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STONE-CECH COMPACTIFICATION AND RINGS OF CONTINUOUS FUNCTIONS

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Stone–Čech compactification and rings of continuous functions

by

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Summary

The overall purpose of this thesis is to define and to investigate rings of continuous functions defined on completely regular topological spaces.

One of the objectives, achieved in Chapter 1 and Chapter 3, is to present and discuss relationships between topological properties of a space X and algebraic properties of its corresponding rings of continuous functions. Both rings of continuous functions considered in the thesis (namely $C(\mathrm{X})$ – the ring of continuous functions, and $C^*(\mathrm{X})$ – the ring of bounded continuous functions) are uniquely determined by the space X. Thus, it is natural to examine the converse of this fact, that is, the specification of conditions under which the space X is determined by the algebraic structure of $C(\mathrm{X})$ or that of $C^*(\mathrm{X})$. The theory developed in this thesis will build up to show that, within the class of compact spaces, the ring structure of $C^*(\mathrm{X})$ determines the space X up to homeomorphism; in other words, the ring C^* distinguishes among compact spaces. Analogous results will be also proved for the ring C and realcompact spaces.

Another interesting aspect of the theory of rings of continuous functions, presented in Chapter 1, is the fact that several important properties of the continuous functions on a space X (like order structure and boundedness of functions) are determined by the ring structures of $C(\mathrm{X})$ and $C^*(\mathrm{X})$. The relationship between C–embeddings and C^*–embeddings of various topological spaces is also established therein.

Chapter 2 deals with various types of compactifications and methods of compactifying topologic spaces. Its main purpose is to study the Stone–Čech compactification which, from the point of view of this thesis, is the most important and interesting type of compactification. It is shown that such a compactification exists for every completely regular space; that it is the "largest" compactification and that it is unique. Several of its characteristics are investigated; as well as its use in determining of the relationships between a space and its rings of continuous functions and their sets of maximal ideals. Finally, various techniques of constructing the Stone–Čech compactification are discussed, accompanied by examples thereof.

In Chapter 4, an interesting application of the theory of rings of continuous functions is discussed. The chapter presents a global version of the well–known Cauchy–Kovalevskaia theorem for nonlinear partial differential equations (PDEs). Its goal is to prove the existence of global generalized solutions for arbitrary analytic nonlinear PDEs on the whole of their domains of analyticity, and shows that these solutions are analytic outside of closed, nowhere dense subsets. One global and universal principle which can define sets of "patched up" solutions for arbitrary analytic nonlinear PDEs is also presented in this chapter. The proofs are based on constructions within rings of continuous functions on Euclidean spaces.

Opsomming

Die hoofdoel van hierdie verhandeling is die definisie en ondersoek van ringe van kontinue funksies, gedefinieer op volledig reguliere topologiese ruimtes.

Een van die oogmerke, wat in Hoofstuk 1 en Hoofstuk 3 bereik word, is om die verwantskappe tussen die topologiese eienskappe van 'n ruimte X, en die algebraïese eienskappe van sy ooreenstemmende ringe van kontinue funksies, te bespreek. Albei ringe van kontinue funksies wat in die verhandeling beskou word (naamlik C(X) – die ring van kontinue funksies, en $C^*(X)$ – die ring van begrensde kontinue funksies) word uniek bepaal deur die ruimte X. Dus is dit natuurlik om die teenoorgestelde implikasie te bestudeer, dit wil sê, die spesifikasie van voorwaardes waaronder die ruimte X bepaal word deur die algebraïese struktuur van C(X) of van $C^*(X)$. Die teorie wat in hierdie verhandeling ontwikkel word, bou op om aan te toon dat, binne die klas van kompakte ruimtes, die ruimte X tot op homeomorfisme na deur die ringstruktuur van $C^*(X)$ bepaal word; met ander woorde, die ring C^* onderskei tussen kompakte ruimtes. Analoë resultate word ook bewys vir die ring C en reëelkompakte ruimtes.

'n Ander interessante aspek van die teorie van ringe van kontinue funksies, wat in Hoofstuk 1 aangebied word, is die feit dat verskeie belangrike eienskappe van die kontinue funksies op 'n ruimte X (soos ordestruktuur en begrensdheid van funksies) bepaal word deur die ringstruktuur van C(X) en $C^*(X)$. Die verwantskap tussen C–inbeddings en C^*–inbeddings van verskeie topologiese ruimtes word ook hier vasgestel.

Hoofstuk 2 handel oor verskeie soorte kompaktifiserings en metodes om topologiese

ruimtes te kompaktifiseer. Hier is die hoofdoel om te Stone–Čech kompaktifisering te bestudeer, wat uit die oogpunt van hierdie verhandeling, die belangrikste en interessantste tipe kompaktifisering is. Daar word aangetoon dat hierdie kompaktifisering vir elke volledig reguliere ruimte bestaan, dat dit die "grootste" kompaktifisering is en dat dit uniek is. Verskeie eienskappe van hierdie kompaktifisering word ondersoek, sowel as die gebruik daarvan om die verwantskappe te bepaal tussen 'n ruimte en sy ringe van kontinue funksies en hulle versamelings van maksimale ideale. Laastens word verskeie tegnieke/metodes om die Stone–Čech kompaktifisering te konstrueer, bespreek, met saamgaande voorbeelde.

In Hoofstuk 4 word 'n interessante toepassing van die teorie van ringe van kontinue funksies bespreek. Die hoofstuk bied 'n globale weergawe van die bekende Cauchy–Kovalevskaia–stelling vir nie–lineêre parsiële differensiaalvergelykings (PDV's). Die doel is om die bestaan van globale veralgemeende oplossings vir willekeurige analitiese nie–lineêre PDV's op hul ganse gebied van analitisiteit te bewys, en dit word aangetoon dat hierdie oplossings analities is buiten op geslote nêrens digte deelversamelings. Een globale en universele beginsel wat veramelings van "aanmekaargelapte" oplossings vir willekeurige analitiese nie–lineêre PDV's kan definieer, word ook in hierdie hoofstuk gemeld. Die bewyse is gebaseer op konstruksies binne die teorie van ringe van kontinue funksies op Euklidiese ruimtes.

Notation

Throughout this thesis it is assumed that the reader has some background in abstract algebra and general topology. Consequently, topological and algebraic concepts which are considered basic, are not defined here. It is natural, however, that the notation of certain pre–knowledge concepts will vary with the sources. For this reason, notation assumed in this work will be summarized here.

Mappings

Let A and B be arbitrary sets. Then:

$\varphi[A] = \{\varphi(x) : x \in A\}$, the image of a set under a mapping/function;

$\overleftarrow{\varphi}[A] = \{x : \varphi(x) \in A]$, the inverse image of a set under a mapping/function;

$A \to B$ indicates a mapping of A to B;

$A \hookrightarrow B$ indicates a one–to–one mapping of A into B.

Topology

Let X and B be arbitrary sets. Then:

$cl_X B$ denotes the closure of B in X. If no confusion can result, clB or $\overline{B}$ is written instead;

$\mathbb{R}^X$ denotes the set of all real–valued functions on X;

βX denotes the Stone–Čech compactification of X;

$\beta(f)$ denotes the extension of a function f to βX.

Algebra

Let I be an ideal of a ring and a an element of that ring. Then:

$I(a)$ denotes the residue class of a;

(I, a) denotes the smallest ideal containing I and a;

A^p denotes the prime z–filter corresponding to the point p;

M^p denotes the corresponding maximal ideal.

1. Rings of continuous functions

[Sources: [BW] H. Linda Byun & Saleem Watson, *"Prime and maximal ideals in subrings of C(X)"*;

[GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;

[W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: The purpose of this chapter is to establish the relationship between C–embeddings and C^*–embeddings. Section 1–1 introduces the concept of rings of continuous functions. In Section 1–2 compact spaces are studied within the framework of the theory of rings of continuous functions. Section 1–3 deals with C–embeddings and C^*–embeddings.

Throughout this chapter it will be assumed that the reader is familiar with the following terms: ring, subring, unit, ideal, maximal ideal, prime ideal [the latter three terms are defined and discussed in Appendix D], topological space, compact space, metric space, discrete space, normal space, relation, residue class.

1–1. The rings $C(X)$ and $C^*(X)$

Consider the set $C(X)$ of all continuous functions from the topological space X into the topological space $\mathbb{R}$. The sum of two continuous functions is clearly continuous; so is their product. Further, if $f \in C(X)$, then $-f \in C(X)$.

Therefore, $C(X)$ is a commutative ring. It is also a ring with unity because the constant function **1** belongs to $C(X)$.

The subset $C^*(X)$ of $C(X)$, consisting of all bounded functions in $C(X)$, is also closed under the algebraic operations of addition and multiplication; thus $C^*(X)$ forms a subring of $C(X)$, with the same unity (i.e., the constant function **1**).

This is stated formally in the definition below.

<u>DEFINITION 1–1.1</u>

C(X) is the ring of all real–valued continuous functions defined on a space X.
The subring consisting of all bounded members of $C(X)$ is called ***C*(X)***.

It is, of course, possible that the subring $C^*(X)$ corresponds exactly with the ring $C(X)$. This happens when every function in $C(X)$ is bounded. When this occurs, the space X is said to be *pseudocompact*. It should be clear that every compact space is pseudocompact, as illustrated in the example below.

<u>EXAMPLE 1–1.1</u>

This example deals with a *compact* topological space.
Consider the closed interval [0,1] of $\mathbb{R}$. The interval [0,1] is compact, and thus every real–valued continuous function on [0,1] is also bounded.
Therefore, $C([0,1]) = C^*([0,1])$.

EXAMPLE 1–1.2

This example deals with a *noncompact* topological space.

Consider the noncompact space $\mathbb{N}$ of positive integers. Since $\mathbb{N}$ is discrete, every real–valued function on $\mathbb{N}$ is continuous.

Thus $C(\mathbb{N})$ is the ring of all sequences of real numbers; while $C^*(\mathbb{N})$ is the ring of all bounded sequences of real numbers.

One of the objectives of this thesis is to present and discuss relationships between topological properties of a space X and algebraic properties of the corresponding rings of continuous functions. Obviously, both rings $C(X)$ and $C^*(X)$ of continuous functions are uniquely determined by the space X. It is natural to examine the converse of this fact, i.e. the specification of conditions under which the space X is determined by the algebraic structure of $C(X)$ or that of $C^*(X)$.

In this work, it will be exhibited what restrictions, if any, need to be placed on the spaces X and Y, so that given that the rings of continuous functions on those spaces are isomorphic, will allow us to conclude that the spaces are homeomorphic.

Another interesting aspect of the theory of rings of continuous functions is the fact that several important properties of the continuous functions on a space X are determined by the ring structures of $C(X)$ and $C^*(X)$. These properties include the order structure [see Appendix A for more detail on order] and boundedness of functions. The following two theorems will state the results formally.

THEOREM 1–1.1

Every ring homeomorphism from $C(Y)$ or $C^*(Y)$ into $C(X)$ is a lattice homeomorphism.

PROOF

Let t be an arbitrary ring homeomorphism from $C(Y)$ or $C^*(Y)$ into $C(X)$.

$g=l^2$ implies $tg=(tl)^2$. Thus, t sends nonnegative functions into nonnegative functions; i.e. t is order–preserving.

Further, $(t|g|)^2 = (t|g|^2) = t(g)^2 = (tg)^2$, and since $t|g| \geq \mathbf{0}$, it follows that $t|g| = |tg|$.

But $\sup\{g,h\} + \sup\{g,h\} = g+h+|g-h|$.

Thus, $t\sup\{g,h\} + t\sup\{g,h\} = tg+th+|tg-th|$

$= \sup\{tg,th\} + \sup\{tg,th\}$.

But $t\sup\{g,h\}$ and $\sup\{tg,th\}$ are real–valued functions on X.

Therefore, $t\sup\{g,h\} = \sup\{tg,th\}$.

Thus, t is a lattice homeomorphism. ■

THEOREM 1–1.2

Every ring homeomorphism from $C(Y)$ or $C^*(Y)$ into $C(X)$ takes bounded functions to bounded functions.

PROOF

Let t be an arbitrary ring homeomorphism from $C(Y)$ or $C^*(Y)$ into $C(X)$.

Because t is a homeomorphism, $t1 = t(1\cdot 1) = (t1)(t1)$, so that the function $t1$ in $C(X)$ is an idempotent. Thus, the only values $t1$ can assume on X are 0 or 1.

Hence, for each $n \in \mathbb{N}$, the only values assumed by the function $tn = t1 + \cdots + t1$ on X are 0 or n.

Let $g \in C^*(Y)$ arbitrarily. Then $|g| \leq n$ for some $n \in \mathbb{N}$.

So $|tg| \leq tn \leq n$. ■

COROLLARY 1–1.3

If X is not pseudocompact, then $C(X)$ is not a homeomorphic image of $C^*(Y)$, for any Y.

PROOF

Recall that a space X is defined pseudocompact if every real–valued continuous function with X as domain, is also bounded.

The result now follows immediately from the preceding theorem. ■

In particular, the rings $C(X)$ and $C^*(X)$ are isomorphic only if they are identical.

THEOREM 1–1.4

Let t be a ring homeomorphism from $C(Y)$ into $C(X)$ whose image contains $C^*(X)$. Then t carries $C^*(Y)$ onto $C^*(X)$.

PROOF

Let t be a ring homeomorphism from $C(Y)$ into $C(X)$ whose image contains $C^*(X)$.

Let $k \in C(Y)$ such that $tk = 1$. Then $t1 = (tk)(t1) = t(k \cdot 1) = tk = 1$. Consequently, $tn = n$ for every $n \in \mathbb{N}$.

Let $f \in C^*(X)$. We want to find $g \in C^*(Y)$ such that $tg = f$.

Choose $h \in C(Y)$ such that $th = f$, and choose $n \in \mathbb{N}$ satisfying $|f| \leq n$.

Define $g = \inf\{\sup\{-n, h\}, n\}$.

Then $g \in C^*(Y)$.

Further, by Theorem 1–1.1, $tg = \inf\{\sup\{-n, f\}, n\} = f$. ■

1–2. Compact spaces – an application of the theory of rings of continuous functions

This section presents an important application of the theory of rings of continuous functions. The theory developed here will build up to show that, within the class of compact spaces, the ring structure of $C^*(X)$ determines the space X up to homeomorphism; in other words, the ring C^* distinguishes among compact spaces. In Chapter 3 similar results concerning the ring C and realcompact spaces will be discussed.

Up to this point no separation axioms were assumed for the topological space on which the rings of continuous functions were defined. The class of topological spaces to be considered in future, however, needs to satisfy certain conditions. It has to be wide enough to include all the "interesting" spaces, yet restrictive enough to admit a significant theory of rings of continuous functions. As it turns out, the class of completely regular spaces [discussed in Appendix E] fulfills the above requirements. In fact, Theorem E–2.3 in Appendix E, eliminates any reason for considering rings of continuous functions on topological spaces other that completely regular spaces.

Thus, throughout the remainder of this chapter it will be assumed, unless otherwise stated, that the topological spaces considered are completely regular.

The following definition uses the concept of *zero–sets*. The zero–set of a function f is denoted by $Z(f)$, while the family of all zero–sets in a space X is denoted Z(X) [see Appendix B for a detailed discussion]. These sets are of use when studying the relation between topological properties of X and algebraic properties of $C(X)$.

DEFINITION 1–2.1

Let I be any ideal in $C(X)$ or $C^*(X)$. If $\cap Z[I]$ is nonempty, then I is said to be a ***fixed*** ideal; otherwise I is called a ***free*** ideal.

In other words, I is a free ideal if and only if, for every point $x \in X$ there exists a function in I that does not vanish at x.

EXAMPLE 1–2.1

Let S be a nonempty subset of Y. The set

$$I = \{f \in C(X) : f[S] = \{0\}\}$$

is an ideal in $C(X)$, as can be easily verified.

Clearly $\cap Z[I]$ is nonempty, because it contains S.

Thus I is a fixed ideal.

Further, it can be established that $I \cap C^*(X)$ is a fixed ideal in $C^*(X)$.

EXAMPLE 1–2.2

Consider the ring $C^*(\mathbb{N})$ and let the sequence $(1/n)_{n \in \mathbb{N}}$ be denoted by **j**.

Any ideal in $C^*(\mathbb{N})$ containing **j** is an example of a free ideal (such ideals do exist, since **j** is not a unit in $C^*(\mathbb{N})$).

EXAMPLE 1–2.3

Consider the set $C_K(\mathbb{N})$ of all functions on $\mathbb{N}$ that vanish at all but a finite number of points.

Evidently $C_K(\mathbb{N})$ is an example of a free ideal both in $C(\mathbb{N})$ and in $C^*(\mathbb{N})$.

The remainder of this section will be devoted to the fixed maximal ideals in the rings $C(X)$ and $C^*(X)$. The discussion of free maximal ideals is more complex; therefore, it is postponed till Chapter 3.

Consider the fixed ideal I in C. Then the set $S=\cap Z[I]$ is nonempty, and the set

$$I' = \{f\in C : f[S]=\{0\}\}$$

is a fixed ideal containing I. Hence a fixed *maximal* ideal must be of this form. Also, since I' increases in size as S decreases, the only possible candidates for fixed maximal ideals are the ideals I' for which S consists of only one point.

The corresponding statements also hold for the ring $C^*(X)$.

Notation: Let I be an ideal in an arbitrary ring A. Then $I(a)$ will denote the residue class of a, i.e. the coset of I determined by a [see Appendix C for a discussion of cosets and factor rings].

<u>THEOREM 1–2.1</u>

Let X be a completely regular space. Then:

(a) The fixed maximal ideals in $C(X)$ are precisely the sets

$$\mathbf{M}_p = \{f\in C : f(p)=0\} \qquad \text{where } p\in X.$$

The ideals $\mathbf{M}_p$ are distinct for distinct p. For each p, $C/\mathbf{M}_p$ is isomorphic with the real field $\mathbb{R}$; moreover, the mapping $\mathbf{M}_p(f)\mapsto f(p)$ is the unique isomorphism of $C/\mathbf{M}_p$ onto $\mathbb{R}$.

(b) The fixed maximal ideals in $C^*(X)$ are precisely the sets

$$\mathbf{M}_p^* = \{f\in C^* : f(p)=0\} \quad \text{where } p\in X.$$

The ideals $\mathbf{M}_p^*$ are distinct for distinct p. For each p, $C^*/\mathbf{M}_p^*$ is isomorphic with the real field $\mathbb{R}$; moreover, the mapping $\mathbf{M}_p^*(f) \mapsto f(p)$ is the unique isomorphism of $C^*/\mathbf{M}_p^*$ onto $\mathbb{R}$.

PROOF

(a) $\mathbf{M}_p$ is the kernel of the homomorphism $f \mapsto f(p)$ of $C(X)$ into $\mathbb{R}$.

$r(p) = r$ for each $r \in \mathbb{R}$, hence the homomorphism is *onto* the field $\mathbb{R}$.

So its kernel $\mathbf{M}_p$ is a maximal ideal.

X is completely regular, consequently p is unique.

On the other hand, let M be any fixed ideal in C. Then there exists a point $p \in \bigcap Z[M]$.

Evidently, M is contained in $\mathbf{M}_p$ which is a proper ideal. Hence if M is maximal, then $M = \mathbf{M}_p$.

Since $\mathbf{M}_p$ is the kernel of the homomorphism onto $\mathbb{R}$, $C/\mathbf{M}_p$ is isomorphic with $\mathbb{R}$. The isomorphism is unique, because the only autotrophism of ρ is the identity [see Appendix A for the proof of this general result].

(b) The proof is identical to that of (a), except for notation. ■

The preceding theorem implies the existence of a one–to–one correspondence between the fixed maximal ideals in C and those in C^*. In fact, this correspondence is immediately obtained from the theorem, namely

$$\mathbf{M}_p \to \mathbf{M}_p^* = \mathbf{M}_p \cap C^*.$$

Moreover, $\mathbf{M}_p$ is the only maximal ideal in C (fixed or free) whose intersection with C^* yields $\mathbf{M}_p^*$. To see this, consider any maximal ideal M in C such that $M \neq \mathbf{M}_p$. There exists $f \in M$ such that $f(p) \neq 0$. Let $g = \inf\{|f|, 1\}$. Then $g \in C^*$, and $Z(g) = Z(f)$. Hence $g(p) \neq 0$, so that $g \notin \mathbf{M}_p^*$. Further, g belongs to the z–ideal M. Thus, although $g \notin \mathbf{M}_p^*$, g belongs to $M \cap C^*(X)$.

Let M be an arbitrary maximal ideal in C. Then $M\cap C^*$ is always a prime ideal in C^* (this follows from general theory: any ideal P in C is prime if and only if $P\cap C^*$ is a prime ideal in C^*).

But $M\cap C^*$ need not be a maximal ideal. Consider, for example, the function $j=(1/n)_{n\in\mathbb{N}}$ in $C^*(\mathbb{N})$. Since j is a unit of $C(\mathbb{N})$, it does not belong to any ideal in $C(\mathbb{N})$. Yet j belongs to every free maximal ideal in $C^*(\mathbb{N})$, as can be proved easily by contradiction. Now, consider any free maximal ideal M in $C(\mathbb{N})$. Then $M\cap C^*(\mathbb{N})$ is not a fixed maximal ideal in $C^*(\mathbb{N})$; and, since it does not contain j, it cannot be a free maximal ideal either.

Moreover, the free maximal ideals in C^* need not be of the form $M\cap C^*$. As an example, consider again the function $j=(1/n)_{n\in\mathbb{N}}$ in $C^*(\mathbb{N})$. Then no free maximal ideal in $C^*(\mathbb{N})$ can assume the form $M\cap C^*(\mathbb{N})$, where M is a maximal ideal in $C(\mathbb{N})$.

<u>THEOREM 1–2.2</u>

Let the space X be compact. Then every ideal I in $C(X) = C^*(X)$ is fixed.

<u>PROOF</u>

Let the space X be compact.

Let I be any ideal in $C(X)$.

Then Z[I] is a family of closed sets with the finite intersection property, i.e. $\cap Z[I]$ is not empty.

Thus, I is a fixed ideal. ■

In the light of Theorem 1–2.1 we now know that if X is compact, then the correspondence $p\mapsto M_p$ is one–to–one from X onto the set of all maximal ideals in $C(X) = C^*(X)$. An important fact about maximal ideals is that they are algebraic invariants, i.e. isomorphic rings have homeomorphic spaces of maximal ideals [see below for a description of the topology imposed on the set $\mathfrak{M}$ of all maximal ideals in $C(X)$]. This implies that the points of a compact space can be recovered from the algebraic structure of the ring. Now, the zero–sets in X form a base for the closed sets. Further, the relation $p \in Z(f)$ is equivalent to the algebraic relation $f \in \mathbf{M}_p$. Thus, the topology of X can also be recovered from $C(X)$.

To describe the above process in more detail, let $\mathfrak{M} = \mathfrak{M}(X)$ denote the set of all maximal ideals in $C(X)$. We transform $\mathfrak{M}$ into a topological space by defining, as a base for the closed sets, all sets of the form

$$B = \{M \in \mathfrak{M} : f \in M\} \text{ where } f \in C(X).$$

For any given f, $\mathbf{M}_p \in B$ if and only if $f(p)=0$. Thus the one–to–one correspondence $p \mapsto \mathbf{M}_p$ carries the zero–sets in X onto the family of all sets B. Therefore, $\mathfrak{M}$ is well defined as a topological space and is homeomorphic to X.

The above–defined topology is called the *Stone topology* on $\mathfrak{M}$. The set, $\mathfrak{M}$ together with the Stone topology, is called the *structure space* of the ring *C*.

This discussion is summarized in the following definition.

DEFINITION 1–2.2

Let A be a commutative ring with unity.

Let I be any ideal in A.

Let $\mathfrak{M}(A)$ denote the collection of all maximal ideals in A.

Consider any subset $\mathfrak{H}$ of $\mathfrak{M}(A)$.

Now the ***kernel*** of $\mathfrak{H}$ is defined to be $\cap \mathfrak{H}$.

The ***hull*** of I is the set $\{M \in \mathfrak{M}(A) : I \subset M\}$.

The ***Stone topology***, or the ***hull–kernel topology***, on the set of maximal ideals $\mathfrak{M}(A)$ is obtained by defining the closure of $\mathfrak{H}$ to be the hull of the kernel of $\mathfrak{H}$:

$$\mathrm{cl}\,\mathfrak{H} = \{M \in \mathfrak{M}(A) : \cap \mathfrak{H} \subset M\}.$$

The space $\mathfrak{M}(A)$ endowed with this topology is referred to as the ***structure space*** of A.

As already mentioned, the ring structure of $C^*(X)$ determines X up to homeomorphism. The following theorem now establishes formally that the ring $C^*(X)$ is an algebraic invariant for the class of compact spaces.

THEOREM 1–2.3

Two compact spaces X and Y are homeomorphic if and only if their rings $C^*(X)$ and $C^*(Y)$ are isomorphic.

PROOF

"$\Rightarrow$": The proof is trivial.

"$\Leftarrow$": Let X and Y be compact spaces.

As described above, X and Y can both be recovered from $C^*(X)$ and $C^*(Y)$ respectively, so the result follows. ■

The next theorem exhibits the relationship among the following concepts: compact spaces, fixed ideals in C, fixed ideals in C^*, fixed maximal ideals in C and fixed maximal ideals in C^*.

THEOREM 1–2.4

The following statements are equivalent:

(1) X is compact;

(2) Every ideal in $C(X)$ is fixed;

(3) Every ideal in $C^*(X)$ is fixed;

(4) Every maximal ideal in $C(X)$ is fixed;

(5) Every maximal ideal in $C^*(X)$ is fixed.

PROOF

(1)⇒(2) This is exactly Theorem 1–2.2.

(1)⇒(3) Let X be compact.

Then $C(X) = C^*(X)$ and the result follows from Theorem 1–2.2.

(3)⇒(2) Let I be a free ideal in C.

Then $I \cap C^*$ is a free ideal in C^*, a contradiction.

(2)⇒(4) Trivial.

(4)⇒(2) Assume I is a free ideal in C.

Then I must be contained in a free maximal ideal in C.

This is a contradiction because all maximal ideals in C are fixed.

(3)⇒(5) Trivial.

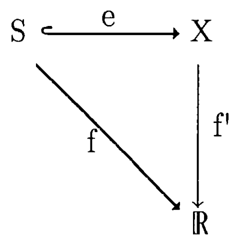
(5)⇒(3) Except for notation, this proof is identical as the proof for (4)⇒(2). ■

1-3. C-embedding and C*-embedding

DEFINITION 1–3.1

A subspace S of X is ***C–embedded* in X** if every member f of $C(S)$ extends to a member g of $C(X)$. Similarly, S is ***C^*–embedded* in X** if every member f of $C^*(S)$ extends to a member g of $C^*(X)$.

The following diagram illustrates C–embedding:



The concept of C^*–embedding is of particular importance in this work, because the Stone–Čech compactification βX is a compact Hausdorff space containing X as a dense C^*–embedded subspace (this will be discussed in Chapter 2).

It is vital to note the following fact: in general, if a subspace is C–embedded, it is also C^*–embedded (this will be proved in Corollary 1–3.2); but the converse is not necessarily true. As will be demonstrated in this section's examples, it is possible for a topological space to be both C^*–embedded and C–embedded, or neither, or only C^*–embedded but not C–embedded. The remainder of this section will be devoted to studying the relationship between these two types of embeddings.

<u>THEOREM 1–3.1</u>

Let S be a subspace of X. S is C^*–embedded in X if and only if every function in $C^*(S)$ can be extended to a function in $C(X)$.

<u>PROOF</u>

"⇒": Let S be C^*–embedded in X. Then, by definition, every $f \in C^*(S)$ extends to a $g \in C^*(X)$.

But $C^*(X) \subset C(X)$, so that every $f \in C^*(S)$ extends to a $g \in C(X)$.

"⇐": Assume every function in $C^*(S)$ can be extended to a function in $C(X)$.

Let $f \in C^*(S)$ arbitrarily. Thus f can be extended to a $g \in C^*(X)$.

But f is bounded, so there exist constants m,M>0 such that $-m < f < M$.

Define the function h as follows:

$$h(x) = \begin{cases} M & \text{if } g(x) \geq M \\ g(x) & \text{if } -m < g(x) < M \\ m & \text{if } g(x) \leq -m \end{cases} \quad \text{for every } x \in X.$$

Then h is continuous and bounded on X, so $h \in C^*(X)$, so S is C^*–embedded in X. ■

The following corollary shows that a C–embedded space is always C^*–embedded.

<u>COROLLARY 1–3.2</u>

Let S be a C–embedded subspace of X. Then S is C^*–embedded in X.

<u>PROOF</u>

Let $f \in C^*(S)$ arbitrarily. Then $f \in C(S)$, because $C^*(S) \subset C(S)$.

Then f can be extended to a $g \in C(X)$ because S is a C–embedded subspace of X.

Now, from the preceding theorem, it follows that S is C^*–embedded in X. ■

<u>THEOREM 1–3.3</u>

(a) Let $S \subset X \subset Y$. Let X be C–embedded in Y. Then S is C–embedded in Y if and only if it is C–embedded in X.

(b) Let $S \subset X \subset Y$. Let X be C^*–embedded in Y. Then S is C^*–embedded in Y if and only if it is C^*–embedded in X.

<u>PROOF</u>

(a) "⇒": Let $f \in C(S)$ arbitrarily.

Then f extends to a $g \in C(Y)$ because S is C–embedded in Y.

Define the function h to be the restriction of g in X.

Then h is continuous on X, so $h \in C(X)$, so S is C–embedded in X.

"$\Leftarrow$": Let $f \in C(S)$ arbitrarily.

Then f extends to a $g \in C(X)$ because S is C–embedded in X.

But g extends to an $h \in C(Y)$ because X is C–embedded in Y.

Thus f extends to $h \in C(Y)$, so S is C–embedded in Y.

(b) The proof is identical as that of part (a), except for notation. ■

Note that the condition "X is C–embedded (or C^*–embedded) in Y" is not actually necessary for the forward direction of parts (a) and (b) to remain true.

EXAMPLE 1–3.1

Consider the subspace $\mathbb{R}-\{0\}$ of $\mathbb{R}$. Let

$$f(r) = \begin{cases} 1 \text{ if } r>0 \\ -1 \text{ if } r<0 \end{cases}.$$

Then f has no continuous extension in $\mathbb{R}$.

Thus the subspace $\mathbb{R}-\{0\}$ is neither C^*–embedded, nor C–embedded in $\mathbb{R}$.

EXAMPLE 1–3.2

Consider the subspace $S=(0,\infty)$ of $\mathbb{R}$. Let

$$f(x) = \sin(1/x).$$

Then f has no continuous extension in $\mathbb{R}$.

Thus the subspace S is neither C^*–embedded, nor C–embedded in $\mathbb{R}$.

EXAMPLE 1–3.3

Consider the subspace $\mathbb{N}$ of positive integers in $\mathbb{R}$.

It is clear that $\mathbb{N}$ is both C^*–embedded and C–embedded in $\mathbb{R}$.

EXAMPLE 1–3.4

This example deals with the space of ordinals [discussed in Appendix F].

The purpose of this example is to show that the space **W** of all countable ordinals is C–embedded (and thus also C^*–embedded) in the space $\mathbf{W}^*$ of all countable ordinals united with the set consisting of the first uncountable ordinal:

Let ω_1 denote the first uncountable ordinal. Then

$$\mathbf{W} = \mathrm{W}(\omega_1) = \{\sigma \colon \sigma < \omega_1\},$$
$$\mathbf{W}^* = \mathrm{W}(\omega_1 + 1) = \{\sigma \colon \sigma \leq \omega_1\}.$$

It is possible to extend $f \in C(\mathbf{W})$ to a function $f^\beta \in C(\mathbf{W}^*)$ by defining $f^\beta(\omega_1)$ to be the final constant value of f. It is trivial that f^β is the unique continuous extension of f. Moreover, given $g \in C(\mathbf{W}^*)$, the restriction of g to **W** belongs to $C(\mathbf{W})$.

Thus, it follows further that $C(\mathbf{W})$ is isomorphic with $C(\mathbf{W}^*)$ under the mapping $f \mapsto f^\beta$.

The above is an example of two spaces which are topologically distinct, but whose rings of continuous functions are isomorphic. Accordingly, neither the algebraic structure of $C^*(\mathrm{X})$, nor even that of $C(\mathrm{X})$, is in general sufficient to determine X as a topological space.

DEFINITION 1–3.2

Subsets A and B of a space X are ***completely separated in X*** if there exists a continuous function f in $C(\mathrm{X})$ such that $f(\mathrm{a})=0$ for every $\mathrm{a} \in \mathrm{A}$ and $f(\mathrm{b})=1$ for every $\mathrm{b} \in \mathrm{B}$.

It is easy to verify that two sets are completely separated if and only if they are contained in disjoint zero–sets [this is proved in Appendix B].

The following theorem presents the basic result about C^*–embeddings, that is, it shows when a subspace is C^*–embedded. The theorem is actually an adaptation of Urysohn's proposition that any closed set in a normal space is C^*–embedded (which will be discussed later in this section).

THEOREM 1–3.4 (Urysohn's Extension Theorem)
A subspace S of a space X is C^*–embedded in X if and only if any two completely separated sets in S are completely separated in X.

PROOF

"$\Rightarrow$": Let A and B be completely separated sets in S.

Then there exists a function $f \in C^*(S)$ such that $f(a)=0$ for every $a \in A$ and $f(b)=1$ for every $b \in B$.

But it is given that S is C^*–embedded in X, thus f has an extension to a function $g \in C^*(X)$.

But $g(a)=0$ for every $a \in A$ and $g(b)=1$ for every $b \in B$.

Thus, A and B are completely separated in X.

"$\Leftarrow$": This part will be proved by induction.

Assume that $f_1 \in C^*(S)$. Then $|f_1| \leq m$ for some $m \in \mathbb{N}$.

Let $r_n = (m/2)(2/3)^n$ for every $n \in \mathbb{N}$.

Then $|f_1| \leq 3r_1$.

Proceeding inductively, suppose an element $f_n \in C^*(S)$ such that $|f_n| \leq 3r_n$, was obtained (induction hypothesis).

Define $A_n = \{s \in S: f_n(s) \leq -r_n\}$ and $B_n = \{s \in S: f_n(s) \geq +r_n\}$.

A_n and B_n are completely separated in S, so it is given that they are completely separated in X.

Hence, there exists $g_n \in C^*(X)$ such that g_n is equal to $-r_n$ on A_n and to $+r_n$ on B_n and $|g_n| \leq r_n$.

Now define $f_{n+1} = f_n - g_n|_S$.

Since $|f_n(s) - g_n(s)| \leq 2r_n$ for every $s \in S$, it is clear that $|f_{n+1}| \leq 3r_{n+1}$.

The induction step is now complete.

The Weierstrass M–test shows that the sequence $\{\sum_{i=1}^{n} g_i\}$ converges uniformly to a constant function on X.

But $g_1 + \cdots + g_n|_S \;= (f_1 - f_2) + \cdots + (f_n - f_{n+1})$

$\qquad\qquad\qquad\;\; = f_1 - f_{n+1}$,

and $f_{n+1}(s)$ converges to zero for every $s \in S$.

Thus the limit of the sequence extends f_1. ■

<u>EXAMPLE 1–3.5</u>

Let X be any metric space.

Then every closed set is a zero–set, so that any two disjoint closed sets are completely separated. Now, if S is closed, then closed sets in S are also closed in X; so that completely separated sets in S have disjoint closures in X. Now by applying Urysohn's Extension Theorem it can be established that every closed set in a metric space is C^*–embedded (this result is stated later in Tietze's Extension Theorem).

<u>EXAMPLE 1–3.6</u>

Consider the space $\Sigma = \mathbf{N} \cup \{\alpha\}$, where $\alpha \notin \mathbf{N}$. Define the topology on Σ as follows: all points of $\mathbf{N}$ are isolated, and the neighborhoods of α are the sets $U \cup \{\alpha\}$ for $U \in \mathfrak{U}$, where $\mathfrak{U}$ is a free ultrafilter [see Appendix D for the definition].

Then **N** is a dense subspace of Σ. Further, **N** is C^*–embedded, but not C–embedded (this can be seen in [GJ] on page 64).

The above example emphasizes an important fact, namely that a C^*–embedded space is not necessarily C–embedded.

The following theorem yields information as to when a C^*–embedded subspace will also be C–embedded.

<u>THEOREM 1–3.5</u>

A C^*–embedded subspace is also C–embedded if and only if it is completely separated from every zero–set disjoint from it.

<u>PROOF</u>

"$\Rightarrow$": Let S be C–embedded in X.

Let $Z(h)$ be a zero–set which misses S.

Define $f \in C(S)$ by $f(s) = 1/h(s)$ for $s \in S$.

Let g be the extension of f to all of X.

Then gh completely separates $Z(h)$ form S.

"$\Leftarrow$": Let S be a C^*–embedded subspace which is completely separated from every zero–set which misses it.

Let $f \in C(S)$.

Then the composition $\arctan \circ f$ belongs to $C^*(S)$ and has an extension to a mapping g in $C(X)$.

The zero–set $Z = \{x \in X: |g(x)| \geq \pi/2\}$ misses S so that there exists a mapping $h: X \to [0,1]$, such that h is constantly equal to 1 on S and to 0 on Z.

Then gh agrees with $\arctan \circ f$ on S. Moreover, $|(gh)(x)| < \pi/2$ for every $x \in X$.

Hence, $\tan\circ(gh)$ is well–defined, continuous, and real–valued on X. Further, it is an extension of f to all of X.

Thus, S is C–embedded in X. ■

EXAMPLE 1–3.7

Consider the topological metric space $\mathbb{R}$. This is a special case of Example 1–3.5, so we can conclude that every closed set in $\mathbb{R}$ is C^*–embedded. Using the preceding theorem we now establish that every closed set in $\mathbb{R}$ is also C–embedded.

This example leads to the following special sufficient condition for a set S in X to be C–embedded.

THEOREM 1–3.6

Let S be a set in X. If there exists a function in $C(X)$ that carries S homeomorphically onto a closed set in $\mathbb{R}$, then S is C–embedded in X.

PROOF

Let h denote a function in $C(X)$ that carries S homeomorphically onto a closed set in $\mathbb{R}$.

Then $\varphi = (h|_S)^{\leftarrow}$ is a continuous mapping from $H=h[S]$ onto S, with $\varphi(h(s)) = s$ for $s \in S$.

Let $f \in C(S)$ arbitrarily.

Then the composition $f\circ\varphi$ belongs to $C(H)$.

But H is closed in $\mathbb{R}$ (by hypothesis), so it is C–embedded. Thus, there exists a function $g \in C(\mathbb{R})$ which agrees with $f\circ\varphi$ on H.

Then $g \circ h$ belongs to $C(X)$. Further, for all $s \in S$,

$$\begin{aligned}(g \circ h)(s) &= f(\varphi(h(s))) \\ &= f(s).\end{aligned}$$

Thus, $g \circ h$ is an extension of f to all of X.

Thus, S is C–embedded in X. ∎

<u>THEOREM 1–3.7</u> (Tietze's Extension Theorem)

Let X be a metric space. Let S be closed subspace of X. Then S is C^*–embedded in X.

<u>PROOF</u>

Let X be a metric space.

Then every closed set is a zero–set, so that any two disjoint closed sets are completely separated [see Appendix B].

Let S be a closed subspace of X, then closed sets in S are also closed in X.

Thus, completely separated sets in S have disjoint closures in X.

It follows now from Theorem 1–3.4 (Urysohn's Extension Theorem) that every closed set in a metric space is C^*–embedded. ∎

<u>THEOREM 1–3.8</u>

Let X be a completely regular space. Let S be compact subspace of X. Then S is C^*–embedded as well as C–embedded in X.

<u>PROOF</u>

Let X be a completely regular space.

Let S be compact subspace of X.

Completely separated sets in S have disjoint closures in S. Because these closures are compact, they are completely separated in X [see Appendix E].

It follows now from Theorem 1–3.4 (Urysohn's Extension Theorem) that S is C^*–embedded in X.

But S is compact, so it is also C–embedded in X. ■

<u>THEOREM 1–3.9</u>

Let X be a Hausdorff space. Then the following statements are equivalent:

(1) X is normal;

(1*) Every two disjoint closed sets in X are completely separated;

(2) Every closed set is C^*–embedded in X;

(3) Every closed set is C–embedded in X.

<u>PROOF</u>

(1)$\Rightarrow$(1*) This is exactly Urysohn's Lemma in Appendix E.

(1*)$\Rightarrow$(1) The proof follows trivially from the definition of completely separated sets.

(1)$\Rightarrow$(2) Let S be any closed set in X.

Consider any A,B$\subseteq$S completely separated in S.

We can assume A and B closed in S. Then A and B are closed in X.

But X is normal, hence it follows from (1*) that A and B are completely separated in X.

Thus, by Theorem 1–3.4, S is C^*–embedded in X.

(3)$\Rightarrow$(2) Follows directly from Corollary 1–3.2.

(2)$\Rightarrow$(3) This follows from Theorem 1–3.5.

(3)$\Rightarrow$(1*) Consider arbitrary disjoint closed A,B$\subseteq$X.

Then C = A$\cup$B is closed.

Define $f:C\to\mathbb{R}$ by $f[A]=0$ and $f[B]=1$.

Then f is continuous on C, thus it extends continuously to X. Thus, A and B are completely separated in X. ∎

What follows now is a summary of the relationships between C–embeddings and C^*–embeddings when dealing with various types of topological spaces and subspaces.

(1) A C–embedded subspace of X is also C^*–embedded in X (Corollary 1–3.2).

(2) If $S\subset X\subset Y$ and X is C–embedded in Y, then S is C–embedded in Y if and only if it is C–embedded in X (Theorem 1–3.3(a)).

(3) If $S\subset X\subset Y$ and X is C^*–embedded in Y, then S is C^*–embedded in Y if and only if it is C^*–embedded in X (Theorem 1–3.3(b)).

(4) A C^*–embedded subspace S of X is also C–embedded in X if and only if S is completely separated from every zero–set disjoint from it (Theorem 1–3.5).

(5) A closed subspace of a metric space X is C^*–embedded in X (Theorem 1–3.7).

(6) A compact subspace of a completely regular space X is both C–embedded and C^*–embedded in X (Theorem 1–3.8).

(7) A closed subspace of a normal space X is both C–embedded and C^*–embedded in X (Theorem 1–3.9).

1–4 Other rings of continuous functions

The study of rings of continuous functions on a completely regular space is by no means limited to the ring C and its subring C^*.

DEFINITION 1–4.1

Let X be a completely regular space. Any subring of $C(X)$ which contains $C^*(X)$ is denoted by $A(X)$.

EXAMPLE 1–4.1

Let X be $\mathbb{R}$, the set of real numbers.

It is easy to check that $A(\mathbb{R}) = C^*((-\infty,0])\cup C([0,\infty))$ is a subring of $C(\mathbb{R})$ which contains $C^*(\mathbb{R})$.

Moreover, $C^*(\mathbb{R}) \subsetneqq A(\mathbb{R}) \subsetneqq C(\mathbb{R})$.

THEOREM 1–4.1

Let X be a completely regular space. Let $f\in A(X)$. Then $|f|\in A(X)$.

PROOF

Let $E = \{x\in X: f(x)\geq 1\}$ and $F = \{x\in X: f(x)\leq -1\}$.

Then E and F are completely separated.

Thus, there exists $g\in C^*(X)$ such that $g(E)=1$ and $g(F)=-1$, and $-1\leq g(x)\leq 1$ for all $x\in X$.

Let $h = gf-|f|$.

If $x\in E\cup F$, then $h(x)=0$.

If $x\notin E\cup F$, then $|h(x)| \leq |g(x)f(x)|+|f(x)| \leq 2|f(x)| \leq 2$.

Hence, $h\in C^*(X)\subset A(X)$ and so $|f| = gf-h\in A(X)$. ∎

COROLLARY 1–4.2

$A(X)$ is a lattice.

PROOF

Choose arbitrary $f,g \in A(X)$.

Then $\sup\{f,g\} = \frac{1}{2}(f+g+|f-g|) \in A(X)$.

The result follows from the preceding theorem ■

In [BW], the subrings of $C(X)$ which contains $C^*(X)$ are studied at great length. Particular attention is paid to characterizing the prime and maximal ideals in $A(X)$ in terms of their residue class rings and in terms of z–filters that correspond to these ideals. It is also shown there, that many of the results investigated in this thesis for the rings $C(X)$ and $C^*(X)$, are also true for the rings $A(X)$. However, the objective of this section is not to present a thorough discussion of the rings $A(X)$, but rather to make the reader aware of the vast scope of the theory of rings of continuous functions.

2. Compactification

[Sources: [GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;
[M] James R. Munkres, *"Topology. A First Course"*;
[W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: This chapter deals with various types of compactifications and methods of compactifying topologic spaces. Section 2–1 defines the concept of compactification. In Section 2–2, the "smallest" compactification of a noncompact space is discussed. Section 2–3 presents the concept of inducing a compactification via an embedding. These sections combined also serve as an introduction to Section 2–4 which, from the point of view of this thesis, deals with the most important type of compactification, namely the Stone–Čech compactification. The importance of this compactification is discussed in Section 2–4.

Throughout the discussion it will be assumed that the reader is familiar with the concept of compact Hausdorff spaces, connected spaces, disconnected spaces, ordering [presented in Appendix A], zero–sets [presented in Appendix B], ideals and maximal ideals [discussed in Appendix D].

2–1. The concept of compactification

DEFINITION 2–1.1

A ***compactification*** of a topological space X is a compact Hausdorff space K containing X such that X is dense in K. Alternatively, we can define it as a compact Hausdorff space K together with an embedding $e:X \to K$ with e[X] dense in K.

It follows immediately from the definition, that if a space X is already compact, then X itself is its only compactification.

In general, however, there may be many ways of compactifying a given space X. At the end of Section 2–3 we will consider examples which deal with the open interval X=(0,1) and its various compactifications.

It is important, however, to note that in order for X to have a compactification, X must be *completely regular*. This means that its one–point sets must be closed in X, and that for each point x and each closed set A not containing x, there must exist a continuous function $f:X \to [0,1]$ such that f(x)=1 and f(A)={0} [please refer to Appendix E, where this concept is discussed formally].

Conversely, we also note that every completely regular space has at least one compactification (namely its Stone–Čech compactification, which is discussed in Section 2–4).

DEFINITION 2–1.2

A space X is said to be ***locally compact at x*** if there exists a compact subset C of X which contains a neighborhood of x. If X is locally compact at each of its points, X is said to be ***locally compact***.

EXAMPLE 2–1.1

It should be clear from the definition that any compact space X is automatically locally compact as well.

To verify it, simply let the compact subset C in the definition of local compactness be equal to the compact space X.

EXAMPLE 2–1.2

We know that the real line $\mathbb{R}$ is not compact. This example, however, shows that $\mathbb{R}$ is locally compact.

This can be verified, with the help of the definition, as follows:

Every point x of $\mathbb{R}$ lies in some interval (a,b) which is the neighborhood of that point x.

The interval (a,b) is in turn contained in the compact set [a,b].

EXAMPLE 2–1.3

The space **Q** of all rationals is not locally compact at any of its points.

Thus, **Q** is not a locally compact space.

2–2. One–point Compactification

DEFINITION 2–2.1

Consider a locally compact Hausdorff space X which is not compact. Consider further some object ∞ outside X. Together they form the set $Y=X\cup\{\infty\}$. Topologize Y by defining the collection of open sets in Y to be all sets of the following types:

(1) U, where U is an open subset of X,

(2) Y–C, where C is a compact subset of X.

The space Y is then called the ***one–point compactification*** of X.

Thus, every locally compact noncompact Hausdorff space has a one–point compactification, which is at the same time its "smallest" compactification (it will be shown in Section 2–4 that the Stone–Čech compactification of a space is its "largest" compactification).

EXAMPLE 2–2.1

The one–point compactification of the real line $\mathbb{R}$ is $\mathbb{R}\cup\{\infty\}$, where the symbol ∞ is some point outside $\mathbb{R}$.

The one–point compactification $\mathbb{R}\cup\{\infty\}$ is actually homeomorphic with the circle.

EXAMPLE 2–2.2

The one–point compactification of the real plane $\mathbb{R}^2$ is homeomorphic with the sphere.

EXAMPLE 2–2.3

Let $\mathbb{N}$ denote the set of all positive integers.

The one–point compactification of the space $\mathbb{N}$ is homeomorphic with the subspace $\{0\}\cup\{1/n\colon n\in\mathbb{N}\}$ of $\mathbb{R}$.

EXAMPLE 2–2.4

This example deals with spaces of ordinals [discussed in Appendix F].

Let ω_1 denote the first uncountable ordinal. By definition:

$$\mathbf{W} = \mathrm{W}(\omega_1) = \{\sigma\colon \sigma<\omega_1\},$$
$$\mathbf{W}^* = \mathrm{W}(\omega_1+1) = \{\sigma\colon \sigma\leq\omega_1\}.$$

Then [as shown in Appendix F], the space $\mathbf{W}^*$ is compact and $\mathbf{W}$ is not.

Moreover, $\mathbf{W}^*$ is the one–point compactification of $\mathbf{W}$.

The basic property of the one point compactification is that any locally compact Hausdorff space can be embedded in another compact Hausdorff space, namely its one–point compactification. This is presented and proved in the following theorem.

THEOREM 2–2.1

Let X be a locally compact Hausdorff space which is not compact; and let Y be the one–point compactification of X.

Then Y is a compact Hausdorff space; X a subspace of Y; the set Y–X consists of a single point; and X is dense in Y.

PROOF

Let X be a locally compact, non–compact Hausdorff space

Let Y be the one–point compactification of X.

Topologize Y by defining the collection of open sets in Y to be all sets of the following types:

(1) U, where U is an open subset of X,

(2) Y–C, where C is a compact subset of X.

X is a subspace of Y because:

Consider any open space A of Y. Then the intersection of A with X is open in X, because $U \cap X = U$ and $(Y-C) \cap X = X-C$, both of which are open in X.

Conversely, any set open in X is a set of type (1) and therefore open in Y.

Thus, the subspace topology of X is inherited from Y.

$cl_Y X = Y$ because:

X is not compact.

Thus, each open set Y–C containing the point ∞ intersects X.

Therefore, ∞ is a limit point of X, so that $cl_Y X = Y$.

Y is compact because:

Let $\mathfrak{A}$ be an open covering of Y.

The collection $\mathfrak{A}$ must contain an open set of type (2), say Y–C, since none of the open sets of type (1) contain the point ∞.

Intersect X with all the members of $\mathfrak{A}$ different from Y–C. This creates a collection of open sets in X covering C.

But C is compact, so finitely many of them cover C. The corresponding finite collection of elements of $\mathfrak{A}$ is also a covering for all of Y.

Y is Hausdorff because:

Let x and y be two points in Y.

Suppose both x and y belong to X. Because X is Hausdorff, there exist disjoint sets U and V open in X which contain x and y respectively.

Suppose that $x \in X$ and $y = \infty$. Because X is locally compact, one can choose a compact set C in X containing a neighborhood U of x. Then U and Y–C are disjoint neighborhoods of x and ∞, respectively, in Y. ■

2–3. Compactification induced by an embedding

Let X be a completely regular space and let Z be a Hausdorff compact space. Consider an embedding h:X→Z of X in Z. Let X_0 denote the subspace h(X) of Z; and let Y_0 denote its closure in Z. Then Y_0 is a compact Hausdorff space and it is the closure of X_0. Therefore, Y_0 is a compactification of X_0.

Let us now construct a space Y containing X such that the pair (X,Y) is homeomorphic to the pair (X_0,Y_0). Consider a set A disjoint from X that is in bijective correspondence with the set (Y_0-X_0) under some map $k:A\to(Y_0-X_0)$. Let $Y=X\cup A$ and define a bijective correspondence $H:Y\to Y_0$ by the rule

$$H(x)=h(x) \text{ for } x\in X,$$

$$H(a)=k(a) \text{ for } a\in A.$$

Now topologize Y by defining a set U to be open in Y if and only if H(U) is open in Y_0. The map H is clearly a homeomorphism. Further the space X is a subspace of Y because H equals the homeomorphism h when restricted to the subset x of Y.

The space Y is called the *compactification of X induced by the embedding h.*

The above construction can be summarized as follows:

If h:X→Z is an embedding of X in the compact Hausdorff space Z, then h induces a compactification Y of X. It has the property that the embedding h can be extended to an embedding H:Y→Z.

To illustrate the fact that one space may have several compactifications, let us consider the open interval (0,1) and three various ways of compactifying it.

EXAMPLE 2–3.1

Consider the unit circle S^1 in $\mathbb{R}^2$. Let $h:(0,1)\rightarrowtail S^1$ be the map

$$h(t)=(\cos 2\pi t)\cdot(\sin 2\pi t).$$

The compactification induced by the embedding h (with $(0,1)=X$, $S^1=Y_o$, $h(X)=X_o$, $\mathbb{R}^2=Z$) is, in fact, equivalent to the one–point compactification of (0,1).

EXAMPLE 2–3.2

Let Y be the closed interval [0,1].

Then Y is a compactification of (0,1) obtained by "adding one point at each end of (0,1)".

EXAMPLE 2–3.3

Consider the square $[-1,1]^2$ in $\mathbb{R}^2$. Let $h:(0,1)\rightarrowtail[-1,1]^2$ be the map

$$h(x)=x\cdot\sin(1/x).$$

The embedding h induces a compactification of (0,1) which is radically different from the above two. It is obtained by adding a point at the right–hand end of (0,1), and a line segment of points at the left–hand end.

2–4. Stone–čech Compactification

This section presents a discussion of the Stone–Čech compactification. It is shown that such a compactification exists for every completely regular space; that it is the "largest" compactification (this concept is explained later), and that it is unique.

Several of its characteristics are investigated; as well as its use in determining of the relationships between a space and its rings of continuous functions and their sets of maximal ideals. Finally, various techniques of obtaining the Stone–Čech compactification are discussed, accompanied by examples thereof.

Existence, Uniqueness and the concept of the "largest" compactification

DEFINITION 2–4.1

Stone–Čech compactification of a topological space X is a compactification in which X is embedded in such a way that every bounded, real–valued function on X can be extended continuously to the Stone–Čech compactification of X. This compactification will be denoted βX.

In the year 1937 M.H. Stone and E. Čech published independent papers which proved the existence of the compactification now known as the Stone–Čech compactification. Čech used βX to investigate properties of a space X by embedding it into βX. Throughout the remainder of this section, βX will prove to be a useful device for the study of relationships between topological characteristics of X and the algebraic structure of the real–valued continuous functions defined on X. We will also find that many topological properties of X can be translated into properties of βX.

The most important characteristic of βX is that it is the "largest" compactification of X. Any other compactification K of X is simply a restriction of βX.

In order to make precise exactly what is meant by stating that βX is the "largest" compactification, consider an arbitrary compactification Y of a space X. As it turns out, there exists a continuous surjective map $g:\beta X \to Y$ which is an identity on X. The map g is actually a quotient map [see Appendix G for a discussion on quotient maps and spaces]. Thus every compactification on X is equivalent to a quotient space of βX.

There are both advantages and drawbacks connected with the size of βX. Among its virtues is the fact that βX is relatively easy to construct (this will be demonstrated towards the end of this section); and that it serves as a representation of all other compactifications. The major disadvantage is that, although X is dense in βX, the Stone–Čech compactification is sometimes too large and complex to investigate (the exact size of βX is investigated in Theorem 2–4.20).

The concept of completely regular spaces [discussed in Appendix E] is particularly important when dealing with the problem of compactification. As already mentioned, a space has to be completely regular in order to have a compactification, and every completely regular space has at least one compactification (i.e. βX). Moreover, the study of properties of a topological space through the method of compactification (in other words, through an embedding into a compact space) is naturally limited to subspaces of compact spaces, namely the completely regular spaces.

Throughout the remainder of this section the spaces considered in the discussion will be presumed completely regular, unless specified otherwise.

[In Appendix E it is shown that no larger class can be studied by means of embeddings into compact Hausdorff spaces. It is further shown that no additional information can be gained by investigating algebraic properties of rings of continuous functions for any larger class of spaces. Appendix E also establishes a very useful relationship between the topology of a completely regular space and the real–valued mappings defined on the space.]

The following theorem establishes the existence of the compactification βX for any completely regular space X. The theorem can also be interpreted algebraically as demonstrating that $C^*(X)$ and $C^*(\beta X)$ are isomorphic.

<u>THEOREM 2–4.1</u> (Stone–Čech)

Let X be any completely regular space. Then X has a compactification βX in which it is C^*–embedded.

<u>PROOF</u>

Let X be any completely regular space.

Choose an arbitrary $f \in C^*(X)$. Let I_f denote the range of f.

f is bounded, so the closure of I_f, $\overline{I}_f$, is compact.

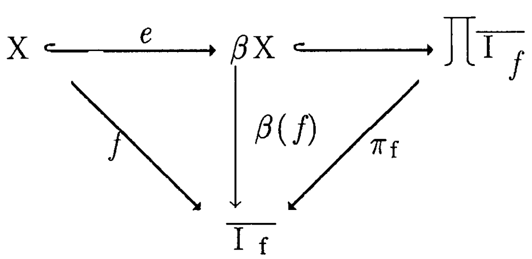
Let $\mathfrak{F} = C^*(X)$ in the Embedding Lemma of Appendix E.

Then X can be embedded into $\prod \overline{I}_f$ by means of the evaluation mapping $e(x)_f = f(x)$.

Let $\beta X = \mathrm{cl}[e(X)]$.

Then the extension $\beta(f) : \beta X \to \overline{I}_f$ is the restriction to βX of the projection π_f.

This is illustrated in the diagram below.



■

The above diagram illustrates one method of representing the Stone–Čech compactification: βX can be obtained as the closure of a copy of X embedded in a product of intervals.

COROLLARY 2–4.2

Let X be a compact space. Then X is homeomorphic to βX.

PROOF

Let X be a compact space.

Consider the above construction.

Because a mapping from a compact space to a Hausdorff space is closed, it follows that e(X) is equal to its closure.

The result follows. ■

In his 1937 paper, Stone showed that bounded continuous functions on X (i.e. members of C^*(X)) can be extended to βX. He demonstrated further that *any* continuous mapping of X into a compact space can be extended to βX. This is stated in the theorem below.

THEOREM 2–4.3 (M.H. Stone)

Let X be a completely regular space. Then X has a compactification βX such that any continuous mapping of X to a compact space K will extend uniquely to βX.

This is presented in the diagram below.

$$\begin{array}{ccc} X & \overset{\eta}{\hookrightarrow} & \beta X \\ & \searrow^{f} & \downarrow \beta(f) \\ & & K \end{array}$$

PROOF

Let X be a completely regular space.

Let K be a compact space.

Choose an arbitrary $g \in C^*(\mathrm{K})$. Let I_g denote the range of g.

Let e be the evaluation map embedding K into their product $\prod \overline{\mathrm{I}_g}$.

$g \circ f$ maps X to $\overline{\mathrm{I}_g}$. Thus, according to Theorem 2–4.1, there exists an extension $\beta(g \circ f)$ of $g \circ f$ to $\beta\mathrm{X}$.

To show that f extends to $\beta\mathrm{X}$, let $h{:}\beta\mathrm{X} \to \prod \overline{\mathrm{I}_g}$ such that $h(\mathrm{p})_g = \beta(g \circ f)(\mathrm{p})$ for all $\mathrm{p} \in \beta\mathrm{X}$. Now the composition of h with each projection is continuous, i.e. $(\pi_g \circ h)(\mathrm{p}) = \beta(g \circ f)(\mathrm{p})$. Thus, h is continuous.

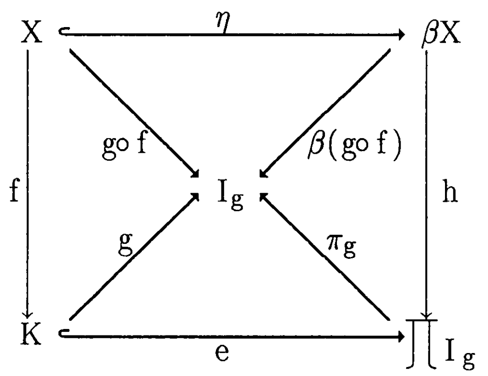
e is an embedding. Thus, it remains to show that the image of $\beta\mathrm{X}$ under h is contained in the image of K under e.

Since $(g \circ f)(\mathrm{X}) = \beta(g \circ f)(\mathrm{p})$ is contained in $g[\mathrm{K}] = e[\mathrm{K}]_g$, it follows that

$$
\begin{aligned}
h[\beta\mathrm{X}] &= h[\mathrm{cl}_{\beta\mathrm{X}}\mathrm{X}] && \text{(because X is compact)} \\
&= \mathrm{cl}(h[\mathrm{X}]) \\
&= \mathrm{cl}(e[\mathrm{K}]) \\
&= \mathrm{cl}(\mathrm{K}) && (e[\mathrm{K}] \cong \mathrm{K} \text{ because } e \text{ is an embedding}) \\
&= \mathrm{K} && \text{(because K is compact).}
\end{aligned}
$$

Further, the extension h is unique because any two extensions of f must agree on the dense subspace X of $\beta\mathrm{X}$.

The situation is presented diagrammatically as follows:



■

COROLLARY 2–4.4

Let X be a completely regular space. Then any compactification of X is a continuous image of βX under a mapping which leaves the points of X fixed.

PROOF

This is just a special case of the preceding theorem and it can be obtained by modifying the proof of the theorem. ■

A pre–ordering [please refer to Appendix A for the definition] can be induced on the set of compactifications of a space X. To achieve it, let L and M be two compactifications of X, and define $L \leq M$ whenever there exists a mapping g of M onto L which leaves the points of X fixed. Then it follows from Corollary 2–4.4 that βX is the maximum element in the set of compactifications of X. This, in turn, enables us to establish that the Stone–Čech compactification is essentially unique, as presented in the following corollary.

COROLLARY 2–4.5

Let X be a completely regular space. Let K be any compactification of X such that every mapping of X to a compact space has an extension to K. Then K is homeomorphic to βX under a homeomorphism which leaves the points of X fixed.

PROOF

Let X be a completely regular space.

Let K be any compactification of X satisfying the stated factorization property.

Then the embedding of X into βX has an extension f to K.

Similarly, the embedding of X into K has an extension g to βX.

The restriction of $f \circ g$ is the identity on X. But X is dense in βX. Thus, $f \circ g$ is the identity on βX.

Similarly, $g \circ f$ is the identity on K.

Therefore, f and g are homeomorphisms leaving points of X fixed.

Moreover, $f = \overleftarrow{g}$. ■

The next theorem yields another characterization of βX, which will prove useful in the study of the construction of βX with the help of zero–sets.

THEOREM 2–4.6 (Čech)

Let X be a completely regular space.

(a) βX is that compactification of X in which completely separated subsets of X have disjoint closures.

(b) βX is that compactification of a *normal* space X in which disjoint closed subsets of X have disjoint closures.

(c) Any two disjoint zero–sets in X have disjoint closures in βX.

PROOF

Let X be a completely regular space.

(a) Let A and B be two subsets of X completely separated by a mapping f.

Then the extension $\beta(f)$ of f to βX completely separates the closures of A and B. Thus, βX satisfies the condition of the theorem.

Now let K be a compactification of X in which completely separated subsets of X have disjoint closures.

By Corollary 2–4.4, there exists a mapping h of βX onto K which leaves the points of X fixed.

βX is compact, so h is a closed mapping, so it is sufficient to show that h is one–to–one.

Let p and q be distinct points of βX. Let $f \in C(\beta X)$ be such that $f(p)=0$ and $f(q)=1$.

Then the sets $A = \{x \in X: f(x) \leq 1/3\}$ and $B = \{x \in X: f(x) \geq 2/3\}$ are completely separated in X and thus have disjoint closures in K.

But $h(p) \in cl_K A$ and $h(q) \in cl_K B$, so $h(p) \neq h(q)$.

Thus, h is a closed continuous bijection.

Therefore, h is a homeomorphism between K and βX.

(b) In a normal space, any two disjoint closed sets are completely separated.

Thus, the result follows from part (a).

(c) As mentioned in Appendix B, any two disjoint zero–sets in X are completely separated.

Thus, the result follows from part (a). ■

Stone–Čech Compactification and Maximal Ideal Spaces

The purpose of this discussion is to determine the relationships between a space X and its rings of continuous functions $C(X)$ and $C^*(X)$.

To begin with, it will be demonstrated that if X is a compact space, then the maximal ideals of $C^*(X)$ are in one–to–one correspondence with the points of X. Further, it will be shown that the maximal ideal corresponding to a point of X is, in fact, the set of continuous mappings which vanish at that point.

In order to investigate the set of maximal ideals as a topological space, a topology has to be defined on it. The topology defined here is identical to the one discussed in Chapter 1.

Let A be a commutative ring with unity.

Let I be any ideal in A.

Let $\mathfrak{M}(A)$ denote the collection of all maximal ideals in A.

Consider any subset $\mathfrak{H}$ of $\mathfrak{M}(A)$.

Now the ***kernel*** of $\mathfrak{H}$ is defined to be $\cap\mathfrak{H}$.

The ***hull*** of I is the set $\{M\in\mathfrak{M}(A)\colon I\subset M\}$.

The ***Stone topology***, or the ***hull–kernel topology***, on the set of maximal ideals $\mathfrak{M}(A)$ is obtained by defining the closure of $\mathfrak{H}$ to be the hull of the kernel of $\mathfrak{H}$:

$$\mathrm{cl}\mathfrak{H} = \{M\in\mathfrak{M}(A)\colon \cap\mathfrak{H}\subset M\}.$$

The space $\mathfrak{M}(A)$ endowed with this topology is referred to as the ***structure space*** of A.

Consider again the commutative ring A with unity. Let $a\in A$. Let $\mathfrak{C}(a)$ denote the set of maximal ideals containing a. Each set $\mathfrak{C}(a)$ is closed because it is the hull of the ideal consisting of all multiples of a. The family $\{\mathfrak{C}(a)\colon a\in A\}$ is a base for the closed sets of $\mathfrak{M}(A)$, since $\mathrm{cl}\mathfrak{H} = \cap\{\mathfrak{C}(a)\colon a\in\cap\mathfrak{H}\}$ when $\mathfrak{H}$ is a subset of $\mathfrak{M}(A)$.

LEMMA 2–4.7

Let A be a commutative ring with unity and let the space $\mathfrak{M}(A)$ be endowed with the Stone topology topology described above. Then $\mathfrak{M}(A)$ is a compact Hausdorff space if and only if for each pair M and N of distinct maximal ideals there exist $a \notin M$ and $b \notin N$ such that ab belongs to every maximal ideal of A.

PROOF

Let A be a commutative ring with unity.

Let $\mathfrak{M}(A)$ be endowed with the Stone topology topology.

"$\Leftarrow$": Consider an arbitrary pair M and N of distinct maximal ideals.

Then, by the hypothesis, there exist $a \notin M$ and $b \notin N$ such that ab belongs to every maximal ideal of A.

Then $U = \mathfrak{M}(A)\backslash\mathfrak{C}(a)$ and $V = \mathfrak{M}(A)\backslash\mathfrak{C}(b)$ are neighborhoods of M and N, respectively.

Further, $U \cap V = (\mathfrak{M}(A)\backslash\mathfrak{C}(a)) \cap (\mathfrak{M}(A)\backslash\mathfrak{C}(b))$

$= \mathfrak{M}(A)\backslash(\mathfrak{C}(a) \cup \mathfrak{C}(b))$.

But every maximal ideal I is prime. Thus,

$$\begin{aligned} I \in \mathfrak{C}(a) \cup \mathfrak{C}(b) &\Leftrightarrow a \in I \text{ or } b \in I \\ &\Leftrightarrow ab \in I \\ &\Leftrightarrow I \in \mathfrak{C}(ab) \qquad (*). \end{aligned}$$

Thus, $U \cap V = \mathfrak{M}(A)\backslash\mathfrak{C}(ab)$.

Hence, since ab belongs to every maximal ideal of A (by hypothesis), $\mathfrak{M}(A) = \mathfrak{C}(ab)$.

Thus, U and V are disjoint. Hence, $\mathfrak{M}(A)$ is Hausdorff.

To demonstrate the compactness of $\mathfrak{M}(A)$, let $\{F_\alpha\}$ be a family of closed sets.

This proof will show that if $\{F_\alpha\}$ has empty intersection, then some finite subfamily has empty intersection. Since each F_α is an intersection of basic closed sets, it is sufficient to assume that each set in the family is a basic set, i.e. that for every α there exists $a_\alpha \in A$ such that $F_\alpha = \mathfrak{C}(a_\alpha)$. The result will then follow from determining when the intersection of a basic family $\{\mathfrak{C}(a_\alpha)\}$ will be empty in terms of the elements $\{a_\alpha\}$.

Now $\cap\mathfrak{C}(a_\alpha) = \emptyset$ exactly when the subset $\{a_\alpha\}$ of A generates A, because:

$$\begin{aligned} \cap\mathfrak{C}(a_\alpha) = \emptyset \quad &\Leftrightarrow \quad \text{for every } M \in \mathfrak{M}(A) \text{ there exists some } a_\alpha \notin M \\ &\Leftrightarrow \quad \text{the only ideal containing } \{a_\alpha\} \text{ is the ring } A \\ &\Leftrightarrow \quad \{a_\alpha\} \text{ generates } A. \end{aligned}$$

Further, there exist $r_i \in A$ such that the identity element of A can be written $1 = r_1 a_{\alpha_1} + \cdots + r_n a_{\alpha_n}$ for some finite family $\{a_{\alpha_i}\}$.

But then the ideal generated by $\{a_{\alpha_i}\}$ contains the identity and hence is equal to A. Thus, the finite subfamily $\{a_{\alpha_i}\}$ generates A so that $\cap\mathfrak{C}(a_{\alpha_i}) = \emptyset$.

Therefore, $\mathfrak{M}(A)$ is compact.

"$\Leftarrow$": Let $\mathfrak{M}(A)$ be a compact Hausdorff space.

Consider an arbitrary pair M and N of distinct maximal ideals. Let $a \notin M$ and $b \notin N$.

Because $\mathfrak{M}(A)$ is Hausdorff, M and N must be disjoint by basic open sets U and V as above.

But,

$$\emptyset = U \cap V = \mathfrak{M}(A) \backslash (\mathfrak{C}(a) \cup \mathfrak{C}(b))$$

implies that $\mathfrak{M}(A) = (\mathfrak{C}(a) \cup \mathfrak{C}(b))$, so that ab belongs to every maximal ideal of A, as proved in (*) above. ■

To determine the relationship between the space X and the ring $C^*(X)$ it is vital to characterize the maximal ideals of the ring $C^*(X)$. This was already done in Chapter 1 (Theorem 1–2.1), but with a different objective in mind. The result was proved by construction in the proof of Theorem 1–2.1; here, on the other hand, a proof by contradiction is used. Bear in mind that it is sufficient to consider a *compact* space X, since $C^*(X)$ and $C^*(\beta X)$ are isomorphic by Theorem 2–4.1.

THEOREM 2–4.8 (M.H. Stone)

Let X be a compact space. Then the maximal ideals of $C^*(X)$ are in one–to–one correspondence with the points of X. For a point p of X they are given by

$$M_p^* = \{f \in C^*(X): f(p)=0\}.$$

PROOF

Let X be a compact space.

Clearly, each M_p^* is an ideal.

Let p and q be distinct points in X. Since distinct points of X are separated by a member of $C^*(X)$, it follows that M_p^* and M_q^* are distinct.

It is now sufficient to show that any proper ideal I is contained in M_p^* for some p.

Assume, on the contrary, that for every point $p \in X$ there exists a member $f_p \in I$ such that $f_p(p) \neq 0$.

Then there exists some neighborhood U(p) of p on which f_p is never equal to zero.

But X is compact, thus the covering $\{U(p)\}$ of X has a finite subcover, $\{U(p_i)\}$.

Let the mapping g be defined as follows: $g = f_{p_1}^2 + \cdots + f_{p_n}^2$.

Then $g \in I$ and is never zero. Thus $g^{\leftarrow} \in C^*(X)$.

Thus, $1 = gg^{\leftarrow}$ belongs to I, so that $I = C^*(X)$.

So I is not a proper ideal, which is a contradiction.

The result follows. ∎

COROLLARY 2–4.9

Let X be a compact space. Then X is homeomorphic with the maximal ideal space $\mathfrak{M}(C^*(X))$.

PROOF

Let X be a compact space.

Let p and q be distinct points in X.

Then there exist f and g in $C^*(X)$ such that $f(p) = g(q) = 1$ and $fg = \mathbf{0}$.

Hence, $f \notin M_p^*$ and $g \notin M_q^*$, although fg belongs to every maximal ideal of $C^*(X)$.

Thus, by Lemma 2–4.7, $\mathfrak{M}(C^*(X))$ is a compact Hausdorff space.

Let τ be the bijection of X with $\mathfrak{M}(C^*(X))$ which maps a point $p \in X$ to the ideal M_p^*.

To prove that τ is a homeomorphism, it is sufficient to show that for a subset S of X, $\mathrm{cl}(\tau[S]) = \tau[\mathrm{cl}(S)]$.

Let $p \in \mathrm{cl}(S)$.

Then every member of $C^*(X)$ which vanishes on all of S also vanishes at p.

By the definition of closure in a space of maximal ideals,

$$\mathrm{cl}(\tau[S]) = \cap\{\mathfrak{C}(f) : f[S] = \{0\}\}.$$

This shows that $\tau(p) = M_p^*$ belongs to $\mathrm{cl}(\tau[S])$.

On the other hand, let $p \notin \mathrm{cl}(S)$.

Then there exists a member of $C^*(X)$ which vanishes on all of S but not at p.

But this means that $\tau(p) = M_p^*$ fails to belong to $\mathrm{cl}(\tau[S])$.

The result follows. ∎

COROLLARY 2–4.10

Let X be a completely regular space. Then its compactification βX is homeomorphic with the maximal ideal space $\mathfrak{M}(C^*(X))$.

PROOF

Let X be a completely regular space.

βX is a compact space, so from the previous theorem it follows that βX is homeomorphic with $\mathfrak{M}(C^*(\beta X))$.

But it is known from Theorem 2–4.1 that $C^*(X)$ is homeomorphic to $C^*(\beta X)$.

Thus, it follows that βX is homeomorphic with $\mathfrak{M}(C^*(X))$. ■

THEOREM 2–4.11

Let X and Y be compact spaces. Then the following statements are equivalent:

(1) $X \cong Y$;

(2) $C^*(X) \cong C^*(Y)$;

(3) $\mathfrak{M}(C^*(X)) \cong \mathfrak{M}(C^*(Y))$.

PROOF

(1)⇒(2): Trivially.

(2)⇒(3): Trivially.

(3)⇒(1): X and Y are compact, so by Corollary 2–4.9 $\mathfrak{M}(C^*(X)) \cong X$ and $\mathfrak{M}(C^*(Y)) \cong Y$.

But it is given that $\mathfrak{M}(C^*(X)) \cong \mathfrak{M}(C^*(Y))$, so $X \cong Y$. ■

It follows from the preceding theorem that the ring $C^*(X)$ determines the space X within homeomorphism (this was already discussed in Chapter 1). It is important to note that, in the above theorem, the rings $C^*(X)$ and $C^*(Y)$ could be replaced by $C(X)$ and $C(Y)$ respectively. This is because the rings C^* and C are equal for a compact space.

The next theorem constitutes the main result of Gelfand and Kolmogoroff's 1939 paper. Here it is used to characterize the maximal ideals of the ring C (compare Stone's result for the ring C^* in Theorem 2–4.8).

THEOREM 2–4.12 (Gelfand and Kolmogoroff)

Let X be a completely regular space. The maximal ideals of the ring $C(X)$ are in one–to–one correspondence with the points of βX. They are given by

$$M^p = \{f \in C(X) \colon p \in \mathrm{cl}_{\beta X} Z(f)\}$$

for a point $p \in \beta X$.

PROOF

Let X be a completely regular space.

Let $p \in \beta X$ arbitrarily.

M^p is a maximal ideal because:

Consider the zero–set on M^p, namely $Z[M^p]$.

Clearly, $Z[M^p]$ is closed under supersets in $Z[X]$ and does not contain the empty set.

Since disjoint zero–sets of X are completely separated, they would have disjoint closures by Theorem 2–4.6.

But p is in the closure of every Z in $Z[M^p]$, thus no two members of $Z[M^p]$ can be disjoint.

Thus, $Z[M^p]$ is a z–filter.

Further, suppose that a zero–set Z meets every member of $Z[M^p]$. Then if $p \notin cl_{\beta X} Z$, there exists a zero–set neighborhood Z' of p in βX which misses Z. But then $Z' \cap X$ is in $Z[M^p]$ and misses Z, which is a contradiction. Thus, $Z[M^p]$ is a z–ultrafilter.

Every maximal ideal is of the form M^p for some $p \in \beta X$, because:

Let M be a maximal ideal.

Then $\{cl_{\beta X} Z : Z \in Z[M]\}$ is a family of closed sets with the finite intersection property in a compact space.

Thus, there exists $p \in \{cl_{\beta X} Z : Z \in Z[M]\}$ so that Z[M] clusters at p.

Then Z[M] converges to p (see Appendix D) and p alone (since βX is Hausdorff).

Thus, $M = M^p$. ■

<u>COROLLARY 2–4.13</u>

Let X be a completely regular space. Then its compactification βX is homeomorphic with the maximal ideal space $\mathfrak{M}(C(X))$.

<u>PROOF</u>

Theorem 2–4.12 establishes a one–to–one correspondence σ between βX and $\mathfrak{M}(C(X))$.

The proof that σ is a homeomorphism is similar to the proof that τ is a homeomorphism in Corollary 2–4.9. ■

We now return to our goal of establishing the relationships between a space X, its compactification βX, their corresponding rings of continuous functions and the sets of their maximal ideals. These relationships can be summarized as follows:

Let X be a completely regular space. Then:

(1) $C^*(X) \cong C^*(\beta X)$ (Theorem 2–4.1);

(2) $\beta X \cong \mathfrak{M}(C^*(X))$ (Corollary 2–4.10);

(3) $\beta X \cong \mathfrak{M}(C(X))$ (Corollary 2–4.13).

If, in addition, the space X is compact, the following relationships exist:

(4) $X \cong \beta X$ (Corollary 2–4.2);

(5) $X \cong \mathfrak{M}(C^*(X))$ (Corollary 2–4.9);

(6) $C(X) = C^*(X)$ (because all continuous functions on a compact space are bounded).

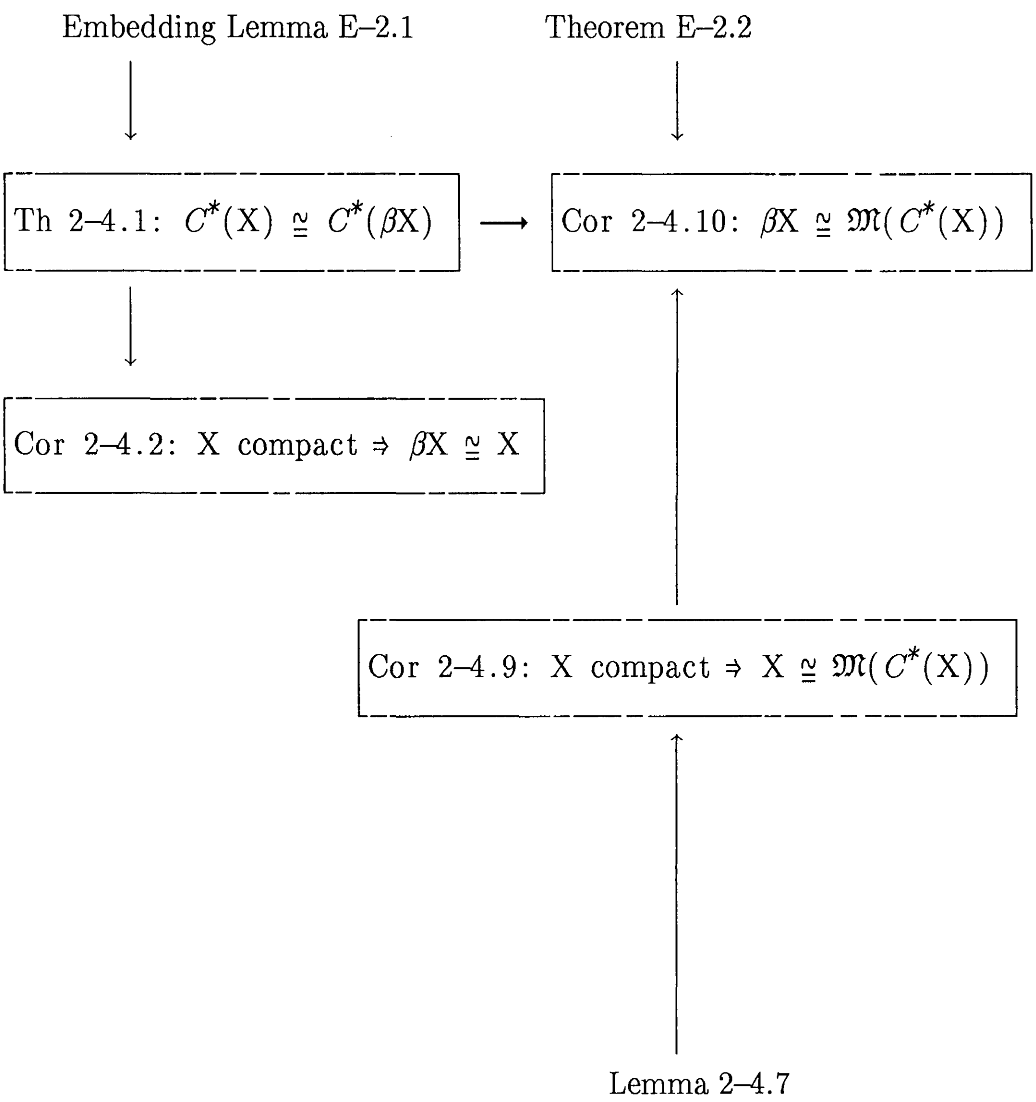
Consider the compact spaces X and Y. Bear in mind that $C^*(X) \cong C(X)$ and $C^*(Y) \cong C(Y)$ because the spaces are compact. Then:

(1) $C^*(X) \cong C^*(Y)$ if and only if $\mathfrak{M}(C^*(X)) \cong \mathfrak{M}(C^*(Y))$ (Theorem 2–4.11);

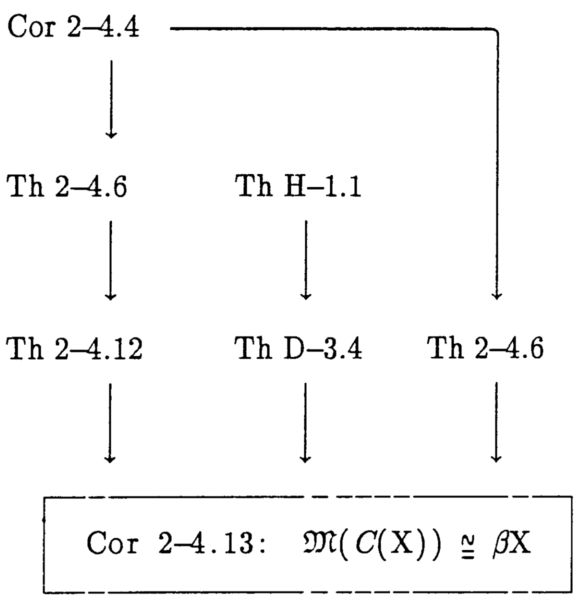
(2) $C^*(X) \cong C^*(Y)$ if and only if $X \cong Y$ (Theorem 2–4.11).

In the diagrams which follow, one can easily see how the above propositions interact (i.e., which theorems are needed in order to prove the main results):

(1)


Embedding Lemma E–2.1
Theorem E–2.2
Th 2–4.1: $C^*(X) \cong C^*(\beta X)$
Cor 2–4.10: $\beta X \cong \mathfrak{M}(C^*(X))$
Cor 2–4.2: X compact $\Rightarrow$ $\beta X \cong X$
Cor 2–4.9: X compact $\Rightarrow$ $X \cong \mathfrak{M}(C^*(X))$
Lemma 2–4.7

(2)



Characterizations of βX

The following theorem is a summary of the equivalent characteristic properties of the Stone–Čech compactification βX.

THEOREM 2–4.14

Let X be a completely regular space. Then X has a unique compactification βX which exhibits the following equivalent properties:

(1) X is C^*–embedded in βX;

(2) Every mapping of X into a compact space K extends uniquely to βX;

(3) Every point of βX is the limit of a unique z–ultrafilter on X;

(4) Let Z_1, Z_2 be zero–sets in X. Then

$$\mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2 = \mathrm{cl}_{\beta X}(Z_1 \cap Z_2);$$

(5) Disjoint zero–sets in X have disjoint closures in βX;

(6) Completely separated sets in X have disjoint closures in βX;

(7) βX is a maximum element of the pre–ordered set of compactifications of X.

PROOF

It is clear that βX satisfies conditions (1), (2), (6) and (7) when one considers βX through embedding X into a product of intervals, as in Theorem 2–4.1.

Condition (3) is obvious in view of the isomorphism of βX with the space of maximal ideals $\mathfrak{M}(C(X))$ demonstrated in Corollary 2–4.13.

Conditions (4) and (5) become transparent on application of Frink's basic result [proved in Appendix H] to the normal base Z[X].

The uniqueness of a compactification satisfying condition (2) was demonstrated in Corollary 2–4.5.

The remainder of the proof shows that the seven properties are equivalent:

(3)⇒(2): Let A^p be the unique z–ultrafilter converging to the point $p \in \beta X$.

Let f be a mapping of X into a compact Hausdorff space Y.

Define $f^{\#}A^p \equiv \{Z \in Z[Y] : f^{\leftarrow}(Z) \in A^p\}$.

Then $f^{\#}A^p$ is a family of closed sets with the finite intersection property. Thus, $\cap f^{\#}A^p$ contains a point $y \in Y$.

It is easy to see that $f^{\#}A^p$ is a prime z–filter converging to y and $\cap f^{\#}A^p = \{y\}$.

This defines a function $\beta(f)$ from βX to Y. If $p \in X$, then $p \in \cap A^p$ and $y = \beta(f)(p)$ is in $\cap f^{\#}A^p$. Thus, $\beta(f)$ is an extension of f.

To show that $\beta(f)$ is continuous, let F be a zero–set neighborhood of $\beta(f)(p)$ and let F' be a zero–set in Y such that Y\F' is contained in F and is a neighborhood of $\beta(f)(p)$.

Then $F \cup F' = Y$.

Let Z and Z' be the inverse images of F and F', respectively, under f.

Then $Z \cup Z' = X$.

Thus, $\mathrm{cl}_{\beta X} Z \cup \mathrm{cl}_{\beta X} Z' = \beta X$.

Since $\beta(f)(p) \notin F'$, $p \notin \mathrm{cl}_{\beta X} Z'$. Thus, $\beta X \setminus \mathrm{cl}_{\beta X} Z'$ is a neighborhood of p.

Every point in this neighborhood belongs to $\mathrm{cl}_{\beta X} Z$. Thus, by the definition of $\beta(f)$, $\beta(f)[\beta X \setminus \mathrm{cl}_{\beta X} Z'] \subset F$, and $\beta(f)$ is continuous.

The uniqueness of the extension is immediate, since any two extensions have to agree on the dense subspace X.

(2)$\Rightarrow$(1): This is clear, because K is compact.

(1)$\Rightarrow$(6): This follows from Urysohn's Extension Theorem (Theorem 1–3.4), which actually shows that (1)$\Leftrightarrow$(6).

(6)$\Rightarrow$(5): This is immediate, since disjoint zero–sets are completely separated.

(5)$\Rightarrow$(4): Clearly, $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) \subset \mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2$.

On the other hand, let $p \in \mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2$.

Then for every zero–set neighborhood V of p in βX, $p \in \mathrm{cl}_{\beta X}(Z_2 \cap V)$.

Then (5) implies that $V \cap Z_1$ and $V \cap Z_2$ cannot be disjoint so that $V \cap (Z_1 \cap Z_2) \neq \emptyset$ and $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$.

(4)$\Rightarrow$(3): X is dense in βX. Thus, the trace on X of the zero–set neighborhoods of a point p in βX is a z–filter $\mathfrak{F}$ on X.

But $\mathfrak{F}$ is contained in a z–ultrafilter $\mathfrak{U}$ and $\mathfrak{U}$ converges to p.

However, distinct z–ultrafilters must contain disjoint zero–sets by Theorem H–1.1. Further, (4) states that any pair of disjoint zero–sets must have disjoint closures in βX.

Hence, exactly one z–ultrafilter converges to p.

(2)$\Rightarrow$(7): This is proved in Corollary 2–4.4.

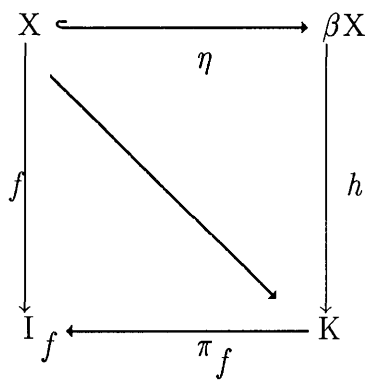
(7)⇒(1): As in Theorem 2–4.1, the evaluation mapping e is can be used to embed X into a product of unit intervals indexed by $C^*(X)$.

Let K be the closure of X in the product.

Then K is a compactification of X and thus is a continuous image of the maximal compactification βX under a mapping h which leaves the points of X fixed.

A member f of $C^*(X)$ extends to K by composing e with the projection π_f. $\pi_f \circ h$ extends f to βX.

This situation is illustrated below.



■

One can observe that in the proof of the conditions (1) to (6) the compactness of βX has not been used. Thus the first six conditions of the above theorem remain equivalent for any space T which contains X as a dense subspace.

The next theorem ties in with Section 1–3, in which C–embeddings and C^*–embeddings were studied. This theorem presents yet another condition under which a subspace will be C^*–embedded.

THEOREM 2–4.15

A subspace S of X is C^*–embedded in X if and only if $\beta S = cl_{\beta X}S$.

PROOF

"⇒": Let a subspace S of X is C^*–embedded in X.

Then clearly S is C^*–embedded in βX.

Thus, $cl_{\beta X}S$ is a compactification of S in which S is C^*–embedded.

"⇐": Let $\beta S = cl_{\beta X}S$.

$cl_{\beta X}S$ is compact and therefore is C^*–embedded in βX.

Since S is C^*–embedded in $cl_{\beta X}S$ (given), it follows that S C^*–embedded in X. ■

The main use of the following theorem lies in the creation of examples.

THEOREM 2–4.16

Let T be a subspace of βX. Let $X \subseteq T$. Then $\beta T = \beta X$.

PROOF

Let T be a subspace of βX.

Let $X \subseteq T$.

Then T is dense in βX.

Further, T is C^*–embedded in βX, because a mapping in $C^*(T)$ can first be restricted to X and then extended to βX.

Thus, $\beta T = \beta X$. ■

We will now investigate the relationship between the zero–sets of a space X and those of βX. Since X is C^*–embedded in its Stone–Čech compactification, one can easily be misled into thinking that a zero–set of βX is just the closure of a zero–set in

X. The following example, however, demonstrates that this is not the case.

<u>EXAMPLE 2–4.1</u>

Let the mapping f on $\mathbb{R}$ be defined by $f(x) = 1/(1+|x|)$.

Let h denote the extension of f to $\beta\mathbb{R}$.

f never vanishes on $\mathbb{R}$, but approaches zero on every noncompact subset of $\mathbb{R}$.

So $h(p)=0$ for every $p \in \beta\mathbb{R}-\mathbb{R}$.

Thus $\beta\mathbb{R}-\mathbb{R}$ is the zero–set of h, yet it is not a closure of any zero–set of $\mathbb{R}$ (the closure of a zero–set is given by $cl_{\beta X}Z = \{p \in \beta X\colon Z \in \mathcal{U}\}$ where $\mathcal{U}$ is an ultrafilter on X).

The actual relationship between the zero–sets of a space X and those of βX is exhibited in the following theorem.

<u>THEOREM 2–4.17</u>

The zero–sets of βX are countable intersections of closures in βX of zero–sets of X.

<u>PROOF</u>

Let $Z \in Z[\beta X]$.

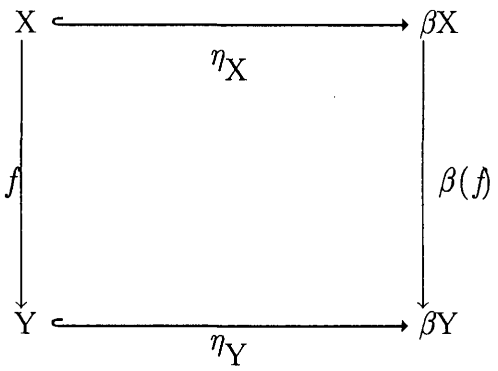
Then $Z = Z(\beta(f))$ for some $f \in C^*(X)$.

$$\text{Thus, } Z(\beta(f)) = \bigcap_{n=1}^{\infty} \{p \in \beta X\colon |\beta(f)(p)| \leq 1/n\}$$

$$= \bigcap_{n=1}^{\infty} cl_{\beta X}\{x \in X\colon |f(x)| \leq 1/n\}.$$

Since $\{x \in X\colon |f(x)| \leq 1/n\}$ is a zero–set for every $n \in \mathbb{N}$, the result follows. ■

Let f be a mapping from X to Y. Let η_Y be the embedding of Y into βY. Then the composition of f and η_Y extends to a mapping $\beta(f)\colon\beta X \to \beta Y$, as illustrated in the following commutative diagram.



<u>THEOREM 2–4.18</u>

Let S be both open and closed in X. Then $cl_{\beta X}S$ and $cl_{\beta X}(X–S)$ are complementary open sets in βX.

<u>PROOF</u>

βX is the union of $cl_{\beta X}S$ and $cl_{\beta X}(X–S)$.

Further, S and X–S are disjoint zero–sets in X, so by Theorem 2–4.6 their closures are disjoint in βX.

Now $cl_{\beta X}S$ is closed in βX, so $cl_{\beta X}(X–S)$is open; and vice–versa. ■

As a special case of the preceding theorem, we have the result that an isolated point in a space X is also isolated in βX.

The next theorem will specify the necessary and sufficient condition for a space X to be open in βX.

<u>THEOREM 2–4.19</u>

X is open in βX if and only if X is locally compact.

PROOF

βX is a Hausdorff space and X is dense in βX.

Thus, the result follows immediately from Appendix E (Theorem E–2.8). ■

The following theorem yields a useful formula to calculate the cardinal number [see Appendix F for the definition of cardinality] of the Stone–Čech compactification. This should give the reader an intuitive understanding of why the Stone–Čech compactification is sometimes considered "too large" for certain purposes.

THEOREM 2–4.20

Let X be an infinite *discrete* space. The cardinal number of βX is given by

$$|\beta X| = 2^{2^{|X|}}$$

PROOF

Let X be an infinite discrete space.

Let $\mathfrak{m} = 2^{2^{|X|}}$.

The problem is to show that there are at least $\mathfrak{m}$ ultrafilters on X.

Consider the following auxiliary sets:

the set $\mathfrak{F}$ of all finite subsets F of X,

and the set Φ of all finite subsets φ of $\mathfrak{F}$.

What follows is a construction of $\mathfrak{m}$ ultrafilters on $\mathfrak{F}\times\Phi$ (since X is infinite, $\mathfrak{F}\times\Phi$ is equipotent with X, so the results are equivalent).

Let S be an arbitrary subset of X, with which a subset $\mathfrak{b}_S$ of $\mathfrak{F}\times\Phi$ is associated as follows:

$$\mathfrak{b}_S = \{(F,\varphi)\in\mathfrak{F}\times\Phi \colon S\cap F\in\varphi\}.$$

Denote the complement of $\mathfrak{b}_S$ in $\mathfrak{F}\times\Phi$ by $-\mathfrak{b}_S$.

Next, for each of the $\mathfrak{m}$ subsets $\mathscr{S}$ of the set of all subsets of X, define the family

$$\mathfrak{B}_{\mathscr{S}} = \{b_S : S \in \mathscr{S}\} \cup \{-b_S : S \notin \mathscr{S}\}$$

of subsets of $\mathfrak{F} \times \Phi$.

$\mathfrak{B}_{\mathscr{S}}$ has the finite intersection property, because:

Let $b_{S_1},\ldots,b_{S_k},-b_{S_{k+1}},\ldots,-b_{S_n}$ be distinct members of $\mathfrak{B}_{\mathscr{S}}$. The indices $S_1,\ldots,S_n$ are distinct subsets of X.

For i<j, choose a single element x_{ij} that belongs to exactly one of S_i, S_j.

The selected elements x_{ij}, for $1 \leq i < j \leq n$, form a finite subset F of X.

For i<j, the sets $S_i \cap F$ and $S_j \cap F$ are distinct, since exactly one of them contains x_{ij}.

Now consider the finite set

$$\varphi = \{S_1 \cap F,\ldots,S_k \cap F\},$$

which is a member of Φ.

Trivially, $S_i \cap F \in \varphi$ for $i \leq k$, and $S_j \cap F \notin \varphi$ for j>k.

Thus, $(F,\varphi) \in b_{S_i}$ for $i \leq k$, and $(F,\varphi) \in -b_{S_j}$ for j>k.

This shows that $\mathfrak{B}_{\mathscr{S}}$ has the finite intersection property.

Thus, each family $\mathfrak{B}_{\mathscr{S}}$ is embeddable in at least one ultrafilter $\mathfrak{U}_{\mathscr{S}}$.

Furthermore, distinct families cannot be contained in the same ultrafilter (for, if $S \in \mathscr{S} - \mathscr{S}'$, then $\mathfrak{B}_{\mathscr{S}}$ contains b_S; while $\mathfrak{B}_{\mathscr{S}'}$ contains $-b_S$).

Since there are $\mathfrak{m}$ sets $\mathscr{S}$, there are $\mathfrak{m}$ ultrafilters $\mathfrak{U}_{\mathscr{S}}$ ■

Various constructions of βX

Although the properties of βX were discussed at length in this section, only two different ways of representing the Stone–Čech compactification were mentioned. First it was illustrated that βX could be obtained as the closure of a copy of X embedded in a product of intervals (Theorem 2–4.1). In this particular representation of βX the C^*–embedding of X shows up most transparently. Later it was proved (in Corollary 2–4.10 and Corollary 2–4.13 respectively) that βX is homeomorphic to the spaces of maximal ideals of the rings $C^*(X)$ and $C(X)$.

A couple of additional constructions of the Stone–Čech compactification will be discussed now. It will shown how to induce βX by an embedding, and also how to construct it using $\mathfrak{Z}$–filters and z–filters.

* This construction of the Stone–Čech compactification uses the idea of inducing the compactification of a completely regular space X by embedding it in some compact Hausdorff space Z, as discussed in Section 2–3.

Let X be a completely regular space. Let $\{f_\alpha\}_{\alpha\in J}$ be the collection of all bounded continuous real–valued functions on X, indexed by some suitable index set J (i.e. $J = C^*(X)$). For each $\alpha\in J$, choose a closed interval I_α in $\mathbb{R}$ containing $f_\alpha(X)$, for instance

$$I_\alpha = [\text{glb} f_\alpha(X), \text{lub} f_\alpha(X)],$$

where glb = greatest lower bound and lub = least upper bound [both terms are defined in Appendix A].

Define the embedding $h: X \to \prod_{\alpha\in J} I_\alpha$ by the rule

$$h(x) = (f_\alpha(x))_{\alpha\in J}.$$

By Tychonoff's theorem [which can be found in Appendix E], $\prod_{\alpha\in J} I_\alpha$ is compact. Because X is completely regular, the collection $\{f_\alpha\}$ separates point from closed sets in X. So h is an embedding [according to the Embedding Lemma of Appendix E].

The compactification of X induced by the embedding h is precisely the Stone–Čech compactification.

* Another representation of the Stone–Čech compactification is in the framework of $\mathfrak{Z}$–spaces. The theory behind $\mathfrak{Z}$–spaces is discussed in detail in Appendix H, so only the result connected with the Stone–Čech compactification is cited here.

Let X be a normal space. Let $\mathfrak{Z}$ be the family of all closed subsets of X. Then $\omega(\mathfrak{Z}) = \beta X$.

[the proof can be found in Appendix H]

* The work of Wallman and Frink made it clear that the construction of the Stone–Čech compactification can be achieved via z–filters [which are investigated in Appendix D].

Let X be a completely regular topological space. The idea behind this construction is to obtain a one–to–one correspondence between the z–ultrafilters on X and the points of βX, such that each z–ultrafilter converges to its corresponding point. But [as discussed in Appendix D] there already exists such a correspondence between the *fixed* z–ultrafilters on X and the points of X. Hence X constitutes an index set for the fixed z–ultrafilters, say I. Using any convenient way, the index set I can be increased to an index set J for the family of *all* (fixed and free) z–ultrafilters.

The points of βX are, in this case, defined to be the elements of this enlarged index set J.

The family of all z–ultrafilters on X is denoted by $(A^p)_{p\in\beta X}$. The topology on βX is defined in such a way that p is the limit of the z–ultrafilter A^p for every $p\in\beta X$.

In particular, let Z be a zero set in the given topological space X. Define

$$B = \{p\in\beta X\colon Z\in A^p\}.$$

Then βX is made into a topological space by taking the family of all sets B as a base for the closed sets.

Examples of the Stone–Čech compactification

<u>EXAMPLE 2–4.2</u>

This example of the Stone–Čech compactification uses the concept of $\mathfrak{Z}$–ultrafilters [discussed in Appendix H]. It was first investigated by H. Wallman in 1938.

Let the space X be discrete. Let the family $\mathfrak{Z}$ be the power set of X.

The collection of all $\mathfrak{Z}$–ultrafilters on X will be denoted by $\omega_X(\mathfrak{Z})$.

Then $\omega_X(\mathfrak{Z})$ is compact and it contains X as a dense subspace. Because X is discrete, it can be embedded in $\omega_X(\mathfrak{Z})$.

Thus $\omega_X(\mathfrak{Z}) = \beta X$.

<u>EXAMPLE 2–4.3</u>

This is an example of the Stone–Čech compactification within the framework of zero–sets [discussed in Appendix B]. As in the previous example, the Stone–Čech compactification is obtained as a Wallman–type compactification.

Let X be a completely regular space and let Z[X] be the collection of its zero–sets.

It is easy to verify that Z[X] is a normal base which is closed under countable intersections.

Further it is known [Appendix B] that any two completely separated subsets of X are contained in disjoint zero–sets.

It is also known [Appendix H] that disjoint zero–sets have disjoint closures in $\omega(Z[X])$.

Thus, completely separated sets of X have disjoint closures in $\omega(Z[X])$.

Hence, by Theorem 2–4.6, $\omega(Z[X])$ is the Stone–Čech compactification βX.

EXAMPLE 2–4.4

This example of the Stone–Čech compactification deals with **W**, the space of all countable ordinals [please refer to Appendix F for a discussion on ordinals].

Let ω_1 be the first uncountable ordinal.

Define $\mathbf{W} = W(\omega_1) = \{\sigma\colon \sigma<\omega_1\}$

and $\mathbf{W}^* = W(\omega_1+1) = \{\sigma\colon \sigma\leq\omega_1\}$.

Then **W** is a noncompact space, while $\mathbf{W}^*$ is compact.

So $\mathbf{W}^*$ is the one–point compactification of **W**.

Moreover, **W** is C^*–embedded in $\mathbf{W}^*$, hence from Theorem 2–4.1 it follows that $\mathbf{W}^*$ is also the Stone–Čech compactification of **W**.

This is an example of a noncompact space whose "largest" compactification (the Stone–Čech compactification) is equal to the "smallest" compactification (the one–point compactification).

EXAMPLE 2–4.5

This example uses the concept of ultrafilters [discussed in Appendix D].

Consider the space $\mathbb{N}$ of all positive integers.

By Theorem 2–4.19, $\mathbb{N}$ is open in $\beta\mathbb{N}$.

More specifically, every point of $\mathbb{N}$ is an isolated point of $\beta\mathbb{N}$. These are the only isolated points of $\beta\mathbb{N}$, because $\mathbb{N}$ is a dense subset of $\beta\mathbb{N}$. Let S be any subset of $\mathbb{N}$. Then by Theorem 2–4.18, $cl_{\beta\mathbb{N}}S$ is open in $\beta\mathbb{N}$. Every point $p\in\beta\mathbb{N}-\mathbb{N}$ is in one–to–one correspondence with the free ultrafilter A^p on $\mathbb{N}$ which converges to p. Hence every neighborhood of p meets $\mathbb{N}$ in a member of A^p. Conversely, if $Z\in A^p$, then the closure of Z is an open neighborhood of p. Further, let p,q be distinct points in $\beta\mathbb{N}-\mathbb{N}$ and choose $Z\in A^p-A^q$, then the closure of Z is an open and closed set containing p but not q; thus $\beta\mathbb{N}$ is totally disconnected.

The subset **O** of odd integers is C^*–embedded in $\mathbb{N}$, so $cl_{\beta\mathbb{N}}\mathbf{O} = \beta\mathbf{O}$ by Theorem 2–4.15; hence $cl_{\beta\mathbb{N}}\mathbf{O}$ is homeomorphic with $\beta\mathbb{N}$. The same argument holds for the subset **E** of even integers. Thus $\beta\mathbb{N}$ can be expressed as the union of two disjoint copies of itself.

It is also possible to decompose $\mathbb{N}$ into infinitely many disjoint infinite sets A_n where $n\in\mathbb{N}$. Then the sets $cl_{\beta\mathbb{N}}A_n$ are disjoint open and closed subsets of $\beta\mathbb{N}$, each one homeomorphic with $\beta\mathbb{N}$. Their union T, though, is not equal to $\beta\mathbb{N}$, because a compact space cannot be equal to a union of infinitely many disjoint open sets.However, T is dense in $\beta\mathbb{N}$ and $\mathbb{N}\subset T\subset\beta\mathbb{N}$, so that $\beta T = \beta\mathbb{N}$.

Choose a point $p_n\in cl_{\beta\mathbb{N}}A_n-\mathbb{N}$ and let $D=\{p_1,p_2,\ldots\}$. D is a discrete subspace of $\beta\mathbb{N}$, so D is homeomorphic with $\mathbb{N}$. Because D is C^*–embedded in T, it is also C^*–embedded in $\beta T = \mathbb{N}$. By Theorem 2–4.15, $cl_{\beta\mathbb{N}}D = \beta D$. Now, D is contained in the closed set $\beta\mathbb{N}-\mathbb{N}$, thus so is $cl_{\beta\mathbb{N}}D$. Thus $\beta D\subset\beta\mathbb{N}-\mathbb{N}$. But βD is homeomorphic with $\beta\mathbb{N}$. Hence $\beta\mathbb{N}-\mathbb{N}$ contains a copy of $\beta\mathbb{N}$.

It can be deduced from Theorem 2–4.20 that the cardinal number of $\beta\mathbb{N}$ is $2^{\mathfrak{c}}$ (c of the continuum).

3. Realcompactification

[Sources: [GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;

[W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: This chapter deals with another interesting topological space: the realcompactification of a completely regular space. Section 3–1 introduces some concepts used in the definitions of realcompact spaces and realcompactification in Section 3–2. Section 3–3 presents realcompact spaces within the framework of real ideals. The work discussed in this chapter builds up to Section 3–4, in which one of the goals of this thesis is reached.

In this chapter it will be assumed that the reader is familiar with the concept of completely regular spaces [discussed in Appendix E], zero–sets [discussed in Appendix B], residue classes [investigated in Appendix C], ideals and z–ultrafilters [discussed in Appendix D], one–point compactification [investigated in Section 2–2], as well as Stone–Čech compactification [investigated in Section 2–4].

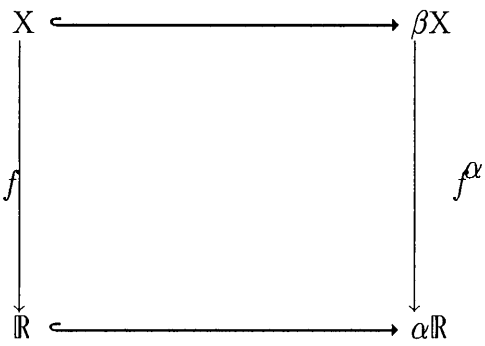
All the spaces considered in this chapter will be assumed completely regular, unless indicated otherwise.

3–1. Introduction

DEFINITION 3–1.1

Let X be a completely regular space and βX its Stone–Čech compactification. The subspace $\beta X - X$, denoted by X^*, is called the ***growth of X***.

Any member $f \in C(X)$ is a mapping of X into $\alpha\mathbb{R} = \mathbb{R} \cup \{\infty\}$ (the one–point compactification of $\mathbb{R}$). Thus, f has an extension f^{α} which maps βX into $\alpha\mathbb{R}$ (this follows from Theorem 2–4.3). The situation is illustrated in the diagram below.



If f is unbounded, then there exists a point in the growth of X at which f^{α} will take the value ∞. (This follows from the fact that f is unbounded on X if and only if $|M(f)|$ is infinitely large – proved in [GJ], section 5.7; and $|M(f)|$ is infinitely large if and only if f^{α} reaches infinity – proved in [GJ], section 7.6.)

DEFINITION 3–1.2

Let X be a completely regular space and βX its Stone–Čech compactification. For each map $f \in C(X)$, define

$$\nu_f X \equiv \beta X - \{p \in \beta X : f^{\alpha}(p) = \infty\}.$$

Thus, $\nu_f X$ is the set of points of βX at which f^α is finite. The set $\nu_f X$ is called the set of ***real points of f***.

3–2. Realcompact spaces

Realcompact spaces were originally defined and investigated as "Q–spaces" by E. Hewitt in 1948. Other names employed by various writers include: e–complete, functionally closed, Hewitt, real–complete, saturated.

The importance of these spaces is demonstrated in Theorem 3–4.1 (Hewitt's Isomorphism Theorem) and by the existence, for any topological space X, of a unique realcompactification in which X is C–embedded (as a comparison, recall that X is C^*–embedded in its Stone–Čech compactification). Hewitt also derived many of the properties of realcompact spaces, although a lot of the theory was independently developed (but not published) by L. Nachbin within the framework of the theory of uniform spaces.

DEFINITION 3–2.1

Let X be a completely regular space and βX its Stone–Čech compactification. Let νX be the subspace of βX consisting of points which are real points for every map $f \in C(X)$; i.e. define

$$\nu X = \cap \{\nu_f X : f \in C(X)\}.$$

The space νX is called the ***realcompactification of X***.

The realcompactification νX of X is sometimes referred to as the *"Nachbin completion"* of X.

DEFINITION 3–2.2

A completely regular space X is said to be ***realcompact*** if $X=\nu X$, i.e. if the only points which are real points for every $f \in C(X)$ are the points of X itself.

EXAMPLE 3–2.1

Let X be a compact space.

Then it is immediate from the definition that X is realcompact.

Evidently, the subspace νX of βX is the largest subspace of βX to which every member f of $C(X)$ can be extended in such a way that no extension of f has the value ∞. In other words, νX is the largest subspace of βX in which X is C–embedded.

Let $\nu(f)$ denote the extension of f to νX. Then the correspondence $f \to \nu(f)$ is an isomorphism of $C(X)$ with $C(\nu X)$.

The subspace νX of βX can be viewed as a "dividing line" between two different types of elements of βX. The distinction between the points in νX and those in $\beta X - \nu X$ can be recognized through the corresponding z–ultrafilters.

DEFINITION 3–2.3

Let p be a point in βX. From Chapter 2 we know that there exists a one–to–one correspondence between maximal ideals of $C(X)$ and z–ultrafilters on X. Consider the z–ultrafilter A^p corresponding with the maximal ideal M^p.

A^p is called a ***real z–ultrafilter*** if the intersection of any countable subfamily of A^p is not empty. The ideal M^p is called a ***real ideal*** if A^p is a real z–ultrafilter.

The characteristic features of realcompact spaces can also be described in terms of zero–sets and z–ultrafilters, as shown in the following theorem.

<u>THEOREM 3–2.1</u> (Hewitt)

The following are equivalent for any completely regular space X:

(1) X is realcompact.

(2) Every point of $\beta X - X$ is contained in a zero–set of βX which misses X.

(Recall from Theorem 2–4.17 that the zero–sets of βX are countable intersections of closures in βX of zero–sets of X.)

(3) Every real z–ultrafilter on X is fixed.

<u>PROOF</u>

(1)$\Leftrightarrow$(2): Let $X = \nu X$.

Then for every $p \in \beta X - X$ there exists a mapping $f \in C(X)$ such that $f > 0$ and $f^{\alpha}(p) = \infty$.

Let g be the reciprocal of f. Let $\beta(g)$ map the space βX to $\beta\mathbb{R}$.

Then $p \in Z(\beta(g)) \subset \beta X - X$.

(The converse is obtained by reversing the steps.)

(2)$\Rightarrow$(3): Let $p \in \beta X - X$.

By (2), there exists a zero–set Z such that $p \in Z \subset \beta X - X$.

But $Z = \cap \mathrm{cl}_{\beta X} Z_n$ and $\cap \mathrm{cl}_{\beta X} Z_n \subset \beta X - X$, thus at $\cap Z_n = \emptyset$.

So the corresponding free z–ultrafilter A^p is not real.

(3)⇒(2): Let $p \in \beta X - X$, so that A^p is not real.

Then there exists a countable family $\{Z(f_n)\}$, contained in A^p, such that $\cap Z(f_n) = \emptyset$ and $f_n \in C^*(X)$ for every n.

Then the zero–set $Z(\beta(f_n))$ of the extension of f_n contains p.

So $\cap Z(\beta(f_n))$ is a zero–set containing p and missing X. ■

<u>EXAMPLE 3–2.2</u>

This example was used in Section 2–4 to demonstrate that $\beta\mathbb{R}-\mathbb{R}$ is a zero–set of $\mathbb{R}$. Here it will be used to show that $\mathbb{R}$ is realcompact.

Let the mapping f on $\mathbb{R}$ be defined by $f(x) = 1/(1+|x|)$. Let h denote the extension of f to $\beta\mathbb{R}$. f never vanishes on $\mathbb{R}$, but approaches zero on every noncompact subset of $\mathbb{R}$.

So $h(p)=0$ for every $p \in \beta\mathbb{R}-\mathbb{R}$.

Thus $\beta\mathbb{R}-\mathbb{R}$ is a zero–set of $\mathbb{R}$.

In other words, every point of $\beta\mathbb{R}-\mathbb{R}$ is contained in a zero–set of βX which misses $\mathbb{R}$.

Thus, by (2) of Theorem 3–2.1, $\mathbb{R}$ is realcompact.

<u>EXAMPLE 3–2.3</u>

A similar argument to that in the preceding example can be used to show that the space of natural numbers, $\mathbb{N}$, is realcompact.

Consider the identity function **i** on $\mathbb{N}$.

Let M be any real maximal ideal in $C(\mathbb{N})$.

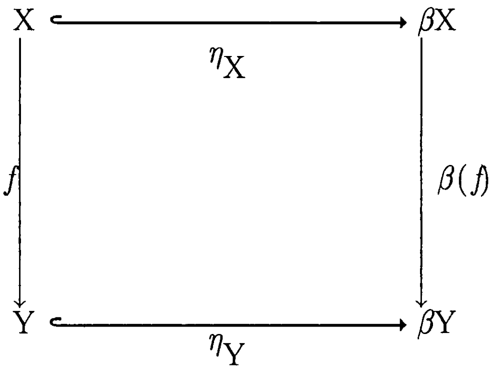
Then $M(\mathbf{i}) \in \mathbb{R}$, say $M(\mathbf{i})=r$. So $\mathbf{i} \equiv \mathbf{r} (\text{mod } M)$.

But then $Z(\mathbf{i}-\mathbf{r})$, which contains at most one point, must belong to $Z[M]$.

This is impossible if M is a free ideal.

Therefore, every real ideal is fixed. Thus, $\mathbb{N}$ is a realcompact space.

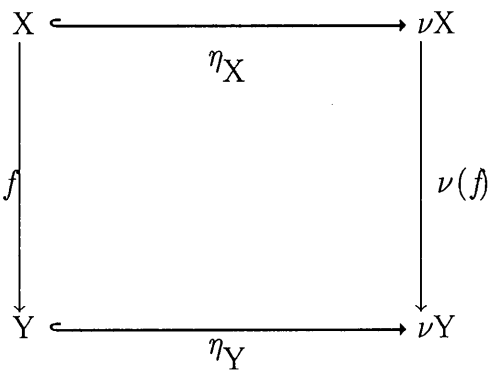
Let f be a mapping of X to Y. As discussed in Section 2–4, there exists an extension $\beta(f)$ of f such that the following diagram commutes:



An analogous result for νX can be established by showing that the image of the restriction $\beta(f)|_{\nu X}$ is a subspace of νY. The result will then follow from the definition of the realcompactification of X.

<u>THEOREM 3–2.2</u>

Let X and Y be completely regular spaces. Let f be a mapping of X to Y and let $\beta(f):\beta X \to \beta Y$ be the extension of f. Let $\nu(f)$ be the restriction $\beta(f)|_{\nu X}$. Then the following diagram commutes:

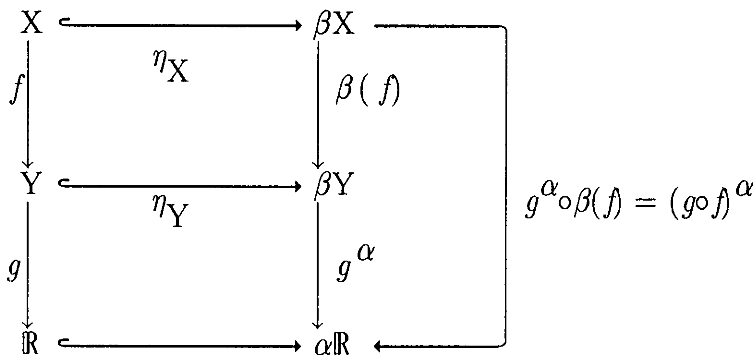


<u>PROOF</u>

Let $g \in C(Y)$, then there exists an extension $g^{\alpha}:\beta Y \to \alpha\mathbb{R}$ of g.

Similarly, $g\circ f \in C(X)$, so there exists an extension $(g\circ f)^{\alpha}:\beta X \to \alpha\mathbb{R}$.

The mappings $(g\circ f)^{\alpha}$ and $g^{\alpha}\circ\beta(f)$ both agree with $g\circ f$ on X, thus $g^{\alpha}\circ\beta(f) = (g\circ f)^{\alpha}$ and the following diagram commutes:



Now let $q \in \nu X$, then from the definition of νX it follows that q is a real point of $g\circ f$ for every $g \in C(Y)$.

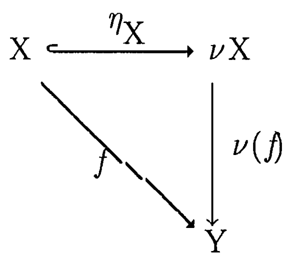
Thus, for every $g \in C(Y)$, $(g^{\alpha}\circ\beta(f))(q) = (g\circ f)^{\alpha}(q) \neq \infty$.

Hence $\beta(f)(q) \in \nu Y = \bigcap\{\nu_g Y : g \in C(Y)\}$ and the result follows. ■

The following proposition a special case of the preceding theorem.

<u>THEOREM 3–2.3</u>

Let X be a completely regular space and Y a realcompact space. Then every mapping of X into Y will extend uniquely to νX. This is demonstrated in the following diagram:



PROOF

Let X be a completely regular space and Y a realcompact space.

Y is realcompact, thus Y=νY.

The result now follows from Theorem 3–2.2. ■

The concept of *pseudocompactness* was mentioned briefly in Chapter 1. It will be now discussed in more detail.

DEFINITION 3–2.4

A space X is called ***pseudocompact*** if every real–valued continuous mapping on X is bounded, i.e. if $C(X) = C^*(X)$.

THEOREM 3–2.4 (Hewitt)

Let X be a completely regular space. Then the following statements are equivalent:

(1) X is pseudocompact.

(2) Every nonempty zero–set of βX meets X.

(3) Every z–ultrafilter on X is real.

(4) Every z–filter on X has the countable intersection property.

PROOF

(1)⇒(2): Assume that Z($\beta(f)$) is contained in βX–X.

Then f does not vanish on X and is not bounded away from 0 on X.

Thus, $\frac{1}{f}$ is an unbounded member of $C(X)$.

But X is realcompact, so $C(X)=C^*(X)$.

Thus $\frac{1}{f}$ is an unbounded member of $C^*(X)$, a contradiction.

(2)⇒(1): Assume that $C(X) \neq C^*(X)$, i.e. that there exists an unbounded member of $C(X)$.

Thus, there exists an unbounded $f \in C(X)$ which does not vanish on X.

So the zero–set of the extension of $\bar{f}$ is not empty and it misses X, a contradiction.

(2)⇔(3): Let $p \in \beta X - X$ and A^p the corresponding free z–ultrafilter. Let $\{Z_n\} \subset A^p$.

Then the zero–sets containing p are of the form $\cap \operatorname{cl}_{\beta X} Z_n$.

Since such a zero–set can miss X exactly when $\cap Z_n = \emptyset$, A^p is a real z–ultrafilter precisely when every zero–set containing p meets X.

(3)⇔(4): The proof is trivial. ■

<u>COROLLARY 3–2.5</u>

(a) A space X is compact if and only if it is realcompact and pseudocompact.

(b) $\nu X = \beta X$ if and only if X is pseudocompact.

<u>PROOF</u>

(a) By Theorem 3–2.4, every z–ultrafilter is real if and only if X is pseudocompact.

By Theorem 3–2.1, every real z–ultrafilter is fixed if and only if X is realcompact.

Thus every z–ultrafilter on X is fixed (so that $X = \beta X$ and X is compact) if and only if X is pseudocompact and realcompact.

(b) νX is the subspace of βX corresponding to the real z–ultrafilters on X.

But by Theorem 3–2.4, every z–ultrafilter is real if and only if X is pseudocompact.

Thus νX corresponds to βX if and only if X is pseudocompact. ■

3–3. Realcompact spaces within the framework of real ideals

Every residue class field of C or C^* (modulo a maximal ideal M) contains a canonical copy of the real field ℝ: the set of images of the constant functions, under the canonical homomorphism. In this discussion, the above–mentioned subfield will be identified with ℝ.

Recall from Chapter 1 (Theorem 1–2.1) that:

(a) *The fixed maximal ideals in $C(X)$ are precisely the sets*

$$\mathbf{M}_p = \{f \in C : f(p)=0\} \qquad \textit{where } p \in X.$$

The ideals $\mathbf{M}_p$ are distinct for distinct p. For each p, $C/\mathbf{M}_p$ is isomorphic with the real field ***R****; moreover, the mapping $\mathbf{M}_p(f) \mapsto f(p)$ is the unique isomorphism of $C/\mathbf{M}_p$ onto* ***R****.*

(b) *The fixed maximal ideals in $C^*(X)$ are precisely the sets*

$$\mathbf{M}_p^* = \{f \in C^* : f(p)=0\} \quad \textit{where } p \in X.$$

The ideals $\mathbf{M}_p^*$ are distinct for distinct p. For each p, $C/\mathbf{M}_p^*$ is isomorphic with the real field ***R****; moreover, the mapping $\mathbf{M}_p^*(f) \mapsto f(p)$ is the unique isomorphism of $C^*/\mathbf{M}_p^*$ onto* ***R****.*

This can now be rewritten as follows:

$$\mathrm{M}_p(f) = f(\mathrm{p}), \text{ for } f \in C(\mathrm{X});$$
$$\mathrm{M}_p^*(f) = f(\mathrm{p}), \text{ for } f \in C^*(\mathrm{X}).$$

<u>DEFINITION 3–3.1</u>

When the canonical copy of ℝ is the entire field C/M (respectively C^*/M), M is called a ***real*** ideal.

THEOREM 3–3.1

Every fixed maximal ideal in C or C^* is real.

PROOF

This follows straight from Theorem 1–2.1. ∎

DEFINITION 3–3.2

When the residue class field modulo M is not real, it is said to be a ***hyper–real*** field and M is called a ***hyper–real*** ideal.

The following theorem contains a complete characterization of the residue class fields of C^*.

THEOREM 3–3.2

(a) Every maximal ideal in C^* is real.

(b) Every maximal ideal in C is real if and only if X is pseudocompact.

PROOF

(a) Let M be an arbitrary maximal ideal in C^*.

Let $|f| \leq n$.

Then $|M(f)| \leq n$.

Thus, C^*/M contains no infinitely large elements, so it is an archimedean field [defined in Appendix A].

Thus, C^*/M is embeddable in $\mathbb{R}$ [proved in Appendix A].

Hence, M is real.

(b) Let M be an arbitrary maximal ideal in C.

Then $|M(f)|$ is infinitely large if and only if f is unbounded on X (the proof requires a number of technicalities which are not relevant to this work. For this reason, it is not cited here, but can be found in chapter 5 of [GJ].).

The result is now immediate. ■

Thus, every maximal ideal in C^*, whether fixed or free, is real by the preceding theorem. Further, every fixed maximal ideal in C is real, according to Theorem 3–3.1. So the only candidates for hyper–real ideals in rings of continuous functions are the free maximal ideals in C. These free maximal ideals in C exist whenever there are unbounded functions in C (this follows from Theorem 3–3.2).

THEOREM 3–3.3

Let X be completely regular. Let M be an arbitrary maximal ideal in C(X). Let Z[M] be the z–ultrafilter corresponding to M.

Then the following statements are equivalent:

(1) M is real.

(2) Z[M] is closed under countable intersection.

(3) Z[M] has the countable intersection property.

PROOF

(1)⇒(2) Let $(Z(f_n))_{n\in\mathbb{N}}$ be a subfamily of Z[M] whose intersection does not belong to Z[M].

Let $g(x) = \Sigma_{n\in\mathbb{N}}\inf\{|f_n(x)|, 2^{-n}\}$, $x\in X$.

Then g is continuous because of uniform convergence, non–negative, and $Z(g) = \cap_n Z(f_n)$ which is not an element of Z[M].

Thus, $M(g)\geq 0$. But $g\notin M$. Therefore, $M(g)>0$.

On the other hand, for any m∈ℕ and for every $x \in Z(f_1) \cap .. \cap Z(f_m)$, which is a member of Z[M], the following is true:

$$g(x) = \Sigma_{n>m} 2^{-n} = 2^{-m}.$$

Thus, $M(g) \leq 2^{-m}$ (this follows from general theory of maximal ideals and zero–sets). This is valid for every m∈ℕ. Hence, M(g) is infinitely small. Thus, C/M is not archimedean [see Appendix A for the definition].

(2)⇒(3) $\emptyset \notin$ Z[M].

Thus, the proof is trivial.

(3)⇒(1) Assume that M is hyper–real.

Then there exists f for which $|{}'M(f)|$ is infinitely large.

Thus, the zero–sets $\{x: |f(x)| \geq n\}_{n \in \mathbb{N}}$ are all in Z[M] (this result is proved in chapter 5 of [GJ], the proof is of no relevance to this work, so it is not cited here). Obviously, their intersection is empty, which is a contradiction. ■

It is natural to try to determine which, if any, spaces satisfy the condition that *every* free maximal ideal in C is hyper–real. As it turns out, the class of spaces satisfying this requirement is precisely the class of realcompact spaces. This is presented in the following theorem.

THEOREM 3–3.4

X is a realcompact space if and only if every free maximal ideal in C(X) is hyper–real.

PROOF

From Theorem 3–2.1 we know that X is realcompact if and only if every real z–ultrafilter on X is fixed.

Thus, X is realcompact if and only if every free z–ultrafilter on X is hyper–real (contrapositive of the above statement).

This is equivalent to saying that X is realcompact if and only if every free maximal ideal in X is hyper–real. ■

EXAMPLE 3–3.1

As already mentioned in Example 3–2.1, compactness implies realcompactness. Presented in this example is another method of demonstrating that a compact space is also realcompact.

Let X be a compact space.

Then, according to Theorem 1–2.4, X contains no free ideals.

Thus X is realcompact.

EXAMPLE 3–3.2

Let X be pseudocompact, but not compact [for example, the space **W** of all countable ordinals, discussed in Appendix F].

Then every maximal ideal in $C(X)$ is real, according to Theorem 3–3.2.

But not every maximal ideal in $C(X)$ is fixed, according to Theorem 1–2.4.

Thus X is not realcompact.

3–4. Realcompact spaces within the framework of rings of continuous functions

Gelfand and Kolmogoroff observed that if $\mathbb{N}$ is the countable discrete space of natural numbers, then the rings $C^*(\mathbb{N})$ and $C^*(\beta\mathbb{N})$ are isomorphic, but $C(\mathbb{N})$ and

$C(\beta\mathbb{N})$ are not. Thus, C is a more sensitive invariant than C^* for distinguishing between topological spaces.

The class of realcompact spaces with respect to C plays a role analogous to that played by the class of compact spaces with respect to C^*.

In Chapter 1, Theorem 1–2.3 established C^* as an algebraic invariant for the class of *compact* spaces. The following theorem shows that C is an algebraic invariant for the class of *realcompact* spaces.

THEOREM 3–4.1 (Hewitt's Isomorphism Theorem)

Let X and Y be realcompact spaces. Then X is homeomorphic to Y if and only if $C(X)$ and $C(Y)$ are isomorphic.

PROOF

"$\Rightarrow$": The proof is trivial.

"$\Leftarrow$": Let X and Y be realcompact spaces.

Let $p \in X$. Then $p \mapsto M^p$ is a one–to–one correspondence from X onto the set of all real maximal ideals in $C(X)$.

Further, the property of being a real maximal ideal is preserved under the ring isomorphism of $C(X)$ with $C(Y)$.

Thus, the points of X can be recovered from the algebraic structure of $C(X)$.

Furthermore, the purely algebraic relation $f \in M^p$ is equivalent to the relation $p \in Z(f)$.

Also, the family of zero–sets is a base for the closed sets in X.

Thus, the topology of X can be recovered from $C(X)$.

A similar argument shows that the topology of Y can be recovered from $C(Y)$, which is given isomorphic to $C(X)$.

Thus X is homeomorphic to Y. ■

Thus, another one of the objectives of this thesis is fulfilled. From Theorem 1–2.3 it follows that two *compact* spaces are homeomorphic if and only if their rings of *continuous bounded* functions are isomorphic. Theorem 3–4.1 now leads to the following conclusion: two *realcompact* spaces are homeomorphic if and only if their rings of *continuous* (bounded and unbounded) functions are isomorphic. Of course, the second condition is a generalization of the first, since every compact space is also realcompact (this does not yield a contradiction, because every continuous function on a compact space is bounded).

4. Nonlinear PDEs – an application of the theory of rings of continuous functions

[Sources: [R] Elemer E. Rosinger, *"Global Version of the Cauchy–Kovalevskaia Theorem for Nonlinear PDEs"*]

Note: This chapter deals with global generalized solutions, as well as sets of "patched up" solutions for arbitrary analytic nonlinear PDEs. Section 4–1 explains further the objective of this work. Section 4–2 defines the necessary concepts and proves some basic results. The crux of the matter, namely the existence of the above–mentioned solutions, is presented in Section 4–3.

For the purposes of this chapter it will be assumed that the reader is familiar with the concept of partial differential equations (PDEs for short), rings of continuous functions [investigated in Chapter 1], nowhere dense sets, linear and nonlinear operators, dual spaces, T–convex sets, rings, ideals [discussed in Appendix D], spaces of all m–times continuously differentiable functions on the space $\mathbb{R}^n$ (denoted by $\mathcal{C}^m(\mathbb{R}^n)$), quotient rings [discussed in Appendix C], noncharacteristic analytic hypersurfaces, ordinals [studied in Appendix F], zero–sets [defined in Appendix B], the original version of the Cauchy–Kovalevskaia theorem.

4–1. Introduction

This chapter presents a global version of the well–known Cauchy–Kovalevskaia theorem for nonlinear partial differential equations. The principal objective of this chapter is to prove the existence of global generalized solutions for arbitrary analytic nonlinear PDEs on the whole of their domains of analyticity. It is also shown that these solutions are analytic outside of closed, nowhere dense subsets.

Another aim is to present one global and universal principle which can define sets of "patched up" solutions for arbitrary analytic nonlinear PDEs. Such a principle of finding general solutions, however, does not necessarily lead to *unique* solutions for given analytic and noncharacteristic Cauchy problems.

The global existence theorem discussed in this chapter naturally has both the strong and the weak points of a general existence result. Its disadvantage is that in the case of *particular* analytic nonlinear PDEs, stronger results may be obtained by using more specific methods. Among the advantages is the fact that this result can be applied to *arbitrary* analytic nonlinear PDEs, by far most of which were not known to have any global solutions whatsoever.

The proofs cited here are based on constructions within rings of continuous functions on Euclidean spaces, together with classical calculus and elements of topology in Euclidean spaces. This is consistent with the fact that the original proof of the Cauchy–Kovalevskaia theorem employs only classical calculus, in spite of the power of the result itself.

It is known that trying to embed the L.Schwartz distributions into differential rings presents major difficulties. That, in turn, implies difficulties in the construction of suitable nonlinear theories of generalized functions. Fortunately, most of these difficulties can be reduced to algebraic problems in rings of continuous functions.

This work will serve as an example of an effective construction which overcomes the above–mentioned difficulties, and at the same time yields a global version of the Cauchy–Kovalevskaia theorem. The construction is based on so–called *nowhere dense ideals*, which are defined later in this chapter.

4–2. Definitions and basic results

Let Ω be a domain in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Consider $T(D): \mathcal{C}^m(\Omega) \to \mathcal{C}^0(\Omega)$, a linear operator of order m. This mapping of $\mathcal{C}^m(\Omega)$ to $\mathcal{C}^0(\Omega)$ is, in general, not necessarily surjective.

However, if $\mathscr{D}'(\Omega)$ denotes the space of L.Schwartz distributions, then we have the following diagram:

$$\begin{array}{ccc} \mathcal{C}^m(\Omega) & \xrightarrow{\quad T(D) \quad} & \mathcal{C}^0(\Omega) \\ \downarrow & & \downarrow \\ \mathscr{D}'(\Omega) & \xrightarrow{\quad T(D) \quad} & \mathscr{D}'(\Omega) \end{array}$$

in which the mapping $T(D): \mathscr{D}'(\Omega) \mapsto \mathscr{D}'(\Omega)$ is surjective if Ω is T–convex.

DEFINITION 4–2.1

Let Ω be a domain in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $\ell \in \mathbb{N} \cup \{\infty\}$.

The set of all sequences

$$s = (\varphi_\nu : \nu \in \mathbb{N}) \in (\mathcal{C}^\ell(\Omega))^{\mathbb{N}}$$

of $\mathcal{C}^\ell$–smooth functions which converge weakly in $\mathscr{D}'(\Omega)$ to a distribution, is denoted by $S^\ell(\Omega)$.

The set of all sequences

$$s = (\varphi_\nu : \nu \in \mathbb{N}) \in (\mathcal{C}^\ell(\Omega))^{\mathbb{N}}$$

of $\mathcal{C}^\ell$–smooth functions which converge weakly in $\mathscr{D}'(\Omega)$ to zero, is denoted by $V^\ell(\Omega)$.

It is known that there exists the vector space isomorphism

$$S^\ell(\Omega)/V^\ell(\Omega) \to \mathscr{D}'(\Omega)$$

defined by

$$s + V^\ell(\Omega) \to S, \ s \in S^\ell(\Omega)$$

where

$$S(\chi) = \lim_{\nu \mapsto \infty} \int_\Omega \varphi_\nu(x)\chi(x)dx, \ \chi \in \mathscr{D}(\Omega).$$

DEFINITION 4–2.2

Let $I_{nd}(\Omega)$ denote the set of all sequences of continuous functions

$$w = (\omega_\nu : \nu \in \mathbb{N}) \in (\mathcal{C}^0(\Omega))^{\mathbb{N}}$$

which satisfy the condition

$\exists\ \Gamma\subset\Omega$ closed, nowhere dense:

$\forall\ x\in\Omega\setminus\Gamma$:

$\exists\ \mu\in\mathbb{N}$:

$\forall\ \nu\in\mathbb{N},\ \nu\geq\mu$:

$\omega_\nu(x) = 0.$

Then $\mathcal{I}_{nd}(\Omega)$ is called the ***nowhere dense ideal*** on Ω.

It is easy to check that $\mathcal{I}_{nd}(\Omega)$ is an ideal in $(\mathcal{C}^0(\Omega))^{\mathbb{N}}$.

Definition 4–2.2 can be replaced by the following equivalent one:

<u>DEFINITION 4–2.3</u>

$\mathcal{I}_{nd}(\Omega)$ denotes the set of all sequences of continuous functions

$$w = (\omega_\nu:\ \nu\in\mathbb{N}) \in (\mathcal{C}^0(\Omega))^{\mathbb{N}}$$

which satisfy the condition

$\exists\ \Gamma\subset\Omega$ closed, nowhere dense:

$\forall\ x\in\Omega\setminus\Gamma$:

$\exists\ \mu\in\mathbb{N},\ V\subset\Omega\setminus\Gamma$ a neighborhood of x:

$\forall\ \nu\in\mathbb{N},\ \nu\geq\mu,\ y\in V$:

$\omega_\nu(y) = 0.$

Let $\ell\in\mathbb{N}\cup\{\infty\}$. Let $p=(p_1,\ldots,p_n)\in\mathbb{N}^n$ be such that $|p| = \sum_{i=1}^{n} p_i \leq \ell$. Let D^p denote the usual p–th order partial derivative, applied termwise to sequences of smooth functions.

Then the following relation is easily obtained from Definition 4–2.3:

$$D^p(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}}) \subset \mathcal{I}_{nd}(\Omega).$$

Consider now a subring $\mathcal{A} \subset (\mathcal{C}^0(\Omega))^{\mathbb{N}}$ which satisfies the following two conditions:

(C1) $\mathcal{I}_{nd}(\Omega) \subset \mathcal{A}$

(C2) $\mathcal{U}(\Omega) \subset \mathcal{A}$, where $\mathcal{U}(\Omega)$ is the subring of all the sequences with identical terms $u(\varphi) = (\varphi,\varphi,\ldots,\varphi,\ldots)$ corresponding to arbitrary continuous functions $\varphi \in \mathcal{C}^0(\Omega)$.

Of course, such rings do exists, take for instance $\mathcal{A} = (\mathcal{C}^0(\Omega))^{\mathbb{N}}$.

Because $\mathcal{C}^0(\Omega)$ is associative and commutative, it follows that the quotient ring

$$A = \mathcal{A}/\mathcal{I}_{nd}(\Omega)$$

is an associative and commutative ring. The unity element of A is

$$1_A = u(1) + \mathcal{I}_{nd}(\Omega) \in A.$$

Moreover, consider the mapping of $\mathcal{C}^0(\Omega)$ into A, defined by mapping every element φ of $\mathcal{C}^0(\Omega)$ to the element $u(\varphi) + \mathcal{I}_{nd}(\Omega)$ of A. Since $\mathcal{I}_{nd}(\Omega) \cap \mathcal{U}(\Omega) = \mathcal{O}$, it follows that the mapping is an embedding of rings.

We now turn our attention to nonlinear partial differential operators on spaces of generalized functions. Let $m \in \mathbb{N}$ represent the order of the nonlinear partial differential operators considered in the remainder of this section.

DEFINITION 4–2.4

Let $\overline{m}$ denote the cardinality of the set $\{p \in \mathbb{N}^n : |p| \leq m\}$.

Consider an arbitrary continuous function $F \in \mathcal{C}^0(\Omega \times \mathbb{R}^{\overline{m}})$.

Then we define the ***m–th order nonlinear partial differential operator***

$$T(D):\mathcal{C}^m(\Omega)\to\mathcal{C}^0(\Omega)$$

by $(T(D)U)(x) = F(x,U(x),\ldots,D^pU(x),\ldots)$, where $p\in\mathbb{N}^n$, $|p|\leq m$, $x\in\Omega$, $U\in\mathcal{C}^m(\Omega)$.

DEFINITION 4–2.5

The following quotient rings can be defined for $\ell\in\mathbb{N}\cup\{\infty\}$:

$\mathcal{A}^\ell = (\mathcal{A}\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}})/(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}})$.

Let $\ell\in\mathbb{N}\cup\{\infty\}$ be given. To see how to extend the partial derivative operators to the quotient rings A^ℓ, suppose that the following condition is additionally satisfied by the subring $\mathcal{A}$:

$$D^p(\mathcal{A}\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}})\subset\mathcal{A},\ p\in\mathbb{N}^n,\ |p|\leq\ell.$$

Given $k\in\mathbb{N}\cup\{\infty\}$, $p\in\mathbb{N}^n$, $|p|+k\leq\ell$, the mapping

$$D^p:A^\ell\to A^k$$

can obviously be defined by

$$s+(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}})\to D^ps+(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^k(\Omega))^{\mathbb{N}})$$

for $s\in\mathcal{A}\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}}$.

DEFINITION 4–2.6

Let $\ell\in\mathbb{N}\cup\{\infty\}$.

Consider $\bar{\mathcal{A}}$, a subring in $(\mathcal{C}^0(\Omega))^{\mathbb{N}}$ containing $\mathcal{I}_{nd}(\Omega)\cup T(D)(\mathcal{A}\cap(\mathcal{C}^\ell(\Omega))^{\mathbb{N}})$.

Define the quotient ring $\bar{A}=\bar{\mathcal{A}}/\mathcal{I}_{nd}(\Omega)$.

PROPOSITION 4–2.1

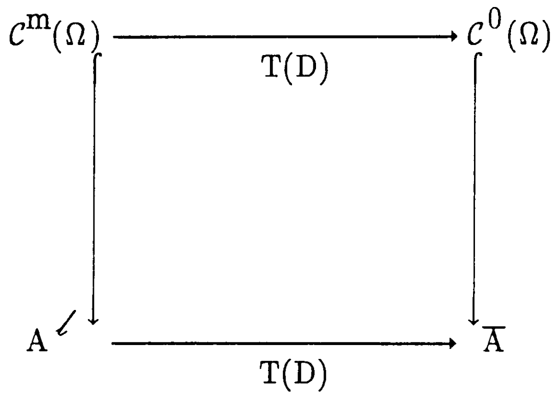
Let $\ell\in\mathbb{N}\cup\{\infty\}$ be such that $\ell\geq m$.

Then the mapping $T(D):\mathcal{C}^m(\Omega)\to\mathcal{C}^0(\Omega)$ in Definition 4–2.4 can be extended to a mapping $T(D):A^{\checkmark}\to\overline{A}$. This is achieved by defining

$$s+(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}})\to T(D)s+\mathcal{I}_{nd}(\Omega)$$

for all $s=(\varphi_\nu:\nu\in\mathbb{N})\in\mathcal{A}\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}}$, where $T(D)s=(T(D)\varphi_\nu:\nu\in\mathbb{N})$.

This is illustrated in the diagram below:



PROOF

Recall that $D^p(\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}})$.

This, together with $v\in\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}}$, implies that $T(D)v\in\mathcal{I}_{nd}(\Omega)$.

Thus, $T(D)(t+v)\in T(D)t+\mathcal{I}_{nd}(\Omega)$.

It follows that $T(D)(t+v)+\mathcal{I}_{nd}(\Omega)\in T(D)t+\mathcal{I}_{nd}(\Omega)$.

Thus, $T(D)(t+v)-T(D)t\in\mathcal{I}_{nd}(\Omega)$.

Therefore, the following property is obtained:

$$\forall\ t\in\mathcal{A}\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}},\ v\in\mathcal{I}_{nd}(\Omega)\cap(\mathcal{C}^{\checkmark}(\Omega))^{\mathbb{N}}:$$
$$T(D)(t+v)-T(D)t\in\mathcal{I}_{nd}(\Omega).$$

Now it follows from general abstract ring theory that the above property is equivalent to the fact that the extension $T(D):A^{\checkmark}\to\overline{A}$ is well–defined. ■

The importance of the extension mentioned in the preceding proposition arises from the fact that the global version of the Cauchy–Kovalevskaia theorem (proved later in this chapter) will yield generalized solutions which are elements in A^{ℓ}.

4–3. The global version of the Cauchy–Kovalevskaia theorem for nonlinear PDEs

DEFINITION 4–3.1

Let G be an arbitrary analytic function. Consider the domain Ω in $\mathbb{R}^n$. On Ω, consider the ***m–th order analytic nonlinear partial differential equation***

$$D_t^m U(t,y) = G(t,y,\ldots,D_t^p D_y^q U(t,y),\ldots)$$

with $t\in\mathbb{R}$, $y\in\mathbb{R}^{n-1}$, such that $x=(t,y)\in\Omega$, $m\geq 1$, $p\in\mathbb{N}$, $0\leq p<m$, $q\in\mathbb{N}^{n-1}$, $p+|q|\leq m$.

Let the following be given:

the noncharacteristic analytic hypersurface

$$S = \{x=(t,y)\in\Omega\colon t=t_o\} \neq \emptyset;$$

and the analytic Cauchy data

$$D_t^p U(t_o,y) = g_p(y),$$

where $0\leq p<m$, g_p is an analytic function, $(t_o,y)\in S$.

The result presented in the following lemma is an existence result.

LEMMA 4–3.1

There exists $\Gamma\subset\Omega$ closed, nowhere dense and an analytic function $\varphi:\Omega\backslash\Gamma\to\mathbb{C}$ which is a solution of the above–defined PDE on $\Omega\backslash\Gamma$ and satisfies the given analytic Cauchy data.

PROOF

Let $x=(t_o,y) \in S$ arbitrarily.

Then the Cauchy–Kovalevskaia theorem yields a nonvoid open set

$$\Omega_x \in \Omega, \text{ with } x \in \Omega_x$$

and an analytic function which is a solution of our PDE on Ω_x.

By analytic continuation, we obtain an analytic solution of the PDE on the nonvoid open set

$$\Omega^1 = \bigcup_{x \in S} \Omega_x \qquad (*).$$

There are two possibilities: Ω^1 is dense in Ω, or it is not. These cases will be handled separately.

Case 1: Ω^1 is dense in Ω.

Let $\Gamma = \Omega \backslash \Omega^1$.

Then Ω^1 because is open, it follows that Γ is closed.

Further, consider an arbitrary open set in Ω, say U.

Because Ω^1 is dense in Ω, it follows that $cl_\Omega \Omega^1 = \Omega$.

Thus, $U \cap \Omega^1 \neq \emptyset$.

So U is not contained in $\Omega \backslash \Omega^1 = \Gamma = cl_\Omega \Gamma$.

Thus, the interior of $cl_\Omega \Gamma$ is empty.

Therefore, Γ is nowhere dense in Ω, and the proof is completed.

Case 2: Ω^1 is not dense in Ω.

Let Ω_1 be the interior of $\Omega \backslash \Omega^1$. So Ω_1 is nonvoid and open.

Let Γ_1 be closed, nowhere dense.

Then the following partition is obtained:

$$\Omega = \Omega_1 \cup \Gamma_1 \cup \Omega^1.$$

Consider $x_1 = (t_1,y_1) \in \Omega_1$.

Then, with $t_1 \in \mathbb{R}$ given by x_1, define

$$S_1 = \{x=(t,y) \in \Omega_1 : t=t_1\} \neq \emptyset$$

which is a noncharacteristic analytic hypersurface for the PDE.

On S_1, consider any given analytic Cauchy data

$$D_t^p U(t_1,y) = g_{1p}(y), \text{ where } p \in \mathbb{N},\ p<m,\ (t_1,y) \in S_1.$$

In this manner, the original problem of proving the lemma for Ω and S is reduced to the problem of proving it for Ω_1 and S_1. This reduction yields an iterative process, which leads to one of the following situations: either after a finite number of iterations Case 1 is reached, or not. These are discussed below.

Alternative 1: Case 1 reached after a finite number of iterations.

In other words, for some $h \geq 1$, the following sequence of partitions is obtained:

$$\Omega_o = \Omega_1 \cup \Gamma_1 \cup \Omega^1 \ (= \Omega)$$

....

$$\Omega_{h-1} = \Omega_h \cup \Gamma_h \cup \Omega^h$$

$$\Omega_h = \Gamma_{h+1} \cup \Omega^{h+1} \text{ (i.e., } \Omega_{h+1} = \emptyset)$$

where $\Omega_o,\ldots,\Omega_h$ and $\Omega^1,\ldots,\Omega^{h+1}$ are nonvoid open; while $\Gamma_1,\ldots,\Gamma_{h+1}$ are closed, nowhere dense.

Thus, we obtained an analytic solution of the PDE on the nonvoid open set

$$\Omega' = \Omega^1 \cup \ldots \cup \Omega^{h+1}.$$

$$\begin{aligned} \text{Now, } cl_\Omega \Omega' &= \bigcup_{i=1}^{h+1} cl_\Omega \Omega^i && \text{(because h+1 is finite)} \\ &= \bigcup_{i=1}^{h+1} cl_\Omega((\Omega_{i-1} \backslash \Omega_i) \backslash \Gamma_i) \\ &= \bigcup_{i=1}^{h+1} \Omega_{i-1} \backslash \Omega_i && \text{(because } \Omega^i = \text{interior}(\Omega_{i-1} \backslash \Omega_i)) \end{aligned}$$

$= \Omega_0 \backslash \Omega_{h+1}$

$= \Omega$ (because $\Omega_{h+1} = \emptyset$).

Therefore, Ω' is dense in Ω.

Thus, we can take $\Gamma = \Omega \backslash \Omega'$, and the proof is completed.

Alternative 2: Case 1 not reached after a finite number of iterations.

Suppose Case 1 is not reached after any finite number of iterations.

Consider an arbitrary ordinal number $\alpha \geq 1$.

Then the open set $\Omega_\alpha \subset \Omega$ is obtained in one of the two ways described below.

(1) If $\alpha = \beta+1$ for a suitable ordinal number β, and the nonvoid open set $\Omega_\beta \subset \Omega$ has already been obtained, then construct the nonvoid open set $\Omega^\alpha \subset \Omega$ according to (*).

Further, define the nonvoid open set

$$\Delta_\alpha = \bigcup_{\gamma \leq \alpha} \Omega^\gamma$$

and take Ω_α equal to the interior of $\Omega \backslash \Delta_\alpha$.

(2) If $\alpha \neq \beta+1$ for any ordinal number β, define the nonvoid open set

$$\Delta_\alpha = \bigcup_{\beta < \alpha} \Omega^\beta$$

and again take Ω_α equal to the interior of $\Omega \backslash \Delta_\alpha$.

This process can be continued until $\Omega_\alpha = \emptyset$ is reached, i.e., the interior of $\Omega \backslash \Delta_\alpha$ is empty.

Thus, we obviously have an analytic solution of the PDE on the nonvoid open set $\Omega' = \Delta_\alpha$.

This means that $\mathrm{cl}_\Omega \Omega' = \mathrm{cl}_\Omega \Delta_\alpha$

$= \Omega \backslash \mathrm{interior}(\Omega \backslash \Delta_\alpha)$

$= \Omega \backslash \Omega_\alpha$

$= \Omega$ (because $\Omega_\alpha = \emptyset$).

Thus, Ω' is dense in Ω, so the proof is completed by taking $\Gamma = \Omega \backslash \Omega'$. ■

Consider again the m–th order analytic nonlinear partial differential equation in Definition 4–3.1:

$$D_t^m U(t,y) = G(t,y,\dots,D_t^p D_y^q U(t,y),\dots)$$

with $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, such that $x=(t,y) \in \Omega$, $m \geq 1$, $p \in \mathbb{N}$, $0 \leq p < m$, $q \in \mathbb{N}^{n-1}$, $p+|q| \leq m$, and with the analytic Cauchy data

$$D_t^p U(t_0,y) = g_p(y),\ 0 \leq p < m,\ (t_0,y) \in S$$

on the noncharacteristic analytic hypersurface

$$S = \{x=(t,y) \in \Omega: t=t_0\} \neq \emptyset.$$

Let $\varphi:\Omega \backslash \Gamma \to \mathbb{C}$ be an analytic solution of the PDE as given by Lemma 4–3.1. Then $\Gamma \subset \Omega$ is closed and nowhere dense. Because Γ is closed, it follows that there exists $\gamma \in \mathcal{C}^\infty(\Omega)$ such that $\Gamma = \{x \in \Omega: \varphi(x) = 0\}$, in other words, Γ is the zero–set of γ.

Suppose we are given a $\mathcal{C}^\infty$–smooth function $\alpha:\mathbb{R} \to [0,1]$, such that

$$\alpha(x) = 0 \text{ for } |x| < a$$

$$\alpha(x) = 1 \text{ for } |x| > b$$

for certain $0 < a < b \infty$.

Then the following definition is obtained.

<u>DEFINITION 4–3.2</u>

Let s be the sequence of $\mathcal{C}^\infty$–smooth functions

$$s = (\varphi_\nu: \nu \in \mathbb{N}) \in (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$$

where, for $\nu \in \mathbb{N}$, the following is valid:

$$\varphi_\nu(x) = \begin{cases} \alpha((\nu+1)\gamma(x))\varphi(x) & \text{if } x\in\Omega\backslash\Gamma, \text{ i.e. } \gamma(x)\neq 0 \\ 0 & \text{if } x\in\Gamma, \text{ i.e. } \gamma(x)=0 \end{cases} .$$

Then the sequence s of $\mathcal{C}^\infty$–smooth functions, defined above, has the property

$\forall\ x\in\Omega\backslash\Gamma$:

$\exists\ \mu\in\mathbb{N}$, $V\subset\Omega\backslash\Gamma$ a neighborhood of x:

$\forall\ \nu\in\mathbb{N}$, $\nu\geq\mu$, $y\in V$:

$\varphi_\nu(y) = \varphi(y)$.

As before, define the nonlinear partial differential operator

$$T(D):\mathcal{C}^m(\Omega)\mapsto\mathcal{C}^0(\Omega)$$

by $(T(D)U)(t,y) = D_t^m U(t,y) - G(t,y,\ldots,D_t^p D_t^q U(t,y),\ldots)$
with $x=(t,y)\in\Omega$, $0\leq p<m$, $q\in\mathbb{N}^{n-1}$, $p+|q|\leq m$.

With the above notation, our PDE can be rewritten as

$$T(D)U(x) = 0,\ x\in\Omega.$$

<u>PROPOSITION 4–3.2</u>

Consider the usual PDE, defined in Definition 4–3.1, and the above–mentioned sequence s of $\mathcal{C}^\infty$–smooth functions.

Then $T(D)s \in \mathcal{I}_{nd}(\Omega) \cap (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$.

<u>PROOF</u>

As already mentioned, $\varphi:\Omega\backslash\Gamma\rightarrow\mathbb{C}$ is an analytic solution of the PDE.

Thus, $T(D)\varphi(x) = 0$ for $x\in\Omega\backslash\Gamma$.

So,
$$\varphi_\nu(x) \xrightarrow[\infty]{\nu} \begin{cases} \varphi(x) & \text{if } x\in\Omega\backslash\Gamma \\ 0 & \text{if } x\in\Gamma \end{cases}$$

Therefore, $T(D)\varphi_\nu(x) \xrightarrow[\infty]{\nu} 0$ for $x\in\Omega$.

Thus, $T(D)s \in \mathcal{I}_{nd}(\Omega)$.

But from the definition of s it is clear that $T(D)s \in (\mathcal{C}^{\infty}(\Omega))^{\mathbb{N}}$.

Hence, $T(D)s \in \mathcal{I}_{nd}(\Omega) \cap (\mathcal{C}^{\infty}(\Omega))^{\mathbb{N}}$. ■

With these preparations, it is now possible to construct the rings of generalized functions within which our analytic nonlinear partial differential equation has a *global* generalized solution for the noncharacteristic analytic Cauchy data cited above.

Consider a subring $\mathcal{A} \subset (\mathcal{C}^{0}(\Omega))^{\mathbb{N}}$ which satisfies the conditions

(CC1) $s \in \mathcal{A}$;

(CC2) $\mathcal{I}_{nd}(\Omega) \cup S^{0}(\Omega) \subset \mathcal{A}$ ($S^{0}(\Omega)$ as defined in Definition 4–2.1).

Clearly, such subrings do exists: for example, $\mathcal{A} = (\mathcal{C}^{0}(\Omega))^{\mathbb{N}}$ satisfies (CC1) and (CC2).

Consider again the m–th order analytic nonlinear partial differential equation in Definition 4–3.1:

$$D_t^m U(t,y) = G(t,y,\ldots,D_t^p D_y^q U(t,y),\ldots)$$

with $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, such that $x=(t,y) \in \Omega$, $m \geq 1$, $p \in \mathbb{N}$, $0 \leq p < m$, $q \in \mathbb{N}^{n-1}$, $p+|q| \leq m$, and with the analytic Cauchy data

$$D_t^p U(t_0,y) = g_p(y),\ 0 \leq p < m,\ (t_0,y) \in S$$

on the noncharacteristic analytic hypersurface

$$S = \{x=(t,y) \in \Omega: t=t_0\} \neq \emptyset.$$

(As shown before, the above PDE is equivalent to $T(D)U(x) = 0$, $x \in \Omega$.)

The theory presented in this chapter culminates in the next theorem, which can be referred to as the *global existence* result for the above PDE.

THEOREM 4–3.3

For $\ell \in \mathbb{N}$, $m \leq \ell$, the m–th order analytic nonlinear partial differential equation

$$T(D)U(x) = 0,\ x=(t,y) \in \Omega$$

with the noncharacteristic analytic Cauchy data

$$D_t^p U(t_0,y) = g_p(y),\ 0 \leq p < m,\ (t_0,y) \in S$$

and
$$S = \{x=(t,y) \in \Omega : t=t_0\} \neq \emptyset$$

has generalized solutions

$$U \in A^\ell$$

defined on the whole of Ω.

These solutions U are analytic functions $\varphi : \Omega \backslash \Gamma \to \mathbb{C}$ when restricted to open dense subsets $\Omega \backslash \Gamma$, for suitable closed, nowhere dense $\Gamma \subset \Omega$.

PROOF

Recall that: $\mathcal{A}^\ell = (\mathcal{A} \cap (\mathcal{C}^\ell(\Omega))^{\mathbb{N}})/(\mathcal{I}_{nd}(\Omega) \cap (\mathcal{C}^\ell(\Omega))^{\mathbb{N}})$;

$\mathcal{A}$ is a subring in $\mathcal{C}^0(\Omega))^{\mathbb{N}}$ containing $\mathcal{I}_{nd}(\Omega) \cup T(D)(\mathcal{A} \cap (\mathcal{C}^\ell(\Omega))^{\mathbb{N}})$;

$\overline{A} = \mathcal{A}/\mathcal{I}_{nd}(\Omega)$.

As described earlier,

$$T(D) : \mathcal{C}^m(\Omega) \to \mathcal{C}^0(\Omega)$$

can be extended to

$$T(D) : A^\ell \to \overline{A}.$$

Because $s \in \mathcal{C}^\ell(\Omega))^{\mathbb{N}}$ and $s \in \mathcal{A}$, it is possible to define

$$U = s + (\mathcal{I}_{nd}(\Omega) \cap (\mathcal{C}^\ell(\Omega))^{\mathbb{N}}) \in A^\ell.$$

It remains to prove that the mapping $T(D) : A^\ell \to \overline{A}$ yields

$$T(D)U = 0 \ \in \overline{A}.$$

But because of the definition of U, $T(D)U = T(D)s + \mathcal{I}_{nd}(\Omega)$.

Further, Proposition 4–3.2 yields $T(D)s \in \mathcal{I}_{nd}(\Omega) \cap (\mathcal{C}^\infty(\Omega))^{\mathbb{N}}$.

Hence, $T(D)s \in \mathcal{I}_{nd}(\Omega)$.

So $T(D)U = \mathcal{I}_{nd}(\Omega)$

$= 0 \in \quad \mathcal{A}/\mathcal{I}_{nd}(\Omega) = \overline{A}$. ∎

From the freedom of choice in the proof of Lemma 4–3.1 and the preceding theorem, it should be clear that, in general, many solutions of the form $\varphi{:}\Omega\backslash\Gamma\to\mathbb{C}$ (where $\Gamma\subset\Omega$ is closed and nowhere dense) can be obtained in the manner presented above.

APPENDIX A
Ordered Sets, Axiom of Choice

[Sources: [GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;

[W] Erwin Kreyszig, *"Introductory functional analysis with applications"*]

A–1. Partially ordered sets

DEFINITION A–1.1

A ***partial ordering*** is a binary relation which is written $\leq$ and satisfies the conditions

(1) $a \leq a$ for every $a \in M$ (reflexivity);

(2) If $a \leq b$ and $b \leq a$, then $a = b$ (anti–symmetry);

(3) If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

DEFINITION A–1.2

A ***partially ordered set*** is a set M on which a partial ordering is defined.

The word "partially" emphasizes the fact that M may contain elements a and b which are incomparable, i.e. neither $a \leq b$ nor $b \leq a$ holds.

EXAMPLE A–1.1

Let $M = \mathbb{N}$, the set of all positive integers. Let $m \leq n$ mean that m divides n. This defines a partial ordering on $\mathbb{N}$. $\mathbb{N}$ is now called a partially ordered set.

DEFINITION A–1.3

A mapping φ from a partially ordered set A into a partially ordered set E is said to ***preserve order*** if $a \leq b$ in A implies $\varphi a \leq \varphi b$ in E.

DEFINITION A–1.4

A ***maximal*** element of A is an element a (not necessarily unique) such that, for any $x \in A$, $x \geq a$ implies $x = a$.

A ***minimal*** element of A is an element a (not necessarily unique) such that, for any $x \in A$, $x \leq a$ implies $x = a$.

It is important to note the difference between the maximal and the largest element. The largest element of A, unique if it exists, is the element c such that $c \geq x$ for all $x \in A$. Naturally, there exists a corresponding difference between the minimal and the smallest element of a set.

EXAMPLE A–1.2

Let $M = \mathscr{P}(X)$, the set of all subsets of a given set X. Let $A \leq B$ mean that A is a subset of B. Then $\mathscr{P}(X)$ is a partially ordered set. The only maximal element of $\mathscr{P}(X)$ is X itself.

DEFINITION A–1.5

A ***pre–ordering*** is a binary relation which is written $\leq$ and satisfies the conditions

(1) $a \leq a$ for every $a \in M$ (reflexivity);

(2) If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

DEFINITION A–1.6

A ***pre–ordered set*** is a set M on which a pre–ordering is defined.

Thus a partially ordered set is actually a pre–ordered set which, in addition, satisfies the anti–symmetry property. An example of a pre–ordered set which is not partially ordered is presented in Section 2–4.

A–2. Lattices

DEFINITION A–2.1

Let A be a partially ordered set. If both $\sup\{a,b\}$ and $\inf\{a,b\}$ exist for all $a,b \in$ A, then A is said to be a ***lattice.***

DEFINITION A–2.2

A subset S of A is a ***sublattice*** provided that, for all $a,b \in$ S, the elements $\sup\{a,b\}$ and $\inf\{a,b\}$ of A belong to S (i.e. it is not sufficient that a and b have a supremum and infimum in S).

DEFINITION A–2.3

A mapping φ from a lattice A into a lattice E is a ***lattice homeomorphism into*** E provided that

(1) $\varphi(\sup\{a,b\}) = \sup\{\varphi a,\varphi b\}$ and

(1) $\varphi(\inf\{a,b\}) = \inf\{\varphi a,\varphi b\}$.

EXAMPLE A–2.1

Let φ be a lattice homeomorphism from A into E. Then $\varphi[A]$ is a sublattice of E.

A–3. Totally ordered sets

DEFINITION A–3.1

Let M be a partially ordered set such that every two elements of the set are comparable. Then M is called a ***totally ordered set*** or a ***chain***.

EXAMPLE A–3.1

Let M be the set of all real numbers and let $x \leq y$ have its usual meaning (i.e. x is less than or equal to y). Then M is a totally ordered set and M has no maximal elements.

DEFINITION A–3.2

Let M be a totally ordered set and let B be a subset of M. If there exists $b \in M$ such that $x \leq b$ for every $x \in B$, then B is said to be ***bounded above***.

The element b is called an ***upper bound*** for B.

If the set of all upper bounds for B has a smallest element, that element is called the ***least upper bound*** of B (***lub*** for short).

The terms ***bounded below***, ***lower bound*** and ***greatest lower bound*** (or ***glb***) are defined correspondingly.

DEFINITION A–3.3

A totally ordered field F is said to be ***archimedean*** if for every element $a \in F$ there exists $n \in \mathbb{N}$ such that $n \geq a$; otherwise F is said to be ***non–archimedean***.

Thus, a non–archimedean field is characterized by the presence of *infinitely large* elements (i.e. elements a such that $n<a$ for every $n\in\mathbb{N}$).

DEFINITION A–3.4

Let A be a partially ordered set.

Then (L,R) is called a ***Dedekind cut*** provided that:

(1) $L = \{x\in A\colon$ every $r\in R$ is such that $x<r\}$;

(1) $R = \{x\in A\colon$ every $l\in L$ is such that $l<x\}$.

EXAMPLE A–3.2

Let $A = \mathbb{R}$ in the preceding definition.

Choose any $a\in\mathbb{R}$.

Let $L = (-\infty,a)$ and $R = (a,+\infty)$.

Then (L,R) is a Dedekind cut in $\mathbb{R}$.

(Note that the example remains true even if a is included in *either* L *or* R, but not in both.)

EXAMPLE A–3.3

Let $A = \mathbb{Q}$ in the preceding definition.

Then $\mathbb{R} = \{(L,R)\colon (L,R)$ is a Dedekind cut in $\mathbb{Q}\}$.

THEOREM A–3.1

An ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field $\mathbb{R}$.

PROOF

"$\Leftarrow$": Clearly, every subfield of $\mathbb{R}$ is archimedean.

"$\Rightarrow$": Let F be any archimedean field.

Let $x<y\in$ F.

Choose $n\in\mathbb{N}$ such that $n > 1/(y-x)$.

Let m be the smallest integer larger than nx.

Then $x < m/n < y$.

Thus, $\mathbb{Q}$ is dense in F, so that every element of F is uniquely determined by a Dedekind cut of $\mathbb{Q}$.

Consequently, F is embeddable in $\mathbb{R}$ in a unique way as an ordered set.

Now let r and s belong to the ordered field F. Let a,b,c,d$\in\mathbb{Q}$ satisfying a $\leq$ r < b and c $\leq$ s < d. Then a+c $\leq$ r+s < b+d.

Thus, addition in F (just like addition in $\mathbb{R}$) are uniquely determined by Dedekind cuts of $\mathbb{Q}$. A similar argument shows that products are also determined in this manner.

Therefore, the embedding of F is an isomorphism. ■

THEOREM A–3.2

The only non–zero homomorphism of $\mathbb{R}$ into itself is the identity.

PROOF

A real number is non–negative if and only if it is a square.

Since any homomorphism takes squares to squares, it takes non–negative numbers to non–negative numbers. Thus, it is order–preserving.

Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a non–zero homomorphism.

Then φr = (φr)(φ1) for every r$\in\mathbb{R}$. Thus, $\varphi 1 = 1$.

Consequently, φ is the identity on $\mathbb{Q}$.

But $\mathbb{Q}$ is dense in $\mathbb{R}$ and φ preserves order.

Thus, φ is the identity on $\mathbb{R}$ as well. ∎

COROLLARY A–3.3

(a) There exists at most one isomorphism from a ring onto $\mathbb{R}$.

(b) Any homomorphism onto $\mathbb{R}$ is uniquely determined by its kernel.

PROOF

(a) Let u and v be isomorphisms from the same ring onto $\mathbb{R}$.

Then $u \circ \overleftarrow{v}$ is an automorphism of $\mathbb{R}$, and hence the identity.

Thus, $u=v$.

(b) Let s and t be homomorphisms from a ring A onto $\mathbb{R}$, with common kernel I.

Consider the associated isomorphisms $\bar{s}$ and $\bar{t}$ from A/I onto $\mathbb{R}$ (i.e., such that $s = \bar{s} \circ h$ and $t = \bar{t} \circ h$, where h is the canonical homomorphism of A onto A/I).

Since $\bar{s} = \bar{t}$, it follows that $s = t$. ∎

A–4. Axiom of Choice

HAUSDORFF'S MAXIMAL PRINCIPLE

Every partially ordered set contains a maximal chain (i.e. maximal in the class of all chains as partially ordered by set inclusion).

This proposition is equivalent to the axiom of choice and to the well–ordering theorem. It is also equivalent to Zorn's Lemma (stated below).

ZORN'S LEMMA

Let M be a nonempty partially–ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then M has at least one maximal element.

APPENDIX B
Zero–sets, Completely separated sets

[Sources: [*GJ*] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*]

B–1. Introduction to zero–sets

The following definition introduces the concept of *zero–sets*, which was originally discussed in E. Hewitt's paper in 1948. These sets are of use when studying the relation between topological properties of X and algebraic properties of $C(X)$.

DEFINITION B–1.1

Consider a space X and a function $f \in C(X)$. Consider further the subset of X of the form

$$f^{\leftarrow}(0) = \{x \in X : f(x) = 0\}.$$

Notation: The set $f^{\leftarrow}(0)$ will be called the ***zero–set*** of f. For convenience it will also be denoted by $Z(f)$ or, for clarity, $Z_X(f)$.

Let $C' \subset C(X)$. The family of zero–sets $\{Z(f) : f \in C'\}$ will be denoted $Z[C']$.

For simplicity, $Z(X)$ or $Z[X]$ will designate the family $Z[C(X)]$ of *all* zero–sets in X.

EXAMPLE B–1.1

Consider $\mathbb{N}$, the set of all positive integers. Let $\mathbf{i}(n)=n$ for all $n\in\mathbb{N}$.

Then $Z(\mathbf{i}) = \emptyset$.

EXAMPLE B–1.2

Let **0** and **1** denote the constant functions on some topological space X.

Clearly $Z(\mathbf{0}) = \{x\in X: \mathbf{0}(x)=0\} = X$

and $Z(\mathbf{1}) = \{x\in X: \mathbf{1}(x)=0\} = \emptyset$.

B–2. Points to note

(1) It should be clear that zero–sets are closed in X.

(2) Any set that is a zero–set of some function in $C(X)$ is called a zero–set in X. Thus, Z is actually a mapping from the ring C onto the set of all zero–sets in X.

(3) $Z(f) = Z(|f|) = Z(f^n)$ for all $n\in N$.

(4) Let $f\in C(X)$ and $g=|f|\wedge 1$. Then $g\in C^*(X)$. Further $Z(f)=Z(g)$ because $Z(\mathbf{1}) = \emptyset$. Hence C and C^* yield the same zero–sets.

(5) In a metric space, every closed set is a zero–set, because it consists precisely of all the points whose distance from it is zero.

(6) $Z(X)$ is closed under finite unions.

(7) $Z(X)$ is closed under countable intersections. For: given $f_n\in C$, let $g_n = \inf\{|f_n|, 2^{-n}\}$ and let $g(x) = \Sigma_{n\in\mathbb{N}} g_n(x)$ where $x\in X$. Since $|g_n| \leq 2^{-n}$, the series converges uniformly, and thus g is continuous. Clearly,

$$Z(g) = \cap_{n\in\mathbb{N}} Z(g_n) = \cap_{n\in\mathbb{N}} Z(f_n).$$

(8) Z(X) is a base for the closed sets of X.

(9) Z(X) is disjunctive, i.e.if a closed subset A of X does not contain a point $x \in X$, then there exists a member of Z(X) containing x and missing A. This is because the zero–set neighborhoods form a base for the neighborhood of a point.

B–3. Completely separated sets

DEFINITION B–3.1

Two subsets A and B of X are ***completely separated in*** X if there exists a function f in $C^*(X)$ such that

$$f[A] = \{0\}, \quad f[B] = \{1\} \quad \text{and} \quad \mathbf{0} \leq f \leq \mathbf{1}.$$

It is actually enough to find a function $g \in C(X)$ satisfying: $g(x) \leq 0$ for all $x \in A$ and $g(x) \geq 1$ for all $x \in B$. If such a function g is found, then $\inf\{\mathbf{1}, \sup\{\mathbf{0}, g\}\}$ has the required properties.

Of course, the numbers 0 and 1 may be replaces in the definition by any numbers $r, s \in \mathbb{R}$ with $r < s$.

It is obvious that two sets contained respectively in two completely separated sets are also completely separated. It also plain that two sets are completely separated if and only if their closures are.

THEOREM B–3.1

Two sets are completely separated if and only if they are contained in disjoint zero–sets. Moreover, completely separated sets have disjoint zero–set neighborhoods (i.e. they have disjoint neighborhoods which are zero sets).

PROOF

"$\Leftarrow$": Let $Z(f) \cap Z(g) = \emptyset$, then $|f| + |g|$ has no zeros, so we may define

$$h(x) = |f(x)| / (|f(x)| + |g(x)|) \quad \text{for } x \in X.$$

Then $h \in C(X)$, $h[Z(f)] = \{0\}$, $h[Z(g)] = \{1\}$.

"$\Rightarrow$": Conversely, let A and B be completely separated.

Then there exists $f \in C(X)$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

The (disjoint!) sets

$$F = \{x: f(x) \leq 1/3\}, \quad G = \{x: f(x) \leq 2/3\}$$

are zero–set neighborhoods of A and B respectively. ■

APPENDIX C
Residue class Rings

[Sources: [P] Charles C.Pinter, *"A book of Abstract Algebra"*;
[S] R.Y.Sharp, *"Steps in Commutative Algebra"*]

Note: Throughout Appendix C it will be assumed that the reader is familiar with a few basic concepts from group and ring theory, namely group, ring, commutative ring, Abelian group, subgroup, ideal [the latter concept is studied in Appendix D].

C–1. Cosets (residue classes)

DEFINITION C–1.1

Let I be an ideal of the commutative ring R. Let $r \in R$. The ***coset*** (or the ***residue class***) of I in R determined by, or containing, r is the set

$$r+I = \{r+z: z \in I\}.$$

Note that, for $r, s \in R$, the cosets $r+I$ and $s+I$ are equal if and only if $r-s \in I$. In fact, the cosets of I in R are precisely the equivalence classes of the equivalence relation ~ on R defined by

$$a \sim b \text{ if and only if } a-b \in I \text{ for } a, b \in R.$$

The set of all cosets of I in R will be denoted by R/I.

C–2. Factor groups (residue class groups)

Let I and J be ideals of the commutative ring R such that $I \subseteq J$. Clearly, R is an Abelian group with respect to addition. Further, I and J are subgroups of R; thus I is a subgroup of J.

We can now construct the factor group J/I. The elements of this group are the cosets of I in J; thus

$$J/I = \{a+I : a \in J\}$$

and, for $a, b \in J$, we have $a+I = b+I$ if and only if $a-b \in I$.

The addition in J/I is such that

$$(a+I)+(b+I) = (a+b)+I \qquad \text{for all } a,b \in J.$$

To verify that this operation is well–defined (i.e. unambiguous), let $a, a', b, b' \in J$ be such that

$$a+I = a'+I, \qquad b+I = b'+I.$$

Then $a-a', b-b' \in I$, so that $(a+b)-(a'+b') \in I$ and

$$(a+b)+I = (a'+b')+I.$$

It is routine to check that this operation is commutative. Thus J/I is an Abelian group.

DEFINITION C–2.1

Let I and J be ideals of the commutative ring R such that $I \subseteq J$. Construct the Abelian group J/I as described above. Then J/I is called the ***factor group***, or ***residue class group***, of J modulo I.

C–3. Factor rings (residue class rings)

DEFINITION C–3.1

Let I be an ideal of the commutative ring R. Then R/I (the set of all cosets of I in R) is called a ***factor ring***, or a ***residue class ring***, of R modulo I.

To construct R/I and to establish that it is, in fact, a ring, consider again I which is an ideal of the commutative ring R. Of course, R is an ideal of itself, and the construction described in Section C–2 can be applied to form the factor group R/I.

We will now discuss how to put a ring structure on the Abelian group R/I. Let $r, r', s, s \in R$ be such that

$$r + I = r' + I, \quad s + I = s' + I.$$

Then $r - r', s - s \in I$, so that

$$\begin{aligned} rs - r's' &= rs - rs' + rs' - r's' \\ &= r(s - s') + (r - r')s \in I. \end{aligned}$$

This implies that $rs + I = r's' + I$. The operation of multiplication on R/I is thus unambiguously defined by the rule

$$(r + I)(s + I) = rs + I \qquad \text{for all } r, s \in R.$$

It is straightforward to verify that this operation is commutative. Thus R/I is a commutative ring with identity 1+I and the zero element 0+I=I.

EXAMPLE C–3.1

Let $n \in \mathbb{N}$, the set of positive integers. Let $\mathbb{Z}$ be the ring of all integers. Then the set

$$n\mathbb{Z} = \{nr: r \in \mathbb{Z}\}$$

of all integer multiples of n is an ideal of $\mathbb{Z}$, as can be easily verified. Using the construction described above, one can form the residue class ring $\mathbb{Z}/n\mathbb{Z}$. This ring is just the ring of residue classes of integers modulo n, often denoted $\mathbb{Z}_n$.

EXAMPLE C–3.2

Let R={0,1,2,3,4,5}=$\mathbb{Z}_6$ (described in the previous example). Let I={0,3}.
Then the elements of R/I are the three residue classes

$$I = I+0 = \{0,3\},$$
$$I+1 = \{1,4\},$$
$$\text{and } I+2 = \{2,5\}.$$

APPENDIX D
Ideals, z–Filters and z–Ultrafilters

[Sources: [GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;
[P] Charles C. Pinter, *"A book of Abstract Algebra"*;
[W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: For the purposes of this appendix it will be assumed that the reader is familiar with the definitions of commutative rings, discrete spaces, zero–sets [discussed in Appendix B], rings of continuous functions [presented in Chapter 1], completely regular spaces and completely separated sets [the latter two concepts are investigated in Appendix E].

D–1. Ideals

In the study of the relations between algebraic properties of the ring of continuous functions $C(X)$ and topological properties of the space X, it is important to examine the special features of the family of zero–sets of an *ideal* of functions.

DEFINITION D–1.1

A nonempty subset I of a ring A is called an ***ideal*** of A if I is closed with respect to addition and negatives (i.e. if it is a subring of A), and if I absorbs products in A (i.e. if $gf \in I$ whenever $f \in I$, for any $g \in A$).

EXAMPLE D–1.1

Consider the ring $\mathbb{Z}$ of all integers and its subset **E** of all even integers. Clearly **E** is an ideal of the ring $\mathbb{Z}$: the sum of two even integers is even, the negative of any integer is even, and the product of an even integer by *any* integer is always even.

It should be noted that the intersection of any nonempty family of ideals is an ideal.

DEFINITION D–1.2

Let A be a ring and let I be an ideal of A. If I is the zero ideal $0\cdot A$ or I=A, then I is called a ***trivial*** ideal of A, otherwise I is called a ***proper*** ideal of A.

DEFINITION D–1.3

Let I be a proper ideal in a ring A. If I is not strictly contained in any proper ideal of A (i.e. if $I \subseteq K$, where K is an ideal containing some element not in I, then necessarily K=A), then I is called a ***maximal*** ideal of A.

EXAMPLE D–1.2

Consider the ring $\mathbb{Z}$ of all integers. Then the ideals $\{\ldots,-6,-3,0,3,6,\ldots\}$ and $\{\ldots,-4,-2,0,2,4,\ldots\}$ are examples of maximal ideals in $\mathbb{Z}$.

In fact, let p be any prime number, then $p\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$.

EXAMPLE D–1.3

Consider the ring $\mathbb{Z}$ of all integers. Then the ideal $\{\ldots,-8,-4,0,4,8,\ldots\}$ is *not* a maximal ideal in $\mathbb{Z}$, because it is strictly contained in the proper ideal

$$2\mathbb{Z} = \{\ldots,-4,-2,0,2,4,\ldots\}.$$

DEFINITION D–1.4

Let I be any ideal in $C(X)$ or $C^*(X)$. Let Z[I] be a zero–set of I. If $\cap Z[I]$ is nonempty, I is called a ***fixed*** ideal, otherwise I is called a ***free*** ideal.

DEFINITION D–1.5

Let A be a commutative ring and let I be an ideal of A. If $ab \in I$ implies that $a \in I$ or $b \in I$, then I is called a ***prime*** ideal of A.

EXAMPLE D–1.4

Consider the ring $\mathbb{Z}$ of all integers and its zero ideal I=\{0\}. Let $a,b \in \mathbb{Z}$ such that $ab=0$. This necessarily implies that $a=0$ or $b=0$, and naturally $0 \in I$. Thus I is a prime ideal in $\mathbb{Z}$.

Remark: Let A be a commutative ring and M a maximal ideal in A. Then M is necessarily a prime ideal in A. The converse, however, is not true: for example \{0\} is a prime ideal of the ring $\mathbb{Z}$ (above), but not a maximal ideal, since $\{0\} \subset 2\mathbb{Z} \subset \mathbb{Z}$.

D–2. Filters and Ultrafilters

DEFINITION D–2.1

Let $\mathfrak{F}$ be a nonempty family of subsets on X, such that

(1) $\phi \notin \mathfrak{F}$;

(2) if $F_1, F_2 \in \mathfrak{F}$, then $F_1 \cap F_2 \in \mathfrak{F}$; and

(3) if $F \in \mathfrak{F}$, $F' \in X$, and $F' \supset F$, then $F' \in \mathfrak{F}$;

then $\mathfrak{F}$ is called a ***filter*** on X.

Thus a filter is a nonempty family of subsets of X, not containing the empty set, closed under the formation of finite intersections and of supersets.

DEFINITION D–2.2

A subcollection $\mathfrak{F}_0$ of $\mathfrak{F}$ is called a ***filter base*** if each element of $\mathfrak{F}$ contains some element of $\mathfrak{F}_0$, i.e. if $\mathfrak{F} = \{F \subset S\colon F_0 \subset F \text{ for some } F_0 \in \mathfrak{F}_0\}$.

DEFINITION D–2.3

Let $\mathfrak{F}$ be a filter on X. If $\cap\{F\colon F \in \mathfrak{F}\} \neq \phi$ we call $\mathfrak{F}$ a ***fixed filter***, otherwise we call $\mathfrak{F}$ a ***free filter***.

EXAMPLE D–2.1

Let A be a nonempty subset of a space X.

Then $\mathcal{A} = \{Y \subseteq X\colon A \subseteq Y\}$ is a filter on X. Moreover, it is a fixed filters because $\cap\{F\colon F \in \mathcal{A}\} = A \neq \phi$.

DEFINITION D–2.4

A filter $\mathfrak{F}$ is an ***ultrafilter*** provided that there is no strictly finer filter $\mathfrak{G}$ than $\mathfrak{F}$. Thus the ultrafilters are the maximal filters.

THEOREM D–2.1

A filter $\mathfrak{F}$ on X is an ultrafilter if and only if for each $E \subset X$, either $E \in \mathfrak{F}$ or $X - E \in \mathfrak{F}$.

PROOF

"⇒": Let $\mathfrak{F}$ be an ultrafilter on X. Let $E \subset X$.

Every element F of $\mathfrak{F}$ meets either E or X–E.

Hence, since no two elements of $\mathfrak{F}$ have empty intersections, they must all meet one or the other. Say $F \cap E \neq \emptyset$ for all $F \in \mathfrak{F}$.

Then $\{F \cap E: F \in \mathfrak{F}\}$ is a filter base for a filter $\mathfrak{G}$ finer than $\mathfrak{F}$ which contains E.

But $\mathfrak{G}$ cannot be strictly finer than $\mathfrak{F}$, so it must be equal to $\mathfrak{F}$, and hence $E \in \mathfrak{F}$.

"⇐": Let $\mathfrak{F}$ contain E or X–E for each $E \subset X$.

Assume that $\mathfrak{G}$ is a strictly finer filter than $\mathfrak{F}$.

Then, for some $A \in \mathfrak{G}$, $A \notin \mathfrak{F}$.

But then $X - A \in \mathfrak{F}$ (from the condition). So $X - A \in \mathfrak{G}$ (because $\mathfrak{F} \subset \mathfrak{G}$).

Thus $X - A \in \mathfrak{G}$ and $A \in \mathfrak{G}$, a contradiction.

Hence $\mathfrak{F}$ is maximal. ■

DEFINITION D–2.5

A filter $\mathfrak{F}$ is called a ***fixed ultrafilter*** if there exists an $x \in X$ such that $\mathfrak{F} = \{F \subset X: x \in F\}$, otherwise $\mathfrak{F}$ is called a ***free ultrafilter***.

EXAMPLE D–2.2

A filter $\mathfrak{F}$ on X is a fixed ultrafilter if and only if $\mathfrak{F} = \{F \subset X: x \in F\}$ for some $x \in F$. By the criterion given in Theorem D–2.1, each filter of this form is an ultrafilter.

On the other hand, if $\mathfrak{F}$ is a fixed ultrafilter, say $\cap \mathfrak{F} = A \neq \emptyset$, then $\mathfrak{F}$ must be the filter of all sets containing A (since this is a filter containing $\mathfrak{F}$). Further, A must be a single point, since the filter of all sets containing $x \in A$ is finer than $\mathfrak{F}$.

D–3. z–Filters and z–Ultrafilters

For the remainder of this appendix, all the topological spaces will be assumed completely regular. The concepts of z–filters, z–ultrafilters as well as their convergence will be discussed in that context.

z–Filters

This section introduces the concept of *z–filters*. The definition of a *z–filter* is analogous to the definition of a *filter*. A z–filter, however, is a topological object, while a filter is a purely set–theoretical one.

DEFINITION D–3.1

Let $\mathfrak{F}$ be a nonempty subfamily of Z(X) such that

(1) $\phi \notin \mathfrak{F}$;

(2) if $Z_1, Z_2 \in \mathfrak{F}$, then $Z_1 \cap Z_2 \in \mathfrak{F}$; and

(3) if $Z \in \mathfrak{F}$, $Z' \in Z(X)$, and $Z' \supset Z$, then $Z' \in \mathfrak{F}$;

then $\mathfrak{F}$ is called a ***z–filter*** on X.

From the definition's third property it is easy to see that the space X belongs to every z–filter. The second property can also be rewritten as

(2') if $Z_1, Z_2 \in \mathfrak{F}$, then $Z_1 \cap Z_2$ contains a member of $\mathfrak{F}$.

Every family $\mathfrak{B}$ of zero–sets possessing the intersection property is contained in a z–filter. The smallest such z–filter is the family $\mathfrak{F}$ of all zero–sets containing finite intersections of members of $\mathfrak{B}$. $\mathfrak{B}$ is then said to *generate* the z–filter $\mathfrak{F}$. When $\mathfrak{B}$ is also closed under finite intersections, it is called a *base* for $\mathfrak{F}$.

It is worthwhile to note that in a discrete space every set is a zero set, so that the objects filter and z–filter are identical in discrete spaces.

In any space X, the intersection of any filter with Z(X) is a z–filter. Conversely, if $\mathfrak{F}'$ is the smallest filter containing a given z–filter $\mathfrak{F}$, then $\mathfrak{F}' \cap Z(X) = \mathfrak{F}$.

The relationships between z–filters and ideals of $C(X)$ were originally investigated by E. Hewitt in his 1948 paper on rings of continuous functions.

The following theorem shows that the image of a proper ideal under the mapping $Z: C(X) \mapsto Z(X)$ is a z–filter. It also establishes that the pre–image of a z–filter is an ideal.

Thus, the result relates z–filters and z–ultrafilters to the ring $C(X)$.

<u>THEOREM D–3.1</u>

(a) If I is a proper ideal in $C(X)$, then the family

$$Z[I] = \{Z(f) : f \in I\}$$

is a z–filter on X.

(b) If $\mathfrak{F}$ is a z–filter on X, then the family

$$Z^{\leftarrow}[\mathfrak{F}] = \{f : Z(f) \in \mathfrak{F}\}$$

is a proper ideal in $C(X)$.

PROOF

(a) A proper ideal contains no unit and the units of $C(X)$ are those maps which have void zero–sets. Thus all members of Z[I] are nonempty.

Let $f_1, f_2 \in I$, then $f_1^2 + f_2^2 \in I$. Since $Z(f_1) \cap Z(f_2) = Z(f_1^2 + f_2^2)$, Z[I] is closed under finite intersections.

Let $Z(f)$ be in Z[I] for $f \in I$. Let $Z(g)$ contain $Z(f)$. Then $Z(g) = Z(f) \cup Z(g) = Z(fg)$ belongs to Z[I].

Thus, Z[I] is a z–filter.

(b) Let $\mathfrak{J} = Z^{\leftarrow}[\mathfrak{F}]$.

Since $\emptyset \notin \mathfrak{F}$, $\mathfrak{J}$ does not contain a unit.

Let $f, g \in \mathfrak{J}$. Then $Z(f-g) \supset Z(f) \cap Z(g)$. Further, $f-g \in \mathfrak{J}$ since $\mathfrak{F}$ is closed under supersets in Z[X] and finite intersections. Thus, $\mathfrak{J}$ is an additive subgroup.

Let $f \in \mathfrak{J}$ and $g \in C(X)$. Then $Z(fg) \supset Z(f)$ and $fg \in \mathfrak{J}$ since $\mathfrak{F}$ is closed under supersets in Z[X]. ■

It is important to note that, in general, the C^* analogue of the preceding theorem is not true. If J is an ideal in C^*, then Z[J] need not satisfy the first property of the definition of a z–filter. This is illustrated in the example below.

EXAMPLE D–3.1

Consider the space $\mathbb{N}$ of all positive integers. The set J of all sequences that converge to zero is obviously an ideal in $C^*(\mathbb{N})$. Now $j = \{1/n : n \in \mathbb{N}\}$ is an element of J, and Z(j) is empty; thus $\emptyset \in Z[J]$ and hence Z[J] is the family $Z(\mathbb{N})$ of all subsets of $\mathbb{N}$.

We note that one can see Z and $Z^{\leftarrow}$ as mappings, and that, for $\mathfrak{F} \subset Z(X)$, they satisfy the relations

$$Z[Z^{\leftarrow}[\mathfrak{F}]] = \mathfrak{F} \quad \text{and} \quad Z^{\leftarrow}[Z[I]] \supseteq I.$$

$Z[Z^{\leftarrow}[\mathfrak{F}]] = \mathfrak{F}$ implies that every z–filter is of the form Z[J] for some ideal J in the ring C. The second relation is illustrated in the following example.

EXAMPLE D–3.2

Consider the space $\mathbb{R}$ and let **i** denote the identity function on $\mathbb{R}$. Let I=(**i**) be the principal ideal (i.e. the ideal generated by **i**) in $C(\mathbb{R})$. Then I consists of all functions $f \in C(\mathbb{R})$ such that $f(x)=xg(x)$ for some $g \in C(\mathbb{R})$. Now in particular, every function in I vanishes at 0; thus every zero–set in Z[I] contains the point 0. Furthermore, since Z[I] is a z–filter that includes the set {0}, it must be the family of *all* zero–sets containing 0.

The ideal $M=Z^{\leftarrow}[Z[I]]$ consists of all functions in $C(\mathbb{R})$ which vanish at 0. Hence the principal ideal I is certainly contained in M. However $M \neq I$ (because, for example, $\mathbf{i}^{1/3} \in M-I$). Yet Z[M]=Z[I]. It is important to note that M is actually a maximal ideal in $C(\mathbb{R})$.

Convergence of z–filters

The discussion turns now to convergence of z–filters on completely regular spaces.

DEFINITION D–3.2

Let $\mathfrak{F}$ be a z–filter on a completely regular space X. A point $p \in X$ is called a ***cluster point*** of $\mathfrak{F}$ if every neighborhood of p meets every member of $\mathfrak{F}$.

Thus, since the members of $\mathfrak{F}$ are closed sets, p is a cluster point of $\mathfrak{F}$ if and only if $p \in \cap \mathfrak{F}$.

DEFINITION D–3.3

Let $\mathfrak{F}$ be a z–filter on a completely regular space X. $\mathfrak{F}$ is said to ***converge*** to the ***limit*** $p \in X$ if every neighborhood of p contains a member of $\mathfrak{F}$.

If S is a nonempty subset of X, then $cl_X S$ is the set of all cluster points of the z–filter $\mathfrak{F}$ of all zero–sets containing S. This is because the zero–sets in the completely regular space X form a base for the closed sets.

Obviously, if $\mathfrak{F}$ converges to p, then p is a cluster point of $\mathfrak{F}$. Further, in a completely regular space X, every neighborhood of p contains a zero–set neighborhood of p. Hence, $\mathfrak{F}$ converges to p if and only if $\mathfrak{F}$ contains the z–filter of all zero–set neighborhoods of p.

z–Ultrafilters

DEFINITION D–3.4

Let $\mathfrak{F}$ be a z–filter on X such that $\mathfrak{F}$ is not contained in any other z–filter on X. Then $\mathfrak{F}$ is a maximal z–filter, and is referred to as a ***z–ultrafilter***.

Thus, a z–ultrafilter is a maximal subfamily of Z(X) with the finite intersection property. From Hausdorff's maximal principle [described in Appendix A], it follows that every subfamily of Z(X) with the finite intersection property is contained in some z–ultrafilter.

As was the case with z–filters, in a discrete space the concept of z–ultrafilters is the same as that of maximal filters (i.e. ultrafilters).

The following result yields a one–to–one correspondence between the z–ultrafilters on X and the maximal ideals of the ring $C(X)$.

THEOREM D–3.2

(a) Let M be a maximal ideal in $C(X)$. Then Z[M] is a z–ultrafilter on X.

(b) Let $\mathfrak{A}$ be a z–ultrafilter on X. Then $Z^{\leftarrow}[\mathfrak{A}]$ is a maximal ideal in $C(X)$.

The mapping Z is one–to–one from the set of all maximal ideals in C onto the set of all z–ultrafilters.

PROOF

Both Z and $Z^{\leftarrow}$ preserve inclusion.

Thus the result follows immediately from Theorem D–3.1. ■

As already illustrated in Example D–3.2, an ideal with a maximal z–filter, i.e. a z–ultrafilter, is not necessarily maximal itself.

THEOREM D–3.3

(a) Let M be a maximal ideal in $C(X)$. If $Z(f)$ meets every member of Z[M], then $f \in M$.

(b) Let $\mathfrak{A}$ be a z–ultrafilter on X. If a zero–set Z meets every member of $\mathfrak{A}$, then $Z \in \mathfrak{A}$.

PROOF

The two statements are equivalent by Theorem D–3.2. Further, in part (b), $\mathfrak{A} \cup \{Z\}$ generates a z–filter. As this contains the maximal z–filter $\mathfrak{A}$, it must be $\mathfrak{A}$. ■

The properties exhibited in the preceding theorem are actually characteristic of maximal ideals and z–filters: if a z–filter $\mathfrak{A}$ contains every zero–set that meets all members of $\mathfrak{A}$, then $\mathfrak{A}$ is also a z–ultrafilter.

Convergence of z–ultrafilters

THEOREM D–3.4

Let p be a cluster point of a z–filter $\mathfrak{F}$. Then at least one z–ultrafilter containing $\mathfrak{F}$ converges to p.

PROOF

Let $\mathfrak{E}$ denote the z–filter of all zero–set neighborhoods of p.

Then $\mathfrak{F} \cup \mathfrak{E}$ has the finite intersection property, and thus it is embeddable in a z–ultrafilter $\mathfrak{A}$. Since $\mathfrak{A}$ contains $\mathfrak{E}$, it converges to p. ■

If $\mathfrak{F}$ converges to p in a completely regular space, then $\cap\mathfrak{F} = \{p\}$; thus, a z–filter has at most one limit. The converse, however, is not valid, as demonstrated in the following example.

EXAMPLE D–3.3

Let $\mathfrak{F}$ consist of all subsets of $\mathbb{N}$ that contain the point 1.

Then $\cap\mathfrak{F} = \{1\}$, yet $\mathfrak{F}$ does not converge to 1.

The family of all zero–sets containing a given point p is denoted by A_p. Evidently, A_p is a z–filter. Moreover, A_p is a z–ultrafilter, because any zero–set not containing p is completely separated from $\{p\}$.

Also, because p is a cluster point of $\mathfrak{F}$ if and only if p belongs to every member of $\mathfrak{F}$, it follows that p is a cluster point of $\mathfrak{F}$ if and only if $\mathfrak{F} \subset A_p$. Immediate consequences of the above statement are that A_p is the unique z–ultrafilter converging to p; and that distinct z–ultrafilters cannot possess a common cluster point.

In contrast, it can happen that distinct *ultrafilters* have a common cluster point. Consider the space $\mathbb{N}^* = \mathbb{N} \cup \{\omega\}$. The topology on $\mathbb{N}^*$ is defined as follows: a set is open in $\mathbb{N}^*$ if it is open in $\mathbb{N}$, or if it is of the form $\mathbb{N}^* - C$ where C is a compact subset of $\mathbb{N}$. On the space $\mathbb{N}^*$, consider the filter $\mathfrak{F}$ of all sets that contain all but finitely many even integers, and the filter $\mathfrak{F}'$ of all sets that contain all but finitely many odd integers. Any ultrafilters $\mathfrak{U}$ and $\mathfrak{U}'$ containing $\mathfrak{F}$ and $\mathfrak{F}'$ respectively, are distinct, yet both converge to ω.

Note that $\mathfrak{F}$ and $\mathfrak{F}'$ are not z–filters, because they have members which are not zero–sets, namely those containing ω. To prove this, let F be a member $\mathfrak{F}$ which does not contain ω. Then F is not compact (this follows from the definition of members of $\mathfrak{F}$). Thus, $\mathbb{N}^* - F$ is not open in $\mathbb{N}^*$ and so F is not closed in $\mathbb{N}^*$. Therefore, F is not a zero–set in $\mathbb{N}^*$.

Any z–ultrafilter containing a z–filter converging to p also converges to p. Therefore, if $\mathfrak{F}$ is a z–filter converging to p, then A_p is the unique z–ultrafilter containing $\mathfrak{F}$.

APPENDIX E
Completely regular spaces

[Sources: [GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*;
[M] James R. Munkres, *"Topology. A First Course"*;
[W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: Throughout this appendix it will be assumed that the reader is familiar with the concept of compact Hausdorff spaces, regular spaces, normal spaces, locally compact spaces [defined in Section 2–1].

E–1. Introduction

DEFINITION E–1.1

Two subsets A and B of a space X are said to be ***completely separated in X*** if there exists a continuous function f such that $f(a)=0$ for all $a \in A$ and $f(b)=1$ for all $b \in B$.

DEFINITION E–1.2

A space X is ***completely regular*** if every closed subset F of X is completely separated from any point x not in F; and if each point is closed.

EXAMPLE E–1.1

Let X be any metric space with a metric d defined on it.

Choose an arbitrary closed subset F of X. Choose an arbitrary point $x \in X-F$.

For any set B in X and any point c in X, define

$$D(c,B) = \inf\{d(c,b): b \in B\}.$$

Now let $f: X \to \mathbb{R}$ be such that

$$f(y) = D(y,F)/D(x,F), \text{ for } y \in X.$$

Then f is zero on F and $f(x)=1$.

Thus, F is completely separated from x, so that X is a completely regular space.

EXAMPLE E–1.2

Let X be any locally compact Hausdorff space.

Thus, X is homeomorphic to an open subset of a compact Hausdorff space (this result can be found in [M], p.186), which in turn is homeomorphic to a completely regular space (by [M], p.237).

Therefore, X is a completely regular space.

The concept of completely regular spaces was originally introduced by Tychonoff in the year 1930. Completely regular spaces play an important role in dealing with the problem of compactification [investigated in Chapter 2]. For example, a space has to be completely regular in order to have a compactification, and every completely regular space has at least one compactification.

Moreover, the study of properties of a topological space through the method of compactification is naturally limited to investigating the completely regular spaces.

Theorem E–2.2 (Tychonoff Theorem) will show that no larger class can be studied by means of compactification. In Theorem E–2.3 we will see that no additional information can be gained by investigating algebraic properties of rings of continuous functions for any larger class of spaces. Theorem E–2.5 will establish a very useful relationship between the topology of a completely regular space and the real–valued mappings defined on the space.

<u>THEOREM E–1.1</u>

Let X be a completely regular space. Then every subspace of X is completely regular.

<u>PROOF</u>

Let Y be a subspace of X.

Let x_0 be a point of Y, and let A be a closed set of Y disjoint from x_0.

Obviously, $A = cl_X A \cap Y$. Thus $x_0 \notin cl_X A$.

Since X is completely regular, it is possible to choose a continuous function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(cl_X A) = \{0\}$.

The restriction of f to Y is the desired continuous function on Y. ■

<u>EXAMPLE E–1.3</u>

The set of real numbers, $\mathbb{R}$, is a metric space. Thus it is completely regular by Example E–1.1.

Moreover, by the preceding theorem, every subspace of $\mathbb{R}$ is completely regular.

<u>THEOREM E–1.2</u>

A product of completely regular spaces is completely regular.

PROOF

Let $X = \prod X_\alpha$ be a product of completely regular spaces.

Let $\mathbf{b} = (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from **b**.

Choose a basis element $\prod U_\alpha$ containing **b** that does not intersect A. Then $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha_1,\ldots,\alpha_n$.

Given $i \in \{1,\ldots,n\}$, choose a continuous function $f_i : X_{\alpha_i} \to [0,1]$ such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X - U_{\alpha_i}) = \{0\}$.

Let $\varphi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$. Then φ_i maps X continuously into $\mathbb{R}$ and vanishes outside $\pi_{\alpha_i}^{\leftarrow}(U_{\alpha_i})$.

The product $f(\mathbf{x}) = \varphi_1(\mathbf{x}) \cdot \ldots \cdot \varphi_n(\mathbf{x})$ equals 1 at **b** and vanishes outside $\prod U_\alpha$.

Thus the above product is the desired continuous function on X. ∎

E–2. Important results

DEFINITION E–2.1

Let $\mathfrak{F}$ be a family of mappings on X. Then $\mathfrak{F}$ is said to ***distinguish points*** if for every pair of distinct points $x,y \in X$ there exists a mapping f in $\mathfrak{F}$ such that $f(x) \neq f(y)$.

The following result, adapted from J.L. Kelly's text, *"General Topology"*, will be used to prove Tychonoff's theorem.

LEMMA E–2.1 (Embedding Lemma)

Let $\mathfrak{F}$ be a family of mappings such that every member f of $\mathfrak{F}$ maps the space X to a space Y_f. Then

(a) The evaluation mapping $e:X \to \prod Y_f$ defined by $e(x)_f = f(x)$ for all points x of X is continuous.

(b) The mapping e is an open mapping onto $e[X]$ if $\mathfrak{F}$ distinguishes points and closed sets.

(c) The mapping e is one–to–one if and only if $\mathfrak{F}$ distinguishes points.

(d) The mapping e is an embedding if $\mathfrak{F}$ distinguishes points and if $\mathfrak{F}$ distinguishes points and closed sets.

PROOF

(a) The composition of the function e with each projection is continuous, i.e. $\pi_f \circ e = f$.

Thus e is continuous.

(b) Let U be an open set in X. Let $x \in U$.

Choose $f \in \mathfrak{F}$ such that $f(x) \notin \mathrm{cl}\, f[X-U]$.

Then the set of all z in $e[X]$ such that $z_f \notin \mathrm{cl}\, f[X-U]$ is a neighborhood of $e(x)$ and is contained in $e[U]$.

Thus, $e[U]$ is open in $e[X]$.

(c) "$\Rightarrow$": Assume that $\mathfrak{F}$ does not distinguish points.

Then there exist distinct points x and y in X such that, for every $f \in \mathfrak{F}$, $f(x) = f(y)$.

Thus, $e(x) = e(y)$, so that e is not one–to–one, which contradicts the hypothesis.

"⇐": Assume that e is not one–to–one.

Then there exist distinct points x and y in X such that $e(x) = e(y)$.

Thus, $\mathfrak{F}$ does not distinguish points, a contradiction.

(d) This result follows from (b) and (c). ■

As mentioned earlier, the following theorem will now establish that completely regular spaces are precisely the class of spaces which can be studied through the method of compactification.

THEOREM E–2.2 (Tychonoff Theorem)

The completely regular spaces are precisely those spaces which can be embedded in a product of copies of the closed unit interval I=[0,1].

PROOF

Let X be a completely regular space.

Let $\mathfrak{F}$ be the family of all continuous mappings from X into the interval [0,1].

Consider the product $[0,1]^{\mathfrak{F}}$ of copies of the closed unit interval I=[0,1].

Because X is completely regular, $\mathfrak{F}$ separates points from closed sets in X.

Then, from part (d) of the Embedding Lemma, it follows that X is embedded in $[0,1]^{\mathfrak{F}}$.

Conversely, let any space X be embedded in $[0,1]^{\mathfrak{F}}$.

$[0,1]^{\mathfrak{F}}$ is compact Hausdorff, and thus completely regular.

X is embedded in $[0,1]^{\mathfrak{F}}$, thus it is homeomorphic to a subspace of $[0,1]^{\mathfrak{F}}$.

But every subspace of a completely regular space is completely regular, by Theorem E–1.1.

Therefore, X is completely regular. ■

DEFINITION E–2.2

Let X be an abstract set and consider an arbitrary subfamily C' of $\mathbb{R}^X$. The ***weak topology induced by*** **C'** ***on*** X is defined to be the smallest topology on X such that all functions in C' are continuous.

In order to understand what the preceding definition implies, consider an arbitrary function $f:X\to\mathbb{R}$. For f to be continuous it is both necessary and sufficient that the preimage under f of each open set in $\mathbb{R}$ be open. Hence, for every function in C' to be continuous, it is both necessary and sufficient that all such preimages, for all $g\in$ C', be open. Let $\mathfrak{J}$ denote the collection of all these preimages. A subset U of X belongs to $\mathfrak{J}$ if and only if there exists $g\in$ C', and an open set V in $\mathbb{R}$, such that $U=\overleftarrow{g}[V]$. If C' is not empty, then $X\in\mathfrak{J}$, because $X=\overleftarrow{g}[\mathbb{R}]$ for any $g\in$ C'.

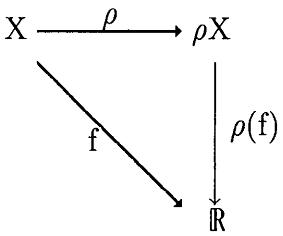
But, in general, $\mathfrak{J}$ is neither a topology nor a base for a topology for X. The weak topology generated by C' turns out to be the smallest topology containing the family $\mathfrak{J}$.

The following theorem was obtained independently by E. Čech and M.H. Stone in 1937. It establishes that completely regular spaces are not distinguishable from more general spaces through the algebraic properties of rings of their continuous functions.

THEOREM E–2.3

For any topological space X, there exists a completely regular space ρX which is a continuous image of X such that any real–valued mapping from X factors through ρX.

This situation is illustrated in the following diagram:



PROOF

Define two points x and y of X to be equivalent if $f(x) = f(y)$ for all $f \in C(X)$. This relation partitions X into equivalence classes.

Let the set of equivalence classes on X be denoted by ρX. Let $\eta: X \to \rho X$ assign to each point of X its equivalence class.

Since every $f \in C(X)$ is constant on each equivalence class, define $\rho(f): \rho X \to \mathbb{R}$ by $\rho(f)(\eta(x)) = f(x)$. This definition of $\rho(f)$ makes the above diagram commute.

Make ρX into a topological space by providing it with the smallest topology under which each $\rho(f)$ is continuous (namely the weak topology defined above).

Then the closed sets of ρX are of the form $F = \cap \rho(f_\alpha)^{\leftarrow}(F_\alpha)$, where each F_α is closed in $\mathbb{R}$. Further, ρX is Hausdorff under this topology, because points of X which are not separated by some member of $C(X)$ are identified in ρX.

If F is closed in ρX and $y \notin F$, then there exists α such that $y \notin \rho(f_\alpha)^{\leftarrow}(F_\alpha)$.

The point $\rho(f_\alpha)(y)$ is completely separated from F_α by some $g \in C(\mathbb{R})$. Further, $g \circ \rho(f_\alpha)$ completely separates y from F. Thus ρX is completely regular.

To show that η is continuous, let F be a closed subset of ρX. Then

$$\eta^{\leftarrow}(F) = \eta^{\leftarrow}(\cap \rho(f_\alpha)^{\leftarrow}(F_\alpha)) = \cap f_\alpha^{\leftarrow}(F_\alpha)$$

because the diagram commutes. But each f_α is continuous, so $\eta^{\leftarrow}(F)$ is the intersection of closed sets and is thus closed. ■

The importance of this theorem lies in the fact that the correspondence $f \to \rho(f)$ preserves both the ring and lattice structure of $C(X)$. Further, the correspondence is an algebraic and lattice isomorphism between $C(X)$ and $C(\rho X)$. Thus, algebraic and lattice properties of $C(X)$ which are valid for an arbitrary space X, also hold for $C(\rho X)$, where ρX is completely regular.

The next theorem introduces a property which characterizes the class of completely regular spaces in terms of zero–sets [discussed in Appendix B].

THEOREM E–2.4

A Hausdorff space X is completely regular if and only if the family of all zero–sets Z(X) is a base for the closed sets.

PROOF

"$\Rightarrow$": Let X be completely regular.

Let F be a closed set in X and let $x \in X-F$.

Then there exists $f \in C(X)$ such that $f(x)=1$ and $f(F)=\{0\}$.

So $F \subset Z(f)$, and $x \notin Z(f)$.

Thus, Z(X) is a base for the closed sets.

"$\Leftarrow$": Let Z(X) be a base for the closed sets.

Let F be a closed set in X and let $x \in X-F$.

Then there exists a zero–set, say $Z(g)$, such that $F \subset Z(g)$ and $x \notin Z(g)$.

Let $r=g(x)$.

Then $r \neq 0$, and the function $f=g/r$ belongs to $C(X)$.

Clearly, $f(x)=1$ and $f(F)=\{0\}$.

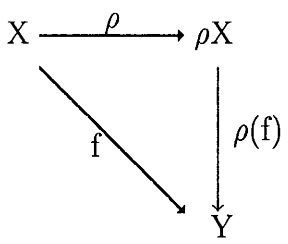
Thus X is completely regular. ■

As a matter of fact, every closed set F in a completely regular space is an intersection of zero–set neighborhoods of F. Also, every neighborhood of a point in a completely regular space contains a zero–set neighborhood of the point.

By utilizing Tychonoff's characterization of complete regularity exhibited in Theorem E–2.2, we can establish that any mapping of a space X into a completely regular space will factor through ρX. This is formulated as follows:

<u>THEOREM E–2.5</u>

If f is a mapping of the space X into a completely regular space Y, then there exists a mapping $\rho(f)$ of ρX into Y such that the following diagram commutes:



<u>PROOF</u>

For each $g \in C(Y)$, let $\mathbb{R}_g$ be a copy of the real line. Let e be the evaluation mapping embedding Y into their product, $\prod \mathbb{R}_g$.

Theorem E–2.3 yields a mapping $\rho(g \circ f)$ of ρX into $\mathbb{R}_g$ such that $g \circ f = \rho(g \circ f) \circ \eta$, because the composition $g \circ f$ maps X into $\mathbb{R}_g$.

To prove that f factors through ρX, define a mapping h of ρX to $\prod \mathbb{R}_g$ by

$$h(z)_g = \rho(g \circ f)(z) \quad \text{for all } z \in \rho X.$$

Then h is continuous.

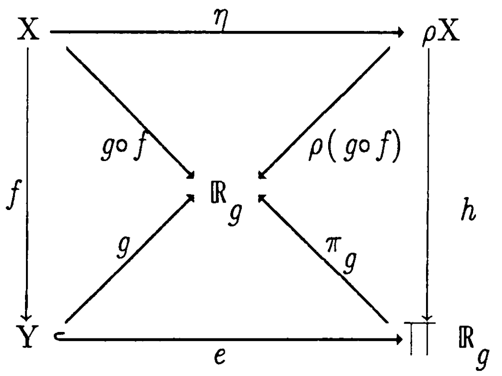
Further, $h[\rho X] \subset e[Y]$, because for each projection map π_g:

$$(\pi_g \circ h)(z) = \rho(g \circ f)(z)$$

so that $(\pi_g \circ h)[\rho X] \subset g[Y]$.

But e is an embedding and $h[\rho X]$ is contained in $e[Y]$. Thus, putting $\rho(f) = e^{\leftarrow} \circ h$ gives the required factorization of f through ρX.

This is illustrated in the following diagram:



■

The separation properties in a completely regular space yield the following technical result.

THEOREM E–2.6

In a completely regular space, any two disjoint closed sets, one of which is compact, are completely separated.

PROOF

Let X be a completely regular space.

Let K and F be disjoint closed subsets of X, with K compact.

For each $x \in K$, choose a zero–set neighborhood Z_x of x, and a zero–set Z'_x containing F and missing Z_x.

The cover $\{Z_x\}$ of K has a finite subcover $\{Z_{x_i}\}$.

Then $\cup Z_{x_i}$ and $\cap Z'_{x_i}$ are disjoint zero–sets containing K and F respectively.

Thus, K and F are completely separated. ■

The fundamental theorem about normal spaces is Urysohn's Lemma, presented next. In this work Urysohn's Lemma is used mainly to show that every compact space is completely regular.

<u>THEOREM E–2.7</u> (Urysohn's Lemma)

Any two disjoint closed sets in a normal space are completely separated. Hence every normal space is completely regular.

<u>PROOF</u>

<u>Helping–Lemma</u>

Let X be an arbitrary space, let R_0 be any dense subset of $\mathbb{R}$*. Suppose that the open sets U_r of X are defined, for all $r \in R_0$, such that*

$$\cup_r U_r = X, \quad \cap_r U_r = \emptyset,$$

and

$$clU_r \subseteq U_s \text{ whenever } r<s.$$

Let $f(x) = \inf\{r \in R_0 : x \in U_r\}$ for $x \in X$.

Then f is a continuous function on X.

<u>Proof</u>

The proof of this lemma is technical and does not provide much insight into the theory of completely regular spaces. It can be found in [GJ], section 3.12.

Let X be a normal space. Let A and B be disjoint closed sets in X.

Define open sets U_r as follows. Let $U_r=\emptyset$ for all $r<0$. Let $U_r=X$ for all $r>1$. Let $U_1=X-B$, then U_1 is a neighborhood of A. Because X is normal, U_1 contains a closed neighborhood of A. Choose U_0 open, so that $A \subset U_0$ and $clU_0 \subset U_1$. Enumerate the rationals belonging to [0,1] in a sequence $9r_n)_{n\in\mathbb{N}}$ with $r_1=1$ and $r_2=0$. Inductively, for each n>2, choose U_{r_n} open so that $clU_{r_k} \subset U_{r_n}$ and $clU_{r_n} \subset U_{r_l}$ where $r_k<r_n<r_l$ and k,l<n.

Then the sets U_r $(r\in\mathbb{Q})$ satisfy the hypotheses of the Helping–Lemma. Evidently, the continuous function f (provided by the Help–Lemma) is equal to 0 on A and to 1 on B. ■

The following example is an application of Urysohn's Lemma. It demonstrates that every subspace of a compact space is completely regular.

EXAMPLE E–2.1

Let X be a compact space and let A be any subspace of X.

Then X is also a normal space, so by Urysohn's Lemma it is completely regular.

Thus A is also completely regular by Theorem E–1.1.

It was mentioned earlier that every locally compact Hausdorff space is completely regular. These spaces, however, have several other important properties which are summarized in the following theorem.

THEOREM E–2.8

Let X be a subspace of a Hausdorff space T.

(a) Let T be locally compact, let X be open in T. Then X is locally compact.

(b) Let X be dense in T. Then every compact neighborhood in X of a point $p\in X$ is a neighborhood in T of p.

(c) Let X be dense in T, let p be an isolated point of X. Then p in isolated in T.

(d) Let X be locally compact and dense in T. Then X is open in T.

PROOF

(a) This follows from the fact that for each $x \in X$, the neighborhood X of x contains a compact neighborhood of x.

(b) Let U be the interior of a compact neighborhood of $p \in X$.

Then $cl_X U$ is compact and hence closed in T. Thus $cl_T U = cl_X U$.

Let V be an open set in T such that $V \cap X = U$.

Because X is dense, $cl_T V = cl_T U \subset X$, so that $V = U$.

(c) The result follows from (b).

(d) The result follows from (b). ■

APPENDIX F
Cardinals and Ordinals

[Sources: [CN] W.Wistar Comfort & Stylianos Negrepontis, *"The Theory of Ultrafilters"*;
[GJ] Leonard Gillman & Meyer Jerison, *"Rings of Continuous Functions"*]

Note: In this appendix it is assumed that the reader is familiar with the definitions of rings of continuous functions [discussed in Chapter 1], order [discussed in Appendix A], countably compact spaces and countable sets.

Cardinals and ordinals are concepts introduced formally by Georg Cantor at the end of nineteenth century. In his best known work, *"Contributions to the Founding of the Theory of Transfinite Numbers"*, he enriched the concept of infinity by developing an arithmetic of transfinite numbers that was analogous to finite arithmetic. The transfinite cardinal numbers $\aleph_0, \aleph_1, \ldots$ (discussed in the following section) also originate from that specific work.

F-1. Cardinals

Informally speaking, the magnitude (whether finite or infinite) of a set is called its ***cardinal number*** or its ***cardinality***.

Cardinal numbers use the same symbols as natural numbers for finite sets. They also include more than one infinite number. The cardinality of an infinite countable set (i.e., any set which can be placed in one–to–one correspondence with natural numbers), for example, is denoted by $\aleph_0$; while the cardinality of the set of all the real numbers, $\mathbb{R}$, is denoted by $\aleph_1$, or $\aleph$, or c (power of the continuum).

The definitions of addition and multiplication are still valid for cardinal numbers; so are the proofs of basic laws of arithmetics and laws of order. Exponentiation can be defined for both finite and infinite numbers as follows: a^b is the cardinal number of of the class of all functions f, assigning to each element x in a class of b elements a value $y=f(x)$ in a class of a elements. There exists an interesting relationship between two of the infinite cardinals ($\aleph_0$ and c), namely that $2^{\aleph_0} = c$.

The main loss in dealing with infinite cardinal numbers is that the laws of cancellation for addition and multiplication, as defined for finite cardinals, are no longer valid.

Notation: From now on, the cardinality of a set A will be denoted by $|A|$.

To define cardinality in mathematical language, it is necessary to understand what is meant by cardinal equivalence.

DEFINITION F–1.1

Consider any two sets A and B. ***Cardinal equivalence*** between A and B is defined as follows: $A \sim B$ if and only if there exists a bijective $\varphi: A \to B$.

In particular, consider the special case when the sets A and B consist of finitely many elements, as in the next definition.

DEFINITION F–1.2

Let A and B consist of finitely many elements, say $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ (i.e., $|A| = n$ and $|B| = m$). Then $A \sim B$ if and only if $n = m$.

It is now possible to formulate a precise definition of finite and infinite sets. The definition below uses the concept of cardinal equivalence.

DEFINITION F–1.3

Consider any two sets A and B.

$|A| + |B| = |A \uplus B|$, where $\uplus$ symbolizes the disjoint union of A and B.

$|A| \cdot |B| = |A \times B|$, where $\times$ symbolizes the Cartesian product of A and B.

DEFINITION F–1.4

Let A be an arbitrary set. A is said to be ***infinite*** if there exists a set B, strictly contained in A, such that $A \sim B$. Alternatively, one can say that the cardinal $\mathfrak{m}$ is infinite provided that $\mathfrak{m} + 1 = \mathfrak{m}$.

EXAMPLE F–1.1

Let $A = \mathbb{N}$, the set of natural numbers.

Let $B = 2\mathbb{N}$, the set of even numbers.

Then clearly B is strictly contained in A.

However, $A \sim B$ because there exists a bijective mapping $\varphi: A \to B$ such that $\varphi(n) = 2n$ for every $n \in \mathbb{N}$.

Thus, by the previous definition, A is an infinite set.

DEFINITION F–1.5

Let A and B be two arbitrary sets. Then $|A| \leq |B|$ if there exists an injective mapping $\varphi: A \rightarrow B$.

The above definition clearly suggests that if $A \subseteq B$, then $|A| \leq |B|$.

EXAMPLE F–1.2

Let $A = \mathbb{N}$, the set of natural numbers.

Let $B = \mathbb{R}$, the set of real numbers.

Then $A \subseteq B$, $|A| = \aleph_0$ and $|B| = c$, as mentioned earlier.

It now follows that $\aleph_0 \leq c$.

As it turns out, the collection of all cardinal numbers is not a set (Burali–Forti paradox). For every given cardinal $\mathfrak{m}$, however, it is possible to define the set $\{\mathfrak{n}: \mathfrak{n} < \mathfrak{m}\}$.

F–2. Ordinals

In layman terminology, one can define ***ordinal numbers*** as the quantities which denote positions in an ordered set (i.e. first, second, third,...).

Mathematically, the concept of ordinals can be described as in the following definition.

DEFINITION F–2.1

Consider any two well–ordered sets A and B. ***Ordinal equivalence*** between A and B is defined as follows: A~B if and only if there exists an order isomorphism φ:A→B.

The *well–ordering property* is the name ascribed to a certain characteristic of an ordered set which is manifested in the fact that its every nonempty subset has a first member.

Two well–ordered sets have the same ordinal number if there exists an order isomorphism between them. Finite ordinals are denoted by natural numbers. The ordinal number of the set of finite ordinals is symbolized by ω, successive ordinals by $\omega+1$, $\omega+2$,...

It was observed by Cantor that the sequence of non–negative integers can be extended uniquely without losing the well–ordering property. Indeed, there must be a first infinite ordinal number after all the integers: the number ω mentioned above, followed by $\omega+1$, $\omega+2$,...

Immediately after the sequence ω, $\omega+1$, $\omega+2$,..., there must be a first ordinal 2ω, followed by $2\omega+1$, etc. Each ordinal number α in this sequence can be identified with the set κ preceding α, in order. For example, 3 corresponds to the ordered set $\{0,1,2\}$; and the ordinal ω to the ordered sequence of all non–negative integers.

Proceeding in this manner, one can define $\alpha+\beta$ as the result of laying the sequence β after the sequence α; $\alpha\beta$ as the result of substituting the sequence β for each term of the sequence α; and α^2 as $\alpha\alpha$. Note that both addition and multiplication are associative, although neither is commutative: $1+\omega = \omega \neq \omega+1$ and $\omega 2 = \omega \neq 2\omega$.

Recall that the cardinal $\mathfrak{m}$ is said to be infinite provided that $\mathfrak{m}+1 = \mathfrak{m}$, although this is not true for ordinals, as mentioned above.

Ordinal numbers can be divided into two classes, described in the definition below.

DEFINITION F–2.2

Consider an arbitrary ordinal α. Then α is called a ***nonlimit ordinal*** (or an ordinal of class 1) if there exist an ordinal β such that $\alpha=\beta+1$; otherwise α is said to be a ***limit ordinal*** (or an ordinal of class 2).

EXAMPLE F–2.1

$2\omega+1$ is a nonlimit ordinal.

0, ω, 2ω and ω^2 are examples of limit ordinals.

Ordinal numbers are, in a way, more powerful than cardinal numbers, since in addition to magnitude, they also represent order. They are of particular use in Chapter 4, in proofs by transfinite induction.

F–3. Spaces of Ordinals

DEFINITION F–3.1

The space of all countable ordinals is denoted by **W**.

The set of all ordinals less than a given ordinal α is written as:

$$\mathrm{W}(\alpha) = \{\sigma\colon \sigma<\alpha\}.$$

$W(\alpha)$ is well–ordered. It can be made into a topological space by taking as a subbase for the open sets the family of all rays $\{x: x>\iota\in W(\alpha)\}$ and $\{x: x<\eta\in W(\alpha)\}$. Thus, 0 is an isolated point (if $\alpha>0$), and for any point $\tau>0$, the set of all open–and–closed intervals

$$[\sigma+1,\tau] = \{x: \sigma<x<\tau+1\} \quad \text{where } \sigma<\tau$$

is a system of basic neighborhoods of τ. Evidently, a point of $W(\alpha)$ is an isolated point if and only if it is not a limit ordinal (i.e., it is 0 or has an immediate predecessor).

The space $W(\omega)$ of all finite ordinals is homeomorphic with the set $\mathbb{N}$ of natural numbers.

It should be clear that if $\sigma<\alpha$, then $W(\sigma)$ is a subspace of $W(\alpha)$.

Every nonempty set of ordinals is well–ordered, so it has a least element. According to the theory of ordinals, every set of ordinals has an upper bound, and hence a supremum.

We say that $A\subset W(\alpha)$ is *bounded* in $W(\alpha)$ if there exists $\sigma<\alpha$ such that $x\leq\sigma$ for all $x\in A$. We say that a subset A is *cofinal* in $W(\alpha)$ if, for all $\sigma<\alpha$, there exists $x\in A$ such that $x\geq\sigma$. Now, if $A\subset W(\alpha)$, then $\sup A\in W(\alpha)$ if and only if A is bounded. Thus, if $W(\alpha)$ has a greatest element, then any subset containing this element is both bounded and cofinal.

EXAMPLE F–3.1

Let $W(\alpha)$ be not bounded in itself.

Let $\alpha=\omega$, so that $W(\alpha)=\mathbb{N}$.

$A\subset W(\alpha)$ is bounded if and only if there exists $\sigma\in\mathbb{N}$ such that, for every $x\in A$, $x\leq\sigma$.

EXAMPLE F–3.2

Let $W(\alpha)$ be bounded in itself.

Let $\alpha=\omega+1$, so that $W(\alpha)=\mathbb{N}\cup\{\omega\}$.

Then every $A\subset W(\alpha)$ is bounded (because we can let $\sigma=\omega$).

THEOREM F–3.1

Let α be any ordinal. Then $W(\alpha)$ is a normal space.

PROOF

(Verification of the Hausdorff axiom is trivial and will not be discussed here.)

Now, let H and K be disjoint closed sets.

For each $\tau\in H$, let U_τ be an open interval of the form $[\sigma,\tau]$ that does not meet K.

Define V_τ correspondingly for $\tau\in K$.

Then $\cup_{\tau\in H}U_\tau$ and $\cup_{\tau\in K}V_\tau$ are disjoint open sets, which contain H and K respectively. ■

DEFINITION F–3.2

A ***tail*** in $W(\alpha)$ is a set of the form

$$W(\alpha)\backslash W(\sigma) = \{x\in W(\alpha)\colon x\geq\sigma\} \quad \text{for } \sigma<\alpha.$$

THEOREM F–3.2

$W(\alpha)$ is compact if and only if α is a nonlimit ordinal.

PROOF

"$\Rightarrow$": Let $W(\alpha)$ be compact.

Every tail in $W(\alpha)$ is a closed set. Further, the family of all tails has the finite intersection property.

Assume that α is a limit ordinal.

Then the intersection of the family of all tails is empty, so that W(α) is not compact: a contradiction.

"$\Leftarrow$": Note that W(0) is empty, hence compact.

Consider any nonlimit ordinal $\alpha' = \alpha+1$.

Assume, inductively, that W(τ) is compact for every nonlimit ordinal $\tau < \alpha'$.

Let $\mathfrak{U}$ be any cover of W(α') by basic open sets. Since the point α is covered, there exists $\sigma < \alpha$ such that $[\sigma+1, \alpha] \in \mathfrak{U}$.

By the induction hypothesis, W(σ+1) is compact (because W(σ+1) is a subspace of W(α')). So a finite subcollection $\mathfrak{F}$ of $\mathfrak{U}$ covers W(σ+1).

Then $\mathfrak{F} \cup \{[\sigma+1, \alpha]\}$ is a finite subfamily of $\mathfrak{U}$ that covers W(α+1). ■

From the preceding theorem it is clear that W(α) is always locally compact. Further, if W(α) is not compact, then W(α+1) is its one–point compactification.

DEFINITION F–3.3

Let ω_1 denote the first uncountable ordinal. Consider

$$\mathbf{W} = \mathrm{W}(\omega_1) = \{\sigma : \sigma < \omega_1\},$$
$$\mathbf{W}^* = \mathrm{W}(\omega_1+1) = \{\sigma : \sigma \leq \omega_1\}.$$

As mentioned above, the space $\mathbf{W}^*$ is compact and **W** is not. In fact, $\mathbf{W}^*$ is the one–point compactification of **W**.

Clearly every uncountable set in **W** is cofinal, because ω_1 is the smallest uncountable ordinal. The converse of this statement is presented in the following theorem.

THEOREM F–3.3

No countable set in **W** is cofinal.

PROOF

Assume that there exists a cofinal set S in **W**.

Then, by definition, $\mathbf{W} = \cup_{\sigma \in S} W(\sigma)$.

So the cardinal of S, $|S|$, can be represented as follows:

$$\begin{aligned} |S| &\leq |\mathbf{W}| \\ &= \Sigma_{\sigma\in S} |\sigma| \\ &\leq |S| \cdot \aleph_0. \end{aligned}$$ ■

An immediate consequence of the preceding theorem is that every countable set A in **W** is contained in a compact subspace (let $\alpha = \sup A \in \mathbf{W}$, A is contained in the compact subspace $W(\alpha+1)$). This, in turn, implies that every countable closed set in **W** is compact.

THEOREM F–3.4

Of any two disjoint closed sets in **W**, one is bounded; and hence countable and compact.

PROOF

Let H and K be cofinal closed sets.

Choose an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in H$ for n odd, and $\alpha_n \in K$ for n even.

Then $\sup_n \alpha_n \in H \cap K$. ■

THEOREM F–3.5

Every function $f \in C(\mathbf{W})$ is constant on a tail $\mathbf{W} - W(\alpha)$, where α depends on f.

PROOF

Every tail $\mathbf{W}-\mathrm{W}(\alpha)$ is countably compact (moreover, it is homeomorphic with $\mathbf{W}$ itself).

Thus, each image set $f[\mathbf{W}-\mathrm{W}(\alpha)]$ is a countably compact subset of $\mathbb{R}$, and hence compact, so that $\cap_{\sigma\in\mathbf{W}} f[\mathbf{W}-\mathrm{W}(\alpha)]$ is not empty.

Choose any number $r\in\cap_{\sigma\in\mathbf{W}} f[\mathbf{W}-\mathrm{W}(\alpha)]$.

Then the closed set $f^{\leftarrow}(r)$ is cofinal in $\mathbf{W}$.

Now, for every $n\in\mathbb{N}$, the closed set $\{x\in\mathbf{W}: |f(x)-r| \geq 1/n\}$ is disjoint from $f^{\leftarrow}(r)$, so (by Theorem F–3.4) it has an upper bound α_n in $\mathbf{W}$.

For any countable ordinal $\alpha>\sup_n\alpha_n$, $f[\mathbf{W}-\mathrm{W}(\alpha)] = \{r\}$. ■

From the preceding theorem it follows immediately that $\mathbf{W}$ is C–embedded in $\mathbf{W}^*$. This is achieved by extending $f\in C(\mathbf{W})$ to a function $f^\beta\in C(\mathbf{W}^*)$ where $f^\beta(\omega_1)$ is the final constant value of f. It follows that $C(\mathbf{W})$ and $C(\mathbf{W}^*)$ are isomorphic under the mapping $f\mapsto f^\beta$.

Note that $\mathrm{W}(\omega)$ is not C^*–embedded in $\mathrm{W}(\omega+1)$.

APPENDIX G
Quotient Spaces

[Sources: [M] James R. Munkres, *"Topology. A First Course"*]

The theory presented in this appendix is needed in Section 2–4, in a discussion concerning the Stone–Čech compactification's being the "largest" compactification. Specifically, the concept of quotient spaces is necessary for the intuitive understanding of the fact that every compactification on X is equivalent to a quotient space of its Stone–Čech compactification.

G–1. The Quotient Topology

In geometry one often has occasion to use "cut–and–paste" techniques to construct various geometric objects. The sphere, for example, can be constructed by taking a disk and collapsing its entire boundary to a single point; while the torus can be obtained by appropriate "pasting" of a rectangle's edges. Formalizing such constructions involves the concept of quotient topology.

DEFINITION G–1.1

Let X and Y be topological spaces; let p:X→Y be a surjective map. The map p is said to be a ***quotient map***, provided a subset U of Y is open in Y if and only if $p^{\leftarrow}(U)$ is open in X.

The above condition is stronger than continuity. An equivalent condition is to require that a subset A of Y be closed in Y if and only if $p^{\leftarrow}$(A) is closed in X (this equivalence follows from the fact that $f^{\leftarrow}$(Y–B) = X–$f^{\leftarrow}$(B)).

EXAMPLE G–1.1

Reminder: A map f is an *open map* if for each open set U of X, the set f(U) is open in Y.

Let p:X→Y be a surjective continuous open map.

Then p is a quotient map.

DEFINITION G–1.2

Let X be a space; let A be a set; let p:X→A be a surjective map. Then there exists exactly one topology $\mathfrak{T}$ on A relative to which p is a quotient map. The topology $\mathfrak{T}$ is called the ***quotient topology*** induces by p.

Of course, the topology $\mathfrak{T}$ is defined by letting it consist of those subsets U of A such that $p^{\leftarrow}$(U) is open in X.

EXAMPLE G–1.2

Let p be the map of the real line $\mathbb{R}$ onto the three–point set A={a,b,c} defined by

$$p(x) = \left| \begin{array}{ll} a & \text{if } x>0 \\ b & \text{if } x<0 \\ c & \text{if } x=0 \end{array} \right.$$

Then the quotient topology on A induced by p is such that the following subsets of A are open in it:

$$\emptyset,\ \{a\},\ \{b\},\ \{a,b\},\ \{a,b,c\}.$$

G–2. The Quotient Space

DEFINITION G–2.1

Let X be a topological space. Let X' be a partition of X, that is, a set of disjoint subsets in X whose union is X. Let p:X→X' be the surjective map that carries each point of X to the element of X' containing it. In the quotient topology induced by p, the space X' is called a ***quotient space*** of X.

X' is sometimes called a *decomposition space* or an *identification space* of X, because X' is obtained by "identifying all the elements in each partition class to a single point".

There is another way of describing the topology on X'. A subset U of X' is a collection of equivalence classes belonging to U. Hence the typical open set of X' is a collection of equivalence classes whose union is an open set of X.

EXAMPLE G–2.1

Define an equivalence relation on the plane X as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ if } x_0 + y_0^2 = x_1 + y_1^2$$

Let X' be the collection of equivalence classes, in the quotient topology.

Then X' is a quotient space homeomorphic to ℝ.

EXAMPLE G–2.2

Define an equivalence relation on the plane X as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Let X' be the collection of equivalence classes, in the quotient topology.

Then X' is a quotient space homeomorphic to the interval $[0,\infty)$ of ℝ.

APPENDIX H
Spaces of $\mathfrak{Z}$–Ultrafilters

[Sources: [W] Russel C. Walker, *"The Stone–Čech Compactification"*]

Note: For the purposes of this appendix it will be assumed that the reader is familiar with the definitions of topological spaces, normal spaces, discrete spaces, T_1–spaces, compact Hausdorff spaces.

H–1. $\mathfrak{Z}$–Ultrafilters

The only difference between the concept of z–filters [discussed in Appendix A] and that of $\mathfrak{Z}$–filters is the fact that instead of the family of zero–sets Z(X) in the definition, we now consider some arbitrary family $\mathfrak{Z}$ of subsets of a space X.

DEFINITION H–1.1

Let $\mathfrak{F}$ be a nonempty subfamily of $\mathfrak{Z}$ such that

(1) $\phi \notin \mathfrak{F}$;

(2) if $Z_1, Z_2 \in \mathfrak{F}$, then $Z_1 \cap Z_2 \in \mathfrak{F}$; and

(3) if $Z \in \mathfrak{F}$, $Z' \in \mathfrak{Z}$, and $Z' \supset Z$, then $Z' \in \mathfrak{F}$;

then $\mathfrak{F}$ is called a ***$\mathfrak{Z}$–filter*** on X.

DEFINITION H–1.2

Let $\mathfrak{F}$ be a $\mathfrak{Z}$–filter on X such that $\mathfrak{F}$ is not contained in any other $\mathfrak{Z}$–filter on X. Then $\mathfrak{F}$ is a maximal $\mathfrak{Z}$–filter, and is referred to as a ***$\mathfrak{Z}$–ultrafilter***.

An alternative characterization of $\mathfrak{Z}$–ultrafilters is stated in the following theorem.

<u>THEOREM H–1.1</u>

A $\mathfrak{Z}$–filter $\mathfrak{U}$ is a $\mathfrak{Z}$–ultrafilter if and only if any member of $\mathfrak{Z}$ which meets (intersects) every member of $\mathfrak{U}$ is in $\mathfrak{U}$.

<u>PROOF</u>

"$\Rightarrow$": Let $\mathfrak{U}$ be a $\mathfrak{Z}$–ultrafilter.

Consider an arbitrary $\mathcal{A} \in \mathfrak{Z}$ which intersects every member of $\mathfrak{U}$.

Then $\mathfrak{U} \cup \{\mathcal{A}\}$ generates a $\mathfrak{Z}$–filter $\mathfrak{V}$ which contains the $\mathfrak{Z}$–ultrafilter $\mathfrak{U}$.

But then $\mathfrak{V}$ must equal $\mathfrak{U}$, because $\mathfrak{U}$ is a $\mathfrak{Z}$–ultrafilter.

Thus, $\mathcal{A}$ is in $\mathfrak{U}$.

"$\Leftarrow$": Let $\mathfrak{U}$ be a $\mathfrak{Z}$–filter containing all members of $\mathfrak{Z}$ which intersect every member of $\mathfrak{U}$.

Then $\mathfrak{U}$ must clearly be maximal, i.e. $\mathfrak{U}$ is a $\mathfrak{Z}$–ultrafilter. ■

<u>DEFINITION H–1.3</u>

$\omega_X(\mathfrak{Z})$, or $\omega(\mathfrak{Z})$ for short, denotes the collection of all $\mathfrak{Z}$–ultrafilters on X.

Let Z belong to $\mathfrak{Z}$. Then Z^{ω} denotes those members of $\omega(\mathfrak{Z})$ which contain Z.

Then $\{Z^{\omega}: Z \in \mathfrak{Z}\}$ is a base for closed sets and thus induces a topology on $\omega(\mathfrak{Z})$.

H–2. Spaces of $\mathfrak{Z}$–ultrafilters

The remainder of this appendix presents concepts and theorems which are necessary in Section 2–4, in order to prove the following important result: for a normal space X, βX is homeomorphic to the space of $\mathfrak{Z}$–ultrafilters on X when $\mathfrak{Z}$ is the set of closed subsets of X (i.e. a subset of $\mathscr{P}(X)$).

Collections of $\mathfrak{Z}$–filters, in particular $\mathfrak{Z}$–ultrafilters, can be used to construct topological spaces.

Recall that $\omega_X(\mathfrak{Z})$ denotes the collection of all the $\mathfrak{Z}$–ultrafilters on X (when no confusion can result, $\omega(\mathfrak{Z})$ is used). For Z in $\mathfrak{Z}$, Z^{ω} denotes the members of $\omega(\mathfrak{Z})$ which contain Z. A topology on $\omega(\mathfrak{Z})$ is imposed by taking $\{Z^{\omega}: Z \in \mathfrak{Z}\}$ as a base for the closed sets.

In the year 1964 O. Frink introduced the concept of a normal base, stated formally in the definition below.

<u>DEFINITION H–2.1</u>

A collection $\mathfrak{Z}$ of closed subsets of a T_1–space X is called a ***normal base*** for X if $\mathfrak{Z}$ satisfies the following conditions:

(1) $\mathfrak{Z}$ is a ring of sets.

(2) $\mathfrak{Z}$ is ***disjunctive***, i.e. if a closed subset A of X does not contain a point $x \in X$, then there exists a member of $\mathfrak{Z}$ containing x and missing A.

(3) $\mathfrak{Z}$ is a base for the closed sets of X, i.e. every closed set is an intersection of the members of $\mathfrak{Z}$.

(4) $\mathfrak{Z}$ is ***normal***, i.e. disjoint members of $\mathfrak{Z}$ are contained in disjoint complements of members of $\mathfrak{Z}$.

The following theorem will be used to prove that, for $\mathfrak{Z}$ any ring of sets, $\omega(\mathfrak{Z})$ is compact.

THEOREM H–2.1

Let $\mathfrak{Z}$ be any ring of sets. Then every $\mathfrak{Z}$–filter is contained in a $\mathfrak{Z}$–ultrafilter.

PROOF

The result can be obtained by a straightforward application of Zorn's Lemma [stated in Appendix A]. ∎

The next result will allow X to be embedded into $\omega(\mathfrak{Z})$.

THEOREM H–2.2

Let $\mathfrak{Z}$ be a disjunctive ring of sets. Let x be a point in X. Let $\varphi(x) = \{Z \in \mathfrak{Z}: x \in \mathfrak{Z}\}$. Then every $\varphi(x)$ is a $\mathfrak{Z}$–ultrafilter.

PROOF

Assume that Z belongs to $\mathfrak{Z}$ and $x \notin Z$.

Z is closed and $\mathfrak{Z}$ is disjunctive, hence there exists Z' containing x and missing Z.

Then Z does not belong to $\varphi(x) = \{Z \in \mathfrak{Z}: x \in \mathfrak{Z}\}$ and every element of $\mathfrak{Z}$ which meets every member of $\varphi(x)$ contains x.

Hence $\varphi(x)$ is a $\mathfrak{Z}$–ultrafilter. ∎

The closure of any subset of $\omega(\mathfrak{Z})$ is the intersection of all basic closed sets containing the given set. The following description of the closures of sets in $\mathfrak{Z}$ is useful in relating X to $\omega(\mathfrak{Z})$.

THEOREM H–2.3

Let $Z \in \mathfrak{Z}$. Then Z^{ω} is the closure of Z in $\omega(\mathfrak{Z})$.

PROOF

Clearly $cl_{\omega(\mathfrak{Z})}Z \subseteq Z^{\omega}$, because $Z \subset Z^{\omega}$.

Suppose that Z_0^{ω} is a basic set containing Z.

Then $Z_0 = Z_0^{\omega} \cap X \supset Z$, so that $Z_0 \subset Z_0^{\omega}$.

Thus, $Z^{\omega} \subseteq cl_{\omega(\mathfrak{Z})}Z$.

Hence, $Z^{\omega} = cl_{\omega(\mathfrak{Z})}Z$. ■

THEOREM H–2.4

(a) Let $Z_1, Z_2 \in \mathfrak{Z}$. Then $(Z_1 \cap Z_2)^{\omega} = Z_1{}^{\omega} \cap Z_2{}^{\omega}$.

(b) Let $Z_1, Z_2 \in \mathfrak{Z}$. Then $(Z_1 \cup Z_2)^{\omega} = Z_1{}^{\omega} \cup Z_2{}^{\omega}$.

PROOF

(a) Two members Z_1, Z_2 of $\mathfrak{Z}$ both belong to a $\mathfrak{Z}$–ultrafilter exactly when their intersection belongs to the $\mathfrak{Z}$–ultrafilter.

Thus the result follows.

(b) Similar to (a). ■

The basic open sets of $\omega(\mathfrak{Z})$ are identified by taking complements of the basic closed sets:

$$\omega(\mathfrak{Z}) - Z^{\omega} = \{\mathfrak{U} \in \omega(\mathfrak{Z}) : Z \notin \mathfrak{U}\}$$

$$= \{\mathfrak{U} \in \omega(\mathfrak{Z}) : Z' \subset X - Z \text{ for some } Z' \in \mathfrak{U}\}.$$

For U=X–Z, the basic open set obtained from U will be denoted by

$${}^{\omega}U = \{\mathfrak{U} \in \omega(\mathfrak{Z}) : Z' \subset U \text{ for some } Z' \in \mathfrak{U}\}.$$

The following theorem is the basic result obtained by Frink.

THEOREM H–2.5

Let $\mathfrak{Z}$ be a normal base for a T_1–space X. Then $\omega(\mathfrak{Z})$ is a compact Hausdorff space and φ is a dense embedding of X into $\omega(\mathfrak{Z})$.

PROOF

Induce $\omega(\mathfrak{Z})$ with the usual topology defined above, i.e. let $\{Z^{\omega}: Z\in\mathfrak{Z}\}$ be a base for the closed sets, where $Z^{\omega} = \{\mathfrak{U}\in\omega(\mathfrak{Z}): Z\in\mathfrak{U}\}$.

$\omega(\mathfrak{Z})$ is Hausdorff because:

Let $\mathfrak{U}$ and $\mathfrak{V}$ be distinct points of $\omega(\mathfrak{Z})$.

Then there exists Z_1 in $\mathfrak{U}$ and Z_2 in $\mathfrak{V}$ such that $Z_1\cap Z_2$ is empty.

Since $\mathfrak{Z}$ is normal, there exist A_1 and A_2 in $\mathfrak{Z}$ such that $Z_1\subset X-A_1$, $Z_2\subset X-A_2$ and $(X-A_1)\cap(X-A_1)$ is empty.

But then ${}^{\omega}(X-A_1)$ and ${}^{\omega}(X-A_2)$ are disjoint basic neighborhoods of $\mathfrak{U}$ and $\mathfrak{V}$ respectively.

$\omega(\mathfrak{Z})$ is compact because:

It is sufficient to prove that any family A^{ω} of basic closed sets with the final intersection property has non–empty intersection.

Let $A = \{Z\in\mathfrak{Z}: Z^{\omega}\in A^{\omega}\}$.

Then A has finite intersection property and is contained in a $\mathfrak{Z}$–ultrafilter $\mathfrak{U}$.

But $\mathfrak{U}\supset A$, so $\mathfrak{U}$ is in Z^{ω} for every Z in A. Further, $\mathfrak{U}$ belongs to $\cap A^{\omega}$.

φ is an embedding because:

The function $\varphi: X\to\omega(\mathfrak{Z})$ is obtained by associating to each point $x\in X$ the $\mathfrak{Z}$–ultrafilter $\varphi(x)$ consisting of all members of $\mathfrak{Z}$ containing x.

X is a T_1 space and $\mathfrak{Z}$ is a base for the closed sets of X. Thus the function φ is one–to–one.

Identify X with its image $\varphi[X]$ and regard X as a subspace of $\omega(\mathfrak{Z})$.

Recall that the topology on $\omega(\mathfrak{Z})$ is defined by letting $\{Z^{\omega}: Z\in\mathfrak{Z}\}$ be a base for the closed sets, where $Z^{\omega} = \{\mathfrak{U}\in\omega(\mathfrak{Z}): Z\in\mathfrak{U}\}$.

$\mathfrak{Z}$ is a base for the closed sets of X and $Z^{\omega}\cap X = Z$. Thus, φ is an embedding.

$\varphi[X]$ is dense in $\omega(\mathfrak{Z})$ because:

Let ${}^{\omega}U$ be a non–empty basic open set of $\omega(\mathfrak{Z})$.

Then there exists $\mathfrak{U}$ in ${}^{\omega}U$.

Let Z be a member of $\mathfrak{U}$ such that Z is a subset of U.

Then the image under φ of a point of Z belongs to ${}^{\omega}U\cap\varphi[X]$. ■

The following result forms the core of this appendix. It is used in Section 2–4 as an example of a space with its Stone–Čech compactification.

<u>COROLLARY H–2.6</u>

Let X be a normal space. Let $\mathfrak{Z}$ be a family of all closed subsets of X.

Then $\omega(\mathfrak{Z}) = \beta X$.

<u>PROOF</u>

Let X be a normal space.

Let $\mathfrak{Z}$ be the normal base of all closed subsets of X.

As already stated in Theorem H–2.3, for $Z\in\mathfrak{Z}$, the closure of Z in $\omega(\mathfrak{Z})$ is given by Z^{ω}.

By Theorem H–2.4(a), for $Z_1,Z_2\in\mathfrak{Z}$, $(Z_1\cap Z_2)^{\omega} = Z_1{}^{\omega}\cap Z_2{}^{\omega}$.

Finally, recall from Section 2–4 that for a normal space X, βX is that compactification of X in which disjoint closed subsets of X have disjoint closures.

The result now follows. ■

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