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A stochastic water-market pricing model for an emerging city in the
case of South Africa.

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A stochastic water-market pricing model for an emerging city in the case of South Africa.

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae in Financial Engineering at the University of Pretoria, is my own work and has not been submitted by me for a degree at this or any other tertiary institution.

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Abstract

Water scarcity in South-Africa is a major discussion point in most circles. The most recent drought in South Africa, especially in Cape Town and other emerging markets, emphasised the need for water in South Africa.

Australia has recently opted for a water trading market and this has helped them to control the use of water and ensure that it is used more responsibly. Measures will have to be implemented in emerging cities such as Cape Town to ensure food safety and a stable economy where droughts are believed to be a major risk factor.

Proposing realistic pricing dynamics for water and establishing a market, will encourage people to use this resource more sparingly. It will give water users a method to reduce the risk of water scarcity by using water derivatives and other trading instruments. It will provide financial benefits to all water users and municipalities. This will allow municipalities the opportunity to manage water supply infrastructure under new dynamics that affect urban growing cities.

In the dissertation I will investigate, explicitly derive and simulate pricing dynamics for water. The pricing will be based on assumptions from the Australian water market that can be adjusted for South African seasonality and climate. I will use the classical uniform method to simulate the Lévy Process based on a Compound Poisson approximation to allow for jumps in water prices, with a Monte Carlo Simulation of the Brownian motion. We will therefore, have a price process for water modelled by a continuous stochastic process. This process will be driven by a Lévy Process with independent increments.

The derivations made from the Australian water market will be assumed from previous research done on their water trading market due to a lack of raw South African data, although the seasonality and mean-reversion coefficients will be derived from South African fresh water supply historical data.

In the dissertation we assume there are no transaction costs or storage costs involved. This will make the derivation of option prices more trivial.

The proposed water market is an incomplete market, and the restrictions on hedging mechanisms and arbitrage theory will be discussed at the end of the dissertation.

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‘The two most important days in your life are the day you are born, and the day you find out why.’
- **Mark Twain**

For opportunitites, wisdom, motivation and determination, I can only thank The Lord as He has gracefully blessed me with these all my life.

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List of Symbols and Abbreviations

Symbol & Abbreviations	Definition
COAG	Council of Australian Government
OTC	Over the Counter
WMS	Water Management System
μ	Drift / Mean Rate of Return
S_t	Spot Price at time t
dW_t	Brownian Motion
σ	Volatility
Ω	Sample Space
F	Set of events
P	Probability of Events Occurring
$N(t)$	Independent Poisson Process
$Q(t)$	Compound Poisson Process
$F(t)$	Filtration Generated by the Brownian Motion
Y_i	I-th jump in the Compound Poisson Process
i.i.d.	Independent and Identically Distributed
β	Beta
SDE	Stochastic Differential Equation
ρ	Correlation Coefficient
$J(t-)$	Value of Jump Process directly before a jump
S_0	Initial Spot Price
λ	Jump intensity / Intensity of Poisson Process
r	Risk Free Rate
K	Strike Price
$\tilde{E}$	Expectation Operator
MSE	Mean-Square Error
DF	Degrees of Freedom
L^p	Functional space of measurable functions whose p-norm is finite.

Table 1: List of Symbols and Abbreviations

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Chapter 1

Introduction

1.1 Overview

Managing water supply and its infrastructure in urban growing cities has been an impossible task over the last couple of years. New dynamics of a growing population, environmental stress and inequality has added additional pressure to the already declining water supply globally. Solutions to maintain and grow a well-functioning water supply system are detrimental to the survival of emerging cities globally. This can be seen as a problem, as well as an opportunity.

We will focus on emerging cities by taking Cape Town, South Africa, as an example. In South Africa, water supplies still have to recover from an El Nino triggered drought that began four years ago. The risk of this shortage hit urban and industrial sectors and will affect the industrial output in the near future. Climatologists believe that the climate variability will only get worse in the years to come [1].

The global climate change will increase the severity and duration of droughts and indirectly worsen the existing pressure on water supplies [2].

By assessing the water supply, it is obvious to state that freshwater supply worldwide is decreasing daily. Most growing cities, such as Cape Town, take their current water supply for granted. The critical significance of water resources, for the South African economy does not require detailed justification.

Better water management practice, and more efficient use of water resources, is a matter of survival. It is crucial for the sustainable development of economies, the environment and social infrastructure

for emerging cities worldwide.

Australia has a similar climate to South Africa [3], and their worst drought that recently affected them occurred between 2003 and 2012 [3]. Many regions of Australia are still experiencing significant drought. Rainfall records had shown a decrease in precipitation levels since 1994, with many scientists blaming climate change and global warming [14].

In the early 1980s the fundamental reforms that enabled water markets in Australia were developed. Limited water trading in Victoria, as well as the National Competition were introduced in the early 1990s. In 1994, water rights and land rights were separated across Australia when the Council of Australian Governments (COAG) water reforms were initiated nationally [14].

The national adoption of water reform packages commenced between 1994 and 2004 and the water markets expanded. The Australian water market is an internationally recognised water reform success story and they now have an annual turnover of between \$1 and \$3 billion. This allows water to be used more economically and productively across Australia [3].

The trading of water in Australia allows the generation of economic benefits as well as risk mitigation for water users and municipalities. Water can now be sold, bought and traded through over the counter instruments as well as on different types of exchanges [3].

In Australia, pre- COAG reforms, the value of water was vastly underestimated and unrealistically priced. This led to an irrational use of water as a resource. In South Africa, we experience a similar situation where the price of water is solely based on the price of water infrastructure and cleanliness.

Water and water sources are therefore, not priced on the value or the supply and demand of water itself. Unsustainable allocation and use of water is something all emerging cities experience, especially during droughts.

The value of water in Australia is now driven by market conditions defined by marginal value and the value is related to the equilibrium between supply and demand [14]. Although the relation to new market conditions is a move in the right direction, water trading has now become a managerial problem for each area in the market region.

The determination of more efficient water trading strategies has risen as one of the key challenging research problems in water markets and research on proposed trading strategies will have to be in place before any water market will be deemed rational in any market.

1.2 Literature Review

Commodity options can be traded for different reasons such as diversification, hedging and speculation [13]. In growing cities, it can also be used for increased income and as a measure of sustainability.

Schreider & Cui, 2009 [20], gives some warning in their article that a system should be in place to monitor water price distortion and speculative activities. Municipalities can diversify their income and resources by adding commodity options to their standard portfolios, and also for risk management purposes [13].

Schreider & Cui, 2009 [20], mentions the usefulness of options as well, especially the benefits of profitability, water insurance and diversifications. In addition to the use of options they proposed the setup of a water trading information system, that will allow consumers access to information that they can use to take part and benefit from the market themselves. This will address one of the main disadvantages, an information gap, that Schreider & Cui, 2009 [20], mentions.

The pricing of water as a commodity and the uses of water instruments have been seen in numerous literature.

Alexander and Venkatraman, 2008 [13], suggested inventory options that will be useful for a market that has a good base and movement. They mentioned that spot price modelling of water will have to be the way we go, as there is not enough existing water pricing data. A variety of commodity options can be traded over-the-counter (OTC) and in our case, the underlying will be the spot price instead of the futures price.

Alexander and Venkatraman, 2008 [13], admits that modelling the spot price, instead of the futures price will be much more challenging. Spot prices display mean-reversion, which is related to seasonality and the equilibrium of supply and demand, and we will have to consider this when modelling the price of water.

The modelling of spot prices, instead of future prices, are useful for path dependent options as the price already takes into account mean reverting processes such as seasonality, which is an important aspect of water supply movements [13].

Alexander and Venkatraman, 2008 [13], made mention of the use of double jump processes to allow for downward spikes in the water price, which provides a more realistic pricing system. This unfortunately causes the water market to be incomplete, and therefore hinders the ability to hedge this market.

In addition to mean reversion, stochastic volatility, as used in Hestons model, can also be taken into account to allow the model to capture skewed and leptokurtic price densities of commodity futures. GARCH models were suggested by Alexander and Venkatraman, 2008 [13], but this will be difficult in growing cities, as historical water prices are not realistic and therefore cannot be used.

Schreider & Cui, 2009 [20], suggests jumps in the modelling of water prices. They did an analysis of water prices over five seasons in Australia and they suggest that jumps are an idiosyncratic characteristic of water price series. They suggested the use of a jump diffusion modelling approach through a combination of a Brownian motion and a Compound Poisson process.

Many corporations in growing cities are dependent on water for their respective businesses. Passinsky, Grubb et.al , 2012 [43], mention how a water market can play a large role in mitigating corporate water risk. The business sectors affected by water shortages, and in turn water risk, as mentioned by Passinsky, Grubb et.al , 2012 [43], includes food and agriculture, mining, forestry, steel, shipping, water, utilities, manufacturing, chemical, pharmaceutical and energy sectors. They add in that the current water risk mitigation tools can be grouped into four categories:

- Water use accounting.
- Business risk assessment frameworks.
- Reporting and disclosure tools.
- Standards and certification frameworks.

These tools are used in most growing cities, as well as in Cape Town. It is evident that these tools do not provide enough risk mitigation to address the full picture of the water scarcity issue.

Passinsky, Grubb et.al , 2012 [43], suggest the use of insurance and weather derivatives to mitigate water risk. These tools are not available in most emerging economies, and therefore an easier approach, or a single approach, will be more beneficial to these cities.

The use of a water market will be an easily accessible mechanism and the pricing of water and its options work as a combination of the above-suggested methods. This will allow a consumer/corporation to trade one instrument that takes into account weather, supply and demand and numerous other factors.

Passinsky, Grubb et.al , 2012 [43], suggest that water options will only have to be traded on monthly contracts. This will simplify the modelling approach as well as the admin on the trades. They mentioned the separate trade of wastewater between corporations, that will allow for more sustainable use of water and decrease the unnecessary waste of water from corporations.

Burrage & Barbu, 2003 [25], suggest a stochastic seasonal model based on mean reversion for the natural factor and an extension of the geometric Brownian motion for commodity spot prices.

They developed a more general two-factor commodity-pricing model through the introduction of a stochastic factor based on a weather indicator that influences commodity supply and demand, instead of the normal convenience yield. This model will have two sources of uncertainty; the spot price and the weather or water supply.

Through the modelling process, Burrage & Barbu, 2003 [25], was able to derive an analytical expression for the futures price and used this to price European options.

Mo, Daniel et. al, 2016 [24], suggests a time change approach in pricing electricity. The pricing of electricity is generally applicable for water pricing uses, as it has very similar distinctive characteristics. They developed a stochastic model within a one-dimensional Markovian Framework in view of the fact that models that are based on one-dimensional Markov processes are often more tractable. Their model exhibits jumps that contribute to mean reversion and it is able to capture the trajectories and statistical properties of the commodity prices. They chose to use the Cox-Ingersoll-Ross diffusion interspersed with a compound poisson jump process (JCIR) with exponentially distributed jump sizes as a background process.

The JCIR process provided them with a unique solution to the stochastic differential equation, and since the JCIR process has a state dependent and mean-reverting component, it also allows for upward as well as downward jumps. This is also necessary when modelling water prices. Therefore, the mean reversion in their model are realised via jumps as well as a continuous drift.

Hart, Hayes & Lence, 2010 [28], acknowledges, all of the above and the importance of seasonality in all commodity price/option modelling. They developed a Bayesian MCMC algorithm to estimate their model. They estimated the Schwartz model, and applied appropriate restrictions to factor in different characteristics of commodities. This method was possible as they used commodities with distinguished historical prices. As we are focusing on a commodity that has zero to none historical data, this method would not work for us.

Dharmawan, 2017 [15], highlights the importance and applicability of using mean-reversion and jump-diffusion on the pricing of commodities and their respective instruments. Their modelling method has a combination between mean-reversion features and jump and seasonal components that are applied to model the behaviour of commodity prices. They calibrate their deterministic seasonal part by making use of the least-squared method and the stochastic section is calibrated by making use of historical data. The parameters of the model are calibrated by discretising the model and allows them to perform Monte Carlo simulations on the commodity prices under real-world

probabilities.

Leonhardt, 2017 [23], suggests an integrated regime switching approach with jumps for commodity prices, focusing on natural gas. This is a commodity influenced by seasonality and certain weather attributes. They argue that some commodities and their futures show integrated price movements. Their paper allows for integration of the term structure within a multi-factor model. This includes seasonality and jumps that will be applicable to our pricing model.

They allow for joint and individual jumps, where the jumps are modelled using a compound Poisson process and the seasonality using a deterministic function. They argue for joint jumps caused by a shortage in supply and demand, but in our case individual jumps are more likely due to the fact that water cannot easily be stored or produced. They add an additional component that allows for regime-switching to their model that separates them from other literature. This approach is modelled through a Markov Chain and extends to the class of geometric models.

Schreider & Cui, 2009 [20], models water prices and their derivatives. Their model is based on stochastic processes with jumps. Their developed model is justified by a complete analysis done on the Australian Water Market. For the option modelling, they use a simple Brownian motion with an addition of a compound Poisson process. Their model does not allow for any seasonality or other attributes that will logically affect water prices and their derivatives.

Jury, 2017 [22], did a thorough analysis on the climate trends across South Africa. This analysis is based on historical data since 1980. Rainfall and temperatures move in conjunction and there are jumps that move coinciding in both attributes. Rainfall movements are seasonally distributed. This finding justifies the use of jumps and seasonality in a water-pricing model for a growing city in South Africa. This will be used as an example for growing cities worldwide.

Michelsen, Taylor et. al, 1998 [32], did a comprehensive study on the effectiveness of conservation water pricing programs. They concluded that the increase of water prices (adequate pricing of water) led to water conservation depending on the price elasticity of demand. Increasing water prices reduced water usage and increased the efficiency of water use.

This justifies the use of a realistic water market in growing cities, especially with the focus on big water using corporations. They suggested a tiered water pricing system. This will be useful in our research as we can separate the pricing of fresh and wastewater and this will in turn, as seen in above mentioned research, trade water quality for water quantity and will realise a more efficient and sustainable water usage network.

1.3 Objectives

The goal for this dissertation is to simulate a water market for growing cities, taking Cape Town as a proxy. This market is proposed as a measure to reduce the irresponsible use of water as well as to address the risks that consumers, the public and corporations experience due to drought and water shortages. This dissertation is only a drop in the water market bucket and will still require further research to propose a more thorough water market without limitations.

One of the options to facilitate the efficiency of a water market is to make use of water derivatives, such as options, to help develop the associated water market. The advantages of water options in a market range from increased cost efficiency to the potential to deliver higher percentage returns. It is important to note that we are proposing a method to advocate for sustainable water use that will financially benefit the public, corporations as well as municipalities in growing cities. **We are not proposing a water market to create large investment opportunities for the private sector, asset managers or investors. We are proposing a market that is fair, accessible and well monitored.**

The use of options will allow for greater price certainty. In the way market water allocation and management is set up in Australia, as well as in South Africa, there are doubts as to whether the market conditions are ripe enough to introduce the use of hedging protocols. The water market can be termed as a quasi-market or incomplete market [42]. In the final section of this report, we will focus on this issue as well as the possibilities for incomplete hedges in the water market and the arbitrage theory applicable to an incomplete market.

We will follow the model of Schreider & Cui, 2009 [20], but we will add in seasonality and mean reversion from Hart, Hayes & Lence, 2010 [28], and additional modelling techniques and attributes from Burrage & Barbu, 2003 [25]. It can be seen that the majority of literature vouches for the use of mean-reversion, seasonality and jumps, simulated by a compound poisson process. We want to propose theoretically sound pricing dynamics for municipal water in emerging cities by using Cape Town as a proxy.

South Africa currently separates their water use between agricultural and municipal use. There are six water use categories, where 62% forms part of agricultural use and 24% part of municipal use. In this dissertation, we will focus on Municipal water and the method can later be expanded to the agricultural sector as well. This allows us to focus on water risk mitigation and sustainable water use in growing cities. The regulation and method will obviously differ between different cities, but the pricing and option modelling will be consistent throughout, with separate parameter estimates.

1.4 Structure of dissertation

In Chapter 2 we discuss the necessary theorems and definitions needed to finalise the modelling of our pricing framework as well as definitions used in the dissertation. It provides a background of mathematical jargon and theorems to strengthen the mathematical and statistical arguments and modelling assumptions in the dissertation.

Chapter 3 and 4 of the dissertation provide an overview of what a water market entails based on examples from existing water markets in the world. It provides an overview on what we intend to implement in the water market to ensure an efficient market that has financial benefits for municipalities and the population of emerging cities.

Chapter 5 is a thorough analysis of existing data for the underlying component of our pricing framework. We analyse the water supply from Cape Town's largest water resource and use these data findings to enhance and accurately represent our pricing model for water.

The pricing and water market framework is thoroughly explained in Chapter 6. This section outlines our proposal for the use of mathematical and financial tools to advance the field of environmental and resource economics. The use of this framework allows for the maintenance of water supply infrastructure in an emerging city and also allows water market users to financially benefit from the proposal. A look at the regulatory changes and implementation for a water market will be dealt with in this section.

Chapter 7 encompasses the derivations for our pricing model and the pricing of derivatives on the commodity. These derivations are used to simulate the sample paths for the spot price of water and the derivatives in Chapter 8. Chapter 8 talks to the benefit for the user and municipality the proposed market provides, and the viability of setting up such a market.

Chapter 9 and 10 provide a short discussion on the impact of options on the water market and hedging protocols in an incomplete market, including a short discussion of arbitrage theory.

We do acknowledge, since we have numerous noise increments and only one price process, that the water market will be an incomplete market. This makes hedging practices in the water market very difficult and slightly inefficient. We will discuss this in greater detail in Chapter 10.

Chapter 2

Concepts

For the purpose of the dissertation there are numerous concepts that we need to clarify for completeness.

2.1 Definitions

2.1.1 Financial Concepts

From a financial markets perspective the following concepts need to be explained.

Definition 2.1 (Incomplete Market) *In reality, markets are incomplete, meaning that some payoffs cannot be replicated by trading in marketed securities.*

The classic no-arbitrage theory of valuation in a complete market, based on the unique price of a self-financing replicating portfolio, is not adequate for non-replicable payoffs in incomplete markets. Staum, 2008 [39].

Definition 2.2 (Mean-Reversion) *Mean-reversion is a phenomena where prices, of a commodity in our instance, tend to pull back to a central/mean value.*

So where a risk-neutral process r is high, mean-reversion causes the process to have a negative drift; when r is low, mean-reversion tends to cause it to have a positive drift. Hull, 2012 [19].

Definition 2.3 (Arbitrage) *Trading strategy that takes advantage of two or more securities/instruments being mis-priced relative to each other.*

An arbitrage opportunity can be seen as an opportunity to make a profit from zero risk or zero initial investment.

This can be considered as a serious case of mis-pricing on the market.
Hull, 2012 [19].

Definition 2.4 (European Call Option) *It is a contract written at time $t = 0$ where:*

- *The contract holder has the right to buy a specific instrument at the exercise date $t = T$ at the strike/exercise price K .*
- *The option holder has the right, but not the obligation to buy the instrument.*

Hull, 2012 [19].

Definition 2.5 (Market Price of Risk) *Let the following equation be:*

$$\alpha_i(t) - r(t) = \sum_{j=1}^K \sigma_{ij}(t) \lambda_j(t).$$

*On the left hand side of the equation, we have excess rate of return over the risk free rate of return (r) for asset (i). On the right we have a linear combination of volatilities σ_{ij} of asset (i) with respect to the Wiener Processes $W_1, \dots, W_K$. Thus λ_j is the factor loading for the individual risk factor W_j . This is referred to as the **Market Price of Risk** for factor j .*

Björk, 2009 [12].

2.1.2 Modelling concepts

In the dissertation we will model pricing dynamics for the water market. The following concepts have to be explained for the modelling discussion.

Definition 2.6 (Probability Measure Space) *We have a probability space (Ω, F, P) , where Ω is the set of all possible outcomes. F is a sigma-algebra, that is a family of subsets of Ω closed under any countable collection of set operations, containing all sets. P gives the probability that an event in such a set of F will happen.*

Schoutens, 2003 [34].

Definition 2.7 (Stochastic Process) *A stochastic process, $X = [X_t, 0 \leq t \leq T]$, is a family of random variables defined on a complete probability space (Ω, F, P) .*

Schoutens, 2003 [34].

Definition 2.8 (Stochastic Differential Equation) *We say that an R^n -valued stochastic process $X(\cdot)$ is a solution of the Itô stochastic differential equation:*

$$dX = b(X, t)dt + B(X, t)dW, \text{ where } X_0 = x_0$$

for $0 \leq t \leq T$, provided

(i) $X(\cdot)$ is progressively measurable with respect to $F(\cdot)$,

(ii) $M := b(X, t) \in L_n^1(0, T)$

(iii) $G := B(X, t) \in L_{n \times m}^2(0, T)$,

and

(iv) $X(t) = \int_0^t x_0 ds + b(X(s), s)ds + B(X(s), s)dWs$ a.s. for all $0 \leq t \leq T$.

Schoutens, 2003 [34].

Definition 2.9 (Wiener Process/Brownian Motion) *A stochastic process is called a Wiener Process/ or Brownian Motion if the following conditions hold:*

- $W(0) = 0$.
- Process W has independent increments, i.e. if $r < s \leq t < u$ and $W(s) - W(t)$ are independent stochastic variables
- For $s < t$ the stochastic variable $W(s) - W(t)$ has the Gaussian Distribution , where $N \sim \sqrt{t - s}$
- W has continuous trajectories

Schoutens, 2003 [34].

Definition 2.10 (Geometric Brownian Motion) *GBM can be written in a simple form as:*

$$\dot{X}_t = (\alpha + \sigma \dot{W}_t)X_t$$

where, $\dot{W}$ is the time derivative of the Wiener Process. Therefore, the GBM can be viewed as a linear ordinary differential equation with a stochastic coefficient driven by white noise.

Björk, 2009 [12].

Definition 2.11 (Independent and Identically Distributed) *A set of variables are i.i.d if and only if:*

- *The variables are all mutually independent.*
- *The variables are all drawn from the same probability distribution.*

Björk, 2009 [12].

Definition 2.12 (Filtration and a model that is F_t^X measurable) Symbol F_t^X denotes all information generated by process X on the interval $[0, t]$.

Based upon the trajectory of a process X , we can see whether a given event B has occurred or not. We can therefore, say:

$$\begin{aligned} B \text{ is } F_t^X \text{ measurable} \\ \text{or } B \in F_t^X. \end{aligned}$$

So if we have a stochastic process Y , and the value of the variables of the process can be determined by the trajectory of X , then:

$$Y(t) \in F_t^X, \forall t \geq 0.$$

Then we have that Y is adapted to the filtration $\{F_t^X\}_{t \geq 0}$.

Björk, 2009 [12].

Definition 2.13 (Martingale) A stochastic process $X = X_t, t \geq 0$ is a martingale relative to (P, F) if:

- (i) X is F - adapted,
- (ii) $E[|X_t|] < \infty$ for all $t \geq 0$,
- (iii) $E[X_t | F_s] = X_s, P - a.s. (0 \leq s \leq t)$.

A martingale is 'constant on average' and models a fair game. A fair game is in relation to (iii) where the best forecast of the unobserved future value X_t based on information at time s, F_s , is the value X_s known at time s . Therefore, the expected value of a martingale X at some time T equals its initial value X_0 :

$$E[X_T | F_0] = X_0,$$

Björk, 2009 [12].

Definition 2.14 (Equivalent Martingale Measure) We say that a probability measure Q , defined on the probability measure space (Ω, F_T) is an equivalent martingale measure if:

- Q is equivalent to P , i.e. they have the same null sets, therefore stating that the events which cannot happen under P can also not happen under Q .
- The discounted stock-price process $\tilde{S} = [\tilde{S}_t = \exp(-rt)S_t, t \geq 0]$ is a martingale under Q .

The existence of an equivalent martingale measure is related to the absence of arbitrage, while the uniqueness of the equivalent martingale measure is related to market completeness.

Björk, 2009 [12].

Definition 2.15 (Lévy Process) Suppose $\phi(u)$ is the characteristic function of a distribution. If, for every positive integer n , $\phi(u)$ is also the n -th power of a characteristic function, we say that the distribution is infinitely divisible.

We can define for every such infinitely divisible distribution a stochastic process, $X = \{X_t, t \geq 0\}$, called a Lévy Process, which starts at zero and has independent and stationary increments such that the distribution of an increment over $[s, s + t]$, $s, t \geq 0$, i.e. $X_{t+s} - X_s$, has $(\phi(u))^t$ as its characteristic function.

Every Lévy Process has a càdlàg modification which is itself a Lévy Process.

Schoutens, 2003 [34].

Definition 2.16 (Càdlàg) Assume for all $t \in (a, b]$ the function f is right continuous and has a left limit. Any continuous function is càdlàg.

If f is càdlàg, we will denote the left limit at each point $t \in (a, b]$ as $f(t-) = \lim_{s \rightarrow t} f(s)$. We stress that $f(t-) = f(t)$ if and only if f is continuous at t . The jump at t is denoted by

$$\Delta f(t) = f(t) - f(t-).$$

Schoutens, 2003 [34].

Definition 2.17 (Predictable Representation Property) *In probability theory a martingale M is said to have the PRP if, for any square-integrable random variable $H \in F_T$, we have*

$$H = E[H] + \int_0^T a_s dM_s,$$

for some predictable process $a = \{a_s, 0 \leq s \leq T\}$. If we have such a representation, the predictable process a will give us our necessary self-financing admissible strategy.

Schoutens, 2003 [34].

2.1.3 Statistical Concepts

For our data analysis, statistical methods and concepts are used and need to be clarified.

Definition 2.18 (Poisson Process) *The Poisson Process is the simplest Lévy Process. It is based on the Poisson (λ), $\lambda > 0$, distribution, which has*

$$\phi_{Poisson}(u; \lambda) = \exp(\lambda(\exp(iu) - 1))$$

as characteristic function. The Poisson distribution lives on the nonnegative integers $\{0, 1, 2, \dots\}$; the probability mass at point j equals

$$\exp(-\lambda) \frac{\lambda^j}{j!}.$$

Since the Poisson (λ) distribution is infinitely divisible, we can define a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $\lambda > 0$ as the process which starts at zero, has independent and stationary increments and where the increment over a time interval of length $s > 0$ follows a Poisson (λs) distribution. The Poisson Process can also be defined as an increasing Pure Jump Process, with jump sizes always equal to 1.

Schoutens, 2003 [34].

Definition 2.19 (Compound Poisson Process) *A Poisson process, where each random variable independently holds a random number with some probability mass function $P(k) = P(B = k), k \geq 0$. Letting $B_1, B_2, \dots$ denote the i.i.d sequential random numbers, the number of Poisson Processes to arrive by time t , $X(t)$ is given by:*

$$X(t) = \sum_{n=1}^{N(t)} B_n,$$

where $N(t)$ is the counting process for the Poisson process.

Schoutens, 2003 [34].

Definition 2.20 (Stationarity) *Strong stationarity requires the shift-in variance (in time) of the finite-dimensional distributions of a stochastic process. This means that the distribution of a finite sub-sequence of random variables of the stochastic process remains the same as we shift it along the time index axis.*

Hull, 2012 [19].

Definition 2.21 (Cross-Seasonal Trading) *Cross seasonal trading refers to the selling/or buying of a security or an instrument over the maturity of a specific season as stipulated by the model.*

In our data model we incorporated a seasonal component. Therefore, the prices of the underlying commodity as well as the prices of the instruments reflect seasonality. To ensure that no additional arbitrage opportunities arise, the trading of instruments or the commodity itself, over a specific modelled season, is not permitted.

Hull, 2012 [19].

Definition 2.22 (P-Value for Regression) *The P-Value of the $\tilde{\chi}^2$ statistic is defined as:*

$$P = P(\tilde{\chi}_{m-1-h}^2 > \tilde{\chi}^2).$$

This value is the most important factor for the acceptance/or rejection of a null-hypothesis during statistical testing.

The P-Value is the probability of obtaining results at least as extreme as the observed results of the hypothesis test, assuming the null-hypothesis is correct. We will reject a null-hypothesis with a P-Value less than our level of significance, which is 0.05 in our case.

Schoutens, 2003 [34].

*The use of the **Dickey-Fuller** and **Augmented Dickey Fuller Testing** solutions are used during the dissertation for model fitting and regression purposes. The exact methods behind the testing and the use of these tests will be explained in Chapter 5. Therefore, these methods are not discussed in this chapter.*

***Regression** [4] and **Maximum Likelihood Estimation** [4] is used to determine the coefficients that allow for seasonality in the model. For comfort purposes these tools are also explained in their relevant chapters and will not be discussed in this chapter.*

2.2 Theorems and Propositions

Whilst deriving the price dynamics in the dissertation, theorems and rules were used from a mathematical and statistical perspective.

The Ito Product Rule for Jump Processes and Itô's Lemma was used during the derivation of the pricing process.

Theorem 2.23 (Itô Lemma) *Assume X has a stochastic differential equation given by:*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

where, μ and σ are adapted processes. Then let f be a $C^{1,2}$ - function.

Define process Z by $Z(t) = f(t, X(t))$. Then Z has a stochastic differential equation given by:

$$df(t, X(t)) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t),$$

With the necessary assumptions, as made above, df is given by:

$$df = \left\{ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 \right\},$$

where we can implement the following multiplication table:

$$\begin{cases} (dt)^2 = 0, \\ dt.dW = 0, \\ (dW)^2 = dt. \end{cases}$$

Björk, 2009 [12].

Theorem 2.24 (Itô Product Rule) *Suppose the value of x follows the Itô process:*

$$dx = a(x, t)dt + b(x, t)dz$$

where dz is a Wiener Process and a and b are function of x and t . The Itô product rule for $G = G(t, x)$:

$$dG = \left(\frac{\delta G}{\delta x} a + \frac{\delta G}{\delta t} + \frac{1}{2} \frac{\delta^2 G}{\delta x^2} b^2 \right) dt + \frac{\delta G}{\delta x} b dz$$

Björk, 2009 [12].

To set up our pricing model for the European Call Options the following definition of an European Call Option was used.

Definition 2.25 (European Call Option) *Price of an European Call Option with Strike Price K and time of maturity T , is given by:*

$$\Pi(t) = F(t, S(t))$$

where,

$$F(t, s) = s \cdot \mu[d_1(t, s)] - e^{-r(T-t)} \cdot K \cdot N[d_2(t, s)],$$

μ is the cumulative distribution for the standard normal distribution $N[0, 1]$ and

$$d_1(t, s) = \frac{1}{\sigma \sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}$$

$$d_2(t, s) = d_1(t, s) - \sigma \sqrt{T-t}$$

Hull, 2012 [19].

To extract an equivalent Martingale Measure mention was made of the following transforms and theorems.

Theorem 2.26 (Girsanov's Theorem) *Let W^P be a d -dimensional standard P -Wiener Process on (Ω, F, P) and let φ be any d -dimensional adapted column vector process. Choose a fixed T and define the process L on $[0, T]$ by:*

$$\begin{aligned} dL_t &= \varphi_t^* L_t dW_t^P \\ L_0 &= 1 \end{aligned}$$

i.e.

$$L_t = e^{\int_0^t \varphi_s^* dW_s^P - \frac{1}{2} \int_0^t \|\varphi_s\|^2 ds}.$$

Assume that:

$$E^P[L_t] = 1$$

and define the new probability measure Q on F_T by:

$$L_T = \frac{dQ}{dP}, \text{ on } F_T.$$

Then:

$$dW_t^P = \varphi_t dt + dW_t^Q$$

where, W^Q is a Wiener Process.

Björk, 2009 [12].

Theorem 2.27 (Esscher Transform) *Let $f_t(x)$ be the density of our model's distribution of X_t . For some real number $\theta \in \{\theta \in \mathbb{R} \mid \int_{-\infty}^{+\infty} \exp(\theta y) f_t(y) dy < \infty\}$ we can define a new density*

$$f_t^{(\theta)}(x) = \frac{\exp(\theta x) f_t(x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f_t(y) dy}.$$

Now we choose θ such that the discounted price process $\{\exp(-(r-q)t)S_t, t \geq 0\}$ is a martingale, i.e.

$$S_0 = [e^{-(r-q)t} E^\theta[S_t]], \quad (2.1)$$

where expectation is taken with respect to the law with density $f_t^{(\theta)}(x)$. Let $\phi(u) = E[\exp(iuX_1)]$ denote the characteristic function of X_1 . Then from equation (2.1) it can be shown that in order to let the discounted price process to be a martingale, we need to have

$$\exp(r-q) = \frac{\phi(-i(\theta+1))}{\phi(-i\theta)}. \quad (2.2)$$

The solution of this equation, θ^* say, gives us the Esscher transform martingale measure through the density function $f_t^{(\theta^*)}(x)$.

Björk, 2009 [12].

Chapter 3

Water Market

3.1 Australian Water Market

The use of water, as well as water institutions in Australia have undergone remarkable changes. Water reforms were initiated in 1995 and they undertook comprehensive fiscal incentives and well organised time-bounded targets. The changes made major advances in the water market sector worldwide, but they still experience issues and challenges that are inherent to water institutional reforms.

The recent reforms of the Australian water market provide useful insight into the theory, practice and understanding of water reforms that can be applied to countries similar to Australia.

The quality standards and legal system's regional diversity intensifies the risk of unequal water allocation and management. Conflicts still arise between private interest and public welfare concerning the trading of water. This will be an issue on the South African front, due to the inequality experienced between different regions in our country. As a result, water institutional reforms are being implemented in each of the Australian states separately, in an effort to achieve a new balance in inter-sectoral water allocation, consistent with the changing economic, environmental and social needs for water [14].

The water institutions and water laws of Australia initially had features more suitable to a better water-endowed region than to the dry reality of Australia. Both South Africa and Australia's previous water laws were based on European laws and were proven insufficient due to the European climate being moister.

Since the initiation of water reforms in 1995, water in Australia has become a tradable commodity and the water market has grown significantly.

According to the Australian water market reports for the year 2016-2017, the water allocation trades across Australia totalled 7035 GL, an increase of 1219 GL from 2015-2016. The report stated that new types of water products such as carryover parking and futures became very popular in 2016-2017 [3].

3.1.1 Water Trading and WaterFind Platform

Waterfind, founded in 2003, is currently one of Australia's leading water exchange platforms. They enable water users to trade water through a free, easy to use, online water exchange, or through an experienced water broker. The company offers water users and water authorities an efficient solution to manage their water requirements. Their online trading platform has been used by more than 12000 customers and to date they have completed 13, 000 trades [11].

Waterfind facilitates the buying and selling of temporary and permanent water. They also convey water transfers and provide free water market information and industry news. The water market information is limited and unfortunately, at this stage, no historical raw data is provided due to instructions by management.

There are currently three trading instruments available on Waterfind. Temporary trade, permanent trade as well as a future trade. Water allocations are updated on the Waterfind website as soon as authorities release them. They are usually released on the 1st and 15th of each month and trades would commence reliant on these water allocations [11].

Trading of water in Australia takes place seasonally. Seasons differ for each sector of Australia. When the allocation for a season is sold to another party, that allocation can be used for that season only. The license holder will regain the allocation for use the following season. This is also known as a Water Lease [11].

The trade process can take place by either using a water allocation transfer or a seasonal assignment. The water allocation transfer can take approximately 120 days, where the seasonal transfer can take approximately seven business days.

Waterfind conducts water exchanges for all water trading zones in Australia. Trading zones define the physical boundaries to, from or within which water may be traded in specific regions. The Murray-Darling Basin is divided into numerous trading zones, such as the Greater 1A Victoria Goulburn region. Waterfind determines a pool water price for each trading zone where trading can

occur. These zones are set up in such a way that they experience the same climate and rainfall. This allows the price of water to be fair and reliant.

In terms of the trading of water, successful sellers will receive a price less than or equal to their offer price. Successful buyers in return will receive a price less than or equal to their offer price. Sellers are considered on an ascending price basis and buyers on a descending basis. The lowest price seller, and the highest price buyer will be the first within the trading zone eligible to trade. The pool price for a trading zone would be greater or equal to the sell price offered by successful sellers and less than or equal to the buy price offered by successful buyers [20].

3.2 Urban Water Sector of South Africa

From mid-2017 until the end of 2018, Cape Town experienced severe water shortages. January 2018 USA Today reported that Cape Town could be one of the first major cities in the world to run out of water. At a certain stage in 2018, the Cape Town metro had less than 90 days' worth of water for a population of over 3.7 million people [2].

Cape Town implemented a strategy to decrease water use for all residents and visitors. They allowed for 50 litres of water use per day. If this limit was reached, all additional water use was charged at a special levy three times higher than the current rate [2]. This levy did not scare off most water users, and the daily limit of water use was breached daily. This shows that using penalties, do not always work as well as rather providing benefits.

An article from globe and mail made the following statement 'Cape Town is just a forewarning of a widening global trend' [1]. This makes Cape Town the perfect example for implementation of a water market to promote sustainable water use and efficiency for most emerging cities.

The city of Cape Town is mostly an urban area, where most people have access to municipal water or boreholes. There are twelve areas in the Western Cape that are defined as rural areas, where people will have limited access to municipal water. Only one of these areas, Mamre, falls under the Cape Town municipality.

The Cape Flats are seen as a partially informal township, and therefore the majority of its residents do have access to municipal water, but they fall into the lower and lower middle-income bracket and adjustments to the principles of the water market will have to be made for the rural and partially informal parts of the city. This will benefit the use of our proposal, as it will be useful for other growing cities that have numerous income classes and area classifications that have to be taken into account. This makes our proposal accessible for all income and area classes.

Cape Town currently has a very large restriction protocol in place. From 1 November 2020 all restrictions were provoked, but they are still in place should the dam levels decrease to a certain level [5].

Cape Town has a formal system in place where you can apply for the following:

- Water restriction exemption.
- Use of alternative water sources (Borehole/Well).
- Water allocation increase.
- Water Management Device.

The city also allows a person to apply for a Water Management Device. This currently allows indigent households to manage water consumption and in some cases avoid having any water charges. The device switches on at a fixed time each morning and restricts water usage during the day, as agreed between yourself and the City of Cape Town. Once a household reaches its daily level the device switches off. Any amount of water allocation not used in a specific day carries over to the following day [5].

Water in Cape Town is currently billed at a fixed charge according to your water meter reading [5]. Therefore, water is not realistically priced or based on supply and demand.

3.2.1 Water Allocation in South Africa

Water allocation and tariffs in Cape Town are divided into four groups [9]:

- Indigent households.
- Residential Households (Domestic Full & Cluster).
- Commercial/Industrial.
- Schools/Sport Bodies/Religious Institutions /Charities.

Each of these groups are given different tariffs, and on a level 1 restriction level no water allocation is given to water users, except for indigent households.

Indigent Households with a direct water connection get their first 6000 litres of water free of charge. Residential households pay a fixed charge per four water steps ranging from 0 kl to > 35 kl. Commercial / Industrial water users pay a fixed amount of R 31.10 p/ kl, and finally Schools/Sport Bodies/Religious Institutions /Charities users pay a fixed amount of R31.10 p/ kl [5].

Therefore, only residential users are tarified at water allocation amounts and indigent household

will be charged the same tariffs as residential users if they exceed their 6000 litre free water limit.

This water allocation/ tariff system is well set out and follows stringent rules. This system will have to be updated for the implementation of a water market, but the current systems in place already sets Cape Town up as a very viable option for a functioning water trading market.

3.3 Implementation of Water Market in Emerging Cities

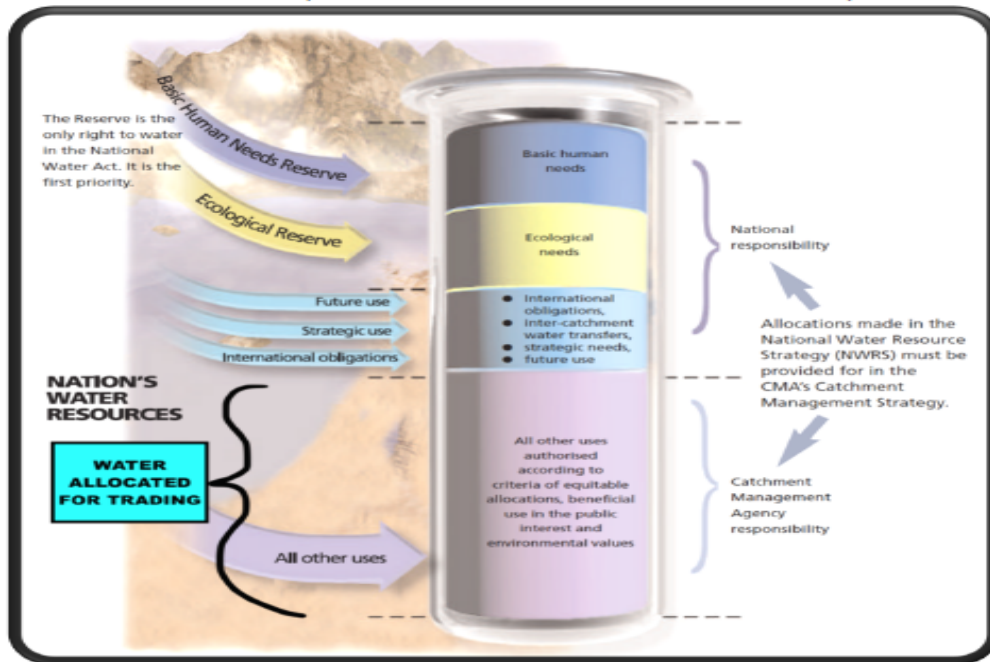
The current laws and acts with respect to water allocation will have to be adjusted to allow for the implementation of a sufficient water market. Water entitlements will have to be allocated based on the area, the necessity, the water resource levels as well as the climate/season for that region. The allocation of water will have to be researched thoroughly and not allocated based primarily on the previous year's allocations. The trading of water as a commodity will have to be legalised in all emerging cities.

In the context of Cape Town, The Water Amendment By-Law was passed in 2018 and specifies that water allocation and water systems have to be registered per property. If the rights for the property are transferred, the water allocation and systems will also have to be transferred with the property, and the City of Cape Town will have to be notified.

In the new National Water Act, national water usage and allocation is dealt into different sectors. The first allocations for water is for basic human needs, ecological reserve, future use, strategic use and international obligations. Surplus water is then made available for all other uses, such as agriculture.

The below graph outlays the water sectors for national water use (National Water Act of South Africa, 2010) [29]:

Figure 3.1: National Water Act of South Africa



The National Water Act and its amendments will have to be taken into account for all water pricing/trading systems to ensure national inclusivity.

Water trading/pricing and allocation will be city specific. City specific will be defined as all areas that receive their water allocation from the same reservoir. This will take into account a single value for transport and storage costs and ensure that the pricing model is solely based on supply of a single reservoir.

The regulatory implementations and further context is given in Chapter 6.

3.3.1 Chapter Summary

We conclude that we can learn a lot from the Australian Water Market and base a lot of our regulation and modelling assumptions on their methods and data. This is due to the fact that Australia has a very similar regulatory framework and climate than South Africa.

South Africa, specifically Cape Town, already has a very well regulated Water system. This will be very helpful in the quick implementation of a Water Market. The fact that the water users are divided into different groups on the existing framework makes the implementation for the market based on these groups very inclusive and sustainable for all users

The WaterFind platform has to proxy as a platform that we need to strive for in all emerging cities if a water market wants to be applied.

Chapter 4

Derivatives

4.1 Explanation and uses in a Commodity Market

In financial markets, derivatives are used to gain profit and reduce risk. Since the water market is risk-prone, and the factors effecting the risk unpredictable, it is important and necessary to allow derivatives in such a market to help traders and users of the market to efficiently position themselves in order to reduce their exposure to risk.

Derivatives will help consumers, municipalities and corporations to align a strategy to defend them against volatile and unpredicted market movements and water scarcity.

The introduction of derivatives will benefit the sustainable use of water and allow resources to be used optimally and efficiently.

Typical derivatives that will be applicable to a sustainable water market are forwards, futures and options. All these instruments are known as derivatives because they depend on the value of the underlying, in this case the spot price of water [19]. These water derivatives will be able to trade over the counter (OTC) or through an exchange.

If exchanges are constructed for the derivative trades, it allows the trades to be monitored and regulated; On the other hand, the beneficial aspect of OTC derivatives is that the derivatives can be tailored for your personal needs, where the derivatives on the exchange are homogenised and standard.

Forward and future contracts work in the same way, the only foremost difference between them is that forwards are traded through the OTC market, where futures are traded through an exchange. Both are defined as a contract between two parties, where there is an agreement to buy or sell a specified asset, at a specified time, at a specified price in the future [19].

These contracts allow the holder to reduce the risk of adverse price movements in the future. They are guaranteed agreements that have to be executed at maturity. Options alternatively, give the holder the option to buy or sell a specified asset at a specified time in the future at a specified price. The option does not have to be executed at maturity and the holder can choose to let the option lapse.

Options require a premium to be paid to the writer whilst the holder has the option, where futures and forwards do not require any payment of premiums upfront. Futures and forwards allow users to have huge potential gain, but also exposes them to huge potential losses. These losses can occur, because the outstanding amount for futures and forwards have to be settled daily.

Options limit the holders loss to the amount paid for the option itself but allow the holder to benefit from favourable price movements. In the dissertation, we will focus on the spot price for water and European options, as we assume that growing cities will at the start of a market only fully benefit from the above.

A European call option provides the holder of the option the right to buy the underlying asset by a certain date for a certain price [19]. The European put option is the sell option of the above mentioned.

Derivatives in the past decade has become an extremely important part of financial markets and have been expanded to other underlying variables similar to water, such as weather and energy.

The core purpose of derivatives is to reduce the exposure to risk and to partially hedge against possible losses in the market. Derivative instruments have positive effects on the market, in most cases, and allow traders to adjust their portfolio to their risk appetite and capital holding. These instruments have colossal advantages in the water trading market and will allow the financial burden of water scarcity to be condensed to smaller, more controlled losses.

We have to emphasise the following: **Derivatives in this dissertation will be used to promote sustainable water use, risk mitigation for municipalities and corporations, and to financially benefit sustainable water users.**

Chapter 5

Data Analysis

Water prices in South Africa and Cape Town specifically, have always been calculated using a fixed amount that includes storage, transport and maintenance cost with an added margin. **Therefore, the pricing is unrealistic and not based on a marginal supply and demand.**

We will not be able to analyse the historical pricing of water and a water price will have to be simulated by using an assumed initial water price determined by logical assumptions.

Different data attributes will be used in the modelling of our water price. The seasonality and water supply will be taken into account. We therefore have to do a thorough analysis on the above mentioned to justify the modelling techniques and coefficients we will use during our model derivation.

Since we are using a stochastic process driven by a Lévy Process¹, with a linear deterministic part, we need to estimate the necessary coefficients.

The estimation of the coefficients can only take part through necessary analysis of the underlying data driving the linear deterministic process.

¹Definition 2.15.

5.1 Analysis of Empirical Data

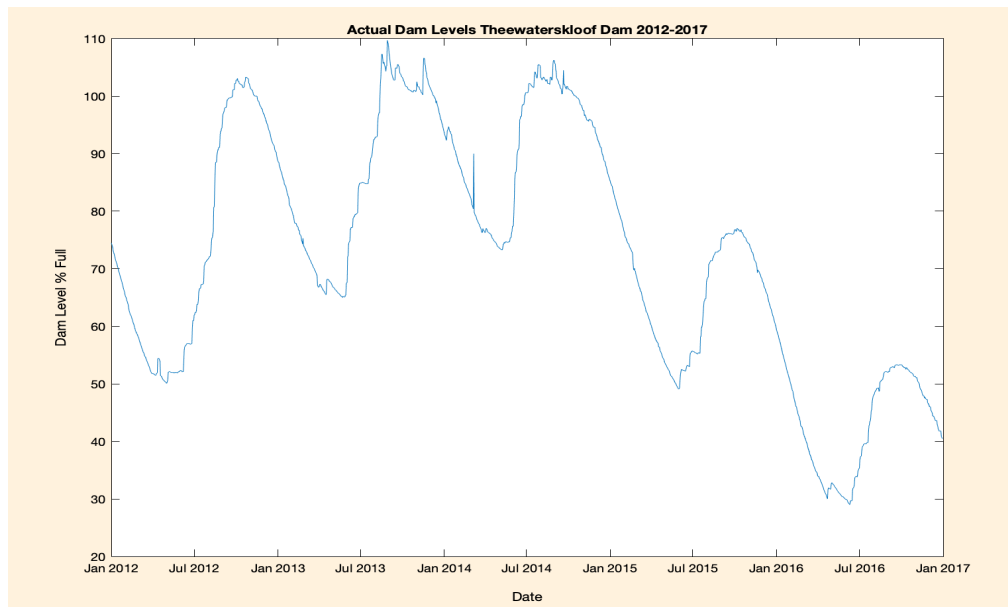
The water supply distribution will be analysed by using the largest water resource in Cape Town. The main water resource for the City of Cape Town is Theewaterskloof Dam. Water will be transported from the dam into reservoirs and thereafter follow the water infrastructure that distributes to each household or water user.

We will assume that the supply of water is in direct correlation to the dam levels. There will logically be a delay between the movement in supply and dam levels, but this delay can be incorporated in later research.

Historical data from 2012-2019 were analysed [6].

For the statistical testing we used the dam levels over 5 years, 2012-2016. From the below graph we can see that a seasonal trend is forming over the 5 years ²:

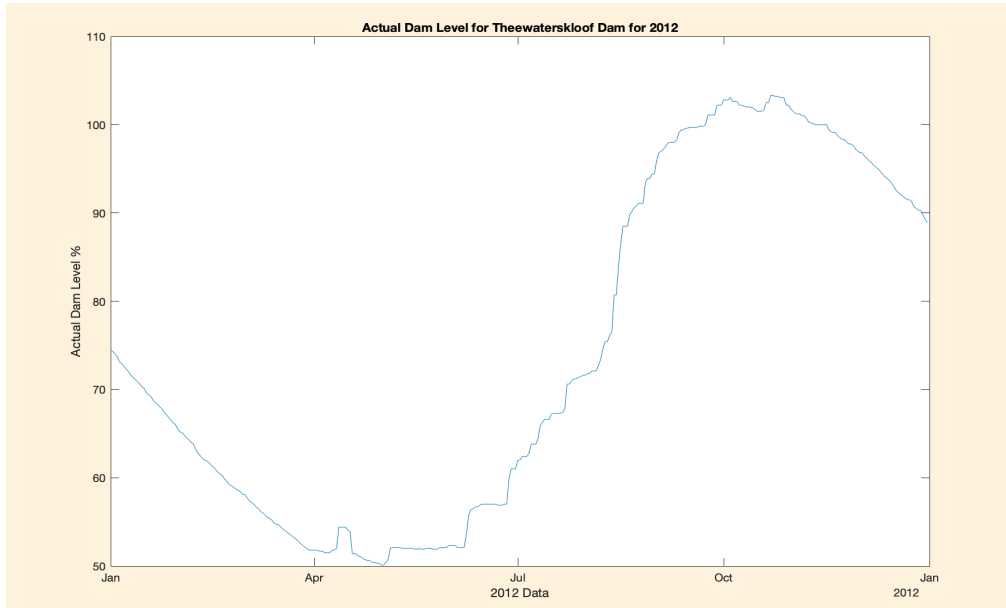
Figure 5.1: Actual Dam average Levels from 2012-2019



²We do take into account from 2015 onwards there was a severe drought in Cape Town, and we are therefore, seeing a decline in dam levels during this time.

In a single year the seasonal trend is as follows:

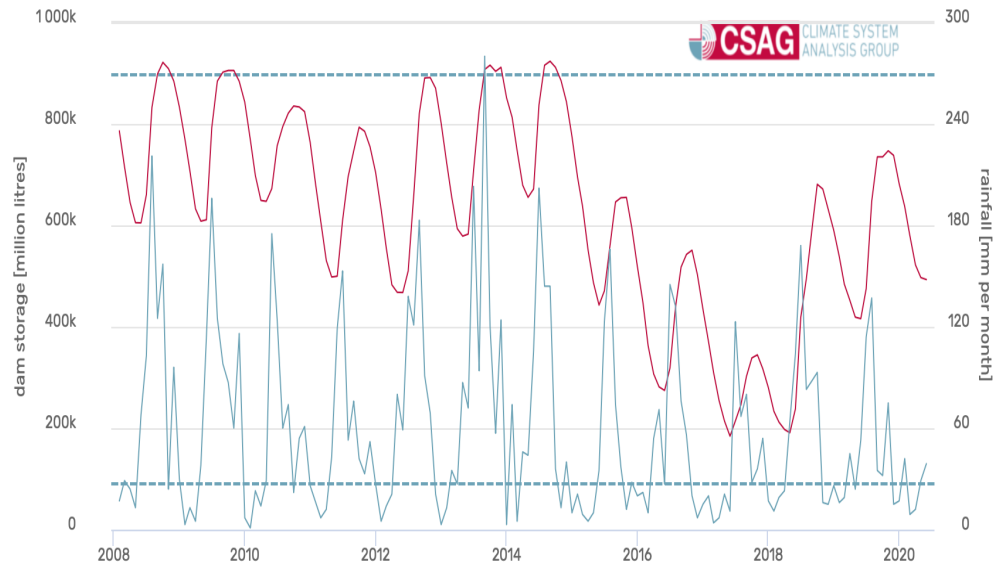
Figure 5.2: Actual Dam Levels over two seasons



From the above graphs, it is obvious that the supply of water follows a trigonometric path. The maximum supply on average is between August and December and the minimum supply takes part during the months of March and July.

Data from the Climate System Analysis Group show the following movement between rainfall and dam levels from the big six dams in Cape Town [5].

Figure 5.3: Rainfall VS Dam Levels



The red curve represents the average dam storage level, and the blue curve the rainfall. From this graph, we can clearly see a strong correlation with a slight delay between dam storage and rainfall. Seasonality and jumps accompany the rainfall data as can be seen from the graph.

An interesting result from the above data analysis shows that the dam storage levels follow the same seasonality as rainfall with a slight delay, but the storage of the water eliminates the jumps from the rainfall.

I took the data I collected from the Theewaterskloof dam and performed statistical analysis to test for the following:

- Seasonality (Trend).
- Mean Reversion (Stationarity).
- Jumps.

The **Augmented Dickey-Fuller test** was used to test for stationarity, definition in Chapter 2, with a constant and a trend. This test is used to test for the presence of a unit root in a time series data sample. The alternative hypothesis for this test can be broken up into three types:

- **Type 0:**
Stationarity with **no constant & no trend**.
- **Type 1:**
Stationarity with **a constant & no trend**.
- **Type 2:**
Stationarity with **a constant & a trend**.

If the null hypothesis is rejected in any of the three scenarios mentioned above, it shows the presence of stationarity in the model and therefore proves the data is mean reverting. The type 1 and 2 alternative hypothesis will allow us to prove that the mean reverting process follows a constant and a trend.

We are including lags to the ADF formulation to allow for higher-autoregressive processes. We therefore, determined the lag length before we applied the testing to the data. We carried out the unit root test under the null hypothesis and computed a value for the test statistic.

The test statistic was compared to the critical value from the Dickey Fuller table for data with a sample size larger than 500 with an assumed alpha of 0.05.

If the test statistic is less than the critical value, we can reject the null hypothesis and the alternative hypothesis, of stationarity, should be proven true. The larger the difference between the critical value and the calculated test statistic shows a stronger rejection of the null hypothesis.

After the completion of the tests on the time series data our results were as follows:

Stationarity test 1: (Type 0: **No Constant/No Trend.**)

Table 5.1: Stationarity Test 1

Tau-Statistical Value	-5.894
Tau-Critical Value	-1.941
Stationary	Yes
Lags	11
Coefficient	-0.394
P-value	< .01

Stationarity test 2: (Type 1: **Constant/No Trend.**)

Table 5.2: Stationarity Test 2

Tau-Statistical Value	-5.897
Tau-Critical Value	-2.864
Stationary	Yes
Lags	11
Coefficient	-0.394
P-value	< .01

Stationarity test 3: (Type 2: **Constant/Trend.**)

Table 5.3: Stationarity Test 3

Tau-Statistical Value	-5.9106
Tau-Critical Value	-3.414
Stationary	Yes
Lags	11
Coefficient	-0.396
P-value	< .01

From the results on the previous page, we see that the calculated test statistic was deemed smaller than the Dickey Fuller Critical value in each of the tests. This shows a strong case that the data is stationary with a constant and a trend.

This proves our data is mean-reverting³ and has a seasonal (trend) component with a constant. This allows us to proceed with modelling the price of water using a seasonal mean-reverting model. Therefore, we need to estimate coefficients for the linear deterministic part of the Lévy Process to allow for seasonality and mean-reversion⁴.

Jumps are an important part of a regression function and the accurate detection of jumps are crucial for understanding the structural changes and for the correct estimation of the model [20]. From the data graphs above it is evident that there are jumps in the data analysed. The jumps can be modelled by the jump process part of the Lévy Process⁵.⁶

By using the dam levels as a proxy underlying for the pricing model, we can take into account, seasonality, mean reversion and jumps in our Lévy Process. This will allow our model to take three data factors into account from one data set analysis.

5.2 Assumptions from Data Analysis

From Chapter 5.1, we will make the following **modelling assumptions**:

- We assume that the price of water will move in conjunction with the dam levels from the largest water resource in the city. Logically the price should have a delay concerning the movements of dam levels, and this can be applied and researched in follow up research.
- We will assume that the water price will follow the same seasonal and jump pattern as the chosen dams level and additionally we assume that the price will be mean reverting.

In addition to the pricing model assumption, we will make assumptions for the modelling of the derivatives. In order to simulate the prices for the options, some restrictions need to be made on the model in order to account for the stylised structures of the option pricing of water. The **restrictions on the options model** is as follows:

- Options are only available seasonally. Each option will therefore reach its maturity at the

³Definition 2.2.

⁴Definition 2.2.

⁵Definition 2.15.

⁶This will be further explained in the model derivation.

end of each season. Since there are two visible seasons in a year, options can only be bought twice a year, and each of these options will mature after six months. This helps us avoid any arbitrage opportunities arising due to the seasonal changes in the water price levels.

- We assume that option trading will only commence after a short while of trading due to the option writer not having a good enough grasp on the strike price of the option from the get-go. This will avoid the use of an unrealistic strike price.
- The options maturity date is restricted to before the seasonal ending date to prevent a pattern emerging in the options price and additional arbitrage⁷ in the market. ⁸
- We assume no transport/storage costs.

The **classical assumptions** from option trading will apply [19]:

- No transaction costs and taxes.
- No restrictions on short selling.
- The risk-free rate is constant throughout the life of the option.
- The market is efficient, and the water prices reflect all available water data.
- There are no transport cost and no storage cost for the commodity.

Following the harsh assumptions that we mentioned above, we need to realise that we are placing some limitations on the model.

In the future, there might arise a need for cross-seasonal options. The framework set out for our modelling of options make it difficult, as the calculation of cross-seasonal options are not possible, as mentioned in the restrictions on the previous page.

The final limitation is that we propose jumps diffusion to calculate or simulate the price of water. Since our jumps are random, a unique option price is not possible with the common non-arbitrage argument. Therefore, only bounds on options can be obtained in the incomplete market [20].

The use of hedging mechanisms therefore, become much more tedious and fairly less accurate.

⁷Definition 2.3.

⁸Further explanations to follow in Chapter 10.

Hedging in an incomplete market⁹ will be discussed in Chapter 10.

5.3 Chapter Summary

We made use of the Augmented Dickey Fuller test to test the data for stationarity with a constant and a trend. The test proved the alternative hypothesis to be correct in all three tests and proved to us that a mean reverting seasonal model will be applicable for our pricing process.

Data assumptions made in this section was based on empirical data analysis and we therefore assume these assumptions to be less volatile than the assumptions we grab from the Australian Water Market.

⁹Definition 2.1.

Chapter 6

Pricing and Trading Framework

The setup of a trading framework will be a tedious process, but will allow all water users to reap the benefits. The current framework existing in Cape Town is already very well outlined and beneficial to our proposed system. Therefore, Cape Town is a good proxy for all emerging cities.

There will be different frameworks and margins for each of the different water user groups, currently existing in Cape Town [9]:

- Indigent households.
- Residential Households (Domestic Full & Cluster).
- Commercial/Industrial.
- Schools/Sport Bodies/Religious Institutions /Charities.

In addition to the freshwater pricing framework for the above, a wastewater framework for commercial/industrial users will also be implemented.

6.1 Summary of new implementations to the existing regulation

The current water regulations, on a district and national level, are outdated and will not entirely align with our proposal.

Therefore, we suggest certain amendments to the separate regulations.

6.1.1 Water-By-Law of Cape Town

Water By-law is the Cape Town specific water regulation framework. This is a framework uniquely setup for the city of Cape Town. As mentioned above, some aspects of this framework will have to be amended [9].

This framework was approved by the Cape Town Council on the 27th of October 2010, and amendments to this act was made on the 31st of May 2018, following the severe droughts in Cape Town [9].

Necessary amendments for the Water By-law:

- All water users that want to make use of the water market will have to install the Water Management System (WMS). These systems are currently only installed in indigent households.
- Water trading can only take place between users inside the jurisdiction of the city of Cape Town.
- The WMS regulation, currently applied to indigent households, will be imposed on all water users that want to partake in the market. The installation costs and maintenance costs of the systems will, unlike for indigent users, be the responsibility of the water user.
- Indigent households will still receive their first 6000 litres of water at a free tariff, where after they can partake in the market for all additional purchases, at rates equal to the rates of residential classified users.

6.2 Urban Trading Framework

Urban users will be separated into two categories:

- Indigent Users
- Residential users

Both of these categories will have access to the water trading market and will be able to purchase water and instruments on water. A margin process will be part of the pricing for water. The margin process will be explained in Chapter 6.4.

Both the above mentioned users will have their own centralised purchasing system, but the water can be bought from a larger centralised pool of all water users registered in the area, domestic or corporate. Users will be able to sell their surplus allocated water, that they are not going to use, into the centralised pool and all users that have a shortage on their current allocated water can buy from this pool.

A certain amount of water will be allocated to each household at the spot price with added municipal margins each month. The price will stay constant throughout the month for this allocated water. This allocation will depend on the size of the property and the inhabitants of that property.

There will be distinguished between indigent and residential users. Indigent users will not pay for the first 6000 litre allocated distribution of their water. As soon as these allocated amounts are reached, a household can apply to buy additional water allocation on the water market. The water market allocations will have an added municipal margin as well as an additional user margin¹.

Users will eventually also have the option to buy instruments on water if they know that they will under/over use in a certain amount of time.

Examples of the water market will be given in Section 8.3.1 to explain the benefits to users of the water market.

¹Explained in Section 6.4.

6.3 Corporate and Other Sector Framework

Corporate users encompass the following:

- Commercial/Industrial.
- Schools and sport bodies.
- Religious institutions.
- Charities.

These users will receive their additional bought water allocation from the same centralised pool as explained in Chapter 6.2. The water allocation and allowances to buy water/instruments will work in the same manner as for residential/indigent households as mentioned in 6.2. These users will also have access to the wastewater framework.

6.3.1 Wastewater Framework

Commercial and industrial users can in some cases rather make use of wastewater than fresh water.

Therefore, we allow them to buy wastewater.

The transport and storage of wastewater will have a large impact on this process. We will therefore, make assumptions about the existence of such resources, but additional regulation, infrastructure and research will be needed before this plan can be implemented.

Wastewater will not have a fully functional market with added instruments. These users will only be allowed to buy wastewater on its spot price with a very small constant margin and added municipal margin.

The market will rely on users in the commercial/industrial sectors that get rid of wastewater, as they can only use fresh water for their operations. This will in turn allow users that have operations independent on the type of water they use, to buy water on this platform at a smaller cost and sell their surplus fresh water that is allocated to them.

Wastewater will be sold at 65% less than the spot price for normal commercial clean water, and the margin added to this will go to the municipality and the user that sells the wastewater to the centralised pool.

Wastewater uses

The following industries can make use of wastewater [9]:

- Battery manufacturing.
- Electric power plants.
- Food industry.
- Iron and steel industry.
- Mines and quarries.
- Nuclear industry.
- Oil and gas extraction.
- Organic chemicals manufacturing.
- Petroleum refining and petrochemicals.
- Pulp and paper industry.
- Textile mills.

These industries can rather make use of wastewater than fresh water and will be key players in the wastewater market. This will allow for more sustainable use of water. Instead of using fresh water they can buy wastewater at a smaller cost and sell their fresh water allocation for a small profit.

Their surplus freshwater will be added to the centralised fresh water pool mentioned in Chapter 6.2, and the wastewater from other industries will have its own centralised pool. All industries that have wastewater can sell their wastewater into the centralised pool, also at a small profit.

The municipal margins from the above trading strategy will be used for the setup and maintenance of wastewater storage and transport infrastructure issued by the municipality.

6.4 Pricing and margins for the Water Market

Water users will pay for water at the spot price set at the beginning of the month, with an added municipal margin. A separation will be made between a municipal margin and a market margin for residential/indigent and corporate ² water users.

A municipal margin will be added onto initial spot prices as well as spot prices on the water trading market. This margin will be set up to allow the municipality to financially benefit from the water market in order to maintain water infrastructure and maintenance/administration costs.

The market margin will only be added on the water trading environment and this will allow the water users that sell their allocated water to benefit from the trading framework and also as an added benefit for saving and using water in a resourceful and sustainable manner. This market margin will encourage water users to use less than their allocated amount of water each month and encourage sustainable water use in emerging cities.

A municipal margin and a wastewater market margin will be added to the wastewater framework.

The margins will be as follows:

Table 6.1: Margins for use in Water Market

Water User	Applied Margin for Water Trading
Municipal	10%
Residential/Indigent Trader	5%
Corporate/Industrial Trader	10%
Schools, Sport, Religious institutions & Charities	5%
Wastewater Trader	3%

These margins and their benefits to the municipality and user will be further explained in Chapter 8.3.1.

²Commercial/Industrial, Schools, Sport Bodies, Religious institutions and Charities.

6.5 Chapter Summary

This chapter outlined the Market Framework for each of the water users as outlined in Chapter 3.2.1.

We concluded that a centralised water allocation system will be used for the storing of all water allocation entering the market through the trading process, but each user group will have their own margin and pricing framework.

The framework allows for a separate centralised pool for wastewater and this contributes to the existing sustainability of the Water Market Framework, where Corporate users can turn to wastewater instead of wasting additional fresh water that could have been put to better use.

The framework also outlines the financial benefit for municipalities and users that partake in the water trading system. This will further be discussed in Chapter 8.3.1.

Chapter 7

Derivations

For stocks as underlying asset, the following normal Geometric Brownian Motion holds:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (7.1)$$

where S_t is the price of the underlying stock at time t , μ is the mean rate of return, and the volatility is denoted by σ [19].

We want to compute the price of the allocated water in Cape Town. From our data analysis of the water reservoir in Chapter 5, the standard Geometric Brownian Motion might not be the most accurate model to simulate the water price process.

We will, as justified by the data analysis, use a model with a combination of the following two processes:

- Stochastic mean-reverting process with seasonal component.
- Compound Poisson Process.

The stochastic process will have a variable that encompasses mean-reversion and seasonality. The Compound Poisson Process will allow for jumps in our data. These processes will be combined to form our final bivariate stochastic differential equation that we will use for our water and derivative pricing.

Due to the fact that we need to combine both above mentioned processes, we will derive a pricing model by making use of the Lévy Process methods [34].

7.1 Derivation for analytical expression of Water Pricing

7.1.1 Lévy Processes

Since the use of a simple Geometric Brownian Motion¹ will not be sufficient for the modelling of pricing dynamics of water, we introduce the concept of Lévy Processes.

We will model pricing data for water stochastically by using a stochastic process driven by a Lévy Measure.

A Lévy Process² was defined in Chapter 2. Every Lévy Process also has a C  dl  g³ attribute and this comes in handy in the implementation of jumps in our model [34].

With the incorporation of a jump process in the modelling framework, we will work with the c  dl  g version of the Lévy Process. This will allow our process to follow the cadlag process [34].

For the definition of the Lévy Process we have a cumulative characteristic function, as in Schoutens, 2003 [34], of:

$$\omega(u) = \log \phi(u),$$

where $\phi(u)$ is the original characteristic function and $\omega(u)$ follows the following equation:

$$\omega(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{(|x|<1)})\gamma(dx), \quad (7.2)$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and γ is a measure on $\mathbb{R} \setminus \{0\}$ with

$$\int_{+\infty}^{-\infty} \inf\{1, x^2\}\gamma(dx) = \int_{-\infty}^{+\infty} (1 \wedge x^2)\gamma(dx) < \infty \quad (7.3)$$

then $\omega(u)$ satisfies the Lévy - Khintchine formula, and γ is the Lévy Measure of the stochastic process X [34].

From the above formula we see the characteristic function encompasses three different elements all independent of each-other.

First we have a linear deterministic and Brownian aspect, followed by a seasonal and mean-reverting⁴ element. Secondly, a jump process that will be simulated using a compound poisson process⁵.

¹Definition 2.10.

²Definition 2.15.

³Definition 2.16.

⁴Definition 2.2.

⁵Definition 2.19.

7.1. DERIVATION FOR ANALYTICAL EXPRESSION OF WATER PRICING⁷

We will use a compound poisson approximation for the jump events where the jump sizes will be randomly based on expected jump sizes from underlying data. The Lévy Measure $\gamma(dx)$ will determine how the jumps occur in the stochastic process⁶ $X[34]$.

7.1.2 Compound Poisson Process

We make use of a compound poisson process⁷, since the arrival of each jump in the ordinary Poisson Process comes with an associated real-value random variable that represents the size of the jump.

These variables are independent and identically distributed⁸ and are also independent of the underlying Poisson Process.

If we have a sample space of Ω , a set of events F and the probability of these events occurring as P . We let (Ω, F, P) be the probability measure space⁹ on which the Brownian Motion $W(t)$ is defined for $0 \leq t \leq T$, and M independent Poisson Processes: $N_1(t), \dots, N_M(t)$ defined for $0 \leq t \leq T$.

Let $F(t)$ be the filtration¹⁰ generated by the Brownian Motion and the M Poisson Processes. Let $\lambda_m \geq 0$ be the intensity of the m -th Poisson Process and let $-1 \leq y_1 \leq \dots \leq y_M$ be non-zero numbers.

We now get:

$$N(t) = \sum_{m=1}^M N_m(t)$$

which is a Poisson Process with intensity $\lambda = \sum_{m=1}^M \lambda_m$

and

$$Q(t) = \sum_{m=1}^M y_m(t) N_m(t)$$

which is a Compound Poisson Process.

It is important to note that the jumps in $Q(t)$ and $N(t)$ happens at exactly the same time. The only difference is the size of the jumps. The jump size of $N(t)$ is always equal to one and the jump sizes in $Q(t)$ are random sizes denoted by Y_i .

⁶Definition 2.6.

⁷Definition 2.18.

⁸Definition 2.11.

⁹Definition 2.8.

¹⁰Definition 2.12.

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Let Y_i denote the size of the i -th jump in $Q(t)$, where Y_i random variables take on the values in the set $\{y_1, \dots, y_m\}$.

We can write $Q(t)$ as follows:

$$Q(t) = \sum_{i=1}^{N(t)} y_i$$

where $P(Y_i = y_m) = p(y_m) = \frac{\lambda_m}{\lambda}$.

Since random variables $Y_1, Y_2, \dots$ are i.i.d we set:

$$\begin{aligned} \beta &= E(Y_i) \\ &= \sum_{m=1}^M y_m p(y_m) \\ &= \frac{1}{\lambda} \sum_{m=1}^M \lambda_m y_m. \end{aligned}$$

Then:

$$Q(t) - \beta \lambda_t = Q(t) - \frac{\lambda_t}{\lambda} \sum_{m=1}^M \lambda_m y_m$$

and the size of the jumps are $\Delta Q(t) = Y_i$.

For $0 \leq s < t$ the increment $Q(t) - Q(s)$ is independent of $F(s)$ and has a mean of $\beta \lambda(t - s)$.

Therefore, we can compute the following:

$$\begin{aligned} E[Q(t) - \beta \lambda_t | F(s)] &= E[Q(t) - Q(s) + Q(s) - \beta \lambda_t | F(s)] \\ &= E[Q(t) - Q(s) | F(s)] + Q(s) - \beta \lambda_t \\ &= \beta \lambda_t - \beta \lambda_s + Q(s) - \beta \lambda_t \\ &= Q(s) - \beta \lambda_s. \end{aligned}$$

This shows that $Q(t) - \beta \lambda_t$ is martingale¹¹ and the most essential properties of sums of i.i.d.¹² random variables are inherited.

¹¹Definition 2.13.

¹²Definition 2.11.

7.1.3 Stochastic Process

We will use a stochastic differential equation to capture the seasonality and mean-reversion of the water reservoir levels, since the price of water allocation will be dependent on the supply of the water reservoir. This SDE will proxy as our linear deterministic and Brownian aspect of our Lévy Process.

From the data analysis on the water reservoir levels in Chapter 5 the data appears to take on a sine function over which a stochastic noise has been applied. So we will model the water supply as a SDE based on Mean-Reversion, seasonality and additive noise.

The modelling of time-series using sinusoidal components is referred to as Spectral Analysis. We will use a very simple appropriate model for spectral analysis that has a mean plus a sinusoidal wave including white noise [19], and the addition of an error term ϵ_t :

$$\begin{aligned} X_t &= \mu + \alpha(\sin(\omega_t + \delta)) + \epsilon_t \\ &= \mu + \alpha(\sin(\delta) \cos(\omega_t) + \cos(\delta) \sin(\omega_t)) + \epsilon_t. \end{aligned} \tag{7.4}$$

Here, we applied the formula for the sine of two angles. The function $\mu + \alpha(\sin(\omega_t + \delta))$ oscillates between $(\mu - \alpha)$ and $(\mu + \alpha)$ in a smooth and periodic function. α is the amplitude, δ the phase angle and ω the frequency. The error term ϵ_t , can be populated from a set of uniform random variables.

Now, if we indicate the water supply variable as X_t , based on data from Theewaterskloof Dam, we get the following SDE:

$$dX_t = (c_1 + c_2 \sin(2\pi t) - aX_t)dt + b dW_t \tag{7.5}$$

where:

- W_t is the additive noise represented by a Brownian Motion.
- c_1 , and c_2 are the coefficients of the seasonal components that are estimated using Maximum Likelihood estimation, and
- a is the drift coefficient that we can estimate from a simple regression process, and this represents the property of mean-reversion.

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From Kloeden & Platen, 2011 [30], the solution of a linear Itô¹³ stochastic differential equation:

$$dX_t = (a(t)X_t + c(t))dt + (b(t)X_t + k(t))dW_t, \quad (7.6)$$

where

$$X(0) = X_0$$

is:

$$X_t = \phi(X_0 + \int_0^t \phi_{s,0}^{-1}(c(s) - b(s)k(s))ds + \int_0^t \phi_{s,0}^{-1}ds dW_s) \quad (7.7)$$

where

$$\phi(t, 0) = \exp\left(\int_0^t (a(s) - \frac{1}{2}b(s)^2)ds + \int_0^t b(s)dW_s\right).$$

So if we write equation (7.5) in the form of equation (7.6), and expand to the formula in equation (7.7) we get:

$$\begin{aligned} X_t = & e^{-at} \left(X_0 + \frac{c_1}{4\pi^2 + a^2} (ae^{at} \sin(2\pi t) - 2\pi e^{at} \cos(2\pi t) + 2\pi) \right. \\ & \left. + \frac{c_2}{a} (e^{at} - 1) + b \int_0^t e^{as} dW_s \right). \end{aligned} \quad (7.8)$$

Formula (7.8) is difficult to use for parameter estimation. Therefore, we discretise equation (7.8) based on the Euler method to get a time-discrete version to estimate the seasonal and mean-reverting parameters [19].

From equation (7.5) we get:

$$dX_t = (c_1 + c_2 \sin(2\pi t) - aX_t)dt + b dW_t$$

and from equation (7.8) we get:

$$\begin{aligned} X_t = & e^{-at} \left(X_0 + \frac{c_1}{4\pi^2 + a^2} (ae^{at} \sin(2\pi t) - 2\pi e^{at} \cos(2\pi t) + 2\pi) \right. \\ & \left. + \frac{c_2}{a} (e^{at} - 1) + b \int_0^t e^{as} dW_s \right), \end{aligned}$$

where $\int_0^t e^{as} dW_s$ is an Itô integral, therefore:

$$\int_0^t e^{as} dW_s = \lim_{t \rightarrow 0} \sum_{i=1}^n e^{s(s_i-1)} \cdot [W_{s_i} - W_{s_{i-1}}].$$

¹³Theorem 2.23.

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So equation (7.5) becomes:

$$X_{t+\Delta t} = (1 - a\Delta t)X_t + (c_2 \sin(2\pi t) + c_2 \cos(2\pi t) + c_1)\Delta t + b\epsilon_t$$

where $\epsilon_t \sim N(0, \Delta t)$ is a random error term. We will use this equation for parameter estimation.

We now need to introduce the seasonal, mean-reverting stochastic process in equation (7.5) into the final Lévy price process that we can use for our final pricing model.

We consider the spot option price as the following Lévy Process:

$$dS_t = (\alpha + \beta X_t)S_t dt + \sigma S_t dW_{1t}, \quad (7.9)$$

where X_t is the above modelled stochastic process:

$$dX_t = (c_1 + c_2 \sin(2\pi t) - aX_t)dt + b dW_{2t}.$$

We will allow for an assumed correlation between the two noise increments dW_{1t} & dW_{2t} , with the coefficient:

$$\text{corr}(dW_{1t}, dW_{2t}) = \rho dt$$

and:

- $E[W_t] = 0, \forall t$
- $\text{var}[W_t] = t, \forall t$

and for $j = 0, 1$ we have that:

$$\frac{W_{t_{j+1}} - W_{t_j}}{\sqrt{t_{j+1} - t_j}}$$

are i.i.d., and follow a normal distribution $\sim N(0, 1), \forall t$.

7.1.4 Final Stochastic differential equation for pricing

For our final pricing model we will be making use of two processes, both derived in the previous section, that represent all three increments of the Lévy Process.

So we have the following two processes, where the compound poisson process represents the jump increment of the Lévy Process, and the stochastic process that allows for the Linear deterministic and Brownian motion increment:

1. Compound Poisson Process

$$dJ(t) = J(t-)dQ(t)$$

where $J(t-)$ is the value directly before the jump at time t .

Therefore $J(t) = J(t-) \cdot (Y_{i+1})$. Where $J(t)$ is the value of the process directly after the jump, $J(t-)$ the value before the jump and Y_{i+1} the jump size. This can be achieved since a Lévy Process has a càdlàg¹⁴ modification and will allow for $\Delta J(t) = J(t) - J(t-)$ [34].

2. Stochastic Process including the mean-reverting seasonal component

$$dX(t) = (\alpha - \beta Y_t)X(t)dt + \sigma X(t)dW_{1t}$$

where

$$dY_t = (c_1 + c_2 \sin(2\pi t) - aY_t)dt + b dW_{2t}$$

and $\text{corr}(dW_{1t}, dW_{2t}) = \rho dt$.

From Shreve, 2005 [37], Schreider & Cui, 2009 [20] and Burrage & Barbu, 2003 [25], we are combining all the above modelling methods using a Lévy Process to formulate a pricing formula.

¹⁴Definition 2.16.

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Since $J(t)$ is a jump process and $X(t)$ a Stochastic Process we can use the Itô product rule¹⁵ for jump processes such that we eventually get a complete Lévy Process:

$$\begin{aligned} dS(t) &= S(t-)dQ(t) + \alpha S(t)dt - \beta \lambda S(t)Y(t)dt + \sigma S(t)dW_{1t} \\ &= (\alpha - \beta \lambda Y_t)S_t dt + \sigma S_t dW_{1t} + S(t-)dQ_t \end{aligned} \quad (7.10)$$

where $\text{corr}(dW_{1t}, dW_{2t}) = \rho dt$.

The solution to equation 7.10 is:

$$S(t) = S(0) \exp\left\{\sigma W_i(t) + (\alpha - \beta \lambda Y_t - \frac{1}{2}\sigma^2)t\right\} \prod_{i=1}^{N(t)} (Y_i + 1). \quad (7.11)$$

Now in order to ensure that equation 7.11 satisfies the SDE in equation 7.10 we remember that:

$$dX(t) = (\alpha - \beta \lambda Y(t))X(t)dt + \sigma X(t)dW_1(t)$$

and

$$dJ(t) = J(t-)dQ(t).$$

So from the Itô product rule for jump processes [34]:

$$\begin{aligned} S(t) &= X(t)J(t) \\ &= S(0) + \int_0^t X(s-)J(s-)dQ(s) + \int_0^t J(s)((\alpha - \beta \lambda Y(s))X(s)ds + \sigma X(s)dW_1(s)) \\ &= S(0) + \int_0^t X(s-)J(s-)dQ(s) + \int_0^t J(s)((\alpha X(s)ds - \beta \lambda Y(s))X(s)ds + \sigma X(s)dW_1(s)) \\ &= S(0) + \int_0^t X(s-)J(s-)dQ(s) + \alpha \int_0^t J(s)X(s)ds - \beta \lambda \int_0^t J(s)Y(s)X(s)ds + \sigma \int_0^t J(s)X(s)dW_1(s). \end{aligned}$$

¹⁵Theorem 2.24.

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So the differential form of the above is:

$$\begin{aligned} dS(t) &= X(t-)J(t-)dQ(t) + \alpha J(t)X(t)dt - \beta\lambda Y(t)J(t)X(t)dt + \sigma S(t)dW_1(t) \\ &= S(t-)dQ(t) + (\alpha - \beta\lambda Y(t))S(t)dt + \sigma S(t)dW_1(t). \end{aligned}$$

This proves that equation 7.11 is a solution to equation 7.10, where:

- $Q(t)$ is a Compound Poisson Process.
- β is the expected value of the jump size.
- λ is the intensity of the Poisson Process.
- $Y(t)$ is the stochastic mean reversion process.

$W_1(t)$ still represents the Brownian Motion correlated to the noise increment from the Mean-Reverting Seasonal component $W_2(t)$ with ρ .

We denote values of jumps y_i as $-1 \leq y_1 \leq \dots \leq y_m$ to guarantee that the water prices can jump down and up, but not to a negative value. Therefore we allow $Y_i + 1 > 0$, $\forall$ i's.

7.2 Risk-Neutral Measure

We need to construct a risk-neutral/ or equivalent martingale measure¹⁶ for our price process.

If an equivalent martingale measure exists we know the pricing structure is arbitrage¹⁷ free [34].

Since we have an incomplete market¹⁸ our pricing process will not have a predictable representation property, and we have to choose our own equivalent martingale measure [34].

It is important to note that if we have a fully functioning market, the market can choose the best fit martingale measure for our process [19].

A risk-neutral pricing method in an incomplete market is only meaningful when [12]:

- The current portfolio is empty.

Therefore when:

- You have no underlying position.
- No existing option inventory.

This is the case in our study. So we let θ be a constant and let $\tilde{\lambda}_1, \dots, \tilde{\lambda}_m$ also be positive constants.

The use of a risk-neutral provides the necessary and sufficient conditions for the market to be arbitrage free. Since it will not be an unique measure, we unfortunately will have to deal with an incomplete market [34].

Girsanov's Theorem¹⁹ comes in handy so that the Brownian motion with drift, becomes a Brownian Motion without drift that is easier to compute [12]. The change of measure affects only the intensity and distribution of the jump sizes.

¹⁶Definition 2.14.

¹⁷Definition 2.3.

¹⁸Definition 2.1.

¹⁹Theorem 2.26.

We define the following:

$$\begin{aligned} Z_0(t) &= e^{-\theta W_1(t) - \frac{1}{2}\theta^2 t} \\ Z_m(t) &= e^{((\lambda_m - \tilde{\lambda}_m)t)(\frac{\tilde{\lambda}_m}{\lambda_m})^{N_m(t)}} \text{ for } m=1, \dots, M \\ Z(t) &= Z_0(t) \prod_{m=1}^M Z_m(t) \end{aligned}$$

and

$$\tilde{P}(A) = \int_A Z(T) dP \text{ for all } A \in \mathcal{F}.$$

Under the probability measure $\tilde{P}$:

- The process $\tilde{W}_1(t) = W_1(t) + \theta t$ is a Brownian motion.
- Each N_m is a Poisson Process with intensity $\tilde{\lambda}_m$.
- $\tilde{W}_1$ and $N_1, \dots, N_m$ are independent of one another.

and we define:

$$\tilde{\lambda} = \sum_{m=1}^M \tilde{\lambda}_m \text{ and } \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}.$$

In summary, under the probability measure $\tilde{P}$, $N(t) = \sum_{m=1}^M N_m(t)$ is a Compound Poisson Process with intensity $\tilde{\lambda}$.

The jump sizes $Y_1, Y_2, \dots$ are i.i.d with $\tilde{p}(Y_i = y_m) = \tilde{p}(y_m)$ and $Q(t) - \tilde{\beta}\tilde{\lambda}t$ is a martingale.

Therefore:

$$\tilde{\beta} = \tilde{E}(Y_i) = \sum_{m=1}^M y_m \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^M \tilde{\lambda}_m y_m.$$

So:

$$\begin{aligned} dS(t) &= (\alpha - \beta\lambda Y_t)S(t)dt + \sigma S(t)dW_1(t) + S(t-)dQ(t) \\ &= rS(t)dt + \sigma S(t)d\tilde{W}_1(t) + S(t-)d(Q(t) - \tilde{\beta}\tilde{\lambda}t) \\ &= (r + \sigma\theta - \tilde{\beta}\tilde{\lambda}Y(t))S(t)dt + \sigma S(t)d\tilde{W}_1(t) + S(t-)dQ(t). \end{aligned} \tag{7.12}$$

Here $(r + \sigma\theta - \tilde{\beta}\tilde{\lambda}Y(t))$ is equivalent to $(\alpha - \beta\lambda Y(t))$ since we chose θ as a constant to allow for this.

Therefore the market price of risk/or equivalent martingale measure for the commodity(water) is:

$$r + \sigma\theta - \tilde{\beta}\tilde{\lambda}Y(t). \tag{7.13}$$

7.3 Pricing of European options on the Spot Price

From Chapter 7.1 we have the following spot price for water:

$$S(t) = S(0) \exp\{\sigma W_1(t) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)t\} \times \prod_{i=1}^{N(t)} (Y_i + 1). \quad (7.14)$$

Therefore, for $0 \leq t < T$, since $S(t)$ is $F(t)$ measurable, we have the price for a European Call option given as:

$$c(t, S(t)) = \tilde{E}[e^{(-r(T-t))}(S(t) - K)^+ | F(t)]$$

where $c(t, S(t))$ is the expected value of the discounted payoff of the European Call Option with respect to the new probability measure $\tilde{P}$.

Let $t \in [0, T)$, then from equation (7.14) it follows that:

$$S(T) = S(t) e^{\{\sigma(\tilde{W}_1(T) - \tilde{W}_1(t)) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)(T-t)\}} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1). \quad (7.15)$$

As the term $S(T)$ is $F(t)$ measurable and the terms on the right hand side of equation (7.15) are independent of $F(t)$, we find that:

$$\begin{aligned} c(t, S(t)) &= \tilde{E}[e^{(-r(T-t))} [x e^{\{\sigma(\tilde{W}_1(T) - \tilde{W}_1(t)) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)(T-t)\}} \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K]^+] \\ &= \tilde{E}[\tilde{E}[e^{(-r(T-t))} (x e^{\{\sigma(\tilde{W}_1(T) - \tilde{W}_1(t)) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)(T-t)\}} \\ &\quad \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) - K)^+ | \sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i + 1))]]]. \end{aligned} \quad (7.16)$$

We let $z = \frac{\tilde{W}_1(T) - \tilde{W}_1(t)}{\sqrt{T-t}}$, therefore z is a standard normal random variable under $\tilde{P}$ where $Z \sim N(0, 1)$. The conditioning σ - algebra is generated by the random variable $\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)$.

Now, since $\prod_{i=N(t)+1}^{N(T)} (Y_i+1)$ is $\sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i+1))$ measurable and z is independent of $\sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i+1))$, we can re-write equation (7.16) by making use of independence:

$$\begin{aligned}
c(t, S(t)) &= \tilde{E}[\tilde{E}[e^{-r(T-t)}](S(t)e^{\{\sigma(W_1(T)-W_1(t))+(r-\tilde{\beta}\tilde{\lambda}Y(t)-\frac{1}{2}\sigma^2)(T-t)\}} \\
&\quad \times \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K)^+ \mid \sigma(\prod_{i=N(t)+1}^{N(T)} (Y_i+1))]] \\
&= \tilde{E}[\exp(-r(T-t))(S(t) \exp\{-\sigma(\sqrt{T-t}z) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)(T-t)\} \\
&\quad S(t) \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K)^+] \\
&= \tilde{E}[\exp(-r(T-t))(S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \exp\{-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t)\} \\
&\quad S(t) \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K)^+].
\end{aligned}$$

The integrand of:

$$S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \exp\{-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t)\} \times \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K$$

will be positive if we find that:

$$S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \exp\{-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t)\} \times \prod_{i=N(t)+1}^{N(T)} (Y_i+1) > K$$

$$\exp\{-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t)\} > \frac{K}{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}$$

$$-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t) > \log\left(\frac{K}{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}\right)$$

$$\sigma(\sqrt{T-t})z + (r - \frac{1}{2}\sigma^2)(T-t) < \log\left(\frac{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{K}\right)$$

$$\sigma(\sqrt{T-t})z < \log\left(\frac{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{K}\right) - (r - \frac{1}{2}\sigma^2)(T-t)$$

$$z < \frac{\log\left(\frac{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{K}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

therefore:

$$z < d2,$$

where:

$$d2 = \frac{\log\left(\frac{S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{K}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

We can now further derive the equation:

$$\begin{aligned} & \tilde{E}[\exp(-r(T-t))(S(t) \exp\{-\sigma(\sqrt{T-t}z) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)(T-t)\}) \\ & \times \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K]^+ \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d2} \exp(-r(T-t))(S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \exp\{-\sigma\sqrt{T-t}z + (r - \frac{1}{2}\sigma^2)(T-t)\}) \\ & \times \prod_{i=N(t)+1}^{N(T)} (Y_i+1) - K) \cdot \exp(-\frac{z^2}{2}) dz \\ &= \frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{\sqrt{2\pi}} \int_{-\infty}^{d2} \exp\{\frac{-z^2}{2} - \sigma\sqrt{T-t}z - \frac{1}{2}\sigma^2(T-t)\} dz \\ & - \frac{K \exp(-r(T-t))}{\sqrt{2\pi}} \int_{-\infty}^{d2} \exp(-\frac{z^2}{2}) dz \\ &= \frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i+1)}{\sqrt{2\pi}} \int_{-\infty}^{d2} \exp\{-\frac{1}{2}(z^2 + 2\sigma\sqrt{T-t}z + \sigma^2(T-t))\} dz \\ & - K \exp(-r(T-t)) N(d2) \\ &= \frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i+1) \cdot \exp(-\frac{1}{2}\sigma^2(T-t))}{\sqrt{2\pi}} \\ & \int_{-\infty}^{d2} \exp\{-\frac{1}{2}(z^2 + 2\sigma\sqrt{T-t}z) - K \exp(-r(T-t)) N(d2) \end{aligned} \tag{7.17}$$

Now by completing the square:

$$\begin{aligned} & -\frac{1}{2}(z^2 + 2\sigma\sqrt{T-t}z) \\ &= -\frac{1}{2}(z^2 + 2\sigma\sqrt{T-t}z - \sigma^2(T-t) + \sigma^2(T-t)) \\ &= -\frac{1}{2}(z + \sigma\sqrt{T-t})^2 + \frac{1}{2}\sigma^2(T-t). \end{aligned} \tag{7.18}$$

So from equation (7.18) we can write equation (7.17) further as:

$$\begin{aligned}
&= \frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)}{\sqrt{2\pi}} \int_{-\infty}^{d2} \exp\left\{-\frac{1}{2}(z + \sigma\sqrt{T-t})^2\right\} dz \\
&\quad - K \exp(-r(T-t))N(d2) \\
&= S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)N(d2 + \sigma\sqrt{T-t}) - K \exp(-r(T-t))N(d2) \\
&= S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)N(d1) - K \exp(-r(T-t))N(d2)
\end{aligned} \tag{7.19}$$

where:

$$d2 = \frac{\log\left(\frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d1 = d2 + \sigma\sqrt{T-t}.$$

For the final European Call option price we add an expectation operator as the jump sizes in the derived price are still random [20].

As conditioned on $N(T) - N(t) = j$, the random variable $\prod_{i=N(t)+1}^{N(T)} (Y_i + 1)$ has the same distribution as $\prod_{i=1}^j (Y_i + 1)$ and $P(N(T) - N(t) = j) = \exp(-\tilde{\lambda}(T-t)) \frac{\tilde{\lambda}^j (T-t)^j}{j!}$, therefore we find:

$$c(t, S(t)) = \sum_{j=0}^{\infty} \exp(-\tilde{\lambda}(T-t)) \frac{\tilde{\lambda}^j (T-t)^j}{j!} \cdot \tilde{E}(\text{Equation 7.19}) \tag{7.20}$$

as the European Option Price for our spot pricing process.

Chapter 8

Simulations and Analysis

We have our finalised pricing structure as follows:

- **Spot Price for water**

$$S(t) = S(0) \exp\{\sigma W_1(t) + (r - \tilde{\beta}\tilde{\lambda}Y(t) - \frac{1}{2}\sigma^2)t\} \times \prod_{i=1}^{N(T)} (Y_i + 1)$$

where

$$Y(t) = e^{-at}(Y(0) + \frac{c_1}{4\pi^2 + a^2}(ae^{at}\sin(2\pi t) - 2\pi e^{at}\cos(2\pi t) + 2\pi) + \frac{c_2}{a}(e^{at} - 1) + b \int_0^t e^{as}dW_2(s)).$$

- **European Call Option Price**

$$c(t, S(t)) = \sum_{j=0}^{\infty} \exp(-\tilde{\lambda}(T-t)) \frac{\tilde{\lambda}_j(T-t)^j}{j!} \cdot \tilde{E}[S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\} \\ \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)N(d_1) - K \exp(-r(T-t))N(d_2)]$$

where:

$$d_2 = \frac{\log(\frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

8.1 Spot Price Simulation and Model Calibration

We have the following pricing structure:

$$\begin{aligned} dS(t) &= S(t-)dQ(t) + \alpha S(t)dt - \beta \lambda Y(t)S(t)dt + \sigma S(t)dW_1(t) \\ &= (\alpha + \beta \lambda Y(t))S(t)dt + \sigma dW_1(t) + S(t-)dQ(t) \end{aligned} \quad (8.1)$$

and

$$dY(t) = (c_1 + c_2 \sin(2\pi t) - aY(t))dt + b dW_2(t) \quad (8.2)$$

where $\text{corr}(dW_1(t), dW_2(t)) = \rho dt$.

8.1.1 Parameter Estimation

We will first estimate the parameters for the mean-reversion, seasonal component of the Lévy Process.

Therefore, if we discretise equation (8.2) using the Euler-Maruyama method, we get:

$$Y_{t+\Delta t} = (1 - a\Delta t)Y_t + (c_2 \sin(2\pi t) + c_2 \cos(2\pi t) + c_1)\Delta t + b\epsilon_{(2,t)} \quad (8.3)$$

where $\epsilon_{(2,t)}$ is the random error term $\sim N(0, \Delta t)$.

Now let:

$$\begin{aligned} a &= \frac{1 - \alpha_1}{\Delta t} \\ c_2 &= \sqrt{\frac{\alpha_2^2 + \alpha_3^2}{\Delta t^2}} \\ c_1 &= \frac{\alpha_4}{\Delta t} \\ b &= \sqrt{\frac{\sum_{i=1}^N e_2 i^2}{N - 4} \cdot \frac{1}{\Delta t}} \end{aligned}$$

We can apply least square estimation for regression using SAS software with lagged variables to obtain the estimated parameters for the following equation [21]:

$$Y_{t+\Delta t} = \alpha_1 Y_t + \alpha_2 \sin(\pi t) + \alpha_3 \cos(2\pi t) + \alpha_4 + b\epsilon_2 t.$$

Table 8.1: Least Square Parameter Estimation

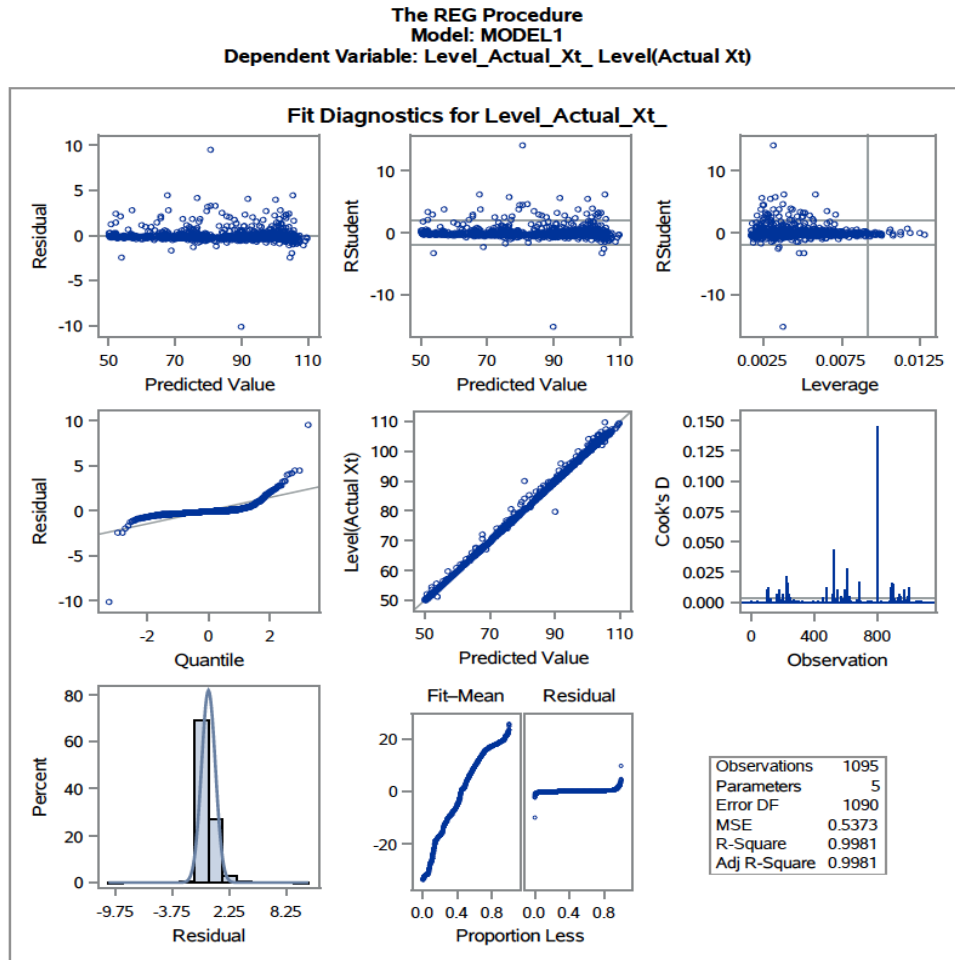
Parameter	Estimate	Parameter	Estimate
α_1	0.9978	a	0.7821
α_2	2.4328	c ₁	-5185.43
α_3	14.257	c ₂	5279.17
α_4	-14.206	b	0.0480
b	0.0480		

With these estimated parameters, and their results on the following page, we can see that the estimates fit well around the model and that all the coefficients are relevant and suitable in the model. They all have a P-Value¹, discussed in Chapter 2, less than 0.05 and the regression graphs are accurate for model representation [21].

Root MSE	0.73300	R-Square	0.9981
Dependent Mean	83.95635	Adj R-Sq	0.9981
Coeff Var	0.87307		

Parameter Estimates						
Variable	Label	DF	Parameter Estimate	Standard Error	t Value	Pr > t
Intercept	Intercept	1	-14.20667	5.67632	-2.50	0.0125
Xt_1	Xt-1	1	0.99786	0.00171	584.35	<.0001
SIN	SIN	1	2.43289	1.03179	2.36	0.0186
COS	COS	1	14.25740	5.65539	2.52	0.0118
E1	E1	1	0.04801	0.02248	2.14	0.0329

¹Definition 2.22.



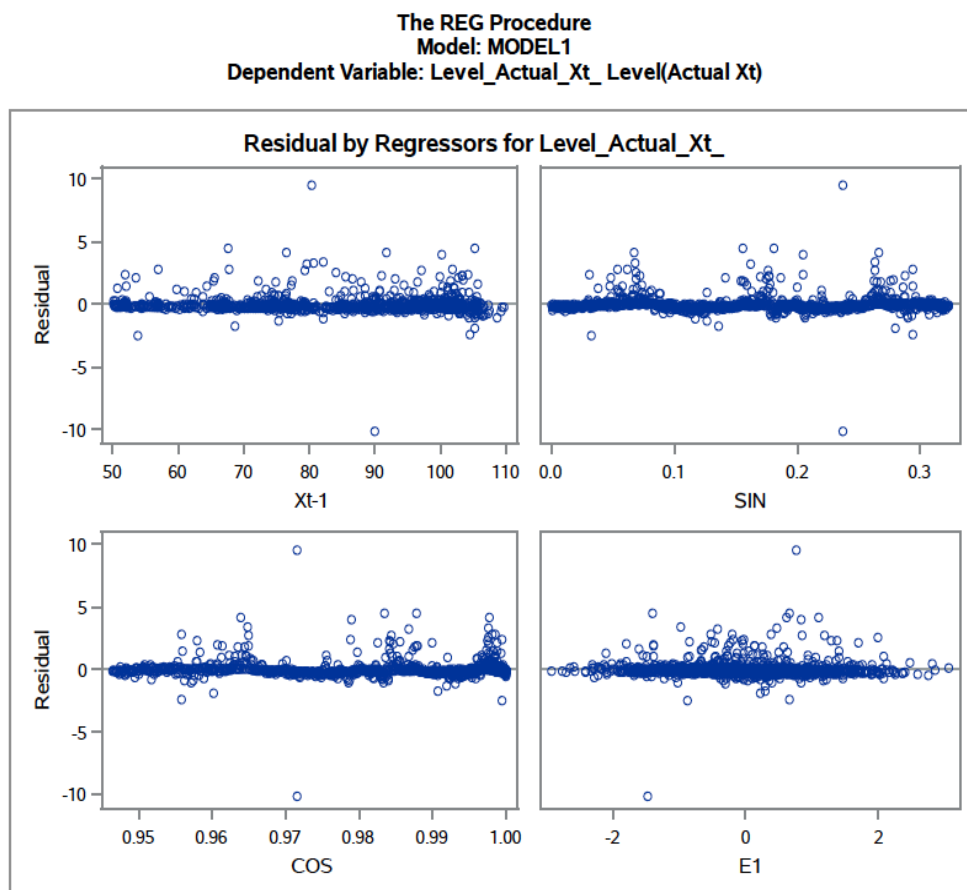
The graphs above, generated from SAS regression analysis [21], show the fit of the parameters to the model during regression. For the first three graphs we can see that most of the observations fall on the same line, and we have very few outliers. This proves an accurate result during regression.

The graph in the lower left corner shows that all results follow an error of a normal distribution with a slight left tail, and this proves that we can use a normally distributed random error term.

The R-square of our regression analysis is very close to a 100%.

R-Square is a goodness of fit measure for regression models, such as ours. This variable measures the strength of the relationship between our chosen estimators and the dependent model on a scale of 0-100% [4].

A R-Square measure of 99% is therefore, a very good result.



The above graphs indicate the residual/outliers of each estimated parameter. As we can see each estimation fell very close nit on the regression line and we do not experience a lot of outliers during our statistical testing.

This indicates that our model estimators are a very good fit and are accurately representing the seasonality and mean-reversion of our model.

8.1.2 Price Process Simulation

If we want to simulate the entire spot price process, we have to discretise (8.1) in order to get:

$$\Delta S_t = (\alpha - \beta \lambda X_t) S_t \Delta t + \sigma S_t \Delta W_{1t} + S(t-) \Delta Q_t \quad (8.4)$$

and then

$$S_{t+1} = (1 + \alpha - \beta \lambda X_t) S_t \Delta t + \sigma S_t \epsilon_1 \Delta t + S(t-) \Delta Q_t$$

where ϵ_1 is the standard normal random variate generated in correlation with ϵ_2 of the seasonal process with correlation coefficient $\rho = 0.01$.

We do not have an existing water price process in South Africa that is priced realistically. Therefore, we have to base our parameters β , λ and σ on the parameters estimated in [20].

These parameters are based on the Australian Water Market that we now assume should be similar to the South African equivalent market. We also assume a risk-free rate of 0.05.

We therefore assume:

Table 8.2: Parameter Estimation Assumed

Parameter	Estimate
β	0.03
σ	0.20
λ	0.05
r	0.05

We will also require $S(0)$ to simulate the spot price from an initial/starting price point. Since South Africa does not have a thorough pricing process we need to make numerous assumptions regarding the initial price of water on the trading market.

These water prices will then be given margins based on the water user that is entering the market as described in Chapter 6.4.

We will assume, $S(0)$ to be R 400 p /kl throughout this dissertation.

8.1.3 Seasonal, Mean-Reverting process simulation results

By using the parameters estimated above and the discretised price process in equation 8.4, we can configure the results of the process by using Matlab [26].

The results are as follows:

Figure 8.1: Cape Town Theewaterskloof Water Levels

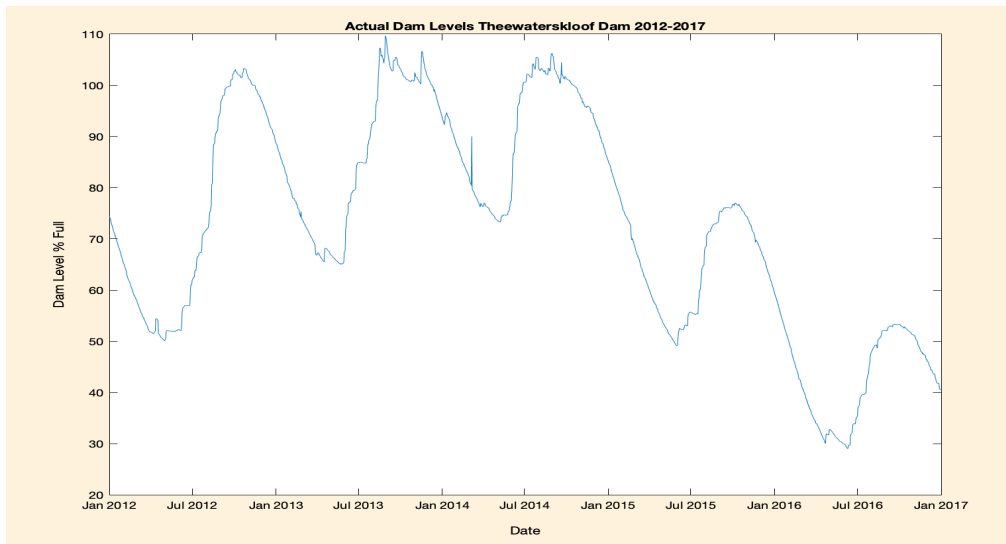
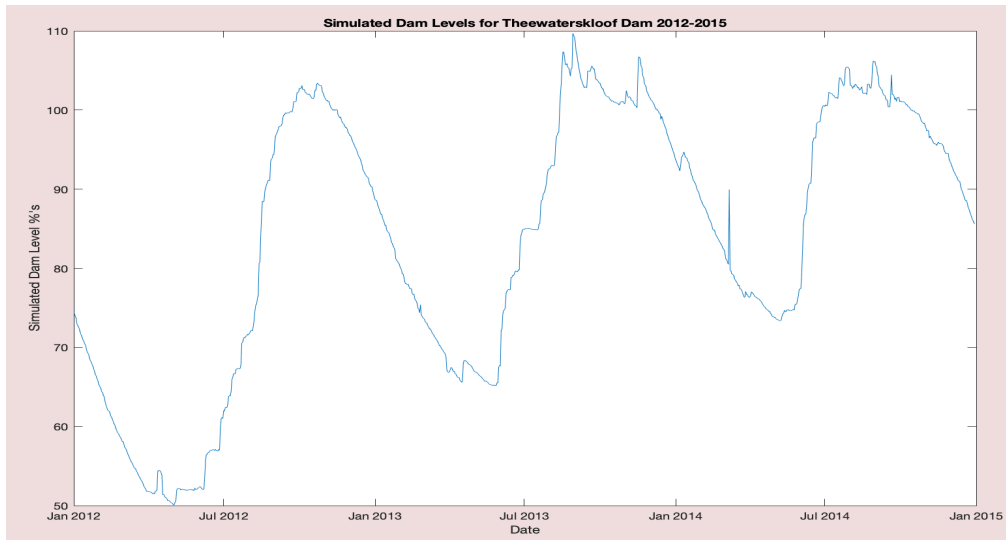


Figure 8.2: Simulated Mean-Reverting Seasonal Component



Comparing the plots in Figures 8.1 & 8.2, we observe that our approximated simulation fits reasonably well with the behaviour of the actual dam water levels over the specified time frame.

We therefore, justify the use of a mean-reverting, seasonal stochastic increment in our final Lévy Process for the pricing structure of water.

8.1.4 Final Lévy Process simulation

For the simulation of the final complete spot price process for water the following assumptions were made:

- **Parameters Estimated**

- $\beta = 0.03$
- $\lambda = 0.05$
- $\sigma = 0.01$

- **Jump sizes**

The size of the jumps are assumed to follow a uniform distribution where the probability and corresponding average jump sizes are as follows: [Modelling of pricing and market impacts]

- (-0.10) with probability (0.2)
- (-0.05) with probability (0.2)
- (0.05) with probability (0.2)
- (0.10) with probability (0.2)
- (0.15) with probability (0.2)

These jump assumptions can be applied and assumed for the South African Market since we experience a warm and tropical climate as most parts of Australia. [3]

8.1.4.1 Simulation process for the final pricing process

To simulate our full Lévy Process we use the compound poisson approximation method as discussed in Schoutens, 2003 [34].

We let S_t be our Lévy Process where:

$$S_{t+1} = (1 + \alpha - \beta \lambda X_t) S_t \Delta t + \sigma S_t \epsilon_1 \Delta t + S(t-) \Delta Q_t. \quad (8.5)$$

Therefore, we have a Lévy Triplet, $[(1 + \alpha - \beta \lambda X_t) S_t, \sigma S_t \epsilon_1, Q_t]$, with a linear discrete process, a brownian motion and a jump process.

Then we choose ϵ such that $0 < \epsilon < 1$ and a partition exists such that $\mathbb{R} \setminus [-\epsilon, \epsilon]$ of real numbers where:

$$a_0 < a_1 < \dots < a_k = \epsilon$$

and

$$\epsilon = a_{k+1} < a_{k+2}, \dots, a_{d+1}.$$

Jumps $> \epsilon$ are approximated by using the summation of poisson processes that are independent of each other. We do this by taking independent poisson processes for each interval with intensity λ_i , given by the Lévy measure of each interval. We assumed all jump intensities to be equal to 1 for all intervals of this part of the process.

In order to allow for random jump sizes we simply have to replace each of the above mentioned jump sizes with their expected values.

8.1.5 Final price process simulation results

We simulated the following processes with $S(0) = 400$, as on pg.77:

Figure 8.3: Stochastic Geometric Brownian Process

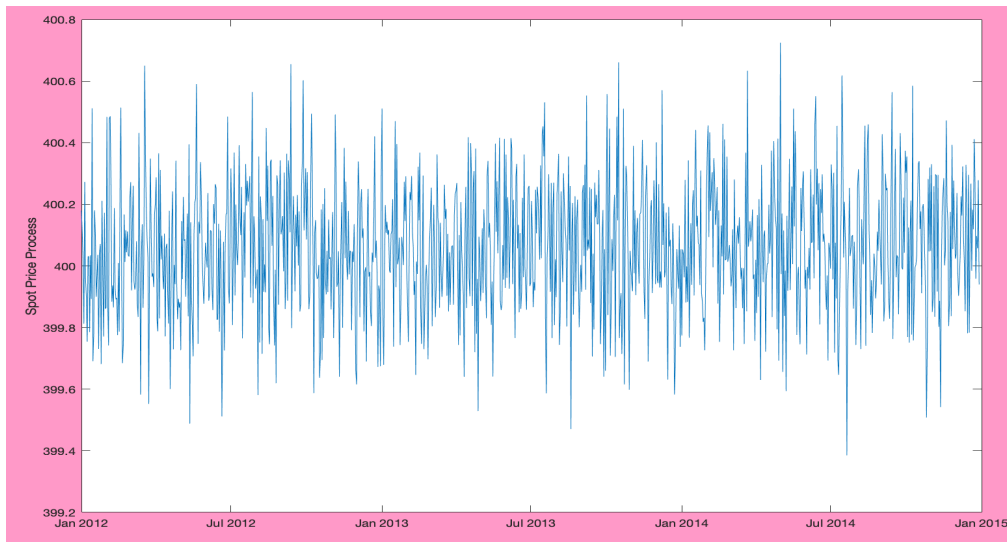
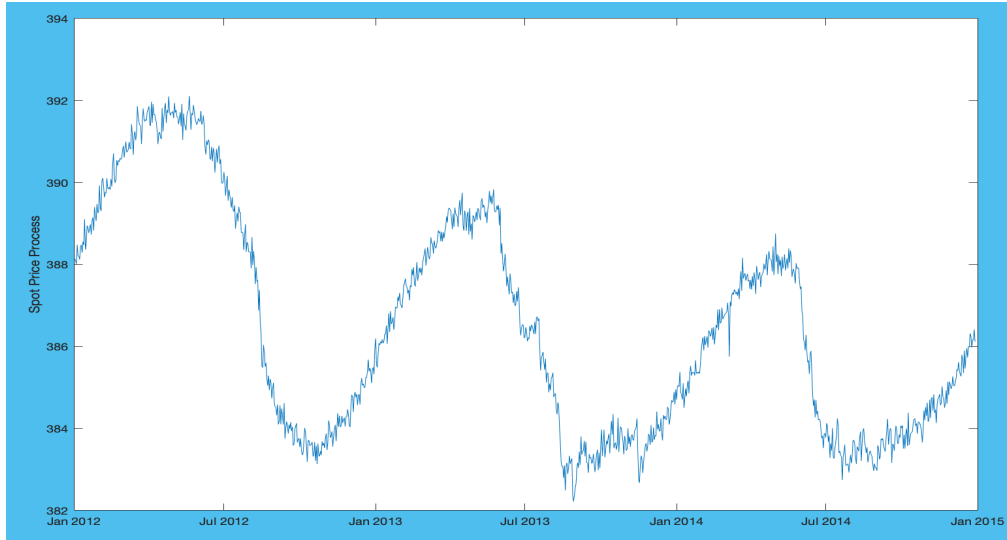


Figure 8.4: Stochastic Process with Mean-Reverting Seasonal Component



On figure 8.3 we see a simple GBM simulation using our estimated parameters. On figure 8.4 we use the same underlying GBM as on figure 8.3 , but we now add our mean-reverting seasonal components.

Figure 8.5: Stochastic Process with Mean-Reverting Seasonal Component & Jump Process

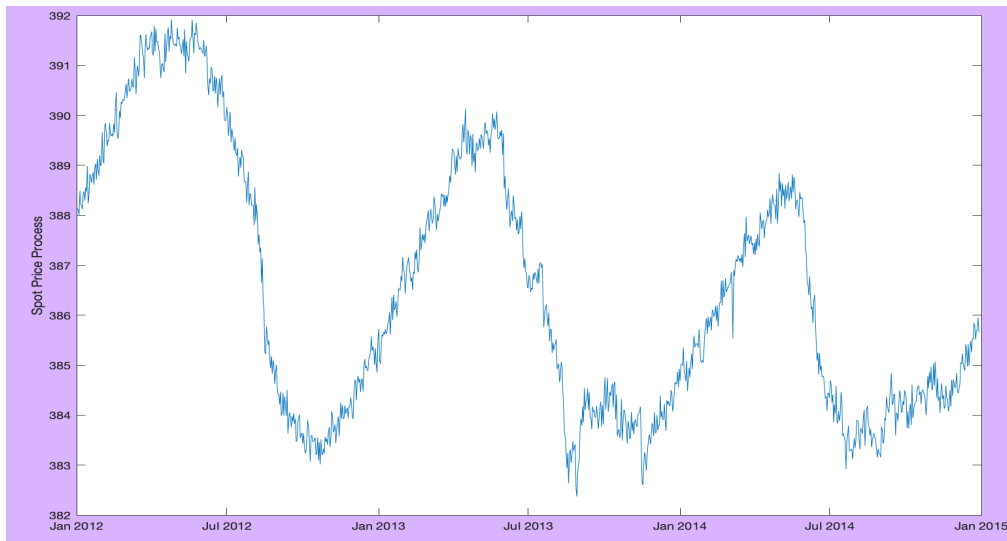


Figure 8.5 is our final model estimation that includes the GBM, the Mean-Reverting Seasonal component as well as the Compound Poisson Jumps.

8.2 Option Price Simulation

We will only price options on water allocation for one season. We assume one season to be 32 weeks, 220 days, as in Schreider & Cui, 2009 [20]. Since seasonality follows a trigonometric pattern, we cannot allow option trading on a cross-seasonal basis. This will increase arbitrage in the market.

From Chapter 7, we know the European Call Option price for water, based on the spot price, is as follows:

$$c(t, S(t)) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+ | F(t)]$$

where $c(t, S(t))$ is the expected value of the discounted payoff of the European Call Option with respect to the new probability measure $\tilde{P}$ and where $\tilde{E}$ is the expectation operator.

When expanding and manipulating the equation, as in Chapter 7.3, we came to the following equation that is in a format that we can use for simulation purposes:

$$\begin{aligned} c(t, S(t)) = & \sum_{j=0}^{\infty} \exp(-\tilde{\lambda}(T-t)) \frac{\tilde{\lambda}_j (T-t)^j}{j!} \cdot \tilde{E}[S(t) \exp\{-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)\}] \\ & \times \prod_{i=N(t)+1}^{N(T)} (Y_i + 1) N(d_1) - K \exp(-r(T-t)) N(d_2) \end{aligned}$$

where:

$$d_2 = \frac{\log\left(\frac{S(t) \exp(-\tilde{\beta}\tilde{\lambda}Y(t)(T-t)) \cdot \prod_{i=N(t)+1}^{N(T)} (Y_i + 1)}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

We can simulate this by using Monte Carlo simulation methods, using 30 000 iterations, and extract the spot price values from the simulated spot prices above.

We will use the following parameters:

Table 8.3: Parameters for Option Price Simulation

Parameter	Estimate	Parameter	Estimate
$S(0)$	400	β	0.03
r	0.05	a	0.7821
σ	0.01	c_1	-5185.43
λ	0.05	c_2	5279.17
Weeks to Maturity	32	b	0.0480
Strike Price	550	Simulations	30 000
Call Prices computed		40	

8.2.1 Option Price Simulation Results

Figure 8.6: European Call Option Price for Spot Price Process

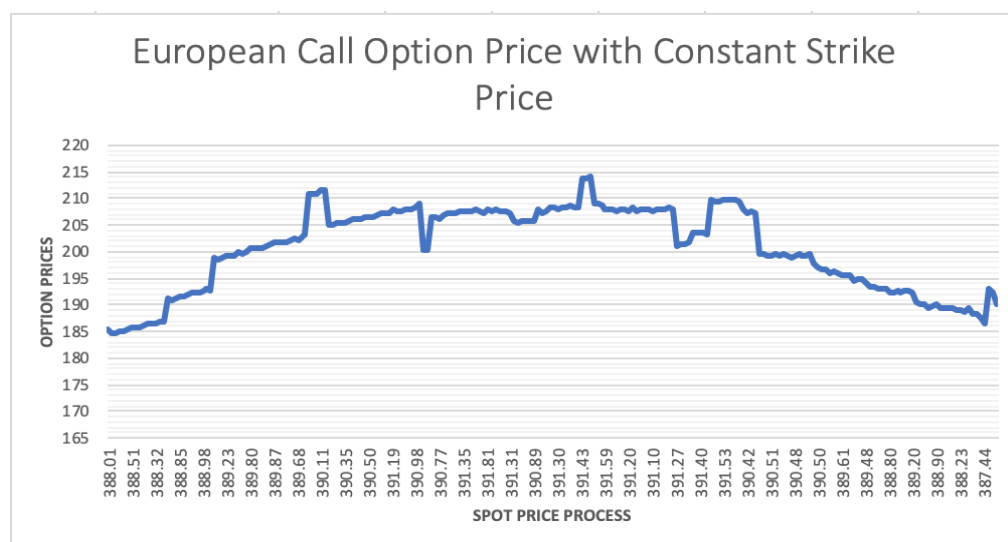


Figure 8.7: European Call Option Price increasing Strike Price



8.3 Summary of simulations and results

We successfully simulated a Lévy price process with a mean-reverting seasonal component as well as a component for the jump process. Figure 8.6 and Figure 8.7 accurately show the results from our simulations.

The parameter estimation for the Stochastic Mean-Reverting Seasonal process was accurate and based on relevant empirical data from the city of Cape Town. A similar analysis can be done for any water supply reservoir in other emerging cities to complete a similar pricing simulation.

In these simulations we made an assumption that the initial spot price of water would be R 400 / kl. This is not based on any relevant data, and more accurate simulations with viable examples will be illustrated in Chapter 8.3.1.

8.3.1 Viability of Water Market

We will make use of an example for a residential water user living in the city bowl of Cape Town.

User A lives in Tamboerskloof in Cape Town. User A surpassed their water allocation and will now need to buy additional water allocation from the centralised water trading framework. **User B** lives in Gardens, Cape Town and followed strict water use protocols during the previous month. User B now has surplus water allocation and decided to sell the allocation to the **centralised water framework**.

At the time of sale the spot price of water is R 36.50/kl. User B has 300 kilo litre surplus water allocation. User B decides to sell this water allocation to the centralised framework. The **municipality** receives a 10% margin on this sale, and User B receives a 5% margin. User A is now allowed to buy this allocation from the centralised framework at a price of **R 41.98/kl = R36.50/kl + 10% + 5%**.

If User A decides to buy 200 kl of the 300 kl available water allocation the benefit for the municipality and User B will be as follows:

- **Municipality**

The municipality will receive an additional **R 730 = (R 36.50/kl * 10%) * 200 kl** for the single trade on the centralised framework. The municipality will use this money for maintenance of infrastructure and the administration duties relating to the water market.

- **User B**

User B will receive **R 365 = (R 36.50/kl * 5%) * 200 kl** for the sale of their surplus water allocation.

If we assume on average 500 residents sell an average of 100 kl of surplus water allocation on the trading platform per month the municipality of Cape Town can make an **income of R 182 500 = (R 36.50/kl * 10%) * 100 kl * 500 Residential Users) per month** only from the residential trades. In addition to this there will be trades for Corporates and other institutions and a completely separate centralised framework for wastewater.

This proves that the water market is a very viable proposal for effective use of a scarce resource with an added financial benefit for both water users and the municipality.

8.4 Impact of options on the water market

As there exists no current water market in South Africa, and the water derivatives market in Australia is not properly embedded in the market the following assumptions are only likely assumptions and can deem to be different in practice.

The most beneficial part of the water derivatives market, especially with the use of options and embedded options, is that water users can increase their leverage in the market without having large amounts of capital. This will allow them to make more use of the profit from such a market.

The option market will allow consumers, that are not part of the actual water trading sector, to take part in the trading of water without physical ownership of water allocation. This gives them similar exposure to the market then a person who owns the underlying asset.

With an increase in market activity the price of water will become less volatile and more trustworthy. The yearly decrease in fresh-water emphasises the need for a market, since water is slowly becoming one of the scarcest commodities worldwide [1].

The option market allows traders and options holders to partially hedge away any adverse water price movements in advance and in unison profiting from a favourable movement in the market. This will encourage the sustainable use of a very scarce resource and allow the municipalities of emerging cities to gain access to additional funds to increase infrastructure and advocate for a sustainable future for water.

We should remember that the exact effect of the introduction of water derivatives into the market cannot be established thus far. Nevertheless, I believe that it will have a positive effect, maybe not right away, but definitely in our water scarce future.

8.5 Chapter Summary

From the simulations we performed on the Spot Price Process and the European Call Option process the results were accurate and we concluded that our initial estimation of parameters were efficient and accurate for model simulation purposes.

The Spot Price process follows the mean-reversion to the seasonal function and jumps are visible in the Option and Spot Price processes.

The Option price process follows our initial hypothesis that it will reach a maximum mid season and decrease as the season ends. The Option price also decreases as expected with an increase in Strike Price.

The findings from the simulation justifies our methods and assumptions.

Chapter 9

Hedging in the incomplete water market

The research done by Wang, 2000 [42], marked the water market as a "quasi market" or an incomplete market. From the book by T. Björk, 2009 [12], a complete market is defined as a market in which every contingent claim can be replicated, therefore completely hedged. It is also mentioned that such a market will have a unique martingale measure with a unique arbitrage free price for a given claim.

From our derivations in Chapter 7 as well as our understanding of the water markets, we can obtain the conclusion that in our model we will experience numerous martingale measures due to the randomness of the Q . We will therefore, not experience a unique arbitrage free price for each claim.

In our research we saw that we are dealing with more random sources than traded assets and the underlying object, water, are in some cases un-tradable which are both key aspects of an incomplete market [12].

In cases where we are dealing with an incomplete market we need to assume that there exists a liquid market for each contingent claim such that there exists no arbitrage possibilities so that we are allowed to trade the contract/derivative. Due to the randomness and incompleteness of our water market simulation we will not be able to replicate our claims exactly and therefore cannot expect to obtain a unique arbitrage free price process. We therefore, experience numerous martingale measures and numerous market prices of risk.

In order to tackle this problem, we had to fix a specific martingale measure and determine a specific market measure of risk. Usually market measure of risk is determined by the market itself, but due to the fact that we have a non-existent market we had to fix and choose this measure ourselves. (Chapter 7)

Our martingale measure (Market Price of Risk), was derived in section 7.2. This measure was deemed appropriate and can be used for hedging part of the risk. The measure was derived and calculated as:

$$r + \sigma\theta - \beta\lambda Y(t).$$

In an existing market it is important to mention that the determination of the market price of risk is not a theoretical one, but an empirical one. We are not expecting this market to work in such a way that a user can completely hedge out his risk with regards to the availability of water. We will only use this market as a method for partial hedging of risk and in order to address the more resourceful use and allocation of water. This market will be able to alleviate the losses from drought and water scarcity.

9.1 Arbitrage Theory

Numerous methods can be employed, apart from our simple solution in Chapter 7.2, to construct a risk neutral measure for a pricing process in an incomplete market.

Risk-neutral measures are also referred to as equivalent martingale measures and martingales are therefore, very important in our pricing process.

9.1.1 Martingale Measures for Lévy Processes

Since our Lévy Process does not have a PRP ¹, our market model is incomplete and we have different equivalent risk neutral/martingale measures [34].

We can find a martingale measure, although it will not be unique, by using an Esscher Transform² or a mean-correcting method. In this dissertation, in Chapter 7.2, we made use of the latter, but the choice of rather using an Esscher Transform can be justified by an optimisation argument as Gerber and Shiu in [44].

The Esscher Transform is a generalisation of the Girsanov Transform³ [34].

¹Definition 2.17.

²Theorem 2.27.

³Through Theorem 2.26.

Esscher Transform

Let $f_t(x)$ be the density of our model's distribution of X_t . For some real number $\theta \in \{\theta \in \mathbb{R} \mid \int_{-\infty}^{+\infty} \exp(\theta y) f_t(y) dy < \infty\}$ we can define a new density

$$f_t^{(\theta)}(x) = \frac{\exp(\theta x) f_t(x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f_t(y) dy}$$

Now we choose θ such that the discounted price process $\{\exp(-(r-q)t)S_t, t \geq 0\}$ is a martingale, i.e.

$$S_0 = [\exp(-(r-q)t)E^{(\theta)}|S_t], \quad (9.1)$$

where expectation is taken with respect to the law with density $f_t^{(\theta)}(x)$. Let $\phi(u) = E[\exp(iuX_1)]$ denote the characteristic function of X_1 . Then from equation 9.1 it can be shown that in order to let the discounted price process to be a martingale, we need to have

$$\exp(r-q) = \frac{\phi(-i(\theta+1))}{\phi(-i\theta)}. \quad (9.2)$$

The solution of this equation, θ^* say, gives us the Esscher transform martingale measure through the density function $f_t^{(\theta^*)}(x)$.

Björk, 2009 [12].

Chapter 10

Conclusion

South Africa experiences similar water scarcity problems then can be seen in Australia. The past couple of years emphasised the need for better protocols and systems to ensure sustainable use and allocation of water in the future as well as the need for a measure to reduce the risk of water scarcity.

We have seen massive amounts of economic losses in the recent droughts across South Africa. Australia managed to set up a working water market and the use of the market increased vastly since its adoption in the early 2000's. The use of options and derivatives in this market is still a work in progress and numerous research has been done to ensure the addition of derivatives in the Australian water market as soon as possible. The use of these derivatives allow for a more stable and user-friendly water market.

We use some of the research and data derived for Australia to simulate a water market scenario for South Africa, with the hope that a water market similar to the Australian market can benefit South Africa in the future. We made some assumptions for the sake of this project that limits the use of the simulated data, but we will focus on decreasing these limitations in a follow up research project.

The data analysis showed numerous jumps, seasonality and mean-reversion. We therefore, integrated a compound poisson process and a mean-reverting, seasonal stochastic process to create a Lévy Process that will accurately represent the price of water.

In the graphs we compared the use of a pure geometric Brownian motion for water as well as the Lévy Process. You can immediately see the changes in the simulated process by the implementation

of the latter. The simulated Lévy Process had a great fit to the profile of the water supply historical data and we therefore, could justify the accuracy and reliability of our parameter estimation.

The empirical evidence on the water price series suggests that there is large variation in the water prices even for a specific season.

The water prices will mostly be determined by supply and demand and therefore manual price adjustments in the market will be impractical. For this reason, the use of derivatives in the market can be utilised to ensure water price certainty. The European call option price for the water market experience numerous jumps, seasonality and mean-reversion as it follows the path of the water price closely. We can also see that the option price will decrease exponentially towards zero as the strike price increases.

Our model assumes constant volatility and a constant risk-free rate, but further research can be introduced with the use of stochastic volatility.

We used Matlab [26], to simulate the numerous variables and then made use of Excel [17], to perform the graphs and final option and water price calculations. SAS was used for calibration of the model and parameter estimation.

In follow up research, the dependence on weather data will be researched further as it is a trivial argument that the dependence of a water market on the above-mentioned factor is much more than assumed in this project. Cross-seasonal derivatives will have to be examined further in the future.

The use of a water market and derivatives have numerous advantages to emerging cities. The water market can operate as a form of a risk reducer for water scarcity and droughts and allow financial benefit to municipalities to improve water infrastructure.

The market encompasses some negative aspects as there will exist quite a large information gap for the users, especially for those with little to no market experience. In a young market, such as the water market, there is a potential for water price distortion as the supply and demand cannot be assumed perfectly before the release of the market. There is no knowledge of the amount of participation in the derivatives market vs the actual trade market.

There is a potential for speculative activity in the market and the government and financial institutions will have to monitor the market activity closely. The model suffers from market incompleteness as mentioned in Chapter 10. The incompleteness proposes a threat to the use of the water market model, but since most market scenarios are incomplete in practice, our model is not

inferior to most, and can still prove to be reliable and useful.

The strike price can be chosen in such a way to best suite user's risk appetite. We suggested in the study that the market has to be closely examined for an initial period before option trading takes place in order to determine a realistic strike pricing.

We assumed that the maturity date of the option will be before the end of the season to address the possibility of speculation of a known drop in water prices at the end of the season. This eliminates arbitrage as a pattern cannot be seen. This assumption also allows for a more stable market and stable water option price.

In the future a more realistic price can be determined by making use of macroeconomic analysis.

The use of water markets and water derivatives will have major advantages on emerging markets and we will experience a greater need for such a trading market as water scarcity increases worldwide. We emphasise the fact that the introduction of water markets will not necessarily reduce the use of water significantly, but that it will help us control the uses of water and ensure that it is used in the most economical and resourceful way. This emphasises the goal of the proposal to be an advocate for sustainability of a scarce resource and not for financial gain.

Appendix

1. Code

1.1 MATLAB Code [26]

Figure 10.1: Brownian Motion Simulation

```

randn('state',100)

T= 32;
N=32;

delta_t=1;
W = zeros(1,N);
dW = zeros(1,N);
W(1)=0;
dW(2) = W(1) + sqrt(delta_t)*randn;      % first approximation outside
the loop ...
W(2) = dW(1); % since W(0) = 0 is not allowed

k=[N,];
i=1;

for j = 3:N
    dW(j) = sqrt(delta_t)*randn;      % general increment
    W(j) = W(j-1) + dW(j);
    k(i,:)=W(j);
    i=i+1;
end

plot([0:delta_t:T],[0,W],'r-')      % plot W against t
xlabel('t','FontSize',16)
ylabel('W(t)','FontSize',16,'Rotation',0)

```

Figure 10.2: Poisson Process Simulation

```

lambda=5;
T=32;
PRV=lambda*T;
Prob_Range = (-0.1*0.2) + (-0.05 * 0.2) + (0.05 * 0.2) + (0.1 * 0.2) +
(0.15 * 0.2);

R = poissrnd(PRV); %number jumps

u = rand(R,1); %Uniform random variables for each jump

Jump_P =u * T; %Jump Occurences according to Poisson

positivenegativerandom = (rand(R,1) * 2) - 1; %Insure no jumps to
negative values

Jump_Size = positivenegativerandom * Prob_Range ; %Jump sizes for
each occurence
Jump_Size
Jump_Points = sort(Jump_P);
Jump_Points

plot(Jump_Points,Jump_Size,'r-') % plot W against t
xlabel('Jump Occurences','FontSize',16)
ylabel('Jump Size','FontSize',16,'Rotation',0)

```

1.2 VBA Code [17]

Figure 10.3: Integration of Brownian and Compound Poisson Process

```

Option Explicit

Private Sub CommandButton1_Click()
    Call mysub
End Sub

Sub mysub() 'Divides the timeframe 0-32 weeks in a chosen amount of intervals (in our case 1000000)
    Dim i As Long
    Range("C2:C" & Range("B:B").End(xlDown).Row).Clear
    Range("A2:A" & Range("A:A").End(xlDown).Row).Clear
    Range("B2:B" & Range("B:B").End(xlDown).Row).Clear

    For i = 1 To 5
        Range("A" & i + 1) = i - 1
    Next i

    Range("A2:A6").Select
    Selection.AutoFill Destination:=Range("A2:A" & Range("H1") + 2)

    Range("B2") = 0
    Range("B3") = "=B2 + 32/$H$1"
    Range("B3").Select
    Selection.AutoFill Destination:=Range("B3:B" & Range("H1") + 2)

    MsgBox "Done"
End Sub

```

```
Sub mysub2() 'Attached a 0 for no jump and a 1 for a jump occurence during the time interval 0 - 32 weeks
    Dim i As Long
    Dim found As Range

    Range("C2") = 0
    Range("C2").Select
    Selection.AutoFill Destination:=Range("C2:C" & Range("H1") + 2)

    For i = 2 To Range("I:I").End(xlDown).Row
        Set found = Nothing
        Set found = Range("B2:B" & Range("H1") + 2).Find(Range("I" & i), LookIn:=xlValues, Lookat:=xlPart)

        If Not found Is Nothing Then
            Range("C" & found.Row) = 1
            Range("J" & i) = ""
        Else
            Range("J" & i) = "Not Found"
        End If
    Next i

    MsgBox "Done"
End Sub

Private Sub CommandButton2_Click()
    Call mysub2
End Sub
```

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