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GROUP INVARIANCE OF GLOBAL GENERALIZED SOLUTIONS OF NONLINEAR PDEs IN COLOMBEAU ALGEBRAS AND IN THE DEDEKIND ORDER COMPLETION METHOD

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Group Invariance of Global Generalized Solutions of Nonlinear PDEs in Colombeau Algebras and in the Dedekind Order Completion Method

by

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Summary

In this thesis a theoretical framework is provided within which large classes of *nonlinear* Lie groups of transformations are defined on spaces of generalized functions which yield *global* generalized solutions for large classes of nonlinear partial differential equations (PDEs). Although these function spaces contain the L Schwartz distributions, this theory stays within finite dimensional manifolds. It is shown how Lie symmetry groups for classical solutions of nonlinear PDEs can be extended to symmetry groups for global generalized solutions. For the first time in the literature, applications are given of the nonlinear group invariance of global weak solutions of nonlinear PDEs, under transformations defined on the whole domains of definition of those solutions.

The major difficulty with the study of Lie group invariance of solutions of nonlinear PDEs has been the absence of a theory for the existence of solutions for general nonlinear PDEs. Also, many of the known classical solutions of specific nonlinear PDEs are not $\mathcal{C}^\infty$–smooth on the whole domain of definition of these PDEs, typically exhibiting singularities. In the last few decades, functional analytic methods have produced existence results concerning global generalized solutions for several particular classes of nonlinear PDEs. However, these generalized solutions are usually linear functionals, such as the L Schwartz distributions, making the study of nonlinear group invariance particularly difficult, even though such transformations arise naturally with nonlinear PDEs. Furthermore, these linear functionals are defined on infinite dimensional vector spaces, making the computation of their Lie symmetry groups highly nontrivial.

All of the above–mentioned difficulties are bypassed by two recent nonlinear theories of generalized functions. The first, based on algebraic solution methods, was developed by E E Rosinger, and in a particular and central case, in the independent work of J F Colombeau. The second is the more powerful, as yet unpublished, Dedekind order completion method of M Oberguggenberger and E E Rosinger. Particularly the algebraic

method of Rosinger and the order completion method of Oberguggenberger and Rosinger, have made available global existence results for solutions of large classes of nonlinear PDEs. In addition, the group transformations of the relevant generalized function spaces can be defined in such a way as to stay within finite dimensional manifolds which are, in fact, the original spaces of independent and dependent variables of the PDEs.

It is shown here how to extend the concept of projectable group actions from classical function spaces to generalized function spaces, specifically to Colombeau's algebra of generalized functions, and to the order completion context of Oberguggenberger and Rosinger. In the case of Colombeau algebras, the same is done for more general groups. The concepts of invariant solutions and symmetry groups are also extended to include global generalized solutions. Finally, in the case of the order completion method, examples are given of the nonlinear group invariance of delta wave solutions of semilinear hyperbolic equations with rough initial data, and of Riemann solvers of the nonlinear shock wave equation.

Opsomming

In hierdie proefskrif word 'n teoretiese raamwerk daargestel waarbinne breë klasse *nie–lineêre* Lie–groepe van transformasies gedefinieer word op ruimtes van veralgemeende funksies wat *globale* veralgemeende oplossings lewer vir groot klasse nie–lineêre parsiële differensiaalvergelykings (PDVs). Alhoewel hierdie funksieruimtes die L Schwartz–distribusies bevat, bly die teorie binne die bestek van eindig–dimensionele menigvoude. Daar word aangetoon hoe Lie simmetriegroepe van klassieke oplossings vir nie–lineêre PDVs uitgebrei kan word na simmetriegroepe van globale veralgemeende oplossings. Vir die eerste maal in die vakletterkunde, word toepassings gegee van die nie–lineêre groepinvariansie, van globale swak oplossings vir nie–lineêre PDVs, onder transformasies wat op die oplossings se hele definisiegebiede gedefinieer is.

Tot dusver was die grootste probleem met die studie van Lie–groepinvariansie van oplossings vir nie–lineêre PDVs, die gebrek aan 'n teorie vir die bestaan van oplossings vir algemene nie–lineêre PDVs. Daarby is baie van die bekende klassieke oplossings van spesifieke nie–lineêre PDVs nie nie orals $\mathcal{C}^{\infty}$–glad nie, en bevat tipies singulariteite. In die laaste paar dekades het funksionaal–analitiese metodes wel resultate gelewer oor die bestaan van globale veralgemeende oplossings vir verskeie spesifieke klasse nie–lineêre PDVs. Ongelukkig is hierdie veralgemeende oplossings gewoonlik lineêre funksionale, soos byvoorbeeld die L Schwartz–distribusies, wat die bestudering van nie–lineêre groepinvariansie heelwat bemoeilik, ondanks die feit dat sulke transformasies algemeen voorkom in die geval van nie–lineêre PDVs. Voorts word hierdie lineêre funksionale op oneindig–dimensionele vektorruimtes gedefinieer, wat weer die berekening van hul Lie simmetriegroepe omskep in 'n hoogs nie–triviale probleem.

Al die bogenoemde struikelblokke word omseil deur twee onlangse nie–lineêre teorieë van veralgemeende funksies. Die eerste een, gebaseer op algebraïese oplossingstegnieke, is ontwikkel deur E E Rosinger, en in 'n middegeval, in die onafhanklike navorsing van J F Colombeau. Die tweede, en meer verreikende teorie is die, tot dusver ongepubliseerde,

Dedekind–ordevervoledigingstegniek van M Oberguggenberger en E E Rosinger. Veral die algebraïese metode van Rosinger en die ordevervoledigingsmetode van Oberguggenberger en Rosinger, het resultate gelewer oor die bestaan van globale veralgemeende oplossings vir groot klasse nie–lineêre PDVs. Daarby kan groep–transformasies van die betrokke veralgemeende funksieruimtes op só 'n manier gedefinieer word dat hulle binne die bestek van eindig–dimensionele menigvoude bly. Om die waarheid te sê, die oorspronklike ruimtes van onafhanklike en afhanklike veranderlikes van die PDVs word gebruik.

Dit word hier aangetoon hoe om projekteerbare groepaksies op klassieke funksieruimtes uit te brei na veralgemeende funksieruimtes, spesifiek na Colombeau–algebras en na die ordevervolledigingskonteks van Oberguggenberger en Rosinger. In die geval van Colombeau–algebras word dieselfde gedoen vir meer algemene groepe. Die definisies van invariante oplossings en simmetriegroepe word ook uitgebrei om veralgemeende oplossings in te sluit. Om mee af te sluit, word daar in die geval van die ordevervolledigingsmetode, voorbeelde gegee van die nie–lineêre groepinvariansie van deltagolf–oplossings van semilineêre hiperboliese vergelykings met growwe beginwaardes, en van Riemann–oplossers van die nie–lineêre skokgolf vergelyking.

CONTENTS

INTRODUCTION

In 1874, the Norwegian mathematician Sophus Lie pioneered the use of continuous groups of transformations in the study of the symmetry properties of partial differential equations, with the view of their solution. The basic idea of symmetry groups of a system of PDEs is to find groups whose elements transform solutions of the system to other solutions of the system, that is, groups with respect to which the set of solutions of the system is invariant. Traditionally, in the classical framework of Lie, these groups are continuous groups of pointwise geometric transformations on the space of independent and dependent variables of the system. The group elements act on solutions by transforming their graphs, which are subsets of the space of independent and dependent variables.

These continuous transformation groups often act nonlinearly, and the geometric transformations may be defined only locally, within some neighborhood of the identity. Yet there are explicit computational methods for computing all the continuous symmetry groups of almost any system of PDEs. The fundamental discovery of Lie was that the complicated nonlinear conditions of invariance of the system under the group transformations are equivalent to much simpler, linear conditions of infinitesimal invariance of the system under the generators of the group. This is due to the fact that there is a one–to–one correspondence between finite dimensional local Lie groups and finite dimensional Lie algebras. In fact, there is a one–to–one correspondence between the connected one–parameter subgroups of such a Lie group, and the one dimensional subspaces of its associated Lie algebra.

Symmetry groups of systems of PDEs have various applications. For example, one can use the defining properties of the groups directly to compute new solutions from known ones. By making use of the most general symmetry group of a given system of PDEs with known solutions, whole classes of other solutions of the system can be generated in this way. Solutions can also be classified into symmetry equivalence classes, where two solutions are considered to be equivalent if the one can be obtained by transforming the other with some group element.

Secondly, families of differential equations, depending on arbitrary parameters or functions, can be classified according to their symmetry groups. Often there are are good reasons, both from a mathematical and physical point of view, to prefer those equations with a higher degree of symmetry. This approach includes taking a given group of symmetries and using the infinitesimal methods of the theory of differential invariants to determine which types of differential equations admit the group. In fact, the continuous symmetry properties of systems of PDEs is fundamental in physics. For example, Einstein's theory of general relativity prescribes coordinate invariance of the space of independent variables under $\mathcal{C}^2$–smooth diffeomorphisms. This question of general coordinate invariance does not end with classical physics. It is just as important in quantum mechanics, where we still witness the most unsatisfactory difference with general relativity. We quote from Dyson's paper 'Missed Opportunities', which is as relevant today as it was in 1972:

'The most glaring incompatibility of concepts in contemporary physics is that between Einstein's principle of general coordinate invariance and all the modern schemes for a quantum–mechanical description of nature. Einstein based his theory of general relativity on the principle that God did not attach any preferred labels to the points of space–time. This principle requires that the laws of physics should be invariant under the Einstein group E, which consists of all one–to–one and twice differentiable transformations of the coordinates. By making full use of invariance under E, Einstein was able to deduce the precise form of his law of gravitation from general requirements of mathematical simplicity without any arbitrariness. He was also able to reformulate the whole of classical physics (electromagnetism and hydrodynamics) in E–invariant fashion, and so determine unambiguously the mutual interactions of matter, radiation and gravitation within the classical domain. There is no part of physics more coherent mathematically and more satisfying aesthetically than this classical theory of Einstein based upon E–invariance.

On the other hand, all the currently viable formalisms for describing nature quantum–mechanically use a much smaller invariance group. ... In practice all serious quantum–mechanical theories are based either on the Poincaré group P or the Galilei group G. This means that a class of preferred inertial coordinate–systems is postulated a priori, in flat contradiction to Einstein's principle. The contradiction is particularly uncomfortable, because Einstein's principle of general coordinate invariance has such an attractive quality of absoluteness. A physicist's intuition tells him that, if Einstein's principle is valid at all, it ought to be valid for the whole of physics, quantum–mechanical as well as classical. If the principle is not universally valid, it is difficult to understand why Einstein achieved such deeply coherent insights into nature by assuming it to be so.'

Thirdly, and also very important from the point of view of physics, is the fact that, in 1918, Emmy Noether proved a one–to–one correspondence between symmetry groups and conservation laws, in the case of PDEs derived from Langrangean systems. She showed that each one–parameter variational symmetry group gives rise to a conservations law of the Euler–Lagrange equations. For example, conservation of energy corresponds to a time translational invariance of the system, and conservation of linear and angular momenta reflect spatial translational and rotational invariance respectively. Her more general result, in which the infinitesimal generators of symmetry groups depend also on derivatives of the dependent variables, leads to a one–to–one correspondence between nontrivial variational symmetry groups of a normal system (most physically important systems satisfy the criterion of normality), and nontrivial conservation laws. Conservation laws have many important physical and mathematical applications, including the study of existence results, shock waves, scattering theory, stability, relativity, fluid mechanics, elasticity and soliton solutions.

Finally, one of the original fundamental, and so far, least successful of Lie's aims in the history of continuous symmetry groups, was the solving of general systems of PDEs. Lie was inspired in this attempt by the success of Galois group theory on the question of solving polynomial equations by radicals, as well as Lie's own discovery that the various special techniques available for solving particular, seemingly unrelated classes of differential equations were, in fact, all special cases of a general integration procedure based on the invariance of these equations under continuous symmetry groups. Although Lie's approach never provided a general theory for the existence of solutions for general nonlinear PDEs, it did unify and significantly extend the available integration techniques. Ordinary differential equations with symmetry groups can have their order reduced, and even in the case of certain partial differential equations, general group symmetries can be used in determining special solutions which are invariant with respect to some subgroup of the full symmetry group. For many nonlinear systems these are the only explicit exact solutions available.

In the case of nonlinear PDEs, there have been some major difficulties with the study of Lie group invariance of solutions. The main difficulty has been the absence of a theory for the existence of solutions for general nonlinear PDEs. Thus, even though it was possible to compute the most general symmetry group of most nonlinear PDEs, it was not even known in general, whether there were any solutions whose invariance could be studied in this way. Furthermore, many of the known classical solutions of specific nonlinear PDEs are not $\mathcal{C}^\infty$–smooth on the whole domain of definition of these PDEs. In fact the following example shows that smooth solutions of nonlinear PDEs may fail to exist outside of arbitrary small neighborhoods of points in the domain of definition of the PDEs. Take the nonlinear ODE

$$u'(t) + u(t)^2 = 0 \quad , \ t \in \mathbb{R}$$

with initial value

$$u(t_0) = u_0$$

for some t_0 ,$u_0 \in \mathbb{R}$, $u_0 \neq 0$. This initial value problem has solution

$$u(t) = u_0 \;/\; (1 + u_0(t - t_0)) \quad , \; t \in \mathbb{R} \setminus \{t_0 - 1/u_0\}$$

This means that for any fixed $t_0 \in \mathbb{R}$, the solution has a singularity arbitrarily near to t_0 , depending on the size of $|u_0|$.

With the advent, in the last few decades, of existence results concerning global generalized solutions for several particular classes of nonlinear PDEs based on functional analytic methods, these difficulties have to an extent been replaced by new ones. The generalized functions used are linear functionals, such as the L Schwartz distributions, so that it becomes particularly difficult to study nonlinear group invariances, since spaces of linear functionals are not closed under nonlinear transformations, yet such transformations arise naturally with nonlinear PDEs. Furthermore, these linear functionals are defined on infinite dimensional vector spaces, so that the transformation groups would have to act on infinite dimensional manifolds, in which case the convenient one–to–one correspondence between Lie groups and Lie algebras, which is the basis for the infinitesimal group calculation methods, ceases to hold.

Two recent nonlinear theories of generalized functions have bypassed all of these difficulties: the algebraic solution method developed by E E Rosinger, and in a particular and central case, independently by J F Colombeau; and the more powerful, as yet unpublished, Dedekind order completion method of M Oberguggenberger and E E Rosinger. The algebraic method of Rosinger and the order completion method of Oberguggenberger and Rosinger, in particular, have made available global existence results for solutions of large classes of nonlinear PDEs. In addition, the group transformations of the relevant generalized function spaces can be defined in such a way as to stay within finite dimensional manifolds, in fact, these are still the original spaces of independent and dependent variables of the PDEs.

It will be shown here for the first time in the literature, how to extend the concepts of group actions from classical function spaces to the generalized context, specifically to Colombeau's algebra of generalized functions, and to the order completion context of Oberguggenberger and Rosinger. The concepts of invariant solutions and symmetry groups will also be extended to include global generalized solutions and, in the case of the order completion method, examples will be given of the generalized global nonlinear group invariance of global weak solutions of a semilinear hyperbolic wave equation, and of the nonlinear shock wave equation.

In 1900, Hilbert posed as the fifth of his twenty three problems the following, see Hilbert (1902) or Browder (1976, pp 12–14):

'How far Lie's concept of continuous groups of transformations of manifolds is approachable in our investigations without the assumption of differentiability of the functions?'

Generally, see Yang, this problem has been interpreted in its narrowest sense, that is, it has been reduced to asking whether the manifolds of independent and dependent variables, upon which the Lie group transformations are defined, and which can be defined merely by continuity conditions, must necessarily also have a differential structure, as has been required in the development of the theory of Lie group transformations since its inception, see Hermann (1966), Ibragimov (1985), Olver (1986). However, as mentioned, Lie's main aim was the study of the group invariance of solutions of nonlinear PDEs, thus going beyond the mere study of the group transformations of finite dimensional manifolds. This led to Hilbert's drawing attention to the following more extended version of his fifth problem, relevant in the context of equations from the calculus of variations, see Hilbert (1902) or Browder (1976, p 14):

How far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?'

As mentioned, we will define large classes of nonlinear Lie groups of transformations on spaces of generalized functions which yield global generalized solutions for large classes of nonlinear PDEs. Since these solutions need not even be continuous, this does present an answer to Hilbert's fifth problem, in its broader formulation.

Returning to the inconsistency between E–invariance in general relativity and invariance under the smaller groups P and G in quantum mechanics, this is largely due to the fact that the study of the group invariance of classical, smooth solutions which often fail to be global, and the more recent generalized solutions given by linear functionals has been fraught with the mentioned difficulties. Certainly, these new nonlinear theories of generalized functions have opened up new possibilities for resolving that difference.

SECTION 1: LIE GROUP TRANSFORMATIONS ON COLOMBEAU ALGEBRAS

The nonlinear theory of Colombeau occupies a natural central position within all the possible algebraic theories presented and characterised in Rosinger [1–8]. It has not produced global existence results which are as general and type independent with regard to the PDEs considered as the 'global Cauchy–Kovalevskaia' theorem for arbitrary analytic nonlinear PDEs, Rosinger [7], or the new order completion theory, which has already proved the existence of global generalized solutions for large, not necessarily monotonous, classes of continuous nonlinear PDEs – the most general, type independent existence result to date. Yet Colombeau's theory does have a number of important advantages, particularly in its strong connections with distribution theory, and so far, it has been the most widely studied of the various possible algebraic methods. It can also give global generalized solutions to large classes of linear $\mathcal{C}^\infty$–smooth coefficient PDEs which were proved to be unsolvable within distributions or hyperfunctions, and to various classes of nonlinear wave equations.

Extending Projectable Group Actions to $\mathcal{G}(X)$

Consider the m–th order, $\mathcal{C}^\infty$–smooth nonlinear PDE

$$T(x,D)u(x) = 0 \quad , \; x \in X$$

where

$$\begin{aligned} T(x,D) : \mathcal{C}^\infty(X) &\longrightarrow \mathcal{C}^\infty(X) \\ u(\cdot) &\longmapsto F(\cdot\, , u, \dots, D^p u, \dots) \end{aligned}$$

is an arbitrary partial differential operator (Appendix 1), and the unknown function is

$$u : X \longrightarrow U$$

Here the open subset $X \subset \mathbb{R}^n$ denotes the space of independent variables, and $U = \mathbb{R}$ the space of dependent variables. In order to study the group invariance of such a PDE, as in classical Lie group theory, we will consider group transformations on the manifold $M = X \times U$.

The Lie group G is a global group of transformations acting on the manifold M if we have a mapping

$$\begin{aligned} G \times M &\longrightarrow M \\ (g,(x,u)) &\longmapsto g\cdot(x,u) = (g_1(x,u),g_2(x,u)) \end{aligned}$$

where $g_1 \in \mathcal{C}^\infty(M,X)$, $g_2 \in \mathcal{C}^\infty(M,U)$, and the mapping has the following properties:

(a) $$\forall\ g,g' \in G,\ (x,u) \in M :$$
$$g\cdot(g'\cdot(x,u)) = (g.\,g')\cdot(x,u)$$

(b) $$\forall\ (x,u) \in M :$$
$$e\cdot(x,u) = (x,u)$$

where $e \in G$ is the identity element in G.

(c) $$\forall\ g \in G,\ (x,u) \in M :$$
$$g^{-1}\cdot(g\cdot(x,u)) = (x,u)$$

where $g^{-1} \in G$ is the inverse element of g.

Property (c) and the fact that all mappings here are assumed $\mathcal{C}^\infty$–smooth, give us

$$\forall\, g \in G :$$
$$\text{the mapping} \quad g_1 \times g_2 : M \longrightarrow M \quad \text{is a } \mathcal{C}^\infty\text{–diffeomorphism}$$

The actions of elements of G on M are simply coordinate transformations involving both the space of independent and the space of dependent variables of the PDE. To be able to talk about the group invariance of solutions which, in the classical sense, are functions from X to U, we obviously need to extend the definition of group actions on the manifold $M = X \times U$ of dependent and independent variables, to actions on functions from X to U. As in classical Lie group theory, we will make use of the fact that there is a one to one correspondence between functions from X to U, and their graphs, which are subsets of $X \times U$.

Before we proceed, we will first restrict the types of group transformations under consideration to the projectable groups

$$\begin{aligned} G \times M &\longrightarrow M \\ (g,(x,u)) &\longmapsto (g_1(x), g_2(x,u)) \end{aligned}$$

where $g_2 \in \mathcal{C}^\infty(M,U)$ and

$$g_1 : X \longrightarrow X$$

is a $\mathcal{C}^\infty$–diffeomorphism. This restriction simplifies the following constructions significantly by avoiding invertibility problems which would generally limit us to only local transformations even of functions which are defined on the whole of X. Since the projectable groups of transformations include groups of the form

$$\begin{aligned} G \times M &\longrightarrow M \\ (g,(x,u)) &\longmapsto (g_1(x),u) \end{aligned}$$

where g_1 is a $\mathcal{C}^\infty$–diffeomorphism, which are the groups considered in General Relativity, they are of great interest in themselves. However, we will later consider the general case of local groups of transformations as well.

In order to extend the actions of the projectable groups from the manifold M to a space of functions from X to U, we first extend them to an action on $\mathcal{P}(M)$ in the following natural, pointwise manner:

$$\begin{aligned} G \times \mathcal{P}(M) &\longrightarrow \mathcal{P}(M) \\ (g, S) &\longmapsto g \cdot S = \{g \cdot (x,u) \mid (x,u) \in S\} \end{aligned}$$

Note that this in itself defines a group action since properties (a), (b) and (c) will hold trivially.

We now proceed to induce, from the action of G on $\mathcal{P}(M)$, an action of G on the function space $\mathcal{C}^\infty(X)$. Given a function $f : X \longrightarrow U$, with domain $\mathcal{D}_f = X$, we denote the graph of f by

$$\Gamma_f = \{(x,f(x)) \mid x \in \mathcal{D}_f\}$$

Obviously $\Gamma_f \in \mathcal{P}(X \times U)$, and we have an injective mapping

$$\begin{aligned} \mathcal{C}^{\infty}(X) = \mathcal{C}^{\infty}(X,U) &\longrightarrow \mathcal{P}(X \times U) = \mathcal{P}(M) \\ f &\longmapsto \Gamma_f \end{aligned}$$

In the case of the projectable groups of transformations, the induced action on a space of functions from X to U may be defined in a very simple way. Given the graph $\Gamma_f \in \mathcal{P}(M)$ of some function $f : X \longrightarrow U$, and given $g \in G$, when we apply the pointwise defined group action $g \cdot \Gamma_f$, the resulting subset of $M = X \times U$ will always be the graph $\Gamma_{g \cdot f}$ of another function $g \cdot f : X \longrightarrow U$, as illustrated in the following commutative diagram

$$\begin{tikzcd}
G \times \mathcal{C}^{\infty}(X) \arrow[r, dashed] & \mathcal{C}^{\infty}(X) \\
(g, f) \arrow[r, mapsto, dashed] \arrow[d, mapsto] & g \cdot f = [g_2 \circ (\mathrm{id}_X \times f)] \circ g_1^{-1} \\
(g, \Gamma_f) \arrow[r, mapsto] & g \cdot \Gamma_f = \Gamma_{g \cdot f} \arrow[u, mapsto] \\
G \times \mathcal{P}(M) \arrow[r] & \mathcal{P}(M)
\end{tikzcd}$$

Again this defines a valid group action, with properties (a), (b) and (c) following trivially from the corresponding properties for G acting on $\mathcal{P}(M)$.

The case of more general groups of transformations is similar, but somewhat complicated by the fact that $g \cdot \Gamma_f$ may in that case not be the graph of a well defined function, or may turn out to be the graph of a function which is well defined only on some open subset of X, even if the function f is defined on the whole of X. Furthermore, the induced action of more general transformation groups on a space of functions from X to U may also happen to be local, that is, only defined for group elements g in some neighborhood of the identity $e \in G$. An example will also be given of a function which may not be transformed by any group elements of a certain group, except trivially, by the identity element.

We recall (Appendix 1) that Colombeau's differential algebra of generalized functions is given by the associative, commutative quotient algebra

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{A} \,/\, \mathcal{I}$$

where

$$\mathcal{A} = \left\{ f \in \mathcal{E}[\mathbb{R}^n] \;\middle|\; \begin{array}{l} \forall\ K \subset X \text{ compact},\ p \in \mathbb{N}^n : \\ \exists\ m \in \mathbb{N}_+ : \\ \forall\ \phi \in \Phi_m : \\ \exists\ \eta,\ c > 0 : \\ \forall\ x \in K,\ \varepsilon \in (0,\eta) : \\ \quad |D_x^p f(\phi_\varepsilon, x)| \le c/\varepsilon^m \end{array} \right\}$$

and

$$\mathcal{I} = \left\{ f \in \mathcal{A} \;\middle|\; \begin{array}{l} \forall\ K \subset X \text{ compact},\ p \in \mathbb{N}^n : \\ \exists\ l \in \mathbb{N}_+,\ \beta \in B : \\ \forall\ m \in \mathbb{N}_+,\ m \ge l,\ \phi \in \Phi_m : \\ \exists\ \eta,\ c > 0 : \\ \forall\ x \in K,\ \varepsilon \in (0,\eta) : \\ \quad |D_x^p f(\phi_\varepsilon, x)| \le c\varepsilon^{\beta(m)-l} \end{array} \right\}$$

Clearly we first need to extend the mapping

$$\begin{aligned} G \times \mathcal{C}^\infty(X) &\longrightarrow \mathcal{C}^\infty(X) \\ (g, f) &\longmapsto g\cdot f = [g_2 \circ (id_X \times f)] \circ g_1^{-1} \end{aligned}$$

in the following natural way to

$$\begin{aligned} G \times \mathcal{E}[X] &\longrightarrow \mathcal{E}[X] \\ (g, f(\phi,\cdot)) &\longmapsto [g_2 \circ (\mathrm{id}_X \times f(\phi,\cdot))] \circ g_1^{-1} \quad ,\phi \in \Phi \end{aligned}$$

Note that the group properties (a), (b) and (c) hold trivially, since they hold for every fixed $\phi \in \Phi$.

Recall (Appendix 1) that we call $F \in \mathcal{C}^\infty(\mathbb{R}^l)$, $l \in \mathbb{N}_+$, slowly increasing iff

$$\begin{aligned} &\forall\, q \in \mathbb{N}^l : \\ &\exists\, c,\ d > 0 : \\ &\forall\, \xi \in \mathbb{R}^l : \\ &\quad |D_\xi^q F(\xi)| \le c(1+|\xi|)^d \end{aligned}$$

and that the PDE can also be written as

$$F(x,u(x),\ldots,D_x^p u(x),\ldots) = 0$$

where $x \in X$, $p \in \mathbb{N}^n$, $|p| \le m$ and F is $\mathcal{C}^\infty$–smooth, so that we have the mapping

$$\begin{aligned} T(x,D) : \mathcal{C}^\infty(X) &\longrightarrow \mathcal{C}^\infty(X) \\ u(\cdot) &\longmapsto F(\cdot, u,\ldots,D^p u,\ldots) \end{aligned}$$

Provided that F is slowly increasing, we recall that the operator $T(x,D)$ can be extended to Colombeau's algebra $\mathcal{G}(\mathbb{R}^n)$ of generalized functions as follows

$$
\begin{aligned}
T(D) : \mathcal{G}(X) &\longrightarrow \mathcal{G}(X) \\
u + \mathcal{I} &\longmapsto (T(x, D_x)u) + \mathcal{I}
\end{aligned}
$$

In order to extend the actions of the projectable groups to $\mathcal{G}(\mathbb{R}^n)$, we will need to impose the same restriction on the mappings

$$g_2 : M \longrightarrow U \quad , g \in G$$

namely that they be slowly increasing. We will call a transformation group for which every element satisfies this condition, slowly increasing. Note that this still includes the groups under consideration in General Relativity, where

$$g_2 = \mathrm{id}_U : U \longrightarrow U$$

is the identity mapping of the space of dependent variables.

Assuming then that g_2 is slowly increasing, it follows from Theorem 1 (Appendix 1) that the action of G on $\mathcal{E}[\mathbb{R}^n]$ can be extended to $\mathcal{G}(X)$ canonically

$$
\begin{aligned}
G \times \mathcal{G}(X) &\longrightarrow \mathcal{G}(X) \\
(g, f + \mathcal{I}) &\longmapsto (g \cdot f) + \mathcal{I}
\end{aligned}
$$

Again this defines a proper group action, with properties (a), (b) and (c) induced from the corresponding properties in the case of the action of G on $\mathcal{E}[X]$.

Extending General Group Actions to $\mathcal{G}(X)$

The case of local groups of transformations is similar to that of global groups of transformations, or specifically projectable groups, as considered above. The Lie group G is a local group of transformations acting on the manifold M if we have an open set

$$\{e\} \times M \subset \mathcal{U} \subset G \times M$$

and a mapping

$$\begin{array}{c} \mathcal{U} \longrightarrow M \\ (g,(x,u)) \longmapsto g\cdot(x,u) = (g_1(x,u),g_2(x,u)) \end{array}$$

where g_1 ,g_2 are $\mathcal{C}^\infty$–smooth, and the mapping has the following properties:

(a)
$$\forall\ (g,(x,u)),\ (g',g\cdot(x,u)),\ (g'\cdot g,(x,u)) \in \mathcal{U} :$$
$$g'\cdot(g\cdot(x,u)) = (g'\cdot g)\cdot(x,u)$$

(b)
$$\forall\ (x,u) \in M :$$
$$e\cdot(x,u) = (x,u)$$

where $e \in G$ is the identity element in G.

(c)
$$\forall\ (g,(x,u)) \in \mathcal{U} :$$
$$(g^{-1},g\cdot(x,u)) \in \mathcal{U} ,$$
$$g^{-1}\cdot(g\cdot(x,u)) = (x,u)$$

where $g^{-1} \in G$ is the inverse element of g.

We denote

$$G_{(x,u)} = \{g \in G \mid (g,(x,u)) \in \mathcal{U}\} \quad , (x,u) \in M$$

and

$$M_g = \{(x,u) \in M \mid (g,(x,u)) \in \mathcal{U}\} \quad , g \in G$$

Clearly we have, for $g \in G$ and $(x,u) \in M$

$$(x,u) \in M_g \Leftrightarrow g \in G_{(x,u)} \Leftrightarrow (g,(x,u)) \in \mathcal{U}$$

and

$$M = M_e$$

Thus $g_1 \in \mathcal{C}^\infty(M_g, X)$ and $g_2 \in \mathcal{C}^\infty(M_g, U)$.

As in the case of the projectable groups, it follows from property (c) that

$$\begin{aligned} &\forall\, g \in G : \\ &\quad \text{the mapping } \; g_1 \times g_2 : M_g \longrightarrow M_{g^{-1}} \; \text{ is a } \mathcal{C}^\infty\text{–diffeomorphism} \end{aligned}$$

As before, in order to extend the actions of local groups from the manifold M to a space of functions from X to U, we first extend them to an action on $\mathcal{P}(M)$ in the pointwise manner. We define the set

$$\mathcal{V} = \{(g,S) \in G \times \mathcal{P}(M) \mid \{g\} \times S \subset \mathcal{U}\}$$

which obviously has the property that

$$\{e\} \times \mathcal{P}(M) \subset \mathcal{V}$$

Now we define the mapping

$$\begin{aligned} \mathcal{V} &\longrightarrow \mathcal{P}(M) \\ (g,S) &\longmapsto g\cdot S = \{g\cdot(x,u) \mid (x,u) \in S\} \end{aligned}$$

This defines a group action since properties (a), (b) and (c) will hold. Indeed, (b) is trivial, since $\{e\} \times \mathcal{P}(M) \subset \mathcal{V}$. (a) follows from the fact that

(1) $$(g,S) \in \mathcal{V} \iff \begin{array}{l} \forall\ (x,u) \in S : \\ \quad (g,(x,u)) \in \mathcal{U} \end{array}$$

and the fact that (a) holds for the local action of G on M. (c) follows from (a), (b) and (1). It is also useful to note that

$$\{g\} \times S \subset \mathcal{U} \iff S \subset M_g$$

We now induce, as before, an action of G on a space of functions from X to U, by making use of the graphs of functions. There is, however, in the case of non–projectable groups, a certain complication, which results in a locality which is entirely independent of any PDE under consideration. Given a function $f : \mathcal{D}_1 \longrightarrow U$, $\mathcal{D}_1 \subset X$ open, and $g \in G$, such that $(g,\Gamma_f) \subset \mathcal{V}$, when we apply the pointwise defined group action $g\cdot\Gamma_f$, we would expect the resulting subset of $M = X \times U$ to be the graph $\Gamma_{g\cdot f}$ of another function $g\cdot f : \mathcal{D}_2 \longrightarrow U$,

$\mathcal{D}_2 \subset X$ open. However, even in the case where $\mathcal{D}_1 = X$, it is possible that $\mathcal{D}_2 \neq X$, or even that $\mathcal{D}_2 = \emptyset$, unless $g = e$, the neutral element. For example, if the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = ax$, for some $a > 0$, and $G = S^1$, the group of rotations in the plane, then Γ_f can only be rotated for angles θ with $|\theta + \tan^{-1} a| < \pi/2$. Secondly, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ cannot by rotated at all, since its gradient is unbounded. Of course, if we take the same function, but defined only on some bounded open subset of $\mathbb{R}$, then certain rotations will again yield a valid function, defined on some other bounded open subset of $\mathbb{R}$. Furthermore, a translation group, for example, will transform the same function defined on the whole of $\mathbb{R}$ into another function defined on the whole of $\mathbb{R}$.

The reason for these difficulties in the case of non–projectable groups of transformations is the fact that the set

$$g \cdot \Gamma_f = \{(\tilde{x},\tilde{u}) = g \cdot (x,u) \mid (x,u) \in \Gamma_f\}$$

is given parametrically by the equations

$$\left.\begin{aligned} \tilde{x} &= g_1(x,f(x)) = [g_1 \circ (\mathrm{id}_X \times f)](x) \\ \tilde{u} &= g_2(x,f(x)) = [g_2 \circ (\mathrm{id}_X \times f)](x) \end{aligned}\right. \quad , \; x \in \mathcal{D}_f$$

For $g \cdot \Gamma_f$ to be the graph of some other function

$$\begin{aligned} g \cdot f : \mathcal{D}_{g \cdot f} &\longrightarrow U \\ \tilde{x} &\longmapsto \tilde{u} \end{aligned}$$

we need to be able to solve the above system of equations. In the case where $g = e \in G$, we have

$$[e_1 \circ (\mathrm{id}_X \times f)] = \mathrm{id}_X$$

Thus, provided g is sufficiently near the identity $e \in G$, the Jacobian matrix of $[g_1 \circ (\mathrm{id}_X \times f)]$ is nonsingular, so that, by the inverse function theorem, we can locally solve for x, giving

$$x = [g_1 \circ (\mathrm{id}_X \times f)]^{-1}(\tilde{x})$$

It follows by substitution into the second equation that the general formula for the transform of $f : \mathcal{D}_f \longrightarrow U$ by $g \approx e \in G$ is

$$g \cdot f = [g_2 \circ (\mathrm{id}_X \times f)] \circ [g_1 \circ (\mathrm{id}_X \times f)]^{-1}$$

Clearly then, the invertibility of $[g_1 \circ (\mathrm{id}_X \times f)]$ is a sufficient condition for $g \cdot f$ to be a well defined function on some open subset of X.

Let us define, for $g \in G$, $f \in \mathcal{C}^\infty(X)$, the open set

$$\mathcal{D}_{g,f} = \left\{ x \in X \;\middle|\; \begin{array}{l} \exists\, B \subset X, \text{ open neighborhood of } x : \\ \text{the function } [g_1 \circ (\mathrm{id}_X \times f)]^{-1} \text{ exists on } B \end{array} \right\}$$

From the above discussion it is clear that

$$\mathcal{D}_{e,f} = X \quad , f \in \mathcal{C}^\infty(X)$$

Also, in the special case of the projectable groups, we have

$$[g_1 \circ (id_X \times f)] = g_1 : X \longrightarrow X$$

which is a $\mathcal{C}^\infty$–diffeomorphism, so that

$$\mathcal{D}_{g,f} = X \quad ,g \in G,\ f \in \mathcal{C}^\infty(X)$$

Since we are interested in group invariance of global solutions (classical or generalized) of nonlinear PDEs, we are only interested in such cases, that is, when

$$\mathcal{D}_{g,f} = \mathcal{D}_f = X$$

Thus we need to define the set

$$\mathcal{W} = \left\{ (g,f) \in G \times \mathcal{C}^\infty(X) \;\middle|\; \begin{array}{l} (g,\Gamma_f) \in \mathcal{V} , \\ \mathcal{D}_{g,f} = X \end{array} \right\}$$

By the above remarks, we clearly have

$$\{e\} \times \mathcal{C}^\infty(X) \subset \mathcal{W}$$

and in the case of the projectable groups,

$$\mathcal{W} = G \times \mathcal{C}^\infty(X)$$

As in the projectable case, we have defined the induced local action of G on $\mathcal{C}^\infty(X)$ by the commutative diagram

$$\begin{array}{ccc}
\mathcal{W} & \dashrightarrow & \mathcal{C}^{\infty}(X) \\
(g,f) & \dashrightarrow & g\cdot f \\
\downarrow & & \uparrow \\
(g,\Gamma_f) & \longmapsto & g\cdot\Gamma_f = \Gamma_{g\cdot f} \\
\mathcal{V} & \longrightarrow & \mathcal{P}(M)
\end{array}$$

To show that this is a valid group action, we need to prove that properties (a) and (c) hold, since property (b) is trivial. We start with (a). Given $(g,f),(g',g\cdot f),(g'\cdot g,f) \in \mathcal{W}$, it follows that $(g,\Gamma_f),(g',g\cdot\Gamma_f),(g'\cdot g,\Gamma_f) \in \mathcal{V}$. Since (a) holds for $\mathcal{V}$, this gives $g'\cdot(g\cdot\Gamma_f) = (g'\cdot g)\cdot\Gamma_f$, and hence $g'\cdot(g\cdot f) = (g'\cdot g)\cdot f$. To prove (c), take $(g,f) \in \mathcal{W}$, so that $(g,\Gamma_f) \in \mathcal{V}$. Since (c) holds for $\mathcal{V}$, we have $(g^{-1},g\cdot\Gamma_f) \in \mathcal{V}$ and $g^{-1}\cdot(g\cdot\Gamma_f) = \Gamma_f$, so that $\mathcal{D}_{g^{-1},g\cdot f} = \mathcal{D}_f = X$, and $g^{-1}\cdot(g\cdot f) = f$.

By defining the set

$$\mathcal{M} = \left\{(g,f) \in G \times \mathcal{E}[X] \;\middle|\; \begin{array}{l} \forall\, \phi \in \Phi : \\ \quad (g,f(\phi,\cdot)) \in \mathcal{W} \end{array}\right\}$$

and the mapping

$$\begin{array}{l} \mathcal{M} \longrightarrow \mathcal{E}[X] \\ (g,f(\phi,\cdot)) \longmapsto [g_2 \circ (id_X \times f(\phi,\cdot))] \circ [g_1 \circ (id_X \times f(\phi,\cdot))]^{-1} \quad ,\phi \in \Phi \end{array}$$

we automatically obtain a valid local group action of G on $\mathcal{E}[X]$. We have

$$\{e\} \times \mathcal{E}[X] \subset \mathcal{M}$$

and the group properties (a), (b) and (c) hold trivially, because they hold for every fixed $\phi \in \Phi$.

Analogous to M_g , it is convenient to define the set

$$\mathcal{M}_g = \{f \in \mathcal{E}[X] \mid (g,f) \in \mathcal{M}\} \quad , g \in G$$

As before, provided g_2 is slowly increasing, this local action of G on $\mathcal{E}[X]$ can be extended to $\mathcal{G}(X)$ canonically, by assuming the existence of a set

$$\{e\} \times \mathcal{G}(X) \subset \mathcal{N} \subset G \times \mathcal{G}(X)$$

with the property that

(2) $$\mathcal{N} \subset \{(g,f + \mathcal{I}) \in G \times \mathcal{G}(X) \mid (f + \mathcal{I}) \cap \mathcal{M}_g \neq \emptyset\}$$

and a mapping

$$\begin{aligned} &\mathcal{N} \longrightarrow \mathcal{G}(X) \\ &(g,f + \mathcal{I}) \longmapsto (g \cdot f') + \mathcal{I} \quad , \text{ where } f' \in f + \mathcal{I} \cap \mathcal{M}_g \end{aligned}$$

In order for this to define a valid local group action, the set $\mathcal{N}$ has to be chosen in such a way that property (a) will hold, since (b) follows trivially from the fact that $\{e\} \times \mathcal{G}(X) \subset \mathcal{N}$, and (c) follows from (2). Indeed, $(g,f + \mathcal{I}) \in \mathcal{N}$ implies $g \cdot f + \mathcal{I} = (g \cdot f') + \mathcal{I}$ for some $f' \in f + \mathcal{I}$. But then $g \cdot f' \in g \cdot f + \mathcal{I}$, $(g^{-1},g \cdot f') \in \mathcal{M}$ and $g^{-1} \cdot (g \cdot f') = f'$, because (c) holds for $\mathcal{M}$, so that $(g^{-1},g \cdot f + \mathcal{I}) \in \mathcal{N}$ and $g^{-1} \cdot (g \cdot f + \mathcal{I}) = f + \mathcal{I}$. A sufficient condition for (a) to hold, is that

$$\text{(3)} \qquad \forall\, g,g' \in G,\ f \in \mathcal{A} : \quad \begin{bmatrix} (g,f),(g',f') \in \mathcal{N}, \\ f - f' \in \mathcal{I} \end{bmatrix} \Longrightarrow (g',f) \in \mathcal{N}$$

because this would mean that given $(g,f + \mathcal{I}),(g',g\cdot f + \mathcal{I}),(g'\cdot g,f + \mathcal{I}) \in \mathcal{N}$, we would have a representant $f' \in f + \mathcal{I}$ with $(g,f'),(g',g\cdot f'),(g'\cdot g,f') \in \mathcal{N}$, so that $g'\cdot(g\cdot f') = (g'\cdot g)\cdot f'$ because (a) holds for $\mathcal{M}$, giving $g'\cdot(g\cdot f + \mathcal{I}) = (g'\cdot g)\cdot f + \mathcal{I}$. Condition (3) does not preclude nontrivial group actions, that is, $\mathcal{N} \neq \{e\} \times \mathcal{G}(X)$, as it is clear that both (2) and (3) hold for the set

$$\mathcal{N}_1 = \{(g,f + \mathcal{I}) \in G \times \mathcal{G}(X) \mid (f + \mathcal{I}) \subset \mathcal{M}_g\}$$

and that $\{e\} \times \mathcal{G}(X) \subsetneq \mathcal{N}_1$. Obviously, the larger $\mathcal{N}$ is, the less localised the action of G on $\mathcal{G}(X)$ will be. For example, in the case of the projectable groups, it is clear that $\mathcal{N} = G \times \mathcal{G}(X)$.

Extending the Concept of Invariance to $\mathcal{G}(X)$

In the classical context, we call a solution $u \in \mathcal{C}^\infty(X)$ of the nonlinear PDE

$$T(x,D)u(x) = 0 \quad , x \in X$$

invariant with respect to a given projectable group G of transformations of the space $M = X \times U$ of independent and dependent variables if and only if, for every $g \in G$, the transformed function $g \cdot u \in \mathcal{C}^\infty(X)$ is still a solution to the PDE, that is iff

$$\forall\ g \in G : \\ T(x,D)g \cdot u(x) = 0 \quad , x \in X$$

We call the transformation group G a symmetry group of the PDE if and only if every solution of the PDE is invariant with respect to G.

The definitions are similar in the case of generalized functions. We call a generalized solution $u + \mathcal{I} \in \mathcal{G}(X)$ of the PDE, that is, one which satisfies

$$T(D)u \in \mathcal{I}$$

invariant with respect to a given projectable, slowly increasing group G of transformations on M if and only if, for every $g \in G$, the transformed generalized function $g \cdot u + \mathcal{I} \in \mathcal{G}(X)$ is still a generalized solution to the PDE, that is iff

$$\forall\ g \in G : \\ T(D)g \cdot u \in \mathcal{I}$$

We call the transformation group G a symmetry group of the PDE in the generalized context if and only if every generalized solution of the PDE is invariant with respect to G.

These concepts of group invariance and symmetry groups in the generalized context are coherent with the corresponding classical concepts, in the sense that the classical situation can be viewed as a special case of the generalized situation. This follows clearly from the commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(X) & \xrightarrow{T(x,D)} & \mathcal{C}^\infty(X) \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}(X) & \xrightarrow{T(D)} & \mathcal{G}(X) \end{array}$$

where the classical function space $\mathcal{C}^\infty(X)$ is embedded as a differential subalgebra into Colombeau's algebra $\mathcal{G}(X)$ of generalized functions.

Both the classical and generalized concepts of local group invariance and local symmetry groups are simply an extension of the corresponding concepts for global groups such as the projectable groups. In the classical context, we call a solution $u \in \mathcal{C}^\infty(X)$ of the nonlinear PDE

$$T(x,D)u(x) = 0 \quad , x \in X$$

invariant with respect to a given local group G of transformations of the space $M = X \times U$ of independent and dependent variables if and only if, for every $g \in G$ which can act on u, the transformed function $g \cdot u \in \mathcal{C}^\infty(X)$ is still a solution to the PDE, that is iff

$$\begin{aligned} &\forall\ g \in G,\ (g,u) \in \mathcal{W} : \\ &\quad T(x,D)g\cdot u(x) = 0 \quad , x \in X \end{aligned}$$

We call the transformation group G a symmetry group of the PDE if and only if every solution of the PDE is invariant with respect to G.

We call a generalized solution $u + \mathcal{I} \in \mathcal{G}(X)$ of the PDE, that is, one which satisfies

$$T(D)u \in \mathcal{I}$$

invariant with respect to a given local, slowly increasing group G of transformations on M if and only if, for every $g \in G$ which can act on $u + \mathcal{I}$, the transformed generalized function $g\cdot u + \mathcal{I} \in \mathcal{G}(X)$ is still a generalized solution to the PDE, that is iff

$$\begin{aligned} &\forall\ g \in G,\ (g,u + \mathcal{I}) \in \mathcal{N} : \\ &\quad T(D)g\cdot u \in \mathcal{I} \end{aligned}$$

In the generalized context, we call the transformation group G a symmetry group of the PDE if and only if every generalized solution of the PDE is invariant with respect to G.

The corresponding concepts of global group invariance and global symmetry groups in, for example, the case of the projectable groups, are clearly special cases of the above definitions, namely the situations where $\mathcal{W} = G \times \mathcal{C}^\infty(X)$ and $\mathcal{N} = G \times \mathcal{G}(X)$.

SECTION 2: LIE GROUP TRANSFORMATIONS IN THE ORDER COMPLETION METHOD

Extending Projectable Group Actions

We consider again the m–th order, nonlinear PDE of the form

$$T(x,D)u(x) = f(x) \quad , \; x \in X$$

where $f \in \mathcal{C}^\infty(X)$ is given, $u : X \longrightarrow U$ is the unknown function, $X \subset \mathbb{R}^n$, $U = \mathbb{R}$.

Let G be a projectable group of transformations acting on the manifold $M = X \times U$

$$\begin{aligned} G \times M &\longrightarrow M \\ (g,(x,u)) &\longmapsto (g_1(x), g_2(x,u)) \end{aligned}$$

where $g_2 \in \mathcal{C}^\infty(M,U)$ and

$$g_1 : X \longrightarrow X$$

is a $\mathcal{C}^\infty$–diffeomorphism.

As in the previous section, we extend the action of G from M to $\mathcal{P}(M)$:

$$\begin{aligned} G \times \mathcal{P}(M) &\longrightarrow \mathcal{P}(M) \\ (g, S) &\longmapsto g \cdot S = \{g \cdot (x,u) \mid (x,u) \in S\} \end{aligned}$$

We now proceed to induce, from the action of G on $\mathcal{P}(M)$, an action of G on the function space

$$\mathcal{C}^{\infty}_{nd}(X) = \left\{ f : X \longrightarrow U \;\middle|\; \begin{array}{l} \exists\ \Sigma \subset X \text{ closed, nowhere dense :} \\ f \in \mathcal{C}^{\infty}(X \setminus \Sigma) \end{array} \right\}$$

since this space plays a central role in the order completion generalized solution method for continuous nonlinear PDEs (Appendix 2).

Given $f \in \mathcal{C}^{\infty}_{nd}(X)$, clearly $\Gamma_f \in \mathcal{P}(X \times U) = \mathcal{P}(M)$, and we have an injective mapping

$$\begin{array}{rcl} \mathcal{C}^{\infty}_{nd}(X) & \longrightarrow & \mathcal{P}(M) \\ f & \longmapsto & \Gamma_f \end{array}$$

Since the mapping $g_1 : X \longrightarrow X$ is a diffeomorphism, it must preserve closed, nowhere dense sets, so that we have the commutative diagram

$$\begin{array}{ccc} G \times \mathcal{C}^{\infty}_{nd}(X) & \dashrightarrow & \mathcal{C}^{\infty}_{nd}(X) \\ (g, f) & \longmapsto & g \cdot f = [g_2 \circ (id_X \times f)] \circ g_1^{-1} \\ \downarrow & & \uparrow \\ (g, \Gamma_f) & \longmapsto & g \cdot \Gamma_f = \Gamma_{g \cdot f} \\ G \times \mathcal{P}(M) & \longrightarrow & \mathcal{P}(M) \end{array}$$

This defines a group action, which can clearly be extended to the space (see Appendix 2)

$$\mathcal{M}^{\infty}(X) = \mathcal{C}^{\infty}_{nd}(X) \ / \sim$$

by the mapping

$$\begin{aligned} G \times \mathcal{M}^{\infty}(X) &\longrightarrow \mathcal{M}^{\infty}(X) \\ (g\,,\, f/\sim) &\longmapsto g\cdot f/\sim \;=\; (g\cdot f)/\sim \end{aligned}$$

In view of the commutative diagram (see Appendix 2)

(1)
$$\begin{tikzcd} \mathcal{C}^{\infty}_{nd}(X) \arrow[r, "T(x,D)"] \arrow[d, hook] & \mathcal{C}^{\infty}_{nd}(X) \arrow[d, hook] \\ \mathcal{M}^{\infty}_{T}(X) \arrow[r, hook, "T"] \arrow[d, hook] & \mathcal{M}^{\infty}(X) \arrow[d, hook] \\ \hat{\mathcal{M}}^{\infty}_{T}(X) \arrow[r, hook, "\hat{T}"] & \hat{\mathcal{M}}^{\infty}(X) \end{tikzcd}$$

of extensions of the partial differential operator T(x,D), it would appear that we now need to extend the group action of G from $\mathcal{C}^{\infty}_{nd}(X)$ to the domains of the operators T and $\hat{T}$, that is, to $\mathcal{M}^{\infty}_{T}(X) = \mathcal{C}^{\infty}_{nd}(X) \,/ \sim_T$ and $\hat{\mathcal{M}}^{\infty}_{T}(X)$ respectively. However, a problem immediately arises concerning the equivalence relation $\sim_T$ which is defined as follows (Appendix 2): given $u,v \in \mathcal{C}^{\infty}_{nd}(X)$,

$$u \sim_T v \Leftrightarrow T(x,D)u \sim T(x,D)v$$

The problem is that given $U,V \in \mathcal{M}^{\infty}(X)$ with $U \sim_T V$, it does not necessarily follow that for every $g \in G$, we have $g\cdot U \sim_T g\cdot V$. That is, we do not necessarily have $T(x,D)U \sim T(x,D)V$ implying that for every $g \in G$, $T(x,D)g\cdot U \sim T(x,D)g\cdot V$. The reason for this is that the equivalence relation $\sim_T$ depends only on the operator T(x,D), and does not necessarily take any account of the group G. For example, let us take $X = \mathbb{R}$, $g \in G = (\mathbb{R},+)$, and the projectable transformation

$$(2) \qquad (g,(x,u)) \longmapsto (x,\ ((x+u)^3+g)^{1/3}-x)$$

with $u,v \in \mathcal{C}^\infty(X) \subset \mathcal{C}^\infty_{nd}(X)$ given by $u : x \longmapsto a$ and $v : x \longmapsto b$, where $a,b \in \mathbb{R}$, $a \neq b$. If we take $T(x,D) = D_x$, then $T(x,D)u = T(x,D)v = 0$ on X, but for $0 \neq g \in G$, we have $g \cdot u : x \longmapsto ((x+a)^3+g)^{1/3}-x$ and $g \cdot v : x \longmapsto ((x+b)^3+g)^{1/3}-x$, so that $T(x,D)g \cdot u \neq T(x,D)g \cdot v$ on X.

Let us therefore define the following equivalence relation: given $u,v \in \mathcal{C}^\infty_{nd}(X)$,

$$(3) \qquad u \sim_{T,G} v \Leftrightarrow \left[\begin{array}{l} \forall\ g \in G : \\ \quad T(x,D)g \cdot u \sim T(x,D)g \cdot v \end{array}\right]$$

Since we clearly have

$$\begin{array}{l} \forall\ u,v \in \mathcal{C}^\infty_{nd}(X) : \\ \quad u \sim_{T,G} v \Longrightarrow u \sim_T v \end{array}$$

it follows (Appendix 2) that by defining

$$\mathcal{M}^\infty_{T,G}(X) = \mathcal{C}^\infty_{nd}(X) \,/\, \sim_{T,G}$$

we obtain the surjective mapping

$$\begin{array}{c} \mathcal{M}^\infty_{T,G}(X) \longrightarrow \mathcal{M}^\infty_T(X) \\ u/\sim_{T,G} \longmapsto u/\sim_T \end{array}$$

resulting in the commutative diagram

$$\begin{tikzcd} \mathcal{M}^{\infty}_{T,G}(X) \arrow[rr, "T_G"] \arrow[dr, "sur"'] & & \mathcal{M}^{\infty}(X) \\ & \mathcal{M}^{\infty}_{T}(X) \arrow[ur, "T"'] & \end{tikzcd}$$

with

$$\begin{aligned} T_G : \mathcal{M}^{\infty}_{T,G}(X) &\to \mathcal{M}^{\infty}(X) \\ u/\sim_{T,G} &\mapsto T_G u/\sim_{T,G} = (T(x,D)u)/\sim \end{aligned}$$

From (3) it also follows that the mapping

$$\begin{aligned} G \times \mathcal{M}^{\infty}_{T,G}(X) &\to \mathcal{M}^{\infty}_{T,G}(X) \\ (g, u/\sim_{T,G}) &\mapsto g \cdot u/\sim_{T,G} = (g \cdot u)/\sim_{T,G} \end{aligned}$$

is well defined, extending the group action of G from $\mathcal{C}^{\infty}_{nd}(X)$ to $\mathcal{M}^{\infty}_{T,G}(X)$.

We now define a partial order $\leq_{T,G}$ on $\mathcal{M}^{\infty}_{T,G}(X)$ as follows: given $U,V \in \mathcal{M}^{\infty}_{T,G}(X)$,

$$U \leq_{T,G} V \Leftrightarrow \left[\begin{array}{l} *)\ U = V \\ \text{or} \\ **)\ \forall\ g \in G : \\ \qquad T_G g \cdot U \lneqq T_G g \cdot V \ \text{ in } \mathcal{M}^{\infty}(X) \end{array} \right]$$

so that (Appendix 2) we get the commutative diagram of increasing mappings

$$\begin{tikzcd}
(\mathcal{M}^{\infty}_{T,G}(X), \leq_{T,G}) \arrow[rr, "T_G"] \arrow[dr, dashed, "sur"'] & & (\mathcal{M}^{\infty}(X), \leq) \\
& (\mathcal{M}^{\infty}_{T}(X), \leq_{T}) \arrow[ur, dashed, "T"'] &
\end{tikzcd}$$

with T an order isomorphical embedding, and $(\mathcal{M}^{\infty}_{T,G}(X), \leq_{T,G})$ with no minimum or maximum, so that we can replace (1) with the following more general commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{\infty}_{nd}(X) \arrow[r, "T(x\,,\,D)"] \arrow[d, hook] & \mathcal{C}^{\infty}_{nd}(X) \arrow[d, hook] \\
\mathcal{M}^{\infty}_{T,G}(X) \arrow[r, hook, "T_G"] \arrow[d, hook] & \mathcal{M}^{\infty}(X) \arrow[d, hook] \\
\hat{\mathcal{M}}^{\infty}_{T,G}(X) \arrow[r, hook, "\hat{T}_G"] & \hat{\mathcal{M}}^{\infty}(X)
\end{tikzcd}$$

where both T_G and $\hat{T}_G$ are increasing mappings.

The partial order $\leq_{T,G}$ is clearly not defined in the same way as the partial order $\leq_{T,*}$ (Appendix 2) when we restrict ourselves from $\mathcal{C}^{m}_{nd}(X)$ and $\mathcal{M}^{m}_{T,*}(X)$, to $\mathcal{C}^{\infty}_{nd}(X)$ and $\mathcal{M}^{\infty}_{T,*}(X)$ respectively. It is actually always smaller. Of course, this implies that it is also not in general the pull–back through T_G of the partial order $\leq$ on $\mathcal{M}^{\infty}(X)$, as is the case with the partial order $\leq_T$, which is generally larger than $\leq_{T,*}$.

The partial order $\leq_{T,G}$ is in fact too small to be admissible (Appendix 2), so that it remains an open problem whether one could prove the general existence result, that is the inclusion $\hat{T}_G(\hat{\mathcal{M}}^{\infty}_{T,G}(X)) \supset \mathcal{C}^{\infty}(\overline{X})$, using this partial order. The following example shows that $\leq_{T,G}$ is not admissible: let us take $X = \mathbb{R}$, $g \in G = (\mathbb{R},+)$, and the projectable transformation $(g,(x,u)) \longmapsto (x, e^{\alpha gx}u)$, so that $g \cdot u : x \longmapsto e^{\alpha gx}u(x)$. Take $T(x,D) = D_x$, so that $T(x,D)g \cdot u : x \longmapsto e^{\alpha gx}(\alpha gu(x) + u'(x))$, and let $u,v \in \mathcal{C}^\infty(X)$ be given by $u : x \longmapsto (1 - \epsilon)x$ and $v : x \longmapsto x$, where $\epsilon > 0$. Then $T(x,D)u + \epsilon = T(x,D)v = 1$ on X. But for any $g \in G$, we have $g \cdot u : x \longmapsto e^{\alpha gx}(1 - \epsilon)x$ and $g \cdot v : x \longmapsto e^{\alpha gx}x$, so that $T(x,D)g \cdot u : x \longmapsto e^{\alpha gx}(\alpha g(1 - \epsilon)x + (1 - \epsilon))$, and $T(x,D)g \cdot v : x \longmapsto e^{\alpha gx}(\alpha gx + 1)$. Thus $T(x,D)g \cdot u \leq T(x,D)g \cdot v$ on X iff $(1 - \epsilon)(\alpha gx + 1) \leq \alpha gx + 1$, $x \in X$. But for $g \neq 0$, there exists $x \in X$ such that $\alpha gx + 1 \leq 0$, which implies that $\epsilon \leq 0$, a contradiction.

Let $\mathcal{A}n_{nd}(X)$ denote the set of analytic functions defined on X except for closed nowhere dense subsets. The global version of the Cauchy–Kovalevskaia theorem (Rosinger [7]) for arbitrary analytic nonlinear PDEs

$$T(D)u(x) = f(x) \quad , x \in X$$

with noncharacteristic analytic Cauchy data, gives us the existence of generalized solutions $u \in \mathcal{A}n_{nd}(X) \subset \mathcal{C}^\infty_{nd}(X)$, that is, satisfying

$$T(D)u \sim f$$

This is clearly equivalent to satisfying

$$T_G u/\sim_{T,G} = f/\sim \quad \text{in } \mathcal{M}^\infty(X)$$

which implies that

$$\hat{T}_G\{u/\sim_{T,G}\}^{ul} = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^\infty(X)$$

We do therefore, have the inclusions

$$T_G(\mathcal{M}^{\infty}_{T,G}(X)) \supset \mathcal{A}n(X)$$

$$\hat{T}_G(\hat{\mathcal{M}}^{\infty}_{T,G}(X)) \supset \mathcal{A}n(X)$$

Before continuing, let us consider the semilinear hyperbolic equation

$$u_t(t,x) + \lambda u_x(t,x) = F(u(t,x))$$

where $t,x \in \mathbb{R}$, $0 \neq \lambda \in \mathbb{R}$, $F \in \mathcal{C}^{\infty}(\mathbb{R})$, which can be associated with the partial differential operator

$$T(D) = D_t + \lambda D_x - F()$$

The classical symmetry groups of this PDE are generated by the following two nonlinear one–dimensional Lie groups

(4)
$$\begin{aligned}(\mathbb{R},+) \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (\epsilon,(t,x,u)) &\longmapsto (t(\epsilon),x,u)\end{aligned}$$

(5)
$$\begin{aligned}(\mathbb{R},+) \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (\epsilon,(t,x,u)) &\longmapsto (t,x(\epsilon),u)\end{aligned}$$

where

$$t'(\epsilon) = a(x - \lambda t(\epsilon)) \quad , \epsilon \in \mathbb{R}$$

$$x'(\epsilon) = b(x(\epsilon) - \lambda t) \quad , \epsilon \in \mathbb{R}$$

for arbitrary a,b $\in \mathcal{C}^\infty(\mathbb{R})$, so that

$$t(\epsilon) = \lambda^{-1}(x - \alpha^{-1}(\alpha(x - \lambda t) + \epsilon)) \quad , \epsilon \in \mathbb{R}$$

$$x(\epsilon) = \lambda t + \beta^{-1}(\beta(x - \lambda t) + \epsilon) \quad , \epsilon \in \mathbb{R}$$

for $\alpha,\beta : \mathbb{R} \longrightarrow \mathbb{R}$ arbitrary $\mathcal{C}^\infty$–smooth diffeomorphisms.

Clearly (4) and (5) are projectable transformation groups. In addition, taking an instance of (5), with $b : \xi \longmapsto \xi^2$, we get

$$x(\epsilon) = \lambda t + (x - \lambda t)/(1 - (x - \lambda t)\epsilon) \quad , \epsilon \in \mathbb{R}$$

which is nonlinear and local, since we cannot have $\epsilon = 1/(x - \lambda t)$ when $x - \lambda t \neq 0$. However, if we take $b : \xi \longmapsto \xi$, for example, we get a global linear transformation group with

$$x(\epsilon) = \lambda t + (x - \lambda t)e^\epsilon \quad , \epsilon \in \mathbb{R}$$

What is most interesting about transformation (5) is the fact that, for $u \in \mathcal{C}^\infty(\mathbb{R}^2)$, $\epsilon \in (\mathbb{R},+)$, we have

$$\epsilon \cdot u(t,x) = u(t,\beta^{-1}(\beta(x - \lambda t) - \epsilon)) \quad , t,x \in \mathbb{R}$$

so that we get the invariance relation

$$(6) \qquad T(D)\epsilon\cdot u(t,x) = T(D)u(t,\beta^{-1}(\beta(x-\lambda t)-\epsilon)) \quad , \ t,x \in \mathbb{R}$$

Thus

$$\forall \ u,v \in \mathcal{C}^{\infty}_{nd}(\mathbb{R}^2) \ :$$
$$u \sim_{T,G} v \Leftrightarrow u \sim_T v$$

so that $\mathcal{M}^{\infty}_{T,G}(\mathbb{R}^2) = \mathcal{M}^{\infty}_{T}(\mathbb{R}^2)$, $T_G = T$, and

$$\forall \ U,V \in \mathcal{M}^{\infty}_{T}(\mathbb{R}^2) \ :$$
$$U \leq_{T,G} V \Leftrightarrow U \leq_T V$$

The same results hold for transformation (4). However, invariance relations such as (6) do not hold in general. That is, given a PDE

$$T(x,D)u = f \quad \text{on } X$$

and a projectable classical symmetry group G of the PDE, so that

$$\forall \ g \in G, \ u \in \mathcal{C}^{\infty}(X), \ T(x,D)u = f \quad \text{on } X \ :$$
$$T(x,D)g\cdot u = f \quad \text{on } X$$

it does not necessarily follow that

$$\forall \ g \in G, \ u \in \mathcal{C}^{\infty}(X) \ :$$
$$T(x,D)g\cdot u(x) = T(g_1^{-1}(x),D)u(g_1^{-1}(x)) \quad , \ x \in X$$

For example, the heat equation

$$u_t(t,x) = u_{xx}(t,x) \quad , \ t > 0, \ x \in \mathbb{R}$$

which we can associate with the partial differential operator

$$T(D) = D_t - D_{xx}$$

has a symmetry group (Olver 1986, p 122)

$$\begin{aligned} (\mathbb{R},+) \times \mathbb{R}_+ \times \mathbb{R}^2 &\longrightarrow \mathbb{R}_+ \times \mathbb{R}^2 \\ (\epsilon,(t,x,u)) &\longmapsto (t\,e^{2\epsilon},xe^{\epsilon},u) \end{aligned}$$

so that, for $u \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R})$, $\epsilon \in (\mathbb{R},+)$,

$$\epsilon \cdot u(t,x) = u(te^{-2\epsilon},xe^{-\epsilon})$$

and, as opposed to (6), we have

$$T(D)\epsilon \cdot u(t,x) = e^{-2\epsilon}T(D)u(te^{-2\epsilon},xe^{-\epsilon})$$

However, this still leads to

$$\begin{aligned} &\forall\ u,v \in \mathcal{C}^\infty_{nd}(\mathbb{R}_+ \times \mathbb{R}) : \\ &u \sim_{T,G} v \Leftrightarrow u \sim_T v \end{aligned}$$

so that $\mathcal{M}^{\infty}_{T,G}(\mathbb{R}_+ \times \mathbb{R}) = \mathcal{M}^{\infty}_{T}(\mathbb{R}_+ \times \mathbb{R})$, $T_G = T$, and

$$\begin{aligned} &\forall\ U,V \in \mathcal{M}^{\infty}_{T}(\mathbb{R}_+ \times \mathbb{R}) \ : \\ &\quad U \leq_{T,G} V \Leftrightarrow U \leq_T V \end{aligned}$$

This does not happen in general though, as illustrated in the example of the transformation (2). Finally, another symmetry group of the heat equation is

$$\begin{aligned} &(\mathbb{R},+) \times \mathbb{R}_+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+ \times \mathbb{R}^2 \\ &\quad (\epsilon,(t,x,u)) \longmapsto (t, x + 2t\epsilon, ue^{-x\epsilon - t\epsilon^2}) \end{aligned}$$

for which it certainly does not hold in general that

$$\begin{aligned} &\forall\ \epsilon \in (\mathbb{R},+),\ u,v \in \mathcal{C}^{\infty}(\mathbb{R}_+ \times \mathbb{R}) \ : \\ &\quad T(D)u = T(D)v \Rightarrow T(D)\,\epsilon\cdot u = T(D)\epsilon\cdot v \end{aligned}$$

We can now simply extend the group action of G from $\mathcal{M}^{\infty}_{T,G}(X)$ to $\hat{\mathcal{M}}^{\infty}_{T,G}(X)$ by means of the following mapping

$$\begin{aligned} &G \times \hat{\mathcal{M}}^{\infty}_{T,G}(X) \longrightarrow \hat{\mathcal{M}}^{\infty}_{T,G}(X) \\ &\quad (g,A) \longmapsto g\cdot A = \{g\cdot U \mid U \in A\} \end{aligned}$$

We show that $g\cdot A$ is indeed an element of $\hat{\mathcal{M}}^{\infty}_{T,G}(X)$: First we prove in $(\mathcal{M}^{\infty}_{T,G}(X), \leq_{T,G})$ the relation

$$U \leq_{T,G} V \Leftrightarrow \left[\begin{array}{l} \forall\ g \in G \ : \\ \quad g\cdot U \leq_{T,G} g\cdot V \end{array} \right] \tag{7}$$

The implication ($\Longleftarrow$) is trivial, by taking $g = e \in G$. We prove the implication ($\Longrightarrow$): By the definition of the partial order $\leq_{T,G}$, we can have $U = V$, in which case ($\Longrightarrow$) must hold, or we can have

$$\forall\, h \in G :$$
$$T_G h \cdot U \lneqq T_G h \cdot V \quad \text{in } \mathcal{M}^\infty(X)$$

But G is a group, so that, given $g \in G$, we have $h \cdot g \in G$ for all $h \in G$, so that

$$\forall\, h \in G :$$
$$T_G h \cdot g \cdot U \lneqq T_G h \cdot g \cdot V \quad \text{in } \mathcal{M}^\infty(X)$$

($\Longrightarrow$) again must hold.

Next we show that in $(\mathcal{M}^\infty_{T,G}(X), \leq_{T,G})$ we have the following relations:

$$\forall\, g \in G,\ A \subset \mathcal{M}^\infty_{T,G}(X) :$$
$$g \cdot A^l = (g \cdot A)^l, \quad g \cdot A^u = (g \cdot A)^u$$

We prove the first; the second follows similarly.

$$\begin{aligned}
g \cdot A^l &= \{ g \cdot U \mid U \in A^l \} \\
&= \{ g \cdot U \mid U \leq_{T,G} V,\ V \in A \} \\
&= \{ g \cdot U \mid g \cdot U \leq_{T,G} g \cdot V,\ V \in A \} \quad \text{, by (7)} \\
&= \{ W \mid W \leq_{T,G} Z,\ Z \in g \cdot A \} \\
&= \{ W \mid W \in (g \cdot A)^l \} \\
&= (g \cdot A)^l
\end{aligned}$$

Thus, for $g \in G$, $A \in \hat{M}^{\infty}_{T,G}(X)$, we have

$$g \cdot A = g \cdot A^{ul} = (g \cdot A^{u})^{l} = (g \cdot A)^{ul} \in \hat{M}^{\infty}_{T,G}(X)$$

so that we have a well defined group action of G on $\hat{M}^{\infty}_{T,G}(X)$.

We note that it also follows from relation (7) that, for every $g \in G$, the (bijective) group transformation

$$\begin{aligned} M^{\infty}_{T,G}(X) &\longrightarrow M^{\infty}_{T,G}(X) \\ U &\longmapsto g \cdot U \end{aligned}$$

is an order isomorphism.

Recall (Appendix 2) that the partial order on $\hat{M}^{\infty}_{T,G}(X)$ is defined as set inclusion. Similar to relation (7), we have for $A, B \in (\hat{M}^{\infty}_{T,G}(X), \subseteq)$, the relation

$$A \subseteq B \Leftrightarrow \left[\begin{array}{l} \forall \ g \in G : \\ \quad g \cdot A \subseteq g \cdot B \end{array} \right]$$

This clearly follows from the fact that the group transformations of elements of $\hat{M}^{\infty}_{T,G}(X)$ are defined elementwise. Thus it again follows that, for every $g \in G$, the bijective group transformation

$$\begin{aligned} \hat{M}^{\infty}_{T,G}(X) &\longrightarrow \hat{M}^{\infty}_{T,G}(X) \\ A &\longmapsto g \cdot A \end{aligned}$$

is an order isomorphism.

Extending the Concept of Invariance

As in the previous section, the definitions of group invariance of solutions, and symmetry groups of PDEs, are similar to the classical concepts, in the case of generalized functions. We call a global generalized solution $A \in \hat{\mathcal{M}}^{\infty}_{T,G}(X)$ of the PDE, that is, one which satisfies

$$\hat{T}_G A = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^{\infty}(X)$$

invariant with respect to a given projectable group G of transformations on M if and only if, for every $g \in G$, the transformed generalized function $g \cdot A \in \hat{\mathcal{M}}^{\infty}_{T,G}(X)$ is still a generalized solution to the PDE, that is iff

$$\begin{array}{l} \forall \; g \in G : \\ \quad \hat{T}_G g \cdot A = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^{\infty}(X) \end{array}$$

We call the transformation group G a symmetry group of the generalized solutions of the PDE if and only if every generalized solution of the PDE is invariant with respect to G, that is, iff

$$\begin{array}{l} \forall \; g \in G, \; A \in \hat{\mathcal{M}}^{\infty}_{T,G}(X), \; \hat{T}_G A = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^{\infty}(X) : \\ \hat{T}_G g \cdot A = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^{\infty}(X) \end{array}$$

This implies that

$$\begin{array}{l} \forall \; g \in G, \; U^{ul} \in \hat{\mathcal{M}}^{\infty}_{T,G}(X), \; T_G U = f/\sim \quad \text{in } \mathcal{M}^{\infty}(X) : \\ \hat{T}_G (g \cdot U)^{ul} = \{f/\sim\}^{ul} \quad \text{in } \hat{\mathcal{M}}^{\infty}(X) \end{array}$$

In other words, G is a symmetry group of the class of generalized solutions of the form $U^{ul} \in \hat{\mathcal{M}}^{\infty}_{T,G}(X)$, with $T_G U = f/\sim$, which is equivalent to

$$\forall\ g \in G,\ U \in \mathcal{M}^{\infty}_{T,G}(X),\ T_G U = f/\sim \text{ in } \mathcal{M}^{\infty}(X) :$$
$$T_G g \cdot U = f/\sim \quad \text{in } \hat{\mathcal{M}}^{\infty}(X)$$

which is again equivalent to

$$\forall\ g \in G,\ u \in \mathcal{C}^{\infty}_{nd}(X),\ T(x,D)u \sim f :$$
$$T(x,D)g \cdot u \sim f$$

and this obviously implies that

$$\forall\ g \in G,\ u \in \mathcal{C}^{\infty}(X),\ T(x,D)u = f \text{ on } X :$$
$$T(x,D)g \cdot u = f \quad \text{on } X$$

so that G is also a symmetry group of the PDE in the classical sense.

Applications to Delta Waves and Shock Waves

As mentioned, at this stage it is an open problem whether the general existence result, that is the inclusion $\hat{T}_G(\hat{\mathcal{M}}^{\infty}_{T,G}(X)) \supset \mathcal{C}^{\infty}(\overline{X})$, holds for the partial order $\leq_{T,G}$ as we have defined it. However, the global version of the Cauchy–Kovalevskaia theorem does give us the inclusion $T_G(\mathcal{M}^{\infty}_{T,G}(X)) \supset An(X)$ for arbitrary analytic PDEs, which has been the most general, type independent existence result for global solutions of nonlinear PDEs available apart from the general order completion result. For this reason, and for simplicity, in the two examples which follow, it suffices to work in the context of the operator T_G instead of $\hat{T}_G$. In addition, we know from the previous discussion that we must find our generalized symmetry groups among the known, classical ones. We will see how some of these nonlinear symmetries can be extended in a simple way to global weak solutions of nonlinear equations with rough initial data, on their entire domains, leading to new insights concerning the role played by the closed nowhere dense singularity sets of those solutions.

We begin by considering the delta wave solutions of the semilinear hyperbolic equation

$$u_t(t,x) + \lambda u_x(t,x) = F(u(t,x))$$

where $t,x \in \mathbb{R}$, $0 \neq \lambda \in \mathbb{R}$, $F \in \mathcal{C}^{\infty}(\mathbb{R})$, F,F' bounded on $\mathbb{R}$. Recall that the PDE can be associated with the partial differential operator

$$T(D) = D_t + \lambda D_x - F()$$

and rewritten in the form

$$T(D)u = 0 \quad \text{on } \mathbb{R}^2$$

Briefly, given any initial value condition

$$u(0,x) = \mu(x) \quad , \ x \in \mathbb{R}$$

with $\mu \in \mathcal{D}'(\mathbb{R})$ finitely supported, we know (Oberguggenberger 1992) that there exists a delta wave solution

$$u = v + w \quad \text{on } \mathbb{R}^2$$

where $w \in \mathcal{C}^\infty(\mathbb{R}^2)$ is the unique classical solution of the semilinear initial value problem

$$\begin{aligned} & w_t(t,x) + \lambda w_x(t,x) = F(u(t,x)) \quad , \ t,x \in \mathbb{R} \\ & w(0,x) = 0 \quad , \ x \in \mathbb{R} \end{aligned}$$

and $v \in \mathcal{D}'(\mathbb{R}^2)$ is the distributional solution of the linear initial value problem

$$\begin{aligned} & v_t(t,x) + \lambda v_x(t,x) = 0 \quad , \ t,x \in \mathbb{R} \\ & v(0,x) = \mu(x) \quad , x \in \mathbb{R} \end{aligned}$$

If we take, for example, the rough initial value

$$\mu = c\delta \in \mathcal{D}'(\mathbb{R}) \quad , \ 0 \neq c \in \mathbb{R}$$

as a multiple of the Dirac delta distribution with support $\{0\}$, then we get

$$v(t,x) = c\delta(x - \lambda t) \quad , \ t,x \in \mathbb{R}$$

so that we have a solution $u \in \mathcal{C}^\infty_{nd}(\mathbb{R}^2)$, given by

$$u = v + w \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \Sigma)$$

where the closed, nowhere dense singularity set

$$\Sigma = \{(t,x) \in \mathbb{R}^2 \mid x = \lambda t\}$$

is the characteristic curve of the initial condition

$$u(0,x) = \mu(x) \quad , \ x \in \mathbb{R}$$

Thus we have

$$T(D)u = T(D)w = 0 \quad \text{on } \mathbb{R}^2 \setminus \Sigma$$

consequently,

$$T(D)u \sim T(D)w \sim 0$$

As mentioned, one of the two classical one–dimensional symmetry groups of this PDE is given by the nonlinear projectable group transformation

$$(5) \qquad \begin{aligned} (\mathbb{R},+) \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (\epsilon,(t,x,u)) &\longmapsto (t,x(\epsilon),u) \end{aligned}$$

where

$$x(\epsilon) = \lambda t + \beta^{-1}(\beta(x - \lambda t) + \epsilon) \quad , \epsilon \in \mathbb{R}$$

for $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ an arbitrary $\mathcal{C}^\infty$–smooth diffeomorphism. Recall that the invariance relation

$$\text{(6)} \qquad T(D)\epsilon \cdot u(t,x) = T(D)u(t,\beta^{-1}(\beta(x - \lambda t) - \epsilon)) \quad , t,x \in \mathbb{R}$$

for $\epsilon \in (\mathbb{R},+)$, $u \in \mathcal{C}^\infty(\mathbb{R}^2)$, results in T_G , $\sim_{T,G}$, $\mathcal{M}^\infty_{T,G}(\mathbb{R}^2)$ and $\leq_{T,G}$ reducing to T, $\sim_T$, $\mathcal{M}^\infty_T(\mathbb{R}^2)$ and $\leq_T$ respectively. Thus we have a generalized solution

$$U = u/\sim_T = w/\sim_T \in \mathcal{M}^\infty_T(\mathbb{R}^2)$$

with

$$TU = 0/\sim$$

Since, for $\epsilon \in (\mathbb{R},+)$, the singularity set Σ is transformed into another closed, nowhere dense singularity set

$$\begin{aligned} \epsilon \cdot \Sigma &= \{(t,x(\epsilon)) \mid (t,x) \in \Sigma\} \\ &= \{(t,x(\epsilon)) \mid x = \lambda t\} \\ &= \{(t,x) \mid x = \lambda t + \beta^{-1}(\beta(0) + \epsilon)\} \\ &= \{(t,x) \mid \beta(x - \lambda t) = \beta(0) + \epsilon\} \end{aligned}$$

it also follows from invariance relation (6) that

$$T(D)\epsilon\cdot u = 0 \quad \text{on } \mathbb{R}^2 \setminus \epsilon\cdot\Sigma$$

which gives us a simple geometric interpretation for the consequence

$$T(D)\epsilon\cdot u \sim 0$$

that is,

$$T\epsilon\cdot U = 0/\sim$$

so that $U \in \mathcal{M}_T^\infty(\mathbb{R}^2)$ is an invariant generalized solution.

The group invariance results in this example can easily be extended to arbitrary rough initial values $\mu \in \mathcal{D}\,'(\mathbb{R})$ with finite support. In addition, these results obviously also hold for nonlinear projectable one–dimensional symmetry groups of the form (4).

Next we consider the Riemann solvers of the nonlinear shock wave equation

$$u_t(t,x) + u(t,x)u_x(t,x) = 0$$

where $0 < t \in \mathbb{R}$, $x \in \mathbb{R}$. The PDE can also be written in the form

$$T(D)u = D_t u + uD_x u = 0 \quad \text{on } \Omega = \mathbb{R}_+ \times \mathbb{R}$$

The Riemann solvers correspond to the discontinuous initial value conditions

$$u(0,x) = u_l + (u_r - u_l)H(x - x_0) \quad , \ x \in \mathbb{R}$$

where $x_0, u_l, u_r \in \mathbb{R}$, $u_l > u_r$, while $H : \mathbb{R} \longrightarrow \mathbb{R}$ is the Heaviside function, defined by

$$H(x) = \left| \begin{array}{l} 0 \Longleftarrow x \leq 0 \\ 1 \Longleftarrow x > 0 \end{array} \right.$$

The initial value is therefore decreasing, so that the global weak solution of the initial value problem will contain shocks. This weak solution, which satisfies both the Rankine–Hugoniot jump condition and the Entropy condition, is given by the Riemann solver

$$u(t,x) = u_l + (u_r - u_l)H(x - x_0 - (u_l + u_r)t/2) \quad , \ (t,x) \in \Omega$$

with the closed, nowhere dense set of shock points

$$\Sigma = \{(t,x) \in \Omega \mid x - x_0 = (u_l + u_r)t/2\}$$

We can define the open subsets of Ω

$$\Omega_l = \{(t,x) \in \Omega \mid x - x_0 < (u_l + u_r)t/2\}$$

$$\Omega_r = \{(t,x) \in \Omega \mid x - x_0 > (u_l + u_r)t/2\}$$

thereby partitioning $\Omega \setminus \Sigma$, so that

$$u(t,x) = \left| \begin{array}{l} u_l \Longleftarrow (t,x) \in \Omega_l \\ u_r \Longleftarrow (t,x) \in \Omega_r \end{array} \right.$$

Clearly we have $u \in \mathcal{C}^\infty(\Omega \setminus \Sigma)$, with

$$T(D)u = 0 \quad \text{on } \Omega_l \cap \Omega_r = \Omega \setminus \Sigma$$

so that

$$u \in C^\infty_{nd}(\Omega)$$

$$T(D)u \sim 0$$

The classical symmetry groups of the nonlinear shock wave equation are generated by the following four nonlinear one–dimensional Lie groups

(8)
$$\begin{array}{l} (\mathbb{R},+) \times \Omega \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R} \\ (\epsilon,(t,x,u)) \longmapsto (t + a(u)\epsilon,x,u) \end{array}$$

(9)
$$\begin{array}{l} (\mathbb{R},+) \times \Omega \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R} \\ (\epsilon,(t,x,u)) \longmapsto (t,x + b(u)\epsilon,u) \end{array}$$

$$\begin{array}{l} (\mathbb{R},+) \times \Omega \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R} \\ (\epsilon,(t,x,u)) \longmapsto (te^{c(u)\epsilon},xe^{c(u)\epsilon},u) \end{array}$$

$$\begin{array}{l} (\mathbb{R},+) \times \Omega \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R} \\ (\epsilon,(t,x,u)) \longmapsto (t,x + t(d^{-1}(d(u) + \epsilon) - u),d^{-1}(d(u) + \epsilon)) \end{array}$$

for $a,b,c,d : \mathbb{R} \longrightarrow \mathbb{R}$ arbitrary $\mathcal{C}^\infty$–smooth functions.

Clearly, none of these four nonlinear one–dimensional Lie groups happens to be projectable, unless functions a, b, c or d are constant. Thus we cannot apply the projectable theory we have developed. However, we shall try to go beyond that theory in the particular case of shock wave solutions given by the Riemann solvers, by examining their possible group invariance with respect to the classical symmetry groups of the form (8). If we assume that

$$a(u_l) = a(u_r) = a_0$$

for some fixed $a_0 \in \mathbb{R}$ (without asking that $a : \mathbb{R} \longrightarrow \mathbb{R}$ be constant), then for any $\epsilon \in (\mathbb{R},+)$, and for the Riemann solver $u \in C^\infty_{nd}(\Omega)$, we obtain the transformation

$$\epsilon \cdot u(t,x) = u(t - a_0\epsilon, x) \quad , \; (t,x) \in \Omega \setminus \epsilon \cdot \Sigma$$

where

$$\begin{aligned} \epsilon \cdot \Sigma &= \{\epsilon \cdot (t,x) \mid (t,x) \in \Sigma\} \\ &= \{\epsilon \cdot (t,x) \mid x - x_0 = (u_l + u_r)t/2\} \\ &= \{(t,x) \mid t = 2(x - x_0)/(u_l + u_r) + a_0\epsilon\} \end{aligned}$$

is the closed, nowhere dense singularity set of $\epsilon \cdot u \in C^\infty_{nd}(\Omega)$, since

$$\begin{aligned} \Omega \setminus \Sigma &\longrightarrow \Omega \setminus \epsilon \cdot \Sigma \\ (t,x) &\longmapsto (t + a_0\epsilon, x) \end{aligned}$$

is a diffeomorphism. Clearly we have

$$T(D)\epsilon\cdot u = 0 \quad \text{on} \quad \Omega \setminus \epsilon\cdot\Sigma$$

so that

$$T(D)\epsilon\cdot u \sim 0$$

Thus the Riemann solver $u \in C^{\infty}_{nd}(\Omega)$ with shocks Σ is invariant with respect to the group (8) with $a(u_l) = a(u_r)$, and is transformed by any $\epsilon \in (\mathbb{R},+)$ into another Riemann solver $\epsilon\cdot u \in C^{\infty}_{nd}(\Omega)$ with shocks $\epsilon\cdot\Sigma$ having the same slope as Σ, but translated along the t–axis. It can be shown that under the assumption that $b(u_l) = b(u_r)$, the same result holds for the group (9), with the corresponding translation of Σ being along the x–axis.

Extending General Group Actions

In the example of the shock wave equation we had little difficulty in considering non–projectable group actions in the particular case of the Riemann solvers. This suffices to illustrate the possibility that the theory of generalized projectable group actions in the order completion context may be extended to more general group actions. We recall that this exercise has in principle already been undertaken for the extension of projectable to local group actions on Colombeau's algebra of generalized functions. The approach used there can also be used in the case of the order completion method, that is, restricting consecutive extensions of group actions on $\mathcal{P}(M)$, and on the domains $\mathcal{C}^{\infty}_{nd}(X)$, $\mathcal{M}^{\infty}_{T,G}(X)$, $\hat{\mathcal{M}}^{\infty}_{T,G}(X)$, to suitable subsets of $G \times \mathcal{P}(M)$, $G \times \mathcal{C}^{\infty}_{nd}(X)$, $G \times \mathcal{M}^{\infty}_{T,G}(X)$ and $G \times \hat{\mathcal{M}}^{\infty}_{T,G}(X)$, respectively. As in the case of Colombeau algebras, the main technical difficulties are showing that the local group properties hold for these extensions. Clearly this more general theory is not vacuous, as we already have the special cases of the projectable group actions and of the non–projectable invariance of the Riemann solvers for the shock wave equation.

CONCLUSION

We have shown how Lie group actions defined on classical function spaces can be extended to two generalized function spaces which yield global generalized solutions for large classes of nonlinear PDEs. This has been done in such a way as to stay within finite dimensional manifolds, thereby avoiding the automatic loss of infinitesimal symmetry group calculation methods for these generalized solutions. In the case of Colombeau algebras, we have defined the concepts of invariant generalized solutions and their symmetry groups, for completely general group transformations. There is scope for future research in that we have not yet developed explicit symmetry group calculation methods for generalized solutions in Colombeau algebras. The extension of the classical theory has also been made for projectable group transformations in the order completion context of Oberguggenberger and Rosinger, and we have shown that a symmetry group of a PDE in the order completion generalized context must necessarily also be a symmetry group in the classical sense.

An indication has been given of how the same extensions in the order completion context can be done for completely general group transformations, by using local group actions, as in the case of Colombeau algebras. The feasibility of this avenue of future research is supported by the example given of the nonlinear group invariance of Riemann solvers of the nonlinear shock wave equation under non–projectable group actions.

We have also presented an example of the application of the projectable theory to the group invariance of delta wave solutions of a semilinear hyperbolic equation with rough initial data, without using the full power of the order completion solution method. It is still an open problem whether the general existence result of this method, the inclusion $\hat{\mathrm{T}}_{\mathrm{G}}(\hat{\mathcal{M}}^{\infty}_{\mathrm{T,G}}(\mathrm{X})) \supset \mathcal{C}^{\infty}(\overline{\mathrm{X}})$, holds for the non–admissible partial order $\leq_{\mathrm{T,G}}$ as we have defined it, or for any suitable partial order $\leq^{\mathrm{G}}_{\mathrm{T},*}$.

The basic constructions used in this thesis open up the possibility of applying similar extensions to other parts of the theory of continuous symmetry groups, for example, the theory of approximate groups and stability of symmetries (Baikov, Gazizov, Ibragimov, 1989). On the horizon, we have the possibility of developing an E–invariant quantum–mechanical description of nature through the study of the nonlinear Lie group invariance of global generalized solutions of arbitrary continuous nonlinear PDEs.

APPENDIX 1: Colombeau Algebras

We consider the class of m–th order ($m \geq 1$) nonlinear partial differential equations of the form

$$F(x,u(x),\ldots,D_x^p u(x),\ldots) = 0$$

where $x \in \mathbb{R}^n$, $p \in \mathbb{N}^n$, $|p| \leq m$, F is $\mathcal{C}^\infty$–smooth, slowly increasing (which will be defined later), and $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the unknown function. Thus the PDE defines a nonlinear partial differential operator

$$\begin{aligned} T(x,D) : \mathcal{C}^\infty(\mathbb{R}^n) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \\ u(\cdot) &\longmapsto F(\cdot,u,\ldots,D^p u,\ldots) \end{aligned}$$

so that the PDE can be rewritten as

$$T(x,D)u(x) = 0, \quad x \in \mathbb{R}^n$$

The mapping

$$T(x,D) : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$$

is in general not surjective: given an arbitrary $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, there does not necessarily exist a $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that $T(\cdot,D)u = f$ on $\mathbb{R}^n$. This means that the PDE $T(\cdot,D)u = 0$ does not necessarily have a classical solution. We therefore turn to spaces of generalized functions, embedding $\mathcal{C}^\infty(\mathbb{R}^n)$ into these spaces, and extending the nonlinear partial differential operator

T(x,D) to these spaces in such a way that in the generalized context, the PDE does have a solution. This solution is in general not a classical function, but a generalized function, i.e. an element of the generalized function space.

We now proceed to define Colombeau's algebra of generalized functions, within which global generalized solutions can be given, both for large classes of linear $\mathcal{C}^\infty$–smooth coefficient PDE's which have been proved to be unsolvable within distributions or hyperfunctions, and for various classes of nonlinear wave equations. We deal here only with generalized functions and solutions defined on the whole $\mathbb{R}^n$. However, generalized functions defined on open subsets $\Omega \subset \mathbb{R}^n$ can be handled similarly. This more general construction is summarised at the end of this appendix.

Let

$$\mathcal{D}(\mathbb{R}^n) = \left\{ \phi \in \mathcal{C}^\infty(\mathbb{R}^n) \;\middle|\; \begin{array}{l} \exists\ K \subset \mathbb{R}^n \text{ compact :} \\ \forall\ x \in \mathbb{R}^n \setminus K : \\ \quad \phi(x) = 0 \end{array} \right\}$$

denote the set of all compactly supported $\mathcal{C}^\infty$–smooth real valued functions on $\mathbb{R}^n$.

Let us define, for $k \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$, the set

$$\Phi_k = \left\{ \phi \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; \begin{array}{l} *)\ \displaystyle\int_{\mathbb{R}^n} \phi(x)\,dx = 1 \\ **)\ \forall\ p \in \mathbb{N}^n,\ 1 \leq |p| \leq k : \\ \qquad \displaystyle\int_{\mathbb{R}^n} x^p \phi(x)\,dx = 0 \end{array} \right\}$$

Let $\Phi = \Phi_1$. Let us define

$$\mathcal{E}[\mathbb{R}^n] = \left\{ f : \Phi \times \mathbb{R}^n \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \forall\ \phi \in \Phi : \\ \quad f(\phi,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^n) \end{array} \right\}$$

and its subalgebra

$$\mathcal{A} = \left\{ f \in \mathcal{E}[\mathbb{R}^n] \;\middle|\; \begin{array}{l} \forall\ K \subset \mathbb{R}^n \text{ compact},\ p \in \mathbb{N}^n : \\ \exists\ k \in \mathbb{N}_+ : \\ \forall\ \phi \in \Phi_k : \\ \exists\ \eta,\ c > 0 : \\ \forall\ x \in K,\ \varepsilon \in (0,\eta) : \\ \quad |D_x^p f(\phi_\varepsilon, x)| \leq c/\varepsilon^k \end{array} \right\}$$

where, for $\varepsilon > 0$, $\phi_\varepsilon \in \Phi_k$ is defined by

$$\phi_\varepsilon : x \longmapsto \phi(x/\varepsilon)/\varepsilon^n$$

Note that we have the algebra isomorphism

$$\mathcal{E}[\mathbb{R}^n] \cong \mathcal{C}^\infty(\mathbb{R}^n)^\Phi$$

that is, $\mathcal{E}[\mathbb{R}^n]$ can be seen as the algebra of sequences of $\mathcal{C}^\infty$–smooth real valued functions defined on $\mathbb{R}^n$, with index set Φ.

Denote

$$B = \left\{ \beta : \mathbb{N}_+ \longrightarrow (0,\infty) \;\middle|\; \begin{array}{l} *)\ \beta \text{ increasing} \\ **)\ \lim\limits_{k\to\infty} \beta(k) = \infty \end{array} \right\}$$

Let us define the ideal $\mathcal{I}$ in $\mathcal{A}$ as

$$\mathcal{I} = \left\{ f \in \mathcal{A} \;\middle|\; \begin{array}{l} \forall\ K \subset \mathbb{R}^n \text{ compact},\ p \in \mathbb{N}^n : \\ \exists\ l \in \mathbb{N}_+,\ \beta \in B : \\ \forall\ k \in \mathbb{N}_+,\ k \geq l,\ \phi \in \Phi_k : \\ \exists\ \eta,\ c > 0 : \\ \forall\ x \in K,\ \varepsilon \in (0,\eta) : \\ \quad |D_x^p f(\phi_\varepsilon, x)| \leq c\varepsilon^{\beta(k)-l} \end{array} \right\}$$

Colombeau's algebra of generalized functions on $\mathbb{R}^n$ is given by the associative, commutative quotient algebra

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{A} \,/\, \mathcal{I}$$

that is, the generalized functions are equivalence classes modulo $\mathcal{I}$ of sequences in $\mathcal{A}$ of $\mathcal{C}^\infty$–smooth functions.

We show that $\mathcal{G}(\mathbb{R}^n)$ is a differential algebra by first extending, for $p \in \mathbb{N}^n$, the partial derivative operator

$$D^p : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$$

in the following natural manner

$$\begin{aligned} D^p : \mathcal{E}[\mathbb{R}^n] &\longrightarrow \mathcal{E}[\mathbb{R}^n] \\ f &\longmapsto D_x^p f \end{aligned}$$

Clearly we have $D^p\mathcal{A} \subset \mathcal{A}$ and $D^p\mathcal{I} \subset \mathcal{I}$, so we may define canonically,

$$\begin{aligned} D^p : \mathcal{G}(\mathbb{R}^n) &\longrightarrow \mathcal{G}(\mathbb{R}^n) \\ f + \mathcal{I} &\longmapsto (D_x^p f) + \mathcal{I} \end{aligned}$$

These partial derivative operators on $\mathcal{G}(\mathbb{R}^n)$ are obviously linear, and satisfy the Leibnitz rule for the derivative of products.

Thus we get the following embedding of differential algebras

$$\begin{aligned} \mathcal{C}^\infty(\mathbb{R}^n) &\longrightarrow \mathcal{G}(\mathbb{R}^n) \\ f &\longmapsto f + \mathcal{I} \end{aligned}$$

where

$$\begin{aligned} f : \Phi \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (\phi, x) &\longmapsto f(x) \end{aligned}$$

Define, for $y \in \mathbb{R}^n$, the translation operator $\tau_y : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^n)$ by

$$\tau_y\phi(x) = \phi(x - y) \quad , \ \phi \in \mathcal{D}(\mathbb{R}^n), \ x \in \mathbb{R}^n$$

Then the space $\mathcal{D}'(\mathbb{R}^n)$ of L Schwartz distributions (the topological dual space of $\mathcal{D}(\mathbb{R}^n)$) can be embedded as a vector subspace of $\mathcal{G}(\mathbb{R}^n)$ as follows

$$\begin{aligned} \mathcal{D}'(\mathbb{R}^n) &\longrightarrow \mathcal{G}(\mathbb{R}^n) \\ T &\longmapsto f_T + \mathcal{I} \end{aligned}$$

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$$|F(f_1(\phi_\varepsilon,x),\dots,f_l(\phi_\varepsilon,x))| \le c''/\varepsilon^{kd}$$

where $c'' > 0$ depends only on c and c'.

Since by the chain rule

$$D_{x_i}[F(f_1(\phi,x),\dots,f_l(\phi,x))] = \sum_{j=1}^{l} (D_{x_j}F)(f_1(\phi,x),\dots,f_l(\phi,x))\cdot(D_{x_i}f_j)(\phi,x)$$

the same kind of bounds adapt immediately for $p = \dot{1} \in \mathbb{N}^n$. The same method can be used for higher order derivatives. ■

Proposition 2: For $l \in \mathbb{N}_+$, if $F \in \mathcal{C}^\infty(\mathbb{R}^l)$ is slowly increasing, and $r_i, r_i' \in \mathcal{A}$, $r_i - r_i' \in \mathcal{I}$, $1 \le i \le l$, then $F(r_1,r_2,\dots,r_l) - F(r_1',r_2',\dots,r_l') \in \mathcal{I}$.

Proof: Again we start with the identity operator D_x^p, $p = 0 \in \mathbb{N}^n$. The mean value theorem gives

$$\begin{aligned}&|F(r_1(\phi_\varepsilon,x),\dots r_l(\phi_\varepsilon,x)) - F(r_1'(\phi_\varepsilon,x),\dots,r_l'(\phi_\varepsilon,x))|\\ &\le \sum_{j=1}^{l} \sup_{0\le\theta\le1}\left|D_{x_j}\left[\{r_i(\phi_\varepsilon,x)+\theta(r_i'-r_i)(\phi_\varepsilon,x)\}_{i=1}^{l}\right]\right|\cdot|(r_j-r_j')(\phi_\varepsilon,x)|\end{aligned}$$

Furthermore, given a compact subset $K \subset \mathbb{R}^n$, there are $\lambda,q \in \mathbb{N}_+$ and $\beta \in B$ such that if $\phi \in \Phi_k$ for k large enough, then there exist $\eta,c > 0$ such that, for all $x \in K$, $\varepsilon \in (0,\eta)$ and $j \in \{1,\dots,l\}$, we have

$$\forall\ y \in \mathbb{R}^{l}:$$
$$|D_{x_j}F(y)| \leq c(1 + |y|^{q})$$

$$|r_j(\phi_\varepsilon,x)| \leq c/\varepsilon^{\lambda}$$

$$|(r_j - r_j')(\phi_\varepsilon,x)| \leq \varepsilon^{\beta(k)-\lambda}$$

Thus we have

$$\left|F\left[\{r_i(\phi_\varepsilon,x)\}_{i=1}^{l}\right] - \left[\{r_i'(\phi_\varepsilon,x)\}_{i=1}^{l}\right]\right| \leq c'\varepsilon^{\beta(k)-\lambda-q\lambda}$$

where c' depends only on c, which is the required bound for this case.

Next we consider the case of $p = 1 \in \mathbb{N}^n$. Let $(\{r_i\})$ denote $\left[\{r_i(\phi_\varepsilon,x)\}_{i=1}^{l}\right]$. For $h \in \{1,\ldots,n\}$ we have

$$D_{x_h}[f(\{r_i\})] = \sum_{j=1}^{l}(D_{x_j}F)(\{r_i\})\cdot(D_{x_i}r_j)(\phi_\varepsilon,x)$$

so that

$$D_{x_h}[F(\{r_i\})] - D_{x_h}[F(\{r_i'\})]$$
$$= \sum_{j=1}^{l}\left\{\left[(D_{x_j}F)(\{r_i\}) - (D_{x_j}F)(\{r_i'\})\right]\cdot(D_{x_i}r_j)(\phi_\varepsilon,x)\right.$$
$$\left. + (D_{x_h}F)(\{r_i'\})\cdot(D_{x_h}(r_j - r_j'))(\phi_\varepsilon,x)\right\}$$

Corresponding bounds applied to each of the terms similar to $F(\{r_i\}) - F(\{r_i'\})$ above give the required bound. The method can clearly be extended to $D_x^p[F(\{r_i\}) - F(\{r_i'\})]$ for higher order ($p \in \mathbb{N}^n$) derivatives. ∎

Theorem 1: For $l \in \mathbb{N}_+$, if $F \in \mathcal{C}^\infty(\mathbb{R}^l)$ is slowly increasing, and $f_i + \mathcal{I} \in \mathcal{G}(\mathbb{R}^n)$, $1 \leq i \leq l$, then we can define the mapping

$$\begin{aligned} F : \mathcal{G}(\mathbb{R}^n)^l &\longrightarrow \mathcal{G}(\mathbb{R}^n) \\ (f_i + \mathcal{I} \mid 1 \leq i \leq l) &\longmapsto F(f_1, f_2, \ldots, f_l) + \mathcal{I} \end{aligned}$$

Proof: Follows directly from Propositions 1 and 2. ∎

The way in which we extend the partial differential operator $T(x,D)$ to $\mathcal{E}[\mathbb{R}^n]$ is the same as for partial derivative operators, that is

$$\begin{aligned} T(x,D) : \mathcal{E}[\mathbb{R}^n] &\longrightarrow \mathcal{E}[\mathbb{R}^n] \\ f &\longmapsto T(x,D_x)f \end{aligned}$$

Clearly it follows from Theorem 1 that if the $\mathcal{C}^\infty$–smooth F in our PDE is slowly increasing, then $T(x,D)$ can also be extended to $\mathcal{G}(\mathbb{R}^n)$ in the familiar canonical way

$$\begin{aligned} T(D) : \mathcal{G}(\mathbb{R}^n) &\longrightarrow \mathcal{G}(\mathbb{R}^n) \\ f + \mathcal{I} &\longmapsto (T(x,D_x)f) + \mathcal{I} \end{aligned}$$

We thus get the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^n) & \xrightarrow{T(x,D)} & \mathcal{C}^\infty(\mathbb{R}^n) \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}(\mathbb{R}^n) & \xrightarrow{T(D)} & \mathcal{G}(\mathbb{R}^n) \end{array}$$

In this generalized context, $u + \mathcal{I} \in \mathcal{G}(\mathbb{R}^n)$ is a solution to the PDE if and only if we have

$$T(D)u \in \mathcal{I}$$

Generalized Functions on Open Sets $\Omega \subset \mathbb{R}^n$

We summarise the construction of Colombeau's algebra $\mathcal{G}(\Omega)$ if generalized functions in the more general case of an open set $\Omega \subset \mathbb{R}^n$. Define the subset $U(\Omega) \subset \Phi \times \Omega$ by

$$U(\Omega) = \{(\phi,x) \in \Phi \times \Omega \mid \tau_x\phi \in \mathcal{D}(\Omega)\}$$

and the subset $\Omega_\phi \subset \Omega$ by

$$\Omega_\phi = \{x \in \Omega \mid (\phi,x) \in U(\Omega)\}$$

Then we can define

$$\mathcal{E}[\Omega] = \left\{ f : U(\Omega) \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \forall\ \phi \in \Phi : \\ \quad f(\phi,\cdot) \in \mathcal{C}^\infty(\Omega_\phi) \end{array} \right\}$$

The set $U(\Omega)$ has the property that

$$\begin{aligned}
&\forall\ K \subset \Omega \text{ compact}, \ \phi \in \Phi : \\
&\exists\ \eta > 0 : \\
&\forall\ x \in K,\ \varepsilon \in (0,\eta) : \\
&\quad (\phi_{\varepsilon}, x) \subset \Omega
\end{aligned}$$

so that we may simply define the quotient algebra

$$\mathcal{G}(\Omega) = \mathcal{A} \ / \ \mathcal{I}$$

after altering the definitions of the subalgebra $\mathcal{A}$ of $\mathcal{E}[\Omega]$ and the ideal $\mathcal{I}$ in $\mathcal{A}$ by requiring that the compact set $K \subset \Omega$.

APPENDIX 2: Dedekind Order Completion Solution Methods

Introduction

We present here the basic order completion solution method of Oberguggenberger and Rosinger, for arbitrary continuous (not necessarily monotonous) nonlinear partial differential equations

(1) $$T(x,D)u(x) = f(x), \quad x \in \Omega$$

where $\Omega \subset \mathbb{R}^n$ is nonvoid open and bounded, $f \in \mathcal{C}^0(\Omega)$ is given, and $u : \Omega \longrightarrow \mathbb{R}$ is the unknown function, while $T(x,D)$ is an m^{th}–order nonlinear partial differential operator

$$T(x,D) : \mathcal{C}^m(\Omega) \longrightarrow \mathcal{C}^0(\Omega)$$

of the form

(2) $$T(x,D)u(x) = F(x,u(x),\ldots,D_x^p u(x),\ldots) \quad ,x \in \Omega$$

where $p \in \mathbb{N}^n$, $|p| \leq m$, and F is jointly continuous in all of its variables.

Classical and Generalized Solutions

A classical solution of (1) for a given $f \in \mathcal{C}^0(\Omega)$, is a function $u \in \mathcal{C}^m(\Omega)$ satisfying

$$T(x,D)u = f \quad \text{on } \Omega$$

and its existence for all $f \in \mathcal{C}^0(\Omega)$ would require that the operator $T(x,D)$ define a surjective mapping, that is, satisfying

$$T(x,D)\mathcal{C}^m(\Omega) \supset \mathcal{C}^0(\Omega)$$

which, we know, is not in general the case, even when $T(x,D)$ is linear. Thus we turn to generalized solutions: we will embed $\mathcal{C}^m(\Omega)$ in some space X of generalized functions, extending $T(x,D)$ to X, so that the extension $\hat{T}(x,D)$ satisfies

$$\hat{T}(x,D)X \supset \mathcal{C}^0(\Omega)$$

The required spaces of generalized functions will essentially be obtained by means of the Dedekind order completion of spaces which contain order isomorphical copies of $\mathcal{C}^m(\Omega)$.

Approximation of Solutions

In order to see precisely what the spaces of generalized solutions that will be constructed must look like, we first need to state a basic result on the approximation of solutions of nonlinear PDEs (1).

For simplicity, we first consider the particular case when Ω is bounded, and the operator $T(x,D)$ is of the polynomial nonlinear form, with continuous coefficients

(3) $$T(x,D) = D_t^m + \sum_{1\le i\le h} c_i(x) \prod_{1\le j\le k_i} D_t^{p_{ij}} D_y^{q_{ij}}, \quad x = (t,y) \in \Omega$$

where $x = (t,y)$, $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, $c_i \in \mathcal{C}^0(\overline{\Omega})$. $p_{ij} \in \mathbb{N}$, $q_{ij} \in \mathbb{N}^{n-1}$, $p_{ij} < m$ and $p_{ij} + |q_{ij}| \le m$. We can denote, for $l \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, according to Tietze's extension theorem,

$$\mathcal{C}^l(\overline{\Omega}) = \mathcal{C}^l(\mathbb{R}^n)\big|_{\overline{\Omega}}$$

and in general

$$\mathcal{C}^l(\overline{\Omega}) \subsetneq \mathcal{C}^l(\Omega)$$

Although the particular case considered in (1),(3) covers a large class of evolution type equations of interest in applications, the order completion method of solving nonlinear PDEs is not restricted to bounded domains Ω, as the reader will see. Nor is it restricted to equations with $T(x,D)$ of the type in (3). This particular case is presented first, and these initial restrictions are made, simply because Proposition 1, and particularly Lemma 1, are far easier to prove under these conditions. This gives us a clearer insight into the mathematics involved without getting bogged down initially by technicalities.

Note that for all of the following results on the approximation of solutions of continuous polynomial nonlinear PDEs of the form (1),(3), the PDE does not have to be monotonous.

Proposition 1: If $f \in \mathcal{C}^0(\overline{\Omega})$, then

$$\begin{aligned}
&\forall\ \epsilon > 0\ : \\
&\exists\ \Gamma_\epsilon \subset \Omega \text{ closed, nowhere dense, } u_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon) : \\
&\quad f - \epsilon \leq T(x,D)u_\epsilon \leq f \text{ on } \Omega \setminus \Gamma_\epsilon
\end{aligned}$$

where $\epsilon \in \mathcal{C}^0(\Omega)$ denotes the constant function with value $\epsilon \in \mathbb{R}$.

Proof: Since $\Omega \subset \mathbb{R}^n$ is bounded, we can take $a > 0$, such that $\Omega \subset [-a,a]^n$. Assume by continuous extension that $f, c_i \in \mathcal{C}^0([-a,a]^n)$. Then we can take $\epsilon > 0$ and from Lemma 1 below, we get $\delta > 0$. If we now subdivide $[-a,a]$ into intervals $I_1,\ldots,I_\mu$, such that each n–dimensional interval $J = I_{\lambda_1} \times \ldots \times I_{\lambda_n}$ with $\lambda_1,\ldots,\lambda_n \leq \mu$, has a diameter $\leq \delta$, then Lemma 1 gives us a polynomial P_{a_J} such that, for $x \in J^\circ$, we have

$$f(x) - \epsilon \leq T(x,D)P_{a_J}(x) \leq f(x)$$

where $a_J \in J$ is the center of the n–dimensional interval J, and J° denotes the interior of J. Define the set

$$\Gamma_\epsilon = ([-a,a]^n \setminus \bigcup J_\alpha{}^\circ) \cap \Omega$$

where the union ranges over all n–dimensional intervals $J_\alpha = I_{\lambda_1} \times \ldots \times I_{\lambda_n}$ with $\lambda_1,\ldots,\lambda_n \leq \mu$. Γ_ϵ is clearly closed in Ω, and since

$$(\overline{\Gamma_\epsilon})^\circ = \Gamma_\epsilon{}^\circ = (\Omega \setminus \bigcup J_\alpha{}^\circ)^\circ \subset (\bigcap (J_\alpha \setminus J_\alpha{}^\circ))^\circ = \emptyset$$

it is nowhere dense. By defining the function $u_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)$ piecewise as

$$u_\epsilon = P_{a_{J_\alpha}} \quad \text{on } \Omega \cap J_\alpha{}^\circ$$

for every J_α , which we can do since the $J_\alpha{}^\circ$ are pairwise disjoint, and

$$\Omega \setminus \Gamma_\epsilon = \Omega \cap \bigcup J_\alpha{}^\circ$$

we obviously get

$$f - \leq T(x,D)u_\epsilon \leq f + \epsilon \quad \text{on } \Omega \setminus \Gamma_\epsilon$$

■

Notice that, by a similar construction, one can obtain a version of Proposition 1 in which

$$f \leq T(x,D)u_\epsilon \leq f + \epsilon \quad \text{on } \Omega \setminus \Gamma_\epsilon$$

Lemma 1: If $f \in \mathcal{C}^0(\overline{\Omega})$, then

$$\begin{aligned}
&\forall\ \epsilon > 0 \ : \\
&\exists\ \delta > 0 \ : \\
&\forall\ x_0 \in \Omega : \\
&\exists\ P_{x_0} \text{ polynomial in } x \in \mathbb{R}^n : \\
&\forall\ x \in \Omega : \\
&\quad \|x - x_0\| \leq \delta \implies f(x) - \epsilon \leq T(x,D)P_{x_0}(x) \leq f(x)
\end{aligned}$$

Proof: Since $\overline{\Omega}$ is compact, $f \in \mathcal{C}^0(\overline{\Omega})$ is uniformly continuous on $\overline{\Omega}$. Therefore, given $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\forall\ x, x_0 \in \Omega\ ,\ \|x - x_0\| \leq \delta_1 :$$
$$|f(x) - f(x_0)| \leq \epsilon/4$$

Define, for $x_0 = (t_0, y_0) \in \Omega$, the polynomial P_{x_0} in $x = (t,y) \in \mathbb{R}^n$ by

$$P_{x_0} = a(t - t_0)^m$$

where we let $a = (f(x_0) - \epsilon/2)/m!$. Now, for $x = (t,y) \in \mathbb{R}^n$,

$$T(x,D) = D_t^m + \sum_i c_i(x) \prod_j D_t^{p_{ij}} D_y^{q_{ij}}$$

see (3), so that

$$T(x,D)P_{x_0}(x) = am! + \sum_i c_i(x) \prod_j D_t^{p_{ij}}(a(t - t_0)^m)$$

Now $p_{ij} \leq m$, therefore

$$|t - t_0| < 1 \implies |D_t^{p_{ij}}(t - t_0)^m| \leq m!|t - t_0|$$

so that, for $x = (t,y) \in \mathbb{R}^n$, $|t - t_0| < 1$, we have

$$|T(x,D)P_{x_0}(x) - am!| \leq C|t - t_0|$$

where C depends only on m, $\|f\|_\infty$, $\|c_i\|_\infty$, and the structure of the products in T(x,D). Thus there exists a $\delta_2 \in (0,1]$, so that

$$|t - t_0| < \delta_2 \implies |T(x,D)P_{x_0}(x) - am!| \leq C|t - t_0| \leq \epsilon/4$$

Hence, if we let $\delta = \min\{\delta_1,\delta_2\}$, then

$$\|x - x_0\| < \delta \implies |f(x) - f(x_0)| \leq \epsilon/4,\ |T(x,D)P_{x_0}(x) - am!| \leq \epsilon/4$$

In addition, our choice of a gives us

$$am! - f(x_0) = -\epsilon/2.$$

so that, for $x = (t,y) \in \mathbb{R}^n$, $\|x - x_0\| < \delta$, we have

$$\begin{aligned} & T(x,D)P_{x_0}(x) - f(x) = \\ & = (T(x,D)P_{x_0}(x) - am!) + (am! - f(x_0)) + (f(x_0) - f(x)) \\ & \leq \quad \epsilon/4 \quad - \quad \epsilon/2 \quad + \quad \epsilon/4 \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} & |T(x,D)P_{x_0}(x) - f(x)| = \\ & = |T(x,D)P_{x_0}(x) - am!| + |am! - f(x_0)| + |f(x_0) - f(x)| \\ & \leq \quad \epsilon/4 \quad + \quad \epsilon/2 \quad + \quad \epsilon/4 \\ & = \epsilon \end{aligned}$$

so that $f(x) - \epsilon \leq T(x,D)P_{x_0}(x) \leq f(x)$. Because $T(x,D)P_{x_0}(x_0) = am! = f(x_0) - \epsilon/2$, we also have $f(x_0) - \epsilon \leq T(x,D)P_{x_0}(x_0) \leq f(x_0)$. ■

We see that Γ_ϵ is obtained as a rectangular grid generated by a finite ($\leq \mu^n$) number of hyperplanes, so that Γ_ϵ has zero Lebesgue measure. It is a natural and unavoidable lack of global regularity to have the closed, nowhere dense singularity Γ_ϵ in Proposition 1, since even when both T(x,D) and f are analytic, one does not necessarily get classical solutions on the entire Ω. In fact, as can be seen in Rosinger [6], the best general global existence result even for the analytic case still has a closed, nowhere dense singularity set, which can have zero Lebesgue measure. Finally, from the form of T(x,D) in (3) and the proof of Lemma 1, it follows that u_ϵ can be chosen on each interval J_α as an m^{th}–order polynomial only in $t \in \mathbb{R}$, that is, independent of $y \in \mathbb{R}^{n-1}$, where $x = (t,y)$.

Returning to the general case of PDEs (1),(2) defined on an arbitrary, possibly unbounded domain Ω, for $x \in \Omega$, let us denote the range of $F(x,\cdot,\ldots)$ in $\mathbb{R}$ by the set

$$R_x = \left\{ F(x,\xi_0,\ldots,\xi_p,\ldots) \;\middle|\; \xi_p \in \mathbb{R},\ p \in \mathbb{N}^n,\ |p| \le m \right\}$$

Since F is assumed jointly continuous in all of its arguments, R_x must be a connected subset of $\mathbb{R}$. The case where $R_x = \mathbb{R}$ is easier to deal with, and happens in all linear PDEs, as well as for most nonlinear PDEs of interest in applications. It obviously also happens with the particular case of (1),(3) considered above.

For $x \in \Omega$, an obvious necessary condition for the existence of a classical solution of (1),(2) in a neighborhood of x is that $f(x) \in R_x$. Instead, we shall make the blanket assumption that we have satisfied the somewhat stronger condition

$$f(x) \in R_x^{\,\circ} \quad , \ x \in \Omega$$

Both of these conditions obviously hold when $R_x = \mathbb{R}$, so that the results which follow also apply to the particular case of (1),(3).

We present the general version of Lemma 1, which is valid for continuous nonlinear PDEs (1),(2) on possibly unbounded domains Ω:

Lemma 2: If $f \in \mathcal{C}^0(\Omega)$, and for $x \in \Omega$, $f(x) \subset R_x^{\circ}$, then

$$\begin{aligned}
&\forall\ x_0 \in \Omega,\ \epsilon > 0 : \\
&\exists\ \delta > 0,\ P \text{ polynomial in } x \in \mathbb{R}^n : \\
&\forall\ x \in \Omega : \\
&\quad \|x - x_0\| \leq \delta \implies f(x) - \epsilon \leq T(x,D)P(x) \leq f(x)
\end{aligned}$$

Proof: Since $f(x) \subset R_x^{\circ}$, $x \in \Omega$, we have, for $x_0 \in \Omega$, and $\epsilon > 0$ small enough,

$$\begin{aligned}
&\exists\ \xi_p \in \mathbb{R},\ p \in \mathbb{N}^n,\ |p| < m : \\
&\quad F(x_0, \xi_0, \ldots, \xi_p, \ldots) = f(x_0) - \epsilon/2
\end{aligned}$$

Take a polynomial P in $x \in \mathbb{R}^n$, which satisfies the condition

$$\begin{aligned}
&\forall\ p \in \mathbb{N}^n,\ |p| < m : \\
&\quad D_x^p P(x_0) = \xi_p
\end{aligned}$$

then obviously

$$T(x_0,D)P(x_0) - f(x_0) = -\epsilon/2$$

and the lemma follows from the continuity in $x \in \Omega$ of both $T(x,D)P$ and f. ∎

We can now prove the more general version of Proposition 1, which is valid for continuous, not necessarily monotonous, nonlinear partial differential operators of type (2), on possibly unbounded domains Ω.

Proposition 2: If $f \in \mathcal{C}^0(\Omega)$, then

$$
(4) \qquad \begin{array}{l} \forall \ \epsilon > 0 \ : \\ \exists \ \Gamma_\epsilon \subset \Omega \text{ closed, nowhere dense, } u_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon) \ : \\ \quad f - \epsilon \le T(x,D)u_\epsilon \le f \text{ on } \Omega \setminus \Gamma_\epsilon \end{array}
$$

Proof: We can write

$$\Omega = \bigcup_{v \in \mathbb{N}} K_v$$

where, for $v \in \mathbb{N}$, the compact sets K_v are assumed:

(*) to be n–dimensional intervals

$$K_v = [a_v, b_v]$$

with $a_v = (a_{v,1},\ldots,a_{v,n})$, $b_v = (b_{v,1},\ldots,b_{v,n}) \in \mathbb{R}^n$, $a_{v,i} \le b_{v,i}$, $i = 1,\ldots,n$.

(**) to be locally finite, that is

$$
\begin{array}{l} \forall \ x \in \Omega \ : \\ \exists \ V_x \subset \Omega, \ V_x \text{ neighborhood of } x : \\ \quad \{v \in \mathbb{N} \mid K_v \cap V_x \neq \emptyset\} \text{ is finite} \end{array}
$$

(***) to have pairwise disjoint interiors.

Now take any fixed $\epsilon > 0$, and any $v \in \mathbb{N}$. We can apply Lemma 2 to each $x_0 \in K_v$: to obtain $\delta_{x_0} > 0$ and P_{x_0} polynomial in $x \in \mathbb{R}^n$, such that, for $x \in \Omega \cap \overline{B}(x_0, \delta_{x_0})$, we have

$$f(x) - \epsilon \leq T(x,D)P_{x_0}(x) \leq f(x)$$

The compactness of K_v implies that

$$\begin{aligned}
&\exists\ \delta > 0 \ : \\
&\forall\ x_0 \in K_v \ : \\
&\exists\ P_{x_0} \text{ polynomial in } x \in \mathbb{R}^n \ : \\
&\forall\ x \in \Omega \ : \\
&\quad \|x - x_0\| \leq \delta \implies f(x) - \epsilon \leq T(x,D)P_{x_0}(x) \leq f(x)
\end{aligned}$$

so that, within the compact K_v , as in the proof of Proposition 1, we can obtain $\Gamma_{v,\epsilon} \subset K_v$ a rectangular grid generated by a finite number of hyperplanes. We can thus define $u_{v,\epsilon} \in \mathcal{C}^m(K_v \setminus \Gamma_{v,\epsilon})$ such that

$$f - \epsilon \leq T(x,D)u_{v,\epsilon} \leq f \quad \text{on } K_v \setminus \Gamma_{v,\epsilon}$$

But since the K_v , $v \in \mathbb{N}$, are locally finite, with pairwise disjoint interiors, it follows that we can again define $u_\epsilon \in \mathcal{C}^m(\Omega \setminus (\Gamma_\epsilon \cup \Gamma))$ piecewise as

$$u_\epsilon = u_{v,\epsilon} \quad \text{on } \Omega \cap \overset{\circ}{K_v} \setminus \Gamma_{v,\epsilon}$$

where the singularity sets

$$\Gamma_\epsilon = \bigcup_{v \in \mathbb{N}} \Gamma_{\epsilon,v} \subset \Omega$$

$$\Gamma = \Omega \setminus \bigcup_{\nu \in \mathbb{N}} K_\nu^{\circ}$$

are closed, nowhere dense, and

$$f - \leq T(x,D)u_\epsilon \leq f + \epsilon \quad \text{on } \Omega \setminus (\Gamma_\epsilon \cup \Gamma)$$

■

As in the case of Proposition 1, we can obtain a version of (4) in which

(5) $$f \leq T(D)u_\epsilon \leq f + \epsilon \quad \text{on } \Omega \setminus \Gamma_\epsilon$$

Constructing Spaces of Generalized Functions

Bearing in mind the approximation result in Proposition 2, we now consider specifically the spaces of real–valued functions denoted by

$$\mathcal{C}^l_{\mathrm{nd}}(\Omega) = \left\{ u \;\middle|\; \begin{array}{l} \exists\ \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ u \in \mathcal{C}^l(\Omega \setminus \Gamma) \end{array} \right\}$$

for $l \in \overline{\mathbb{N}}$. Obviously we have

$$\mathcal{C}^l(\overline{\Omega}) \subset \mathcal{C}^l(\Omega) \subset \mathcal{C}^l_{\mathrm{nd}}(\Omega)$$

We now proceed to define an equivalence relation $\sim$ on $\mathcal{C}^0_{\mathrm{nd}}(\Omega)$: For $f,g \in \mathcal{C}^0_{\mathrm{nd}}(\Omega)$ define

$$(6) \qquad f \sim g \quad \Leftrightarrow \quad \left[\begin{array}{l} \exists\ \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ \quad f,g \in \mathcal{C}^0(\Omega \setminus \Gamma), \\ \quad f = g \quad \text{on } \Omega \setminus \Gamma \end{array}\right]$$

Now we construct the quotient space

$$\mathcal{M}^0(\Omega) = \mathcal{C}^0_{nd}(\Omega) \ / \sim$$

denoting the $\sim$ equivalence class of $f \in \mathcal{C}^0_{nd}(\Omega)$ by $F \in \mathcal{M}^0(\Omega)$, and giving $\mathcal{M}^0(\Omega)$ the partial order $\leq$ defined, for $F,G \in \mathcal{M}^0(\Omega)$, by

$$(7) \qquad F \leq G \quad \Leftrightarrow \quad \left[\begin{array}{l} \exists\ f \in F,\ g \in G,\ \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ \quad f,g \in \mathcal{C}^0(\Omega \setminus \Gamma), \\ \quad f \leq g \quad \text{on } \Omega \setminus \Gamma \end{array}\right]$$

It is clear that $\mathcal{M}^0(\Omega)$ has no minimum or maximum elements, so we can apply Mac Neille's construction of the basic Dedekind order completion (Appendix 3) of $(\mathcal{M}^0(\Omega),\leq)$ to obtain the order complete space

$$(\hat{\mathcal{M}}^0(\Omega),\leq)$$

with the order isomorphical embedding which preserves infima and suprema:

$$\begin{array}{rcl} \varepsilon_0 : \mathcal{M}^0(\Omega) & \hookrightarrow & \hat{\mathcal{M}}^0(\Omega) \subset \mathcal{P}(\mathcal{M}^0(\Omega)) \\ F & \longmapsto & <F] \end{array}$$

The usual order on $\mathcal{C}^0(\Omega)$ is defined, for $f,g \in \mathcal{C}^0(\Omega)$, by

$$f \leq g \quad \Leftrightarrow \quad \left[\begin{array}{l} \forall \; x \in \Omega \; : \\ \quad f(x) \; \leq g(x) \end{array} \right]$$

and in this context we can state:

Lemma 3: The mapping

$$\begin{array}{ccc} \mathcal{C}^0(\Omega) & \longrightarrow & \mathcal{M}^0(\Omega) \\ f & \longmapsto & F \end{array}$$

where F is the $\sim$ equivalence class of f, is an order isomorphical embedding.

Proof: The definition of $\mathcal{M}^0(\Omega)$ ensures that the mapping is surjective. Let us show that it is injective: Take $f,g \in \mathcal{C}^0(\Omega)$ such that $f \sim g$, then according the definition of $\sim$, $f = g$ on $\Omega \setminus \Gamma$, for some $\Gamma \subset \Omega$ closed, nowhere dense. Hence $\Omega \setminus \Gamma$ is open and dense in Ω, so that the continuity of f,g implies that $f = g$ on Ω.

Take $f,g \in \mathcal{C}^0(\Omega)$ such that $f \leq g$ on Ω, and let F,G be their respective $\sim$ equivalence classes in $\mathcal{M}^0(\Omega)$. Then according the definition of $\leq$ on $\mathcal{M}^0(\Omega)$, $F \leq G$, since we can take $\Gamma = \emptyset$. Conversely, take $F,G \in \mathcal{M}^0(\Omega)$ such that $F \leq G$, and assume that $f(x) > g(x)$ for some $x \in \Omega$. Then the continuity of f,g implies that $f > g$ on some open neighborhood $V \subset \Omega$ of x. But $F \leq G$, so that $f \leq g$ on $\Omega \setminus \Gamma$, for some $\Gamma \subset \Omega$ closed, nowhere dense, which implies that $V \subset \Gamma$, a contradiction. ∎

This gives us the following sequence of order isomorphical embeddings:

(8)
$$\begin{array}{ccccc} \mathcal{C}^0(\overline{\Omega}) \subset \mathcal{C}^0(\Omega) \subset \mathcal{C}^0_{nd}(\Omega) & \xhookrightarrow{\text{can}} & \mathcal{M}^0(\Omega) & \xhookrightarrow{\varepsilon_0} & \hat{\mathcal{M}}^0(\Omega) \subset \mathcal{P}(\mathcal{M}^0(\Omega)) \\ f & \longmapsto & F & \longmapsto & <F] \end{array}$$

where the embedding $\varepsilon_0 : \mathcal{M}^0(\Omega) \hookrightarrow \hat{\mathcal{M}}^0(\Omega)$ preserves infima and suprema, and the canonical mapping is denoted by 'can'.

Extending T(x,D) to Spaces of Generalized Functions

Because of the form of the nonlinear partial differential operator $T(x,D) : \mathcal{C}^m(\Omega) \longrightarrow \mathcal{C}^0(\Omega)$ in (3), and because $\mathcal{C}^l(\Omega) \subset \mathcal{C}^l_{nd}(\Omega)$ for $l \in \overline{\mathbb{N}}$, we can clearly extend T(x,D) as follows:

$$\begin{array}{ccc} \mathcal{C}^m(\Omega) & \xrightarrow{T(x,D)} & \mathcal{C}^0(\Omega) \\ \Big\downarrow \mathrm{id} & & \mathrm{id} \Big\downarrow \\ \mathcal{C}^m_{nd}(\Omega) & \xrightarrow{\mathbf{T}(x,D)} & \mathcal{C}^0_{nd}(\Omega) \end{array}$$

Since, $u \in \mathcal{C}^m(\Omega \setminus \Gamma)$ implies that $\mathbf{T}(x,D)u \in \mathcal{C}^0(\Omega \setminus \Gamma)$, it follows that the closed, nowhere dense singularity $\Gamma \subset \Omega$ of $u \in \mathcal{C}^m_{nd}(\Omega)$ need not be changed by the application of $\mathbf{T}(x,D)$.

Similarly to the definition of our space $\mathcal{M}^0(\Omega) \hookleftarrow \mathcal{C}^0(\Omega)$ of generalized functions, related to $\mathcal{C}^0_{nd}(\Omega)$, we can now define a space of generalized functions, related to $\mathcal{C}^m_{nd}(\Omega)$ and $\mathbf{T}(x,D)$. Define an equivalence relation $\sim_T$ on $\mathcal{C}^m_{nd}(\Omega)$: For $u,v \in \mathcal{C}^m_{nd}(\Omega)$ define

$$u \sim_T v \quad \Leftrightarrow \quad \mathbf{T}(x,D)u \sim \mathbf{T}(x,D)v$$

That is, by (6), and the fact that the singularity $\Gamma \subset \Omega$ stays the same,

$$u \sim_T v \quad \Leftrightarrow \quad \left[\begin{array}{l} \exists\ \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ \quad u,v \in \mathcal{C}^0(\Omega \setminus \Gamma), \\ \quad \hat{T}(x,D)u = \hat{T}(x,D)v \quad \text{on } \Omega \setminus \Gamma \end{array} \right]$$

Now we construct the quotient space

$$\mathcal{M}_T^m(\Omega) = \mathcal{C}_{nd}^m(\Omega) \ / \sim_T$$

denoting the $\sim_T$ equivalence class of $u \in \mathcal{C}_{nd}^m(\Omega)$ by $U \in \mathcal{M}_T^m(\Omega)$.

We can now extend $\hat{T}(x,D)$ further, as shown by the commutative diagram

(9)
$$\begin{array}{ccc} \mathcal{C}^m(\Omega) & \xrightarrow{T(x,D)} & \mathcal{C}^0(\Omega) \\ \downarrow \text{id} & & \text{id} \downarrow \\ \mathcal{C}_{nd}^m(\Omega) & \xrightarrow{\hat{T}(x,D)} & \mathcal{C}_{nd}^0(\Omega) \\ \downarrow \text{can} & & \text{can} \downarrow \\ \mathcal{M}_T^m(\Omega) & \xhookrightarrow{T} & \mathcal{M}^0(\Omega) \\ U & \longmapsto & F \end{array}$$

where F is the unique $\sim$ equivalence class in $\mathcal{M}^0(\Omega)$ of any $f = T(x,D)u$ where u belongs to the $\sim_T$ equivalence class U in $\mathcal{M}_T^m(\Omega)$. Note that T is injective, since $TU = TV$ in $\mathcal{M}^0(\Omega)$, iff $T(x,D)u \sim T(x,D)v$ in $\mathcal{C}^0(\Omega)$, iff $u \sim_T v$ in $\mathcal{C}_{nd}^m(\Omega)$, iff $U = V$ in $\mathcal{M}_T^m(\Omega)$.

If we give $\mathcal{M}_T^m(\Omega)$ the partial order $\leq_T$ defined, for $U,V \in \mathcal{M}_T^m(\Omega)$, by

$$U \leq_T V \quad \Leftrightarrow \quad TU \leq TV$$

that is, by (7), and the fact that the singularity $\Gamma \subset \Omega$ stays the same,

$$(10) \qquad U \leq_T V \quad \Leftrightarrow \quad \left[\begin{array}{l} \exists\, f \in TU,\ g \in TV,\ \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ f,g \in \mathcal{C}^0(\Omega \setminus \Gamma), \\ f \leq g \quad \text{on } \Omega \setminus \Gamma \end{array} \right]$$

then the mapping $T : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ is clearly an order isomorphical embedding. Thus, even though $T(x,D)$ need not be monotonous, the extension T always is.

However, the mapping $T : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ is still not surjective, so we will extend it to the Dedekind order completions of both its domain and its range. Since we have already obtained the order completion $\hat{\mathcal{M}}^0(\Omega)$ of $\mathcal{M}^0(\Omega)$, we can now do the same for $\mathcal{M}^m_T(\Omega)$. It is not immediately clear that $\mathcal{M}^m_T(\Omega)$ has no minimum or maximum elements, as required by Mac Neille's theorem (Appendix 3), so this has to be confirmed first.

Lemma 4: The partial ordered set $(\mathcal{M}^m_T(\Omega),\leq_T)$ has no minimum or maximum element.

Proof: We will show that $\mathcal{M}^m_T(\Omega)$ contains no minimum; the proof that $\mathcal{M}^m_T(\Omega)$ contains no maximum is similar. Given any $U \in \mathcal{M}^m_T(\Omega)$, we shall construct a $W \in \mathcal{M}^m_T(\Omega)$, $W \neq U$, such that $W \leq_T U$: Take $u \in U$, such that $u \in \mathcal{C}^m(\Omega \setminus \Gamma)$, for some $\Gamma \subset \Omega$ closed, nowhere dense. Since the singularity $\Gamma \subset \Omega$ of $u \in \mathcal{C}^m_{nd}(\Omega)$ need not be changed by the application of $\hat{T}(x,D)$, we have

$$f = \hat{T}(x,D)u \in \mathcal{C}^0(\Omega \setminus \Gamma)$$

Let us take a nonvoid, open subset $\Delta \subset \Omega \setminus \Gamma$, such that $\overline{\Delta} \subset \Omega \setminus \Gamma$, then the fact that Ω is bounded implies that $\overline{\Delta}$ is compact, so that we have

$$-\infty < a = \inf_{\mathbb{R}} \{f(x) \mid x \in \overline{\Delta}\}$$

By Lemma 2 (with $\Omega = \Delta$, $f = a - 1$), for any given $x_0 \in \Delta$, there exists an open neighborhood $V \subset \Delta$ of x_0, and a polynomial P_{x_0} in $x \in \mathbb{R}^n$, such that

$$T(x,D)P_{x_0}(x) = a - 1 \quad \forall\, x \in V$$

Finally, if we define $w \in \mathcal{C}^m_{nd}(\Omega)$ by

$$w(x) = \begin{cases} P_{x_0}(x) & \Leftarrow x \in V \quad (\subset \Delta \subset \Omega \setminus \Gamma) \\ u(x) & \Leftarrow x \in (\Omega \setminus \Gamma) \setminus \overline{V} \end{cases}$$

then clearly $w \in \mathcal{C}^m(\Omega \setminus (\Gamma \cup \partial V))$, where we denote $\partial V = \overline{V} \setminus V$, and in addition, we have $\hat{T}(x,D)w = a - 1 < a \leq f = \hat{T}(x,D)u$ on V, and $\hat{T}(x,D)w = \hat{T}(x,D)u$ on $\Omega \setminus (\Gamma \cup \overline{V})$, so that, for the $\sim_T$ equivalence class $W \in \mathcal{M}^m_T(\Omega)$ of w, we have $W \neq U$, $W \leq_T U$. ∎

Since $\mathcal{M}^m_T(\Omega)$ has no minimum or maximum elements, we can apply Mac Neille's construction of the basic Dedekind order completion of $(\mathcal{M}^m_T(\Omega), \leq_T)$ to obtain the order complete space

$$(\hat{\mathcal{M}}^m_T(\Omega), \leq_T)$$

with the order isomorphical embedding which preserves infima and suprema:

$$\begin{aligned} \varepsilon_m : \mathcal{M}^m_T(\Omega) &\hookrightarrow \hat{\mathcal{M}}^m_T(\Omega) \subset \mathcal{P}(\mathcal{M}^m_T(\Omega)) \\ U &\longmapsto\ < U] \end{aligned}$$

Because Mac Neille's construction gives us $\hat{\mathcal{M}}^m_T(\Omega) \subset \mathcal{P}(\mathcal{M}^m_T(\Omega))$, we define an extension of $T : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ by restricting the mapping (Appendix 3)

$$\begin{aligned} \mathcal{P}(\mathcal{M}^{\mathrm{m}}_{\mathrm{T}}(\Omega)) &\longrightarrow \hat{\mathcal{M}}^0(\Omega) \\ \mathrm{A} &\longmapsto <\mathrm{TA}] \end{aligned}$$

to $\hat{\mathcal{M}}^{\mathrm{m}}_{\mathrm{T}}(\Omega) \subset \mathcal{P}(\mathcal{M}^{\mathrm{m}}_{\mathrm{T}}(\Omega))$, thereby obtaining the commutative diagram

$$\begin{tikzcd}
\mathrm{U} \arrow[rrr, mapsto] \arrow[ddd] & & & \mathrm{T}(\mathrm{U}) \arrow[ddd] \\
& \mathcal{M}^{\mathrm{m}}_{\mathrm{T}}(\Omega) \arrow[r, hook, "\mathrm{T}"'] \arrow[d, hook, "\varepsilon_{\mathrm{m}}"] & \mathcal{M}^0(\Omega) \arrow[d, hook', "\varepsilon_0"'] & \\
& \hat{\mathcal{M}}^{\mathrm{m}}_{\mathrm{T}}(\Omega) \arrow[r, hook, "\hat{\mathrm{T}}"] & \hat{\mathcal{M}}^0(\Omega) & \\
<\mathrm{U}] \arrow[rrr, mapsto] & & & <\mathrm{T}(\mathrm{U})]
\end{tikzcd}$$

where, according to the Proposition in Appendix 3, the extension $\hat{\mathrm{T}} : \hat{\mathcal{M}}^{\mathrm{m}}_{\mathrm{T}}(\Omega) \longrightarrow \hat{\mathcal{M}}^0(\Omega)$ is an order isomorphical embedding, because the mapping $\mathrm{T} : \mathcal{M}^{\mathrm{m}}_{\mathrm{T}}(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ is an order isomorphical embedding.

We can now state the basic result on the existence and uniqueness of generalized solutions of continuous nonlinear PDEs of the form (1),(2) obtained by means of the Dedekind order completion of spaces of smooth functions:

Theorem 1: We have an order isomorphical embedding

$$\begin{aligned} \mathrm{C}^0(\overline{\Omega}) &\hookrightarrow \hat{\mathrm{T}}(\hat{\mathcal{M}}^{\mathrm{m}}_{\mathrm{T}}(\Omega)) \\ \mathrm{f} &\longmapsto <\mathrm{F}] \end{aligned}$$

That is, for every $\mathrm{f} \in \mathrm{C}^0(\overline{\Omega})$, there exists a unique $U \in \hat{\mathcal{M}}^{\mathrm{m}}_{\mathrm{T}}(\Omega)$, such that

(11) $$\hat{\mathrm{T}}U = <\mathrm{F}] \quad \text{in } \hat{\mathcal{M}}^0(\Omega)$$

where $F \in \mathcal{M}^0(\Omega)$ is the $\sim$ equivalence class of f, which means that U is the unique generalized solution of the continuous nonlinear PDE in (1),(2).

Before proving the theorem, we first summarize the construction and the result in the following commutative diagram:

(12)
$$\begin{tikzcd}
\mathcal{C}^m(\overline{\Omega}) \subset \mathcal{C}^m(\Omega) \arrow[rr, "T(x,D)"] \arrow[d, phantom, "\cap"] & & \mathcal{C}^0(\Omega) \supset \mathcal{C}^0(\overline{\Omega}) \arrow[d, phantom, "\cap"] \\
\mathcal{C}^m_{nd}(\Omega) \arrow[rr, "T(x,D)"] \arrow[d, "can"] & & \mathcal{C}^0_{nd}(\Omega) \arrow[d, "can"] \\
\mathcal{C}^m_{nd}(\Omega)/\sim_T \; = \mathcal{M}^m_T(\Omega) \arrow[r, leftrightarrow, "T"] \arrow[d, "\varepsilon_m"] & T(\mathcal{M}^m_T(\Omega)) \arrow[r, phantom, "\subset"] & \mathcal{M}^0(\Omega) = \mathcal{C}^0_{nd}(\Omega)/\sim \arrow[d, "\varepsilon_0"] \\
\hat{\mathcal{M}}^m_T(\Omega) \arrow[r, leftrightarrow, "\hat{T}"] & \hat{T}(\hat{\mathcal{M}}^m_T(\Omega)) \arrow[r, phantom, "\subset"] & \hat{\mathcal{M}}^0_T(\Omega) \\
\exists ! \; U \arrow[r, mapsto] & <F] & \\
 & f \arrow[u] & \mathcal{C}^0(\overline{\Omega}) \subset \mathcal{C}^0(\Omega) \arrow[ul, hook]
\end{tikzcd}$$

Proof: Since $\hat{T} : \hat{\mathcal{M}}^m_T(\Omega) \longrightarrow \hat{\mathcal{M}}^0(\Omega)$ is injective, U is clearly unique.

Let $\epsilon > 0$. By the comment (5) concerning Proposition 2, there exists a closed, nowhere dense $\Gamma_\epsilon \subset \Omega$, and $u_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)$, such that $f \leq T(x,D)u_\epsilon \leq f + \epsilon$ on $\Omega \setminus \Gamma_\epsilon$. Let $U_\epsilon \in \mathcal{M}^m_T(\Omega)$ be the $\sim_T$ equivalence class of u_ϵ, and $F, F - \epsilon, F + \epsilon \in \mathcal{M}^0(\Omega)$ be the $\sim$ equivalence classes of $f, f - \epsilon, f + \epsilon$ respectively, then $F \leq TU_\epsilon \leq F + \epsilon$ in $\mathcal{M}^0(\Omega)$, since by Lemma 3 and (10), we have the order isomorphical embeddings $\mathcal{C}^0(\Omega) \hookrightarrow \mathcal{M}^0(\Omega)$, $T : \mathcal{M}^m_T(\Omega) \hookrightarrow \mathcal{M}^0(\Omega)$.

Let $A = \{U_\epsilon \in \mathcal{M}^m_T(\Omega) \mid \epsilon > 0\}$ and let

$$U = A^l = \left\{ U \in \mathcal{M}^m_T(\Omega) \;\middle|\; \begin{array}{l} \forall\ \epsilon > 0 : \\ U \leq_T U_\epsilon \end{array} \right\}$$

Since $A^l = A^{lul}$ (Appendix 3), we have, by Mac Neille's construction, that $\mathcal{U} \in \hat{\mathcal{M}}_T^m(\Omega)$. From Proposition 2, there exists a $V_\epsilon \in \mathcal{M}_T^m(\Omega)$, such that $F - \epsilon \leq TV_\epsilon \leq F$ in $\mathcal{M}^0(\Omega)$, similarly to the argument above. But then $TV_\epsilon \leq F \leq TU_\epsilon$ in $\mathcal{M}^0(\Omega)$, so that $V_\epsilon \leq_T U_\epsilon$ in $\mathcal{M}_T^m(\Omega)$. It follows that $\mathcal{U} \neq \emptyset$.

It only remains to prove that $\hat{T}\mathcal{U} = \langle F]$ in $\hat{\mathcal{M}}^0(\Omega)$. For this, we will need the following property of $\mathcal{U}$:

$$\mathcal{U} = \{U \in \mathcal{M}_T^m(\Omega) \mid TU \leq F \text{ in } \mathcal{M}^0(\Omega)\} \tag{13}$$

We prove the inclusion $\supset$: Take $U \in \mathcal{M}_T^m(\Omega)$ such that $TU \leq F$ in $\mathcal{M}^0(\Omega)$, then there exists $u \in U$ with $T(x,D)u \leq f$ on $\Omega \setminus \Gamma$, for some $\Gamma \subset \Omega$ closed, nowhere dense. But then we have $T(x,D)u \leq T(x,D)u_\epsilon$ on $\Omega \setminus (\Gamma \cup \Gamma_\epsilon)$, so that $U \leq_T U_\epsilon$ in $\mathcal{M}_T^m(\Omega)$ for all $\epsilon > 0$, by the argument above concerning (5). Hence $U \in A^l = \mathcal{U}$.

Conversely, we prove $\subset$: Take $u \in U$ for some $U \in \mathcal{U}$, and assume that $TU \nleq F$ in $\mathcal{M}^0(\Omega)$. Then for every closed, nowhere dense $\Gamma \subset \Omega$, there exists $x \in \Omega \setminus \Gamma$, such that $T(x,D)u(x) > f(x)$. But $T(x,D),u,f$ are continuous, so that $T(x,D)u > f + \delta$ on some open neighborhood $\Delta \subset \Omega \setminus \Gamma$ of x, for some $\delta > 0$. But $U \in \mathcal{U} = A^l$, so that, for every $\epsilon > 0$ there exists a closed, nowhere dense $\Gamma_\epsilon \subset \Omega$ such that $T(x,D)u \leq T(x,D)u_\epsilon \leq f + \epsilon$ on $\Omega \setminus \Gamma_\epsilon$, where the second inequality follows from the argument above concerning (5). This is a contradiction, since $\Omega \setminus (\Gamma \cup \Gamma_\epsilon) \neq \emptyset$.

We proceed to prove that $\hat{T}\mathcal{U} = \langle F]$ in $\hat{\mathcal{M}}^0(\Omega)$: Note that from (13) it follows that $\hat{T}\mathcal{U} \subset \langle F]$, so that, from the definition of $\hat{T}$ and Appendix 3, $\hat{T}\mathcal{U} = (T\mathcal{U})^{ul} \subset \langle F]^{ul} = \langle F]$. Thus it only remains to prove that $\langle F] \subset \hat{T}\mathcal{U}$, that is, $\langle F] \leq \hat{T}\mathcal{U}$ in $\hat{\mathcal{M}}^0(\Omega)$.

Take $g \in C^0(\overline{\Omega})$ such that $g < f$ on $\overline{\Omega}$, so that the continuity of f,g implies that $g \leq f - \epsilon$ on $\overline{\Omega}$, for some $\epsilon > 0$. Then, by Proposition 2, we can find $v \in \mathcal{C}^m(\Omega \setminus \Gamma)$ for some $\Gamma \subset \Omega$ closed, nowhere dense, such that $g \leq f - \epsilon \leq T(x,D)v \leq f$ on $\Omega \setminus \Gamma$. Thus, if $V \in \mathcal{M}^m_T(\Omega)$ is the $\sim_T$ equivalence class of v, and $G \in \mathcal{M}^0(\Omega)$ is the $\sim$ equivalence class of g, then $G \leq TV \leq F$ in $\mathcal{M}^0(\Omega)$, since $T : \mathcal{M}^m_T(\Omega) \hookrightarrow \mathcal{M}^0(\Omega)$ is an order isomorphical embedding. Thus (13) implies that $V \in \mathcal{U}$.

Now, by Lemma 5 below, we have

$$F = \sup_{\mathcal{M}^0(\Omega)} \{g \in \mathcal{C}^0(\overline{\Omega}) \mid g < f \text{ on } \overline{\Omega}\}$$

and by (8), the order isomorphical embedding

$$\begin{array}{ccc} \mathcal{M}^0(\Omega) & \xhookrightarrow{\varepsilon_0} & \hat{\mathcal{M}}^0(\Omega) \\ F & \longmapsto & <F] \end{array}$$

preserves infima and suprema, so that

$$\begin{aligned} <F] &= \sup_{\hat{\mathcal{M}}^0(\Omega)} \left\{ <G] \;\middle|\; \begin{array}{l} \exists\, g \in G : \\ g < f \text{ on } \overline{\Omega} \end{array} \right\} \\ &\leq \sup_{\hat{\mathcal{M}}^0(\Omega)} \left\{ <TV] \;\middle|\; \begin{array}{l} \exists\, g \in G \leq TV \text{ in } \mathcal{M}^0(\Omega) : \\ g < f \text{ on } \overline{\Omega} \end{array} \right\} \\ &\leq \sup_{\hat{\mathcal{M}}^0(\Omega)} \{<TU] \mid U \in \mathcal{U}\} \end{aligned}$$

where the last inequality follows from (13). But from the proof of Mac Neille's theorem we know that

$$\sup_{\hat{M}^0(\Omega)} \{<TU] \mid U \in \mathcal{U}\} = \left[\bigcup_{U \in \mathcal{U}} <TU]\right]^{ul} .$$

Furthermore, in any partially ordered set X, and for any family $\{x_i \mid i \in I\} \subset X$, we have

$$\left[\bigcup_{i \in I} <x_i]\right]^{u} = \left\{ x \in X \;\middle|\; \begin{array}{l} \forall\ a \in \bigcup_{i \in I} <x_i] \ : \\ a \leq x \end{array} \right\}$$

$$= \left\{ x \in X \;\middle|\; \begin{array}{l} \forall\ i \in I \ : \\ x_i \leq x \end{array} \right\} = \{x_i \mid i \in I\}^{u}$$

Therefore $<F] \leq (\bigcup_{U \in \mathcal{U}} <TU])^{ul} = (T\,\mathcal{U})^{ul} = \hat{T}\mathcal{U}$ in $\hat{M}^0(\Omega)$, and we are done. ∎

Lemma 5: For every $f \in \mathcal{C}^0(\overline{\Omega})$ we have

$$F = \sup_{M^0(\Omega)} \left\{ g \in \mathcal{C}^0(\overline{\Omega}) \;\middle|\; \begin{array}{l} \forall\ x \in \overline{\Omega} \ : \\ g(x) \ < \ f(x) \end{array} \right\}$$

where $F \in M^0(\Omega)$ is the $\sim$ equivalence class of f.

Proof: The inequality $\geq$ is obvious. We prove $\leq$: Assume the contrary, then there exists an $H \in M^0(\Omega)$ such that

$$\begin{array}{l} \forall\ g \in \mathcal{C}^0(\overline{\Omega}) \ : \\ \quad \left[\begin{array}{l} \forall\ x \in \overline{\Omega} \ : \\ g(x) \ < f(x) \end{array} \right] \Longrightarrow g \leq H \ \text{ in } M^0(\Omega) \end{array}$$

That is,

$$H \in \left\{ g \in \mathcal{C}^0(\overline{\Omega}) \;\middle|\; \begin{array}{l} \forall\ x \in \overline{\Omega} \ : \\ \quad g(x) \ < \ f(x) \end{array} \right\}^u .$$

Note however, that $F \nleq H$ in $\mathcal{M}^0(\Omega)$, so that for any $h \in H$ such that $h \in \mathcal{C}^0(\Omega \setminus \Gamma)$ for some $\Gamma \subset \Omega$ closed, nowhere dense, we have

$$\begin{array}{l} \exists\ x_0 \in \Omega \ \setminus \Gamma \ : \\ \quad h(x_0) \ < \ f(x_0) \end{array}$$

Thus, since h,f are continuous on $\Omega \setminus \Gamma$, there exists an open neighborhood $\Delta \subset \Omega \setminus \Gamma$ of x_0, and a $\delta > 0$, such that $h \leq f - \delta$ on Δ. Thus, if we take $g = f - \delta/2$, then $h < g$ on Δ. But at the same time $g \in \mathcal{C}^0(\overline{\Omega})$ and $g < f$ on $\overline{\Omega}$, so that

$$g \in \left\{ g' \in \mathcal{C}^0(\overline{\Omega}) \;\middle|\; \begin{array}{l} \forall\ x \in \overline{\Omega} \ : \\ \quad g'(x) \ < \ f(x) \end{array} \right\}$$

which implies that $g \leq h$ on $\Omega \setminus \Gamma'$ for some $\Gamma' \subset \Omega$ closed, nowhere dense. This is a contradiction, since $\Omega \setminus (\Gamma \cup \Gamma') \neq \emptyset$. ∎

Some Remarks

From the proof of Theorem 1 it is apparent that the essential property of the unique generalized solution $\mathcal{U} \in \hat{\mathcal{M}}^m_T(\Omega)$ in (11) is that it satisfies (13), namely

$$\mathcal{U} = \{U \in \mathcal{M}^m_T(\Omega) \mid TU \leq F \ \text{ in } \mathcal{M}^0(\Omega)\}$$

where $F \in \mathcal{M}^0(\Omega)$ is the unique $\sim$ equivalence class of $f \in \mathcal{C}^0(\overline{\Omega})$. Thus $\mathcal{U}$ can be viewed as the totality of classes of subsolutions of the continuous nonlinear PDE in (1),(2) given by Lemma 2. Obviously, the abundance of these local, classical subsolutions, given by Lemma 2, is the crucial property of the PDE in (1),(2) which leads to the existence result in Theorem 1. After constructing the polynomials which provide sufficiently many local, classical subsolutions, it only remains to make a simple and natural construction related to partial orders: the Dedekind order completion of spaces whose elements are generalized functions patched up from locally smooth functions.

From the commutative diagram (12) we obtain the following coherence property between classical and generalized solutions: Whenever classical solutions $u \in \mathcal{C}^m(\Omega)$ exist for arbitrary continuous nonlinear PDEs in (1),(2) then, for their $\sim_T$ equivalence classes $U \in \mathcal{M}_T^m(\Omega)$, the corresponding $<U] \in \hat{\mathcal{M}}_T^m(\Omega)$ will always be generalized solutions in the sense of Theorem 1. In fact, the following more general coherence between classical and generalized solutions can be obtained in a similar way: Given a classical $u \in \mathcal{C}^m(\Omega \setminus \Gamma)$ of any given continuous nonlinear PDE in (1),(2) for some $\Gamma \subset \Omega$ closed, nowhere dense, then the corresponding $<U] \in \hat{\mathcal{M}}_T^m(\Omega)$ will be a generalized solution in the sense of Theorem 1, on the whole of Ω.

We give a simple, ring theoretic characterization of the closed, nowhere dense subsets $\Gamma \subset \Omega$ which are so fundamental in the definition of the spaces $\mathcal{C}_{nd}^{l}(\Omega)$, for $l \in \overline{\mathbb{N}}$, the spaces $\mathcal{M}^0(\Omega)$, $\hat{\mathcal{M}}^0(\Omega)$, and subsequent spaces $\mathcal{M}_T^m(\Omega)$, $\hat{\mathcal{M}}_T^m(\Omega)$ of generalized functions. Recall (Gillman and Jerison 1960) that the zero–set of a function $f \in \mathcal{C}^0(\Omega)$ is the set $Z(f) = \{x \in \Omega \mid f(x) = 0\}$, and that the family $Z(\Omega)$ of zero–sets in Ω are a base for the closed sets in Ω iff Ω is completely regular, which is the case for $\Omega \subset \mathbb{R}^n$. Now we have the following characterization:

$$\left[\begin{array}{l} Z(f) \text{ is} \\ \text{nowhere dense} \\ \text{in } \Omega \end{array}\right] \Leftrightarrow \left[\begin{array}{l} f \text{ is not a} \\ \text{zero divisor} \\ \text{in } \mathcal{C}^0(\Omega) \end{array}\right]$$

Indeed, for '$\Longrightarrow$', take $g \in \mathcal{C}^0(\Omega)$ such that $fg = 0$ on Ω , then clearly $Z(g) \supset \Omega \setminus Z(f)$, so that $Z(g)$ is dense in Ω . Thus the continuity of g implies that $g = 0$ on Ω . Conversely, for '$\Longleftarrow$', assume that $Z(f)$ has some nonvoid open subset Δ, then there exists some compactly supported $g \in \mathcal{C}^\infty(\Omega)$, such that $Z(g) \supset \Omega \setminus \Delta$, so that $fg = 0$ on Ω , since $Z(fg) = Z(f) \cup Z(g) \supset \Delta \cup \Omega \setminus \Delta = \Omega$.

Now as it turns out (Narasimhan, 1973) for $\Omega \subset \mathbb{R}^n$, not only is the family of zero–sets a base for the closed sets, but in fact for a given closed set $\Gamma \subset \Omega$, there exists $h \in \mathcal{C}^\infty(\Omega)$, such that $\Gamma = Z(h)$. Thus, for any $l \in \overline{\mathbb{N}}$, the closed nowhere dense subsets $\Gamma \subset \Omega$ are precisely those sets which are the zero sets of functions $f \in \mathcal{C}^l(\Omega)$, which are not zero divisors in $\mathcal{C}^l(\Omega)$.

Partial Orders Compatible with a Nonlinear Partial Differential Operator

In the commutative diagram

$$(14) \qquad \begin{array}{ccc} \mathcal{C}^m_{nd}(\Omega) & \xrightarrow{\hat{T}(x,D)} & \mathcal{C}^0_{nd}(\Omega) \\ \downarrow & & \downarrow \\ \mathcal{M}^m_T(\Omega) & \xrightarrow{T} & \mathcal{M}^0(\Omega) \\ \downarrow & & \downarrow \\ \hat{\mathcal{M}}^m_T(\Omega) & \xrightarrow{\hat{T}} & \hat{\mathcal{M}}^0(\Omega) \end{array}$$

the poset $(\mathcal{M}^m_T(\Omega), \le_T)$ is the pull–back through the partial differential operator $T(x,D)$ of a natural poset $(\mathcal{M}^0(\Omega), \le)$ which is independent of $T(x,D)$. In the resulting order isomorphical embedding

$$T : (\mathcal{M}^m_T(\Omega), \leq_T) \longrightarrow (\mathcal{M}^0(\Omega), \leq)$$

both the set $\mathcal{M}^m_T(\Omega)$ and the partial order $\leq_T$ on it, will therefore depend on T(x,D), and clearly $\leq_T$ is the largest (Appendix 3) partial order on $\mathcal{M}^m_T(\Omega)$ for which this mapping will be an order isomorphism.

Let us suppose that we have some other equivalence relation $\sim_{T,*}$ defined on $\mathcal{C}^m_{nd}(\Omega)$ so that

$$\forall\ u,v \in \mathcal{C}^m_{nd}(\Omega) :$$
$$u \sim_{T,*} v \implies u \sim_T v$$

This means that $\sim_{T,*}$ is smaller than $\sim_T$. If we define

$$\mathcal{M}^m_{T,*}(\Omega) = \mathcal{C}^m_{nd}(\Omega) \ / \sim_{T,*}$$

then we obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{M}^m_{T,*}(\Omega) \arrow[rr, "T_*"] \arrow[dr, "\sigma"'] & & \mathcal{M}^0(\Omega) \\
& \mathcal{M}^m_T(\Omega) \arrow[ur, "T"'] &
\end{tikzcd}$$

where σ is the canonical surjective mapping between the two quotient spaces, and T_* is the canonical quotient mapping

$$\begin{array}{ccc} \mathcal{M}^{m}_{T,*}(\Omega) & \xrightarrow{T_*} & \mathcal{M}^{0}(\Omega) \\ U & \longmapsto & T_*U \end{array}$$

where T_*U is the ~ equivalence class of any T(x,D)u, with $u \in U$.

The existence of σ is guaranteed by the fact that $\sim_{T,*}$ is smaller than $\sim_T$, so that we have

$$\begin{array}{ccc} \mathcal{M}^{m}_{T,*}(\Omega) & \xrightarrow{\sigma} & \mathcal{M}^{m}_{T}(\Omega) \\ U & \longmapsto & U' \end{array}$$

where U' is the $\sim_T$ equivalence class of any $u \in U$.

We can now define the following partial order $\leq_{T,*}$ on $\mathcal{M}^{m}_{T,*}(\Omega)$: Given $U,V \in \mathcal{M}^{m}_{T,*}(\Omega)$,

$$U \leq_{T,*} V \Leftrightarrow \left[\begin{array}{l} *)\ U = V \\ \text{or} \\ **)\ T_*U \lneqq T_*V \quad \text{in } \mathcal{M}^{0}(\Omega) \end{array} \right]$$

so that we get the commutative diagram of increasing mappings

(15)
$$\begin{tikzcd} (\mathcal{M}^{m}_{T,*}(\Omega), \leq_{T,*}) \arrow[rr, "T_*"] \arrow[dr, dashed, "\sigma"'] & & (\mathcal{M}^{0}(\Omega), \leq) \\ & (\mathcal{M}^{m}_{T}(\Omega), \leq_T) \arrow[ur, "T"'] & \end{tikzcd}$$

with T an order isomorphical embedding. It is also easy to see that $(\mathcal{M}^{m}_{T,*}(\Omega), \leq_{T,*})$ has no minimum or maximum.

Obviously, in the case where $\sim_{T,*}$ and $\sim_{T}$ are identical, we have $(\mathcal{M}^m_{T,*}(\Omega), \leq_{T,*}) = (\mathcal{M}^m_{T}(\Omega), \leq_{T})$, $T_* = T$, and $\sigma = \mathrm{id} : \mathcal{M}^m_T(\Omega) \longrightarrow \mathcal{M}^m_T(\Omega)$. Furthermore, in general the partial order $\leq_{T,*}$ on $\mathcal{M}^m_{T,*}(\Omega)$ is not the pull–back through T_* of the partial order $\leq$ on $\mathcal{M}^0(\Omega)$. Indeed, let $\dashv$ denote that pull–back order, then for $U,V \in \mathcal{M}^m_{T,*}(\Omega)$ we have

$$U \dashv V \Leftrightarrow T_*U \leq T_*V$$

But T_* is not necessarily injective, so that $\dashv$ may fail to be antisymmetric, and will thus in general not be a partial order. Also, since clearly

$$\begin{aligned} &\forall\ U,V \in \mathcal{M}^m_{T,*}(\Omega)\ : \\ &\quad U \leq_{T,*} V \Rightarrow U \dashv V \end{aligned}$$

it follows that in general $\leq_{T,*}$ is a smaller binary relation than $\dashv$.

In view of the Proposition in Appendix 3, the diagram (14) is actually a particular case of the following more general class of commutative diagrams

$$\begin{tikzcd} \mathcal{C}^m_{nd}(\Omega) \arrow[r, "T(x,D)"] \arrow[d, hook] & \mathcal{C}^0_{nd}(\Omega) \arrow[d, hook] \\ \mathcal{M}^m_{T,*}(\Omega) \arrow[r, hook, "T_*"] \arrow[d, hook] & \mathcal{M}^0(\Omega) \arrow[d, hook] \\ \hat{\mathcal{M}}^m_{T,*}(\Omega) \arrow[r, hook, "\hat{T}_*"] & \hat{\mathcal{M}}^0(\Omega) \end{tikzcd}$$

where both T_* and $\hat{T}_*$ are increasing mappings.

This more general construction allows us the following extension of the existence result in Theorem 1, the more general result being:

Theorem 2: We have an order isomorphical embedding

$$\begin{array}{ccc} C^0(\overline{\Omega}) & \hookrightarrow & \hat{T}_*(\hat{\mathcal{M}}^m_{T,*}(\Omega)) \\ f & \longmapsto & <F] \end{array}$$

Moreover, for every $f \in C^0(\overline{\Omega})$, we have

$$\mathcal{U} = \{U \in \mathcal{M}^m_{T,*}(\Omega) \mid T_*U \leq F \text{ in } \mathcal{M}^0(\Omega)\} \in \hat{\mathcal{M}}^m_{T,*}(\Omega)$$

$$\hat{T}_*\mathcal{U} = <F] \text{ in } \mathcal{M}^0(\Omega)$$

where $F \in \mathcal{M}^0(\Omega)$ is the $\sim$ equivalence class of f, and $\mathcal{U}$ is the maximum generalized solution of the continuous nonlinear PDE in (1),(2) that is

(16) $$\forall\ \mathcal{U}' \in \hat{\mathcal{M}}^m_{T,*}(\Omega),\ \hat{T}_*\mathcal{U}' = <F] \text{ in } \mathcal{M}^0(\Omega) : \\ \mathcal{U}' \leq U \text{ in } \mathcal{M}^0(\Omega)$$

Proof: The proof is similar to that of Theorem 1.

Let $\epsilon > 0$. By (5), there exists a closed, nowhere dense $\Gamma_\epsilon \subset \Omega$, and $u_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)$, such that $f + \epsilon/2 \leq T(x,D)u_\epsilon \leq f + \epsilon$ on $\Omega \setminus \Gamma_\epsilon$. Let $U_\epsilon \in \mathcal{M}^m_{T,*}(\Omega)$ be the $\sim_{T,*}$ equivalence class of u_ϵ, and $F, F - \epsilon, F + \epsilon/2, F + \epsilon \in \mathcal{M}^0(\Omega)$ be the $\sim$ equivalence classes of $f, f - \epsilon, f + \epsilon/2, f + \epsilon$ respectively, then $F + \epsilon/2 \leq T_*U_\epsilon \leq F + \epsilon$ in $\mathcal{M}^0(\Omega)$, by Lemma 2 and (15).

Let $A = \{U_\epsilon \in \mathcal{M}^m_{T,*}(\Omega) \mid \epsilon > 0\}$ and let

$$\mathcal{U} = A^l = \left\{ U \in \mathcal{M}^m_{T,*}(\Omega) \;\middle|\; \begin{array}{l} \forall\ \epsilon > 0 : \\ U \leq_{T,*} U_\epsilon \end{array} \right\}$$

Since $F + \epsilon/2 \leq T_*U_\epsilon \leq F + \epsilon$ in $\mathcal{M}^0(\Omega)$, we have $A \neq \emptyset$, so that $\mathcal{U} = A^l \neq \mathcal{M}^m_{T,*}(\Omega)$ (Appendix 3). Since $A^l = A^{lul}$, we have, by Mac Neille's construction, that $\mathcal{U} \in \hat{\mathcal{M}}^m_{T,*}(\Omega)$. From Proposition 2, there exists a $V_\epsilon \in \mathcal{M}^m_{T,*}(\Omega)$, such that $F - \epsilon \leq T_*V_\epsilon \leq F$ in $\mathcal{M}^0(\Omega)$, similarly to the argument above. But then $T_*V_\epsilon \leq F < F + \epsilon/2 \leq T_*U_\epsilon$ in $\mathcal{M}^0(\Omega)$, so that $V_\epsilon \leq_{T,*} U_\epsilon$ in $\mathcal{M}^m_{T,*}(\Omega)$. It follows that $\mathcal{U} \neq \emptyset$.

It only remains to prove that $\hat{T}_*\mathcal{U} = \langle F]$ in $\hat{\mathcal{M}}^0(\Omega)$. For this, we will need the following property of $\mathcal{U}$:

(17) $$\mathcal{U} = \{U \in \mathcal{M}^m_{T,*}(\Omega) \mid T_*U \leq F \quad \text{in } \mathcal{M}^0(\Omega)\}$$

We prove the inclusion $\supset$: Take $U \in \mathcal{M}^m_{T,*}(\Omega)$ such that $T_*U \leq F$ in $\mathcal{M}^0(\Omega)$, then there exists $u \in U$ with $T(x,D)u \leq f$ on $\Omega \setminus \Gamma$, for some $\Gamma \subset \Omega$ closed, nowhere dense. But then we have $T(x,D)u < f + \epsilon/2 \leq T(x,D)u_\epsilon$ on $\Omega \setminus (\Gamma \cup \Gamma_\epsilon)$, so that $U \leq_{T,*} U_\epsilon$ in $\mathcal{M}^m_{T,*}(\Omega)$ for all $\epsilon > 0$. Hence $U \in A^l = \mathcal{U}$.

Conversely, we prove $\subset$: Take $u \in U$ for some $U \in \mathcal{U}$, and assume that $T_*U \nleq F$ in $\mathcal{M}^0(\Omega)$. Then for every closed, nowhere dense $\Gamma \subset \Omega$, there exists $x \in \Omega \setminus \Gamma$, such that

$T(x,D)u(x) > f(x)$. But $T(x,D),u,f$ are continuous, so that $T(x,D)u > f + \delta$ on some open neighborhood $\Delta \subset \Omega \setminus \Gamma$ of x, for some $\delta > 0$. But $U \in \mathcal{U} = A^{l}$, so that, for every $\epsilon > 0$ there exists a closed, nowhere dense $\Gamma_\epsilon \subset \Omega$ such that $T(x,D)u \leq T(x,D)u_\epsilon \leq f + \epsilon$ on $\Omega \setminus \Gamma_\epsilon$. This is a contradiction, since $\Omega \setminus (\Gamma \cup \Gamma_\epsilon) \neq \emptyset$.

We proceed to prove that $\hat{T}\mathcal{U} = {<}F]$ in $\hat{\mathcal{M}}^0(\Omega)$: Note that from (17) it follows that $\hat{T}_*\mathcal{U} \subset {<}F]$, so that, from the definition of $\hat{T}_*$ and Appendix 3, $\hat{T}_*\mathcal{U} = (T_*\mathcal{U})^{ul} \subset {<}F]^{ul} = {<}F]$. Thus it only remains to prove that ${<}F] \subset \hat{T}_*\mathcal{U}$, that is, ${<}F] \leq \hat{T}_*\mathcal{U}$ in $\hat{\mathcal{M}}^0(\Omega)$.

Take $g \in C^0(\overline{\Omega})$ such that $g < f$ on $\overline{\Omega}$, so that the continuity of f,g implies that $g \leq f - \epsilon$ on $\overline{\Omega}$, for some $\epsilon > 0$. Then, by Proposition 2, we can find $v \in \mathcal{C}^m(\Omega \setminus \Gamma)$ for some $\Gamma \subset \Omega$ closed, nowhere dense, such that $g \leq f - \epsilon \leq T(x,D)v \leq f$ on $\Omega \setminus \Gamma$. Thus, if $V \in \mathcal{M}^m_{T,*}(\Omega)$ is the $\sim_{T,*}$ equivalence class of v, and $G \in \mathcal{M}^0(\Omega)$ is the $\sim$ equivalence class of g, then $G \leq T_*V \leq F$ in $\mathcal{M}^0(\Omega)$, by (15). Thus (17) implies that $V \in \mathcal{U}$.

Now, by Lemma 5, we have

$$F = \sup_{\mathcal{M}^0(\Omega)} \{g \in \mathcal{C}^0(\overline{\Omega}) \mid g < f \text{ on } \overline{\Omega}\}$$

and by (8), the order isomorphical embedding

$$\begin{array}{ccc} \mathcal{M}^0(\Omega) & \xhookrightarrow{\varepsilon_0} & \hat{\mathcal{M}}^0(\Omega) \\ F & \longmapsto & {<}F] \end{array}$$

preserves infima and suprema, so that

$$
\begin{aligned}
<F] &= \sup_{\hat{M}^0(\Omega)} \left\{ <G] \;\middle|\; \begin{array}{l} \exists\; g \in G : \\ g < f \quad \text{on } \overline{\Omega} \end{array} \right\} \\
&\leq \sup_{\hat{M}^0(\Omega)} \left\{ <T_*V] \;\middle|\; \begin{array}{l} \exists\; g \in G \leq T_*V \text{ in } M^0(\Omega) : \\ g < f \quad \text{on } \overline{\Omega} \end{array} \right\} \\
&\leq \sup_{\hat{M}^0(\Omega)} \{<T_*U] \mid U \in \mathcal{U}\}
\end{aligned}
$$

where the last inequality follows from (17). But from the proof of Mac Neille's theorem we know that

$$
\sup_{\hat{M}^0(\Omega)} \{<T_*U] \mid U \in \mathcal{U}\} = \left[\bigcup_{U \in \mathcal{U}} <T_*U] \right]^{ul}
$$

Furthermore, in any poset X, and for any family $\{x_i \mid i \in I\} \subset X$, we have

$$
\begin{aligned}
\left[\bigcup_{i \in I} <x_i] \right]^{u} &= \left\{ x \in X \;\middle|\; \begin{array}{l} \forall\; a \in \bigcup_{i \in I} <x_i] : \\ a \leq x \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \forall\; i \in I : \\ x_i \leq x \end{array} \right\} = \{x_i \mid i \in I\}^{u}
\end{aligned}
$$

Therefore $<F] \leq (\bigcup_{U \in \mathcal{U}} <T_*U])^{ul} = (T_*\mathcal{U})^{ul} = \hat{T}_*\mathcal{U}$ in $\hat{M}^0(\Omega)$, so that $\hat{T}_*\mathcal{U} = <F]$ in $\hat{M}^0(\Omega)$

We prove (16): Take any $\mathcal{U}' \in \hat{M}^m_{T,*}(\Omega)$ such that $\hat{T}_*\mathcal{U}' = <F]$ in $\hat{M}^0(\Omega)$. But then, as above

$$
\sup_{\hat{M}^0(\Omega)} \{<T_*U'] \mid U' \in \mathcal{U}'\} = (T_*\mathcal{U}')^{ul} = \hat{T}_*\mathcal{U}'
$$

so that (17) and the fact that $\hat{T}_*\mathcal{U}' \leq F$ in $\hat{M}^0(\Omega)$, imply that $\mathcal{U}' \subset \mathcal{U}$. ∎

It is clear that the existence result in Theorem 2 is nontrivial, since it will not necessarily hold for any partial order on $\mathcal{M}^{m}_{T,*}(\Omega)$ which is smaller than the pull–back order through T_* of the natural order on $\mathcal{M}^0(\Omega)$. For example, the smallest possible partial order on any given set is equality, and obviously for the poset $(\mathcal{M}^{m}_{T,*}(\Omega),=)$ we have $\hat{\mathcal{M}}^{m}_{T,*}(\Omega) = \mathcal{M}^{m}_{T,*}(\Omega)$ and $\hat{T}_* = T_*$, yet we recall that in general, even for linear partial differential operators, the mapping

$$\mathcal{C}^{m}_{nd}(\Omega) \xrightarrow{T(x,D)} \mathcal{C}^0(\Omega)$$

is not surjective, so that we may have $\hat{T}_* U \neq \langle F]$ regardless of the equivalence relation $\sim_{T,*}$. It is thus of interest to see how small a partial order $\dashv_{T,*}$ may be taken, so that we still have $\hat{T}_* U = \langle F]$. A closer look at the proof of Theorem 2 suggests that the partial order should satisfy the following two conditions: Given $U,V \in \mathcal{M}^{m}_{T,*}(\Omega)$,

$$U \dashv_{T,*} V \Rightarrow T_*U \leq T_*V \quad \text{in } \mathcal{M}^0(\Omega)$$

$$\left[\begin{array}{l} \exists\ \epsilon > 0 : \\ \quad T_*U + \epsilon \leq T_*V \quad \text{in } \mathcal{M}^0(\Omega) \end{array}\right] \Rightarrow U \dashv_{T,*} V$$

We call a partial order which satisfies these two conditions admissible. The partial order $\leq_{T,*}$ is clearly admissible, and admissibility of the partial order on $\mathcal{M}^{m}_{T,*}(\Omega)$ is clearly sufficient for the proof of Theorem 2. Therefore the theorem can be extended to any admissible partial order on $\mathcal{M}^{m}_{T,*}(\Omega)$, with the only difference being that we obtain a different Dedekind order completion $\hat{\mathcal{M}}^{m}_{T,*}(\Omega)$ and a different increasing mapping $\hat{T}_*$. If, for instance, we define the partial order $\dashv_{T,*}$ on $\mathcal{M}^{m}_{T,*}(\Omega)$ by, for $U,V \in \mathcal{M}^{m}_{T,*}(\Omega)$

$$U \dashv_{T,*} V \Leftrightarrow \left[\begin{array}{l} *)\ U = V \\ \text{or} \\ **)\ \exists\ \epsilon > 0 : \\ \qquad T_* U + \epsilon \leq T_* V \quad \text{in } \mathcal{M}^0(\Omega) \end{array}\right]$$

then $\dashv_{T,*}$ is clearly admissible, and strictly smaller than $\leq_{T,*}$ in general, so that this is a genuine strengthening of Theorem 2.

APPENDIX 3: Order Structure and Dedekind Order Completion

Given a set X and a binary relation $\leq$ on X, consider four properties that the binary relation $\leq$ may satisfy, namely given any $x,y,z \in X$:

Reflexivity: $x \leq x$

Transitivity: $x \leq y,\ y \leq z \implies x \leq z$

Antisymmetry: $x \leq y,\ y \leq x \implies x = y$

Connectedness: $x \leq y$ or $y \leq x$

If $\leq$ is reflexive and transitive then we call $\leq$ a pre–order. A pre–order that is antisymmetric is called a partial order. A partial order that is connected is called a total or linear order. We call $(X,\leq)$ a pre–ordered set, a partially ordered set, or a totally ordered set, respectively. A totally ordered subset of a partially ordered set X is called a chain in X.

Given a set X and two partial orders $\leq$ and $\dashv$ defined on it, we call $\dashv$ larger than $\leftarrow$, or equivalently, $\leq$ smaller than $\dashv$, iff

$$\begin{aligned} &\forall\ x,y \in = : \\ &\quad x \leq y \implies x \dashv y \end{aligned}$$

so that the identity mapping of X gives the increasing bijection

$$(X,\leq) \xrightarrow{id_X} (X,\dashv)$$

Clearly the concept of larger (and smaller) can be defined for any two binary relations, for instance, for two equivalence relations.

An upper bound of a subset C of a partially ordered set X is any $u \in X$, such that

$$\forall\, c \in C : c \leq u$$

The (unique) supremum of a subset C of a partially ordered set X, if it exists, is defined as the least upper bound of C, and denoted by $\sup_X C$. That is, if we denote

$$C^u = \left\{ x \in X \;\middle|\; \begin{array}{c} \forall\, c \in C : \\ c \leq x \end{array} \right\}$$

$$C^l = \left\{ x \in X \;\middle|\; \begin{array}{c} \forall\, c \in C : \\ x \leq c \end{array} \right\}$$

so that C^u, C^l are respectively the sets of upper and lower bounds of C, and if $\sup_X C$ exists, then $\sup_X C \in C^u$ and

$$\begin{array}{l} \forall\, u \in C^u : \\ \quad \sup_X C \leq u \end{array}$$

The (unique) greatest or maximum element of a subset C of a partially ordered set X, if it exists, is that element $\max C \in C$, satisfying

$$\begin{array}{l} \forall\, c \in C : \\ \quad c \leq \max C \end{array}$$

A maximal element of a partially ordered set X is any $m \in X$, such that

$$\begin{aligned} &\forall\, x \in X : \\ &\quad x \leq m \Rightarrow x = m \end{aligned}$$

Similarly, we can define a lower bound, the infimum (greatest lower bound) denoted $\inf_X C$, the least or minimum element denoted min C, and a minimal element.

In a partially ordered set, the symbol $x \vee y$ denotes $\sup_X \{x,y\}$, and $x \wedge y$ denotes $\inf_X \{x,y\}$. When both $x \vee y$ and $x \wedge y$ exist, for all $x,y \in X$, then X is called a lattice. A subset C of a lattice X is a sublattice if $x \vee y \in C$ and $x \wedge y \in C$, for all $x,y \in C$. A partially ordered set in which every nonvoid subset has both a supremum and an infimum is said to be a complete lattice, or order complete.

A totally ordered set X is called well–ordered if every nonvoid subset of X has a minimum element. A subset C of a totally ordered set X is said to be cofinal (resp. coinitial) if

$$\begin{aligned} &\forall\, x \in X : \\ &\exists\, c \in C : \\ &\quad x \leq c \qquad (\text{resp. } c \leq x) \end{aligned}$$

A partially ordered set X is said to be Dedekind complete if every nonvoid subset of X with an upper bound has a supremum or equivalently, if every nonvoid subset with a lower bound has an infimum. We show that the former implies the latter; the converse is similar: Assume that for every $A \subset X$ nonvoid, the existence of some upper bound $a \in A^u$ grants the existence of $\sup_X A$. Given $B \subset X$ nonvoid, we wish to prove that the existence of some lower bound $b \in B^l$ grants the existence of $\inf_X B$. Take $A = B^l$, then $b \in A$, so that $A \neq \emptyset$. But $B \neq \emptyset$

and clearly $B \subset A^u$, so that there exists some upper bound $a \in A^u$, granting the existence of $\sup_X A = \sup_X B^l = \inf_X B$.

A mapping $\varphi : X \longrightarrow Y$ between two partially ordered sets is said to be order preserving (or increasing) if, given $x,x' \in X$, we have

$$x \leq x' \implies \varphi(x) \leq \varphi(x')$$

If φ is bijective and

$$x \leq x' \iff \varphi(x) \leq \varphi(x')$$

then φ is called an order isomorphism between X and Y, and we say that X and Y are order isomorphic. If φ is only injective, but it is an order isomorphism between X and $\varphi(X)$, then we call φ an order isomorphical embedding.

An injective mapping $\varphi : X \longrightarrow Y$ between two lattices is called a lattice homomorphism if, given $x,x' \in X$, we have

$$\varphi(x \vee x') = \varphi(x) \vee \varphi(x') \quad \text{and} \quad \varphi(x \wedge x') = \varphi(x) \wedge \varphi(x')$$

so that $\varphi(X)$ is a sublattice of Y.

A mapping $\varphi : X \longrightarrow Y$ between two partially ordered sets is said to preserve infima and suprema if, given $x \in X$ and $C \subset X$, we have

$$x = \sup_X C \implies \varphi(x) = \sup_Y \varphi(C)$$

$$x = \inf_X C \implies \varphi(x) = \inf_Y \varphi(C)$$

that is, $\varphi(\sup_X C) = \sup_Y \varphi(C)$ and $\varphi(\inf_X C) = \inf_Y \varphi(C)$.

Given $A,B \subset X$, where X is a partially ordered set, we call (A,B) a Dedekind pair if both the following conditions hold:

$$A = B^l$$

$$B = A^u$$

where we recall that A^u denotes the set of upper bounds of A, and B^l denotes the set of lower bounds of B.

An order isomorphism always preserves infima and suprema. Indeed, let $\varphi : X \longrightarrow Y$ be an order isomorphism between partially ordered sets X and Y. For arbitrary $A \subset X$, we have $\sup_X A = \min A^u$, with A^u defined as above. But clearly $\varphi(A^u) = \varphi(A)^u$, since the order isomorphism φ satisfies

$$x \leq x' \text{ in } X \iff \varphi(x) \leq \varphi(x') \text{ in } \varphi(X) = Y$$

Therefore

$$\varphi(\sup_X A) = \varphi(\min A^u) = \min \varphi(A)^u = \sup_Y \varphi(A)$$

The proof for infima is similar.

Note that the properties of a mapping being an order isomorphical embedding or preserving infima and suprema are in general independent:

Example 1: The identity mapping

$$\mathrm{id} : (\mathbb{R},=) \longrightarrow (\mathbb{R},<)$$

is clearly not an order isomorphical embedding, yet it preserves infima and suprema. ■

Example 2: Consider the inclusion mapping

$$i : \mathcal{C}(X) \longrightarrow \mathbb{R}^X$$

where $X = [0,1]$, and the partial order on $\mathcal{C}(X)$ is induced from the partial order on $\mathbb{R}^X$, which is defined pointwise. Then clearly, the mapping i is an order isomorphical embedding. However, i does not preserve infima: Consider the family $\{f_n \mid n \in \mathbb{N}\} \subset \mathcal{C}(X) \subset \mathbb{R}^X$, where for every $n \in \mathbb{N}$, $f_n : x \mapsto x^n$. Then

$$\inf_{C(X)} \{f_n \mid n \in \mathbb{N}\} = 0 \in \mathcal{C}(X)$$

and

$$\inf_{\mathbb{R}^X} \{f_n \mid n \in \mathbb{N}\} = f : x \mapsto \left| \begin{array}{l} 0 \Leftarrow x \in [0,1) \\ 1 \Leftarrow x = 1 \end{array} \right.$$

so that $\mathbf{0} \neq f \in \mathbb{R}^X \setminus \mathcal{C}(X)$. ■

Also note that an injective order preserving mapping need not be an order isomorphical embedding:

Example 3: Let X be the partially ordered set of three elements

$$\begin{tikzcd} & 2 \\ 1 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd}$$

and Y the totally ordered set of three elements

$$a \longrightarrow b \longrightarrow c$$

where the arrows indicate the order relations between elements, and define $\varphi : X \longrightarrow Y$ by $\varphi(1) = a$, $\varphi(2) = b$, $\varphi(3) = c$, then φ is bijective and order preserving, but since $\varphi(2) < \varphi(3)$, whereas 2 and 3 are not even comparable, φ is not an order isomorphical embedding, let alone an order isomorphism. ■

If X is a lattice and Y a partially ordered set, and the mapping $\varphi : X \longrightarrow Y$ preserves infima and suprema, then φ is an order isomorphical embedding. Indeed, we have

$$x \leq x' \text{ in } X \quad \Leftrightarrow \quad \left[\begin{array}{l} \forall\, a \in X : \\ \quad a \vee x \leq a \vee x', \\ \quad a \wedge x \leq a \wedge x' \end{array} \right]$$

But since φ preserves infima and suprema, we have

$$\left[\begin{array}{l}\forall\ a \in X: \\ \quad a \vee x \leq a \vee x', \\ \quad a \wedge x \leq a \wedge x'\end{array}\right] \Leftrightarrow \left[\begin{array}{l}\forall\ a \in X: \\ \quad \varphi(a) \vee \varphi(x) \leq \varphi(a) \vee \varphi(x'), \\ \quad \varphi(a) \wedge \varphi(x) \leq \varphi(a) \wedge \varphi(x')\end{array}\right]$$

$$\Leftrightarrow \varphi(x) \leq \varphi(x') \text{ in } \varphi(X) \subset Y$$

and we are done.

If X and Y are both lattices, and for $y \in Y$ we have

$$y = \sup_Y \{\varphi(x) \mid \varphi(x) \leq y\} = \inf_Y \{\varphi(x) \mid y \leq \varphi(x)\}$$

then φ preserves infima and suprema iff φ is an order isomorphical embedding. Due to the previous result, we only need to prove sufficiency: For arbitrary $A \subset X$, we have $\sup_X A = \min A^u$, as before. But clearly $\varphi(A^u) = \varphi(A)^u \cap \varphi(X)$, since φ satisfies

$$x \leq x' \text{ in } X \Leftrightarrow \varphi(x) \leq \varphi(x') \text{ in } \varphi(X) \subset Y$$

Therefore

$$\varphi(\sup_X A) = \varphi(\min A^u) = \min(\varphi(A)^u \cap \varphi(X)) = \sup_{\varphi(X)} \varphi(A)$$

But our additional property of the lattice Y gives us

$$\sup_{\varphi(X)} \varphi(A) = \sup_Y \varphi(A)$$

The proof for infima is similar.

We now proceed with some definitions and notation needed before we can state the Mac Neille theorem used in the order completion solution method for nonlinear PDEs (Appendix 2). To begin with, let us take a nonvoid partially ordered set $(X,\leq)$, assuming that X has no largest or smallest element. Further, we denote the half unbounded intervals in X as follows

$$<a] = \{x \in X \mid x \leq a\}$$

$$[a> = \{x \in X \mid a \leq x\}$$

where $a \in X$.

In addition, we define the two dual operators on $\mathcal{P}(X)$:

$$\begin{array}{lll} \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X) \\ A & \longmapsto & A^u = \bigcap\limits_{a \in A} [a> = \left\{ x \in X \;\middle|\; \begin{array}{c} \forall\ a \in A : \\ a \leq x \end{array} \right\} \end{array}$$

$$\begin{array}{lll} \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X) \\ A & \longmapsto & A^l = \bigcap\limits_{a \in A} <a] = \left\{ x \in X \;\middle|\; \begin{array}{c} \forall\ a \in A : \\ x \leq a \end{array} \right\} \end{array}$$

Since we have assumed that X has no largest or smallest element, it follows that $|X| \neq 1$, so that obviously, for $A,B \subset X$, we have

$$A^u = X \iff A^l = X \iff A = \emptyset$$

$$A^u = \emptyset \iff A \text{ unbounded from above}$$

$$A^l = \emptyset \iff A \text{ unbounded from below}$$

$$A \subset B \implies B^u \subset A^u,\ B^l \subset A^l$$

Note that by applying the two dual operators consecutively on $A \subset X$ we can construct

$$\begin{aligned} A^{ul} &= \left\{ x \in X \;\middle|\; \begin{matrix} \forall\ \bar{a} \in A^u : \\ x \leq \bar{a} \end{matrix} \right\} \\ &= \left\{ x \in X \;\middle|\; \begin{matrix} \forall\ \bar{a} \in \left\{\bar{x} \in X \;\middle|\; \begin{matrix} \forall\ a \in A : \\ a \leq \bar{x} \end{matrix} \right\} : \\ x \leq \bar{a} \end{matrix} \right\} \end{aligned}$$

A subset $A \subset X$ which satisfies the following condition

$$A = A^{ul}$$

is called a cut in X, and we define the subset $\hat{X} \subset \mathcal{P}(X)$ of cuts of X by

$$\hat{X} = \{A \subset X \mid A = A^{ul}\}$$

Since X has no largest or smallest element, it follows that $X^u = X^l = \emptyset$. In addition we know from above that $\emptyset^u = \emptyset^l = X$. Thus $\emptyset^{ul} = X^l = \emptyset$ and $X^{ul} = \emptyset^l = X$, so that we have

$$\emptyset, X \in \hat{X}$$

and $\hat{X} \neq \emptyset$.

Given $a \in A$, for $A \subset X$, we have $a \leq \bar{a}$ for all $\bar{a} \in A^u = \{\bar{x} \in X \mid \forall\ a \in A : a \leq \bar{x}\}$, so it follows that $a \in A^{ul} = \{x \in X \mid \forall\ \bar{a} \in A^u : x \leq \bar{a}\}$, and therefore

$$A \subset A^{ul}$$

By a similar argument we also have

$$A \subset A^{lu}$$

From this last inclusion it follows that, for $A \subset X$, we have $A^u \subset (A^u)^{lu}$. Since $A \subset A^{ul}$, we also have $(A^{ul})^u \subset A^u$. Hence

$$A^{ulu} = A^u$$

By a similar argument we also have

$$A^{lul} = A^l$$

Thus we get $(A^{ul})^{ul} = (A^{ulu})^l = (A^u)^l = A^{ul}$, so that

$$\forall\ A \subset X : A^{ul} \in \hat{X}$$

An often used argument is the following:

(*)
$$\begin{aligned} A \subset B \quad &\Longrightarrow \quad B^u \subset A^u \\ &\Longrightarrow \quad A^{ul} \subset B^{ul} \end{aligned}$$

A direct result of this is that

$$(**) \qquad \begin{array}{l} \forall\ A \subset X,\ B \in \hat{X}\ : \\ A \subset B \Longrightarrow A^{ul} \subset B \\ B \subset A \Longrightarrow B \subset A^{ul} \end{array}$$

Let us denote, for $x \in X$, the sets $\{x\}^u = x^u$, $\{x\}^l = x^l$, $\{x\}^{ul} = x^{ul}$, $\{x\}^{lu} = x^{lu}$. Then we obviously have the following:

$$x^u = [x>$$

$$x^l = <x]$$

$$[x>^l = <x]$$

$$<x]^u = [x>$$

so that

$$x^{ul} = <x]$$

$$x^{lu} = [x>$$

We can embed X into $\hat{X} \subset \mathcal{P}(X)$ by means of the following mapping, which is clearly injective from the above equalities:

$$\begin{array}{l} \varepsilon_X : X \longrightarrow \hat{X} \\ \qquad x \longmapsto x^{ul} = x^l = <x] \end{array}$$

We also define the following natural partial order on $\hat{X}$: Given $A,B \in \hat{X}$ let

$$A \le B \text{ in } \hat{X} \iff A \subset B \text{ in } X$$

We are now ready to state the Mac Neille theorem which is the basic result on Dedekind order completion:

Theorem (Mac Neille): If the nonvoid partially ordered set X has no smallest or largest element, then the partially ordered set $\hat{X}$ is order complete. In addition, the embedding $\varepsilon_X : X \longrightarrow \hat{X}$ preserves infima and suprema, and is an order isomorphical embedding (ie. an order isomorphism between X and $\varepsilon_X(X) \subset \hat{X}$). We also have, for $A \in \hat{X}$, the relations

$$\begin{aligned} A &= \sup_{\hat{X}} \{\varepsilon_X(x) \mid x \in X,\ \varepsilon_X(x) \subset A\} \\ &= \inf_{\hat{X}} \{\varepsilon_X(x) \mid x \in X,\ A \subset \varepsilon_X(x)\} \end{aligned}$$

Proof: To show that $\hat{X}$ is order complete, we need to show that an arbitrary subset $\{A_i \mid i \in I\} \subset \hat{X}$ has an infimum and a supremum in $\hat{X}$. Now

$$\begin{aligned} \sup_{\hat{X}} \{A_i \mid i \in I\} &= \inf_{\hat{X}} \{A \in X \mid \bigcup_{i \in I} A_i \subset A\} \\ &= (\bigcup_{i \in I} A_i)^{ul} \end{aligned}$$

Indeed, the first equality is obvious from the definition of the order (set inclusion) on $\hat{X}$. In the second equality, $\geq$ follows from (**), and $\leq$ from the fact that $\bigcup_{i \in I} A_i \subset (\bigcup_{i \in I} A_i)^{ul}$.

Furthermore, we know that $A \subset A^{ul}$ for every $A \subset X$, and that $\bigcap_{i \in I} A_i \subset A_i$ for every $i \in I$, so by (**), we have

$$\bigcap_{i\in I} A_i \le \Big(\bigcap_{i\in I} A_i\Big)^{ul} \le \bigcap_{i\in I} A_i$$

so that

$$\inf_{\hat{X}} \{A_i \mid i \in I\} = \bigcap_{i\in I} A_i$$

To show that $\varepsilon_X : X \longrightarrow \hat{X}$ preserves infima and suprema, note that we have already proved that, for an arbitrary $E \subset X$ with $\sup_X E \in X$, we have

$$\sup_{\hat{X}} \varepsilon_X(E) = \sup_{\hat{X}} \{<x] \mid x \in E\} = \Big(\bigcup_{x\in E} <x]\Big)^{ul}$$

But we recall that

$$\sup_X E = \min E^u = \min \Big(\bigcup_{x\in E} <x]\Big)^u$$

so that

$$\Big(\bigcup_{x\in E} <x]\Big)^{ul} = \big(\sup_X E\big)^l = \varepsilon_X\big(\sup_X E\big) \quad .$$

and we have proved that $\sup_{\hat{X}} \varepsilon_X(E) = \varepsilon_X(\sup_X E)$.

Take $E \subset X$ with $\inf_X E \in X$, then we have already proved that

$$\inf_{\hat{X}} \varepsilon_X(E) = \bigcap_{x \in E} <x]$$

But obviously

$$\begin{bmatrix} \forall\ x \in E : \\ z \in <x] \end{bmatrix} \Leftrightarrow z \in <\inf_X E] = \varepsilon_X(\inf_X E)$$

so that $\inf_{\hat{X}} \varepsilon_X(E) = \varepsilon_X(\inf_X E)$.

We show that ε_X is an order isomorphical embedding, that is,

$$x \leq y \text{ in } X \Leftrightarrow \varepsilon_X(x) \leq \varepsilon_X(y) \text{ in } \varepsilon_X(X) \subset \hat{X}$$

But $x \leq y$ iff $x \in <y]$ iff $\varepsilon_X(x) = <x] \subset <y] = \varepsilon_X(y)$, so we are done.

Next we prove that, for $A \in \hat{X}$, we have the relation

$$A = \sup_{\hat{X}} \{<x] \mid x \in X, <x] \subset A\}$$

which holds trivially if $A = \emptyset$.

A related result is that, for $A \subset X$ nonvoid, we have

$$A^{ul} = \sup_{\hat{X}} \{<x] \mid x \in A\}$$

Indeed, if $x \in A$ then $<x] = x^{ul} \subset A^{ul}$, so we have the inequality $\geq$. Clearly

$A \subset \sup_{\hat{X}} \{<x] \mid x \in A\} \in \hat{X}$, so from (**) we get $A^{ul} \subset \sup_{\hat{X}} \{<x] \mid x \in A\}$, giving us $\leq$.

Thus, if $\emptyset \neq A \in \hat{X}$, then

$$A = A^{ul} = \sup_{\hat{X}} \{<x] \mid x \in A\} = \sup_{\hat{X}} \{<x] \mid x \in X,\ <x] \subset A\}$$

since $x \in A$ iff $<x] \subset A$.

For $A \in \hat{X}$, the relation

$$A = \inf_{\hat{X}} \{<x] \mid x \in X,\ A \subset <x]\}$$

holds trivially if $A = X$, so we assume that $A \neq X$. Now $A \subset <x]$ iff $x \in A^u$, so that

$$A = A^{ul} = \bigcap \{<x] \mid x \in A^u\} = \bigcap \{<x] \mid x \in X,\ A \subset <x]\}$$

and we are done. ■

If the partially ordered set X is already order complete, then our order isomorphical embedding

$$\begin{aligned} \varepsilon_X : X &\longrightarrow \hat{X} \\ x &\longmapsto <x] \end{aligned}$$

which preserves infima and suprema, is also surjective, so it becomes an order isomorphism between X and $\hat{X}$. Indeed, we prove that, given $A \in \hat{X}$, there exists an $x \in X$, such that

$A = \langle x]$: From Mac Neille's theorem we have

$$\begin{aligned} A &= \sup_{\hat{X}} \{\langle x] \mid x \in X,\ \langle x] \subset A\} \\ &= \sup_{\hat{X}} \{\langle x] \mid x \in A\} \\ &= \langle \sup_X A] \end{aligned}$$

since $x \in A$ iff $\langle x] \subset A$, and because $\varepsilon_X : x \mapsto \langle x]$ preserves infima and suprema. Thus

$$A = \langle x]$$

where $x = \sup_X A \in X$.

To understand the order completion procedure in Mac Neille's theorem in a broader context, recall that, for $A,B \subset X$, we call (A,B) a Dedekind pair iff $A = B^l$ and $B = A^u$. Thus, for a Dedekind pair (A,B) we have

$$A^{ul} = B^l = A$$

$$B^{lu} = A^u = B$$

so that A is a cut, that is $A \in \hat{X}$. Since, for $A \subset X$, we have $A^{ulu} = A^u$ and $A^{lul} = A^l$, it follows that in general that (A^{ul},A^u) and (A^l,A^{lu}) are Dedekind pairs. From this fact it is clear that the construction of order completion given in Mac Neille's theorem may as well have been done in the following (dual) way:

$$\hat{X} = \{A \subset X \mid A^{lu} = A\}$$

$$\begin{aligned} \varepsilon_X : X &\longrightarrow \hat{X} \\ x &\longmapsto x^{lu} = x^u = [x> \end{aligned}$$

with $A \in \hat{X}$ satisfying

$$\begin{aligned} A &= \inf_{\hat{X}} \{x^u \mid x \in X,\ A \subset x^u\} \\ &= \sup_{\hat{X}} \{x^u \mid x \in X,\ x^u \subset A\} \end{aligned}$$

and the order on $\hat{X}$ is defined as before, that is, set inclusion.

We can also extend order isomorphical embeddings to the order completions of the sets involved. This is done by means of the following construction: Given an arbitrary mapping

$$\varphi : X \longrightarrow Y$$

between two partially ordered sets X,Y, both of which we again assume not to have a minimum or a maximum element, let $\hat{X}, \hat{Y}$ dencte the Dedekind order completions of X,Y resp., constructed according to Mac Neille's theorem. Define the mapping

$$\begin{aligned} \hat{\varphi} : \mathcal{P}(X) &\longrightarrow \hat{Y} \\ A &\longmapsto \varphi(A)^{ul} = \sup_{\hat{Y}} \{<\varphi(x)] \mid x \in A\} \end{aligned}$$

which is well defined, since $\hat{Y} = \{B \subset Y \mid B^{ul} = B\}$, and $\{B^{ul} \mid B \subset Y\} \subset \hat{Y}$. Since $y^{ul} = <y]$ for every $y \in Y$, we get the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \big\downarrow & & \big\downarrow \varepsilon_Y \\ \mathcal{P}(X) & \xrightarrow{\hat{\varphi}} & \hat{Y} \\ \{x\} & \longmapsto & <\varphi(x)] \end{array}$$

In this context, we have the following useful result, which is needed for solving nonlinear PDEs by means of order completion (Appendix 2).

Proposition: If we define the partial order relation on $\mathcal{P}(X)$ as set inclusion (as for $\hat{X} \subset \mathcal{P}(X)$), then the mapping $\hat{\varphi} : \mathcal{P}(X) \longrightarrow \hat{Y}$ is increasing (where $\varphi : X \longrightarrow Y$ is an arbitrary mapping). If φ is increasing, then $\hat{\varphi}$ is an extension of φ, in the sense of the following commutative diagram:

$$\begin{array}{cccc} & X & \xrightarrow{\varphi} & Y \\ & \big\downarrow \varepsilon_X & & \big\downarrow \varepsilon_Y \\ \mathcal{P}(X) \supset & \hat{X} & \xrightarrow{\hat{\varphi}|_{\hat{X}}} & \hat{Y} \\ & <x] & \longmapsto & <\varphi(x)] \end{array}$$

That is, $\hat{\varphi}(<x]) = <\varphi(x)]$. If φ is an order isomorphical embedding, then $\hat{\varphi}|_{\hat{X}}$ is also an order isomorphical embedding.

Proof: We show that $\hat{\varphi} : \mathcal{P}(X) \longrightarrow \hat{Y}$ is always increasing: Since the order on both $\mathcal{P}(X)$ and $\hat{Y}$ is set inclusion, it follows from (*) that

$$\begin{aligned} A \leq B \text{ in } \mathcal{P}(X) &\Leftrightarrow A \subset B \subset X \\ &\Rightarrow \varphi(A) \subset \varphi(B) \subset Y \\ &\Rightarrow \varphi(A)^{ul} \subset \varphi(B)^{ul} \subset Y \\ &\Leftrightarrow \hat{\varphi}(A) \leq \hat{\varphi}(B) \text{ in } \hat{Y} \end{aligned}$$

Next we show that if φ is increasing, then $\hat{\varphi}(<x]) = <\varphi(x)]$, for $x \in X$: Since

$$x' \leq x \text{ in } X \implies \varphi(x') \leq \varphi(x) \text{ in } Y$$

it follows that

$$\varphi(<x]) = \varphi(\{x' \mid x' \leq x\}) \subset \{\varphi(x') \mid \varphi(x') \leq \varphi(x)\} = <\varphi(x)]$$

so that, by (*), we have

$$\hat{\varphi}(<x]) = \varphi(<x])^{ul} \subset <\varphi(x)]^{ul} = <\varphi(x)]$$

Since we have $A \subset A^{ul} \in \hat{Y}$ for every $A \subset Y$, it follows that

$$\{\varphi(x)\} \subset \varphi(<x]) \subset \varphi(<x])^{ul} \in \hat{Y}$$

so that, by (**), we have

$$<\varphi(x)] = \varphi(x)^{ul} \subset \varphi(<x])^{ul} = \hat{\varphi}(<x])$$

because $<y] = y^{ul}$ for every $y \in Y$. Thus we have shown that $\hat{\varphi}(<x]) = <\varphi(x)]$.

Finally, we show that if φ is an order isomorphical embedding, then $\hat{\varphi}|_{\hat{X}}$ is also an order isomorphical embedding: We need only show that $\hat{\varphi}|_{\hat{X}}$ is injective, and that the inverse mapping $\hat{\varphi}|_{\hat{X}}^{-1}$ is increasing, since we already know $\hat{\varphi}$ to be increasing.

To show that $\hat{\varphi}|_{\hat{X}}$ is injective, take $A,B \in \hat{X}$ with $A \neq B$. Since either $A \setminus B = \emptyset$ or $B \setminus A = \emptyset$, we can assume that $A \setminus B = \emptyset$. Since $B^{ul} = B \in \hat{X}$, this means that there exists an $x \in A \setminus B^{ul}$, so that we have

$$\exists\, z \in B^{u} : x \nleq z$$

But $z \in B^{u}$ implies that

$$\forall\, y \in B : y \leq z$$

and since φ is an isomorphical embedding, we have

$$\varphi(x) \nleq \varphi(z) \quad \text{and} \quad \forall\, y \in B : \varphi(y) \leq \varphi(z)$$

Thus $\varphi(z) \in \varphi(B)^{u}$, so that

$$\varphi(x) \in \varphi(A) \setminus \varphi(B)^{ul} \subset \varphi(A)^{ul} \setminus \varphi(B)^{ul} = \hat{\varphi}(A) \setminus \hat{\varphi}(B)$$

since we have $A \subset A^{ul} \in \hat{Y}$ for every $A \subset Y$. But this means that $\hat{\varphi}(A) \neq \hat{\varphi}(B)$, so we have proved that $\hat{\varphi}|_{\hat{X}}$ is injective.

To show that $\hat{\varphi}|_{\hat{X}}^{-1}$ is increasing, or equivalently that, for $A,B \in \hat{X}$, we have

$$\hat{\varphi}(A) \leq \hat{\varphi}(B) \text{ in } \hat{Y} \implies A \leq B \text{ in } \hat{X}$$

we will take $A,B \in \hat{X}$, with $A < B$ (ie. $A \leq B$ and $A \neq B$, since trivially, $\hat{\varphi}(A) = \hat{\varphi}(B)$ implies $A = B$). Since the order on $\hat{X}$ is set inclusion, we can take $x \in B \setminus A \neq \emptyset$. The same argument used to prove injectivity can again be used to show that

$$\varphi(x) \in \hat{\varphi}(B) \setminus \hat{\varphi}(A) \neq \emptyset$$

so that $\hat{\varphi}(A) < \hat{\varphi}(B)$, since the order on $\hat{Y}$ is set inclusion. ∎

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