $\widehat{\mu}_{FB}$, can be obtained by substituting a proper estimator of μ into the solution to get

$$\widehat{\mu}_{FB}^{(1)} = \mu'$$

Another possible solution is to use two sets of available information contained in the prior distribution for the unknown parameter μ and the density function $X|\,\mu$, which contains the information of the original sample.

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In this case define

$$\lambda \equiv L\left(\mu,\sigma^2\right).p\left(\mu\right) \propto e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2-\frac{\left(\mu-\mu'\right)^2}{2\sigma'^2}}$$

Then taking the derivative with respect to μ, equating to zero and substituting the final result into (4.20) the feasible estimator is obtained

$$\widehat{\mu}_{FB}^{(2)} = \frac{\left(\sum_{i=1}^{n} x_i + \frac{\mu'\sigma^2}{\sigma'^2}\right)}{\left(\frac{\sigma^2}{\sigma'^2}+n\right)} \tag{4.22}$$

Note that λ is only a way to produce an estimator of μ and is totally independent of the loss function. The result of λ was substituted into (4.19), that was initiated from BLINEX loss and the sketches were generated using simulation exercises based on (4.19). This feasible Bayes estimator is exactly the same as the Bayes estimator under squared error loss.

The properties of $\widehat{\mu}_{FB}^{(2)}$ are given below

$$E\left[\widehat{\mu}_{FB}^{(2)}\right] = \frac{n\mu+\mu'\left(\frac{\sigma^2}{\sigma'^2}\right)}{\frac{\sigma^2}{\sigma'^2}+n}$$

if $\mu=\mu'$ then $E\left[\widehat{\mu}_{FB}^{(2)}\right] = \frac{\left(\frac{\sigma^2}{\sigma'^2}+n\right)\mu}{\frac{\sigma^2}{\sigma'^2}+n} = \mu$, therefore under these circumstances $\widehat{\mu}_{FB}^{(2)}$ is an unbiased estimator of μ.

$$Var\left[\widehat{\mu}_{FB}^{(2)}\right] = \frac{n\sigma^2}{\left(\frac{\sigma^2}{\sigma'^2}+n\right)^2} = \frac{n\sigma^2\sigma'^2}{\left(\sigma^2+n\sigma'^2\right)^2}$$

if $\sigma^2=\sigma'^2$ then $Var\left[\widehat{\mu}_{FB}^{(2)}\right] = \left[\frac{n}{(n+1)^2}\right]\sigma^2$, it can therefore be seen that when $n\to\infty$, $Var\left[\widehat{\mu}_{FB}^{(2)}\right] \to 0$.

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4.4.2 A Feasible Bayes Preliminary Test estimator

Facing the same problem as in the introduction and using $\widehat{\mu}_{FB}^{(2)}$ as the unrestricted estimator, the feasible Bayes preliminary test estimator is defined by

$$\mu_{FBP} = \left[I_R(F) \times \widehat{\mu}_{FB}^{(2)}\right] + \left[I_{NR}(F) \times \mu_0\right]$$

The performance of FBPTE will be explored under BLINEX loss in the following section.

Comparing PTE with FBPTE under BLINEX loss function

The risk functions for the PTE and FBPTE were calculated for specific choices of the parameters involved and are compared to each other. In all the sketches (see Figure 4.5) the risk function of the PTE is presented by the dashed line curve and the risk function of the FBPTE is given by the solid line curve. Both risk functions of the estimators are functions of δ, where $-1 \leq \delta \leq 1$. In the calculation of the FBPTE only a fixed value for the prior mean, $\mu' = 13$ and prior standard deviation, $\sigma' = 1$ were considered and the performance of the estimators must still be evaluated for different choices of parameters. For these choices it can be seen that depending on the value of δ, σ and n, the one estimator outperforms the other. The SAS program used for these simulations is in Appendix C.

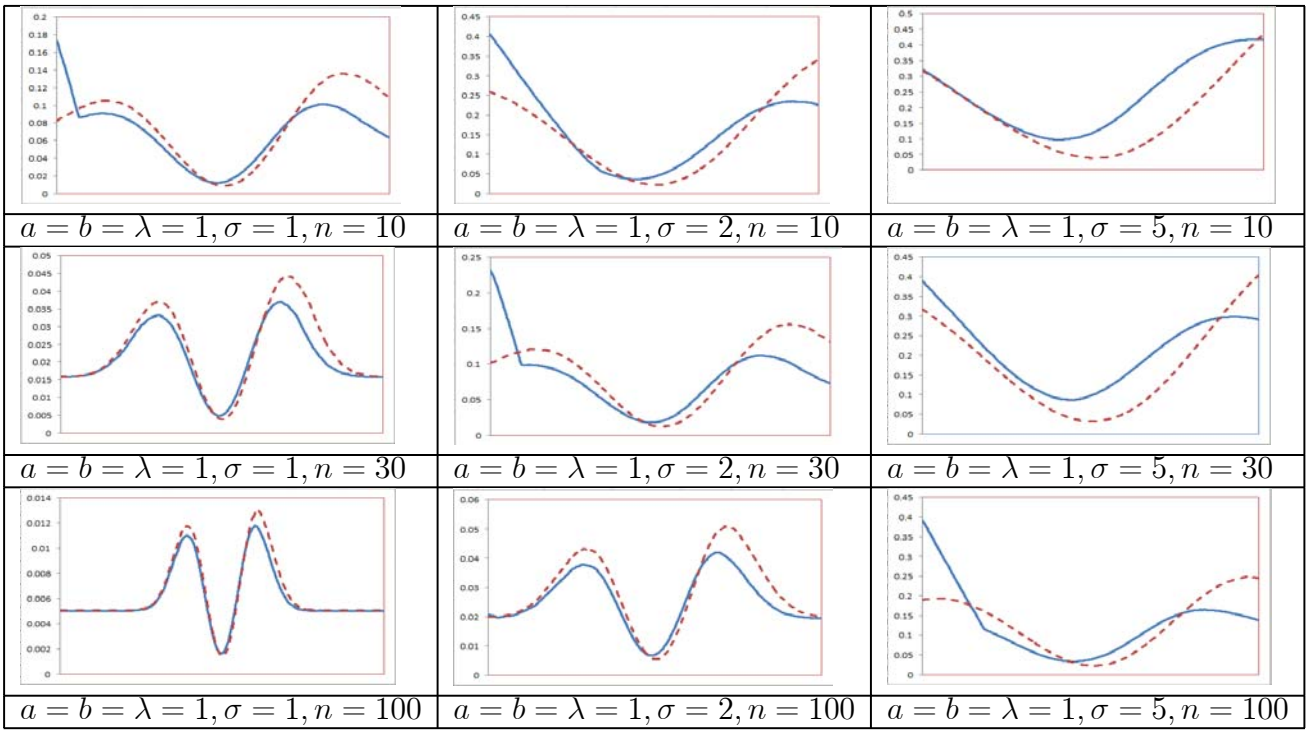

Figure 4.5.: Comparison of PTE and FBPTE for different parameter values.

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The FBPTE was defined and the PTE was computationally compared to the FBPTE. For specific choices of parameters and sample sizes both estimators perform well under specific conditions and for specific values of δ. Since the feasible Bayes estimator, given by (4.22) is nothing more than the Bayes estimator under squared error loss, it cannot be expected that the risk of FBPTE has a special form. It is still unclear how a Bayes estimator under BLINEX loss with stable risk performance can be found, which leaves this problem open for further research. By examining the results obtained, we found that the PTE performs better than FBPTE for some specific parameter choices and vice versa.

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Chapter 5

Economic applications of the preliminary test ridge regression estimator: Cobb-Douglas production function

5.1 Introduction

The Cobb-Douglas production function dates back to the work done by Paul Douglas and Charles Cobb in 1928, which relates production inputs and output for the U.S. manufacturing sector for 1899-1922. It has been noted by Brown (1966), Sandelin (1976) and Samuelson (1979) that Knut Wicksell actually discovered this form in the 19th century and it was only empirically tested by Cobb and Douglas in 1928. The Cobb-Douglas production function is still the most universal functional form in both theoretical and empirical analyses of production growth.

The Cobb-Douglas production function can be defined as follows:

$$Y_i = \beta_0 X_{1i}^{\beta_1} X_{2i}^{\beta_2} e^{u_i}$$

where $Y =$ Production output

$X_1 =$ Labour input

$X_2 =$ Capital input

$u =$ Stochastic error term

$e =$ The base of the natural log.

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By making use of the double log transformation, the model can be written as

$$\ln Y_i = \ln \beta_0 + \beta_1 \ln X_{1i} + \beta_2 \ln X_{2i} + u_i$$

Due to the double log transformation, the regression coefficients β_1 and β_2 of the Cobb-Douglas production function are interpreted as elasticities and the sum of the two coefficients represents the return to scale. The first coefficient β_1 measures the percentage change in production output for a percentage increase in labour input, by keeping capital input constant (partial elasticity), β_2 can be interpreted in a similar way as the percentage change in production output for a percentage increase in capital input, by keeping labour input constant. The sum of the two coefficients, namely $\beta_1 + \beta_2$ measures the return to scale and can be interpreted as the typical response of production output to a proportionate change in the two inputs.

When $\beta_1 + \beta_2 = 1$ it is an indication of constant return to scale which suggests that if the inputs double the production output will double. For $\beta_1 + \beta_2 > 1$, increasing return to scale is observed which suggests that when inputs double, production output will more than double and when $\beta_1 + \beta_2 < 1$, decreasing return to scale is observed which indicates that when inputs double, production output will less than double. It is therefore evident that in order to test for constant return to scale, the vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ should be subjected to a linear restriction.

The use of asymmetric loss functions for econometric modelling has been discussed by Waud (1976), Kunstman (1984), Horowitz (1987) to name a few. Granger (1999) also discussed the importance of using bounded loss function, where he gives different examples of the advantage of penalising over or underestimation of estimators, depending on which of the two is the most serious. Zellner (1986) showed that sample mean and least square regression coefficient estimators are inadmissible relative to asymmetric LINEX loss proposed by Varian (1975). He suggested alternative estimators that dominate in terms of their risk functions and Bayes risks.

Ruge-Murcia (2002) developed and estimated a theoretical model of inflation targeting, similar to the Barro and Gordon model (1983) in which he suggested that the central banker's preferences are asymmetric around the targeted inflation rate. It was further suggested that positive deviations from the inflation target must be weighed more heavily compared to negative deviations in the loss function used by the central banker. In his work he suggested the use of the LINEX loss function, which is a more flexible loss function, but it is not a bounded loss function.

Christofferson and Diebold (1996) suggested that in the field of volatility forecasting, using squared error loss is not the appropriate loss function to use and other approaches to volatility forecasting should be considered. Christofferson and Diebold (1997) examined LINEX forecasts under the assumption that the process is normally distributed.

In the example of the Cobb-Douglas production function it can be deemed that the loss/cost of overestimating production response to capital and labour input is generally more serious than underestimating production response and therefore overestimation should be penalised more heavily. The use of the BLINEX loss function can therefore be considered. Furthermore, the two exploratory variables are deemed to be negatively correlated, which suggest that multicollinearity is a problem that needs to be addressed by making use of ridge regression suggested by Hoerl and Kennard (1970).

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5.2 Application

In this section the Cobb-Douglas production function will be considered under the restriction of constant return to scale and the estimators will be calculated for all the possible estimators of the ridge parameter.

The real data set which for all illustrations consists of following variables:

(1) The South African manufacturing production output as measured by the Gross Domestic product (GDP in real prices) measured in R million in the manufacturing sector.

(2) Labour input as measured by employment figures within the manufacturing sector.

(3) Capital input as measured by fixed capital stock (in real prices) measured in R million in the manufacturing sector.

The definitions of the variables are given below:

Gross Domestic Product: The total market value of all final goods and services produced in a country in a given year, equal to total consumer, investment and government spending, plus the value of exports, minus the value of imports.

*Fixed capital stock:*Fixed capital stock consists of buildings, installations, transmission devices, machinery, equipment, means of transport, tools, production and sales implements, draft animals, and commercial livestock.

*Employment figures:*The number of employees employed within the manufacturing sector.

The manufacturing sector in South Africa provides a platform for stimulating the growth in services, employment creation and economic empowerment. This presents an opportunity to significantly increase the country's growth and development.

The manufacturing sector in South Africa is consists mainly of the following industries (see www.southafrica.info/business/economy/sectors/manufacturing.htm**):**

Agriprocessing industry - In South Africa maize is most widely grown followed by wheat, oats, sugar cane and sunflowers. The government is working to develop small-scale farming in efforts to create more jobs in this sector. Citrus, locally produced wines and flowers are mainly exported.

Automotive industry -The automotive industry is one of South Africa's most important industries. The major multinationals use South Africa to source components and assemble vehicles for both the local and international markets.

Chemical industry - The synthetic coal and natural gas-based liquid fuels and petrochemicals industry is one of the most important industries with South Africa being world leader in coal-based synthesis and gas-to-liquids (GTL) technologies.

ICT and electronics industry -The South African information technology (IT) industry growth is higher than the world average. The country has an established and sophisticated information and communications technology (ICT) and electronics sector which consists of more than 3 000 com-

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panies. The three main sub-sectors in this industry are: telecommunications, electronics and information technology.

Metals industry - South Africa has a well-developed metals industry, with numerous natural resources and a good infrastructure. It consists of basic iron ore, steel, basic non-ferrous metals and metal products. The iron and steel industries involve the manufacturing of primary iron and steel products from smelting to semi-finished stages.

Textiles, clothing and footwear industry - The South African textile and clothing industry aims to use all the natural, human and technological resources at its disposal to make South Africa the preferred domestic and international supplier of South African manufactured textiles and clothing.

The data was obtained from the Reserve bank of South Africa, collected annually from 1985 to 2012. The following graph represents the relationship between the 3 variables:

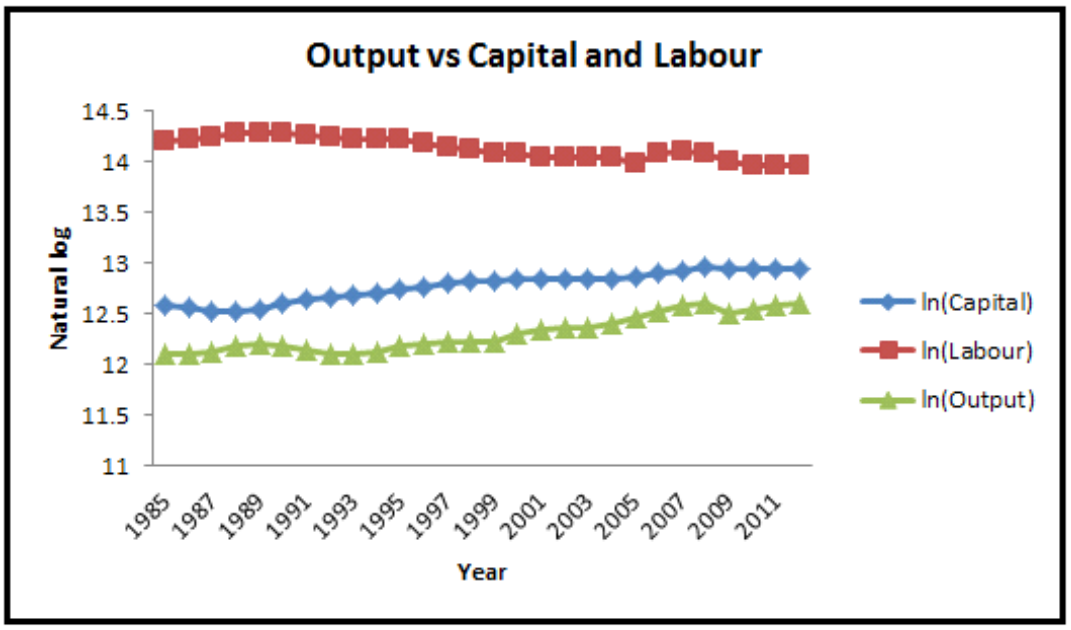

Figure 5.1: Relationship between production output, capital and labour.

Figure 5.1 indicated that there has been a steady increase in production output over the period 1985 to 2012. The two explanatory variables in the analysis is capital input and labour input. The graph illustrates that the manufacturing sector experiences a steady increase in capital input and a steady decrease in labour input over the specified period. This is an indication of a capital intensive market, where more fixed capital stock is used in manufacturing and less labour is used in the production process.

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The linear restriction of constant return to scale $\beta_1 + \beta_2 = 1$

suggests a sub-space formulation

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = 1$$

where the restriction $\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$ holds with $\mathbf{H}_{1\times 3} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{h}_{1\times 1} = 1$.

By using ordinary least squares (OLS) the following estimated regression equation is obtained:

$$\begin{aligned} \widehat{\ln Y_i} &= 10.68 - 0.54 \ln X_{1i} + 0.72 \ln X_{2i} \\ &\quad (8.36)\ (0.36) \qquad (0.27) \\ Adj R^2 &= 0.77 \quad r_{12} = -0.90 \quad VIF = 5.52 \\ cov\,(\beta_1, \beta_2) &= 0.0885 \quad CI = 9793.53 \end{aligned}$$

It is clear from the above model that a degree of multicollinearity is present in the data, since the two explanatory variables are highly negatively correlated, the VIF (Variance inflating factor) and the condition index (CI) also indicates the presence of severe multicollinearity in the model. In this example the HK ridge parameter was used to adjust for multicollinearity in the data.

When estimating using OLS, variances and covariances of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are given by

$$var\left(\widehat{\beta}_1\right) = \frac{\sigma^2}{\sum x_{1i}^2 \left(1 - r_{12}^2\right)}$$

$$var\left(\widehat{\beta}_2\right) = \frac{\sigma^2}{\sum x_{2i}^2 \left(1 - r_{12}^2\right)}$$

and

$$cov\left(\widehat{\beta}_1, \widehat{\beta}_2\right) = \frac{-r_{12}\sigma^2}{\left(1 - r_{12}^2\right)\sqrt{\sum x_{1i}^2 \sum x_{2i}^2}}$$

therefore as $|r_{12}|$ tends to 1 the multicollinearity increases and the two variances and covariance will increase accordingly. The VIF measures the speed at which the variances and covariances increase and is given by

$$VIF = \frac{1}{\left(1 - r_{12}^2\right)}$$

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This measure shows the extend to which the variance of an estimator is inflated due to the presence of multicollinearity, as $r_{12}^2 \to 1$, $VIF \to \infty$. In this sample, when estimating the parameter values using OLS, the variances of the estimated parameters are 5.5 times higher than it would have been if no multicollinearity was present.

Another condition that has not yet been taken into account is the fact that an economic restriction needs to be placed on the model in order to determine whether constant return to scale is present in the South African manufacturing sector for the sample period 1985 to 2012. In order to test the linear economic restriction of $\beta_1 + \beta_2 = 1$ the unrestricted, restricted and preliminary test ridge regression estimators as given by (1.8), (1.11) and (1.13) needs to be calculated.

The estimated regression equation for the unrestricted ridge regression estimators is given by:

$$\begin{aligned} \widehat{\ln Y}_i &= 0.11 - 0.078 \ln X_{1i} + 1.041 \ln X_{2i} \\ &\quad (0.065) \quad (0.063) \qquad (0.069) \\ Adj R^2 &= 0.73 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

The estimated regression equation for the restricted ridge regression estimator is given by

$$\begin{aligned} \widehat{\ln Y}_i &= 0.027 - 0.075 \ln X_{1i} + 1.045 \ln X_{2i} \\ &\quad (< 0.001) \quad (0.063) \qquad (0.069) \\ Adj R^2 &= 0.75 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

Based on the likelihood ratio test given by (1.6) the estimated regression equation for the preliminary test ridge regression estimator is given by:

$$\begin{aligned} \widehat{\ln Y}_i &= 0.027 - 0.075 \ln X_{1i} + 1.045 \ln X_{2i} \\ &\quad (< 0.001) \quad (0.063) \qquad (0.069) \\ Adj R^2 &= 0.75 \qquad cov\left(\beta_1, \beta_2\right) = -0.0043 \end{aligned}$$

therefore the null hypothesis (1.4) was not rejected at a 5% level of significance. It can therefore be concluded that for this sample period under consideration, the South African manufacturing sector from 1985 to 2012 experienced constant return to scale.

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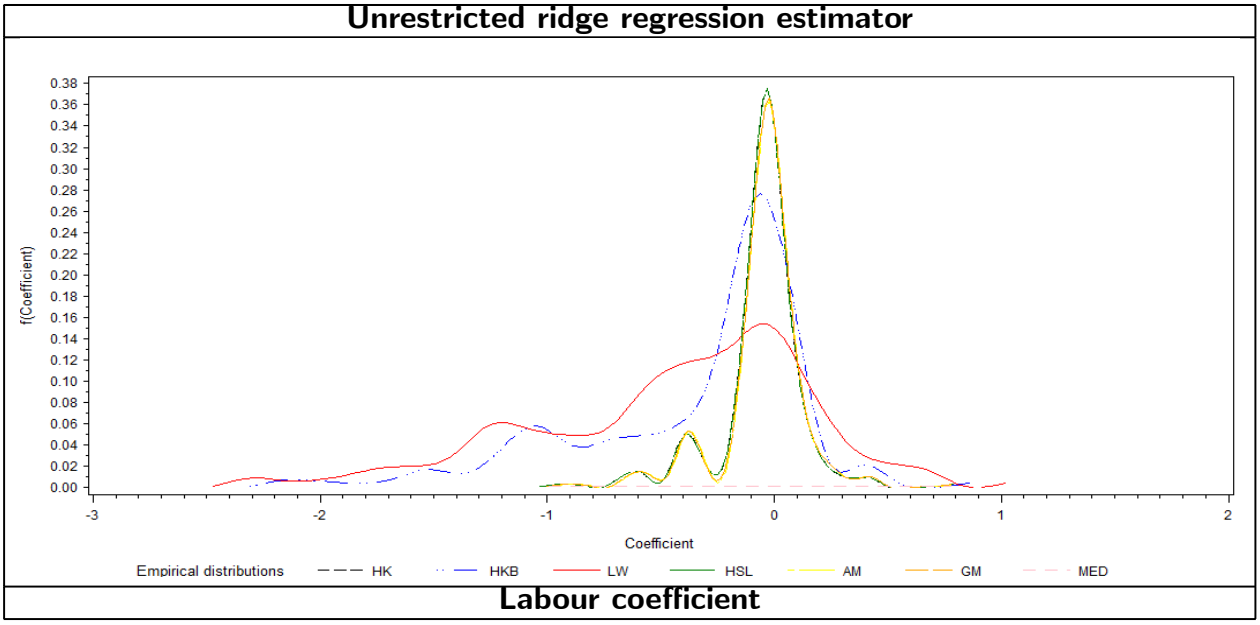
The restriction of constant return to scale could not be rejected, which implies that by doubling the input in the market will result in output doubling as well. Each of the slope coefficients can be interpreted as partial elasticities. The coefficient -0.075 measures the partial elasticity of production output to labour input, i.e. a 1% increase in the labour input, holding capital input constant, will result in a 0.075% decrease in production output. Whereas the coefficient 1.045 measures the partial elasticity of production output to capital input, i.e. a 1% increase in capital input, holding labour input constant, will result in a 1.045% increase in production output.

The following results are based on a simulation study where block bootstrapping was employed to select samples from the original data set used in the above example. Block bootstrapping is the appropriate technique to use when time series data is considered, since the data is more likely to be time dependent. Lahiri (2003) gives a detailed discussion on bootstrapping techniques for time dependent data. Nonoverlapping block bootstrapping was used in this analysis. This procedure involves splitting the sample data into b blocks of size l. (see Santana and Allison, 2013). The value of b is chosen in such a way that $lb \leq T$, where T is the sample size. In the case where $b = T$ this nonoverlapping block bootstrap method reduces to the standard bootstrapping procedure. The SAS program used for the following simulations is in Appendix D.

In Figure 5.2 the two coefficients associated with labour and capital input were estimated using the unrestricted ridge regression estimator and by using the different ridge parameters(k) first discussed in section 2.5. For the LW and HKB ridge parameters it can be seen that the variation in the estimates relative to the URRE's calculated under all the other ridge parameters is relatively high.



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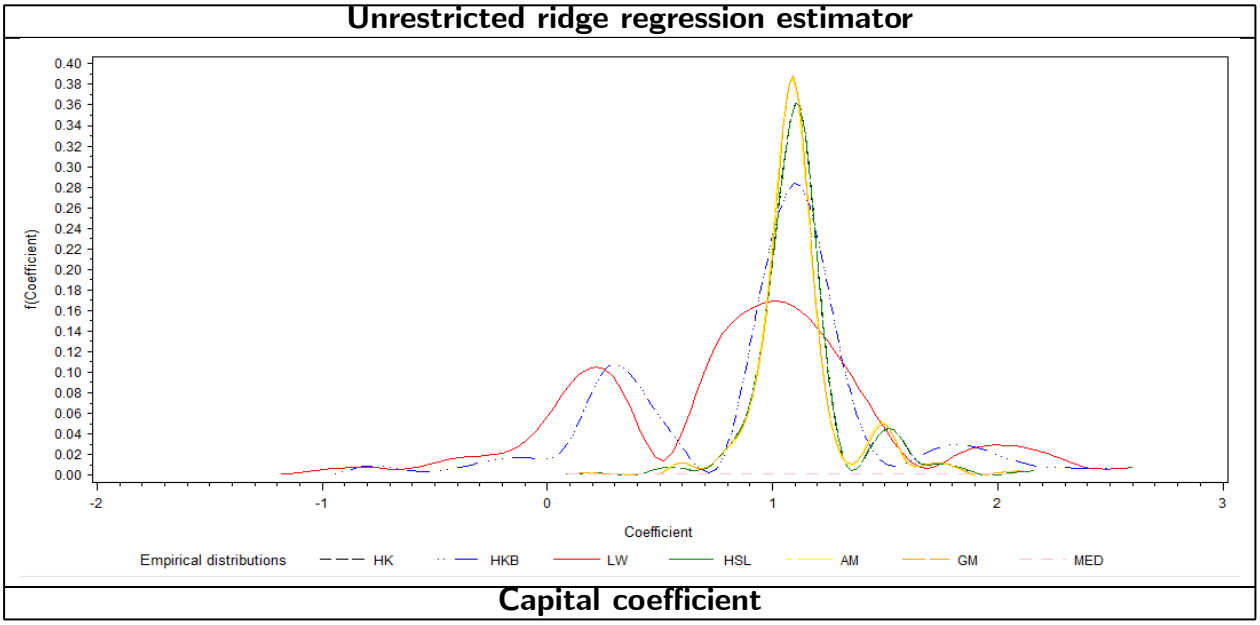

Figure 5.2 : Empirical distributions of slope coefficients of the unrestricted ridge regression estimator.

Table 5.1: Measures of risk performance for the unrestricted ridge regression estimator.

Ridge parameter	HK	HKB	LW	HSL	AM	GM	MED
Parameter estimate 1	**-0.10**	**-0.42**	**-0.53**	**-0.10**	**-0.09**	**-0.10**	**-0.09**
Standard error	0.060	0.181	0.347	0.060	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**0.84**	**0.76**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.151	0.268	0.066	0.065	0.065	0.065
Cov(est1, est2)	-0.0039	0.0188	0.0833	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.73	0.78	0.79	0.73	0.71	0.72	0.71
Risk RNL	0.37	0.79	0.96	0.37	0.35	0.36	0.35
Risk LINEX	0.08	6.19E+19	1.84E+22	0.08	0.08	0.08	0.08
Risk BLINEX	0.07	0.63	0.87	0.07	0.08	0.08	0.08

It is very obvious that the risk associated with the unrestricted ridge regression estimators using the HKB and LW ridge parameter is much higher compared to the risks associated with all the other ridge parameters. When the risk is evaluated under LINEX loss, which is an unbounded loss function it is very high due to the form of the LINEX loss function and the role of exponential term in the loss function. This is one of the disadvantages of working with the LINEX loss function and why a bounded loss function is more preferable.

In Figure 5.3 the two coefficients associated with labour and capital input were estimated using the restricted ridge regression estimator and by using the different ridge parameters(k) first discussed in section 2.5. It can be seen that under all the ridge parameters the empirical distributions of the restricted ridge regression estimates are similar.

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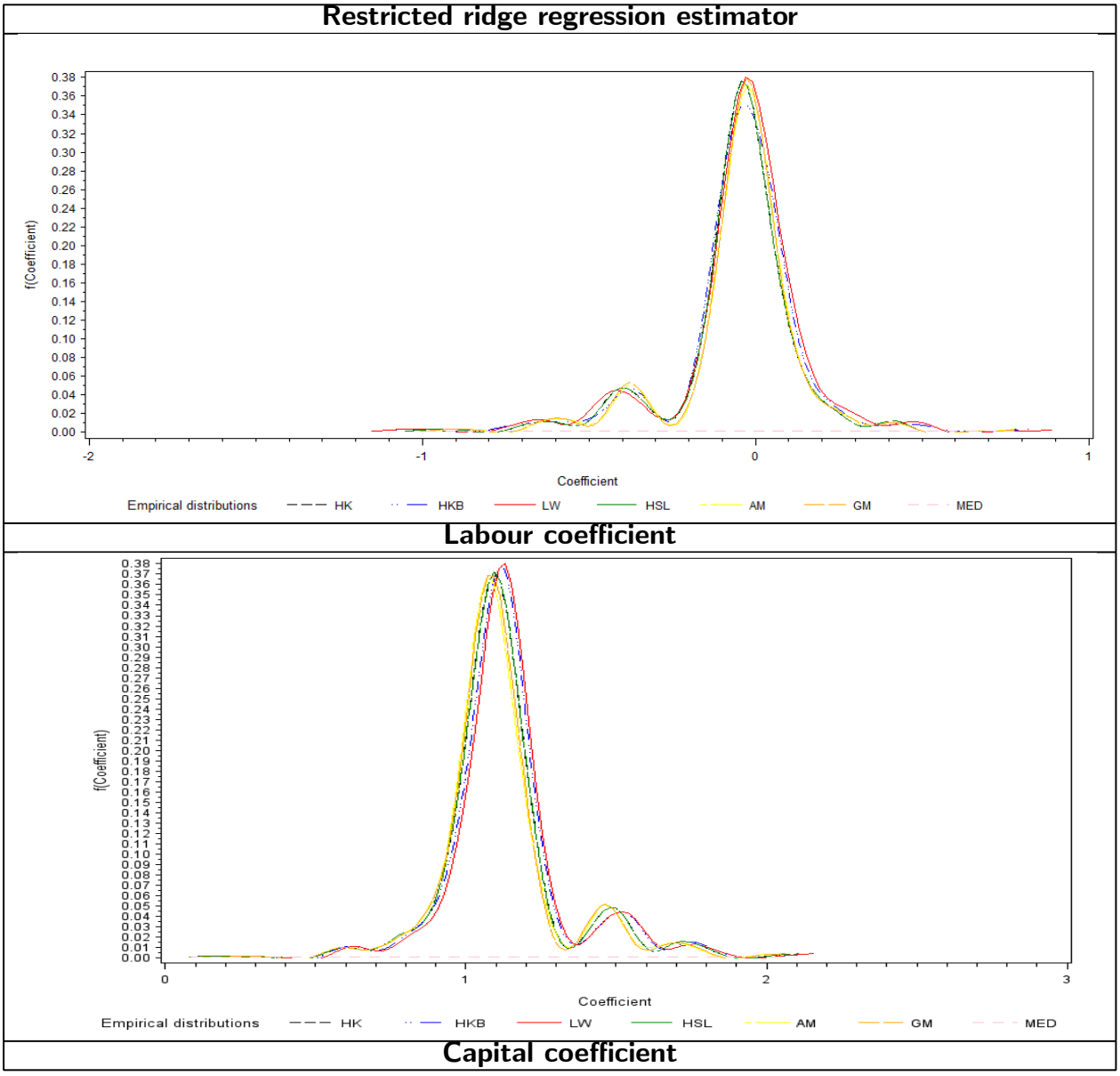

Figure 5.3 : Empirical distributions of slope coefficients of the restricted ridge regression estimator.

Table 5.2: Measures of risk performance for the restricted ridge regression estimator.

Ridge parameter	HK	HKB	LW	HSL	AM	GM	MED
Parameter estimate 1	**-0.10**	**-0.10**	**-0.09**	**-0.10**	**-0.09**	**-0.09**	**-0.09**
Standard error	0.060	0.062	0.065	0.060	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**1.09**	**1.09**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.066	0.065	0.066	0.065	0.065	0.065
Cov(est1,est2)	-0.004	-0.004	-0.004	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.75	0.76	0.76	0.75	0.73	0.73	0.73
Risk RNL	0.36	0.39	0.43	0.36	0.35	0.35	0.35
Risk LINEX	0.09	0.17	0.27	0.09	0.09	0.09	0.09
Risk BLINEX	0.09	0.14	0.21	0.09	0.08	0.08	0.08

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In the Figure 5.4 the two coefficients associated with labour and capital input were estimated using the preliminary test ridge regression estimator and by using the different ridge parameters(k) first discussed in section 2.5. The preliminary test estimator depends on the results of the hypothesis test, therefore the preliminary test estimator will be the appropriate combination of the two component estimators, namely the unrestricted ridge regression estimator and the restricted ridge regression estimator. The effect of the variation in the LW and HKB ridge parameter is therefore again present in the preliminary test estimator, but not to the same extent as in the first figure.

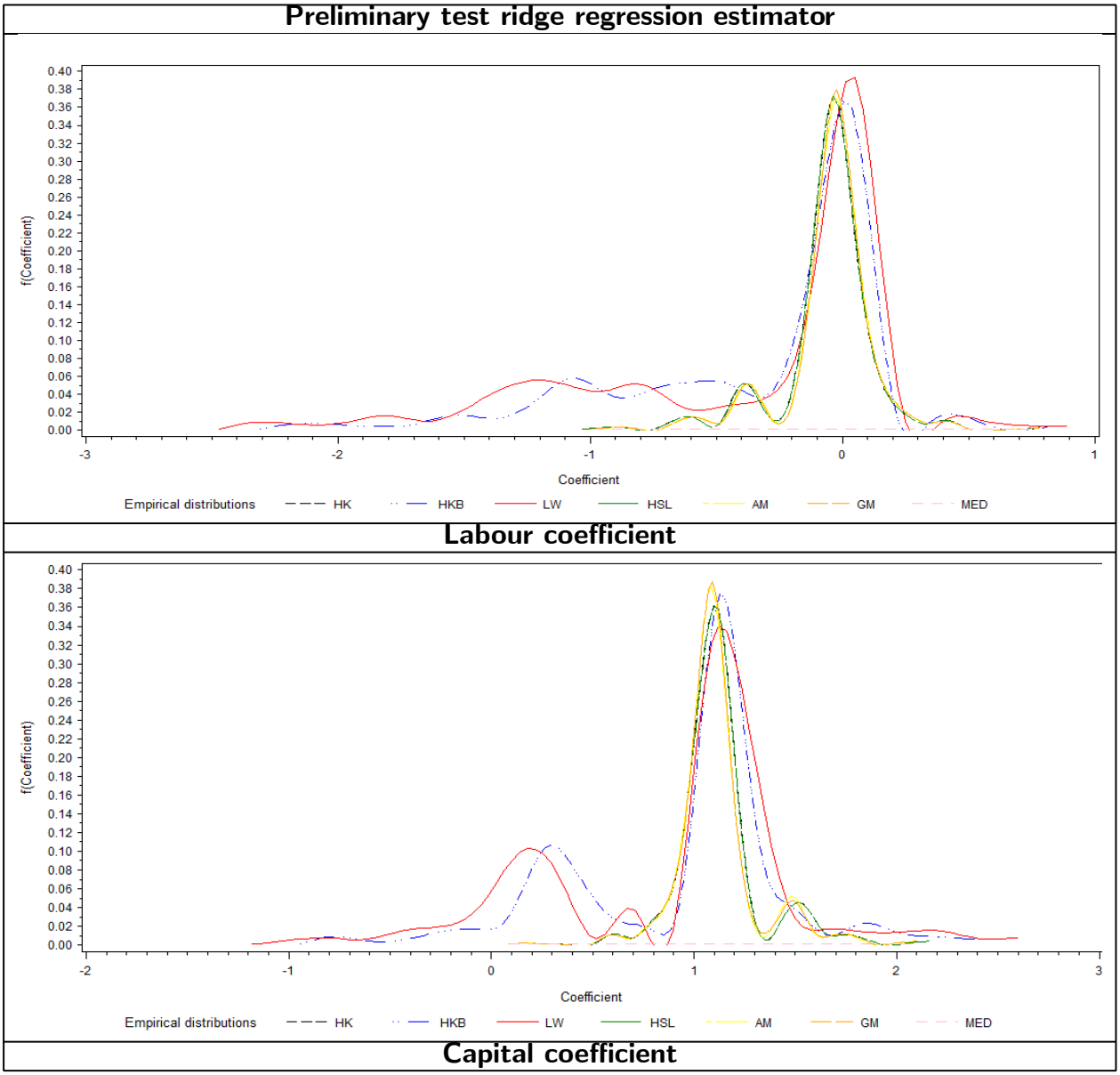

Figure 5.4 : Empirical distributions of slope coefficients of the preliminary test ridge regression estimator.

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Table 5.3: Measures of risk performance for the preliminary test ridge regression estimator.

Ridge parameter	HK	HKB	LW	HSL	AM	GM	MED
Parameter estimate 1	**-0.10**	**-0.40**	**-0.45**	**-0.10**	**-0.09**	**-0.10**	**-0.09**
Standard error	0.06	0.136	0.241	0.06	0.059	0.059	0.059
Parameter estimate 2	**1.07**	**0.86**	**0.82**	**1.07**	**1.06**	**1.06**	**1.06**
Standard error	0.066	0.119	0.192	0.066	0.065	0.065	0.065
Cov(est1, est2)	-0.004	0.010	0.051	-0.0039	-0.0038	-0.0038	-0.0038
Adj R-Square	0.73	0.78	0.79	0.73	0.72	0.72	0.71
Risk RNL	0.37	0.60	0.64	0.37	0.35	0.36	0.35
Risk LINEX	0.08	6.19E+19	1.84E+22	0.08	0.09	0.08	0.09
Risk BLINEX	0.07	0.43	0.49	0.07	0.08	0.08	0.08

By examining all these results it is clear that the risk associated with all the proposed estimators (regardless of ridge parameter) was consistently the lowest under BLINEX loss, which shows that the BLINEX loss function is the most preferable asymmetric loss function to work with.

Lastly, the following figure depicts the empirical distributions of the two parameter estimates of the Cobb-Douglas production function for the preliminary test estimator separately and also the empirical distribution of the restriction placed on the two slope parameters, namely $\beta_1 + \beta_2 = 1$. In this example the HK ridge parameter was used to control for multicollinearity.

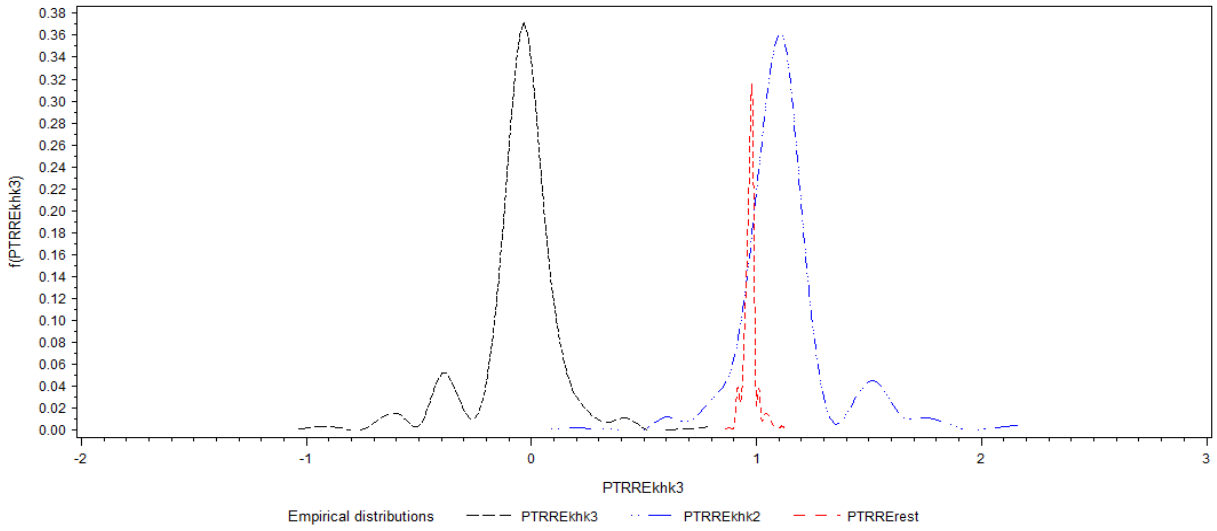

Figure 5.5 : Empirical distribution of the slope parameters and the restriction for the preliminary test estimator.

Table 5.4.: Risk performance of the preliminary test estimator and restriction.

Parameter estimate	**-Standard error**	**R-square**	**Risk RNL**	**Risk LINEX**	**Risk BLINEX**
-0.104	0.06	0.75	0.259	0.024	0.023
1.071	0.066				
0.97	0.03				

By examining the empirical distribution of the PTRRE for the restriction $\beta_1 + \beta_2 = 1$, Figure 5.5 clearly indicates the values are very closely distributed around the value of 1, with a standard error of 0.03.

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Chapter 6

Conclusions

In this section some findings of the research are reviewed together with some new insight into future research which could be conducted in this field.

6.1 Alternative approach

In this thesis, different situations in which the preliminary test estimator is applicable were considered and the performance of the preliminary test estimator was evaluated under different proposed loss functions, such as the reflected normal , linear exponential (LINEX) and bounded LINEX (BLINEX) loss functions. The focus of the study was to estimate regression coefficients of a multiple regression model under two conditions, namely multicollinearity and linear restrictions imposed on the regression coefficients. In order to address the multicollinearity problem, the coefficients were adjusted by making use of Hoerl and Kennard's (1970) approach in ridge regression. The risk function for the preliminary test estimator and its component estimators were derived under reflected normal loss, LINEX loss and BLINEX loss. Chapters 2,3, and 4 showed that the derivation of the risk functions was very cumbersome and therefore different numerical studies were used to compare the performance of the estimators under study. However, in order to move into a theoretical environment an alternative approach for deriving the risk function of the preliminary test estimator in particular is proposed.

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The following property of double expectations is used to derive the risk function of the PTRRE:

$$E[L] \equiv E[E[L|T]]$$

where $L = \mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)$, the reflected normal loss function defined for the PTRRE , $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$ and $T =$ Result of the hypothesis test based on the linear restriction specified in (1.4).

The derivation of the suggested risk function for the PTRRE under reflected normal loss is given below:

$$E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)\right] = cE\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\right]$$

$$= E\left\{E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)\middle| T\right]\right\} = cE\left\{E\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\middle| T\right]\right\}$$

$$= cE\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\middle| H_0 \text{ not rejected}\right] P(H_0 \text{ not rejected})$$

$$+cE\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\middle| H_0 \text{ rejected}\right] P(H_0 \text{ rejected})$$

$$= cE\left[1 - \exp\left(\frac{\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\right] P(H_0 \text{ not rejected})$$

$$+cE\left[1 - \exp\left(\frac{\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)}{-2\gamma^2}\right)\right] P(H_0 \text{ rejected})$$

$$= E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n(k), \boldsymbol{\beta}\right)\right] P(H_0 \text{ not rejected}) + E\left[\mathcal{L}_{RN}\left(\widetilde{\boldsymbol{\beta}}_n(k), \boldsymbol{\beta}\right)\right] P(H_0 \text{ rejected})$$

where both $E\left[\mathcal{L}_{RN}\left(\widehat{\boldsymbol{\beta}}_n(k), \boldsymbol{\beta}\right)\right]$ and $E\left[\mathcal{L}_{RN}\left(\widetilde{\boldsymbol{\beta}}_n(k), \boldsymbol{\beta}\right)\right]$ were calculated in sections 2.2 and 2.3.

With $\pounds_n$ the likelihood ratio test statistic for testing H_0 against H_A as defined in (1.6), $F_{q,m}(\alpha)$ the critical value from the central F-distribution with q and m degrees of freedom and α is the level of significance. Therefore when $\pounds_n \leq F_{q,m}(\alpha)$ the null hypothesis is not rejected, whereas when $\pounds_n > F_{q,m}(\alpha)$ the null hypothesis is rejected.

We now define $L = \mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k), \boldsymbol{\beta}\right)$, the LINEX loss function, the PTRRE , $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$ and $T =$ Result of the hypothesis test based on the linear restriction specified in (1.4).

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In a similar way the risk function of the PTRRE under LINEX loss is defined:

$$E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right] = dE\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]$$

$$= \left\{E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\middle|T\right]\right\} = dE\left\{E\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]\middle|T\right\}$$

$$= dE\left[\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]\middle|H_0 \text{ not rejected}\right]P\left(H_0 \text{ not rejected}\right)$$

$$+dE\left[\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]\middle|H_0 \text{ rejected}\right]P\left(H_0 \text{ rejected}\right)$$

$$= dE\left[\left[\exp\left[\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]\right]P\left(H_0 \text{ not rejected}\right)$$

$$+dE\left[\left[\exp\left[\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right]-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]\right]P\left(H_0 \text{ rejected}\right)$$

$$= E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0 \text{ not rejected}\right)+E\left[\mathcal{L}_{LINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0 \text{ rejected}\right)$$

both $E\left[\mathcal{L}_{LINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ and $E\left[\mathcal{L}_{LINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ were calculated sections 3.2 and 3.3.

Lastly $L = \mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)$, the BLINEX loss function, the PTRRE, $\widehat{\boldsymbol{\beta}}_n^{PT}(k)$ and $T =$ Result of the hypothesis test based on the linear restriction specified in (1.4).

The risk function of the PTRRE under BLINEX loss are defined:

$$E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\right] = \tfrac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\right]$$

$$= \left\{E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k),\boldsymbol{\beta}\right)\middle|T\right]\right\} = \tfrac{1}{\lambda}E\left\{E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|T\right]\right\}$$

$$= \tfrac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|H_0 \text{ not rejected}\right]P\left(H_0 \text{ not rejected}\right)$$

$$+\tfrac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n^{PT}(k)-\boldsymbol{\beta}\right)-1\right]}\middle|H_0 \text{ rejected}\right]P\left(H_0 \text{ rejected}\right)$$

$$= \tfrac{1}{\lambda}E\left[1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widehat{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]}\right]P\left(H_0 \text{ not rejected}\right)$$

$$+\tfrac{1}{\lambda}E\left[=1-\frac{1}{1+b\left[\exp\left(\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)\right)-\mathbf{a}'\left(\widetilde{\boldsymbol{\beta}}_n(k)-\boldsymbol{\beta}\right)-1\right]}\right]P\left(H_0 \text{ rejected}\right)$$

$$= E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0 \text{ not rejected}\right)+E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]P\left(H_0 \text{ rejected}\right)$$

both $E\left[\mathcal{L}_{BLINEX}\left(\widehat{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ and $E\left[\mathcal{L}_{BLINEX}\left(\widetilde{\boldsymbol{\beta}}_n(k),\boldsymbol{\beta}\right)\right]$ were calculated in sections 4.3.1 and 4.3.2.

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6.2 Comparison

In order to evaluate the effect that different ridge parameters will have on the estimation, the risk functions wre calculated under different conditions, for all three ridge regression estimators, namely an increase in variance, an increase in the level of multicollinearity, an increase in the number of parameters to be estimated in the regression model and an increase in the sample size. These results were compared to each other and summarised for all the proposed estimators and proposed loss functions in Table 6.1 (see the graphs in Appendix E). When choosing a specific loss function one must also take into account which of the available ridge parameters will lead to least amount of risk. As expected for all estimators it was found the risk increased when the variance, level of multicollinearity and number of parameters to be estimated increased and the risk of all the estimators decreased when the sample size increased.

Table 6.1: Comparison of estimators under the three proposed loss functions.

Parameter	Reflected normal loss	LINEX loss	BLINEX loss
σ	For all three estimators, namely URRE, RRRE and PTRRE it was found that for relative small values of σ i.e. $0.01 \leq \sigma \leq 4$, the GM and MED ridge parameters lead to the lowest risk, whereas for $\sigma > 4$ the risk for all estimators is relatively small when using the HK and HSL ridge parameter. In all cases the LW ridge parameter performed the worst.	For URRE and PTRRE it was found that the risk is the lowest when using the HK and HSL ridge parameters, whereas for RRRE the LW ridge parameter leads to the lowest risk. In all cases the AM ridge parameter performed the worst.	For RRRE and PTRRE it was found that the LW ridge parameter performs the best, whereas for URRE the HK and HSL ridge parameters performed the best over different values of σ. In all cases the AM ridge parameter performed the worst.
γ	For all three estimators, namely URRE, RRRE and PTRRE it was found that over all the tested levels of multicollinearity, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst.	For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, LW and HKB ridge parameters lead to the lowest risk over all the tested levels of multicollinearity. For the HSL ridge parameter the risks are also low for $\gamma < 0.95$. The AM ridge parameter performed the worst overall.	For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, LW and HKB ridge parameters lead to the lowest risk over different levels of multicollinearity. For the HSL ridge parameter the risks are also low for $\gamma < 0.95$. The AM ridge parameter performed the worst overall.
p	For all three estimators, namely URRE, RRRE and PTRRE it was found that for an increasing number of parameters to be estimated, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst.	For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK and HSL ridge parameters lead to the lowest risks over an increasing number of parameters to be estimated. The HKB ridge parameter also performs well, except for RRRE when $p > 20$. The AM ridge parameter performed the worst overall.	For all three estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB and HSL perform the best over an increasing number of parameters to be estimated. For the LW ridge parameter also performs well, except for RRRE when $p > 20$. The AM ridge parameter performed the worst overall.
n	For all three estimators, namely URRE, RRRE and PTRRE it was found that when the sample size increases, the GM and MED ridge parameters performed the best and the LW ridge parameter performed the worst, except for cases where the sample size is above 50 the AM ridge parameter performs the worst.	For all the estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB, LW and HSL ridge parameters perform the best over different sample sizes and the AM ridge parameter performs the worst overall.	For all the estimators, namely URRE, RRRE and PTRRE it was found that the HK, HKB, LW and HSL ridge parameters perform the best over different sample sizes and the AM ridge parameter performs the worst overall.

In order to motivate the use of the BLINEX loss function rather than the reflected normal loss or the LINEX loss function, the risk for the preliminary test estimator and its component estimators under BLINEX loss is compared to the risk of the preliminary test estimator and its components estimators under both reflected normal loss and LINEX loss. It is shown that both the risk under reflected normal loss and the risk under LINEX loss is higher than the risk under BLINEX loss.

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The reflected normal loss, BLINEX and LINEX risk functions are illustrated in the following sketches (see Figures 6.1 to 6.2) for specific values of the bounding parameter, λ, the asymmetry parameter b and the shape parameter a which influences the direction of the penalisation of the error. In addition to these parameters, different samples sizes were selected and the population variance was also adjusted to determine the effect on the risk functions. The maximum loss, level of multicollinearity and the number of parameters included in the model were kept fixed at $c = 1, \gamma = 0.8$ and $p = 4$ respectively. In each sketch the corresponding risk functions for reflected normal loss, BLINEX and LINEX loss are compared to each other. The dashed (blue) line represents the risk under reflected normal loss, the solid (red) line represents risk under LINEX loss and the dash-dot (green) line represents risk under BLINEX loss. All the risk functions are represented as a function of $\boldsymbol{\delta} = \mathbf{H}\boldsymbol{\beta} - \mathbf{h}$ (in this application δ was a scalar), where $-1 \leq \delta \leq 1$. The first column of the panel indicates the risk functions of the PTRRE under the three loss functions. In the second column the risk functions of the RRRE under the three loss functions are given and in the last column reflects the risk functions of the URRE under all three loss functions.

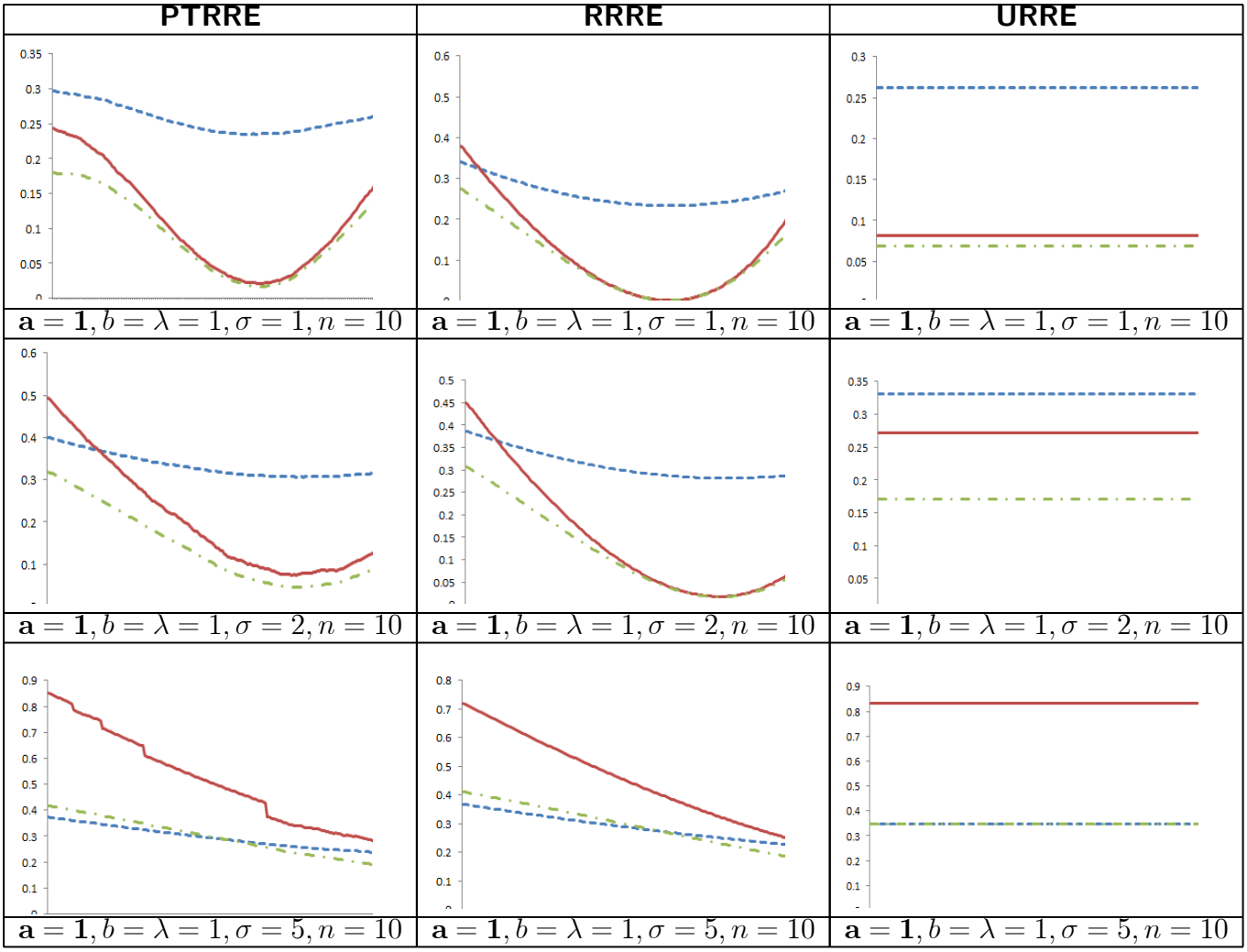

Figure 6.1 : Risk functions under reflected normal, BLINEX and LINEX loss for different choices of the scale parameter and $n = 10$.

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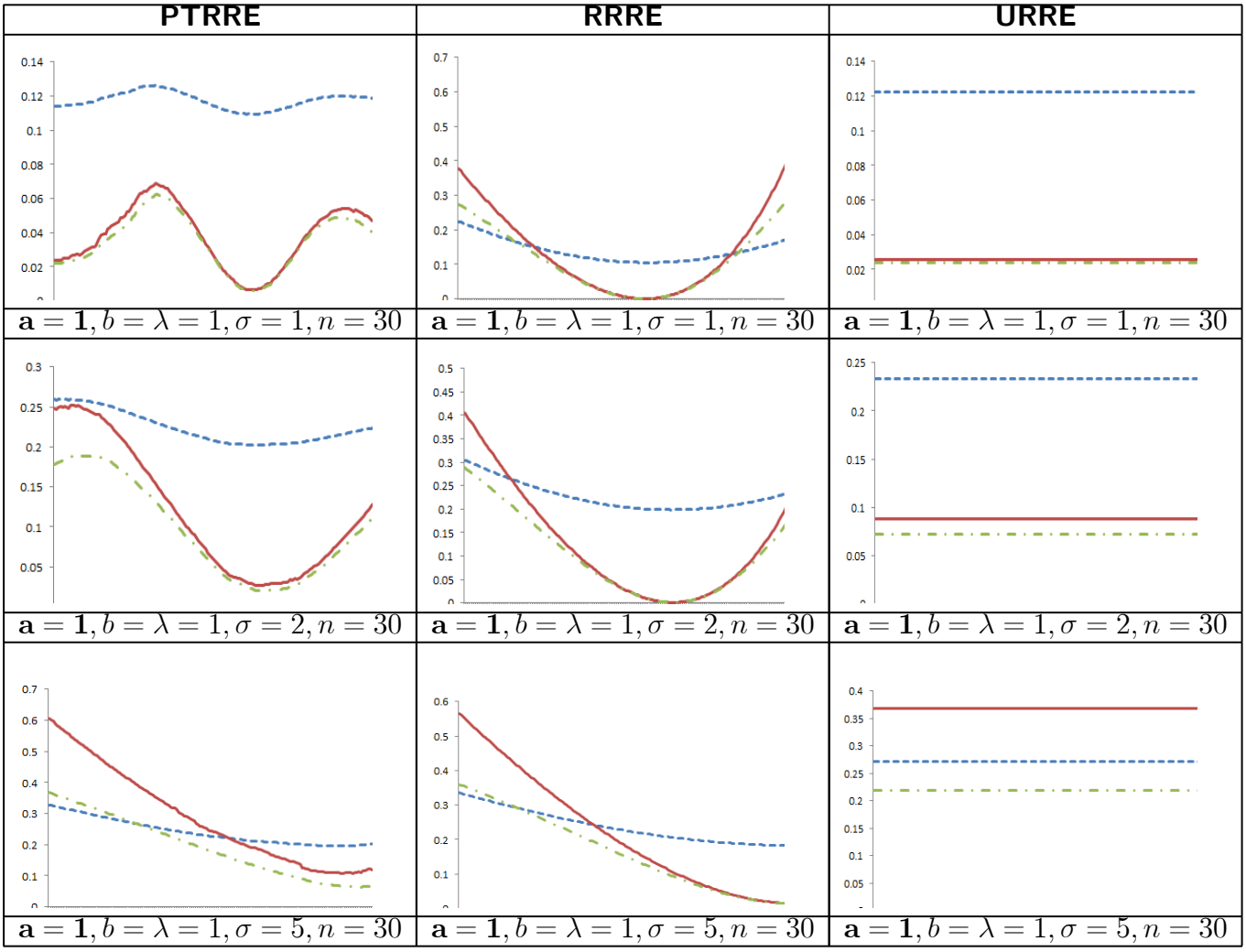

Figure 6.2 : Risk functions under reflected normal, BLINEX and LINEX loss for different choices of the scale parameter and $n = 30$.

Even though only specific choices of the parameters a, b, λ, c, σ and n were investigated, it is clear that the BLINEX risk functions represent lower risk than the corresponding LINEX risk function and reflected normal risk functions across all estimators. These results can be used to motivate why the BLINEX loss function should be used, rather than the other two loss functions, namely LINEX and reflected normal loss function. The risk function of URRE is not a function of δ, therefore the risk is constant over the domain of δ. As shown in Wen and Levy (2001), the BLINEX loss function has the same flexibility as the LINEX, together with the added advantage of being a bounded function. It is also shown that under BLINEX loss all the estimators performs better when compared to LINEX loss. Even though the reflected normal loss function is also a bounded loss function, it doesn't perform as well as the BLINEX loss function under most circumstances.

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6.3 Future directions

Some developments in this research work will now be discussed. It is well-known that the PTE has a discontinuous nature because of the indicator function. Another possible extension could be the study of the Stein-type shrinkage estimator for the parameter $\boldsymbol{\beta}$ as

$$\widehat{\boldsymbol{\beta}}^{S} = \widehat{\boldsymbol{\beta}} + \left(1 - d\mathcal{L}_n^{-1}\right)\left(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}\right), \quad d > 0$$

and its positive part, test statistic $(\mathcal{L}_n)$ and the ridge counterparts.

Throughout this thesis the normality assumption for the error term was assumed. However, in most practical situations this assumption is violated and the distribution departs from normality and heavy tailed distributions can be adopted instead.

Only the multiple regression model was considered. However, the same analysis can be applied to other models, such as seemingly unrelated regression, multivariate regression and semiparametric regression.

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Part I

Appendices

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Appendix A Program 1

Note: Programs can be more streamlined, but that was not the purpose of the study.

```
proc iml;

n=100;

p=4;

newp=p+1;

it=1000;

gamma=0.8;

sigma=1;

Z=j(n,newp,0);

X=j(n,p,0);

lambda=j(p,p,0);

urreRNLkhk=j(it,1,0);

urreRNLkhkb=j(it,1,0);

urreRNLklw=j(it,1,0);

urreRNLkhsl=j(it,1,0);

urreRNLkam=j(it,1,0);

urreRNLkgm=j(it,1,0);

urreRNLkgm1=j(it,1,0);

rrreRNLkhk=j(it,1,0);

rrreRNLkhkb=j(it,1,0);

rrreRNLklw=j(it,1,0);

rrreRNLkhsl=j(it,1,0);

rrreRNLkam=j(it,1,0);

rrreRNLkgm=j(it,1,0);

rrreRNLkgm1=j(it,1,0);

pteRNLkhk=j(it,1,0);
```

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```
pteRNLkhkb=j(it,1,0);
pteRNLklw=j(it,1,0);
pteRNLkhsl=j(it,1,0);
pteRNLkam=j(it,1,0);
pteRNLkgm=j(it,1,0);
pteRNLkgm1=j(it,1,0);
urreLINEXkhk=j(it,1,0);
urreLINEXkhkb=j(it,1,0);
urreLINEXklw=j(it,1,0);
urreLINEXkhsl=j(it,1,0);
urreLINEXkam=j(it,1,0);
urreLINEXkgm=j(it,1,0);
urreLINEXkgm1=j(it,1,0);
rrreLINEXkhk=j(it,1,0);
rrreLINEXkhkb=j(it,1,0);
rrreLINEXklw=j(it,1,0);
rrreLINEXkhsl=j(it,1,0);
rrreLINEXkam=j(it,1,0);
rrreLINEXkgm=j(it,1,0);
rrreLINEXkgm1=j(it,1,0);
pteLINEXkhk=j(it,1,0);
pteLINEXkhkb=j(it,1,0);
pteLINEXklw=j(it,1,0);
pteLINEXkhsl=j(it,1,0);
pteLINEXkam=j(it,1,0);
pteLINEXkgm=j(it,1,0);
pteLINEXkgm1=j(it,1,0);
```

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```
urreBLINEXkhk=j(it,1,0);
urreBLINEXkhkb=j(it,1,0);
urreBLINEXklw=j(it,1,0);
urreBLINEXkhsl=j(it,1,0);
urreBLINEXkam=j(it,1,0);
urreBLINEXkgm=j(it,1,0);
urreBLINEXkgm1=j(it,1,0);
rrreBLINEXkhk=j(it,1,0);
rrreBLINEXkhkb=j(it,1,0);
rrreBLINEXklw=j(it,1,0);
rrreBLINEXkhsl=j(it,1,0);
rrreBLINEXkam=j(it,1,0);
rrreBLINEXkgm=j(it,1,0);
rrreBLINEXkgm1=j(it,1,0);
pteBLINEXkhk=j(it,1,0);
pteBLINEXkhkb=j(it,1,0);
pteBLINEXklw=j(it,1,0);
pteBLINEXkhsl=j(it,1,0);
pteBLINEXkam=j(it,1,0);
pteBLINEXkgm=j(it,1,0);
pteBLINEXkgm1=j(it,1,0);
een=j(n,1,1);
eenp=j(p,1,1);
eenit=j(it,1,1);
Id=I(n);
Idp=I(p);
std=j(p,1,0);
```

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```
stdz=j(newp,1,0);

standx=j(n,p,0);

standz=j(n,newp,0);

stdstand=j(n,p,0);

beta=j(p,1,0);

epsilon=j(n,1,0);

standeps=j(n,1,0);

e=j(n,1,0);

y=j(n,1,0);

H=j(1,p,1);

r=2;

q=1;

c=1;

d=1;

a=j(p,1,1);

gamma1=1;

m=n-p;

los=0.05;

Fcrit=Finv(1-los,q,m);

lamb=1;

b=lamb*d;

*print fcrit;

*Generate standard normal distributions;

do w=1 to it;

do i=1 to n;

do j=1 to newp;

Z[i,j]=normal(0);
```

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```
end;
end;
*print z;
zbar=z'*een/n;
dub=(1/n)*een*een';
varz=(1/(n-1))*z'*(id-dub)*z;
diagz=vecdiag(varz);
do k=1 to newp;
stdz[k]=sqrt(diagz[k]);
end;
*print zbar dub varz diagz stdz;
*Standardised z values;
do i=1 to n;
do j=1 to newp;
standz[i,j]=(z[i,j]-zbar[j])/stdz[j];
end;
end;
*print standz;
*Generate the explanatory variables which are correlated with one another;
do i=1 to n;
do j=1 to p;
X[i,j]=sqrt(1-gamma**2)*standZ[i,j]+gamma*standZ[i,newp];
end;
end;
*print x;
*Calculate x-bar and standard deviation for every distribution;
xbar=x'*een/n;
```

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```
dub=(1/n)*een*een‘;
var=(1/(n-1))*x‘*(id-dub)*x;
diag=vecdiag(var);
do k=1 to p;
std[k]=sqrt(diag[k]);
end;
*print xbar dub var diag std;
*Standardised X values;
do i=1 to n;
do j=1 to p;
standx[i,j]=(X[i,j]-xbar[j])/std[j];
end;
end;
*print standx;
/*Check standardisation;*/
xbarstand=standx‘*een/n;
dub1=(1/n)*een*een‘;
varstand=(1/(n-1))*standx‘*(id-dub)*standx;
diagstand=vecdiag(varstand);
do k=1 to p;
stdstand[k]=sqrt(diagstand[k]);
end;
*print xbarstand diagstand varstand stdstand;
XprimeX=standx‘*standx;
invxpx=inv(XprimeX);
*print xprimex invxpx;
eigvector=eigvec(xprimex);
```

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```
eigvalue=eigval(xprimex);
*print eigvalue eigvector;
do i=1 to p;
lambda[i,i]=eigvalue[i];
end;
*print lambda;
do l=1 to p;
beta[l,1]=eigvector[l,1];
end;
*print beta;
normbet=sqrt(beta'*beta);
*print normbet;
*Generate error terms;
do i=1 to n;
epsilon[i]=normal(0);
end;
*print epsilon;
*Calculate x-bar and standard deviation for epsilon;
xbar1=epsilon'*een/n;
dub2=(1/n)*een*een';
var2=(1/(n-1))*epsilon'*(id-dub)*epsilon;
diag2=vecdiag(var2);
*print xbar1 diag2;
*Standardised epsilon values;
do i=1 to n;
standeps[i]=(epsilon[i]-xbar1[1])/sqrt(diag2[1]);
end;
```

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```
*print standeps;
*Calculate x-bar and standard deviation for standeps;
xbar3=standeps‘*een/n;
dub3=(1/n)*een*een‘;
var3=(1/(n-1))*standeps‘*(id-dub)*standeps;
diag3=vecdiag(var3);
*print xbar3 diag3;
*Calculate e-terms;
do i=1 to n;
e[i]=0+sigma*standeps[i];
end;
*print e;
*Calculate y;
xbeta=standx*beta;
*print xbeta;
do i=1 to n;
y[i]=xbeta[i]+e[i];
end;
transeigvec=eigvector‘;
*print y xbeta e;
alpha=eigvector*beta;
maxalpha=max(alpha);
alphasquare=alpha‘*alpha;
*print transeigvec beta alpha;
*alpha maxalpha;
betahat=invxpx*standx‘*y;
betahatsquare=betahat‘*betahat;
```

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```
*print betahat beta betahatsquare alphasquare;
yhat=standx*betahat;
*print yhat y;
ehat=y-yhat;
*print ehat;
ehatsquare=ehat`*ehat;
*print ehatsquare;
sigmasquarehat=ehatsquare/(n-p);
*print sigmasquarehat;
Khk=sigmasquarehat/(maxalpha**2);
*print Khk;
Khkb=sigmasquarehat*p/betahatsquare;
*print Khkb;
Klw=p*sigmasquarehat/(betahat`*lambda*betahat);
*print klw;
*print eigvalue alpha;
prodeigalpha=eigvalue#alpha;
prodeigalphasq=eigvalue#alpha#alpha;
*print prodeigalpha prodeigalphasq;
sqprodea=prodeigalpha#prodeigalpha;
numhsl=sqprodea`*eenp;
denomhsl=(prodeigalphasq`*eenp)**2;
Khsl=sigmasquarehat*numhsl/denomhsl;
*print sqprodea numhsl denomhsl khsl;
alpsq=alpha#alpha;
eenooralphasq=1/(alpha#alpha);
sumeenooralphasq=eenooralphasq`*eenp;
```

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```
Kam=(1/p)*sigmasquarehat*sumeenooralphasq;
*print eenooralphasq sumeenooralphasq kam;
alphasq=alpha#alpha;
*print alphasq;
v=1;
do i=1 to p;
v=v*alphasq[i,1];
end;
*print v;
denomgm=v**(1/p);
*print denomgm;
kgm=sigmasquarehat/denomgm;
*print kgm denomgm;
sigdivalphasq=sigmasquarehat/alphasq;
*print sigdivalphasq;
kgm1=median(sigdivalphasq);
*print kgm1 sigdivalphasq;
*Unrestricted ridge regression estimator;
Rkhk=inv(Idp+khk*inv(xprimex));
Rkhkb=inv(Idp+khkb*inv(xprimex));
Rklw=inv(Idp+klw*inv(xprimex));
Rkhsl=inv(Idp+khsl*inv(xprimex));
Rkam=inv(Idp+kam*inv(xprimex));
Rkgm=inv(Idp+kgm*inv(xprimex));
Rkgm1=inv(Idp+kgm1*inv(xprimex));
*print rkhk rkhkb rklw rkhsl rkam rkgm rkgm1;
URREkhk=Rkhk*betahat;
```

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```
URREkhkb=Rkhkb*betahat;

URREklw=Rklw*betahat;

URREkhsl=Rkhsl*betahat;

URREkam=Rkam*betahat;

URREkgm=Rkgm*betahat;

URREkgm1=Rkgm1*betahat;

*print beta betahat urrekhk urrekhkb urreklw urrekhsl urrekam urrekgm urrekgm1;

*RRRE estimators;

RRE=betahat-inv(xprimex)*H'*inv(H*inv(xprimex)*H')*(H*betahat-r);

*print rre;

RRREkhk=Rkhk*rre;

RRREkhkb=Rkhkb*rre;

RRREklw=Rklw*rre;

RRREkhsl=Rkhsl*rre;

RRREkam=Rkam*rre;

RRREkgm=Rkgm*rre;

RRREkgm1=Rkgm1*rre;

*s-squared;

ssq=(1/(m))*(y-standx*betahat)'*(y-standx*betahat);

*print ssq;

*likelyhood ratio test ;

lkhd=((H*betahat-r)'*inv(H*inv(xprimex)*H')*(H*betahat-r))/(q*ssq);

if lkhd<fcrit then INRF1=1;

if lkhd>=fcrit then INRF1=0;

*PTE ridge estimators for different k;

pte1=betahat-(betahat-rre)*inrf1;

ptekhk=rkhk*pte1;
```

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```
ptekhkb=rkhkb*pte1;
pteklw=rklw*pte1;
ptekhsl=rkhsl*pte1;
ptekam=rkam*pte1;
ptekgm=rkgm*pte1;
ptekgm1=rkgm1*pte1;
*print beta ptekhk ptekhkb pteklw ptekhsl ptekam ptekgm ptekgm1;
*Calculate reflected normal loss values for different k;
urreRNLkhk[w]=c*(1-exp((urrekhk-beta)'*(urrekhk-beta)/(-2*gamma1**2)));
urreRNLkhkb[w]=c*(1-exp((urrekhkb-beta)'*(urrekhkb-beta)/(-2*gamma1**2)));
urreRNLklw[w]=c*(1-exp((urreklw-beta)'*(urreklw-beta)/(-2*gamma1**2)));
urreRNLkhsl[w]=c*(1-exp((urrekhsl-beta)'*(urrekhsl-beta)/(-2*gamma1**2)));
urreRNLkam[w]=c*(1-exp((urrekam-beta)'*(urrekam-beta)/(-2*gamma1**2)));
urreRNLkgm[w]=c*(1-exp((urrekgm-beta)'*(urrekgm-beta)/(-2*gamma1**2)));
urreRNLkgm1[w]=c*(1-exp((urrekgm1-beta)'*(urrekgm1-beta)/(-2*gamma1**2)));
rrreRNLkhk[w]=c*(1-exp((rrrekhk-beta)'*(rrrekhk-beta)/(-2*gamma1**2)));
rrreRNLkhkb[w]=c*(1-exp((rrrekhkb-beta)'*(rrrekhkb-beta)/(-2*gamma1**2)));
rrreRNLklw[w]=c*(1-exp((rrreklw-beta)'*(rrreklw-beta)/(-2*gamma1**2)));
rrreRNLkhsl[w]=c*(1-exp((rrrekhsl-beta)'*(rrrekhsl-beta)/(-2*gamma1**2)));
rrreRNLkam[w]=c*(1-exp((rrrekam-beta)'*(rrrekam-beta)/(-2*gamma1**2)));
rrreRNLkgm[w]=c*(1-exp((rrrekgm-beta)'*(rrrekgm-beta)/(-2*gamma1**2)));
rrreRNLkgm1[w]=c*(1-exp((rrrekgm1-beta)'*(rrrekgm1-beta)/(-2*gamma1**2)));
pteRNLkhk[w]=c*(1-exp((ptekhk-beta)'*(ptekhk-beta)/(-2*gamma1**2)));
pteRNLkhkb[w]=c*(1-exp((ptekhkb-beta)'*(ptekhkb-beta)/(-2*gamma1**2)));
pteRNLklw[w]=c*(1-exp((pteklw-beta)'*(pteklw-beta)/(-2*gamma1**2)));
pteRNLkhsl[w]=c*(1-exp((ptekhsl-beta)'*(ptekhsl-beta)/(-2*gamma1**2)));
pteRNLkam[w]=c*(1-exp((ptekam-beta)'*(ptekam-beta)/(-2*gamma1**2)));
```

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```
pteRNLkgm[w]=c*(1-exp((ptekgm-beta)'*(ptekgm-beta)/(-2*gamma1**2)));
pteRNLkgm1[w]=c*(1-exp((ptekgm1-beta)'*(ptekgm1-beta)/(-2*gamma1**2)));
*print urrernlkhk urrernlkhkb urrernlklw urrernlkhsl urrernlkam urrernlkgm urrernlkgm1;
*print rrrernlkhk rrrernlkhkb rrrernlklw rrrernlkhsl rrrernlkam rrrernlkgm rrrernlkgm1;
*print pternlkhk pternlkhkb pternlklw pternlkhsl pternlkam pternlkgm pternlkgm1;
*Calculate LINEX loss values for different k;
urreLINEXkhk[w]=d*(exp(a'*(urrekhk-beta))-a'*(urrekhk-beta)-1);
urreLINEXkhkb[w]=d*(exp(a'*(urrekhkb-beta))-a'*(urrekhkb-beta)-1);
urreLINEXklw[w]=d*(exp(a'*(urreklw-beta))-a'*(urreklw-beta)-1);
urreLINEXkhsl[w]=d*(exp(a'*(urrekhsl-beta))-a'*(urrekhsl-beta)-1);
urreLINEXkam[w]=d*(exp(a'*(urrekam-beta))-a'*(urrekam-beta)-1);
urreLINEXkgm[w]=d*(exp(a'*(urrekgm-beta))-a'*(urrekgm-beta)-1);
urreLINEXkgm1[w]=d*(exp(a'*(urrekgm1-beta))-a'*(urrekgm1-beta)-1);
rrreLINEXkhk[w]=d*(exp(a'*(rrrekhk-beta))-a'*(rrrekhk-beta)-1);
rrreLINEXkhkb[w]=d*(exp(a'*(rrrekhkb-beta))-a'*(rrrekhkb-beta)-1);
rrreLINEXklw[w]=d*(exp(a'*(rrreklw-beta))-a'*(rrreklw-beta)-1);
rrreLINEXkhsl[w]=d*(exp(a'*(rrrekhsl-beta))-a'*(rrrekhsl-beta)-1);
rrreLINEXkam[w]=d*(exp(a'*(rrrekam-beta))-a'*(rrrekam-beta)-1);
rrreLINEXkgm[w]=d*(exp(a'*(rrrekgm-beta))-a'*(rrrekgm-beta)-1);
rrreLINEXkgm1[w]=d*(exp(a'*(rrrekgm1-beta))-a'*(rrrekgm1-beta)-1);
pteLINEXkhk[w]=d*(exp(a'*(ptekhk-beta))-a'*(ptekhk-beta)-1);
pteLINEXkhkb[w]=d*(exp(a'*(ptekhkb-beta))-a'*(ptekhkb-beta)-1);
pteLINEXklw[w]=d*(exp(a'*(pteklw-beta))-a'*(pteklw-beta)-1);
pteLINEXkhsl[w]=d*(exp(a'*(ptekhsl-beta))-a'*(ptekhsl-beta)-1);
pteLINEXkam[w]=d*(exp(a'*(ptekam-beta))-a'*(ptekam-beta)-1);
pteLINEXkgm[w]=d*(exp(a'*(ptekgm-beta))-a'*(ptekgm-beta)-1);
pteLINEXkgm1[w]=d*(exp(a'*(ptekgm1-beta))-a'*(ptekgm1-beta)-1);
```

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```
*Calculate BLINEX loss for different k;
urreBLINEXkhk[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhk-beta))-a‘*(urrekhk-beta)-1)));
urreBLINEXkhkb[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhkb-beta))-a‘*(urrekhkb-beta)-1)));
urreBLINEXklw[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urreklw-beta))-a‘*(urreklw-beta)-1)));
urreBLINEXkhsl[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhsl-beta))-a‘*(urrekhsl-beta)-1)));
urreBLINEXkam[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekam-beta))-a‘*(urrekam-beta)-1)));
urreBLINEXkgm[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekgm-beta))-a‘*(urrekgm-beta)-1)));
urreBLINEXkgm1[w]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekgm1-beta))-a‘*(urrekgm1-beta)-1)));
rrreBLINEXkhk[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhk-beta))-a‘*(rrrekhk-beta)-1)));
rrreBLINEXkhkb[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhkb-beta))-a‘*(rrrekhkb-beta)-1)));
rrreBLINEXklw[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrreklw-beta))-a‘*(rrreklw-beta)-1)));
rrreBLINEXkhsl[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhsl-beta))-a‘*(rrrekhsl-beta)-1)));
rrreBLINEXkam[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekam-beta))-a‘*(rrrekam-beta)-1)));
rrreBLINEXkgm[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekgm-beta))-a‘*(rrrekgm-beta)-1)));
rrreBLINEXkgm1[w]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekgm1-beta))-a‘*(rrrekgm1-beta)-1)));
pteBLINEXkhk[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhk-beta))-a‘*(ptekhk-beta)-1)));
pteBLINEXkhkb[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhkb-beta))-a‘*(ptekhkb-beta)-1)));
pteBLINEXklw[w]=1/lamb*(1-1/(1+b*(exp(a‘*(pteklw-beta))-a‘*(pteklw-beta)-1)));
pteBLINEXkhsl[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhsl-beta))-a‘*(ptekhsl-beta)-1)));
pteBLINEXkam[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekam-beta))-a‘*(ptekam-beta)-1)));
pteBLINEXkgm[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekgm-beta))-a‘*(ptekgm-beta)-1)));
pteBLINEXkgm1[w]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekgm1-beta))-a‘*(ptekgm1-beta)-1)));
end;
* Calculate risk functions;
riskurrernlkhk=eenit‘*urrernlkhk/it;
riskurrernlkhkb=eenit‘*urrernlkhkb/it;
riskurrernlklw=eenit‘*urrernlklw/it;
```

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riskurrernlkhsl=eenit‘*urrernlkhsl/it;

riskurrernlkam=eenit‘*urrernlkam/it;

riskurrernlkgm=eenit‘*urrernlkgm/it;

riskurrernlkgm1=eenit‘*urrernlkgm1/it;

riskrrrernlkhk=eenit‘*rrrernlkhk/it;

riskrrrernlkhkb=eenit‘*rrrernlkhkb/it;

riskrrrernlklw=eenit‘*rrrernlklw/it;

riskrrrernlkhsl=eenit‘*rrrernlkhsl/it;

riskrrrernlkam=eenit‘*rrrernlkam/it;

riskrrrernlkgm=eenit‘*rrrernlkgm/it;

riskrrrernlkgm1=eenit‘*rrrernlkgm1/it;

riskpternlkhk=eenit‘*pternlkhk/it;

riskpternlkhkb=eenit‘*pternlkhkb/it;

riskpternlklw=eenit‘*pternlklw/it;

riskpternlkhsl=eenit‘*pternlkhsl/it;

riskpternlkam=eenit‘*pternlkam/it;

riskpternlkgm=eenit‘*pternlkgm/it;

riskpternlkgm1=eenit‘*pternlkgm1/it;

riskurreLINEXkhk=eenit‘*urreLINEXkhk/it;

riskurreLINEXkhkb=eenit‘*urreLINEXkhkb/it;

riskurreLINEXklw=eenit‘*urreLINEXklw/it;

riskurreLINEXkhsl=eenit‘*urreLINEXkhsl/it;

riskurreLINEXkam=eenit‘*urreLINEXkam/it;

riskurreLINEXkgm=eenit‘*urreLINEXkgm/it;

riskurreLINEXkgm1=eenit‘*urreLINEXkgm1/it;

riskrrreLINEXkhk=eenit‘*rrreLINEXkhk/it;

riskrrreLINEXkhkb=eenit‘*rrreLINEXkhkb/it;

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riskrrreLINEXklw=eenit‘*rrreLINEXklw/it;

riskrrreLINEXkhsl=eenit‘*rrreLINEXkhsl/it;

riskrrreLINEXkam=eenit‘*rrreLINEXkam/it;

riskrrreLINEXkgm=eenit‘*rrreLINEXkgm/it;

riskrrreLINEXkgm1=eenit‘*rrreLINEXkgm1/it;

riskpteLINEXkhk=eenit‘*pteLINEXkhk/it;

riskpteLINEXkhkb=eenit‘*pteLINEXkhkb/it;

riskpteLINEXklw=eenit‘*pteLINEXklw/it;

riskpteLINEXkhsl=eenit‘*pteLINEXkhsl/it;

riskpteLINEXkam=eenit‘*pteLINEXkam/it;

riskpteLINEXkgm=eenit‘*pteLINEXkgm/it;

riskpteLINEXkgm1=eenit‘*pteLINEXkgm1/it;

riskurreBLINEXkhk=eenit‘*urreBLINEXkhk/it;

riskurreBLINEXkhkb=eenit‘*urreBLINEXkhkb/it;

riskurreBLINEXklw=eenit‘*urreBLINEXklw/it;

riskurreBLINEXkhsl=eenit‘*urreBLINEXkhsl/it;

riskurreBLINEXkam=eenit‘*urreBLINEXkam/it;

riskurreBLINEXkgm=eenit‘*urreBLINEXkgm/it;

riskurreBLINEXkgm1=eenit‘*urreBLINEXkgm1/it;

riskrrreBLINEXkhk=eenit‘*rrreBLINEXkhk/it;

riskrrreBLINEXkhkb=eenit‘*rrreBLINEXkhkb/it;

riskrrreBLINEXklw=eenit‘*rrreBLINEXklw/it;

riskrrreBLINEXkhsl=eenit‘*rrreBLINEXkhsl/it;

riskrrreBLINEXkam=eenit‘*rrreBLINEXkam/it;

riskrrreBLINEXkgm=eenit‘*rrreBLINEXkgm/it;

riskrrreBLINEXkgm1=eenit‘*rrreBLINEXkgm1/it;

riskpteBLINEXkhk=eenit‘*pteBLINEXkhk/it;

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riskpteBLINEXkhkb=eenit‘*pteBLINEXkhkb/it;

riskpteBLINEXklw=eenit‘*pteBLINEXklw/it;

riskpteBLINEXkhsl=eenit‘*pteBLINEXkhsl/it;

riskpteBLINEXkam=eenit‘*pteBLINEXkam/it;

riskpteBLINEXkgm=eenit‘*pteBLINEXkgm/it;

riskpteBLINEXkgm1=eenit‘*pteBLINEXkgm1/it;

print riskurrernlkhk riskurrernlkhkb riskurrernlklw riskurrernlkhsl riskurrernlkam riskurrernlkgm riskurrernlkgm1;

print riskurrelinexkhk riskurrelinexkhkb riskurrelinexklw riskurrelinexkhsl riskurrelinexkam riskurrelinexkgm riskurrelinexkgm1;

print riskurreblinexkhk riskurreblinexkhkb riskurreblinexklw riskurreblinexkhsl riskurreblinexkam riskurreblinexkgm riskurreblinexkgm1;

print riskrrrernlkhk riskrrrernlkhkb riskrrrernlklw riskrrrernlkhsl riskrrrernlkam riskrrrernlkgm riskrrrernlkgm1;

print riskrrrelinexkhk riskrrrelinexkhkb riskrrrelinexklw riskrrrelinexkhsl riskrrrelinexkam riskrrrelinexkgm riskrrrelinexkgm1;

print riskrrreblinexkhk riskrrreblinexkhkb riskrrreblinexklw riskrrreblinexkhsl riskrrreblinexkam riskrrreblinexkgm riskrrreblinexkgm1;

print riskpternlkhk riskpternlkhkb riskpternlklw riskpternlkhsl riskpternlkam riskpternlkgm riskpternlkgm1;

print riskptelinexkhk riskptelinexkhkb riskptelinexklw riskptelinexkhsl riskptelinexkam riskptelinexkgm riskptelinexkgm1;

print riskpteblinexkhk riskpteblinexkhkb riskpteblinexklw riskpteblinexkhsl riskpteblinexkam riskpteblinexkgm riskpteblinexkgm1;

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Appendix B Program 2

Note: Programs can be more streamlined, but that was not the purpose of the study.

```
proc iml;

n=100;

p=4;

newp=p+1;

it=500;

gamma=0.80;

sigma=5;

Z=j(n,newp,0);

X=j(n,p,0);

lambda=j(p,p,0);

be=1 ;

en=3 ;

step=0.01 ;

dim=(en-be)/step+1 ;

urreRNLkhk=j(it,dim,0);

urreRNLkhkb=j(it,dim,0);

urreRNLklw=j(it,dim,0);

urreRNLkhsl=j(it,dim,0);

urreRNLkam=j(it,dim,0);

urreRNLkgm=j(it,dim,0);

urreRNLkgm1=j(it,dim,0);

rrreRNLkhk=j(it,dim,0);

rrreRNLkhkb=j(it,dim,0);

rrreRNLklw=j(it,dim,0);

rrreRNLkhsl=j(it,dim,0);
```

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```
rrreRNLkam=j(it,dim,0);
rrreRNLkgm=j(it,dim,0);
rrreRNLkgm1=j(it,dim,0);
pteRNLkhk=j(it,dim,0);
pteRNLkhkb=j(it,dim,0);
pteRNLklw=j(it,dim,0);
pteRNLkhsl=j(it,dim,0);
pteRNLkam=j(it,dim,0);
pteRNLkgm=j(it,dim,0);
pteRNLkgm1=j(it,dim,0);
urreLINEXkhk=j(it,dim,0);
urreLINEXkhkb=j(it,dim,0);
urreLINEXklw=j(it,dim,0);
urreLINEXkhsl=j(it,dim,0);
urreLINEXkam=j(it,dim,0);
urreLINEXkgm=j(it,dim,0);
urreLINEXkgm1=j(it,dim,0);
rrreLINEXkhk=j(it,dim,0);
rrreLINEXkhkb=j(it,dim,0);
rrreLINEXklw=j(it,dim,0);
rrreLINEXkhsl=j(it,dim,0);
rrreLINEXkam=j(it,dim,0);
rrreLINEXkgm=j(it,dim,0);
rrreLINEXkgm1=j(it,dim,0);
pteLINEXkhk=j(it,dim,0);
pteLINEXkhkb=j(it,dim,0);
pteLINEXklw=j(it,dim,0);
```

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pteLINEXkhsl=j(it,dim,0);

pteLINEXkam=j(it,dim,0);

pteLINEXkgm=j(it,dim,0);

pteLINEXkgm1=j(it,dim,0);

urreBLINEXkhk=j(it,dim,0);

urreBLINEXkhkb=j(it,dim,0);

urreBLINEXklw=j(it,dim,0);

urreBLINEXkhsl=j(it,dim,0);

urreBLINEXkam=j(it,dim,0);

urreBLINEXkgm=j(it,dim,0);

urreBLINEXkgm1=j(it,dim,0);

rrreBLINEXkhk=j(it,dim,0);

rrreBLINEXkhkb=j(it,dim,0);

rrreBLINEXklw=j(it,dim,0);

rrreBLINEXkhsl=j(it,dim,0);

rrreBLINEXkam=j(it,dim,0);

rrreBLINEXkgm=j(it,dim,0);

rrreBLINEXkgm1=j(it,dim,0);

pteBLINEXkhk=j(it,dim,0);

pteBLINEXkhkb=j(it,dim,0);

pteBLINEXklw=j(it,dim,0);

pteBLINEXkhsl=j(it,dim,0);

pteBLINEXkam=j(it,dim,0);

pteBLINEXkgm=j(it,dim,0);

pteBLINEXkgm1=j(it,dim,0);

een=j(n,1,1);

eenp=j(p,1,1);

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```
eenit=j(it,1,1);

Id=I(n);

Idp=I(p);

std=j(p,1,0);

stdz=j(newp,1,0);

standx=j(n,p,0);

standz=j(n,newp,0);

stdstand=j(n,p,0);

ctr=j(dim,1,0);

beta=j(p,1,0);

epsilon=j(n,1,0);

standeps=j(n,1,0);

e=j(n,1,0);

y=j(n,1,0);

H=j(1,p,1);

*r=2;

q=1;

c=1;

d=1;

a=j(p,1,1);

gamma1=1;

m=n-p;

los=0.05;

Fcrit=Finv(1-los,q,m);

lamb=1;

b=lamb*d;

*print fcrit;
```

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```
*Generate standard normal distributions;
do w=1 to it;
do i=1 to n;
do j=1 to newp;
Z[i,j]=normal(0);
end;
end;
*print z;
zbar=z'*een/n;
dub=(1/n)*een*een';
varz=(1/(n-1))*z'*(id-dub)*z;
diagz=vecdiag(varz);
do k=1 to newp;
stdz[k]=sqrt(diagz[k]);
end;
*print zbar dub varz diagz stdz;
*Standardised z values;
do i=1 to n;
do j=1 to newp;
standz[i,j]=(z[i,j]-zbar[j])/stdz[j];
end;
end;
*print standz;
*Generate the explanatory variables which are correlated with one another;
do i=1 to n;
do j=1 to p;
X[i,j]=sqrt(1-gamma**2)*standZ[i,j]+gamma*standZ[i,newp];
```

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```
end;
end;
*print x;
*Calculate x-bar and standard deviation for every distribution;
xbar=x'*een/n;
dub=(1/n)*een*een';
var=(1/(n-1))*x'*(id-dub)*x;
diag=vecdiag(var);
do k=1 to p;
std[k]=sqrt(diag[k]);
end;
*print xbar dub var diag std;
*Standardised X values;
do i=1 to n;
do j=1 to p;
standx[i,j]=(X[i,j]-xbar[j])/std[j];
end;
end;
*print standx;
/*Check standardisation;*/
xbarstand=standx'*een/n;
dub1=(1/n)*een*een';
varstand=(1/(n-1))*standx'*(id-dub)*standx;
diagstand=vecdiag(varstand);
do k=1 to p;
stdstand[k]=sqrt(diagstand[k]);
end;
```

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```
*print xbarstand diagstand varstand stdstand;
XprimeX=standx'*standx;
invxpx=inv(XprimeX);
*print xprimex invxpx;
eigvector=eigvec(xprimex);
eigvalue=eigval(xprimex);
*print eigvalue eigvector;
do i=1 to p;
lambda[i,i]=eigvalue[i];
end;
*print lambda;
do l=1 to p;
beta[l,1]=eigvector[l,1];
end;
*print beta;
normbet=sqrt(beta'*beta);
*print normbet;
*Generate error terms;
do i=1 to n;
epsilon[i]=normal(0);
end;
*print epsilon;
*Calculate x-bar and standard deviation for epsilon;
xbar1=epsilon'*een/n;
dub2=(1/n)*een*een';
var2=(1/(n-1))*epsilon'*(id-dub)*epsilon;
diag2=vecdiag(var2);
```

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```
*print xbar1 diag2;
*Standardised epsilon values;
do i=1 to n;
standeps[i]=(epsilon[i]-xbar1[1])/sqrt(diag2[1]);
end;
*print standeps;
*Calculate x-bar and standard deviation for standeps;
xbar3=standeps‘*een/n;
dub3=(1/n)*een*een‘;
var3=(1/(n-1))*standeps‘*(id-dub)*standeps;
diag3=vecdiag(var3);
*print xbar3 diag3;
*Calculate e-terms;
do i=1 to n;
e[i]=0+sigma*standeps[i];
end;
*print e;
*Calculate y;
xbeta=standx*beta;
*print xbeta;
do i=1 to n;
y[i]=xbeta[i]+e[i];
end;
transeigvec=eigvector‘;
*print y xbeta e;
alpha=eigvector*beta;
maxalpha=max(alpha);
```

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```
alphasquare=alpha‘*alpha;
*print transeigvec beta alpha;
*alpha maxalpha;
betahat=invxpx*standx‘*y;
betahatsquare=betahat‘*betahat;
*print betahat beta betahatsquare alphasquare;
yhat=standx*betahat;
*print yhat y;
ehat=y-yhat;
*print ehat;
ehatsquare=ehat‘*ehat;
*print ehatsquare;
sigmasquarehat=ehatsquare/(n-p);
*print sigmasquarehat;
Khk=sigmasquarehat/(maxalpha**2);
*print Khk;
Khkb=sigmasquarehat*p/betahatsquare;
*print Khkb;
Klw=p*sigmasquarehat/(betahat‘*lambda*betahat);
*print klw;
*print eigvalue alpha;
prodeigalpha=eigvalue#alpha;
prodeigalphasq=eigvalue#alpha#alpha;
*print prodeigalpha prodeigalphasq;
sqprodea=prodeigalpha#prodeigalpha;
numhsl=sqprodea‘*eenp;
denomhsl=(prodeigalphasq‘*eenp)**2;
```

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```
Khsl=sigmasquarehat*numhsl/denomhsl;
*print sqprodea numhsl denomhsl khsl;
alpsq=alpha#alpha;
eenooralphasq=1/(alpha#alpha);
sumeenooralphasq=eenooralphasq'*eenp;
Kam=(1/p)*sigmasquarehat*sumeenooralphasq;
*print eenooralphasq sumeenooralphasq kam;
alphasq=alpha#alpha;
*print alphasq;
v=1;
do i=1 to p;
v=v*alphasq[i,1];
end;
*print v;
denomgm=v**(1/p);
*print denomgm;
kgm=sigmasquarehat/denomgm;
*print kgm denomgm;
sigdivalphasq=sigmasquarehat/alphasq;
*print sigdivalphasq;
kgm1=median(sigdivalphasq);
*print kgm1 sigdivalphasq;
*Unrestricted ridge regression estimator;
Rkhk=inv(Idp+khk*inv(xprimex));
Rkhkb=inv(Idp+khkb*inv(xprimex));
Rklw=inv(Idp+klw*inv(xprimex));
Rkhsl=inv(Idp+khsl*inv(xprimex));
```

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```
Rkam=inv(Idp+kam*inv(xprimex));

Rkgm=inv(Idp+kgm*inv(xprimex));

Rkgm1=inv(Idp+kgm1*inv(xprimex));

*print rkhk rkhkb rklw rkhsl rkam rkgm rkgm1;

URREkhk=Rkhk*betahat;

URREkhkb=Rkhkb*betahat;

URREklw=Rklw*betahat;

URREkhsl=Rkhsl*betahat;

URREkam=Rkam*betahat;

URREkgm=Rkgm*betahat;

URREkgm1=Rkgm1*betahat;

*print beta betahat urrekhk urrekhkb urreklw urrekhsl urrekam urrekgm urrekgm1;

*RRRE estimators;

ct=0;

do r=be to en by step;

ct=ct+1;

*print r ct;

RRE=betahat-inv(xprimex)*H'*inv(H*inv(xprimex)*H')*(H*betahat-r);

*print rre;

RRREkhk=Rkhk*rre;

RRREkhkb=Rkhkb*rre;

RRREklw=Rklw*rre;

RRREkhsl=Rkhsl*rre;

RRREkam=Rkam*rre;

RRREkgm=Rkgm*rre;

RRREkgm1=Rkgm1*rre;

*print beta betahat rrrekhk rrrekhkb rrreklw rrrekhsl rrrekam rrrekgm rrrekgm1;
```

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```
*s-squared ];
ssq=(1/(m))*(y-standx*betahat)'*(y-standx*betahat);
*likelyhood ratio test;
lkhd=((H*betahat-r)'*inv(H*inv(xprimex)*H')*(H*betahat-r))/(q*ssq);
if lkhd<fcrit then INRF1=1;
if lkhd>=fcrit then INRF1=0;
*print fcrit lkhk inrf;
*PTE ridge estimator;
pte1=betahat-(betahat-rre)*inrf1;
ptekhk=rkhk*pte1;
ptekhkb=rkhkb*pte1;
pteklw=rklw*pte1;
ptekhsl=rkhsl*pte1;
ptekam=rkam*pte1;
ptekgm=rkgm*pte1;
ptekgm1=rkgm1*pte1;
*Calculate reflected normal loss values for different k;
urreRNLkhk[w,ct]=c*(1-exp((urrekhk-beta)'*(urrekhk-beta)/(-2*gamma1**2)));
urreRNLkhkb[w,ct]=c*(1-exp((urrekhkb-beta)'*(urrekhkb-beta)/(-2*gamma1**2)));
urreRNLklw[w,ct]=c*(1-exp((urreklw-beta)'*(urreklw-beta)/(-2*gamma1**2)));
urreRNLkhsl[w,ct]=c*(1-exp((urrekhsl-beta)'*(urrekhsl-beta)/(-2*gamma1**2)));
urreRNLkam[w,ct]=c*(1-exp((urrekam-beta)'*(urrekam-beta)/(-2*gamma1**2)));
urreRNLkgm[w,ct]=c*(1-exp((urrekgm-beta)'*(urrekgm-beta)/(-2*gamma1**2)));
urreRNLkgm1[w,ct]=c*(1-exp((urrekgm1-beta)'*(urrekgm1-beta)/(-2*gamma1**2)));
rrreRNLkhk[w,ct]=c*(1-exp((rrrekhk-beta)'*(rrrekhk-beta)/(-2*gamma1**2)));
rrreRNLkhkb[w,ct]=c*(1-exp((rrrekhkb-beta)'*(rrrekhkb-beta)/(-2*gamma1**2)));
rrreRNLklw[w,ct]=c*(1-exp((rrreklw-beta)'*(rrreklw-beta)/(-2*gamma1**2)));
```

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```
rrreRNLkhsl[w,ct]=c*(1-exp((rrrekhsl-beta)'*(rrrekhsl-beta)/(-2*gamma1**2)));
rrreRNLkam[w,ct]=c*(1-exp((rrrekam-beta)'*(rrrekam-beta)/(-2*gamma1**2)));
rrreRNLkgm[w,ct]=c*(1-exp((rrrekgm-beta)'*(rrrekgm-beta)/(-2*gamma1**2)));
rrreRNLkgm1[w,ct]=c*(1-exp((rrrekgm1-beta)'*(rrrekgm1-beta)/(-2*gamma1**2)));
pteRNLkhk[w,ct]=c*(1-exp((ptekhk-beta)'*(ptekhk-beta)/(-2*gamma1**2)));
pteRNLkhkb[w,ct]=c*(1-exp((ptekhkb-beta)'*(ptekhkb-beta)/(-2*gamma1**2)));
pteRNLklw[w,ct]=c*(1-exp((pteklw-beta)'*(pteklw-beta)/(-2*gamma1**2)));
pteRNLkhsl[w,ct]=c*(1-exp((ptekhsl-beta)'*(ptekhsl-beta)/(-2*gamma1**2)));
pteRNLkam[w,ct]=c*(1-exp((ptekam-beta)'*(ptekam-beta)/(-2*gamma1**2)));
pteRNLkgm[w,ct]=c*(1-exp((ptekgm-beta)'*(ptekgm-beta)/(-2*gamma1**2)));
pteRNLkgm1[w,ct]=c*(1-exp((ptekgm1-beta)'*(ptekgm1-beta)/(-2*gamma1**2)));
*Calculate LINEX loss values for different k;
urreLINEXkhk[w,ct]=d*(exp(a'*(urrekhk-beta))-a'*(urrekhk-beta)-1);
urreLINEXkhkb[w,ct]=d*(exp(a'*(urrekhkb-beta))-a'*(urrekhkb-beta)-1);
urreLINEXklw[w,ct]=d*(exp(a'*(urreklw-beta))-a'*(urreklw-beta)-1);
urreLINEXkhsl[w,ct]=d*(exp(a'*(urrekhsl-beta))-a'*(urrekhsl-beta)-1);
urreLINEXkam[w,ct]=d*(exp(a'*(urrekam-beta))-a'*(urrekam-beta)-1);
urreLINEXkgm[w,ct]=d*(exp(a'*(urrekgm-beta))-a'*(urrekgm-beta)-1);
urreLINEXkgm1[w,ct]=d*(exp(a'*(urrekgm1-beta))-a'*(urrekgm1-beta)-1);
rrreLINEXkhk[w,ct]=d*(exp(a'*(rrrekhk-beta))-a'*(rrrekhk-beta)-1);
rrreLINEXkhkb[w,ct]=d*(exp(a'*(rrrekhkb-beta))-a'*(rrrekhkb-beta)-1);
rrreLINEXklw[w,ct]=d*(exp(a'*(rrreklw-beta))-a'*(rrreklw-beta)-1);
rrreLINEXkhsl[w,ct]=d*(exp(a'*(rrrekhsl-beta))-a'*(rrrekhsl-beta)-1);
rrreLINEXkam[w,ct]=d*(exp(a'*(rrrekam-beta))-a'*(rrrekam-beta)-1);
rrreLINEXkgm[w,ct]=d*(exp(a'*(rrrekgm-beta))-a'*(rrrekgm-beta)-1);
rrreLINEXkgm1[w,ct]=d*(exp(a'*(rrrekgm1-beta))-a'*(rrrekgm1-beta)-1);
pteLINEXkhk[w,ct]=d*(exp(a'*(ptekhk-beta))-a'*(ptekhk-beta)-1);
```

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```
pteLINEXkhkb[w,ct]=d*(exp(a‘*(ptekhkb-beta))-a‘*(ptekhkb-beta)-1);
pteLINEXklw[w,ct]=d*(exp(a‘*(pteklw-beta))-a‘*(pteklw-beta)-1);
pteLINEXkhsl[w,ct]=d*(exp(a‘*(ptekhsl-beta))-a‘*(ptekhsl-beta)-1);
pteLINEXkam[w,ct]=d*(exp(a‘*(ptekam-beta))-a‘*(ptekam-beta)-1);
pteLINEXkgm[w,ct]=d*(exp(a‘*(ptekgm-beta))-a‘*(ptekgm-beta)-1);
pteLINEXkgm1[w,ct]=d*(exp(a‘*(ptekgm1-beta))-a‘*(ptekgm1-beta)-1);
*Calculate BLINEX loss for different k;
urreBLINEXkhk[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhk-beta))-a‘*(urrekhk-beta)-1)));
urreBLINEXkhkb[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhkb-beta))-a‘*(urrekhkb-beta)-1)));
urreBLINEXklw[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urreklw-beta))-a‘*(urreklw-beta)-1)));
urreBLINEXkhsl[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekhsl-beta))-a‘*(urrekhsl-beta)-1)));
urreBLINEXkam[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekam-beta))-a‘*(urrekam-beta)-1)));
urreBLINEXkgm[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekgm-beta))-a‘*(urrekgm-beta)-1)));
urreBLINEXkgm1[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(urrekgm1-beta))-a‘*(urrekgm1-beta)-1)));
rrreBLINEXkhk[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhk-beta))-a‘*(rrrekhk-beta)-1)));
rrreBLINEXkhkb[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhkb-beta))-a‘*(rrrekhkb-beta)-1)));
rrreBLINEXklw[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrreklw-beta))-a‘*(rrreklw-beta)-1)));
rrreBLINEXkhsl[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekhsl-beta))-a‘*(rrrekhsl-beta)-1)));
rrreBLINEXkam[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekam-beta))-a‘*(rrrekam-beta)-1)));
rrreBLINEXkgm[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekgm-beta))-a‘*(rrrekgm-beta)-1)));
rrreBLINEXkgm1[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(rrrekgm1-beta))-a‘*(rrrekgm1-beta)-1)));
pteBLINEXkhk[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhk-beta))-a‘*(ptekhk-beta)-1)));
pteBLINEXkhkb[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhkb-beta))-a‘*(ptekhkb-beta)-1)));
pteBLINEXklw[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(pteklw-beta))-a‘*(pteklw-beta)-1)));
pteBLINEXkhsl[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekhsl-beta))-a‘*(ptekhsl-beta)-1)));
pteBLINEXkam[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekam-beta))-a‘*(ptekam-beta)-1)));
pteBLINEXkgm[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekgm-beta))-a‘*(ptekgm-beta)-1)));
```

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```
pteBLINEXkgm1[w,ct]=1/lamb*(1-1/(1+b*(exp(a‘*(ptekgm1-beta))-a‘*(ptekgm1-beta)-1)));
riskrrrernlkgm=rrrernlkgm[:,] ;
riskrrrernlkgm1=riskrrrernlkgm‘;
riskrrreLINEXkhk=rrreLINEXkhk[:,];
riskrrreLINEXkhk1=riskrrreLINEXkhk‘;
riskrrreBLINEXkhk=rrreBLINEXkhk[:,];
riskrrreBLINEXkhk1=riskrrreBLINEXkhk‘;
ctr[ct]=r;
end;
end;
lm=(ctr||riskrrrernlkhk1 ||riskrrreLINEXkhk1 || riskrrreBLINEXkhk1);
*print lm;
nm1={"r" "riskrrrernlkhk" "riskrrreLINEXkhk" "riskrrreBLINEXkhk"};
create plotdata from lm[colname=nm1];
append from lm;
*print lm[colname=nm1];
quit;
symbol1 value=dot
height=.1;
proc gplot data=plotdata;
plot (riskrrrernlkhk riskrrreLINEXkhk riskrrreBLINEXkhk)*r/overlay;
run;
```

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Appendix C Program 3

Note: Programs can be more streamlined, but that was not the purpose of the study.

```
proc iml worksize=3000000;

a=1;

d=1;

lambda=1;

b=d*lambda;

sigma=5;

priormu=13;

priorsigma=1;

n=100;

alpha=0.05;

mu=15;

p=10000;

df=n-1;

Z=j(n,p,0);

X=j(n,p,0);

een=j(n,1,1);

eenp=j(p,1,1);

Id=I(n);

varvec=j(p,1,0);

tstat=j(p,1,0);

tstatfb=j(p,1,0);

pte=j(p,1,0);

ptefb=j(p,1,0);

count=j(p,1,0);

denom=j(p,1,0);
```

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```
denomfb=j(p,1,0);
umleb=j(p,1,0);
rmleb=j(p,1,0);
umlel=j(p,1,0);
rmlel=j(p,1,0);
fbe=j(p,1,0);
stdfb=j(p,1,0);
std=j(p,1,0);
losspte=j(p,1,0);
lossptefb=j(p,1,0);
lossumle=j(p,1,0);
lossrmle=j(p,1,0);
lossumlel=j(p,1,0);
lossrmlel=j(p,1,0);
linex=j(p,1,0);
tinvl=tinv(alpha/2,df);
tinvr=tinv(1-(alpha/2),df);
*Generate standard normal distributions;
do i=1 to n;
do j=1 to p;
Z[i,j]=normal(10);
end;
end;
*Generate any normal distribution;
do i=1 to n;
do j=1 to p;
X[i,j]=mu+sigma*Z[i,j];
```

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```
end;
end;
*Calculate x-bar and standard deviation for every distribution;
xbar=x'*een/n;
dub=(1/n)*een*een';
var=(1/(n-1))*x'*(id-dub)*x;
diag=vecdiag(var);
do k=1 to p;
std[k]=sqrt(diag[k]);
end;
*Calculate feasible Bayes estimator and sit standard deviation;
do g=1 to p;
fbe[g]=(n*xbar[g]+(priormu*sigma**2/priorsigma**2))/((sigma**2/priorsigma**2)+n);
stdfb[g]=sqrt(n*std[g]**2/((std[g]**2/priorsigma**2)+n));
end;
*Calculate the t-test statistic for every distribution';
do muo=14 to 16 by 0.001;
do l=1 to p;
tstat[l]=((xbar[l]-muo)*sqrt(n))/std[l];
tstatfb[l]=((fbe[l]-muo)*sqrt(n))/stdfb[l];
end;
*Determine in which cases the null hypothesis is rejected and calculate the classical and
feasible Bayes preliminary test estimators;
do m=1 to p;
if tstat[m]>=tinv(1-(alpha/2),df) then pte[m]=xbar[m];
if tstat[m]<=tinv(alpha/2,df) then pte[m]=xbar[m];
if tstat[m]>tinv(alpha/2,df) & tstat[m]<tinv(1-(alpha/2),df) then pte[m]=muo;
```

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```
if tstatfb[m]>=tinv(1-(alpha/2),df) then ptefb[m]=fbe[m];
if tstatfb[m]<=tinv(alpha/2,df) then ptefb[m]=fbe[m];
if tstatfb[m]>tinv(alpha/2,df) & tstat[m]<tinv(1-(alpha/2),df) then ptefb[m]=muo;
end;
*Calculate all the loss functions';
do t=1 to p;
denom[t]=1+b*(exp(a*(pte[t]-mu))-a*(pte[t]-mu)-1);
denomfb[t]=1+b*(exp(a*(ptefb[t]-mu))-a*(ptefb[t]-mu)-1);
linex[t]=d*(exp(a*(pte[t]-mu))-a*(pte[t]-mu)-1);
rmleb[t]=1+b*(exp(a*(muo-mu))-a*(muo-mu)-1);
umleb[t]=1+b*(exp(a*(xbar[t]-mu))-a*(xbar[t]-mu)-1);
rmlel[t]=d*(exp(a*(muo-mu))-a*(muo-mu)-1);
umlel[t]=d*(exp(a*(xbar[t]-mu))-a*(xbar[t]-mu)-1);
end;
do s=1 to p;
losspte[s]=(1/lambda)*(1-(1/denom[s]));
lossptefb[s]=(1/lambda)*(1-(1/denomfb[s]));
lossrmle[s]=(1/lambda)*(1-(1/rmleb[s]));
lossumle[s]=(1/lambda)*(1-(1/umleb[s]));
*print losspte lossptefb;
end;
estblinexpte=losspte`*eenp;
estblinexptefb=lossptefb`*eenp;
estlinexpte=linex`*eenp;
estumleb=lossumle`*eenp;
estrmleb=lossrmle`*eenp;
estrmlel=rmlel`*eenp;
```

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```
estumlel=umlel'*eenp;

*Calculate the average loss (risk function);

delta=muo-mu;

pteblinexrisk=estblinexpte/p;

ptelinexrisk=estlinexpte/p;

ptebriskfb=estblinexptefb/p;

umleriskb=estumleb/p;

rmleriskb=estrmleb/p;

umleriskl=estumlel/p;

rmleriskl=estrmlel/p;

lm=lm//(muo||delta||ptebriskfb||pteblinexrisk||ptelinexrisk||umleriskb||umleriskl||rmleriskb||rmleriskl);

end;

nm1={"muo" "delta" "ptebriskfb" "pteblinexrisk" "ptelinexrisk" "umleriskb" "umleriskl" "rmleriskb"
"rmleriskl"};

create plotdata from lm[colname=nm1];

append from lm;

*print lm[colname=nm1];

quit;

symbol1 value=dot

height=.1;

proc gplot data=plotdata;

plot (pteblinexrisk ptelinexrisk)*delta/overlay;

plot (pteblinexrisk umleriskb rmleriskb)*delta/overlay;

plot (umleriskb umleriskl)*delta/overlay;

plot (rmleriskb rmleriskl)*delta/overlay;

run;
```

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Appendix D Program 4

Note: Programs can be more streamlined, but that was not the purpose of the study.

```
data cobbdouglas;

input output capital labour ;

datalines;
180852      294150      1467259.912
180556      287093      1490590.9
184537      277872      1535956.71
196545      275321      1591691.848
200217      282502      1594284.18
195717      296760      1599468.844
186776      307645      1563176.196
180651      314255      1522995.05
180322      321247      1491887.066
185135      328843      1503552.56
197156      340574      1512625.722
199882      352327      1456890.584
205293      362443      1395970.782
204794      369342      1350604.972
205974      373618      1315608.49
222666      375926      1296166
229698      378609      1262465.684
236133      377934      1254688.688
232581      377046      1250800.19
243965      381602      1248207.858
259101      388486      1185991.89
275782      399450      1297462.166
```

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```
290246      411555      1319496.988
297889      425733      1298758.332
267723      421818      1211915.21
282509      418402      1170437.898
292733      419091      1153587.74
299705      420081      1150995.408
;
data judy;
set cobbdouglas;
logout=log(output);
logcap=log(capital);
loglab=log(labour);
/*
proc reg data=judy outest=b ridge=0 to 1 by 0.01;
model logout=logcap loglab/ vif;
run;
*/
**********IML**********;
proc iml;
use judy;
read all into xy;
n=nrow(xy);
x=j(n,1,1)||xy[,5]||xy[,6];
y=xy[,4];
T=n;
id=I(3);
H1={0 1 1};
```

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```
c=1;
gamma1=1;
h2=1;
los=0.05;
q=1;
m=nrow(x)-ncol(x);
p=ncol(x);
beta=j(p,1,0);
eenp=j(p,1,1);
eenn=j(n,1,1);
eigenvector=j(p,p,0);
lambda=j(p,p,0);
Fcrit=Finv(1-los,q,m);
lamb=1;
d=1;
b=d*lamb;
a=j(p,1,1);
*print los q m fcrit H3;
start estimation;
logoutput= yb;
loginput=      xb;
loginput1= xb1;
ybar=(eenn'*logoutput)/n;
betat=inv(loginput'*loginput)*loginput'*logoutput;
e_h=logoutput - loginput* betat;
samsigma=1/m*(e_h'*e_h);
XprimeX=loginput'*loginput;
```

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```
eigenvector=eigvec(xprimex);
eigvalue=eigval(xprimex);
do i=1 to p;
lambda[i,i]=eigvalue[i];
end;
do l=1 to p;
beta[l,1]=eigenvector[l,1];
end;
***************Ridge parameters*******************;
alpha=eigenvector*beta;
maxalpha=max(alpha);
khk=samsigma/(maxalpha**2);
alphasquare=alpha`*alpha;
betahatsquare=betat`*betat;
ehatsquare=e_h`*e_h;
sigmasquarehat=ehatsquare/(n-p);
Khkb=sigmasquarehat*p/betahatsquare;
Klw=p*sigmasquarehat/(betat`*lambda*betat);
prodeigalpha=eigvalue#alpha;
prodeigalphasq=eigvalue#alpha#alpha;
sqprodea=prodeigalpha#prodeigalpha;
numhsl=sqprodea`*eenp;
denomhsl=(prodeigalphasq`*eenp)**2;
Khsl=sigmasquarehat*numhsl/denomhsl;
alpsq=alpha#alpha;
eenooralphasq=1/(alpha#alpha);
sumeenooralphasq=eenooralphasq`*eenp;
```

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```
Kam=(1/p)*sigmasquarehat*sumeenooralphasq;
alphasq=alpha#alpha;
v=1;
do i=1 to p;
v=v*alphasq[i,1];
end;
denomgm=v**(1/p);
kgm=sigmasquarehat/denomgm;
sigdivalphasq=sigmasquarehat/alphasq;
kgm1=median(sigdivalphasq);
*******URRE*******;
URREkhk=inv(id+khk*inv(loginput'*loginput))*betat;
URREkhkb=inv(id+khkb*inv(loginput'*loginput))*betat;
URREklw=inv(id+klw*inv(loginput'*loginput))*betat;
URREkhsl=inv(id+khsl*inv(loginput'*loginput))*betat;
URREkam=inv(id+kam*inv(loginput'*loginput))*betat;
URREkgm=inv(id+kgm*inv(loginput'*loginput))*betat;
URREkgm1=inv(id+kgm1*inv(loginput'*loginput))*betat;
RsqURREkhk=(urrekhk'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjkhk=1-(1-rsqurrekhk)*(n-1)/(n-p);
RsqURREkhkb=(urrekhkb'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjkhkb=1-(1-rsqurrekhkb)*(n-1)/(n-p);
RsqURREklw=(urreklw'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjklw=1-(1-rsqurreklw)*(n-1)/(n-p);
RsqURREkhsl=(urrekhsl'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjkhsl=1-(1-rsqurrekhsl)*(n-1)/(n-p);
RsqURREkam=(urrekam'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
```

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```
adjkam=1-(1-rsqurrekam)*(n-1)/(n-p);
RsqURREkgm=(urrekgm'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjkgm=1-(1-rsqurrekgm)*(n-1)/(n-p);
RsqURREkgm1=(urrekgm1'*loginput'*logoutput-n*ybar**2)/(logoutput'*logoutput-n*ybar**2);
adjkgm1=1-(1-rsqurrekgm1)*(n-1)/(n-p);
Rkhk=inv(id+khk*inv(loginput'*loginput));
Rkhkb=inv(id+khkb*inv(loginput'*loginput));
Rklw=inv(id+klw*inv(loginput'*loginput));
Rkhsl=inv(id+khsl*inv(loginput'*loginput));
Rkam=inv(id+kam*inv(loginput'*loginput));
Rkgm=inv(id+kgm*inv(loginput'*loginput));
Rkgm1=inv(id+kgm1*inv(loginput'*loginput));
A=inv(loginput'*loginput)-inv(loginput'*loginput)*H1'
*inv(H1*inv(loginput'*loginput)*H1')*H1*inv(loginput'*loginput);
varUrrekhk=samsigma*Rkhk*inv(loginput'*loginput)*Rkhk';
varUrrekhkb=samsigma*Rkhkb*inv(loginput'*loginput)*Rkhkb';
varUrreklw=samsigma*Rklw*inv(loginput'*loginput)*Rklw';
varUrrekhsl=samsigma*Rkhsl*inv(loginput'*loginput)*Rkhsl';
varUrrekam=samsigma*Rkam*inv(loginput'*loginput)*Rkam';
varUrrekgm=samsigma*Rkgm*inv(loginput'*loginput)*Rkgm';
varUrrekgm1=samsigma*Rkgm1*inv(loginput'*loginput)*Rkgm1';
/******************RRRE***************;
betah=betat-inv(loginput'*loginput)*H1'*inv(H1*inv(loginput'*loginput)*H1')*(H1*betat-h2);
RRREkhk=inv(id+khk*inv(loginput'*loginput))*betah;
********************PTE*************************************;
I=((H1*betat-h2)'*inv(H1*inv(loginput'*loginput)*H1')*(H1*betat-h2))/samsigma;
*************code to calculate the critical value*****************;
```

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```
if l<fcrit then INRF=1;
if l>=fcrit then INRF=0;
*******************************;
betapte=betat-(betat-betah)*INRF;
PTRREkhk=inv(id+khk*inv(loginput'*loginput))*betapte;
*/
***************Loss functions for estimators******;
urrekhkRNL=c*(1-exp((urrekhk-beta)'*(urrekhk-beta)/(-2*gamma1**2)));
urrekhkbRNL=c*(1-exp((urrekhkb-beta)'*(urrekhkb-beta)/(-2*gamma1**2)));
urreklwRNL=c*(1-exp((urreklw-beta)'*(urreklw-beta)/(-2*gamma1**2)));
urrekhslRNL=c*(1-exp((urrekhsl-beta)'*(urrekhsl-beta)/(-2*gamma1**2)));
urrekamRNL=c*(1-exp((urrekam-beta)'*(urrekam-beta)/(-2*gamma1**2)));
urrekgmRNL=c*(1-exp((urrekgm-beta)'*(urrekgm-beta)/(-2*gamma1**2)));
urrekgm1RNL=c*(1-exp((urrekgm1-beta)'*(urrekgm1-beta)/(-2*gamma1**2)));
urrekhkLINEXa=d*(exp(a'*(urrekhk-beta))-a'*(urrekhk-beta)-1);
urrekhkbLINEXa=d*(exp(a'*(urrekhkb-beta))-a'*(urrekhkb-beta)-1);
urreklwLINEX=d*(exp(a'*(urreklw-beta))-a'*(urreklw-beta)-1);
urrekhslLINEX=d*(exp(a'*(urrekhsl-beta))-a'*(urrekhsl-beta)-1);
urrekamLINEX=d*(exp(a'*(urrekam-beta))-a'*(urrekam-beta)-1);
urrekgmLINEX=d*(exp(a'*(urrekgm-beta))-a'*(urrekgm-beta)-1);
urrekgm1LINEX=d*(exp(a'*(urrekgm1-beta))-a'*(urrekgm1-beta)-1);
urrekhkBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekhk-beta))-a'*(urrekhk-beta)-1)));
urrekhkbBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekhkb-beta))-a'*(urrekhkb-beta)-1)));
urreklwBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urreklw-beta))-a'*(urreklw-beta)-1)));
urrekhslBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekhsl-beta))-a'*(urrekhsl-beta)-1)));
urrekamBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekam-beta))-a'*(urrekam-beta)-1)));
urrekgmBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekgm-beta))-a'*(urrekgm-beta)-1)));
```

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```
urrekgm1BLINEX=1/lamb*(1-1/(1+b*(exp(a'*(urrekgm1-beta))-a'*(urrekgm1-beta)-1)));
*rrrekhkBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(rrrekhk-beta))-a'*(rrrekhk-beta)-1)));
*ptrrekhkBLINEX=1/lamb*(1-1/(1+b*(exp(a'*(ptrrekhk-beta))-a'*(ptrrekhk-beta)-1)));
finish estimation;
**********Blocks Bootstrap (non-overlapping)**********;
length=4; *read on optimal length selection;
blocks=int(n*1/length);
start bootstrap;
pp=int(uniform(j(blocks,1,999))#blocks)+1;
xbt=j(n,2,9999999999);
ybt=j(n,1,9999999999);

do NBB=1 to blocks;
xbx=x[(pp[NBB,])*length-length+1:(pp[NBB,])*length,2:3];
xbt[nbb*length-length+1:NBB*length,1:2]=xbx;
yby=y[(pp[NBB,])*length-length+1:(pp[NBB,])*length,];
ybt[nbb*length-length+1:NBB*length,1]=yby;
end;
finish bootstrap;
bs=1000;
xbb=j(n,bs,.);
xbb2=j(n,bs,.);
ybb=j(n,bs,.);
do boot=1 to bs;
call bootstrap;
xbb[,boot]=xbt[,1];
xbb2[,boot]=xbt[,2];
```

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```
ybb[,boot]=ybt;
xb=j(n,1,1)||xbb[,boot]||xbb2[,boot];
xb1=xbb[,boot]||xbb2[,boot];
yb=ybb[,boot];
call estimation;
URREkhk2=URREkhk2//URREkhk[2,];
URREkhkb2=URREkhkb2//URREkhkb[2,];
URREklw2=URREklw2//URREklw[2,];
URREkhsl2=URREkhsl2//URREkhsl[2,];
URREkam2=URREkam2//URREkam[2,];
URREkgm2=URREkgm2//URREkgm[2,];
URREkgm12=URREkgm12//URREkgm1[2,];
RsqURREkhkv=RsqURREkhkv//RsqURREkhk[1,];
RsqURREkhkbv=RsqURREkhkbv//RsqURREkhkb[1,];
RsqURREklwv=RsqURREklwv//RsqURREklw[1,];
RsqURREkhslv=RsqURREkhslv//RsqURREkhsl[1,];
RsqURREkamv=RsqURREkamv//RsqURREkam[1,];
RsqURREkgmv=RsqURREkgmv//RsqURREkgm[1,];
RsqURREkgm1v=RsqURREkgm1v//RsqURREkgm1[1,];
adjkhkv=adjkhkv//adjkhk[1,];
adjkhkbv=adjkhkbv//adjkhkb[1,];
adjklwv=adjklwv//adjklw[1,];
adjkhslv=adjkhslv//adjkhsl[1,];
adjkamv=adjkamv//adjkam[1,];
adjkgmv=adjkgmv//adjkgm[1,];
adjkgm1v=adjkgm1v//adjkgm1[1,];
URREkhkRNLv=URREkhkRNLv//URREkhkRNL[1,];
```

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```
URREkhkbRNLv=URREkhkbRNLv//URREkhkbRNL[1,];
URREklwRNLv=URREklwRNLv//URREklwRNL[1,];
URREkhslRNLv=URREkhslRNLv//URREkhslRNL[1,];
URREkamRNLv=URREkamRNLv//URREkamRNL[1,];
URREkgmRNLv=URREkgmRNLv//URREkgmRNL[1,];
URREkgm1RNLv=URREkgm1RNLv//URREkgm1RNL[1,];
URREkhkLINEXva=URREkhkLINEXva//URREkhkLINEXa[1,];
URREkhkbLINEXva=URREkhkbLINEXva//URREkhkbLINEXa[1,];
URREklwLINEXv=URREklwLINEXv//URREklwLINEX[1,];
URREkhslLINEXv=URREkhslLINEXv//URREkhslLINEX[1,];
URREkamLINEXv=URREkamLINEXv//URREkamLINEX[1,];
URREkgmLINEXv=URREkgmLINEXv//URREkgmLINEX[1,];
URREkgm1LINEXv=URREkgm1LINEXv//URREkgm1LINEX[1,];
URREkhkBLINEXv=URREkhkBLINEXv//URREkhkBLINEX[1,];
URREkhkbBLINEXv=URREkhkbBLINEXv//URREkhkbBLINEX[1,];
URREklwBLINEXv=URREklwBLINEXv//URREklwBLINEX[1,];
URREkhslBLINEXv=URREkhslBLINEXv//URREkhslBLINEX[1,];
URREkamBLINEXv=URREkamBLINEXv//URREkamBLINEX[1,];
URREkgmBLINEXv=URREkgmBLINEXv//URREkgmBLINEX[1,];
URREkgm1BLINEXv=URREkgm1BLINEXv//URREkgm1BLINEX[1,];
varurrevkhk=varurrevkhk//varurrekhk[2,3];
varurrevkhkb=varurrevkhkb//varurrekhkb[2,3];
varurrevklw=varurrevklw//varurreklw[2,3];
varurrevkhsl=varurrevkhsl//varurrekhsl[2,3];
varurrevkam=varurrevkam//varurrekam[2,3];
varurrevkgm=varurrevkgm//varurrekgm[2,3];
varurrevkgm1=varurrevkgm1//varurrekgm1[2,3];
```

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```
end;
ng=20; *number of groups;
width1=(max(URREkhk2)-min(URREkhk2))/ng;
width2=(max(URREkhkb2)-min(URREkhkb2))/ng;
width3=(max(URREklw2)-min(URREklw2))/ng;
width4=(max(URREkhsl2)-min(URREkhsl2))/ng;
width5=(max(URREkam2)-min(URREkam2))/ng;
width6=(max(URREkgm2)-min(URREkgm2))/ng;
width7=(max(URREkgm12)-min(URREkgm12))/ng;
do i=0 to bs;
ii1=min(URREkhk2)+width1*i;
ii2=min(URREkhkb2)+width2*i;
ii3=min(URREklw2)+width3*i;
ii4=min(URREkhsl2)+width4*i;
ii5=min(URREkam2)+width5*i;
ii6=min(URREkgm2)+width6*i;
ii7=min(URREkgm12)+width7*i;
emp1=emp1//ii1;
emp2=emp2//ii2;
emp3=emp3//ii3;
emp4=emp4//ii4;
emp5=emp5//ii5;
emp6=emp6//ii6;
emp7=emp7//ii7;
end;
nm=nrow(emp1);
Frequency_URREkhk2=j(nm,1,.);
```

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```
Frequency_URREkhkb2=j(nm,1,.);
Frequency_URREklw2=j(nm,1,.);
Frequency_URREkhsl2=j(nm,1,.);
Frequency_URREkam2=j(nm,1,.);
Frequency_URREkgm2=j(nm,1,.);
Frequency_URREkgm12=j(nm,1,.);
do j=2 to nm;
ckhk=0;
ckhk1=0;
ckhkb=0;
ckhkb1=0;
cklw=0;
cklw1=0;
ckhsl=0;
ckhsl1=0;
ckam=0;
ckam1=0;
ckgm=0;
ckgm1=0;
ckgm1a=0;
ckgm11a=0;
do k=1 to bs;
if URREkhk2[k,]<=emp1[1,] then ckhk1=ckhk1+1;
Frequency_URREkhk2[1,]=ckhk1/bs;
if URREkhkb2[k,]<=emp2[1,] then ckhkb1=ckhkb1+1;
Frequency_URREkhkb2[1,]=ckhkb1/bs;
if URREklw2[k,]<=emp3[1,] then cklw1=cklw1+1;
```

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```
Frequency_URREklw2[1,]=cklw1/bs;
if URREkhsl2[k,]<=emp4[1,] then ckhsl1=ckhsl1+1;
Frequency_URREkhsl2[1,]=ckhsl1/bs;
if URREkam2[k,]<=emp5[1,] then ckam1=ckam1+1;
Frequency_URREkam2[1,]=ckam1/bs;
if URREkgm2[k,]<=emp6[1,] then ckgm1=ckgm1+1;
Frequency_URREkgm2[1,]=ckgm1/bs;
if URREkgm12[k,]<=emp7[1,] then ckgm11a=ckgm11a+1;
Frequency_URREkgm12[1,]=ckgm11a/bs;
if emp1[j-1,]<URREkhk2[k,] & URREkhk2[k,]<=emp1[j,] then ckhk=ckhk+1;
Frequency_URREkhk2[j,]=ckhk/bs;
if emp2[j-1,]<URREkhkb2[k,] & URREkhkb2[k,]<=emp2[j,] then ckhkb=ckhkb+1;
Frequency_URREkhkb2[j,]=ckhkb/bs;
if emp3[j-1,]<URREklw2[k,] & URREklw2[k,]<=emp3[j,] then cklw=cklw+1;
Frequency_URREklw2[j,]=cklw/bs;
if emp4[j-1,]<URREkhsl2[k,] & URREkhsl2[k,]<=emp4[j,] then ckhsl=ckhsl+1;
Frequency_URREkhsl2[j,]=ckhsl/bs;
if emp5[j-1,]<URREkam2[k,] & URREkam2[k,]<=emp5[j,] then ckam=ckam+1;
Frequency_URREkam2[j,]=ckam/bs;
if emp6[j-1,]<URREkgm2[k,] & URREkgm2[k,]<=emp6[j,] then ckgm=ckgm+1;
Frequency_URREkgm2[j,]=ckgm/bs;
if emp7[j-1,]<URREkgm12[k,] & URREkgm12[k,]<=emp7[j,] then ckgm1a=ckgm1a+1;
Frequency_URREkgm12[j,]=ckgm11a/bs;
end;
end;
qqq=emp1[1:ng+1,]||frequency_URREkhk2[1:ng+1,]||emp2[1:ng+1,]||frequency_URREkhkb2[1:ng+1,]
||emp3[1:ng+1,]||frequency_URREklw2[1:ng+1,]||emp4[1:ng+1,]||frequency_URREkhsl2[1:ng+1,]
```

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```
||emp5[1:ng+1,]||frequency_URREkam2[1:ng+1,]||emp6[1:ng+1,]||frequency_URREkgm2[1:ng+1,]
||emp7[1:ng+1,]||frequency_URREkgm12[1:ng+1,];
*print qqq;
qqq1=adjkhkv||adjkhkbv||adjklwv||adjkhslv||adjkamv||adjkgmv||adjkgm1v;
qqq2=URREkhkRNLv||URREkhkbRNLv||URREklwRNLv||URREkhslRNLv||URREkamRNLv
||URREkgmRNLv||URREkgm1RNLv
||URREkhkLINEXva||URREkhkbLINEXva||URREklwLINEXv||URREkhslLINEXv||URREkamLINEXv
||URREkgmLINEXv||URREkgm1LINEXv
||URREkhkBLINEXv||URREkhkbBLINEXv||URREklwBLINEXv||URREkhslBLINEXv
||URREkamBLINEXv||URREkgmBLINEXv||URREkgm1BLINEXv;
qqq3=URREkhk2||URREkhkb2||URREklw2||URREkhsl2||URREkam2||URREkgm2||URREkgm12;
qqq4=varurrevkhk||varurrevkhkb||varurrevklw||varurrevkhsl||varurrevkam||varurrevkgm||varurrevkgm1;
create gogo from qqq[colname= {'URREkhk2' 'relativefreq' 'URREkhkb2' 'freq2' 'URREklw2' 'freq3'
'URREkhsl2' 'freq4' 'URREkam2' 'freq5' 'URREkgm2' 'freq6' 'URREkgm12' 'freq7'  'RsqURREkhk'}];
append from qqq;
create gogo1 from qqq1[colname= {'RsqURREkhk' 'RsqURREkhkb' 'RsqURREklw' 'RsqURREkhsl'
'RsqURREkam'
'RsqURREkgm' 'RsqURREkgm1'}];
append from qqq1;
create gogo3 from qqq3[colname= {'URREkhk2' 'URREkhkb2' 'URREklw2' 'URREkhsl2' 'UR-
REkam2' 'URREkgm2' 'URREkgm12'}];
append from qqq3;
create gogo2 from qqq2[colname= {'URREkhkRNL' 'URREkhkbRNL' 'URREklwRNL' 'URREkhslRNL'
'URREkamRNL' 'URREkgmRNL' 'URREkgm1RNL'
'URREkhkLINEXa' 'URREkhkbLINEXa' 'URREklwLINEX' 'URREkhslLINEX' 'URREkamLINEX'
'URREkgmLINEX' 'URREkgm1LINEX'
'URREkhkBLINEX' 'URREkhkbBLINEX' 'URREklwBLINEX' 'URREkhslBLINEX' 'URREkamBLINEX'
'URREkgmBLINEX' 'URREkgm1BLINEX'}];
```

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```
append from qqq2;

create gogo4 from qqq4[colname= {'URREvarkhk' 'URREvarkhkb' 'URREvarklw' 'URREvarkhsl'

'URREvarkam' 'URREvarkgm' 'URREvarkgm1'}];

append from qqq4;

quit;

goptions i=spline;

axis1 label=(angle=90 'f(Coefficient)') ;

axis2 label=('Coefficient') ;

legend1 label=('Empirical distributions')

value=('HK' 'HKB' 'LW' 'HSL' 'AM' 'GM' 'MED' );

symbol1 color=black l=3;

symbol2 color=blue l=46;

symbol3 color=red l=1;

symbol4 color=green l=5;

symbol5 color=yellow l=10;

symbol6 color=orange l=13;

symbol7 color=pink l=20;

proc gplot data=gogo;

plot relativefreq*URREkhk2 freq2*URREkhkb2 freq3*URREklw2 freq4*URREkhsl2 freq5*URREkam2
freq6*URREkgm2 freq7*URREkgm12/ overlay legend=legend1

vaxis=axis1 haxis=axis2;

run;

proc means data=gogo3 mean stderr min max;

var URREkhk2 URREkhkb2 URREklw2 URREkhsl2 URREkam2 URREkgm2 URREkgm12;

run;

proc means data=gogo1 mean stderr min max;

var rsqURREkhk rsqURREkhkb rsqURREklw rsqURREkhsl rsqURREkam rsqURREkgm rsqURREkgm1;
```

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```
run;

proc means data=gogo2 mean stderr min max;

var URREkhkRNl URREkhkbRNL URREklwRNL URREkhslRNL URREkamRNL URREkgmRNL UR-
REkgm1RNL

URREkhkLINEXa URREkhkbLINEXa URREklwLINEX URREkhslLINEX URREkamLINEX URREkgm-
LINEX URREkgm1LINEX

URREkhkBLINEX URREkhkbBLINEX URREklwBLINEX URREkhslBLINEX URREkamBLINEX UR-
REkgmBLINEX URREkgm1BLINEX;

run;

proc means data=gogo4 mean n;

var urrevarkhk urrevarkhkb urrevarklw urrevarkhsl urrevarkam urrevarkgm urrevarkgm1;

run;
```

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Appendix E Graphs

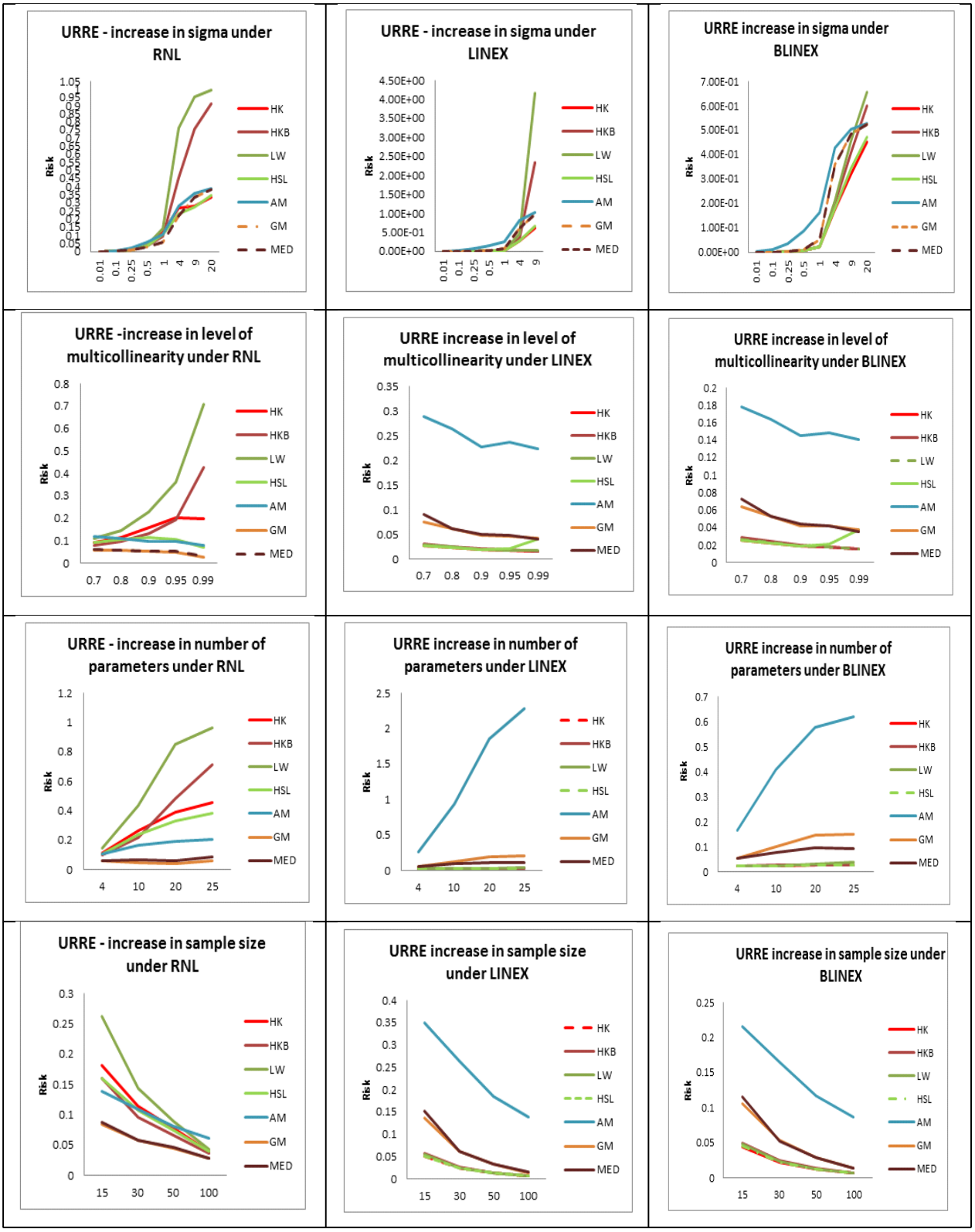

Figure E1 : The performance of the URRE under the three proposed loss function.

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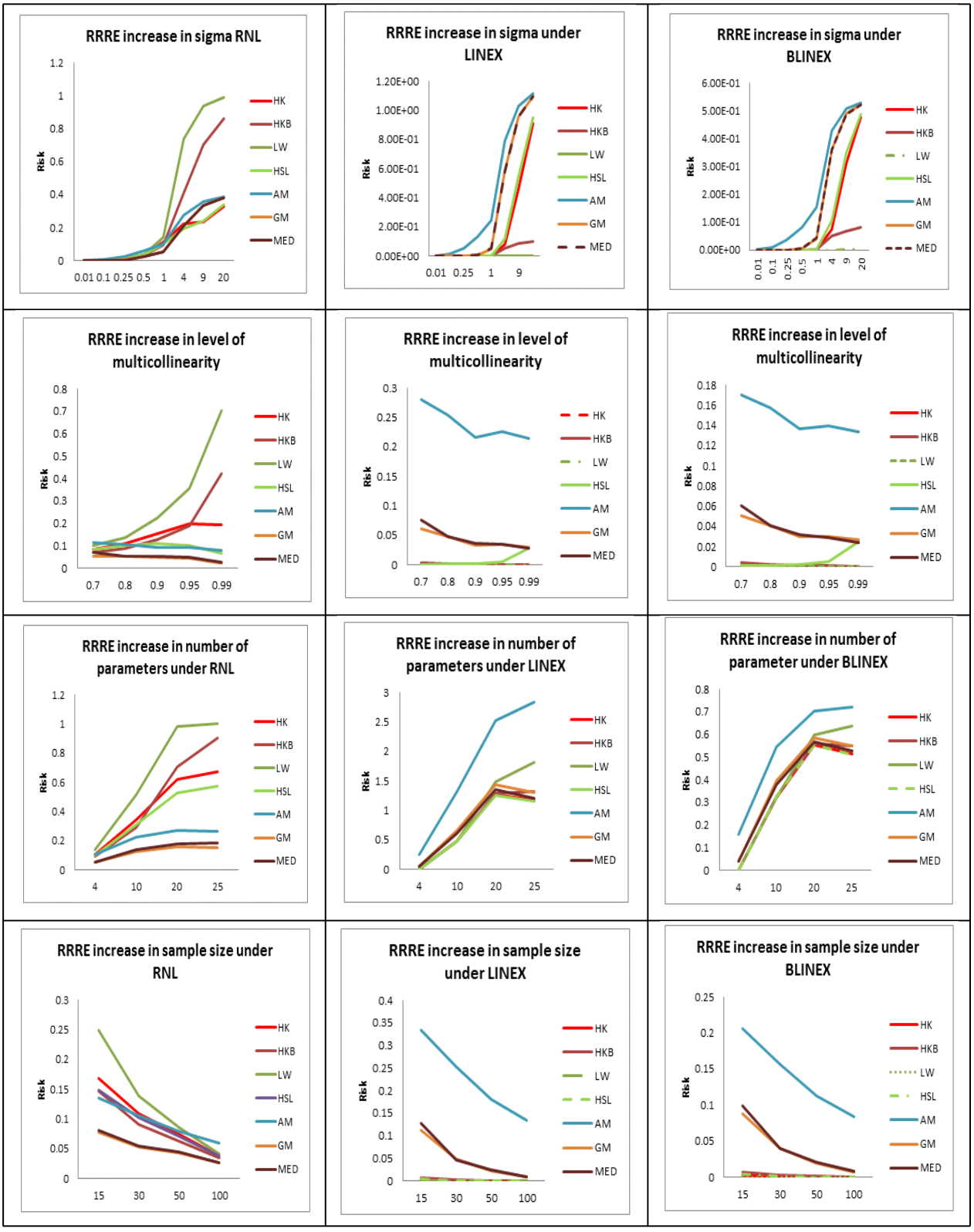

Figure E2.: The performance of the RRRE under the three proposed loss function.

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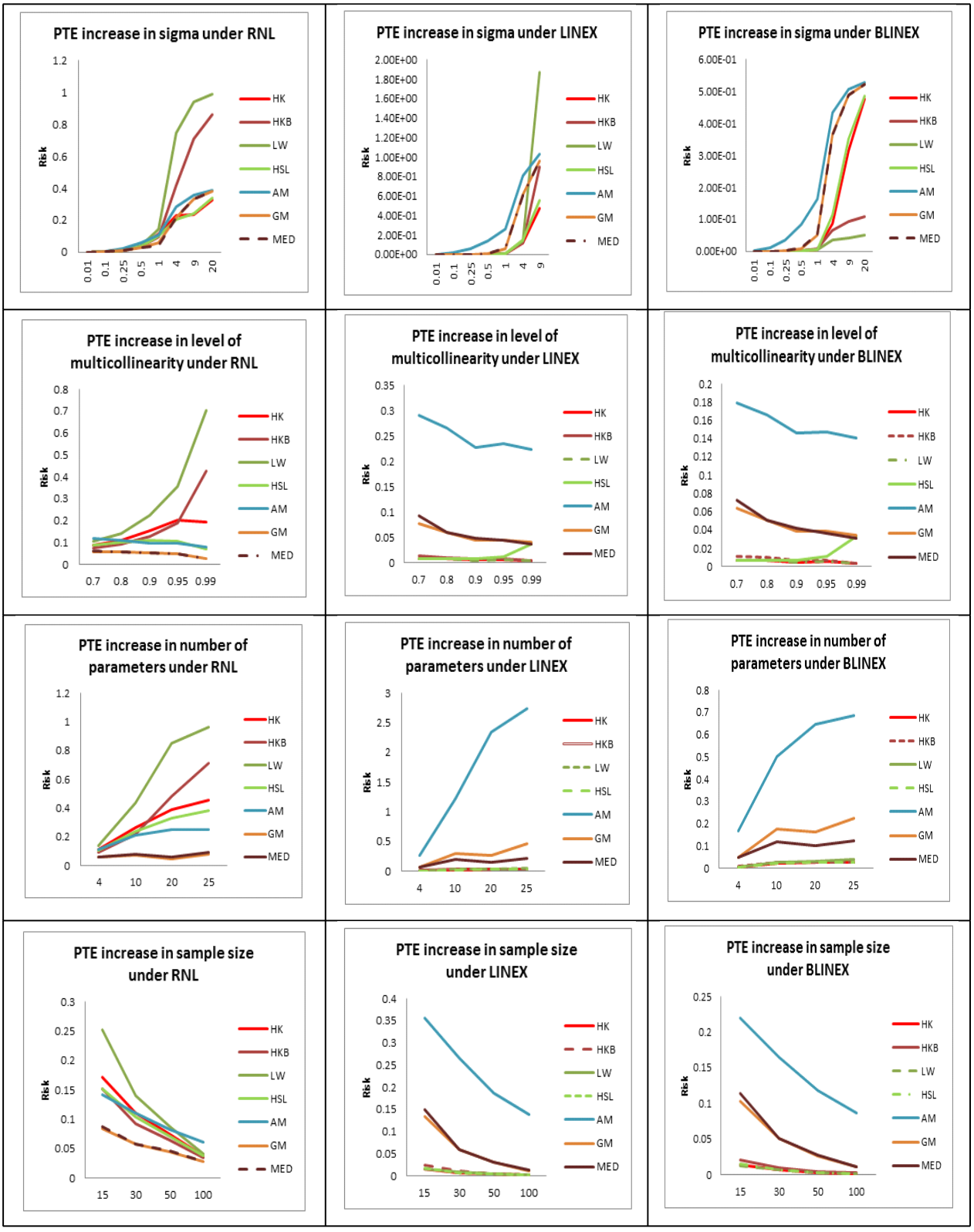

Figure E3 : The performance of the PTRRE under the three proposed loss functions.

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Part II

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