

SUPPLEMENTARY

Advances in Pólya-Aeppli distributions and applications

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Submitted in fulfillment of the requirements for the degree

Philosophiae Doctor

in the [Faculty of Natural and Agricultural Sciences](#)

[University of Pretoria](#)

July 30, 2025



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Chapter 3

Type I bivariate Pólya-Aeppli distributions

In this supplementary material for Chapter 3, it is shown how the MLEs for the Type I bivariate Pólya-Aeppli distributions, referred to in Chapter 3, were obtained. The following results will also be used throughout this supplementary material.

Let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3) and let $x_1 = \frac{-\lambda_1(1-\rho)}{\rho}$. It follows that (A.9) can be written as

$$L_{n-2}^2(x_1) = \frac{n\rho}{\lambda_1(1-\rho)}L_{n-1}^0(x_1) - \frac{\rho}{\lambda_1(1-\rho)}L_{n-1}^1(x_1) \quad (\text{S.3.0.1})$$

and (A.11) can be written as

$$L_{n-2}^3(x) = \frac{(n+1)\rho}{\lambda(1-\rho)}L_{n-1}^1(x) - \frac{2\rho}{\lambda(1-\rho)}L_{n-1}^2(x). \quad (\text{S.3.0.2})$$

3.3 Type I bivariate Pólya-Aeppli distributions

3.3.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 3.3.5 for the BPA_I distribution. The Newton-Raphson algorithm uses the equation in (3.16) for $t \geq 1$ and the MoM estimates in Subsection 3.3.4 are used as initial values of the parameters $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}$ and $\rho^{(0)}$ for the BPA_I distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{\text{th}}$ iterate values is reached or until a specified maximum number of iterations is reached (Balakrishnan et al. [1]).

In order to calculate the gradient vector ∇F in (3.14) and the Hessian matrix in (3.15), such that the Newton-Raphson algorithm in (3.16) can be utilized, the first and second derivatives for each element of the *pmf* in (3.9) need to be calculated with respect to each of the parameters: $\lambda_1, \lambda_2, \lambda_3$ and ρ .

The results of the calculation of the first and second derivatives of the *pmf* in (3.9) for the BPA_I distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f(0, 0)$, $f(n_1, 0)$ where $n_1 \in \{1, 2, \dots\}$, $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$, and $f(n_1, n_2)$ where $\{n_1, n_2\} \in \{1, 2, \dots\}$ with respect to each of the parameters: $\lambda_1, \lambda_2, \lambda_3$ and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_1 = \frac{-\lambda_1(1-\rho)}{\rho}$, $x_2 = \frac{-\lambda_2(1-\rho)}{\rho}$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
 g_1 &= \frac{\partial f(n_1, n_2)}{\partial \lambda_1} + f(n_1, n_2) \\
 &= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
 &\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
 &\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0), \\
 &\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
 \end{aligned} \tag{S.3.0.3}$$

$$\begin{aligned}
 g'_{1,1} &= \frac{\partial g_1}{\partial \lambda_1} \\
 &= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1^2} + \frac{\partial f(n_1, n_2)}{\partial \lambda_1} \\
 &= \left[\frac{(1-\rho) n_1}{\rho \lambda_1} \right] \left[\frac{\lambda_1 \lambda_2 \rho^{n_1+n_2-2} (1-\rho)^2}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
 &\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho)^2 L_{n_1-r-1}^{r+1}(x_1)}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
 &\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - \frac{1}{\rho} g_1, \\
 &\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
 \end{aligned} \tag{S.3.0.4}$$

$$\begin{aligned}
g_2 &= \frac{\partial f(n_1, n_2)}{\partial \lambda_2} + f(n_1, n_2) \\
&= \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho) L_{n_2-r-1}^r(x_2)}{\rho L_{n_2-r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.5}$$

$$\begin{aligned}
g'_{2,2} &= \frac{\partial g_2}{\partial \lambda_2} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_2^2} + \frac{\partial f(n_1, n_2)}{\partial \lambda_2} \\
&= \left[\frac{(1-\rho) n_2}{\rho \lambda_2} \right] \left[\frac{\lambda_1 \lambda_2 \rho^{n_1+n_2-2} (1-\rho)^2}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho)^2 L_{n_2-r-1}^{r+1}(x_2)}{\rho^2 L_{n_2-r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - \frac{1}{\rho} g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g_3 &= \frac{\partial f(n_1, n_2)}{\partial \lambda_3} + f(n_1, n_2) \\
&= \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-1}}{(r-1)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) f(0, 0), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.6}$$

$$\begin{aligned}
g'_{3,3} &= \frac{\partial g_3}{\partial \lambda_3} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3^2} + \frac{\partial f(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=2, n_1, n_2 \neq 1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) f(0, 0) - g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.7}$$

$$\begin{aligned}
g'_{1,2} &= \frac{\partial g_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} + \frac{\partial f(n_1, n_2)}{\partial \lambda_2} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^r (1-\rho)^2}{r! \rho^2} \frac{L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2)}{L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2)} \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.8}$$

$$\begin{aligned}
g'_{2,1} &= \frac{\partial g_2}{\partial \lambda_1} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_2 \partial \lambda_1} + \frac{\partial f(n_1, n_2)}{\partial \lambda_1} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^r (1-\rho)^2}{r! \rho^2} \frac{L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2)}{L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2)} \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.9}$$

$$\begin{aligned}
g'_{1,3} &= \frac{\partial g_1}{\partial \lambda_3} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} + \frac{\partial f(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.10}$$

$$\begin{aligned}
g'_{3,1} &= \frac{\partial g_3}{\partial \lambda_1} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3 \partial \lambda_1} + \frac{\partial f(n_1, n_2)}{\partial \lambda_1} \\
&= \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.11}$$

$$\begin{aligned}
g'_{2,3} &= \frac{\partial g_2}{\partial \lambda_3} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_2 \partial \lambda_3} + \frac{\partial f(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_2-r-1}^r(x_2)}{\rho L_{n_2-r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.12}$$

$$\begin{aligned}
g'_{3,2} &= \frac{\partial g_3}{\partial \lambda_2} \\
&= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3 \partial \lambda_2} + \frac{\partial f(n_1, n_2)}{\partial \lambda_2} \\
&= \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_2-r-1}^r(x_2)}{\rho L_{n_2-r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h_1 &= \frac{\partial f(n_1, 0)}{\partial \lambda_1} + f(n_1, 0) \\
&= \frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} f(n_1, 0), \\
&\quad n_1 \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.13}$$

$$\begin{aligned}
h'_{1,1} &= \frac{\partial h_1}{\partial \lambda_1} \\
&= \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1^2} + \frac{\partial f(n_1, 0)}{\partial \lambda_1} \\
&= \frac{(1-\rho)}{\rho} \frac{n_1}{\lambda_1} f(n_1, 0) - \frac{1}{\rho} h_1, \\
n_1 &\in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.14}$$

$$\begin{aligned}
h_2 &= \frac{\partial f(0, n_2)}{\partial \lambda_2} + f(0, n_2) \\
&= \frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} f(0, n_2), \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{2,2} &= \frac{\partial h_2}{\partial \lambda_2} \\
&= \frac{\partial^2 f(0, n_2)}{\partial \lambda_2^2} + \frac{\partial f(0, n_2)}{\partial \lambda_2} \\
&= \frac{(1-\rho)}{\rho} \frac{n_2}{\lambda_2} f(0, n_2) - \frac{1}{\rho} h_2, \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{1,2} &= \frac{\partial h_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \lambda_2} + \frac{\partial f(n_1, 0)}{\partial \lambda_2} \\
&= -h_1, \\
n_1 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{2,1} &= \frac{\partial h_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f(0, n_2)}{\partial \lambda_2 \partial \lambda_1} + \frac{\partial f(0, n_2)}{\partial \lambda_1} \\
&= -h_2, \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.15}$$

$$\begin{aligned}
h'_{1,3} &= \frac{\partial h_1}{\partial \lambda_3} \\
&= \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \lambda_3} + \frac{\partial f(n_1, 0)}{\partial \lambda_3} \\
&= -h_1, \\
n_1 &\in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.16}$$

$$\begin{aligned}
h'_{2,3} &= \frac{\partial h_2}{\partial \lambda_3} \\
&= \frac{\partial^2 f(0, n_2)}{\partial \lambda_2 \partial \lambda_3} + \frac{\partial f(0, n_2)}{\partial \lambda_3} \\
&= -h_2, \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned}$$

The results for h_2 and $h'_{2,2}$ follow similarly.

It then follows that the first and second derivatives of the pmf in (3.9) are

First derivative with respect to λ_1

$$\begin{aligned}
\frac{\partial f(0, 0)}{\partial \lambda_1} &= -f(0, 0) \\
\frac{\partial f(n_1, 0)}{\partial \lambda_1} &= h_1 - f(n_1, 0), \quad n_1 \in \{1, 2, \dots\}. \\
\frac{\partial f(0, n_2)}{\partial \lambda_1} &= -f(0, n_2), \quad n_2 \in \{1, 2, \dots\}. \\
\frac{\partial f(n_1, n_2)}{\partial \lambda_1} &= g_1 - f(n_1, n_2), \quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.17}$$

First derivative with respect to λ_2

$$\begin{aligned}
\frac{\partial f(0, 0)}{\partial \lambda_2} &= -f(0, 0) \\
\frac{\partial f(n_1, 0)}{\partial \lambda_2} &= -f(n_1, 0), \quad n_1 \in \{1, 2, \dots\}. \\
\frac{\partial f(0, n_2)}{\partial \lambda_2} &= h_2 - f(0, n_2), \quad n_2 \in \{1, 2, \dots\}. \\
\frac{\partial f(n_1, n_2)}{\partial \lambda_2} &= g_2 - f(n_1, n_2), \quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.18}$$

First derivative with respect to λ_3

$$\begin{aligned}
\frac{\partial f(0,0)}{\partial \lambda_3} &= -f(0,0) \\
\frac{\partial f(n_1,0)}{\partial \lambda_3} &= -f(n_1,0), \quad n_1 \in \{1,2,\dots\}. \\
\frac{\partial f(0,n_2)}{\partial \lambda_3} &= -f(0,n_2), \quad n_2 \in \{1,2,\dots\}. \\
\frac{\partial f(n_1,n_2)}{\partial \lambda_3} &= g_3 - f(n_1,n_2), \quad \{n_1,n_2\} \in \{1,2,\dots\}.
\end{aligned} \tag{S.3.0.19}$$

First derivative with respect to ρ

$$\begin{aligned}
\frac{\partial f(0,0)}{\partial \rho} &= 0 \\
\frac{\partial f(n_1,0)}{\partial \rho} &= \frac{n_1}{\rho} f(n_1,0) - \frac{\lambda_1 h_1}{(1-\rho)\rho}, \quad n_1 \in \{1,2,\dots\}. \\
\frac{\partial f(0,n_2)}{\partial \rho} &= \frac{n_2}{\rho} f(0,n_2) - \frac{\lambda_2 h_2}{(1-\rho)\rho}, \quad n_2 \in \{1,2,\dots\}. \\
\frac{\partial f(n_1,n_2)}{\partial \rho} &= \frac{(n_1+n_2)}{\rho} f(n_1,n_2) - \frac{\lambda_1 g_1}{(1-\rho)\rho} - \frac{\lambda_2 g_2}{(1-\rho)\rho} - \frac{2\lambda_3 g_3}{(1-\rho)\rho}, \\
&\quad \{n_1,n_2\} \in \{1,2,\dots\}.
\end{aligned} \tag{S.3.0.20}$$

Second derivative with respect to λ_1

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \lambda_1^2} &= f(0,0) \\
\frac{\partial^2 f(n_1,0)}{\partial \lambda_1^2} &= h'_{1,1} - h_1 + f(n_1,0), \quad n_1 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(0,n_2)}{\partial \lambda_1^2} &= f(0,n_2), \quad n_2 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(n_1,n_2)}{\partial \lambda_1^2} &= g'_{1,1} - g_1 + f(n_1,n_2), \quad \{n_1,n_2\} \in \{1,2,\dots\}.
\end{aligned}$$

Second derivative with respect to λ_2

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \lambda_2^2} &= f(0,0) \\ \frac{\partial^2 f(n_1,0)}{\partial \lambda_2^2} &= f(n_1,0), \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \lambda_2^2} &= h'_{2,2} - h_2 + f(0,n_2), \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \lambda_2^2} &= g'_{2,2} - g_2 + f(n_1,n_2), \quad \{n_1,n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to λ_3

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \lambda_3^2} &= f(0,0) \\ \frac{\partial^2 f(n_1,0)}{\partial \lambda_3^2} &= f(n_1,0), \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \lambda_3^2} &= f(0,n_2), \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \lambda_3^2} &= g'_{3,3} - g_3 + f(n_1,n_2), \quad \{n_1,n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to λ_1 and λ_2

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \lambda_1 \partial \lambda_2} &= f(0,0) \\ \frac{\partial^2 f(n_1,0)}{\partial \lambda_1 \partial \lambda_2} &= f(n_1,0) - h_1, \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \lambda_1 \partial \lambda_2} &= f(0,n_2) - h_2, \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \lambda_1 \partial \lambda_2} &= g'_{1,2} - g_2 + f(n_1,n_2), \quad \{n_1,n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to λ_1 and λ_3

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \lambda_1 \partial \lambda_3} &= f(0,0) \\ \frac{\partial^2 f(n_1,0)}{\partial \lambda_1 \partial \lambda_3} &= f(n_1,0) - h_1, \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \lambda_1 \partial \lambda_3} &= f(0,n_2), \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \lambda_1 \partial \lambda_3} &= g'_{1,3} - g_3 + f(n_1,n_2), \quad \{n_1, n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to λ_2 and λ_3

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \lambda_2 \partial \lambda_3} &= f(0,0) \\ \frac{\partial^2 f(n_1,0)}{\partial \lambda_2 \partial \lambda_3} &= f(n_1,0), \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \lambda_2 \partial \lambda_3} &= f(0,n_2) - h_2, \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \lambda_2 \partial \lambda_3} &= g'_{2,3} - g_3 + f(n_1,n_2), \quad \{n_1, n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to ρ and λ_1

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \rho \partial \lambda_1} &= 0 \\ \frac{\partial^2 f(n_1,0)}{\partial \rho \partial \lambda_1} &= \frac{n_1}{\rho} \frac{\partial f(n_1,0)}{\partial \lambda_1} - \frac{h_1}{(1-\rho)\rho} - \frac{\lambda_1 h'_{1,1}}{(1-\rho)\rho}, \\ & \quad n_1 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \rho \partial \lambda_1} &= \frac{n_2}{\rho} \frac{\partial f(0,n_2)}{\partial \lambda_1} - \frac{\lambda_2 h'_{2,1}}{(1-\rho)\rho}, \quad n_2 \in \{1,2,\dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \rho \partial \lambda_1} &= \frac{(n_1+n_2)}{\rho} \frac{\partial f(n_1,n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} \\ & \quad - \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}, \\ & \quad \{n_1, n_2\} \in \{1,2,\dots\}.\end{aligned}$$

Second derivative with respect to ρ and λ_2

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \rho \partial \lambda_2} &= 0 \\
\frac{\partial^2 f(n_1,0)}{\partial \rho \partial \lambda_2} &= \frac{n_1}{\rho} \frac{\partial f(n_1,0)}{\partial \lambda_2} - \frac{\lambda_1 h'_{1,2}}{(1-\rho)\rho}, \quad n_1 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(0,n_2)}{\partial \rho \partial \lambda_2} &= \frac{n_2}{\rho} \frac{\partial f(0,n_2)}{\partial \lambda_2} - \frac{h_2}{(1-\rho)\rho} - \frac{\lambda_1 h'_{2,2}}{(1-\rho)\rho}, \\
&\quad n_2 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(n_1,n_2)}{\partial \rho \partial \lambda_2} &= \frac{(n_1+n_2)}{\rho} \frac{\partial f(n_1,n_2)}{\partial \lambda_2} - \frac{g_2}{(1-\rho)\rho} \\
&\quad - \frac{\lambda_1 g'_{1,2}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,2}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,2}}{(1-\rho)\rho}, \\
&\quad \{n_1, n_2\} \in \{1,2,\dots\}.
\end{aligned}$$

Second derivative with respect to ρ and λ_3

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \rho \partial \lambda_3} &= 0 \\
\frac{\partial^2 f(n_1,0)}{\partial \rho \partial \lambda_3} &= \frac{n_1}{\rho} \frac{\partial f(n_1,0)}{\partial \lambda_3} - \frac{\lambda_1 h'_{1,3}}{(1-\rho)\rho}, \quad n_1 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(0,n_2)}{\partial \rho \partial \lambda_3} &= \frac{n_2}{\rho} \frac{\partial f(0,n_2)}{\partial \lambda_3} - \frac{\lambda_2 h'_{2,3}}{(1-\rho)\rho}, \quad n_2 \in \{1,2,\dots\}. \\
\frac{\partial^2 f(n_1,n_2)}{\partial \rho \partial \lambda_3} &= \frac{(n_1+n_2)}{\rho} \frac{\partial f(n_1,n_2)}{\partial \lambda_3} - \frac{2g_3}{(1-\rho)\rho} \\
&\quad - \frac{\lambda_1 g'_{1,3}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1-\rho)\rho}, \\
&\quad \{n_1, n_2\} \in \{1,2,\dots\}.
\end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned} \frac{\partial^2 f(0,0)}{\partial \rho^2} &= 0 \\ \frac{\partial^2 f(n_1,0)}{\partial \rho^2} &= \left[\frac{n_1-1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1,0)}{\partial \rho} \\ &\quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1,0)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1,0)}{\partial \rho} \right] - \frac{n_1}{(1-\rho)\rho} f(n_1,0), \\ &\quad n_1 \in \{1, 2, \dots\}. \\ \frac{\partial^2 f(0,n_2)}{\partial \rho^2} &= \left[\frac{n_2-1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(0,n_2)}{\partial \rho} \\ &\quad - \frac{\lambda_2}{(1-\rho)\rho} \left[\frac{\partial^2 f(0,n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f(0,n_2)}{\partial \rho} \right] - \frac{n_2}{(1-\rho)\rho} f(0,n_2), \\ &\quad n_2 \in \{1, 2, \dots\}. \\ \frac{\partial^2 f(n_1,n_2)}{\partial \rho^2} &= \left[\frac{n_1+n_2-1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1,n_2)}{\partial \rho} \\ &\quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1,n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1,n_2)}{\partial \rho} \right] \\ &\quad - \frac{\lambda_2}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1,n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f(n_1,n_2)}{\partial \rho} \right] \\ &\quad - \frac{2\lambda_3}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1,n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f(n_1,n_2)}{\partial \rho} \right] \\ &\quad - \frac{(n_1+n_2)}{(1-\rho)\rho} f(n_1,n_2), \\ &\quad \{n_1, n_2\} \in \{1, 2, \dots\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (3.9) is given here.

First derivative with respect to λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (3.9), it follows that

$$\frac{\partial f(0,0)}{\partial \lambda_1} = -f(0,0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.13) and (S.3.0.17) and using (S.3.0.1), it follows that

$$\begin{aligned}
 & \frac{\partial f(n_1, 0)}{\partial \lambda_1} \\
 &= \frac{(1-\rho)\rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) f(0, 0) + \frac{\lambda_1(1-\rho)^2\rho^{n_1-2}}{n_1} L_{n_1-2}^2(x_1) f(0, 0) \\
 & \quad + \frac{f(n_1, 0)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_1} \\
 &= \frac{\lambda_1(1-\rho)^2\rho^{n_1-2}}{n_1} \left[\frac{n_1\rho}{\lambda_1(1-\rho)} L_{n_1-1}^0(x_1) - \frac{\rho}{\lambda_1(1-\rho)} L_{n_1-1}^1(x_1) \right] f(0, 0) \\
 & \quad - f(n_1, 0) + \frac{f(n_1, 0)}{\lambda_1} \\
 &= \frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} f(n_1, 0) - f(n_1, 0) \\
 &= h_1 - f(n_1, 0).
 \end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (3.9) and (S.3.0.17), it follows that

$$\begin{aligned}
 & \frac{\partial f(0, n_2)}{\partial \lambda_1} \\
 &= \frac{f(0, n_2)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_1} \\
 &= -f(0, n_2).
 \end{aligned}$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.3) and (S.3.0.17) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, n_2)}{\partial \lambda_1} \\
&= \frac{1}{\lambda_1} \left[\frac{\lambda_1 (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) \right] \left[\frac{\lambda_2 (1-\rho) \rho^{n_2-1}}{n_2} L_{n_2-1}^1(x_2) \right] f(0,0) \\
&+ \left[\frac{\lambda_1 (1-\rho)^2 \rho^{n_1-2}}{n_1} L_{n_1-2}^2(x_1) \right] \left[\frac{\lambda_2 (1-\rho) \rho^{n_2-1}}{n_2} L_{n_2-1}^1(x_2) \right] f(0,0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} (1-\rho)^{r+1} \rho^{n_1-r-1} L_{n_1-r-1}^r(x_1) \\
&\times \left[(1-\rho)^r \rho^{n_2-r} L_{n_2-r}^{r-1}(x_2) \right] f(0,0) + \frac{f(n_1, n_2)}{f(0,0)} \frac{\partial f(0,0)}{\partial \lambda_1} \\
&= \frac{1}{\lambda_1} \left[\frac{\lambda_1 (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) \right] \left[\frac{\lambda_2 (1-\rho) \rho^{n_2-1}}{n_2} L_{n_2-1}^1(x_2) \right] f(0,0) \\
&+ \left[\frac{\lambda_1 (1-\rho)^2 \rho^{n_1-2}}{n_1} \left[\frac{n_1 \rho}{\lambda_1 (1-\rho)} L_{n_1-1}^0(x_1) - \frac{\rho}{\lambda_1 (1-\rho)} L_{n_1-1}^1(x_1) \right] \right] \\
&\times \left[\frac{\lambda_2 (1-\rho) \rho^{n_2-1}}{n_2} L_{n_2-1}^1(x_2) \right] f(0,0) + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0,0) - f(n_1, n_2) \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0,0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0,0) - f(n_1, n_2) \\
&= g_1 - f(n_1, n_2).
\end{aligned}$$

The first derivative with respect to λ_2 follows similarly.

First derivative with respect to λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (3.9), it follows that

$$\frac{\partial f(0,0)}{\partial \lambda_3} = -f(0,0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.9) and (S.3.0.19), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, 0)}{\partial \lambda_3} \\
&= \frac{f(n_1, 0)}{f(0,0)} \frac{\partial f(0,0)}{\partial \lambda_3} \\
&= -f(n_1, 0).
\end{aligned}$$

The result for the first derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to λ_3 follows in a similar way.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.6) and (S.3.0.19), it follows that

$$\begin{aligned} & \frac{\partial f(n_1, n_2)}{\partial \lambda_3} \\ &= \sum_{r=1}^{\min(n_1, n_2)} \frac{r \lambda_3^{r-1}}{r!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) f(0, 0) \\ & \quad + \frac{f(n_1, n_2)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_3} \\ &= g_3 - f(n_1, n_2). \end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial f(0, 0)}{\partial \rho} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.13) and (S.3.0.20) and using (S.3.0.1), it follows that

$$\begin{aligned} & \frac{\partial f(n_1, 0)}{\partial \rho} \\ &= \left[\frac{(n_1-1)}{\rho} \right] \left[\frac{\lambda_1 (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) \right] f(0, 0) \\ & \quad - \left[\frac{1}{(1-\rho)} \right] \left[\frac{\lambda_1 (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) \right] f(0, 0) \\ & \quad - \left[\frac{\lambda_1 L_{n_1-2}^2(x_1)}{\rho^2 L_{n_1-1}^1(x_1)} \right] \left[\frac{\lambda_1 (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(x_1) \right] f(0, 0) + \frac{f(n_1, 0)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \rho} \\ &= \left[\frac{n_1}{\rho} - \frac{1}{(1-\rho)\rho} \right] f(n_1, 0) \\ & \quad - \left[\frac{\lambda_1}{\rho^2} \left[\frac{n_1 \rho}{\lambda_1 (1-\rho)} L_{n_1-1}^0(x_1) - \frac{\rho}{\lambda_1 (1-\rho)} L_{n_1-1}^1(x_1) \right] \frac{1}{L_{n_1-1}^1(x_1)} \right] f(n_1, 0) \\ &= \left[\frac{n_1}{\rho} - \frac{1}{(1-\rho)\rho} + \frac{1}{(1-\rho)\rho} \right] f(n_1, 0) - \left[\frac{n_1}{(1-\rho)\rho} \frac{L_{n_1-1}^0(x_1)}{L_{n_1-1}^1(x_1)} \right] f(n_1, 0) \\ &= \frac{n_1}{\rho} f(n_1, 0) - \frac{\lambda_1 h_1}{(1-\rho)\rho}. \end{aligned}$$

The result for the first derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.3), (S.3.0.5), (S.3.0.6) and (S.3.0.20) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{2}{(1 - \rho)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{\lambda_1 L_{n_1 - 2}^2(x_1)}{\rho^2 L_{n_1 - 1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{\lambda_2 L_{n_2 - 2}^2(x_2)}{\rho^2 L_{n_2 - 1}^1(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(n_1 + n_2 - 2r)}{\rho} \right] (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{2r}{(1 - \rho)} \right] (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{\lambda_1 L_{n_1 - r - 1}^r(x_1)}{\rho^2 L_{n_1 - r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^r}{r!} \left[\frac{\lambda_2 L_{n_2 - r - 1}^r(x_2)}{\rho^2 L_{n_2 - r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0) + \frac{f(n_1, n_2)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \rho}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{\lambda_1}{\rho^2} \left[\frac{n_1 \rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^0(x_1) - \frac{\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^1(x_1) \right] \frac{1}{L_{n_1 - 1}^1(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{\lambda_2}{\rho^2} \left[\frac{n_2 \rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^0(x_2) - \frac{\rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^1(x_2) \right] \frac{1}{L_{n_2 - 1}^1(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad + \frac{(n_1 + n_2)}{\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{r!} (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \frac{2}{(1 - \rho)\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{r \lambda_3^r}{r!} (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \frac{\lambda_1}{\rho^2} \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^r}{r!} \frac{L_{n_1 - r - 1}^r(x_1)}{L_{n_1 - r}^{r-1}(x_1)} \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - \frac{\lambda_2}{\rho^2} \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^r}{r!} \frac{L_{n_2 - r - 1}^r(x_2)}{L_{n_2 - r}^{r-1}(x_2)} \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} + \frac{1}{(1 - \rho)\rho} + \frac{1}{(1 - \rho)\rho} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{n_1}{(1 - \rho)\rho} \frac{L_{n_1 - 1}^0(x_1)}{L_{n_1 - 1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad - \left[\frac{n_2}{(1 - \rho)\rho} \frac{L_{n_2 - 1}^0(x_2)}{L_{n_2 - 1}^1(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f(0, 0) \\
&\quad + \frac{(n_1 + n_2)}{\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{r!} (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \frac{2\lambda_3}{(1 - \rho)\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-1}}{(r-1)!} (1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) f(0, 0) \\
&\quad - \frac{\lambda_1}{(1 - \rho)\rho} \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1 - \rho) L_{n_1 - r - 1}^r(x_1)}{\rho L_{n_1 - r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - \frac{\lambda_2}{(1 - \rho)\rho} \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^r}{r!} \left[\frac{(1 - \rho) L_{n_2 - r - 1}^r(x_2)}{\rho L_{n_2 - r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r} L_{n_1 - r}^{r-1}(x_1) L_{n_2 - r}^{r-1}(x_2) \right] f(0, 0) \\
&= \frac{(n_1 + n_2)}{\rho} f(n_1, n_2) - \frac{\lambda_1 g_1}{(1 - \rho)\rho} - \frac{\lambda_2 g_2}{(1 - \rho)\rho} - \frac{2\lambda_3 g_3}{(1 - \rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.17), it follows that

$$\frac{\partial^2 f(0, 0)}{\partial \lambda_1^2} = f(0, 0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.13), (S.3.0.14), (S.3.0.17) and using (A.10), it follows that

$$\begin{aligned}
 & \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1^2} \\
 &= \frac{(1-\rho)n_1}{\rho} \frac{L_{n_1-2}^1(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} f(n_1, 0) - \frac{\partial f(n_1, 0)}{\partial \lambda_1} + \frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_1} h_1 \\
 &= \frac{(1-\rho)n_1}{\rho} \frac{L_{n_1-1}^1(x_1) - L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} f(n_1, 0) - \frac{\partial f(n_1, 0)}{\partial \lambda_1} - h_1 \\
 &= \frac{(1-\rho)n_1}{\rho} \frac{1}{\lambda_1} f(n_1, 0) - \frac{1}{\rho} h_1 - \frac{\partial f(n_1, 0)}{\partial \lambda_1} \\
 &= h'_{1,1} - h_1 + f(n_1, 0).
 \end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (S.3.0.17), it follows that

$$\frac{\partial^2 f(0, n_2)}{\partial \lambda_1^2} = f(0, n_2).$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.9), (S.3.0.3), (S.3.0.4), (S.3.0.17) and using (A.10), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1^2} \\
&= \left[\frac{(1-\rho)n_1}{\rho} \frac{L_{n_1-2}^1(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 \rho^{n_1+n_2-2} (1-\rho)^2}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&+ \sum_{r=1, n_1 \neq 1, 2}^{\min(n_1-2, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho)^2 L_{n_1-r-2}^{r+1}(x_1)}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
&\times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&- \frac{\partial f(n_1, n_2)}{\partial \lambda_1} + \frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_1} g_1 \\
&= \left[\frac{(1-\rho)n_1}{\rho} \frac{L_{n_1-1}^1(x_1) - L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \\
&\times \left[\frac{\lambda_1 \lambda_2 \rho^{n_1+n_2-2} (1-\rho)^2}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho)^2 \left[L_{n_1-r-1}^{r+1}(x_1) - L_{n_1-r-1}^r(x_1) \right]}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
&\times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&- \frac{\partial f(n_1, n_2)}{\partial \lambda_1} - g_1 \\
&= \left[\frac{(1-\rho)n_1}{\rho} \frac{1}{\lambda_1} \right] \left[\frac{\lambda_1 \lambda_2 \rho^{n_1+n_2-2} (1-\rho)^2}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho)^2 L_{n_1-r-1}^{r+1}(x_1)}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
&\times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&- \frac{1}{\rho} g_1 - g_1 + f(n_1, n_2) \\
&= g'_{1,1} - g_1 + f(n_1, n_2).
\end{aligned}$$

The second derivative with respect to λ_2 follows similarly.

Second derivative with respect to λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.19), it follows that

$$\frac{\partial^2 f(0, 0)}{\partial \lambda_3^2} = f(0, 0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.19), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, 0)}{\partial \lambda_3^2} \\ &= -\frac{\partial f(n_1, 0)}{\partial \lambda_3} \\ &= f(n_1, 0). \end{aligned}$$

The result for the second derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to λ_3 follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (S.3.0.6), (S.3.0.7) and (S.3.0.19), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3^2} \\ &= \sum_{r=2, n_1, n_2 \neq 1}^{\min(n_1, n_2)} \frac{(r-1)\lambda_3^{r-2}}{(r-1)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) f(0, 0) \\ & \quad + \frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_3} g_3 - \frac{\partial f(n_1, n_2)}{\partial \lambda_3} \\ &= \sum_{r=2, n_1, n_2 \neq 1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) f(0, 0) \\ & \quad - \frac{\partial f(n_1, n_2)}{\partial \lambda_3} - g_3 \\ &= g'_{3,3} - g_3 + f(n_1, n_2). \end{aligned}$$

Second derivative with respect to λ_1 and λ_2

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.18), it follows that

$$\frac{\partial^2 f(0, 0)}{\partial \lambda_1 \partial \lambda_2} = f(0, 0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.13) and (S.3.0.18), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \lambda_2} \\ &= \frac{1}{f(n_1, 0)} \frac{\partial f(n_1, 0)}{\partial \lambda_2} h_1 - \frac{\partial f(n_1, 0)}{\partial \lambda_2} \\ &= f(n_1, 0) - h_1 \end{aligned}$$

The result for the second derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to λ_1 and λ_2 follows in a similar way.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (S.3.0.3), (S.3.0.8) and (S.3.0.18) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \frac{1}{\lambda_2} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho) L_{n_2-2}^2(x_2)}{\rho L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^r}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \left[\frac{(1-\rho) L_{n_2-r-1}^r(x_2)}{\rho L_{n_2-r}^{r-1}(x_2)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - \frac{\partial f(n_1, n_2)}{\partial \lambda_2} + \frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_2} g_1
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \frac{1}{\lambda_2} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \left[\frac{(1-\rho)}{\rho} \left[\frac{n_2 \rho}{\lambda_2 (1-\rho)} L_{n_2-1}^0(x_2) - \frac{\rho}{\lambda_2 (1-\rho)} L_{n_2-1}^1(x_2) \right] \right. \\
&\quad \left. \frac{1}{L_{n_2-1}^1(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^r}{r!} \frac{(1-\rho)^2}{\rho^2} \frac{L_{n_1-r-1}^r(x_1)}{L_{n_1-r}^{r-1}(x_1)} \frac{L_{n_2-r-1}^r(x_2)}{L_{n_2-r}^{r-1}(x_2)} \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - g_1 - g_2 + f(n_1, n_2)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^r (1-\rho)^2}{r! \rho^2} \frac{L_{n_1-r-1}^r(x_1)}{L_{n_1-r}^{r-1}(x_1)} \frac{L_{n_2-r-1}^r(x_2)}{L_{n_2-r}^{r-1}(x_2)} \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - g_1 - g_2 + f(n_1, n_2) \\
&= g'_{1,2} - g_2 + f(n_1, n_2).
\end{aligned}$$

Second derivative with respect to λ_1 and λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.19), it follows that

$$\frac{\partial^2 f(0, 0)}{\partial \lambda_1 \partial \lambda_3} = f(0, 0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.13) and (S.3.0.19), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \lambda_3} \\
&= \frac{1}{f(n_1, 0)} \frac{\partial f(n_1, 0)}{\partial \lambda_3} h_1 - \frac{\partial f(n_1, 0)}{\partial \lambda_3} \\
&= f(n_1, 0) - h_1.
\end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (S.3.0.19), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(0, n_2)}{\partial \lambda_1 \partial \lambda_3} \\
&= -\frac{\partial f(0, n_2)}{\partial \lambda_3} \\
&= f(0, n_2).
\end{aligned}$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (S.3.0.3), (S.3.0.10) and (S.3.0.19), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} \\
&= \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{r \lambda_3^{r-1}}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) \\
&\quad - \frac{\partial f(n_1, n_2)}{\partial \lambda_3} + \frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda_3} g_1 \\
&= \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] f(0, 0) - \frac{\partial f(n_1, n_2)}{\partial \lambda_3} - g_1 \\
&= g'_{1,3} - g_3 + f(n_1, n_2).
\end{aligned}$$

The second derivative with respect to λ_2 and λ_3 follows in a similar way.

Second derivative with respect to ρ and λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f(0, 0)}{\partial \rho \partial \lambda_1} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. From (S.3.0.13) and (S.3.0.14), it follows that

$$\frac{\partial^2 f(n_1, 0)}{\partial \rho \partial \lambda_1} = \frac{n_1}{\rho} \frac{\partial f(n_1, 0)}{\partial \lambda_1} - \frac{h_1}{(1-\rho)\rho} - \frac{\lambda_1 h'_{1,1}}{(1-\rho)\rho}.$$

Let $n_2 \in \{1, 2, \dots\}$. From (S.3.0.15), it follows that

$$\frac{\partial^2 f(0, n_2)}{\partial \rho \partial \lambda_1} = \frac{n_2}{\rho} \frac{\partial f(0, n_2)}{\partial \lambda_1} - \frac{\lambda_2 h'_{2,1}}{(1-\rho)\rho}.$$

Let $n_1, n_2 \in \{1, 2, \dots\}$. From (S.3.0.4), (S.3.0.9) and (S.3.0.11), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \rho \partial \lambda_1} \\
&= \frac{(n_1 + n_2)}{\rho} \frac{\partial f(n_1, n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} - \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}.
\end{aligned}$$

The second derivative with respect to ρ and λ_2 follows in a similar way.

Second derivative with respect to ρ and λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f(0, 0)}{\partial \rho \partial \lambda_3} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. From (S.3.0.16), it follows that

$$\frac{\partial^2 f(n_1, 0)}{\partial \rho \partial \lambda_3} = \frac{n_1}{\rho} \frac{\partial f(n_1, 0)}{\partial \lambda_3} - \frac{\lambda_1 h'_{1,3}}{(1-\rho)\rho}.$$

The result for the second derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ and λ_3 follows similarly.

Let $n_1, n_2 \in \{1, 2, \dots\}$. From (S.3.0.7), (S.3.0.10) and (S.3.0.12), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \rho \partial \lambda_3} \\ &= \frac{(n_1 + n_2)}{\rho} \frac{\partial f(n_1, n_2)}{\partial \lambda_3} - \frac{2g_3}{(1-\rho)\rho} - \frac{\lambda_1 g'_{1,3}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f(0, 0)}{\partial \rho^2} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.13), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, 0)}{\partial \rho^2} \\ &= \frac{n_1}{\rho} \frac{\partial f(n_1, 0)}{\partial \rho} - \frac{n_1}{\rho^2} f(n_1, 0) + \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \lambda_1 h_1 \\ & \quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1, 0)}{\partial \rho} \right] \\ &= \left[\frac{n_1 - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1, 0)}{\partial \rho} - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, 0)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1, 0)}{\partial \rho} \right] \\ & \quad - \frac{n_1}{(1-\rho)\rho} f(n_1, 0). \end{aligned}$$

The result for the second derivative of $f(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ follows similarly.

Let $\{n_1, n_2\} = \{1, 2, \dots\}$. Substituting from (S.3.0.3), (S.3.0.5) and (S.3.0.6), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, n_2)}{\partial \rho^2} \\
&= \frac{(n_1 + n_2)}{\rho} \frac{\partial f(n_1, n_2)}{\partial \rho} \\
&+ \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] [\lambda_1 g_1 + \lambda_2 g_2 + 2\lambda_3 g_3] \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{\lambda_2}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{2\lambda_3}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] - \frac{(n_1 + n_2)}{\rho^2} f(n_1, n_2) \\
&= \left[\frac{n_1 + n_2 - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1, n_2)}{\partial \rho} \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{\lambda_2}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{2\lambda_3}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f(n_1, n_2)}{\partial \rho} \right] - \frac{(n_1 + n_2)}{(1-\rho)\rho} f(n_1, n_2).
\end{aligned}$$

3.4 Weighted Type I bivariate Pólya-Aeppli distribution Case I

3.4.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 3.4.5 for the $WBPA_I^{(1)}$ distribution. The Newton-Raphson algorithm uses the equation in (3.16) for $t \geq 1$ and the MoM estimates in Subsection 3.4.4 are used as initial values of the parameters $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}$ and $\rho^{(0)}$ for the $WBPA_I^{(1)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached (Balakrishnan et al. [1]).

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (3.17) need to be calculated with respect to each of the parameters: $\lambda_1, \lambda_2, \lambda_3$ and ρ .

The results of the calculation of the first and second derivatives of the *pmf* for the $WBPA_I^{(1)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(n_1, n_2)$ where $\{n_1, n_2\} \in \{0, 1\}$, and $f(n_1, n_2)$ where $\{n_1, n_2\} \in \{2, 3, \dots\}$ with respect to each of the parameters: $\lambda_1, \lambda_2, \lambda_3$ and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_1 = \frac{-\lambda_1(1-\rho)}{\rho}$, $x_2 = \frac{-\lambda_2(1-\rho)}{\rho}$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
 g_1 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} + f^w(n_1, n_2) \\
 &= \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-2} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1}}{(r-2)!} \\
 &\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)}, \\
 &\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
 \end{aligned} \tag{S.3.0.21}$$

$$\begin{aligned}
 g'_{1,1} &= \frac{\partial g_1}{\partial \lambda_1} \\
 &= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
 &= \sum_{r=2, n_1 \neq 2, 3}^{\min(n_1-2, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\
 &\quad \times L_{n_1-r-2}^{r+1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_1, \\
 &\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
 \end{aligned} \tag{S.3.0.22}$$

$$\begin{aligned}
 g_2 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} + f^w(n_1, n_2) \\
 &= \sum_{r=2, n_2 \neq 2}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-2} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1}}{(r-2)!} \\
 &\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)}, \\
 &\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
 \end{aligned} \tag{S.3.0.23}$$

$$\begin{aligned}
 g'_{2,2} &= \frac{\partial g_2}{\partial \lambda_2} \\
 &= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2^2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
 &= \sum_{r=2, n_2 \neq 2, 3}^{\min(n_1, n_2-2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\
 &\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r-2}^{r+1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_2, \\
 &\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
 \end{aligned}$$

$$\begin{aligned}
g_3 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} + f^w(n_1, n_2) \\
&= \sum_{r=3, n_1, n_2 \neq 2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-3} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r-3)!} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)}, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.24}$$

$$\begin{aligned}
g'_{3,3} &= \frac{\partial g_3}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=4, n_1 \neq 2, 3}^{\min(n_1, n_2)} \frac{\lambda_3^{r-4}}{(r-4)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r}, \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_3, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.25}$$

$$\begin{aligned}
g'_{1,2} &= \frac{\partial g_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&= \sum_{r=2, n_1, n_2 \neq 2}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\
&\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_1, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.26}$$

$$\begin{aligned}
g'_{2,1} &= \frac{\partial g_2}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \lambda_1} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \sum_{r=2, n_1, n_2 \neq 2}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\
&\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_2, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.27}$$

$$\begin{aligned}
g'_{1,3} &= \frac{\partial g_1}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=3, n_1 \neq 2, 3, n_2 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_1, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.28}$$

$$\begin{aligned}
g'_{3,1} &= \frac{\partial g_3}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \lambda_1} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \sum_{r=3, n_1 \neq 2, 3, n_2 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_3, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.29}$$

$$\begin{aligned}
g'_{2,3} &= \frac{\partial g_2}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \lambda_3} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \sum_{r=3, n_1 \neq 2, n_2 \neq 2, 3}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_2, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.3.0.30}$$

$$\begin{aligned}
g'_{3,2} &= \frac{\partial g_3}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \lambda_2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&= \sum_{r=3, n_1 \neq 2, n_2 \neq 2, 3}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - g_3, \\
&\quad \{n_1, n_2\} \in \{2, 3, \dots\}.
\end{aligned}$$

The results for $g'_{2,3}$ and $g'_{3,2}$ follow similarly.

It then follows that the first and second derivatives of the pmf in (3.17) are

First derivative with respect to λ_1

$$\frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} = g_1 - f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}. \quad (\text{S.3.0.31})$$

First derivative with respect to λ_2

$$\frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} = g_2 - f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}. \quad (\text{S.3.0.32})$$

First derivative with respect to λ_3

$$\frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} = g_3 - f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}. \quad (\text{S.3.0.33})$$

First derivative with respect to ρ

$$\begin{aligned} \frac{\partial f^w(n_1, n_2)}{\partial \rho} &= \left[\frac{(n_1 + n_2)}{\rho} - \frac{4}{(1 - \rho)\rho} \right] f^w(n_1, n_2) \\ &\quad - \frac{\lambda_1 g_1}{(1 - \rho)\rho} - \frac{\lambda_2 g_2}{\rho(1 - \rho)} - \frac{2\lambda_3 g_3}{(1 - \rho)\rho}, \\ &\quad \{n_1, n_2\} \in \{2, 3, \dots\}. \end{aligned} \quad (\text{S.3.0.34})$$

Second derivative with respect to λ_1

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} = g'_1 - g_1 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to λ_2

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2^2} = g'_2 - g_2 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to λ_3

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} = g'_3 - g_3 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to λ_1 and λ_2

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} = g'_{1,2} - g_2 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to λ_1 and λ_3

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} = g'_{1,2} - g_2 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to λ_2 and λ_3

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \lambda_3} = g'_{2,3} - g_3 + f^w(n_1, n_2), \quad \{n_1, n_2\} \in \{2, 3, \dots\}.$$

Second derivative with respect to ρ and λ_1

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_1} &= \left[\frac{n_1 + n_2}{\rho} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} \\ &\quad - \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}, \\ &\quad \{n_1, n_2\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ and λ_2

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_2} &= \left[\frac{n_1 + n_2}{\rho} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} - \frac{g_2}{(1-\rho)\rho} \\ &\quad - \frac{\lambda_1 g'_{1,2}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,2}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,2}}{(1-\rho)\rho}, \\ &\quad \{n_1, n_2\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ and λ_3

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_3} &= \left[\frac{(n_1 + n_2)}{\rho} - \frac{4}{(1 - \rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - \frac{2g_3}{(1 - \rho)\rho} \\ &\quad - \frac{\lambda_1 g'_{1,3}}{(1 - \rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1 - \rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1 - \rho)\rho}, \\ &\quad \{n_1, n_2\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho^2} &= \left[\frac{n_1 + n_2 - 1}{\rho} + \frac{1}{(1 - \rho)} - \frac{4}{(1 - \rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\ &\quad - \frac{\lambda_1}{(1 - \rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ &\quad - \frac{\lambda_2}{\rho(1 - \rho)} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ &\quad - \frac{2\lambda_3}{(1 - \rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ &\quad - \frac{(n_1 + n_2)}{(1 - \rho)\rho} f^w(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{2, 3, \dots\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the *pmf* in (3.17) are given here.

First derivative with respect to λ_1

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (3.17) and (S.3.0.21), it follows that

$$\begin{aligned} &\frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\ &= \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1 - \rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\ &\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - f^w(n_1, n_2) \\ &= g_1 - f^w(n_1, n_2). \end{aligned}$$

The first derivative with respect to λ_2 follows similarly.

First derivative with respect to λ_3

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (3.17) and (S.3.0.24), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\ &= \sum_{r=3, n_1, n_2 \neq 2}^{\min(n_1, n_2)} \frac{(r-2) \lambda_3^{r-3}}{(r-2)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} \\ & \quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - f^w(n_1, n_2) \\ &= \sum_{r=3, n_1, n_2 \neq 2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} \\ & \quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - f^w(n_1, n_2) \\ &= g_3 - f^w(n_1, n_2). \end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, n_2)}{\partial \rho} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (3.17), (S.3.0.21), (S.3.0.23) and (S.3.0.24), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\ &= \sum_{r=2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} \left[\frac{(n_1+n_2-2r)}{\rho} \right] \\ & \quad \times [(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2)] e^{-(\lambda_1+\lambda_2+\lambda_3)} \\ & \quad - \sum_{r=2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} \left[\frac{2r}{(1-\rho)} \right] \\ & \quad \times [(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2)] e^{-(\lambda_1+\lambda_2+\lambda_3)} \\ & \quad - \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} \left[\frac{\lambda_1}{\rho^2} L_{n_1-r-1}^r(x_1) \right] \\ & \quad \times [(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_2-r}^{r-1}(x_2)] e^{-(\lambda_1+\lambda_2+\lambda_3)} \\ & \quad - \sum_{r=2, n_2 \neq 2}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-2}}{(r-2)!} \left[\frac{\lambda_2}{\rho^2} L_{n_2-r-1}^r(x_2) \right] \\ & \quad \times [(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1)] e^{-(\lambda_1+\lambda_2+\lambda_3)} \end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\
&= \sum_{r=2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} \left[\frac{(n_1 + n_2 - 2r)}{\rho} - \frac{2r}{(1-\rho)} \right] \\
&\quad \times \left[(1-\rho)^{2r} \rho^{n_1+n_2-2r} L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) \right] e^{-(\lambda_1+\lambda_2+\lambda_3)} \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} \\
&\quad - \frac{\lambda_2}{(1-\rho)\rho} \sum_{r=2, n_2 \neq 2}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} \\
&= \left[\frac{(n_1 + n_2)}{\rho} - \frac{4}{(1-\rho)\rho} \right] \sum_{r=2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} \\
&\quad - \frac{2\lambda_3}{(1-\rho)\rho} \sum_{r=2}^{\min(n_1, n_2)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} \\
&\quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} g_1 - \frac{\lambda_2}{\rho(1-\rho)} g_2 \\
&= \left[\frac{(n_1 + n_2)}{\rho} - \frac{4}{(1-\rho)\rho} \right] f^w(n_1, n_2) - \frac{\lambda_1 g_1}{(1-\rho)\rho} - \frac{\lambda_2 g_2}{\rho(1-\rho)} - \frac{2\lambda_3 g_3}{(1-\rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_1

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (3.17), (S.3.0.21), (S.3.0.22) and (S.3.0.31), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} \\
&= \sum_{r=2, n_1 \neq 2, 3}^{\min(n_1-2, n_2)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\
&\quad \times L_{n_1-r-2}^{r+1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} - g_1 \\
&= g'_1 - g_1 + f^w(n_1, n_2).
\end{aligned}$$

The second derivative with respect to λ_2 follows similarly.

Second derivative with respect to λ_3

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (S.3.0.24), (S.3.0.25) and (S.3.0.33), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} \\ &= \sum_{r=4, n_1 \neq 2, 3}^{\min(n_1, n_2)} \frac{\lambda_3^{r-4}}{(r-4)!} (1-\rho)^{2r} \rho^{n_1+n_2-2r} \\ & \quad \times L_{n_1-r}^{r-1}(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - g_3 \\ &= g'_3 - g_3 + f^w(n_1, n_2). \end{aligned}$$

Second derivative with respect to λ_1 and λ_2

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (S.3.0.21), (S.3.0.26) and (S.3.0.32), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\ &= \sum_{r=2, n_1, n_2 \neq 2}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r-2}}{(r-2)!} (1-\rho)^{2r+2} \rho^{n_1+n_2-2r-2} \\ & \quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} - g_1 \\ &= g'_{1,2} - g_2 + f^w(n_1, n_2). \end{aligned}$$

Second derivative with respect to λ_1 and λ_3

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (S.3.0.21), (S.3.0.28) and (S.3.0.33), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} \\ &= \sum_{r=3, n_1 \neq 2, 3, n_2 \neq 2}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r-3}}{(r-3)!} (1-\rho)^{2r+1} \rho^{n_1+n_2-2r-1} \\ & \quad \times L_{n_1-r-1}^r(x_1) L_{n_2-r}^{r-1}(x_2) e^{-(\lambda_1+\lambda_2+\lambda_3)} - \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - g_1 \\ &= g'_{1,3} - g_3 + f^w(n_1, n_2). \end{aligned}$$

The second derivative with respect to λ_2 and λ_3 follows in a similar way.

Second derivative with respect to ρ and λ_1

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_1} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. From (S.3.0.22), (S.3.0.27) and (S.3.0.29), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_1} \\ &= \left[\frac{n_1+n_2}{\rho} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} \\ & \quad - \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}. \end{aligned}$$

The second derivative with respect to ρ and λ_2 follows in a similar way.

Second derivative with respect to ρ and λ_3

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_3} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. From (S.3.0.25), (S.3.0.28) and (S.3.0.30), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_3} \\ &= \left[\frac{(n_1+n_2)}{\rho} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - \frac{2g_3}{(1-\rho)\rho} \\ & \quad - \frac{\lambda_1 g'_{1,3}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, n_2\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \rho^2} = 0.$$

Let $\{n_1, n_2\} \in \{2, 3, \dots\}$. Substituting from (S.3.0.21), (S.3.0.23) and (S.3.0.24), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho^2} \\ &= \left[\frac{n_1 + n_2}{\rho} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\ & \quad - \left[\frac{(n_1 + n_2)}{\rho^2} - \frac{4}{(1-\rho)\rho^2} + \frac{4}{(1-\rho)^2\rho} \right] f^w(n_1, n_2) \\ & \quad + \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] [\lambda_1 g_1 + \lambda_2 g_2 + 2\lambda_3 g_3] \\ & \quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ & \quad - \frac{\lambda_2}{\rho(1-\rho)} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ & \quad - \frac{2\lambda_3}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ &= \left[\frac{n_1 + n_2 - 1}{\rho} + \frac{1}{(1-\rho)} - \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\ & \quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ & \quad - \frac{\lambda_2}{\rho(1-\rho)} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ & \quad - \frac{2\lambda_3}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \rho} + \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\ & \quad - \frac{(n_1 + n_2)}{(1-\rho)\rho} f^w(n_1, n_2). \end{aligned}$$

3.5 Weighted Type I bivariate Pólya-Aeppli distribution Case II

3.5.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 3.5.5 for the $WBPA_I^{(2)}$ distribution. The Newton-Raphson algorithm uses the equation in (3.16) for $t \geq 1$ and where the MoM estimates in Subsection 3.5.4 are used as initial values of the parameters $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}$ and $\rho^{(0)}$ for the $WBPA_I^{(2)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached (Balakrishnan et al. [1]).

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (3.20) need to be calculated with respect to each of the parameters: $\lambda_1, \lambda_2, \lambda_3$ and ρ .

The results of the calculation of the first and second derivatives of the pmf for the $WBPA_I^{(2)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(0,0)$, $f^w(n_1,0)$ where $n_1 \in \{1,2,\dots\}$, $f^w(0,n_2)$ where $n_2 \in \{1,2,\dots\}$, and $f^w(n_1,n_2)$ where $\{n_1,n_2\} \in \{1,2,\dots\}$ with respect to each of the parameters: λ_1 , λ_2 , λ_3 , and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_1 = \frac{-\lambda_1(1-\rho)}{\rho}$, $x_2 = \frac{-\lambda_2(1-\rho)}{\rho}$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
g_1 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} - \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} + \frac{f^w(n_1, n_2)}{\lambda_1} \\
&= \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \quad (S.3.0.35) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g'_{1,1} &= \frac{\partial g_1}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} - \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1 \\
&= \frac{(n_1+1)}{\lambda_1} \left[\frac{(1-\rho)}{\rho} + \frac{1}{\lambda_1} \frac{L_{n_1-1}^1(x_1)}{L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(1-\rho)n_1}{\rho \lambda_1} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_1-r-1}^{r+1}(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&\quad - \frac{1}{\rho} g_1 + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.36}$$

$$\begin{aligned}
g_2 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} - \frac{f^w(n_1, n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} + \frac{f^w(n_1, n_2)}{\lambda_2} \\
&= \left[\frac{(n_2+1)}{\lambda_2} \frac{L_{n_2-1}^1(x_2)}{L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \right] f^w(0,0), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.37}$$

$$\begin{aligned}
g'_{2,2} &= \frac{\partial g_2}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2^2} - \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] g_2 \\
&= \frac{(n_2+1)}{\lambda_2} \left[\frac{(1-\rho)}{\rho} + \frac{1}{\lambda_2} \frac{L_{n_2-1}^1(x_2)}{L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(1-\rho)n_2}{\rho \lambda_2} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_2-r-1}^{r+1}(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \right] f^w(0,0) \\
&\quad - \frac{1}{\rho} g_2 + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g_3 &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - \frac{f^w(n_1, n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} + \frac{f^w(n_1, n_2)}{\lambda_3} \\
&= \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2 (n_1+1)(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{(r+1)!} \frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.38}$$

$$\begin{aligned}
g'_{3,3} &= \frac{\partial g_3}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} - \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] g_3 \\
&= \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{r! \lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.39}$$

$$\begin{aligned}
g'_{1,2} &= \frac{\partial g_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&= \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{(n_2+1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1+1) n_2 (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0,0) + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.40}$$

$$\begin{aligned}
g'_{2,1} &= \frac{\partial g_2}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{(n_2+1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0,0) + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.41}$$

$$\begin{aligned}
g'_{1,3} &= \frac{\partial g_1}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.42}$$

$$\begin{aligned}
g'_{3,1} &= \frac{\partial g_3}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \lambda_1} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1+1) n_2 (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.43}$$

$$\begin{aligned}
g'_{2,3} &= \frac{\partial g_2}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \lambda_3} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{(n_2+1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1+1) n_2 (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \right] f^w(0,0) \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] g_2, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.44}$$

$$\begin{aligned}
g'_{3,2} &= \frac{\partial g_3}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \lambda_2} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&= \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{1}{\lambda_3} \left[\frac{(n_2+1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1+1) n_2 (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2-1)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \right] f^w(0,0) \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] g_3, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h_1 &= \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} - \frac{f^w(n_1, 0)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} + \frac{f^w(n_1, 0)}{\lambda_1} \\
&= \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] f^w(n_1, 0), \\
&\quad n_1 \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.45}$$

$$\begin{aligned}
h'_{1,1} &= \frac{\partial h_1}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1^2} - \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_1 \\
&= \frac{(1-\rho)(n_1+1)}{\rho \lambda_1} f^w(n_1, 0) - \frac{1}{\rho} h_1 \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} + 1 \right] h_1, \\
&\quad n_1 \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.46}$$

$$\begin{aligned}
h_2 &= \frac{\partial f^w(0, n_2)}{\partial \lambda_2} - \frac{f^w(0, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} + \frac{f^w(0, n_2)}{\lambda_2} \\
&= \left[\frac{(n_2 + 1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] f^w(0, n_2), \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{2,2} &= \frac{\partial h_2}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(0, n_2)}{\partial \lambda_2^2} - \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(0, n_2)}{\partial \lambda_2} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] h_2 \\
&= \frac{(1 - \rho)(n_2 + 1)}{\rho} \frac{f^w(0, n_2)}{\lambda_2} - \frac{1}{\rho} h_2 \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} + 1 \right] h_2, \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{1,2} &= \frac{\partial h_1}{\partial \lambda_2} \\
&= \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_2^2} - \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_2} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] h_1 \\
&= \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] h_1 - \frac{f^w(n_1, 0)}{\lambda_2} \\
n_1 &\in \{1, 2, \dots\}.
\end{aligned}$$

$$\begin{aligned}
h'_{2,1} &= \frac{\partial h_2}{\partial \lambda_1} \\
&= \frac{\partial^2 f^w(0, n_2)}{\partial \lambda_1^2} - \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(0, n_2)}{\partial \lambda_1} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_2 \\
&= \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_2 - \frac{f^w(0, n_2)}{\lambda_1} \\
n_2 &\in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.47}$$

$$\begin{aligned}
h'_{1,3} &= \frac{\partial h_1}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_3^2} - \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_3} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] h_1 \\
&= \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] h_1 - \frac{f^w(n_1, 0)}{\lambda_3} \\
&\quad n_1 \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.48}$$

$$\begin{aligned}
h'_{2,3} &= \frac{\partial h_2}{\partial \lambda_3} \\
&= \frac{\partial^2 f^w(0, n_2)}{\partial \lambda_3^2} - \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(0, n_2)}{\partial \lambda_3} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] h_2 \\
&= \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] h_2 - \frac{f^w(0, n_2)}{\lambda_3} \\
&\quad n_2 \in \{1, 2, \dots\}.
\end{aligned}$$

The results for h_2 and $h'_{2,2}$ follow similarly.

It then follows that the first and second derivatives of the pmf in (3.20) are

First derivative with respect to λ_1

$$\begin{aligned}
\frac{\partial f^w(0, 0)}{\partial \lambda_1} &= \left[\frac{1}{\lambda_1} - \frac{1}{(1 - e^{-\lambda_1})} \right] f^w(0, 0) \\
\frac{\partial f^w(n_1, 0)}{\partial \lambda_1} &= h_1 + \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, 0)}{\lambda_1}, \\
&\quad n_1 \in \{1, 2, \dots\} \\
\frac{\partial f^w(0, n_2)}{\partial \lambda_1} &= \frac{f^w(0, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1}, \\
&\quad n_2 \in \{1, 2, \dots\} \\
\frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} &= g_1 + \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, n_2)}{\lambda_1}, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.49}$$

First derivative with respect to λ_2

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \lambda_2} &= \left[\frac{1}{\lambda_2} - \frac{1}{(1-e^{-\lambda_2})} \right] f^w(0,0) \\
\frac{\partial f^w(n_1,0)}{\partial \lambda_2} &= \frac{f^w(n_1,0)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2}, \\
& n_1 \in \{1, 2, \dots\} \\
\frac{\partial f^w(0,n_2)}{\partial \lambda_2} &= h_2 + \frac{f^w(0,n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{f^w(0,n_2)}{\lambda_2}, \\
& n_2 \in \{1, 2, \dots\} \\
\frac{\partial f^w(n_1,n_2)}{\partial \lambda_2} &= g_2 + \frac{f^w(n_1,n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{f^w(n_1,n_2)}{\lambda_2}, \\
& \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.50}$$

First derivative with respect to λ_3

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \lambda_3} &= \left[\frac{1}{\lambda_3} - \frac{1}{(1-e^{-\lambda_3})} \right] f^w(0,0) \\
\frac{\partial f^w(n_1,0)}{\partial \lambda_3} &= \frac{f^w(n_1,0)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3}, \\
& n_1 \in \{1, 2, \dots\} \\
\frac{\partial f^w(0,n_2)}{\partial \lambda_3} &= \frac{f^w(0,n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3}, \\
& n_2 \in \{1, 2, \dots\} \\
\frac{\partial f^w(n_1,n_2)}{\partial \lambda_3} &= g_3 + \frac{f^w(n_1,n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{f^w(n_1,n_2)}{\lambda_3}, \\
& \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.51}$$

First derivative with respect to ρ

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \rho} &= 0 \\
\frac{\partial f^w(n_1,0)}{\partial \rho} &= \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] f^w(n_1,0) - \frac{\lambda_1 h_1}{(1-\rho)\rho}, \\
&\quad n_1 \in \{1, 2, \dots\} \\
\frac{\partial f^w(0,n_2)}{\partial \rho} &= \left[\frac{n_2}{\rho} + \frac{1}{(1-\rho)\rho} \right] f^w(0,n_2) - \frac{\lambda_2 h_2}{(1-\rho)\rho}, \\
&\quad n_2 \in \{1, 2, \dots\} \\
\frac{\partial f^w(n_1,n_2)}{\partial \rho} &= \left[\frac{n_1+n_2}{\rho} + \frac{4}{(1-\rho)\rho} \right] f^w(n_1,n_2) \\
&\quad - \frac{\lambda_1 g_1}{(1-\rho)\rho} - \frac{\lambda_2 g_2}{(1-\rho)\rho} - \frac{2\lambda_3 g_3}{(1-\rho)\rho}, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.52}$$

Second derivative with respect to λ_1

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \lambda_1^2} &= \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{f^w(0,0)}{\lambda_1} \\
\frac{\partial^2 f^w(n_1,0)}{\partial \lambda_1^2} &= h'_{1,1} + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1,0)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_1, \\
&\quad n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \lambda_1^2} &= \frac{f^w(0,n_2)}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_1^2} \\
&\quad n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_1^2} &= g'_{1,1} + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.53}$$

Second derivative with respect to λ_2

$$\begin{aligned} \frac{\partial^2 f^w(0,0)}{\partial \lambda_2^2} &= \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{f^w(0,0)}{\lambda_2} \\ \frac{\partial^2 f^w(n_1,0)}{\partial \lambda_2^2} &= \frac{f^w(n_1,0)}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_2^2}, \\ n_1 &\in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(0,n_2)}{\partial \lambda_2^2} &= h'_{2,2} + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(0,n_2)}{\partial \lambda_2} \\ &\quad - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] h_2, \\ n_2 &\in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_2^2} &= g'_{2,2} + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{2}{\lambda_2} + 1 \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_2} \\ &\quad - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} + 1 \right] g_2, \\ \{n_1, n_2\} &\in \{1, 2, \dots\}. \end{aligned}$$

Second derivative with respect to λ_3

$$\begin{aligned} \frac{\partial^2 f^w(0,0)}{\partial \lambda_3^2} &= \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{f^w(0,0)}{\lambda_3} \\ \frac{\partial^2 f^w(n_1,0)}{\partial \lambda_3^2} &= \frac{f^w(n_1,0)}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_3^2}, \\ n_1 &\in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(0,n_2)}{\partial \lambda_3^2} &= \frac{f^w(0,n_2)}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_3^2}, \\ n_2 &\in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_3^2} &= g'_{3,3} + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_3} \\ &\quad - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] g_3, \\ \{n_1, n_2\} &\in \{1, 2, \dots\}. \end{aligned}$$

Second derivative with respect to λ_1 and λ_2

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \lambda_1 \partial \lambda_2} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} \frac{\partial f^w(0,0)}{\partial \lambda_2} \\
\frac{\partial^2 f^w(n_1,0)}{\partial \lambda_1 \partial \lambda_2} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} \frac{\partial f^w(n_1,0)}{\partial \lambda_1}, \\
& n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \lambda_1 \partial \lambda_2} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} \frac{\partial f^w(0,n_2)}{\partial \lambda_2}, \\
& n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_1 \partial \lambda_2} &= g'_{1,2} + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_2}, \\
& \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.54}$$

Second derivative with respect to λ_1 and λ_3

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \lambda_1 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} \frac{\partial f^w(0,0)}{\partial \lambda_3} \\
\frac{\partial^2 f^w(n_1,0)}{\partial \lambda_1 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} \frac{\partial f^w(n_1,0)}{\partial \lambda_1}, \\
& n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \lambda_1 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} \frac{\partial f^w(0,n_2)}{\partial \lambda_1}, \\
& n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_1 \partial \lambda_3} &= g'_{1,3} + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_3}, \\
& \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.3.0.55}$$

Second derivative with respect to λ_2 and λ_3

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \lambda_2 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} \frac{\partial f^w(0,0)}{\partial \lambda_3} \\
\frac{\partial^2 f^w(n_1,0)}{\partial \lambda_2 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} \frac{\partial f^w(n_1,0)}{\partial \lambda_2}, \\
& n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \lambda_2 \partial \lambda_3} &= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} \frac{\partial f^w(0,n_2)}{\partial \lambda_2}, \\
& n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_2 \partial \lambda_3} &= g'_{2,3} + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_3}, \\
& \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

Second derivative with respect to ρ and λ_1

$$\begin{aligned}\frac{\partial^2 f^w(0,0)}{\partial \rho \partial \lambda_1} &= 0 \\ \frac{\partial^2 f^w(n_1,0)}{\partial \rho \partial \lambda_1} &= \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,0)}{\partial \lambda_1} - \frac{h_1}{(1-\rho)\rho} - \frac{\lambda_1 h'_{1,1}}{(1-\rho)\rho}, \\ & n_1 \in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(0,n_2)}{\partial \rho \partial \lambda_1} &= \left[\frac{n_2}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(0,n_2)}{\partial \lambda_1} - \frac{\lambda'_2 h_{2,1}}{(1-\rho)\rho}, \\ & n_2 \in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(n_1,n_2)}{\partial \rho \partial \lambda_1} &= \left[\frac{n_1+n_2}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} \\ & - \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}, \\ & \{n_1, n_2\} \in \{1, 2, \dots\}.\end{aligned}$$

Second derivative with respect to ρ and λ_2

$$\begin{aligned}\frac{\partial^2 f^w(0,0)}{\partial \rho \partial \lambda_2} &= 0 \\ \frac{\partial^2 f^w(n_1,0)}{\partial \rho \partial \lambda_2} &= \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,0)}{\partial \lambda_2} - \frac{\lambda'_2 h_{1,2}}{(1-\rho)\rho}, \\ & n_1 \in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(0,n_2)}{\partial \rho \partial \lambda_2} &= \left[\frac{n_2}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(0,n_2)}{\partial \lambda_2} - \frac{h_2}{(1-\rho)\rho} - \frac{\lambda_1 h'_{2,2}}{(1-\rho)\rho}, \\ & n_2 \in \{1, 2, \dots\} \\ \frac{\partial^2 f^w(n_1,n_2)}{\partial \rho \partial \lambda_2} &= \left[\frac{n_1+n_2}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_2} - \frac{g_2}{(1-\rho)\rho} \\ & - \frac{\lambda_1 g'_{1,2}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,2}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,2}}{(1-\rho)\rho}, \\ & \{n_1, n_2\} \in \{1, 2, \dots\}.\end{aligned}$$

Second derivative with respect to ρ and λ_3

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \rho \partial \lambda_3} &= 0 \\
\frac{\partial^2 f^w(n_1,0)}{\partial \rho \partial \lambda_3} &= \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,0)}{\partial \lambda_3} - \frac{\lambda_1 h'_{1,3}}{(1-\rho)\rho}, \\
&\quad n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \rho \partial \lambda_3} &= \left[\frac{n_2}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(0,n_2)}{\partial \lambda_3} - \frac{\lambda_2 h'_{2,3}}{(1-\rho)\rho}, \\
&\quad n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \rho \partial \lambda_3} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_3} - \frac{2g_3}{(1-\rho)\rho} \\
&\quad - \frac{\lambda_1 g'_{1,3}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1-\rho)\rho}, \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \rho^2} &= 0 \\
\frac{\partial^2 f^w(n_1,0)}{\partial \rho^2} &= \left[\frac{n_1-1}{\rho} + \frac{1}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1,0)}{\partial \rho} \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1,0)}{\partial \lambda_1 \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1,0)}{\partial \lambda_1} \right] \\
&\quad - \frac{n_1}{(1-\rho)\rho} f^w(n_1,0), \\
&\quad n_1 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(0,n_2)}{\partial \rho^2} &= \left[\frac{n_2-1}{\rho} + \frac{1}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(0,n_2)}{\partial \rho} \\
&\quad - \frac{\lambda_2}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(0,n_2)}{\partial \lambda_2 \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(0,n_2)}{\partial \lambda_2} \right] \\
&\quad - \frac{n_2}{(1-\rho)\rho} f^w(0,n_2), \\
&\quad n_2 \in \{1, 2, \dots\} \\
\frac{\partial^2 f^w(n_1,n_2)}{\partial \rho^2} &= \left[\frac{n_1+n_2-1}{\rho} + \frac{4}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1,n_2)}{\partial \rho} \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_1 \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_1} \right] \\
&\quad - \frac{\lambda_2 g_2}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_2 \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_2} \right] \\
&\quad - \frac{2\lambda_3 g_3}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1,n_2)}{\partial \lambda_3 \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1,n_2)}{\partial \lambda_3} \right] \\
&\quad - \frac{n_1+n_2}{(1-\rho)\rho} f^w(n_1,n_2), \\
&\quad \{n_1, n_2\} \in \{1, 2, \dots\}.
\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the *pmf* in (3.20) are given here.

First derivative with respect to λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (3.20), it follows that

$$\begin{aligned}
 & \frac{\partial f^w(0, 0)}{\partial \lambda_1} \\
 &= \frac{\lambda_3 e^{-\lambda_3}}{(1 - e^{-\lambda_3})} \frac{\lambda_2 e^{-\lambda_2}}{(1 - e^{-\lambda_2})} \frac{e^{-\lambda_1}}{(1 - e^{-\lambda_1})} \left[1 - \lambda_1 - \frac{\lambda_1 e^{-\lambda_1}}{(1 - e^{-\lambda_1})} \right] \\
 &= \left[\frac{1}{\lambda_1} - 1 - \frac{e^{-\lambda_1}}{(1 - e^{-\lambda_1})} \right] f^w(0, 0) \\
 &= \left[\frac{1}{\lambda_1} - \frac{1}{(1 - e^{-\lambda_1})} \right] f^w(0, 0).
 \end{aligned}$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.20) and (S.3.0.45) and using (S.3.0.2), it follows that

$$\begin{aligned}
 & \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
 &= \frac{1}{\lambda_1} \left[\frac{\lambda_1 (1 - \rho) \rho^{n_1 - 1}}{n_1 (n_1 + 1)} L_{n_1 - 1}^2(x_1) \right] f^w(0, 0) \\
 &+ \left[\frac{(1 - \rho) L_{n_1 - 2}^3(x_1)}{\rho L_{n_1 - 1}^2(x_1)} \right] \left[\frac{\lambda_1 (1 - \rho) \rho^{n_1 - 1}}{n_1 (n_1 + 1)} L_{n_1 - 1}^2(x_1) \right] f^w(0, 0) \\
 &+ \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \\
 &= \left[\frac{(1 - \rho)}{\rho} \left[\frac{(n_1 + 1) \rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^1(x_1) - \frac{2\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^2(x_1) \right] \right. \\
 &\quad \left. \times \frac{1}{L_{n_1 - 1}^2(x_1)} \right] f^w(n_1, 0) + \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} + \frac{f^w(n_1, 0)}{\lambda_1} \\
 &= \left[\frac{(n_1 + 1) L_{n_1 - 1}^1(x_1)}{\lambda_1 L_{n_1 - 1}^2(x_1)} \right] f^w(n_1, 0) + \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, 0)}{\lambda_1} \\
 &= h_1 + \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, 0)}{\lambda_1}.
 \end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (3.20), it follows that

$$\frac{\partial f^w(0, n_2)}{\partial \lambda_1} = \frac{f^w(0, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1}.$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.20) and (S.3.0.35) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \frac{1}{\lambda_1} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \left[\frac{(1-\rho) L_{n_1-2}^3(x_1)}{\rho L_{n_1-1}^2(x_1)} \right] \\
&\times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \frac{\lambda_3}{2!} \left[\frac{(1-\rho) L_{n_1-2}^2(x_1)}{\rho L_{n_1-1}^1(x_1)} \right] \\
&\times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&- \frac{1}{\lambda_1} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+2)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&+ \frac{f^w(n_1, n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \frac{1}{\lambda_1} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \left[\frac{(1-\rho)}{\rho} \left[\frac{(n_1+1)\rho}{\lambda_1(1-\rho)} L_{n_1-1}^1(x_1) - \frac{2\rho}{\lambda_1(1-\rho)} L_{n_1-1}^2(x_1) \right] \frac{1}{L_{n_1-1}^2(x_1)} \right] \\
&\times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \frac{\lambda_3}{2!} \left[\frac{(1-\rho)}{\rho} \left[\frac{n_1 \rho}{\lambda_1(1-\rho)} L_{n_1-1}^0(x_1) - \frac{\rho}{\lambda_1(1-\rho)} L_{n_1-1}^1(x_1) \right] \frac{1}{L_{n_1-1}^1(x_1)} \right] \\
&\times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&- \frac{1}{\lambda_1} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&+ \frac{f^w(n_1, n_2)}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&= \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\
&\quad + \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, n_2)}{\lambda_1} \\
&= g_1 + \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(n_1, n_2)}{\lambda_1}.
\end{aligned}$$

The first derivative with respect to λ_2 follows in a similar way.

First derivative with respect to λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Similarly to (S.3.0.49), it follows that

$$\frac{\partial f^w(0, 0)}{\partial \lambda_3} = \left[\frac{1}{\lambda_3} - \frac{1}{(1 - e^{-\lambda_3})} \right] f^w(0, 0).$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.20), it follows that

$$\frac{\partial f^w(n_1, 0)}{\partial \lambda_3} = \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3}.$$

The result for the first derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$, with respect to λ_3 follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.20) and (S.3.0.38), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \frac{1}{2!} \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&+ \sum_{r=1}^{\min(n_1, n_2)} \frac{(r+1) \lambda_3^r (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+2)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, 0) \\
&+ \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} \\
&= \frac{1}{2!} \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&+ \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+1)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, 0) \\
&- \frac{1}{\lambda_3} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+2)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, 0) \\
&+ \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&+ \frac{1}{\lambda_3} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2 (n_1+1) (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&+ \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+1)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, 0) \\
&+ \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{f^w(n_1, n_2)}{\lambda_3} \\
&= g_3 + \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{f^w(n_1, n_2)}{\lambda_3}.
\end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial f^w(0, 0)}{\partial \rho} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (3.20), (S.3.0.45) and (S.3.0.52) and using (S.3.0.2), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, 0)}{\partial \rho} \\ &= \left[\frac{(n_1 - 1)}{\rho} \right] \left[\frac{\lambda_1 (1 - \rho) \rho^{n_1 - 1}}{n_1 (n_1 + 1)} L_{n_1 - 1}^2(x_1) \right] f^w(0, 0) \\ & \quad - \left[\frac{1}{(1 - \rho)} \right] \left[\frac{\lambda_1 (1 - \rho) \rho^{n_1 - 1}}{n_1 (n_1 + 1)} L_{n_1 - 1}^2(x_1) \right] f^w(0, 0) \\ & \quad - \left[\frac{\lambda_1 L_{n_1 - 2}^3(x_1)}{\rho^2 L_{n_1 - 1}^2(x_1)} \right] \left[\frac{\lambda_1 (1 - \rho) \rho^{n_1 - 1}}{n_1 (n_1 + 1)} L_{n_1 - 1}^2(x_1) \right] f^w(0, 0) \\ & \quad + \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \rho}. \\ &= \left[\frac{n_1}{\rho} - \frac{1}{(1 - \rho)\rho} \right] f^w(n_1, 0) \\ & \quad - \left[\frac{\lambda_1}{\rho^2} \left[\frac{(n_1 + 1)\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^1(x_1) - \frac{2\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^2(x_1) \right] \frac{1}{L_{n_1 - 1}^2(x_1)} \right] f^w(n_1, 0) \\ &= \left[\frac{n_1}{\rho} - \frac{1}{(1 - \rho)\rho} + \frac{2}{(1 - \rho)\rho} \right] f^w(n_1, 0) - \left[\frac{(n_1 + 1)}{(1 - \rho)\rho} \right] \frac{L_{n_1 - 1}^1(x_1)}{L_{n_1 - 1}^2(x_1)} f^w(n_1, 0) \\ &= \left[\frac{n_1}{\rho} + \frac{1}{(1 - \rho)\rho} \right] f^w(n_1, 0) - \frac{\lambda_1 h_1}{(1 - \rho)\rho}. \end{aligned}$$

The result for the first derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.20), (S.3.0.35), (S.3.0.37), (S.3.0.38) and (S.3.0.52) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{2}{(1 - \rho)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_1 L_{n_1 - 2}^3(x_1)}{\rho^2 L_{n_1 - 1}^2(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_2 L_{n_2 - 2}^3(x_2)}{\rho^2 L_{n_2 - 1}^2(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(n_1 + n_2 - 2)}{\rho} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{2}{(1 - \rho)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_1 L_{n_1 - 2}^2(x_1)}{\rho^2 L_{n_1 - 1}^1(x_1)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_2 L_{n_2 - 2}^2(x_2)}{\rho^2 L_{n_2 - 1}^1(x_2)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(n_1 + n_2 - 2r)}{\rho} \right] \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{2r}{(1 - \rho)} \right] \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{\lambda_1}{\rho^2} L_{n_1 - r - 1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) \right] f^w(0, 0) \\
&\quad - \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{\lambda_2}{\rho^2} L_{n_2 - r - 1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \right] f^w(0, 0) \\
&\quad + \frac{f^w(n_1, n_2)}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \rho}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_1}{\rho^2} \left[\frac{(n_1 + 1) \rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^1(x_1) - \frac{2\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^2(x_1) \right] \frac{1}{L_{n_1 - 1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_2}{\rho^2} \left[\frac{(n_2 + 1) \rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^1(x_2) - \frac{2\rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^2(x_2) \right] \frac{1}{L_{n_2 - 1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_1}{\rho^2} \left[\frac{n_1 \rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^0(x_1) - \frac{\rho}{\lambda_1 (1 - \rho)} L_{n_1 - 1}^1(x_1) \right] \frac{1}{L_{n_1 - 1}^1(x_1)} \right] \\
&\quad \times \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_2}{\rho^2} \left[\frac{n_2 \rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^0(x_2) - \frac{\rho}{\lambda_2 (1 - \rho)} L_{n_2 - 1}^1(x_2) \right] \frac{1}{L_{n_2 - 1}^1(x_2)} \right] \\
&\quad \times \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \frac{(n_1 + n_2)}{\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{2r}{(1 - \rho)} + \frac{2r}{\rho} \right] \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{\lambda_1}{\rho^2} L_{n_1 - r - 1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) \right] f^w(0, 0) \\
&\quad - \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{\lambda_2}{\rho^2} L_{n_2 - r - 1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \right] f^w(0, 0)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\
&= \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} + \frac{2}{(1 - \rho)\rho} + \frac{2}{(1 - \rho)\rho} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_1}{(1 - \rho)\rho} \frac{(n_1 + 1) L_{n_1 - 1}^1(x_1)}{\lambda_1 L_{n_1 - 1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \left[\frac{\lambda_2}{(1 - \rho)\rho} \frac{(n_2 + 1) L_{n_2 - 1}^1(x_2)}{\lambda_2 L_{n_2 - 1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1 - 1}^2(x_1) L_{n_2 - 1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(n_1 + n_2 - 2)}{\rho} - \frac{2}{(1 - \rho)} + \frac{1}{(1 - \rho)\rho} + \frac{1}{(1 - \rho)\rho} \right] \\
&\quad \times \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_1}{(1 - \rho)\rho} \frac{n_1 L_{n_1 - 1}^0(x_1)}{\lambda_1 L_{n_1 - 1}^1(x_1)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{\lambda_2}{(1 - \rho)\rho} \frac{n_2 L_{n_2 - 1}^0(x_2)}{\lambda_2 L_{n_2 - 1}^1(x_2)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1 + n_2 - 2}}{n_1 n_2} L_{n_1 - 1}^1(x_1) L_{n_2 - 1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \left[\frac{(n_1 + n_2)}{\rho} + \frac{4}{(1 - \rho)\rho} \right] \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \frac{2\lambda_3}{(1 - \rho)\rho} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r}{(r+1)!} \frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \\
&\quad \times \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) f^w(0, 0) \\
&\quad - \frac{\lambda_1}{(1 - \rho)\rho} \sum_{r=1, n_1 \neq 1}^{\min(n_1 - 1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)}{\rho} L_{n_1 - r - 1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_2 - r}^{r-1}(x_2) - \binom{n_2 - 1}{n_2 - r} \right) \right] f^w(0, 0) \\
&\quad - \frac{\lambda_2}{(1 - \rho)\rho} \sum_{r=1, n_2 \neq 1}^{\min(n_1, n_2 - 1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)}{\rho} L_{n_2 - r - 1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1 + n_2 - 2r}}{\lambda_1 \lambda_2} \left(L_{n_1 - r}^{r-1}(x_1) - \binom{n_1 - 1}{n_1 - r} \right) \right] f^w(0, 0) \\
&= \left[\frac{n_1 + n_2}{\rho} + \frac{4}{(1 - \rho)\rho} \right] f^w(n_1, n_2) - \frac{\lambda_1 g_1}{(1 - \rho)\rho} - \frac{\lambda_2 g_2}{(1 - \rho)\rho} - \frac{2\lambda_3 g_3}{(1 - \rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.49), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1^2} \\
&= \left[\frac{e^{-\lambda_1}}{(1 - e^{-\lambda_1})^2} - \frac{1}{\lambda_1^2} \right] f^w(0, 0) + \frac{1}{f^w(0, 0)} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 \\
&= \frac{1}{(1 - e^{-\lambda_1})} \left[1 + \frac{e^{-\lambda_1}}{(1 - e^{-\lambda_1})} - 1 \right] f^w(0, 0) \\
&\quad - \frac{f^w(0, 0)}{\lambda_1^2} + \frac{1}{f^w(0, 0)} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 \\
&= \frac{1}{(1 - e^{-\lambda_1})^2} f^w(0, 0) - \frac{1}{(1 - e^{-\lambda_1})} f^w(0, 0) \\
&\quad - \frac{f^w(0, 0)}{\lambda_1^2} + \frac{1}{f^w(0, 0)} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 \\
&= \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{f^w(0, 0)}{\lambda_1}.
\end{aligned}$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.45) and (S.3.0.46) and using (A.12), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1^2} \\
&= \left[\frac{(1 - \rho)(n_1 + 1)}{\rho} \frac{L_{n_1-2}^2(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] f^w(n_1, 0) \\
&\quad + \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} h_1 + \left[\frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1^2} - \frac{1}{f^w(0, 0)^2} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 + \frac{1}{\lambda_1^2} \right] f^w(n_1, 0) \\
&= \left[\frac{(1 - \rho)(n_1 + 1)}{\rho} \frac{1}{\lambda_1} \left[L_{n_1-1}^2(x_1) - L_{n_1-1}^1(x_1) \right] \frac{1}{L_{n_1-1}^2(x_1)} \right] f^w(n_1, 0) \\
&\quad - \left[-\frac{1}{\lambda_1} + 1 \right] h_1 + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
&= \frac{(1 - \rho)(n_1 + 1)}{\rho} \frac{1}{\lambda_1} f^w(n_1, 0) - \frac{1}{\rho} h_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} + 1 \right] h_1 \\
&\quad + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_1 \\
&= h'_{1,1} + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] h_1.
\end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (S.3.0.49), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, n_2)}{\partial \lambda_1^2} \\ &= \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, n_2)}{\partial \lambda_1} - \frac{f^w(0, n_2)}{f^w(0, 0)^2} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 \\ & \quad + \frac{f^w(0, n_2)}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1^2} \\ &= \frac{f^w(0, n_2)}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1^2}. \end{aligned}$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (3.20), (S.3.0.35) and (S.3.0.36) and using (A.10) and (A.12), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} \\ &= \left[\frac{(1-\rho)(n_1+1)}{\rho} \frac{L_{n_1-2}^2(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\ & \quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\ & \quad - \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1^2 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\ & \quad + \frac{\lambda_3}{2!} \left[\frac{(1-\rho)n_1}{\rho} \frac{L_{n_1-2}^1(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\ & \quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} \frac{1}{\lambda_1} L_{n_1-r-1}^r(x_1) \right] \\ & \quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\ & \quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_1-r-2}^{r+1}(x_1) \right] \\ & \quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\ & \quad + \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\ & \quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1^2} - \frac{1}{f^w(0, 0)^2} \left(\frac{\partial f^w(0, 0)}{\partial \lambda_1} \right)^2 + \frac{1}{\lambda_1^2} \right] f^w(n_1, n_2) \end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} \\
&= \left[\frac{(1-\rho)(n_1+1)}{\rho \lambda_1} \left[L_{n_1-1}^2(x_1) - L_{n_1-1}^1(x_1) \right] \frac{1}{L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad - \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1^2 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(1-\rho)n_1}{\rho \lambda_1} \left[L_{n_1-1}^1(x_1) - L_{n_1-1}^0(x_1) \right] \frac{1}{L_{n_1-1}^1(x_1)} \right] \\
&\quad \times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad - \frac{1}{\lambda_1} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} \left[L_{n_1-r-1}^{r+1}(x_1) - L_{n_1-r-1}^r(x_1) \right] \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&\quad - \left[-\frac{1}{\lambda_1} + 1 \right] g_1 + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1^2} \\
&= \frac{(n_1 + 1)}{\lambda_1} \left[\frac{(1 - \rho)}{\rho} + \frac{1}{\lambda_1} \frac{L_{n_1-1}^1(x_1)}{L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{(1 - \rho) n_1}{\rho \lambda_1} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)^2}{\rho^2} L_{n_1-r-1}^{r+1}(x_1) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\
&\quad - \frac{1}{\rho} g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1 \\
&\quad + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1 \\
&= g'_{1,1} + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{2}{\lambda_1} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} \\
&\quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} + 1 \right] g_1.
\end{aligned}$$

The second derivative with respect to λ_2 follows similarly.

Second derivative with respect to λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Similarly to (S.3.0.53), it follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \lambda_3^2} = \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{f^w(0, 0)}{\lambda_3}.$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.51), it follows that

$$\frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_3^2} = \frac{f^w(n_1, 0)}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_3^2}.$$

The result for the second derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to λ_3 follows similarly.

Let $\{n_1, n_2\} = \{1, 2, \dots\}$. Substituting from (S.3.0.38) and (S.3.0.39), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} \\
&= -\frac{1}{\lambda_3^2} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2 (n_1+1)(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \sum_{r=1}^{\min(n_1, n_2)} \frac{r \lambda_3^{r-1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+1)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0) \\
&+ \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} g_3 + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&+ \left[\frac{1}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_3^2} - \frac{1}{f^w(0,0)^2} \left(\frac{\partial f^w(0,0)}{\partial \lambda_3} \right)^2 + \frac{1}{\lambda_3^2} \right] f^w(n_1, n_2) \\
&= \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{r! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0) \\
&- \frac{1}{\lambda_3} \left[\frac{1}{\lambda_3} \frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2 (n_1+1)(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&- \frac{1}{\lambda_3} \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^r (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{(r+1)! \lambda_1 \lambda_2} \\
&\times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0,0) \\
&- \left[-\frac{1}{\lambda_3} + 1 \right] g_3 + \left[\frac{2}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3^2} \\
&= \sum_{r=1}^{\min(n_1, n_2)} \frac{\lambda_3^{r-1} (1-\rho)^{2r} \rho^{n_1+n_2-2r}}{r! \lambda_1 \lambda_2} \\
& \quad \times \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) f^w(0, 0) \\
& \quad + \frac{1}{\lambda_3} \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
& \quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] g_3 \\
& \quad + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
& \quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] g_3 \\
&= g'_{3,3} + \left[\frac{2}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{2}{\lambda_3} + 1 \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
& \quad - \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} + 1 \right] g_3.
\end{aligned}$$

Second derivative with respect to λ_1 and λ_2

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.49), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1 \partial \lambda_2} \\
&= \left[\frac{1}{\lambda_1} - \frac{1}{(1-e^{-\lambda_1})} \right] \frac{\partial f^w(0, 0)}{\partial \lambda_2} \\
&= \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, 0)}{\partial \lambda_2}.
\end{aligned}$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.45), (S.3.0.50) and (S.3.0.54), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1 \partial \lambda_2} \\
&= \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_2} + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_2} \\
& \quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1 \partial \lambda_2} - \frac{1}{f^w(0, 0)^2} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, 0)}{\partial \lambda_2} \right] f^w(n_1, 0) \\
&= \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} \frac{\partial f^w(n_1, 0)}{\partial \lambda_1}.
\end{aligned}$$

The result for the second derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to λ_1 and λ_2 follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (S.3.0.35), (S.3.0.40) and (S.3.0.54) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \frac{1}{\lambda_2} \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} \right. \\
&\quad \times L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \left. \right] f^w(0, 0) + \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{(1 - \rho) L_{n_2-2}^3(x_2)}{\rho L_{n_2-1}^2(x_2)} \right] \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} \right. \\
&\quad \times L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \left. \right] f^w(0, 0) + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \\
&\quad \times \left[\frac{(1 - \rho) L_{n_2-2}^2(x_2)}{\rho L_{n_2-1}^1(x_2)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} \right. \\
&\quad \times L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \left. \right] f^w(0, 0) - \frac{1}{\lambda_2} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \\
&\quad \times \left[\frac{(1 - \rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right. \\
&\quad \times \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \left. \right] f^w(0, 0) + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \\
&\quad \times \left[\frac{(1 - \rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \left[\frac{(1 - \rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0, 0) + \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} g_1 \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&\quad + \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1 \partial \lambda_2} - \frac{1}{f^w(0, 0)^2} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, 0)}{\partial \lambda_2} \right] f^w(n_1, n_2)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \frac{1}{\lambda_2} \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{(1-\rho)}{\rho} \left[\frac{(n_2+1)\rho}{\lambda_2(1-\rho)} L_{n_2-1}^1(x_2) - \frac{2\rho}{\lambda_2(1-\rho)} L_{n_2-1}^2(x_2) \right] \right] \\
&\times \frac{1}{L_{n_2-1}^2(x_2)} \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&+ \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)}{\rho} \left[\frac{n_2 \rho}{\lambda_2(1-\rho)} L_{n_2-1}^0(x_2) - \frac{\rho}{\lambda_2(1-\rho)} L_{n_2-1}^1(x_2) \right] \right] \\
&\times \frac{1}{L_{n_2-1}^1(x_2)} \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&- \frac{1}{\lambda_2} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0,0) \\
&+ \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_1-r-1}^r(x_1) L_{n_2-r-1}^r(x_2) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0,0) + \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} g_1 \\
&+ \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= -\frac{1}{\lambda_2} \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&\quad - \frac{1}{\lambda_2} \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&\quad - \frac{1}{\lambda_2} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)}{\rho} \frac{L_{n_1-r-1}^r(x_1)}{\left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right)} \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_1-r}^{r-1}(x_1) - \binom{n_1-1}{n_1-r} \right) \right] \\
&\quad \times \left[L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right] f^w(0, 0) \\
&\quad + \left[\frac{(n_1 + 1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{(n_2 + 1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1 + 1) n_2 (n_2 + 1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, 0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{(1 - \rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1 - \rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \left[\frac{(1 - \rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0, 0) \\
&\quad + \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_2} g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_2} \\
&= \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \left[\frac{(n_2+1) L_{n_2-1}^1(x_2)}{\lambda_2 L_{n_2-1}^2(x_2)} \right] \\
&\quad \times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1(n_1+1)n_2(n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0,0) \\
&\quad + \frac{\lambda_3}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{n_2 L_{n_2-1}^0(x_2)}{\lambda_2 L_{n_2-1}^1(x_2)} \right] \\
&\quad \times \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0,0) \\
&\quad + \sum_{r=1, n_1, n_2 \neq 1}^{\min(n_1-1, n_2-1)} \frac{\lambda_3^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \left[\frac{(1-\rho)}{\rho} L_{n_2-r-1}^r(x_2) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \right] f^w(0,0) + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] g_1 \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2} \\
&= g'_{1,2} + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_2}.
\end{aligned}$$

Second derivative with respect to λ_1 and λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.3.0.49), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0,0)}{\partial \lambda_1 \partial \lambda_3} \\
&= \left[\frac{1}{\lambda_1} - \frac{1}{(1-e^{-\lambda_1})} \right] \frac{\partial f^w(0,0)}{\partial \lambda_3} \\
&= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} \frac{\partial f^w(0,0)}{\partial \lambda_3}.
\end{aligned}$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.45), (S.3.0.51) and (S.3.0.55), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1 \partial \lambda_3} \\
&= \frac{1}{f^w(n_1, 0)} \frac{\partial f^w(n_1, 0)}{\partial \lambda_3} h_1 + \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_3} \\
&\quad + \left[\frac{1}{f^w(0,0)} \frac{\partial^2 f^w(0,0)}{\partial \lambda_1 \partial \lambda_3} - \frac{1}{f^w(0,0)^2} \frac{\partial f^w(0,0)}{\partial \lambda_1} \frac{\partial f^w(0,0)}{\partial \lambda_3} \right] f^w(n_1, 0) \\
&= \frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} \frac{\partial f^w(n_1, 0)}{\partial \lambda_1}.
\end{aligned}$$

Let $n_2 \in \{1, 2, \dots\}$. Substituting from (S.3.0.51) and (S.3.0.55), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, n_2)}{\partial \lambda_1 \partial \lambda_3} \\ &= \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, n_2)}{\partial \lambda_3} \\ &+ \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1 \partial \lambda_3} - \frac{1}{f^w(0, 0)^2} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, 0)}{\partial \lambda_3} \right] f^w(0, n_2) \\ &= \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} \frac{\partial f^w(0, n_2)}{\partial \lambda_1}. \end{aligned}$$

Let $\{n_1, n_2\} = \{1, 2, \dots\}$. Substituting from (S.3.0.35), (S.3.0.42) and (S.3.0.55), it follows that

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} &= \frac{1}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\ &+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{(r+1) \lambda_3^r}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\ &\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\ &+ \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\ &+ \left[\frac{1}{f^w(0, 0)} \frac{\partial^2 f^w(0, 0)}{\partial \lambda_1 \partial \lambda_3} - \frac{1}{f^w(0, 0)^2} \frac{\partial f^w(0, 0)}{\partial \lambda_1} \frac{\partial f^w(0, 0)}{\partial \lambda_3} \right] f^w(n_1, n_2) \\ &= \frac{1}{2!} \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\ &+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\ &\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\ &- \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\ &\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\ &+ \frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \lambda_3} \\
&= \left[\frac{n_1 L_{n_1-1}^0(x_1)}{\lambda_1 L_{n_1-1}^1(x_1)} \right] \left[\frac{(1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 n_2} L_{n_1-1}^1(x_1) L_{n_2-1}^1(x_2) \right] f^w(0, 0) \\
&+ \frac{1}{\lambda_3} \left[\frac{(n_1+1) L_{n_1-1}^1(x_1)}{\lambda_1 L_{n_1-1}^2(x_1)} \right] \\
&\times \left[\frac{\lambda_1 \lambda_2 (1-\rho)^2 \rho^{n_1+n_2-2}}{n_1 (n_1+1) n_2 (n_2+1)} L_{n_1-1}^2(x_1) L_{n_2-1}^2(x_2) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, n_2)} \frac{\lambda_3^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_1+n_2-2r}}{\lambda_1 \lambda_2} \left(L_{n_2-r}^{r-1}(x_2) - \binom{n_2-1}{n_2-r} \right) \right] f^w(0, 0) \\
&+ \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] g_1 + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} \\
&= g'_{1,3} + \left[\frac{1}{f^w(0, 0)} \frac{\partial f^w(0, 0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3}.
\end{aligned}$$

The second derivative with respect to λ_2 and λ_3 follows similarly.

Second derivative with respect to ρ and λ_1

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \rho \partial \lambda_1} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. From (S.3.0.45) and (S.3.0.46), it follows that

$$\frac{\partial^2 f^w(n_1, 0)}{\partial \rho \partial \lambda_1} = \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} - \frac{h_1}{(1-\rho)\rho} - \frac{\lambda_1 h'_{1,1}}{(1-\rho)\rho}.$$

Let $n_2 \in \{1, 2, \dots\}$. From (S.3.0.47), it follows that

$$\frac{\partial^2 f^w(0, n_2)}{\partial \rho \partial \lambda_1} = \left[\frac{n_2}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(0, n_2)}{\partial \lambda_1} - \frac{\lambda_2' h_{2,1}}{(1-\rho)\rho}.$$

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. From (S.3.0.36), (S.3.0.41) and (S.3.0.43), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_1} \\
&= \left[\frac{n_1+n_2}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_1} - \frac{g_1}{(1-\rho)\rho} \\
&- \frac{\lambda_1 g'_{1,1}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,1}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,1}}{(1-\rho)\rho}.
\end{aligned}$$

The second derivative with respect to ρ and λ_2 follows similarly.

Second derivative with respect to ρ and λ_3

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \rho \partial \lambda_3} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. From (S.3.0.48), it follows that

$$\frac{\partial^2 f^w(n_1, 0)}{\partial \rho \partial \lambda_3} = \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_3} - \frac{\lambda_1 h'_{1,3}}{(1-\rho)\rho}.$$

The result for the second derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ and λ_3 follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. From (S.3.0.39), (S.3.0.42) and (S.3.0.44), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho \partial \lambda_3} \\ &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \lambda_3} - \frac{2g_3}{(1-\rho)\rho} \\ & \quad - \frac{\lambda_1 g'_{1,3}}{(1-\rho)\rho} - \frac{\lambda_2 g'_{2,3}}{(1-\rho)\rho} - \frac{2\lambda_3 g'_{3,3}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \rho^2} = 0.$$

Let $n_1 \in \{1, 2, \dots\}$. Substituting from (S.3.0.45), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, 0)}{\partial \rho^2} \\
&= \left[\frac{n_1}{\rho} + \frac{1}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, 0)}{\partial \rho} - \left[\frac{n_1}{\rho^2} + \frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] f^w(n_1, 0) \\
&+ \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \lambda_1 h_1 \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \right] \\
&= \left[\frac{(n_1-1)}{\rho} + \frac{1}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1, 0)}{\partial \rho} \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, 0)}{\partial \lambda_1 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, 0)}{\partial \lambda_1} \right] \\
&- \frac{n_1}{(1-\rho)\rho} f^w(n_1, 0).
\end{aligned}$$

The result for the second derivative of $f^w(0, n_2)$ where $n_2 \in \{1, 2, \dots\}$ with respect to ρ follows similarly.

Let $\{n_1, n_2\} \in \{1, 2, \dots\}$. Substituting from (S.3.0.35), (S.3.0.37) and (S.3.0.38), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, n_2)}{\partial \rho^2} \\
&= \left[\frac{n_1+n_2}{\rho} + \frac{4}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \\
&- \left[\frac{n_1+n_2}{\rho^2} + \frac{4}{(1-\rho)\rho^2} - \frac{4}{(1-\rho)^2\rho} \right] f^w(n_1, n_2) \\
&+ \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] [\lambda_1 g_1 + \lambda_2 g_2 + 2\lambda_3 g_3] \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{\lambda_2 g_2}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{2\lambda_3 g_3}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\
&= \left[\frac{n_1+n_2-1}{\rho} + \frac{4}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} - \frac{n_1+n_2}{(1-\rho)\rho} f^w(n_1, n_2) \\
&- \frac{\lambda_1}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_1 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_1} - \frac{1}{\lambda_1} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{\lambda_2 g_2}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_2 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_2} - \frac{1}{\lambda_2} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right] \\
&- \frac{2\lambda_3 g_3}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda_3 \partial \rho} - \left[\frac{1}{f^w(0,0)} \frac{\partial f^w(0,0)}{\partial \lambda_3} - \frac{1}{\lambda_3} \right] \frac{\partial f^w(n_1, n_2)}{\partial \rho} \right].
\end{aligned}$$

Chapter 4

Multivariate extension of Type I bivariate Pólya-Aeppli distributions

This supplementary material for Chapter 4 provides the derivations and calculations for the following results: the *pgfs*, moments, and moment-based functions in Section 4.2; the *pmfs* and selected results for the conditional distributions and expected values; and the MoM estimators and MLEs for the BPA_I , $WBPA_I^{(1)}$, and $WBPA_I^{(2)}$ distributions, as discussed in Sections 4.3, 4.4, and 4.5, respectively.

4.2 Properties

4.2.1 Probability generating functions

Theorem 4.2.1. *Let $(N_1, \dots, N_k) \sim MPA_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$. The joint *pgf* of the MPA_I distribution is*

$$\psi_{N_1, \dots, N_k}(s_1, \dots, s_k) = e^{-\lambda_{k+1}(1 - \prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k e^{-\lambda_i(1 - \psi_1(s_i))}.$$

Proof. As discussed in Section 2.1, the variables U_i , $i \in \{1, \dots, k\}$ in Definition 4.1.1 are PA distributed, specifically $U_i \sim PA(\lambda_i, \rho)$, with their corresponding *pgfs* provided in Table 2.1. This gives the following

$$E\left(s_i^{U_i}\right) = \psi_{U_i}(s_i) = e^{-\lambda_i(1 - \psi_1(s_i))}.$$

Furthermore, since the $W_{i\ell}$ s in Definition 4.1.1 are independent of each other and independent of Z_{k+1} , it follows from (A.39) and the *pgfs* in Table 2.1 that

$$E\left(\prod_{i=1}^k s_i^{W_i}\right) = \psi_2\left(\prod_{i=1}^k \psi_1(s_i)\right) = e^{-\lambda_{k+1}(1 - \prod_{i=1}^k \psi_1(s_i))}.$$

Therefore, the joint pgf of $(N_1, \dots, N_k) \sim \text{MPA}_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is

$$\begin{aligned} & \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ &= \mathbb{E} \left(\prod_{i=1}^k s_i^{N_i} \right) \\ &= \mathbb{E} \left(\prod_{i=1}^k s_i^{U_i} s_i^{W_i} \right) \\ &= \left[\prod_{i=1}^k \mathbb{E} \left(s_i^{U_i} \right) \right] \left[\mathbb{E} \left(\prod_{i=1}^k s_i^{W_i} \right) \right] \\ &= e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k e^{-\lambda_i(1-\psi_1(s_i))}. \end{aligned}$$

□

Theorem 4.2.2. Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$. The joint pgf of the $\text{WMPA}_I^{(1)}$ distribution is

$$\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) = e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))}.$$

Proof. As discussed in Section 2.1, the variables $U_i^w, i \in \{1, \dots, k\}$ in Definition 4.1.2 are $\text{WPA}^{(1)}$ distributed, specifically $U_i^w \sim \text{WPA}^{(1)}(\lambda_i, \rho)$, with their corresponding pgfs provided in Table 2.1. This gives the following

$$\mathbb{E} \left(s_i^{U_i^w} \right) = \psi_{U_i^w}(s_i) = \psi_1(s_i) e^{-\lambda_i(1-\psi_1(s_i))}.$$

Furthermore, since the $W_{i\ell}^w$ s in Definition 4.1.2 are independent of each other and independent of Z_{k+1}^w , it follows from (A.39) and the pgfs in Table 2.1 that

$$\mathbb{E} \left(\prod_{i=1}^k s_i^{W_i^w} \right) = \psi_2 \left(\prod_{i=1}^k \psi_1(s_i) \right) = e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k \psi_1(s_i).$$

Similar to Theorem 4.2.1, the joint pgf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is given by

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \left[\prod_{i=1}^k \mathbb{E} \left(s_i^{U_i^w} \right) \right] \left[\mathbb{E} \left(\prod_{i=1}^k s_i^{W_i^w} \right) \right] \\ &= e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))}. \end{aligned}$$

□

Theorem 4.2.3. Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$. The joint pgf of the $\text{WMPA}_I^{(2)}$ distribution is

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(e^{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} - 1 \right) \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2}. \end{aligned}$$

Proof. As discussed in Section 2.1, the variables $U_i^w, i \in \{1, \dots, k\}$ in Definition 4.1.3 are WPA⁽²⁾ distributed, specifically $U_i^w \sim \text{WPA}^{(2)}(\lambda_i, \rho)$, with their corresponding pgfs provided in Table 2.1. This gives the following

$$\mathbb{E}(s_i^{U_i^w}) = \psi_{U_i^w}(s_i) = \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)}.$$

Since the $W_{i\ell}^w$ s are independent of each other and independent of Z_{k+1}^w , it follows in the same way as in Theorem 4.2.2 that

$$\mathbb{E}\left(\prod_{i=1}^k s_i^{W_i^w}\right) = \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \frac{(e^{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} - 1)}{\prod_{i=1}^k \psi_1(s_i)}$$

and the joint pgf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(e^{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} - 1 \right) \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2}. \end{aligned}$$

□

4.2.2 Moments and moment-based functions

The moments of the multivariate distributions can be obtained in a similar way to the bivariate case presented in Section 3.2.2. This involves calculating the first, second, and joint partial derivatives of the pgfs in (4.1), (4.2), and (4.3) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}, i \neq j$.

For $(N_1, \dots, N_k) \sim \text{MPA}_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (4.1) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}$, $i \neq j$ are

$$\begin{aligned} \frac{\partial \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i} &= \frac{\psi_1(s_i)^2}{(1-\rho)s_i^2} \left[\lambda_i + \frac{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} \right] \psi_{N_1, \dots, N_k}(s_1, \dots, s_k), \\ \frac{\partial^2 \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i \partial s_j} &= \frac{\psi_1(s_i)^2 \psi_1(s_j)^2}{(1-\rho)^2 s_i^2 s_j^2} \left[\lambda_{k+1} + \left[\lambda_i + \frac{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} \right] \right. \\ &\quad \left. \times \left[\lambda_j + \frac{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_j)} \right] \right] \psi_{N_1, \dots, N_k}(s_1, \dots, s_k), \\ \frac{\partial^2 \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i^2} &= \frac{\psi_1(s_i)^2}{(1-\rho)^2 s_i^4} [2(\psi_1(s_i) - (1-\rho)s_i) \\ &\quad + \psi_1(s_i)^2 \left[\lambda_i + \frac{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} \right]] \left[\lambda_i + \frac{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} \right] \\ &\quad \times \psi_{N_1, \dots, N_k}(s_1, \dots, s_k). \end{aligned}$$

For $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (4.2) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}$, $i \neq j$ are

$$\begin{aligned} \frac{\partial \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i} &= \frac{\psi_1(s_i)}{(1-\rho)s_i^2} \left[2 + \lambda_i \psi_1(s_i) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i \partial s_j} &= \frac{\psi_1(s_i) \psi_1(s_j)}{(1-\rho)^2 s_i^2 s_j^2} \left[\left[\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \right. \\ &\quad \left. + \left[2 + \lambda_i \psi_1(s_i) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \right. \\ &\quad \left. \times \left[2 + \lambda_j \psi_1(s_j) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \right] \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i^2} &= \frac{\psi_1(s_i)^2}{(1-\rho)^2 s_i^4} \left[\left[\frac{\psi_1(s_i) - 2(1-\rho)s_i}{\psi_1(s_i)} \right] \right. \\ &\quad \times \left[2 + \lambda_i \psi_1(s_i) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \\ &\quad + \left[\frac{2(1-\rho)s_i^2}{\psi_1(s_i)} + \lambda_i \psi_1(s_i) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right] \\ &\quad \left. + \left[2 + \lambda_i \psi_1(s_i) + \lambda_{k+1} \prod_{i=1}^k \psi_1(s_i) \right]^2 \right] \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \end{aligned}$$

and for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (4.3) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}$, $i \neq j$ are

$$\begin{aligned} & \frac{\partial \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i} \\ &= \frac{\psi_1(s_i)^2}{(1-\rho)s_i^2} \left[\frac{\lambda_i}{(1-e^{-\lambda_i \psi_1(s_i)})} + \frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})} \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} - \frac{2}{\psi_1(s_i)} \right] \\ & \times \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \end{aligned}$$

$$\begin{aligned} & \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i \partial s_j} \\ &= \frac{\psi_1(s_i)^2 \psi_1(s_j)^2}{(1-\rho)^2 s_i^2 s_j^2} \left[\left[\frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{r=1}^k \psi_1(s_r)})} - \frac{\lambda_{k+1}^2 e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} \prod_{i=1}^k \psi_1(s_i)}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})^2} \right] \right. \\ & \times \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i) \psi_1(s_j)} + \left[\frac{\lambda_i}{(1-e^{-\lambda_i \psi_1(s_i)})} + \frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})} \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} - \frac{2}{\psi_1(s_i)} \right] \\ & \times \left. \left[\frac{\lambda_j}{(1-e^{-\lambda_j \psi_1(s_j)})} + \frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})} \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_j)} - \frac{2}{\psi_1(s_j)} \right] \right] \\ & \times \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \end{aligned}$$

$$\begin{aligned} & \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i^2} \\ &= \frac{\psi_1(s_i)^4}{(1-\rho)^2 s_i^4} \left[\frac{2(\psi_1(s_i) - (1-\rho)s_i)}{\psi_1(s_i)^2} \right. \\ & \times \left[\frac{\lambda_i}{(1-e^{-\lambda_i \psi_1(s_i)})} + \frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})} \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} - \frac{2}{\psi_1(s_i)} \right] \\ & - \left[\frac{\lambda_i^2 e^{-\lambda_i \psi_1(s_i)}}{(1-e^{-\lambda_i \psi_1(s_i)})^2} + \frac{\lambda_{k+1}^2 e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})^2} \frac{\prod_{i=1}^k \psi_1(s_i)^2}{\psi_1(s_i)^2} - \frac{2}{\psi_1(s_i)^2} \right] \\ & + \left. \left[\frac{\lambda_i}{(1-e^{-\lambda_i \psi_1(s_i)})} + \frac{\lambda_{k+1}}{(1-e^{-\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)})} \frac{\prod_{i=1}^k \psi_1(s_i)}{\psi_1(s_i)} - \frac{2}{\psi_1(s_i)} \right]^2 \right] \\ & \times \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k). \end{aligned}$$

4.3 Type I multivariate Pólya-Aeppli distribution

4.3.1 Joint probability mass function

Theorem 4.3.1. *The pmf of $(N_1, \dots, N_k) \sim \text{MPA}_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is*

$$\begin{aligned}
 f(0, \dots, 0) &= e^{-\lambda_{k+1}} \prod_{i=1}^k e^{-\lambda_i} \\
 f(n_a, \dots, n_b, 0, \dots, 0) &= \left(\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) f(0, \dots, 0), \\
 \{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
 \{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d \\
 f(n_1, \dots, n_k) &= \left[\left(\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) \right. \\
 &\quad \left. + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right) \right] f(0, \dots, 0), \\
 \{n_1, \dots, n_k\} &\in \{1, 2, \dots\},
 \end{aligned}$$

where $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i \in \{1, \dots, k\}$ and $L_{n_i}^\alpha(x_i)$ is the Laguerre polynomial in (A.3).

Proof. For the first part of the proof, $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows from (A.17) and (4.1) that

$$f(0, \dots, 0) = \psi_{N_1, \dots, N_k}(0, \dots, 0) = e^{-\lambda_{k+1}} \prod_{i=1}^k e^{-\lambda_i}.$$

The second part of the proof is derived for the case where at least one, but not all, of the $\{n_1, \dots, n_k\}$ is zero, with the rest being greater than zero. Therefore, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$ and consider $\{n_1, \dots, n_k\}$ such that $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, where $a \leq b$ and $c \leq d$. Since $U_i \sim PA(\lambda_i, \rho)$, $i \in \{1, \dots, k\}$, from the pgf in Table 2.1, (4.1) and

(A.1), it follows that

$$\begin{aligned}
& \psi_{N_1, \dots, N_k}(s_a, \dots, s_b, 0, \dots, 0) \\
&= \sum_{n_a, \dots, n_b=1}^{\infty} f(n_a, \dots, n_b, 0, \dots, 0) s_a^{n_a} \dots s_b^{n_b} \\
&= e^{-\lambda_{k+1}} \left(\prod_{i=a}^b e^{-\lambda_i(1-\psi_1(s_i))} \right) \left(\prod_{j=c}^d e^{-\lambda_j} \right) \\
&= e^{-\lambda_{k+1}} \left(\prod_{i=a}^b \psi_{U_i}(s_i) \right) \left(\prod_{j=c}^d e^{-\lambda_j} \right) \\
&= \sum_{n_a, \dots, n_b=1}^{\infty} \left[e^{-\lambda_{k+1}} \left(\prod_{i=a}^b P(U_i = n_i) \right) \left(\prod_{j=c}^d e^{-\lambda_j} \right) \right] s_a^{n_a} \dots s_b^{n_b} \\
&= \sum_{n_a, \dots, n_b=1}^{\infty} \left[e^{-\lambda_{k+1}} \left(\prod_{i=a}^b \frac{e^{-\lambda_i} \lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) \left(\prod_{j=c}^d e^{-\lambda_j} \right) \right] s_a^{n_a} \dots s_b^{n_b} \\
&= \sum_{n_a, \dots, n_b=1}^{\infty} \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) f(0, \dots, 0) \right] s_a^{n_a} \dots s_b^{n_b},
\end{aligned}$$

where $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i \in \{a, \dots, b\}$. The result for the second part of the proof follows from this.

In the third and final part of the proof, the *pmf* is derived for $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Let $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$ and $z_i = \rho s_i$, $i \in \{1, \dots, k\}$, then using (A.1) the *pgf* in (4.1) can be written as

$$\begin{aligned}
& \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} f(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\
&= e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k e^{-\lambda_i(1-\psi_1(s_i))} \\
&= \left[e^{\lambda_{k+1} \prod_{i=1}^k \frac{(1-\rho)s_i}{1-\rho s_i}} \prod_{i=1}^k e^{\lambda_i \left(\frac{(1-\rho)s_i}{1-\rho s_i} \right)} \right] f(0, \dots, 0) \\
&= \left[e^{\lambda_{k+1} \prod_{i=1}^k \frac{(1-\rho)}{\rho} \frac{z_i}{1-z_i}} \prod_{i=1}^k e^{\left(\frac{x_i z_i}{z_i - 1} \right)} \right] f(0, \dots, 0) \\
&= \left[\left(\prod_{i=1}^k e^{\left(\frac{x_i z_i}{z_i - 1} \right)} \right) \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \frac{(1-\rho)^r}{\rho^r} \frac{z_i^r}{(1-z_i)^r} \right) \right] f(0, \dots, 0) \\
&= \left[\left(\prod_{i=1}^k e^{\left(\frac{x_i z_i}{z_i - 1} \right)} \right) + \sum_{r=1}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \frac{(1-\rho)^r}{\rho^r} \frac{z_i^r}{(1-z_i)^r} e^{\left(\frac{x_i z_i}{z_i - 1} \right)} \right) \right] f(0, \dots, 0) \\
&= \psi_{N_1, \dots, N_k}^{(1)}(s_1, \dots, s_k) + \psi_{N_1, \dots, N_k}^{(2)}(s_1, \dots, s_k),
\end{aligned}$$

where

$$\begin{aligned}\psi_{N_1, \dots, N_k}^{(1)}(s_1, \dots, s_k) &= \left[\prod_{i=1}^k e^{\left(\frac{x_i z_i}{z_i - 1}\right)} \right] f(0, \dots, 0) \\ \psi_{N_1, \dots, N_k}^{(2)}(s_1, \dots, s_k) &= \left[\sum_{r=1}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \frac{(1-\rho)^r}{\rho^r} \frac{z_i^r}{(1-z_i)^r} e^{\left(\frac{x_i z_i}{z_i - 1}\right)} \right) \right] f(0, \dots, 0).\end{aligned}$$

The expression for $\psi_{N_1, \dots, N_k}^{(1)}(s_1, \dots, s_k)$ can be rewritten in terms of the *pgf's* for $U_i \sim \text{PA}(\lambda_i, \rho)$, $i \in \{1, \dots, k\}$ in Table 2.1 and (2.1), such that

$$\begin{aligned}\psi_{N_1, \dots, N_k}^{(1)}(s_1, \dots, s_k) &= \sum_{n_1, \dots, n_k=1}^{\infty} f^{(1)}(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\ &= \left[\prod_{i=1}^k e^{\left(\frac{x_i z_i}{z_i - 1}\right)} \right] f(0, \dots, 0) \\ &= e^{-\lambda_{k+1}} \prod_{i=1}^k e^{-\lambda_i(1-\psi_1(s_i))} \\ &= e^{-\lambda_{k+1}} \prod_{i=1}^k \psi_{U_i}(s_i) \\ &= \sum_{n_1, \dots, n_k=1}^{\infty} \left[e^{-\lambda_{k+1}} \prod_{i=1}^k P(U_i = n_i) \right] s_1^{n_1} \dots s_k^{n_k}. \\ &= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\left(\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) f(0, \dots, 0) \right] s_1^{n_1} \dots s_k^{n_k}.\end{aligned}$$

Using the generating function for Laguerre polynomials in (A.2), as well as the mathematical results in (A.1), $\psi_{N_1, \dots, N_k}^{(2)}(s_1, \dots, s_k)$ can be rewritten as

$$\begin{aligned}
& \psi_{N_1, \dots, N_k}^{(2)}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} f^{(2)}(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\
&= \left[\sum_{r=1}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \frac{(1-\rho)^r}{\rho^r} \frac{z_i^r}{(1-z_i)^r} e^{\left(\frac{x_i z_i}{z_i-1}\right)} \right) \right] f(0, \dots, 0) \\
&= \left[\sum_{r=0}^{\infty} \frac{\lambda_{k+1}^{r+1}}{(r+1)!} \left(\prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\rho^{r+1}} \frac{z_i^{r+1}}{(1-z_i)^{r+1}} e^{\left(\frac{x_i z_i}{z_i-1}\right)} \right) \right] f(0, \dots, 0) \\
&= \left[\sum_{r=0}^{\infty} \frac{\lambda_{k+1}^{r+1}}{(r+1)!} \prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\rho^{r+1}} z_i^{r+1} \sum_{n_i=0}^{\infty} L_{n_i}^r(x_i) z_i^{n_i} \right] f(0, \dots, 0) \\
&= \left[\sum_{n_1, \dots, n_k=0}^{\infty} \sum_{r=0}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+1)!} \prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\rho^{r+1}} L_{n_i-r}^r(x_i) z_i^{n_i+1} \right] f(0, \dots, 0) \\
&= \left[\sum_{n_1, \dots, n_k=1}^{\infty} \sum_{r=0}^{\min(n_1, \dots, n_k)-1} \frac{\lambda_{k+1}^{r+1}}{(r+1)!} \prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\rho^{r+1}} L_{n_i-r-1}^r(x_i) z_i^{n_i} \right] f(0, \dots, 0) \\
&= \left[\sum_{n_1, \dots, n_k=1}^{\infty} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k \frac{(1-\rho)^r}{\rho^r} L_{n_i-r}^{r-1}(x_i) z_i^{n_i} \right] f(0, \dots, 0) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\left(\sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right) f(0, \dots, 0) \right] s_1^{n_1} \dots s_k^{n_k}.
\end{aligned}$$

The result for the third part of the proof follows, such that

$$\begin{aligned}
& f(n_1, \dots, n_k) \\
&= f^{(1)}(n_1, \dots, n_k) + f^{(2)}(n_1, \dots, n_k) \\
&= \left[\left(\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) \right. \\
&\quad \left. + \sum_{r=1}^{\min(n_1, \dots, n_k)} \left(\frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right) \right] f(0, \dots, 0).
\end{aligned}$$

□

4.3.3 Conditional distributions and expected values

Theorem 4.3.4. *The pgf of N_2, \dots, N_k conditioned on N_1 for $(N_1, \dots, N_k) \sim \text{MPA}_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is*

$$\begin{aligned} \psi_{N_2, \dots, N_k | (N_1=0)}(s_2, \dots, s_k) &= \prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \\ \psi_{N_2, \dots, N_k | (N_1=n_1)}(s_2, \dots, s_k) &= \frac{\left(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)\right) L_{n_1-1}^1(t_1)}{(\lambda_1 + \lambda_{k+1}) L_{n_1-1}^1(y_1)} \prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))}, \\ n_1 &\in \{1, 2, \dots\}, \end{aligned}$$

where $y_1 = \frac{-(\lambda_1 + \lambda_{k+1})(1-\rho)}{\rho}$, $t_1 = \frac{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\rho)}{\rho}$ and $L_n^\alpha(x)$ is the Laguerre polynomial in (A.3).

Proof. Let $y_1 = \frac{-(\lambda_1 + \lambda_{k+1})(1-\rho)}{\rho}$ and $t_1 = \frac{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\rho)}{\rho}$. From (A.35) and using the marginal pmf in (4.7) for $N_1 \sim \text{PA}(\lambda_1 + \lambda_{k+1}, \rho)$, it follows that the pgf in (4.1) can be written as

$$\begin{aligned} &\psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} f(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\ &= \sum_{n_1=0}^{\infty} f(n_1) s_1^{n_1} \left[\sum_{n_2, \dots, n_k=0}^{\infty} f(n_2, \dots, n_k | n_1) s_2^{n_2} \cdots s_k^{n_k} \right] \\ &= \sum_{n_1=0}^{\infty} \psi_{N_2, \dots, N_k | (N_1=n_1)}(s_2, \dots, s_k) f(n_1) s_1^{n_1} \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \right] \left[e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} \right]. \end{aligned}$$

From (4.6) the expression $e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))}$ is the pgf of a PA $(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i), \rho)$ distributed random variable. It then follows from (4.7) that for $n_1 = 0$

$$\begin{aligned} &e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} \\ &= e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} \end{aligned}$$

and for $n_1 \in \{1, 2, \dots\}$

$$\begin{aligned} &e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} \\ &= \sum_{n_1=1}^{\infty} \frac{\left(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)\right) (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(t_1) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} s_1^{n_1}. \end{aligned}$$

Therefore, $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ can also be written as

$$\begin{aligned} & \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \right] e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} \\ & \quad + e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \right] \\ & \quad \times \sum_{n_1=1}^{\infty} \frac{(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(t_1) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} s_1^{n_1} \end{aligned}$$

and it follows that the *pgf* of N_2, \dots, N_k conditioned on N_1 for $n_1 = 0$ is

$$\begin{aligned} & \psi_{N_2, \dots, N_k | (N_1=0)}(s_2, \dots, s_k) \\ &= \frac{e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left(\prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \right) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{e^{-(\lambda_1 + \lambda_{k+1})}} \\ &= \prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \end{aligned}$$

and for $n_1 \in \{1, 2, \dots\}$

$$\begin{aligned} & \psi_{N_2, \dots, N_k | (N_1=n_1)}(s_2, \dots, s_k) \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))} \\ & \quad \times \left(\frac{\left((\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) (1-\rho) \rho^{n_1-1} \right) L_{n_1-1}^1(t_1) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{n_1} \right. \\ & \quad \left. \frac{(\lambda_1 + \lambda_{k+1}) (1-\rho) \rho^{n_1-1}}{n_1} L_{n_1-1}^1(y_1) e^{-(\lambda_1 + \lambda_{k+1})} \right) \\ &= \frac{(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) L_{n_1-1}^1(t_1)}{(\lambda_1 + \lambda_{k+1}) L_{n_1-1}^1(y_1)} \prod_{i=2}^k e^{-\lambda_i(1-\psi_1(s_i))}. \end{aligned}$$

□

4.3.4 Method of moments estimates

Let $(N_1, \dots, N_k) \sim \text{MPA}_I(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$. Furthermore, for

$i, j \in \{1, \dots, k\}, i \neq j$, let

$$\begin{aligned}\bar{n}_i &= \frac{1}{m} \sum_{\ell=1}^m n_{i\ell} \\ s_{ij} &= \frac{1}{(m-1)} \sum_{\ell=1}^m (n_{i\ell} - \bar{n}_i) (n_{j\ell} - \bar{n}_j) \\ s_i^2 &= \frac{1}{(m-1)} \sum_{\ell=1}^m (n_{i\ell} - \bar{n}_i)^2.\end{aligned}$$

The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$, $\text{Cov}(N_i^w, N_j^w)$ and $\text{Var}(N_i^w)$, where $i, j \in \{1, \dots, k\}, i \neq j$ as presented in Table 4.1 in Section 4.2.2.

Therefore, if $\hat{\rho}$ is the MoM estimator for ρ , the MoM estimator for λ_{k+1} is

$$\begin{aligned}\frac{1}{2} \sum_{i,j=1,i \neq j}^k \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})^2} &= \frac{1}{2} \sum_{i,j=1,i \neq j}^k s_{ij} \\ \binom{k}{2} \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})^2} &= \frac{1}{2} \sum_{i,j=1,i \neq j}^k s_{ij} \\ \hat{\lambda}_{k+1} &= (1-\hat{\rho})^2 \frac{(k-2)!}{k!} \sum_{i,j=1,i \neq j}^k s_{ij}.\end{aligned}$$

An MoM estimator for λ_i is

$$\begin{aligned}\frac{\hat{\lambda}_i + \hat{\lambda}_{k+1}}{(1-\hat{\rho})} &= \bar{n}_i \\ \hat{\lambda}_i &= (1-\hat{\rho}) \bar{n}_i - \hat{\lambda}_{k+1} \\ \hat{\lambda}_i &= (1-\hat{\rho}) \bar{n}_i - (1-\hat{\rho})^2 \frac{(k-2)!}{k!} \sum_{i,j=1,i \neq j}^k s_{ij}.\end{aligned}$$

In order to obtain $\hat{\rho}$, the sample moment s_i^2 is equated with the population moment $\text{Var}(N_i^w)$. It follows that

$$\begin{aligned} \sum_{i=1}^k \frac{(1+\hat{\rho})}{(1-\hat{\rho})^2} (\hat{\lambda}_i + \hat{\lambda}_{k+1}) &= \sum_{i=1}^k s_i^2 \\ \sum_{i=1}^k \frac{(1+\hat{\rho})}{(1-\hat{\rho})} \bar{n}_i &= \sum_{i=1}^k s_i^2 \\ (1+\hat{\rho}) \sum_{i=1}^k \bar{n}_i &= (1-\hat{\rho}) \sum_{i=1}^k s_i^2 \\ \sum_{i=1}^k \bar{n}_i + \hat{\rho} \sum_{i=1}^k \bar{n}_i &= \sum_{i=1}^k s_i^2 - \hat{\rho} \sum_{i=1}^k s_i^2 \\ \hat{\rho} \sum_{i=1}^k s_i^2 + \hat{\rho} \sum_{i=1}^k \bar{n}_i &= \sum_{i=1}^k s_i^2 - \sum_{i=1}^k \bar{n}_i \\ \hat{\rho} &= \frac{\sum_{i=1}^k s_i^2 - \sum_{i=1}^k \bar{n}_i}{\sum_{i=1}^k s_i^2 + \sum_{i=1}^k \bar{n}_i}. \end{aligned}$$

4.3.5 Maximum likelihood estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 4.3.5 for the $WMPA_I$ distribution. The Newton-Raphson algorithm uses the equation in (4.13) for $t \geq 1$ and the MoM estimates in Subsection 4.3.4 are used as initial values of the parameters $\lambda_1^{(0)}, \dots, \lambda_k^{(0)}, \lambda_{k+1}^{(0)}$ and $\rho^{(0)}$ for the MPA_I distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F in (4.11) and the Hessian matrix in (4.12), such that the Newton-Raphson algorithm in (4.13) can be utilized, the first and second derivatives for each element of the pmf in (4.5) need to be calculated with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ .

The results of the calculation of the first and second derivatives of the pmf in (4.5) for the $WMPA_I$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. The first and second derivatives of $f(0, \dots, 0)$, $f(n_a, \dots, n_b, 0, \dots, 0)$ where $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, and $f(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
g_j &= \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} + f(n_1, \dots, n_k) \\
&= \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.1}$$

$$\begin{aligned}
g'_{j,j} &= \frac{\partial g_j}{\partial \lambda_j} \\
&= \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j^2} + \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \frac{(1-\rho) n_j}{\rho \lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i \rho^{n_i-1} (1-\rho)}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho)^2 L_{n_j-r-1}^{r+1}(x_j)}{\rho^2 L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - \frac{1}{\rho} g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.2}$$

$$\begin{aligned}
g_{k+1} &= \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} + f(n_1, \dots, n_k) \\
&= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.3}$$

$$\begin{aligned}
g'_{k+1,k+1} &= \frac{\partial g_{k+1}}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} + \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \sum_{r=2, (n_1, \dots, n_k) \neq 1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_{k+1}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.4}$$

$$\begin{aligned}
g'_{j,\ell} &= \frac{\partial g_j}{\partial \lambda_\ell} \\
&= \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} + \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&= \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{n_\ell L_{n_\ell-1}^0(x_\ell)}{\lambda_\ell L_{n_\ell-1}^1(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^r (1-\rho)^2 L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell)}{r! \rho^2 L_{n_j-r}^{r-1}(x_j) L_{n_\ell-r}^{r-1}(x_\ell)} \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.5}$$

$$\begin{aligned}
g'_{j,k+1} &= \frac{\partial g_j}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} + \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.6}$$

$$\begin{aligned}
g'_{k+1,j} &= \frac{\partial g_{k+1}}{\partial \lambda_j} \\
&= \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \lambda_j} + \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_{k+1}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.7}$$

Let $j \in \{a, \dots, b\}$, then

$$\begin{aligned}
h_j &= \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} + f(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
&\quad \{n_a, \dots, n_b\} \in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.8}$$

$$\begin{aligned}
h'_{j,j} &= \frac{\partial h_j}{\partial \lambda_j} \\
&= \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= \frac{(1-\rho) n_j}{\rho \lambda_j} f(n_a, \dots, n_b, 0, \dots, 0) - \frac{1}{\rho} h_j, \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
&\quad \{n_a, \dots, n_b\} \in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.9}$$

$$\begin{aligned}
h'_{j,\ell} &= \frac{\partial h_j}{\partial \lambda_\ell} \\
&= \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell}, \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
&\quad \{n_a, \dots, n_b\} \in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.10}$$

$$\begin{aligned}
h'_{j,k+1} &= \frac{\partial h_j}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}}, \tag{S.4.0.11} \\
\{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
\{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. It then follows that the first and second derivatives of the pmf in (4.5) are

First derivative with respect to λ_j

$$\begin{aligned}
\frac{\partial f(0, \dots, 0)}{\partial \lambda_j} &= -f(0, \dots, 0) \\
\frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} &= h_j - f(n_a, \dots, n_b, 0, \dots, 0), \\
& \quad j \in \{a, \dots, b\} \\
\frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} &= -f(n_a, \dots, n_b, 0, \dots, 0), \tag{S.4.0.12} \\
& \quad j \notin \{a, \dots, b\} \\
\frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} &= g_j - f(n_1, \dots, n_k), \\
& \quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

First derivative with respect to λ_{k+1}

$$\begin{aligned}
\frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} &= -f(0, \dots, 0) \\
\frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} &= -f(n_a, \dots, n_b, 0, \dots, 0) \\
\frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} &= g_{k+1} - f(n_1, \dots, n_k), \tag{S.4.0.13} \\
& \quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

First derivative with respect to ρ

$$\begin{aligned}
\frac{\partial f(0, \dots, 0)}{\partial \rho} &= 0 \\
\frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} &= \frac{\sum_{i=a}^b n_i}{\rho} f(n_a, \dots, n_b, 0, \dots, 0) - \frac{\sum_{i=a}^b \lambda_i h_i}{(1-\rho)\rho}, \\
\frac{\partial f(n_1, \dots, n_k)}{\partial \rho} &= \frac{\sum_{i=1}^k n_i}{\rho} f(n_1, \dots, n_k) - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g_{k+1}}{(1-\rho)\rho}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.14}$$

Second derivative with respect to λ_j

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j^2} &= f(0, \dots, 0) \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} &= h'_{j,j} - h_j + f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} &= f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j^2} &= g'_{j,j} - g_j + f(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

Second derivative with respect to λ_{k+1}

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_{k+1}^2} &= f(0, \dots, 0) \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}^2} &= f(n_a, \dots, n_b, 0, \dots, 0), \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} &= g'_{k+1,k+1} - g_{k+1} + f(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= f(0, \dots, 0) \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= \frac{1}{f(n_a, \dots, n_b, 0, \dots, 0)} h_j h_\ell \\
&\quad - h_j - h_\ell + f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j, \ell \in \{a, \dots, b\} \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= f(n_a, \dots, n_b, 0, \dots, 0) - h_j, \\
&\quad j \in \{a, \dots, b\}, \ell \notin \{a, \dots, b\} \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j, \ell \notin \{a, \dots, b\} \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} &= g'_{j,\ell} - g_\ell + f(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

Second derivative with respect to λ_j and λ_{k+1}

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= f(0, \dots, 0) \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= f(n_a, \dots, n_b, 0, \dots, 0) - h_j, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= f(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} &= g'_{j,k+1} - g_j + f(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

Second derivative with respect to ρ and λ_j

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \rho \partial \lambda_j} &= 0 \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} &= \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{h_j}{(1-\rho)\rho} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} &= \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}, \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} &= \frac{\sum_{i=1}^k n_i}{\rho} \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} - \frac{g_j}{(1-\rho)\rho} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g'_{k+1,j}}{(1-\rho)\rho}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} &= 0 \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} &= \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{\sum_{i=a}^b \lambda_i h'_{i,k+1}}{(1-\rho)\rho} \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} &= \frac{\sum_{j=1}^k n_j}{\rho} \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - \frac{kg_{k+1}}{(1-\rho)\rho} \\
&\quad - \frac{\sum_{j=1}^k \lambda_j g'_{j,k+1}}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g'_{k+1}}{(1-\rho)\rho}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \rho^2} &= 0 \\
\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho^2} &= \left[\frac{\sum_{i=a}^b n_i - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&\quad - \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \\
&\quad \times \left[\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=a}^b n_i}{(1-\rho)\rho} f(n_a, \dots, n_b, 0, \dots, 0) \\
\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho^2} &= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{j=1}^k n_j}{(1-\rho)\rho} f(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (4.5) are given here.

First derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.5), it follows that

$$\frac{\partial f(0, \dots, 0)}{\partial \lambda_j} = -f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (4.5), (S.4.0.8) and (S.4.0.12) and using

(S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= \frac{1}{\lambda_j} \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \left[\frac{(1-\rho) L_{n_j-2}^2(x_j)}{\rho L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \frac{f(n_a, \dots, n_b, 0, \dots, 0)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_j} \\
&= \left[\frac{(1-\rho)}{\rho} \left[\frac{n_j \rho}{\lambda_j (1-\rho)} L_{n_j-1}^0(x_j) - \frac{\rho}{\lambda_j (1-\rho)} L_{n_j-1}^1(x_j) \right] \frac{1}{L_{n_j-1}^1(x_j)} \right] \\
&\quad \times f(n_a, \dots, n_b, 0, \dots, 0) - f(n_a, \dots, n_b, 0, \dots, 0) + \frac{f(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j} \\
&= \frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} f(n_a, \dots, n_b, 0, \dots, 0) - f(n_a, \dots, n_b, 0, \dots, 0) \\
&= h_j - f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (4.5) and (S.4.0.12), it follows that

$$\begin{aligned}
& \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= \frac{f(n_a, \dots, n_b, 0, \dots, 0)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_j} \\
&= -f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.5), (S.4.0.1) and (S.4.0.12) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \frac{1}{\lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \left[\frac{(1-\rho) L_{n_j-2}^2(x_j)}{\rho L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) + \frac{f(n_1, \dots, n_k)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_j} \\
&= \frac{1}{\lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \left[\frac{(1-\rho)}{\rho} \left[\frac{n_j \rho}{\lambda_j (1-\rho)} L_{n_j-1}^0(x_j) - \frac{\rho}{\lambda_j (1-\rho)} L_{n_j-1}^1(x_j) \right] \frac{1}{L_{n_j-1}^1(x_j)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - f(n_1, \dots, n_k)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - f(n_1, \dots, n_k) \\
&= g_j - f(n_1, \dots, n_k).
\end{aligned}$$

First derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (4.5), it follows that

$$\frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} = -f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. Substituting from (4.5) and (S.4.0.13), it follows that

$$\begin{aligned} & \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\ &= \frac{f(n_a, \dots, n_b, 0, \dots, 0)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} \\ &= -f(n_a, \dots, n_b, 0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (4.5), (S.4.0.3) and (S.4.0.13), it follows that

$$\begin{aligned} & \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\ &= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\ & \quad + \frac{f(n_1, \dots, n_k)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} \\ &= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - f(n_1, \dots, n_k) \\ &= g_{k+1} - f(n_1, \dots, n_k). \end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial f(0, \dots, 0)}{\partial \rho} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. Substituting from (4.5), (S.4.0.8) and (S.4.0.14) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=a}^b n_i - b + a - 1 \right)}{\rho} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\frac{(b-a+1)}{(1-\rho)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\sum_{i=a}^b \frac{\lambda_i}{\rho^2} \frac{L_{n_i-2}^2(x_i)}{L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \frac{f(n_a, \dots, n_b, 0, \dots, 0)}{f(0, \dots, 0)} \frac{f(0, \dots, 0)}{\partial \rho} \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} - \frac{(b-a+1)}{(1-\rho)\rho} \right] f(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \sum_{i=a}^b \frac{\lambda_i}{\rho^2} \left[\frac{n_i \rho}{\lambda_i (1-\rho)} L_{n_i-1}^0(x_i) - \frac{\rho}{\lambda_i (1-\rho)} L_{n_i-1}^1(x_i) \right] \\
&\quad \times \frac{1}{L_{n_i-1}^1(x_i)} f(n_a, \dots, n_b, 0, \dots, 0) \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} - \frac{(b-a+1)}{(1-\rho)\rho} + \sum_{i=a}^b \frac{1}{(1-\rho)\rho} \right] f(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \left[\sum_{i=a}^b \frac{n_i}{(1-\rho)\rho} \frac{L_{n_i-1}^0(x_i)}{L_{n_i-1}^1(x_i)} \right] f(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{\sum_{i=a}^b n_i}{\rho} f(n_a, \dots, n_b, 0, \dots, 0) - \frac{\sum_{i=a}^b \lambda_i h_i}{(1-\rho)\rho}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (4.5), (S.4.0.1), (S.4.0.3) and (S.4.0.14) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i - k\right)}{\rho} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\sum_{i=1}^k \frac{\lambda_i L_{n_i-2}^2(x_i)}{\rho^2 L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{\left(\sum_{i=1}^k n_i - kr\right)}{\rho} \right] \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{kr}{(1-\rho)} \right] \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^r}{r!} \left[\frac{\lambda_1 L_{n_1-r-1}^r(x_1)}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad \vdots \\
&\quad - \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^r}{r!} \left[\frac{\lambda_k L_{n_k-r-1}^r(x_k)}{\rho^2 L_{n_k-r}^{r-1}(x_k)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) + \frac{f(n_1, \dots, n_k)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \rho}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i - k \right)}{\rho} - \frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\sum_{i=1}^k \frac{\lambda_i}{\rho^2} \left[\frac{n_i \rho}{\lambda_i (1-\rho)} L_{n_i-1}^0(x_i) - \frac{\rho}{\lambda_i (1-\rho)} L_{n_i-1}^1(x_i) \right] \frac{1}{L_{n_i-1}^1(x_i)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \frac{\sum_{i=1}^k n_i}{\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \frac{k}{(1-\rho)\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{r \lambda_{k+1}^r}{r!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \frac{\lambda_1}{\rho^2} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^r}{r!} \frac{L_{n_1-r-1}^r(x_1)}{L_{n_1-r}^{r-1}(x_1)} \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad \vdots \\
&\quad - \frac{\lambda_k}{\rho^2} \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^r}{r!} \frac{L_{n_k-r-1}^r(x_k)}{L_{n_k-r}^{r-1}(x_k)} \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{k}{(1-\rho)\rho} + \sum_{i=1}^k \frac{1}{(1-\rho)\rho} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad - \left[\sum_{i=1}^k \frac{\lambda_i}{(1-\rho)\rho} \frac{n_i L_{n_i-1}^0(x_i)}{\lambda_i L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&\quad + \frac{\sum_{i=1}^k n_i}{\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&\quad \vdots \\
&\quad - \frac{\lambda_k}{(1-\rho)\rho} \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho) L_{n_k-r-1}^r(x_k)}{\rho L_{n_k-r}^{r-1}(x_k)} \right] \\
&\quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&= \frac{\sum_{i=1}^k n_i}{\rho} f(n_1, \dots, n_k) - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g_{k+1}}{(1-\rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.12), it follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j^2} = f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (S.4.0.8), (S.4.0.9), (S.4.0.12) and using

(A.10), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&= \frac{(1-\rho) n_j}{\rho} \frac{L_{n_j-2}^1(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} f(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} + \frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_j} h_j \\
&= \frac{(1-\rho) n_j}{\rho} \frac{L_{n_j-1}^1(x_j) - L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} f(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - h_j \\
&= \frac{(1-\rho) n_j}{\rho} \frac{1}{\lambda_j} f(n_a, \dots, n_b, 0, \dots, 0) - \frac{1}{\rho} h_j - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= h'_{j,j} - h_j + f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (S.4.0.12), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&= - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.5), (S.4.0.1), (S.4.0.2), (S.4.0.12) and using (A.10), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j^2} \\
&= \left[\frac{(1-\rho) n_j}{\rho} \frac{L_{n_j-2}^1(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i \rho^{n_i-1} (1-\rho)}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1, 2}^{\min(n_j-2, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho)^2 L_{n_j-r-2}^{r+1}(x_j)}{\rho^2 L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&- \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} + \frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_j} g_j \\
&= \left[\frac{(1-\rho) n_j}{\rho} \frac{L_{n_j-1}^1(x_j) - L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i \rho^{n_i-1} (1-\rho)}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho)^2 [L_{n_j-r-1}^{r+1}(x_j) - L_{n_j-r-1}^r(x_j)]}{\rho^2 L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} - g_j \\
&= \frac{(1-\rho) n_j}{\rho} \frac{L_{n_j-1}^1(x_j) - L_{n_j-1}^0(x_j)}{\lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i \rho^{n_i-1} (1-\rho)}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{r!} \left[\frac{(1-\rho)^2 L_{n_j-r-1}^{r+1}(x_j)}{\rho^2 L_{n_j-r}^{r-1}(x_j)} \right] \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - \frac{1}{\rho} g_j - g_j + f(n_1, \dots, n_k) \\
&= g'_{j,j} - g_j + f(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (S.4.0.14), it follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_{k+1}^2} = f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. Substituting from (S.4.0.14), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}^2} \\ &= -\frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\ &= f(n_a, \dots, n_b, 0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (S.4.0.3), (S.4.0.4) and (S.4.0.14), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} \\ &= \sum_{r=2, (n_1, \dots, n_k) \neq 1}^{\min(n_1, \dots, n_k)} \frac{(r-1) \lambda_{k+1}^{r-2}}{(r-1)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\ & \quad - \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} + \frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} g_{k+1} \\ &= \sum_{r=2, (n_1, \dots, n_k) \neq 1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\ & \quad - \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - g_{k+1} \\ &= g'_{k+1, k+1} - g_{k+1} + f(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.12), it follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} = f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j, \ell \in \{a, \dots, b\}$. Substituting from (S.4.0.8) and (S.4.0.12), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\ &= \frac{1}{f(n_a, \dots, n_b, 0, \dots, 0)} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} h_j - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\ &= \frac{1}{f(n_a, \dots, n_b, 0, \dots, 0)} h_j h_\ell - h_j - h_\ell + f(n_a, \dots, n_b, 0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$, $\ell \notin \{a, \dots, b\}$. Substituting from (S.4.0.8) and (S.4.0.12), it

follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{f(n_a, \dots, n_b, 0, \dots, 0)} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} h_j - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\
&= f(n_a, \dots, n_b, 0, \dots, 0) - h_j.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j, \ell \notin \{a, \dots, b\}$. Substituting from (S.4.0.12), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\
&= f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.1), (S.4.0.5) and (S.4.0.12) and using (S.3.0.1), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{\lambda_\ell} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{(1-\rho) L_{n_\ell-2}^2(x_\ell)}{\rho L_{n_\ell-1}^1(x_\ell)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^r (1-\rho) L_{n_j-r-1}^r(x_j)}{r! \rho L_{n_j-r}^{r-1}(x_j)} \\
&\times \left[\frac{(1-\rho) L_{n_\ell-r-1}^r(x_\ell)}{\rho L_{n_\ell-r}^{r-1}(x_\ell)} \right] \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
&- \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_\ell} + \frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_\ell} g_j \\
&= \frac{1}{\lambda_\ell} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{(1-\rho)}{\rho} \left[\frac{n_\ell \rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^0(x_\ell) - \frac{\rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^1(x_\ell) \right] \right] \\
&\times \frac{1}{L_{n_\ell-1}^1(x_\ell)} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^r (1-\rho)^2 L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell)}{r! \rho^2 L_{n_j-r}^{r-1}(x_j) L_{n_\ell-r}^{r-1}(x_\ell)} \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_j - g_\ell + f(n_1, \dots, n_k)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{n_\ell L_{n_\ell-1}^0(x_\ell)}{\lambda_\ell L_{n_\ell-1}^1(x_\ell)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f(0, \dots, 0) \\
&+ \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^r (1-\rho)^2 L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell)}{r! \rho^2 L_{n_j-r}^{r-1}(x_j) L_{n_\ell-r}^{r-1}(x_\ell)} \\
&\times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_j - \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&= g'_{j,\ell} - g_\ell + f(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to λ_j and λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.13), it follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} = f(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (S.4.0.8) and (S.4.0.13), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \frac{1}{f(n_a, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} h_j - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\
&= f(n_a, \dots, n_b, 0, \dots, 0) - h_j.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (S.4.0.13), it follows that

$$\begin{aligned}
& \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&= - \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\
&= f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.1), (S.4.0.6) and (S.4.0.13), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{r \lambda_{k+1}^{r-1}}{r!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
& \quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) \\
& \quad - \frac{f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} + \frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda_{k+1}} g_j \\
&= \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-1}}{(r-1)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
& \quad \times \left[\prod_{i=1}^k (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] f(0, \dots, 0) - g_j - \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= g'_{j, k+1} - g_j + f(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to ρ and λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. It follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \rho \partial \lambda_j} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. From (S.4.0.8) and (S.4.0.9), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} \\
&= \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - \frac{h_j}{(1-\rho)\rho} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. From (S.4.0.10), it follows that

$$\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} = \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}.$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. From (S.4.0.2), (S.4.0.5) and (S.4.0.7), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} \\ &= \frac{\sum_{i=1}^k n_i}{\rho} \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_j} - \frac{g_j}{(1-\rho)\rho} - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g'_{k+1,j}}{(1-\rho)\rho} \end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. From (S.4.0.11), it follows that

$$\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} = \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{\sum_{i=a}^b \lambda_i h'_{i,k+1}}{(1-\rho)\rho}.$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. From (S.4.0.4) and (S.4.0.6), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} \\ &= \frac{\sum_{j=1}^k n_j}{\rho} \frac{\partial f(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - \frac{k g_{k+1}}{(1-\rho)\rho} - \frac{\sum_{j=1}^k \lambda_j g'_{j,k+1}}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g'_{k+1}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial^2 f(0, \dots, 0)}{\partial \rho^2} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. Substituting from (S.4.0.8), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho^2} \\
&= \frac{\sum_{i=a}^b n_i}{\rho} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} - \frac{\sum_{i=a}^b n_i}{\rho^2} f(n_a, \dots, n_b, 0, \dots, 0) \\
&+ \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \sum_{i=a}^b \lambda_i h_i \\
&- \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&= \left[\frac{\sum_{i=a}^b n_i - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&- \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&- \frac{\sum_{i=a}^b n_i}{(1-\rho)\rho} f(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (S.4.0.1) and (S.4.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(n_1, \dots, n_k)}{\partial \rho^2} \\
&= \frac{\sum_{i=1}^k n_i}{\rho} \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} - \frac{\sum_{i=1}^k n_i}{\rho^2} f(n_1, \dots, n_k) \\
&+ \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \left[\sum_{i=1}^k \lambda_i g_i + k\lambda_{k+1}g_{k+1} \right] \\
&- \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right] \\
&- \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right] \\
&= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} - \frac{\sum_{i=1}^k n_i}{(1-\rho)\rho} f(n_1, \dots, n_k) \\
&- \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right] \\
&- \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f(n_1, \dots, n_k)}{\partial \rho} \right].
\end{aligned}$$

4.4 Weighted Type I multivariate Pólya-Aeppli distribution Case I

4.4.1 Joint probability mass function

Theorem 4.4.1. The pmf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is

$$f^w(n_1, \dots, n_k) = 0, \quad \{n_1, \dots, n_k\} \in \{0, 1\}$$

$$f^w(n_1, \dots, n_k) = e^{-\lambda_{k+1}} \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2} (1-\rho)^{kr}}{(r-2)!} \prod_{i=1}^k e^{-\lambda_i} \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i), \quad (\text{S.4.0.15})$$

$$\{n_1, \dots, n_k\} \in \{2, 3, \dots\},$$

where $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i \in \{1, \dots, k\}$ and $L_{n_i}^\alpha(x_i)$ is the Laguerre polynomial in (A.3).

Proof. For the first part of the proof, $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows from (A.17) and (4.2) that

$$f^w(n_1, n_2) = 0.$$

In the second part of the proof, the pmf is derived for $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Let $x_i = -\lambda_i \frac{(1-\rho)}{\rho}$ and $z_i = \rho s_i$, where $i \in \{1, \dots, k\}$ and using the generating function for Laguerre polynomials in

(A.2), as well as the mathematical results in (A.1), the pgf in (4.2) can be written as

$$\begin{aligned}
& \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=2}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\
&= e^{-\lambda_{k+1}(1-\prod_{i=1}^k \psi_1(s_i))} \prod_{i=1}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \\
&= e^{-\lambda_{k+1}} e^{\lambda_{k+1} \prod_{i=1}^k \left(\frac{(1-\rho)s_i}{1-\rho s_i}\right)} \prod_{i=1}^k \left(\frac{(1-\rho)s_i}{1-\rho s_i}\right)^2 e^{-\lambda_i} e^{\lambda_i \left(\frac{(1-\rho)s_i}{1-\rho s_i}\right)} \\
&= e^{-\lambda_{k+1}} e^{\lambda_{k+1} \prod_{i=1}^k \left(\frac{(1-\rho)z_i}{\rho(1-z_i)}\right)} \prod_{i=1}^k \frac{(1-\rho)^2}{\rho^2} \frac{z_i^2}{(1-z_i)^2} e^{-\lambda_i} e^{\left(\frac{\lambda_i z_i}{z_i-1}\right)} \\
&= e^{-\lambda_{k+1}} \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \left(\frac{(1-\rho)z_i}{\rho(1-z_i)}\right) \right)^r \prod_{i=1}^k \frac{(1-\rho)^2}{\rho^2} \frac{z_i^2}{(1-z_i)^2} e^{-\lambda_i} e^{\left(\frac{\lambda_i z_i}{z_i-1}\right)} \\
&= e^{-\lambda_{k+1}} \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k \frac{(1-\rho)^{r+2}}{\rho^{r+2}} \frac{z_i^{r+2}}{(1-z_i)^{r+2}} e^{-\lambda_i} e^{\left(\frac{\lambda_i z_i}{z_i-1}\right)} \\
&= e^{-\lambda_{k+1}} \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k e^{-\lambda_i} \frac{(1-\rho)^{r+2}}{\rho^{r+2}} z_i^{r+2} \sum_{n_i=0}^{\infty} L_{n_i}^{r+1}(x_i) z_i^{n_i} \\
&= e^{-\lambda_{k+1}} \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k e^{-\lambda_i} \frac{(1-\rho)^{r+2}}{\rho^{r+2}} \sum_{n_i=0}^{\infty} L_{n_i}^{r+1}(x_i) z_i^{n_i+r+2} \\
&= \sum_{n_1, \dots, n_k=0}^{\infty} \left[e^{-\lambda_{k+1}} \sum_{r=0}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k e^{-\lambda_i} \frac{(1-\rho)^{r+2}}{\rho^{r+2}} L_{n_i-r}^{r+1}(x_i) \right] z_1^{n_1+2} \cdots z_k^{n_k+2} \\
&= \sum_{n_1, \dots, n_k=2}^{\infty} \left[e^{-\lambda_{k+1}} \sum_{r=0}^{\min(n_1, \dots, n_k)-2} \frac{\lambda_{k+1}^r}{r!} \prod_{i=1}^k e^{-\lambda_i} \frac{(1-\rho)^{r+2}}{\rho^{r+2}} L_{n_i-r-2}^{r+1}(x_i) \right] z_1^{n_1} \cdots z_k^{n_k} \\
&= \sum_{n_1, \dots, n_k=2}^{\infty} \left[e^{-\lambda_{k+1}} \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \prod_{i=1}^k e^{-\lambda_i} \frac{(1-\rho)^r}{\rho^r} L_{n_i-r}^{r-1}(x_i) \right] z_1^{n_1} \cdots z_k^{n_k} \\
&= \sum_{n_1, \dots, n_k=2}^{\infty} \left[e^{-\lambda_{k+1}} \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] s_1^{n_1} \cdots s_k^{n_k}
\end{aligned}$$

The result for the second part of the proof follows from this.

□

4.4.3 Conditional distributions and expected values

Theorem 4.4.4. The pgf of N_2^w, \dots, N_k^w conditioned on N_1^w for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is

$$\begin{aligned} \psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) &= 0, \quad n_1 \in \{0, 1\} \\ \psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) &= \left[\prod_{i=2}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \right] \frac{L_{n_1-2}^1(t_1)}{L_{n_1-2}^1(y_1)}, \quad n_1 \in \{2, 3, \dots\}, \end{aligned}$$

where $y_1 = \frac{-(\lambda_1 + \lambda_{k+1})(1-\rho)}{\rho}$, $t_1 = \frac{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\rho)}{\rho}$ and $L_n^\alpha(x)$ is the Laguerre polynomial in (A.3).

Proof. Let $y_1 = \frac{-(\lambda_1 + \lambda_{k+1})(1-\rho)}{\rho}$ and $t_1 = \frac{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\rho)}{\rho}$. From (A.35) and using the marginal pmf in (4.16), it follows that the pgf in (4.2) can be written as

$$\begin{aligned} &\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\ &= \sum_{n_1=2}^{\infty} f^w(n_1) s_1^{n_1} \left[\sum_{n_2, \dots, n_k=2}^{\infty} f^w(n_2, \dots, n_k | n_1) s_2^{n_2} \dots s_k^{n_k} \right] \\ &= \sum_{n_1=2}^{\infty} \psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) f^w(n_1) s_1^{n_1} \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \right] \left[\psi_1(s_1)^2 e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} \right]. \end{aligned}$$

It follows that the expression $\psi_1(s_1)^2 e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))}$ is the pgf of a random variable with the same distribution as (4.15) and parameters $\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)$ and ρ . It then follows from (4.16) that for $n_1 \in \{0, 1\}$

$$\psi_1(s_1)^2 e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} = 0$$

and for $n_1 \in \{2, 3, \dots\}$

$$\begin{aligned} &\psi_1(s_1)^2 e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))(1-\psi_1(s_1))} \\ &= \sum_{n_1=2}^{\infty} \left[(1-\rho)^2 \rho^{n_1-2} L_{n_1-2}^1(t_1) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} \right] s_1^{n_1}. \end{aligned}$$

Therefore, $\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)$ can also be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \right] \\ & \quad \times \sum_{n_1=2}^{\infty} \left[(1-\rho)^2 \rho^{n_1-2} L_{n_1-2}^1(t_1) e^{-(\lambda_1+\lambda_{k+1}\prod_{i=2}^k \psi_1(s_i))} \right] s_1^{n_1} \end{aligned}$$

and it follows that the pgf of N_2^w, \dots, N_k^w conditioned on N_1^w where $n_1 \in \{0, 1\}$ is

$$\psi_{N_2^w, \dots, N_k^w | (N_1=n_1)}(s_2, \dots, s_k) = 0$$

and for $n_1 \in \{2, 3, \dots\}$

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1=n_1)}(s_2, \dots, s_k) \\ &= e^{-\lambda_{k+1}(1-\prod_{i=2}^k \psi_1(s_i))} \left[\prod_{i=2}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \right] \\ & \quad \times \left(\frac{(1-\rho)^2 \rho^{n_1-2} L_{n_1-2}^1(t_1) e^{-(\lambda_1+\lambda_{k+1}\prod_{i=2}^k \psi_1(s_i))}}{(1-\rho)^2 \rho^{n_1-2} L_{n_1-2}^1(y_1) e^{-(\lambda_1+\lambda_{k+1})}} \right) \\ &= \left[\prod_{i=2}^k \psi_1(s_i)^2 e^{-\lambda_i(1-\psi_1(s_i))} \right] \frac{L_{n_1-2}^1(t_1)}{L_{n_1-2}^1(y_1)}. \end{aligned}$$

□

4.4.4 Method of moments estimates

Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(1)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$.

Let \bar{n}_i , s_{ij} and s_i^2 be the sample moments given in (4.10), where $i, j \in \{1, \dots, k\}$, $i \neq j$. The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$, $\text{Cov}(N_i^w, N_j^w)$ and $\text{Var}(N_i^w)$ as presented in Table 4.1 in Section 4.2.2.

Therefore, if $\hat{\rho}$ is the MoM estimator for ρ , the MoM estimator for λ_{k+1} is

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1, i \neq j}^k \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})^2} &= \frac{1}{2} \sum_{i,j=1, i \neq j}^k s_{ij} \\ \binom{k}{2} \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})^2} &= \frac{1}{2} \sum_{i,j=1, i \neq j}^k s_{ij} \\ \hat{\lambda}_{k+1} &= (1-\hat{\rho})^2 \frac{(k-2)!}{k!} \sum_{i,j=1, i \neq j}^k s_{ij}. \end{aligned}$$

An MoM estimator for λ_i is

$$\begin{aligned}\frac{\hat{\lambda}_i + \hat{\lambda}_{k+1} + 2}{(1 - \hat{\rho})} &= \bar{n}_i \\ \hat{\lambda}_i &= (1 - \hat{\rho}) \bar{n}_i - \hat{\lambda}_{k+1} - 2 \\ \hat{\lambda}_i &= (1 - \hat{\rho}) \bar{n}_i - (1 - \hat{\rho})^2 \frac{(k-2)!}{k!} \sum_{i,j=1, i \neq j}^k s_{ij} - 2.\end{aligned}$$

In order to obtain $\hat{\rho}$, the sample moment s_i^2 is equated with the population moment $\text{Var}(N_i^w)$. It follows that

$$\begin{aligned}\sum_{i=1}^k \frac{(1 + \hat{\rho})(\hat{\lambda}_i + \hat{\lambda}_{k+1} + 2) - 2}{(1 - \hat{\rho})^2} &= \sum_{i=1}^k s_i^2 \\ \sum_{i=1}^k \frac{(1 + \hat{\rho})(1 - \hat{\rho})\bar{n}_i - 2}{(1 - \hat{\rho})^2} &= \sum_{i=1}^k s_i^2 \\ (1 + \hat{\rho})(1 - \hat{\rho}) \sum_{i=1}^k \bar{n}_i - 2k &= (1 - \hat{\rho})^2 \sum_{i=1}^k s_i^2 \\ \hat{\rho}^2 \left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right) - 2\hat{\rho} \sum_{i=1}^k s_i^2 &= \left(\sum_{i=1}^k \bar{n}_i - s_i^2 \right) - 2k \\ \hat{\rho}^2 - 2\hat{\rho} \frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} + \left[\frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} \right]^2 &= \frac{\left(\sum_{i=1}^k \bar{n}_i - s_i^2 \right) - 2k}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} + \left[\frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} \right]^2 \\ \left[\hat{\rho} - \frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} \right]^2 &= \frac{\left(\sum_{i=1}^k \bar{n}_i \right)^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)^2} - \frac{2k}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)}.\end{aligned}$$

It follows that the MoM estimator for ρ is

$$\begin{aligned}\hat{\rho} &= \frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} \pm \sqrt{\frac{\left(\sum_{i=1}^k \bar{n}_i \right)^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)^2} - \frac{2k}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)}} \\ &= \frac{\sum_{i=1}^k s_i^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)} - \sqrt{\frac{\left(\sum_{i=1}^k \bar{n}_i \right)^2}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)^2} - \frac{2k}{\left(\sum_{i=1}^k \bar{n}_i + s_i^2 \right)}}.\end{aligned}$$

4.4.5 Maximum likelihood estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 4.4.5 for the $\text{WMPA}_I^{(1)}$ distribution. The Newton-Raphson algorithm uses the equation in (4.13) for $t \geq 1$ and the MoM estimates in Subsection 4.4.4) are used as initial values of the parameters $\lambda_1^{(0)}, \dots, \lambda_k^{(0)}, \lambda_{k+1}^{(0)}$ and $\rho^{(0)}$ for the $\text{WMPA}_I^{(1)}$ distribution. The iterative process is

repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{\text{th}}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (4.14) need to be calculated with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ .

The results of the calculation of the first and second derivatives of the *pmf* for the $\text{WMPA}_I^{(1)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{0, 1\}$, and $f^w(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$ with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
 g_j &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} + f^w(n_1, \dots, n_k) \\
 &= \sum_{r=2, n_j \neq 2}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
 &\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}}, \\
 &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
 \end{aligned} \tag{S.4.0.16}$$

$$\begin{aligned}
 g'_{j,j} &= \frac{\partial g_j}{\partial \lambda_j} \\
 &= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
 &= \sum_{r=2, n_j \neq 2, 3}^{\min(n_j-2, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho)^2}{\rho^2} \frac{L_{n_j-r-2}^{r+1}(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
 &\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - g_j, \\
 &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
 \end{aligned} \tag{S.4.0.17}$$

$$\begin{aligned}
g_{k+1} &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} + f^w(n_1, \dots, n_k) \\
&= \sum_{r=3, (n_1, \dots, n_k) \neq 2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}}, \quad (\text{S.4.0.18}) \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g'_{k+1, k+1} &= \frac{\partial g_{k+1}}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \sum_{r=4, (n_1, \dots, n_k) \neq 2, 3}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-4}}{(r-4)!} \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - g_{k+1}, \quad (\text{S.4.0.19}) \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g'_{j, \ell} &= \frac{\partial g_j}{\partial \lambda_\ell} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&= \sum_{r=2, n_j, n_\ell \neq 2}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \\
&\quad \times \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \left[\frac{(1-\rho) L_{n_\ell-r-1}^r(x_\ell)}{\rho L_{n_\ell-r}^{r-1}(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - g_j, \quad (\text{S.4.0.20}) \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned}$$

$$\begin{aligned}
g'_{j,k+1} &= \frac{\partial g_j}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \sum_{r=3, n_j \neq 2, 3, (n_1, \dots, n_k) \neq n_j \neq 2}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.4.0.21}$$

$$\begin{aligned}
g'_{k+1,j} &= \frac{\partial g_{k+1}}{\partial \lambda_j} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \lambda_j} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \sum_{r=3, n_j \neq 2, 3, (n_1, \dots, n_k) \neq n_j \neq 2}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - g_{k+1}, \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.4.0.22}$$

It then follows that the first and second derivatives of the pmf in (4.14) are

First derivative with respect to λ_j

$$\begin{aligned}
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} &= g_j - f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.4.0.23}$$

First derivative with respect to λ_{k+1}

$$\begin{aligned}
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} &= g_{k+1} - f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}.
\end{aligned} \tag{S.4.0.24}$$

First derivative with respect to ρ

$$\begin{aligned} \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} &= \left[\frac{\left(\sum_{i=1}^k n_i \right)}{\rho} - \frac{2k}{(1-\rho)\rho} \right] f^w(n_1, \dots, n_k) \\ &\quad - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g_{k+1}}{(1-\rho)\rho}, \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned} \tag{S.4.0.25}$$

Second derivative with respect to λ_j

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} &= g'_{j,j} - g_j + f^w(n_1, \dots, n_k), \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to λ_{k+1}

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} &= g'_{k+1,k+1} - g_{k+1} + f^w(n_1, \dots, n_k), \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} &= g'_{j,\ell} - g_\ell + f^w(n_1, \dots, n_k), \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to λ_j and λ_{k+1}

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} &= g'_{j,k+1} - g_{k+1} + f^w(n_1, \dots, n_k), \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ and λ_j

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\ &\quad - \frac{g_j}{(1-\rho)\rho} - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g'_{k+1,j}}{(1-\rho)\rho}, \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\ &\quad - \frac{kg_{k+1}}{(1-\rho)\rho} - \frac{\sum_{i=1}^k \lambda_i g'_{i,k+1}}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g'_{k+1,k+1}}{(1-\rho)\rho}, \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho^2} &= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{1}{(1-\rho)} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\ &\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\ &\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\ &\quad - \frac{\sum_{i=1}^k n_i}{(1-\rho)\rho} f^w(n_1, \dots, n_k), \\ &\quad \{n_1, \dots, n_k\} \in \{2, 3, \dots\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (4.14) are given here.

First derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (4.14) and (S.4.0.16), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\ &= \sum_{r=2, n_j \neq 2}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho) L_{n_j-r-1}^r(x_j)}{\rho L_{n_j-r}^{r-1}(x_j)} \right] \\ & \quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - f^w(n_1, \dots, n_k) \\ &= g_j - f^w(n_1, \dots, n_k). \end{aligned}$$

First derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (4.14) and (S.4.0.18), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\ &= \sum_{r=3, (n_1, \dots, n_k) \neq 2}^{\min(n_1, \dots, n_k)} \frac{(r-2) \lambda_{k+1}^{r-3}}{(r-2)!} \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\ & \quad - f^w(n_1, \dots, n_k) \\ &= \sum_{r=3, (n_1, \dots, n_k) \neq 2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\ & \quad - f^w(n_1, \dots, n_k) \\ &= g_{k+1} - f^w(n_1, \dots, n_k). \end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (4.14), (S.4.0.16) and (S.4.0.18), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&= \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{\left(\sum_{i=1}^k n_i - kr\right)}{\rho} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad - \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{kr}{(1-\rho)} \right] \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad - \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{\lambda_1 L_{n_1-r-1}^r(x_1)}{\rho^2 L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad \vdots \\
&\quad - \sum_{r=2, n_k \neq 2}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{\lambda_k L_{n_k-r-1}^r(x_k)}{\rho^2 L_{n_k-r}^{r-1}(x_k)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&= \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{\left(\sum_{i=1}^k n_i - kr\right)}{\rho} - \frac{kr}{(1-\rho)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad - \frac{\lambda_1}{(1-\rho)\rho} \sum_{r=2, n_1 \neq 2}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho) L_{n_1-r-1}^r(x_1)}{\rho} - \frac{L_{n_1-r}^{r-1}(x_1)}{L_{n_1-r}^{r-1}(x_1)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad \vdots \\
&\quad - \frac{\lambda_k}{(1-\rho)\rho} \sum_{r=2, n_k \neq 2}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho) L_{n_k-r-1}^r(x_k)}{\rho} - \frac{L_{n_k-r}^{r-1}(x_k)}{L_{n_k-r}^{r-1}(x_k)} \right] \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i\right)}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \\
&\quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \sum_{r=2}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i\right)}{\rho} - \frac{2k}{(1-\rho)\rho} \right] f^w(n_1, \dots, n_k) - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g_{k+1}}{(1-\rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (4.14), (S.4.0.16), (S.4.0.17) and (S.4.0.23), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} \\ &= \sum_{r=2, n_j \neq 2, 3}^{\min(n_j-2, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho)^2 L_{n_j-r-2}^{r+1}(x_j)}{\rho^2 L_{n_j-r}^{r-1}(x_j)} \right] \\ & \quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x) \right] e^{-\lambda_{k+1}} - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} - g_j \\ &= g'_{j,j} - g_j + f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (S.4.0.18), (S.4.0.19) and (S.4.0.24), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} \\ &= \sum_{r=4, (n_1, \dots, n_k) \neq 2, 3}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-4}}{(r-4)!} \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x) \right] e^{-\lambda_{k+1}} \\ & \quad - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - g_{k+1} \\ &= g'_{k+1, k+1} - g_{k+1} + f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (S.4.0.16), (S.4.0.20) and (S.4.0.23), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\ &= \sum_{r=2, n_j, n_\ell \neq 2}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r-2}}{(r-2)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\ & \quad \times \left[\frac{(1-\rho)}{\rho} \frac{L_{n_\ell-r-1}^r(x_\ell)}{L_{n_\ell-r}^{r-1}(x_\ell)} \right] \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} \\ & \quad - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell} - g_j \\ &= g'_{j,\ell} - g_\ell + f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to λ_j and λ_{k+1}

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (S.4.0.16), (S.4.0.21) and (S.4.0.24), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\ &= \sum_{r=3, n_j \neq 2, 3, (n_1, \dots, n_k) \neq n_j \neq 2}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r-3}}{(r-3)!} \left[\frac{(1-\rho)}{\rho} \frac{L_{n_j-r-1}^r(x_j)}{L_{n_j-r}^{r-1}(x_j)} \right] \\ & \quad \times \left[\prod_{i=1}^k e^{-\lambda_i} (1-\rho)^r \rho^{n_i-r} L_{n_i-r}^{r-1}(x_i) \right] e^{-\lambda_{k+1}} - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - g_j \\ &= g'_{j,k+1} - g_{k+1} + f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to ρ and λ_j

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. From (S.4.0.17), (S.4.0.20) and (S.4.0.22), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} \\ &= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} - \frac{g_j}{(1-\rho)\rho} \\ & \quad - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k \lambda_{k+1} g'_{k+1,j}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. From (S.4.0.19) and (S.4.0.21), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} \\ &= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - \frac{k g_{k+1}}{(1-\rho)\rho} \\ & \quad - \frac{\sum_{i=1}^k \lambda_i g'_{i,k+1}}{(1-\rho)\rho} - \frac{k \lambda_{k+1} g'_{k+1,k+1}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, \dots, n_k\} \in \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho^2} = 0.$$

Let $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$. Substituting from (S.4.0.16) and (S.4.0.18), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho^2} \\
&= \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \left[\frac{\sum_{i=1}^k n_i}{\rho^2} - \frac{2k}{(1-\rho)\rho^2} + \frac{2k}{(1-\rho)^2\rho} \right] f^w(n_1, \dots, n_k) \\
&\quad + \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \left[\sum_{i=1}^k \lambda_i g_i + k\lambda_{k+1}g_{k+1} \right] \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{1}{(1-\rho)} - \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=1}^k n_i}{(1-\rho)\rho} f^w(n_1, \dots, n_k).
\end{aligned}$$

4.5 Weighted Type I multivariate Pólya-Aeppli distribution Case II

4.5.1 Joint probability mass function

Theorem 4.5.1. *The pmf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is*

$$\begin{aligned}
 f^w(0, \dots, 0) &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \prod_{i=1}^k \frac{\lambda_i e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \\
 f^w(n_a, \dots, n_b, 0, \dots, 0) &= \left(\prod_{i=a}^b \frac{\lambda_i (1 - \rho) \rho^{n_i - 1}}{n_i (n_i + 1)} L_{n_i - 1}^2(x_i) \right) f^w(0, \dots, 0), \\
 \{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
 \{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d \\
 f^w(n_1, \dots, n_k) &= \left[\left(\prod_{i=1}^k \frac{\lambda_i (1 - \rho) \rho^{n_i - 1}}{n_i (n_i + 1)} L_{n_i - 1}^2(x_i) \right) \right. \\
 &\quad + \frac{\lambda_{k+1}}{2!} \left(\prod_{i=1}^k \frac{(1 - \rho) \rho^{n_i - 1}}{n_i} L_{n_i - 1}^1(x_i) \right) \\
 &\quad + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
 &\quad \times \left. \left(\prod_{i=1}^k \frac{(1 - \rho)^r \rho^{n_i - r}}{\lambda_i} \left(L_{n_i - r}^{r-1}(x_i) - \binom{n_i - 1}{n_i - r} \right) \right) \right] \\
 &\quad \times f^w(0, \dots, 0), \quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\},
 \end{aligned}$$

where $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i \in \{1, \dots, k\}$ and $L_{n_i}^\alpha(x_i)$ is the Laguerre polynomial in (A.3).

Proof. For the first part of the proof, $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. From (A.1), the pgf in (4.3) can be rewritten as

$$\begin{aligned}
 &\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\
 &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(e^{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} - 1 \right) \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \\
 &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left[\sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r \left(\prod_{i=1}^k \psi_1(s_i)^r \right)}{r!} - 1 \right] \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{1}{\psi_1(s_i)^2} \left[\sum_{n_i=0}^{\infty} \frac{\lambda_i^{n_i} \psi_1(s_i)^{n_i}}{n_i!} - 1 \right] \\
 &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \sum_{r=1}^{\infty} \frac{\lambda_{k+1}^r \prod_{i=1}^k \psi_1(s_i)^r}{r!} \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{1}{\psi_1(s_i)^2} \sum_{n_i=1}^{\infty} \frac{\lambda_i^{n_i} \psi_1(s_i)^{n_i}}{n_i!} \\
 &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \sum_{r=1}^{\infty} \frac{\lambda_{k+1}^r \prod_{i=1}^k \psi_1(s_i)^{r-1}}{r!} \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \sum_{n_i=1}^{\infty} \frac{\lambda_i^{n_i} \psi_1(s_i)^{n_i-1}}{n_i!} \\
 &= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left[\lambda_{k+1} + \sum_{r=2}^{\infty} \frac{\lambda_{k+1}^r \prod_{i=1}^k \psi_1(s_i)^{r-1}}{r!} \right] \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \left[\lambda_i + \sum_{n_i=2}^{\infty} \frac{\lambda_i^{n_i} \psi_1(s_i)^{n_i-1}}{n_i!} \right].
 \end{aligned}$$

Therefore, it follows from (A.17) that

$$\begin{aligned} f^w(0, \dots, 0) &= \psi_{N_1^w, \dots, N_k^w}(0, \dots, 0) \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \prod_{i=1}^k \frac{\lambda_i e^{-\lambda_i}}{(1 - e^{-\lambda_i})}. \end{aligned}$$

The second part of the proof is derived for the case where at least one, but not all, of the $\{n_1, \dots, n_k\}$ is zero, with the rest being greater than zero. Therefore, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$ and consider $\{n_1, \dots, n_k\}$ such that $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, where $a \leq b$ and $c \leq d$. Since $U_i^w \sim WPA^{(2)}(\lambda_i, \rho)$, $i \in \{1, \dots, k\}$, from Table 2.1, (2.3), (4.3) and (A.1) it follows that

$$\begin{aligned} &\psi_{N_1^w, \dots, N_k^w}(s_a, \dots, s_b, 0, \dots, 0) \\ &= \sum_{n_a, \dots, n_b=1}^{\infty} f^w(n_a, \dots, n_b, 0, \dots, 0) s_a^{n_a} \dots s_b^{n_b} \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(\prod_{i=a}^b \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)} \right) \left(\prod_{j=c}^d \frac{\lambda_j e^{-\lambda_j}}{(1 - e^{-\lambda_j})} \right) \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(\prod_{i=a}^b \psi_{U_i^w}(s_i) \right) \left(\prod_{j=c}^d \frac{\lambda_j e^{-\lambda_j}}{(1 - e^{-\lambda_j})} \right) \\ &= \sum_{n_a, \dots, n_b=1}^{\infty} \left[\frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(\prod_{i=a}^b P(U_i^w = n_i) \right) \left(\prod_{j=c}^d \frac{\lambda_j e^{-\lambda_j}}{(1 - e^{-\lambda_j})} \right) \right] s_a^{n_a} \dots s_b^{n_b} \\ &= \sum_{n_a, \dots, n_b=1}^{\infty} \left[\left(\prod_{i=a}^b \frac{\lambda_i (1 - \rho) \rho^{n_i - 1}}{n_i (n_i + 1)} L_{n_i - 1}^2(x_i) \right) f^w(0, \dots, 0) \right] s_a^{n_a} \dots s_b^{n_b}, \end{aligned}$$

where $x_i = -\lambda_i \frac{(1-\rho)}{\rho}$, $i \in \{a, \dots, b\}$. The result for the second part of the proof follows from this.

In the third and final part of the proof, the *pmf* is derived for $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. If we let $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$ and $z_i = \rho s_i$, $i \in \{1, \dots, k\}$ then using (A.1) the *pgf* in (4.3) can be written as

$$\begin{aligned}
& \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\
&= \frac{e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left(e^{\lambda_{k+1} \prod_{i=1}^k \psi_1(s_i)} - 1 \right) \prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \\
&= \left[\frac{1}{\lambda_{k+1}} \left(e^{\lambda_{k+1} \prod_{i=1}^k \left(\frac{(1-\rho)s_i}{1-\rho s_i} \right)} - 1 \right) \prod_{i=1}^k \frac{1}{\lambda_i \left(\frac{(1-\rho)s_i}{1-\rho s_i} \right)^2} \left(e^{\lambda_i \left(\frac{(1-\rho)s_i}{1-\rho s_i} \right)} - 1 \right) \right] f^w(0, \dots, 0) \\
&= \left[\frac{1}{\lambda_{k+1}} \left(e^{\lambda_{k+1} \prod_{i=1}^k \left(\frac{(1-\rho)}{\rho} \frac{z_i}{1-z_i} \right)} - 1 \right) \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho^2}{(1-\rho)^2} \frac{(1-z_i)^2}{z_i^2} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] f^w(0, \dots, 0) \\
&= \left[\frac{1}{\lambda_{k+1}} \left(\sum_{r=0}^{\infty} \frac{\lambda_{k+1}^r}{r!} \left(\prod_{i=1}^k \frac{(1-\rho)}{\rho} \frac{z_i}{1-z_i} \right)^r - 1 \right) \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho^2}{(1-\rho)^2} \frac{(1-z_i)^2}{z_i^2} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] \\
&\quad \times f^w(0, \dots, 0) \\
&= \left[\sum_{r=0}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right. \\
&\quad \left. - \frac{1}{\lambda_{k+1}} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho^2}{(1-\rho)^2} \frac{(1-z_i)^2}{z_i^2} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] f^w(0, \dots, 0) \\
&= \left[\sum_{r=1}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] f^w(0, \dots, 0) \\
&= \left[\left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho}{(1-\rho)} \frac{(1-z_i)}{z_i} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right. \\
&\quad \left. + \sum_{r=2}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] f^w(0, \dots, 0) \\
&= \left[\left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho}{(1-\rho)} \frac{(1-z_i)}{z_i} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) + \frac{\lambda_{k+1}}{2!} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right. \\
&\quad \left. + \sum_{r=3}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left(\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1} \right)} - 1 \right) \right) \right] f^w(0, \dots, 0) \\
&= \psi_{N_1^w, \dots, N_k^w}^{(1)}(s_1, \dots, s_k) + \psi_{N_1^w, \dots, N_k^w}^{(2)}(s_1, \dots, s_k) + \psi_{N_1^w, \dots, N_k^w}^{(3)}(s_1, \dots, s_k),
\end{aligned}$$

where

$$\begin{aligned}\psi_{N_1^w, \dots, N_k^w}^{(1)}(s_1, \dots, s_k) &= \left[\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho}{(1-\rho)} \frac{(1-z_i)}{z_i} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\ \psi_{N_1^w, \dots, N_k^w}^{(2)}(s_1, \dots, s_k) &= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\ \psi_{N_1^w, \dots, N_k^w}^{(3)}(s_1, \dots, s_k) &= \sum_{r=3}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0).\end{aligned}$$

The expression for $\psi_{N_1, \dots, N_k}^{(1)}(s_1, \dots, s_k)$ can be rewritten in terms of the pgfs for $U_i^w \sim \text{PA}(\lambda_i, \rho)$, $i \in \{1, \dots, k\}$ in Table 2.1 and (2.3), such that

$$\begin{aligned}\psi_{N_1^w, \dots, N_k^w}^{(1)}(s_1, \dots, s_k) &= \sum_{n_1, \dots, n_k=1}^{\infty} f^{w(1)}(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\ &= \left[\prod_{i=1}^k \frac{1}{\lambda_i} \frac{\rho}{(1-\rho)} \frac{(1-z_i)}{z_i} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left[\prod_{i=1}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)} \right] \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \prod_{i=1}^k \psi_{U_i^w}^w(s_i) \\ &= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \prod_{i=1}^k P(U_i^w = n_i) \right] s_1^{n_1} \dots s_k^{n_k} \\ &= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\left(\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right) f^w(0, \dots, 0) \right] s_1^{n_1} \dots s_k^{n_k}\end{aligned}$$

Using the generating function for Laguerre polynomials in (A.2), as well as the mathematical results in (A.1), $\psi_{N_1, \dots, N_k}^{(2)}(s_1, \dots, s_k)$ and $\psi_{N_1^w, \dots, N_k^w}^{(3)}(s_1, \dots, s_k)$ can be rewritten as

$$\begin{aligned}
& \psi_{N_1^w, \dots, N_k^w}^{(2)}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} f^{w(2)}(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-z_i)}{(1-z_i)} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{(1-z_i)}{\lambda_i} \left(\sum_{n_i=0}^{\infty} L_{n_i}^0(x_i) z_i^{n_i} - \sum_{n_i=0}^{\infty} z_i^{n_i} \right) \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{(1-z_i)}{\lambda_i} \sum_{n_i=0}^{\infty} \left(L_{n_i}^0(x_i) - 1 \right) z_i^{n_i} \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \sum_{n_i=0}^{\infty} \left(L_{n_i}^0(x_i) - 1 \right) z_i^{n_i} - \prod_{i=1}^k \frac{1}{\lambda_i} \sum_{n_i=0}^{\infty} \left(L_{n_i}^0(x_i) - 1 \right) z_i^{n_i+1} \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \sum_{n_i=1}^{\infty} \left(L_{n_i}^0(x_i) - 1 \right) z_i^{n_i} - \prod_{i=1}^k \frac{1}{\lambda_i} \sum_{n_i=1}^{\infty} \left(L_{n_i-1}^0(x_i) - 1 \right) z_i^{n_i} \right] f^w(0, \dots, 0) \\
&= \frac{\lambda_{k+1}}{2!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \sum_{n_i=1}^{\infty} \left(L_{n_i}^0(x_i) - L_{n_i-1}^0(x_i) \right) z_i^{n_i} \right] f^w(0, \dots, 0) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\frac{\lambda_{k+1}}{2!} \left(\prod_{i=1}^k \frac{\rho^{n_i}}{\lambda_i} \left(L_{n_i}^0(x_i) - L_{n_i-1}^0(x_i) \right) \right) f^w(0, \dots, 0) \right] s_1^{n_1} \cdots s_k^{n_k} \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\frac{\lambda_{k+1}}{2!} \left(\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) f^w(0, \dots, 0) \right] s_1^{n_1} \cdots s_k^{n_k}
\end{aligned}$$

and

$$\begin{aligned}
& \psi_{N_1^w, \dots, N_k^w}^{(3)}(s_1, \dots, s_k) \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} f^{w(3)}(n_1, \dots, n_k) s_1^{n_1} \dots s_k^{n_k} \\
&= \sum_{r=3}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{1}{\lambda_i} \frac{(1-\rho)^{r-2}}{\rho^{r-2}} \frac{z_i^{r-2}}{(1-z_i)^{r-2}} \left(e^{\left(\frac{x_i z_i}{z_i-1}\right)} - 1 \right) \right] f^w(0, \dots, 0) \\
&= \sum_{r=3}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{(1-\rho)^{r-2}}{\lambda_i \rho^{r-2}} \frac{z_i^{r-2}}{\rho^{r-2}} \left(\sum_{n_i=0}^{\infty} L_{n_i}^{r-3}(x_i) z_i^{n_i} - \sum_{n_i=0}^{\infty} \binom{n_i+r-3}{n_i} z_i^{n_i} \right) \right] f^w(0, \dots, 0) \\
&= \sum_{r=3}^{\infty} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{(1-\rho)^{r-2}}{\lambda_i \rho^{r-2}} \sum_{n_i=0}^{\infty} \left(L_{n_i}^{r-3}(x_i) - \binom{n_i+r-3}{n_i} \right) z_i^{n_i+r-2} \right] f^w(0, \dots, 0) \\
&= \sum_{r=0}^{\infty} \frac{\lambda_{k+1}^{r+2}}{(r+3)!} \left[\sum_{n_1, \dots, n_k=0}^{\infty} \left(\prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\lambda_i \rho^{r+1}} \left(L_{n_i}^r(x_i) - \binom{n_i+r}{n_i} \right) \right) \right. \\
&\quad \left. \times z_1^{n_1+r+1} \dots z_k^{n_k+r+1} \right] f^w(0, \dots, 0) \\
&= \sum_{n_1, \dots, n_k=0}^{\infty} \left[\sum_{r=0}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+2}}{(r+3)!} \left[\left(\prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\lambda_i \rho^{r+1}} \left(L_{n_i-r}^r(x_i) - \binom{n_i}{n_i-r} \right) \right) \right] \right. \\
&\quad \left. \times f^w(0, \dots, 0) \right] z_1^{n_1+1} \dots z_k^{n_k+1} \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\sum_{r=0}^{\min(n_1, \dots, n_k)-1} \frac{\lambda_{k+1}^{r+2}}{(r+3)!} \left[\left(\prod_{i=1}^k \frac{(1-\rho)^{r+1}}{\lambda_i \rho^{r+1}} \left(L_{n_i-r-1}^r(x_i) - \binom{n_i-1}{n_i-r-1} \right) \right) \right] \right. \\
&\quad \left. \times f^w(0, \dots, 0) \right] z_1^{n_1} \dots z_k^{n_k} \\
&= \sum_{n_1, \dots, n_k=1}^{\infty} \left[\sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left(\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right) \right. \\
&\quad \left. \times f^w(0, \dots, 0) \right] s_1^{n_1} \dots s_k^{n_k}.
\end{aligned}$$

The result for the third part of the proof follows, such that

$$\begin{aligned}
& f^w(n_1, \dots, n_k) \\
&= f^{w(1)}(n_1, \dots, n_k) + f^{w(2)}(n_1, \dots, n_k) + f^{w(3)}(n_1, \dots, n_k) \\
&= \left[\left(\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right) + \frac{\lambda_{k+1}}{2!} \left(\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right) \right. \\
&\quad \left. + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left(\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right) \right] f^w(0, \dots, 0).
\end{aligned}$$

□

4.5.3 Conditional distributions and expected values

Theorem 4.5.4. The pgf of N_2^w, \dots, N_k^w conditioned on N_1^w for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ is

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1=0)}(s_2, \dots, s_k) \\ &= \prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)} \\ & \psi_{N_2^w | (N_1^w = n_1)}(s_2) \\ &= \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \right] \\ & \quad \times \frac{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1 - m_1 + 3} - \lambda_1^{n_1 - m_1 + 3} - (\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1 - m_1 + 3}}{(n_1 - m_1 + 3)!}}{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1})^{n_1 - m_1 + 3} - \lambda_1^{n_1 - m_1 + 3} - \lambda_{k+1}^{n_1 - m_1 + 3}}{(n_1 - m_1 + 3)!}}, \\ & n_1 \in \{1, 2, \dots\}. \end{aligned}$$

Proof. From (A.35) and using the marginal pmf in (4.21), it follows that the pgf in (4.3) can be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\ &= \sum_{n_1=0}^{\infty} f^w(n_1) s_1^{n_1} \left[\sum_{n_2, \dots, n_k=0}^{\infty} f^w(n_2, \dots, n_k | n_1) s_2^{n_2} \cdots s_k^{n_k} \right] \\ &= \sum_{n_1=0}^{\infty} \psi_{N_2^w, \dots, N_k^w | (N_1=n_1)}(s_2, \dots, s_k) f^w(n_1) s_1^{n_1} \\ &= \frac{(e^{\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} - 1)}{(1 - e^{\lambda_{k+1}})} \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \right] \\ & \quad \times \left[\frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) (1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)})} \frac{(e^{\lambda_1 \psi_1(s_1)} - 1) (e^{(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) \psi_1(s_1)} - 1)}{\psi_1(s_1)^2} \right]. \end{aligned}$$

It follows that the expression

$$\frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) (1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)})} \frac{(e^{\lambda_1 \psi_1(s_1)} - 1) (e^{(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) \psi_1(s_1)} - 1)}{\psi_1(s_1)^2}$$

is the pgf of a random variable with the same distribution as (4.20) and parameters λ_1 , $\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)$ and ρ . It then follows from (4.21) that for $n_1 = 0$

$$\begin{aligned} & \frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} \left(e^{\lambda_1 \psi_1(s_1)} - 1 \right) \left(e^{(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) \psi_1(s_1)} - 1 \right)}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right) \psi_1(s_1)^2} \\ &= \frac{\lambda_1 \lambda_{k+1} \left(\prod_{i=2}^k \psi_1(s_i) \right) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right)} \end{aligned}$$

and for $n_1 \in \{1, 2, \dots\}$

$$\begin{aligned} & \frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))} \left(e^{\lambda_1 \psi_1(s_1)} - 1 \right) \left(e^{(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)) \psi_1(s_1)} - 1 \right)}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right) \psi_1(s_1)^2} \\ &= \sum_{n_1=1}^{\infty} \frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right)} \sum_{m_1=1}^{n_1} \binom{n_1 - 1}{m_1 - 1} (1 - \rho)^{n_1 - m_1 + 1} \rho^{m_1 - 1} \\ & \times \frac{(\lambda_1 + \lambda_{k+1})^{n_1 - m_1 + 3} - \lambda_1^{n_1 - m_1 + 3} - \left(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i) \right)^{n_1 - m_1 + 3}}{(n_1 - m_1 + 3)!} s_1^{n_1}. \end{aligned}$$

Therefore, $\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)$ can also be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \frac{\left(e^{\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} - 1 \right)}{(1 - e^{\lambda_{k+1}})} \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{\left(e^{\lambda_i \psi_1(s_i)} - 1 \right)}{\psi_1(s_i)^2} \right] \\ & \times \frac{\lambda_1 \lambda_{k+1} \left(\prod_{i=2}^k \psi_1(s_i) \right) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right)} \\ & + \frac{\left(e^{\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} - 1 \right)}{1 - (e^{\lambda_{k+1}})} \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{\left(e^{\lambda_i \psi_1(s_i)} - 1 \right)}{\psi_1(s_i)^2} \right] \\ & \times \sum_{n_1=1}^{\infty} \frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1}) \left(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} \right)} \sum_{m_1=1}^{n_1} \binom{n_1 - 1}{m_1 - 1} (1 - \rho)^{n_1 - m_1 + 1} \rho^{m_1 - 1} \\ & \times \frac{(\lambda_1 + \lambda_{k+1})^{n_1 - m_1 + 3} - \lambda_1^{n_1 - m_1 + 3} - \left(\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i) \right)^{n_1 - m_1 + 3}}{(n_1 - m_1 + 3)!} s_1^{n_1}. \end{aligned}$$

and it follows that the pgf of N_2^w conditioned on N_1^w for $n_1 = 0$ is

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1=0)}(s_2, \dots, s_k) \\ &= \frac{(1 - e^{-\lambda_1})(1 - e^{-\lambda_{k+1}})}{\lambda_1 \lambda_{k+1} e^{-(\lambda_1 + \lambda_{k+1})}} \frac{(e^{\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} - 1)}{(1 - e^{-\lambda_{k+1}})} \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \right] \\ & \times \frac{\lambda_1 \lambda_{k+1} \left(\prod_{i=2}^k \psi_1(s_i) \right) e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1})(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)})} \\ &= \prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)} \end{aligned}$$

and for $n_1 \in \{1, 2, \dots\}$

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1=n_1)}(s_2, \dots, s_k) \\ &= \frac{(1 - e^{-\lambda_1})(1 - e^{-\lambda_{k+1}})}{e^{-(\lambda_1 + \lambda_{k+1})}} \frac{(e^{\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)} - 1)}{(e^{\lambda_{k+1}} - 1)} \\ & \times \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \right] \frac{e^{-(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))}}{(1 - e^{-\lambda_1})(1 - e^{-\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i)})} \\ & \times \frac{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1-m_1+3} - \lambda_1^{n_1-m_1+3} - (\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1-m_1+3}}{(n_1 - m_1 + 3)!}}{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1})^{n_1-m_1+3} - \lambda_i^{n_1-m_1+3} - \lambda_{k+1}^{n_1-m_1+3}}{(n_1 - m_1 + 3)!}} \\ &= \left[\prod_{i=2}^k \frac{e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \frac{(e^{\lambda_i \psi_1(s_i)} - 1)}{\psi_1(s_i)^2} \right] \\ & \times \frac{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1-m_1+3} - \lambda_1^{n_1-m_1+3} - (\lambda_{k+1} \prod_{i=2}^k \psi_1(s_i))^{n_1-m_1+3}}{(n_1 - m_1 + 3)!}}{\sum_{m_1=1}^{n_1} \binom{n_1-1}{m_1-1} \left(\frac{\rho}{1-\rho}\right)^{m_1} \frac{(\lambda_1 + \lambda_{k+1})^{n_1-m_1+3} - \lambda_i^{n_1-m_1+3} - \lambda_{k+1}^{n_1-m_1+3}}{(n_1 - m_1 + 3)!}}. \end{aligned}$$

□

4.5.4 Method of moments estimates

Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_I^{(2)}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \rho)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$.

Let \bar{n}_i , s_{ij} and s_i^2 be the sample moments given in (4.10), where $i, j \in \{1, \dots, k\}$, $i \neq j$. The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$, $\text{Cov}(N_i^w, N_j^w)$ and $\text{Var}(N_i^w)$ where $i, j \in \{1, \dots, k\}$, $i \neq j$ as presented in Table 4.1 in Section 4.2.2.

It follows that if $\hat{\lambda}_1$ and $\hat{\lambda}_{k+1}$ are estimators for λ_1 and λ_{k+1} respectively, then

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1,i \neq j}^k s_{ij} &= \frac{1}{2} \sum_{i,j=1,i \neq j}^k \frac{\hat{\lambda}_{k+1} (1 - e^{-\hat{\lambda}_{k+1}}) - \hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1 - \hat{\rho})^2 (1 - e^{-\hat{\lambda}_{k+1}})^2} \\ \frac{1}{2} \sum_{i,j=1,i \neq j}^k s_{ij} &= \binom{k}{2} \frac{\hat{\lambda}_{k+1} (1 - e^{-\hat{\lambda}_{k+1}}) - \hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1 - \hat{\rho})^2 (1 - e^{-\hat{\lambda}_{k+1}})^2} \\ (1 - \hat{\rho})^2 &= \frac{k!}{(k-2)!} \frac{\hat{\lambda}_{k+1} (1 - e^{-\hat{\lambda}_{k+1}}) - \hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1 - e^{-\hat{\lambda}_{k+1}})^2 \sum_{i,j=1,i \neq j}^k s_{ij}} \end{aligned}$$

and an estimator for ρ is

$$\begin{aligned} \hat{\rho} &= 1 \pm \sqrt{\frac{k!}{(k-2)!} \frac{\hat{\lambda}_{k+1} (1 - e^{-\hat{\lambda}_{k+1}}) - \hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1 - e^{-\hat{\lambda}_{k+1}})^2 \sum_{i,j=1,i \neq j}^k s_{ij}}} \\ \hat{\rho} &= 1 - \sqrt{\frac{k!}{(k-2)!} \frac{\hat{\lambda}_{k+1} (1 - e^{-\hat{\lambda}_{k+1}}) - \hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1 - e^{-\hat{\lambda}_{k+1}})^2 \sum_{i,j=1,i \neq j}^k s_{ij}}}. \end{aligned}$$

As in Section 3.5.4, explicit expressions for the MoM estimators of the parameters are unavailable, therefore the Newton-Raphson method is employed to estimate λ_1 and λ_{k+1} (Press et al. [3]).

This is done by setting up a system of equations and iteratively refining the parameter estimates. At each iteration, the Jacobian matrix of the system is computed and used to update the estimates until convergence is reached.

Therefore, the Newton-Raphson algorithm can be employed to derive the MoM estimators for λ_1 and λ_{k+1} , such that

$$\begin{bmatrix} \hat{\lambda}_1^{(t+1)} \\ \hat{\lambda}_{k+1}^{(t+1)} \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_1^{(t)} \\ \hat{\lambda}_{k+1}^{(t)} \end{bmatrix} - \left[J_g \left(\hat{\lambda}_1^{(t)}, \hat{\lambda}_{k+1}^{(t)} \right) \right]^{-1} g \left(\hat{\lambda}_1^{(t)}, \hat{\lambda}_{k+1}^{(t)} \right)$$

for $t \geq 1$.

Substituting the estimator for ρ in (4.24) into the expressions for $E(N_1^w) = \hat{\mu}_1$ and $\text{Var}(N_1^w) = \hat{\sigma}_1^2$, the function $g(\hat{\lambda}_1, \hat{\lambda}_{k+1})$ can be defined as

$$\begin{aligned} & g(\hat{\lambda}_1, \hat{\lambda}_{k+1}) \\ &= \begin{bmatrix} g_1(\hat{\lambda}_1, \hat{\lambda}_{k+1}) \\ g_2(\hat{\lambda}_1, \hat{\lambda}_{k+1}) \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mu}_1 - \bar{n}_1 \\ \hat{\sigma}_1^2 - s_1^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{(1-\hat{\rho})} \left[\frac{\hat{\lambda}_1}{(1-e^{-\hat{\lambda}_1})} + \frac{\hat{\lambda}_{k+1}}{(1-e^{-\hat{\lambda}_{k+1}})} - 2 \right] - \bar{n}_1 \\ \frac{(1+\hat{\rho})}{(1-\hat{\rho})^2} \left[\frac{\hat{\lambda}_1}{(1-e^{-\hat{\lambda}_1})} + \frac{\hat{\lambda}_{k+1}}{(1-e^{-\hat{\lambda}_{k+1}})} - 2 \right] - \frac{1}{(1-\hat{\rho})^2} \left[\frac{\hat{\lambda}_1^2 e^{-\hat{\lambda}_1}}{(1-e^{-\hat{\lambda}_1})^2} + \frac{\hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1-e^{-\hat{\lambda}_{k+1}})^2} - 2 \right] - s_1^2 \end{bmatrix}. \end{aligned}$$

The expression for the Jacobian matrix is

$$J_g(\hat{\lambda}_1, \hat{\lambda}_{k+1}) = \begin{bmatrix} \frac{\partial g_1}{\partial \hat{\lambda}_1} & \frac{\partial g_1}{\partial \hat{\lambda}_{k+1}} \\ \frac{\partial g_2}{\partial \hat{\lambda}_1} & \frac{\partial g_2}{\partial \hat{\lambda}_{k+1}} \end{bmatrix}.$$

and the values for the elements of the matrix can be calculated as follows:

$$\begin{aligned} \frac{\partial g_1}{\partial \hat{\lambda}_1} &= \frac{1}{(1-\hat{\rho})} \frac{1}{(1-e^{-\hat{\lambda}_1})} \left[1 - \frac{\hat{\lambda}_1 e^{-\hat{\lambda}_1}}{(1-e^{-\hat{\lambda}_1})} \right] \\ \frac{\partial g_2}{\partial \hat{\lambda}_1} &= \frac{(1+\hat{\rho})}{(1-\hat{\rho})} \frac{\partial g_1}{\partial \hat{\lambda}_1} - \frac{1}{(1-\hat{\rho})^2} \frac{\hat{\lambda}_1 e^{-\hat{\lambda}_1}}{(1-e^{-\hat{\lambda}_1})^2} \left[2 - \hat{\lambda}_1 - \frac{2\hat{\lambda}_1 e^{-\hat{\lambda}_1}}{(1-e^{-\hat{\lambda}_1})} \right] \\ \frac{\partial g_1}{\partial \hat{\lambda}_{k+1}} &= \frac{1}{(1-\hat{\rho})} \frac{1}{(1-e^{-\hat{\lambda}_{k+1}})} \left[1 - \frac{\hat{\lambda}_{k+1} e^{-\hat{\lambda}_{k+1}}}{(1-e^{-\hat{\lambda}_{k+1}})} \right] + \frac{(g_1 + \bar{n}_1)}{(1-\hat{\rho})} \frac{\partial \hat{\rho}}{\partial \hat{\lambda}_{k+1}} \\ \frac{\partial g_2}{\partial \hat{\lambda}_{k+1}} &= \frac{(1+\hat{\rho})}{(1-\hat{\rho})} \frac{\partial g_1}{\partial \hat{\lambda}_{k+1}} - \frac{1}{(1-\hat{\rho})^2} \frac{\hat{\lambda}_{k+1} e^{-\hat{\lambda}_{k+1}}}{(1-e^{-\hat{\lambda}_{k+1}})^2} \left[2 - \hat{\lambda}_{k+1} - \frac{2\hat{\lambda}_{k+1} e^{-\hat{\lambda}_{k+1}}}{(1-e^{-\hat{\lambda}_{k+1}})} \right] \\ &\quad + \frac{(g_1 + \bar{n}_1)}{(1-\hat{\rho})} \frac{\partial \hat{\rho}}{\partial \hat{\lambda}_{k+1}} + \frac{2(g_2 + s_1^2)}{(1-\hat{\rho})} \frac{\partial \hat{\rho}}{\partial \hat{\lambda}_{k+1}} + \frac{2}{(1-\hat{\rho})^3} \frac{\partial \hat{\rho}}{\partial \hat{\lambda}_{k+1}}, \end{aligned}$$

where

$$\frac{\partial \hat{\rho}}{\partial \hat{\lambda}_{k+1}} = -\frac{1}{(1-e^{-\hat{\lambda}_{k+1}}) \sqrt{s_{ij}}} \sqrt{(1-e^{-\hat{\lambda}_{k+1}}) - (3 + \hat{\lambda}_{k+1}) \hat{\lambda}_{k+1} e^{-\hat{\lambda}_{k+1}} + \frac{2\hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1-e^{-\hat{\lambda}_{k+1}})}}.$$

The MoM estimators from the MPA_I distribution in Section 4.3.4 are used as initial values for the

parameters $\hat{\lambda}_1^{(0)}$ and $\hat{\lambda}_{k+1}^{(0)}$. In cases where the MoM estimators in Section 4.3.4 are unavailable, a grid search method, as suggested by Woodford and Phillips [4], can be used as an alternative to determine the starting values. However, this approach will not be discussed further in this thesis.

The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

Lastly, an estimator for ρ can be obtained by substituting the parameter estimators obtained through the iterative process for λ_1 and λ_{k+1} into (4.24). Estimators for the rest of the λ_i s, $i \in \{2, \dots, k\}$ can now be found using the expressions in Table 4.1 as follows:

$$\begin{aligned} \frac{\hat{\lambda}_i}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_i})} + \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_{k+1}})} - \frac{2}{(1-\hat{\rho})} &= \bar{n}_i \\ \bar{n}_i - \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_{k+1}})} + \frac{2}{(1-\hat{\rho})} &= \frac{\hat{\lambda}_i}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_i})} \\ \frac{\hat{\lambda}_i}{(1-\hat{\rho}) \left[\bar{n}_i - \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_{k+1}})} + \frac{2}{(1-\hat{\rho})} \right]} &= (1-e^{-\hat{\lambda}_i}) \end{aligned}$$

and

$$\begin{aligned} s_i^2 &= \frac{(1+\hat{\rho})\hat{\lambda}_i}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_i})} + \frac{(1+\hat{\rho})\hat{\lambda}_{k+1}}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_{k+1}})} \\ &\quad - \frac{\hat{\lambda}_i^2 e^{-\hat{\lambda}_i}}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_i})^2} - \frac{\hat{\lambda}_{k+1}^2 e^{-\hat{\lambda}_{k+1}}}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_{k+1}})^2} - \frac{2\hat{\rho}}{(1-\hat{\rho})^2} \\ s_i^2 &= \frac{\hat{\rho}}{(1-\hat{\rho})}\bar{n}_i + \frac{\hat{\lambda}_i}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_i})} - \frac{\hat{\lambda}_i^2 e^{-\hat{\lambda}_i}}{(1-\hat{\rho})^2(1-e^{-\hat{\lambda}_i})^2} + s_{ij}. \end{aligned}$$

Therefore, an estimator for λ_i is

$$\begin{aligned} \hat{\lambda}_i &= \frac{\hat{\lambda}_i^2 e^{-\hat{\lambda}_i}}{(1-e^{-\hat{\lambda}_i})} + (1-\hat{\rho})^2(1-e^{-\hat{\lambda}_i}) \left[s_i^2 - \frac{\hat{\rho}}{(1-\hat{\rho})}\bar{n}_i - s_{ij} \right] \\ \hat{\lambda}_i &= (1-\hat{\rho}) \left[\bar{n}_i - \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_{k+1}})} + \frac{2}{(1-\hat{\rho})} \right] + \frac{(1-\hat{\rho}) \left(s_i^2 - \sum_{j=1, j \neq i}^k \frac{s_{ij}}{(k-1)} \right) - \hat{\rho}\bar{n}_i}{\left[\bar{n}_i - \frac{\hat{\lambda}_{k+1}}{(1-\hat{\rho})(1-e^{-\hat{\lambda}_{k+1}})} + \frac{2}{(1-\hat{\rho})} \right]} - 1. \end{aligned}$$

4.5.5 Maximum Likelihood Estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 4.5.5 for the $WMPA_I^{(2)}$ distribution. The Newton-Raphson algorithm uses the equation in (4.13) for $t \geq 1$ and the MoM estimates in Subsection 4.5.4 are used as initial values of the

parameters $\lambda_1^{(0)}, \dots, \lambda_k^{(0)}, \lambda_{k+1}^{(0)}$ and $\rho^{(0)}$ for the $\text{WMPA}_I^{(2)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{\text{th}}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (4.19) need to be calculated with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ .

The results of the calculation of the first and second derivatives of the *pmf* for the $\text{WMPA}_I^{(2)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. The first and second derivatives of $f^w(0, \dots, 0)$, $f^w(n_a, \dots, n_b, 0, \dots, 0)$ where $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, and $f^w(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ with respect to each of the parameters: $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ and ρ are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x_i = \frac{-\lambda_i(1-\rho)}{\rho}$, $i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned}
 g_j &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} - \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} + \frac{f^w(n_1, \dots, n_k)}{\lambda_j} \\
 &= \left[\frac{(n_j + 1)}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
 &\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{n_j}{\lambda_j} \frac{L_{n_j-1}^0(x_j)}{L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
 &\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
 &\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \right. \\
 &\quad \left. \times \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0), \\
 &\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
 \end{aligned} \tag{S.4.0.26}$$

$$\begin{aligned}
g'_{j,j} &= \frac{\partial g_j}{\partial \lambda_j} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} \\
&\quad - \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j \\
&= \frac{(n_j + 1)}{\lambda_j} \left[\frac{1}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} + \frac{(1-\rho)}{\rho} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho)}{\rho} \frac{n_j}{\lambda_j} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-1}^{r+1}(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \right. \\
&\quad \times \left. \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) - \frac{1}{\rho} g_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.27}$$

$$\begin{aligned}
g_{k+1} &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} + \frac{f^w(n_1, \dots, n_k)}{\lambda_{k+1}} \\
&= \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{\lambda_{k+1}} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{(r+1)!} \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.28}$$

$$\begin{aligned}
g'_{k+1,k+1} &= \frac{\partial g_{k+1}}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} \\
&\quad - \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} + 1 \right] g_{k+1} \\
&= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \right. \\
&\quad \times \left. \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{\lambda_{k+1}} \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] g_{k+1}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.29}$$

$$\begin{aligned}
g'_{j,\ell} &= \frac{\partial g_j}{\partial \lambda_\ell} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&= \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\frac{(n_\ell + 1) L_{n_\ell-1}^1(x_\ell)}{\lambda_\ell L_{n_\ell-1}^2(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1 - \rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{n_\ell L_{n_\ell-1}^0(x_\ell)}{\lambda_\ell L_{n_\ell-1}^1(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1 - \rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
&\quad \times \left[\frac{(1 - \rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \left[\frac{(1 - \rho)}{\rho} L_{n_\ell-r-1}^r(x_\ell) \right] \\
&\quad \times \left[\frac{(1 - \rho)^{2r} \rho^{n_j+n_\ell-2r}}{\lambda_j \lambda_\ell} \prod_{i=1, i \neq j, \ell}^k \frac{(1 - \rho)^r \rho^{n_i-r}}{\lambda_i} \right] \\
&\quad \times \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) f^w(0, \dots, 0) \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.30}$$

$$\begin{aligned}
g'_{j,k+1} &= \frac{\partial g_j}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{\lambda_{k+1}} \left[\frac{(n_j+1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \right. \\
&\quad \times \left. \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.31}$$

$$\begin{aligned}
g'_{k+1,j} &= \frac{\partial g_{k+1}}{\partial \lambda_j} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \left[\frac{n_j}{\lambda_j} \frac{L_{n_j-1}^0(x_j)}{L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho)\rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{\lambda_{k+1}} \left[\frac{(n_j+1)}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho)\rho^{n_i-1}}{n_i(n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \right. \\
&\quad \left. \times \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] g_{k+1}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.32}$$

Let $j \in \{a, \dots, b\}$, then

$$\begin{aligned}
h_j &= \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j} \\
&= \left[\frac{(n_j+1)}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} \right] f^w(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
&\quad \{n_a, \dots, n_b\} \in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.33}$$

$$\begin{aligned}
h'_{j,j} &= \frac{\partial h_j}{\partial \lambda_j} \\
&= \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&\quad - \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] h_j \\
&= \frac{(1-\rho)(n_j+1)}{\rho \lambda_j} f^w(n_a, \dots, n_b, 0, \dots, 0) - \frac{1}{\rho} h_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} + 1 \right] h_j, \\
\{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
\{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.34}$$

$$\begin{aligned}
h'_{j,\ell} &= \frac{\partial h_j}{\partial \lambda_\ell} \\
&= \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&\quad + \left[\frac{1}{\lambda_j} - \frac{1}{f^w(0, \dots, 0)} \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell}, \\
\{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
\{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.35}$$

$$\begin{aligned}
h'_{j,k+1} &= \frac{\partial h_j}{\partial \lambda_{k+1}} \\
&= \frac{\partial^2 f(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&\quad + \left[\frac{1}{\lambda_j} - \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}}, \\
\{n_1, \dots, n_k\} &\equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}, \\
\{n_a, \dots, n_b\} &\in \{1, 2, \dots\}, a \leq b, c \leq d.
\end{aligned} \tag{S.4.0.36}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. It then follows that the first and second derivatives of the pmf in (4.19) are

First derivative with respect to λ_j

$$\begin{aligned}
\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} &= \left[\frac{1}{\lambda_j} - \frac{1}{(1 - e^{-\lambda_j})} \right] f^w(0, \dots, 0) \\
\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} &= h_j + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j}, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} &= \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j}, \quad (S.4.0.37) \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} &= g_j + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{f^w(n_1, \dots, n_k)}{\lambda_j}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

First derivative with respect to λ_{k+1}

$$\begin{aligned}
\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} &= \left[\frac{1}{\lambda_{k+1}} - \frac{1}{(1 - e^{-\lambda_{k+1}})} \right] f^w(0, \dots, 0) \\
\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} &= \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} &= g_{k+1} + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&\quad - \frac{f^w(n_1, \dots, n_k)}{\lambda_{k+1}}, \quad (S.4.0.38) \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

First derivative with respect to ρ

$$\begin{aligned}
\frac{\partial f^w(0, \dots, 0)}{\partial \rho} &= 0 \\
\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} &= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \frac{\sum_{i=a}^b \lambda_i h_i}{(1-\rho)\rho} \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] f^w(n_1, \dots, n_k) \\
&\quad - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g_{k+1}}{(1-\rho)\rho}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned} \tag{S.4.0.39}$$

Second derivative with respect to λ_j

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2} &= \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{f^w(0, \dots, 0)}{\lambda_j} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} &= h'_{j,j} \\
&\quad + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] h_j, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} &= \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2}, \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} &= g'_{j,j} + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \\
&\quad \times \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned} \tag{S.4.0.40}$$

Second derivative with respect to λ_{k+1}

$$\begin{aligned} \frac{\partial^2 f^w(0, \dots, 0)}{\partial^2 \lambda_{k+1}} &= \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_{k+1}} + 1 \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\ &\quad - \frac{f^w(0, \dots, 0)}{\lambda_{k+1}} \\ \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial^2 \lambda_{k+1}} &= \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_{k+1}^2} \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} &= g'_{k+1, k+1} \\ &\quad + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\ &\quad - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} + 1 \right] g_{k+1}, \\ &\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}. \end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= \frac{1}{f^w(n_a, \dots, n_b, 0, \dots, 0)} h_j h_\ell \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] h_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] h_\ell \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \\
&\quad \times \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] \\
&\quad \times f^w(n_a, \dots, n_b, 0, \dots, 0), \\
&\quad j, \ell \in \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j}, \\
&\quad j \in \{a, \dots, b\}, \ell \notin \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell}, \\
&\quad j, \ell \notin \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} &= g'_{j,\ell} + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \\
&\quad \times \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned} \tag{S.4.041}$$

Second derivative with respect to λ_j and λ_{k+1}

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad \times \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j}, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} &= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}}, \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} &= g'_{j,k+1} + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \\
&\quad \times \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned} \tag{S.4.0.42}$$

Second derivative with respect to ρ and λ_j

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho \partial \lambda_j} &= 0 \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} &= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{h_j}{(1-\rho)\rho} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}, \\
&\quad j \in \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} &= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}, \\
&\quad j \notin \{a, \dots, b\} \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&\quad - \frac{g_j}{(1-\rho)\rho} - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g'_{k+1,j}}{(1-\rho)\rho}, \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}, j \in \{1, \dots, k\}.
\end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

$$\begin{aligned} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} &= 0 \\ \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} &= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\ &\quad - \frac{\sum_{i=a}^b \lambda_i h'_{i,k+1}}{(1-\rho)\rho} \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\ &\quad - \frac{kg_{k+1}}{(1-\rho)\rho} - \frac{\sum_{i=1}^k \lambda_i g'_{i,k+1}}{(1-\rho)\rho} - \frac{k\lambda_{k+1}g'_{k+1}}{(1-\rho)\rho}, \\ &\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}. \end{aligned}$$

Second derivative with respect to ρ

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho^2} &= 0 \\
\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho^2} &= \left[\frac{\sum_{i=a}^b n_i - 1}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \\
&\quad \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&\quad - \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \right. \\
&\quad \left. \times \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=a}^b n_i}{(1-\rho)\rho} f^w(n_a, \dots, n_b, 0, \dots, 0) \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho^2} &= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{2k}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=1}^k n_i}{(1-\rho)\rho} f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{1, 2, \dots\}.
\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (4.19) are given here.

First derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.19), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\ &= \frac{\lambda_{k+1} e^{-\lambda_{k+1}}}{(1 - e^{-\lambda_{k+1}})} \left[\frac{e^{-\lambda_j}}{(1 - e^{-\lambda_j})} - \frac{\lambda_j e^{-\lambda_j}}{(1 - e^{-\lambda_j})} - \frac{\lambda_j (e^{-\lambda_j})^2}{(1 - e^{-\lambda_j})^2} \right] \left[\prod_{i=1, i \neq j}^k \frac{\lambda_i e^{-\lambda_i}}{(1 - e^{-\lambda_i})} \right] \\ &= \left[\frac{1}{\lambda_j} - 1 - \frac{e^{-\lambda_j}}{(1 - e^{-\lambda_j})} \right] f^w(0, \dots, 0) \\ &= \left[\frac{1}{\lambda_j} - \frac{1}{(1 - e^{-\lambda_j})} \right] f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (4.19) and (S.4.0.33) and using (S.3.0.2), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\ &= \frac{1}{\lambda_j} \left[\prod_{i=a}^b \frac{\lambda_i (1 - \rho) \rho^{n_i - 1}}{n_i (n_i + 1)} L_{n_i - 1}^2(x_i) \right] f^w(0, \dots, 0) \\ &+ \left[\frac{(1 - \rho) L_{n_j - 2}^3(x_j)}{\rho L_{n_j - 1}^2(x_j)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1 - \rho) \rho^{n_i - 1}}{n_i (n_i + 1)} L_{n_i - 1}^2(x_i) \right] f^w(0, \dots, 0) \\ &+ \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\ &= \left[\frac{(1 - \rho)}{\rho} \left[\frac{(n_j + 1) \rho}{\lambda_j (1 - \rho)} L_{n_j - 1}^1(x_j) - \frac{2\rho}{\lambda_j (1 - \rho)} L_{n_j - 1}^2(x_j) \right] \frac{1}{L_{n_j - 1}^2(x_j)} \right] \\ &\times f^w(n_a, \dots, n_b, 0, \dots, 0) + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\ &+ \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j} \\ &= \left[\frac{(n_j + 1) L_{n_j - 1}^1(x_j)}{\lambda_j L_{n_j - 1}^2(x_j)} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\ &+ \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j} \\ &= h_j + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{\lambda_j}. \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (4.19), it follows that

$$\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} = \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j}.$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.19) and (S.4.0.26) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \frac{1}{\lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{(1-\rho) L_{n_j-2}^3(x_j)}{\rho L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho) L_{n_j-2}^2(x_j)}{\rho L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\lambda_j} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] \\
&\times f^w(0, \dots, 0) + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j}, \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \\
&= \frac{1}{\lambda_j} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{(1-\rho)}{\rho} \left[\frac{(n_j+1)\rho}{\lambda_j(1-\rho)} L_{n_j-1}^1(x_j) - \frac{2\rho}{\lambda_j(1-\rho)} L_{n_j-1}^2(x_j) \right] \frac{1}{L_{n_j-1}^2(x_j)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho)}{\rho} \left[\frac{n_j \rho}{\lambda_j(1-\rho)} L_{n_j-1}^0(x_j) - \frac{\rho}{\lambda_j(1-\rho)} L_{n_j-1}^1(x_j) \right] \frac{1}{L_{n_j-1}^1(x_j)} \right] \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) - \frac{1}{\lambda_j} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&= \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(n_1, \dots, n_k)}{\lambda_j} \\
&= g_j + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(n_1, \dots, n_k)}{\lambda_j}.
\end{aligned}$$

First derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Similarly to (S.4.0.37), it follows that

$$\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} = \left[\frac{1}{\lambda_{k+1}} - \frac{1}{(1 - e^{-\lambda_{k+1}})} \right] f^w(0, \dots, 0).$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. Substituting from (4.19), it follows that

$$\frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} = \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}}.$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (4.19) and (S.4.0.28), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= \frac{1}{2!} \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{(r+1) \lambda_{k+1}^r}{(r+2)!} \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&= \frac{1}{2!} \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{(r+1)!} \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \frac{1}{\lambda_{k+1}} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&= \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{\lambda_{k+1}} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{f^w(n_1, \dots, n_k)}{\lambda_{k+1}} \\
&= g_{k+1} + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{f^w(n_1, \dots, n_k)}{\lambda_{k+1}}.
\end{aligned}$$

First derivative with respect to ρ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial f^w(0, \dots, 0)}{\partial \rho} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$. Substituting from (4.19), (S.4.0.33) and (S.4.0.39) and using (S.3.0.2), it follows

that

$$\begin{aligned}
& \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&= \left[\frac{(\sum_{i=a}^b n_i - b + a - 1)}{\rho} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \left[\frac{(b-a+1)}{(1-\rho)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \left[\sum_{i=a}^b \frac{\lambda_i L_{n_i-2}^3(x_i)}{\rho^2 L_{n_i-1}^2(x_i)} \right] \left[\prod_{i=a}^b \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \rho}. \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} - \frac{(b-a+1)}{(1-\rho)\rho} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \left[\sum_{i=a}^b \frac{\lambda_i}{\rho^2} \left[\frac{(n_i+1)\rho}{\lambda_i(1-\rho)} L_{n_i-1}^1(x_i) - \frac{2\rho}{\lambda_i(1-\rho)} L_{n_i-1}^2(x_i) \right] \frac{1}{L_{n_i-1}^2(x_i)} \right] \\
&\quad \times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} - \frac{(b-a+1)}{(1-\rho)\rho} + \sum_{i=a}^b \frac{2}{(1-\rho)\rho} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad - \left[\sum_{i=a}^b \frac{(n_i+1)}{(1-\rho)\rho} \frac{L_{n_i-1}^1(x_i)}{L_{n_i-1}^2(x_i)} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) - \frac{\sum_{i=a}^b \lambda_i h_i}{(1-\rho)\rho}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (4.19), (S.4.0.26), (S.4.0.28) and (S.4.0.39) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i - k \right)}{\rho} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \left[\frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \left[\sum_{i=1}^k \frac{\lambda_i L_{n_i-2}^3(x_i)}{\rho^2 L_{n_i-1}^2(x_i)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{\left(\sum_{i=1}^k n_i - k \right)}{\rho} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{\lambda_{k+1}}{2!} \left[\frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{\lambda_{k+1}}{2!} \left[\sum_{i=1}^k \frac{\lambda_i L_{n_i-2}^2(x_i)}{\rho^2 L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{\left(\sum_{i=1}^k n_i - kr \right)}{\rho} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{kr}{(1-\rho)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{\lambda_1}{\rho^2} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_1-r}}{\lambda_1} \prod_{i=2}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad \vdots \\
&\quad - \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{\lambda_k}{\rho^2} L_{n_k-r-1}^r(x_k) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_k-r}}{\lambda_k} \prod_{i=1}^{k-1} \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \rho}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i - k\right)}{\rho} - \frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \left[\sum_{i=1}^k \frac{\lambda_i}{\rho^2} \left[\frac{(n_i+1)\rho}{\lambda_i(1-\rho)} L_{n_i-1}^1(x_i) - \frac{2\rho}{\lambda_i(1-\rho)} L_{n_i-1}^2(x_i) \right] \frac{1}{L_{n_i-1}^2(x_i)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{\left(\sum_{i=1}^k n_i - k\right)}{\rho} - \frac{k}{(1-\rho)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{\lambda_{k+1}}{2!} \left[\sum_{i=1}^k \frac{\lambda_i}{\rho^2} \left[\frac{n_i \rho}{\lambda_i(1-\rho)} L_{n_i-1}^0(x_i) - \frac{\rho}{\lambda_i(1-\rho)} L_{n_i-1}^1(x_i) \right] \frac{1}{L_{n_i-1}^1(x_i)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) + \frac{\sum_{i=1}^k n_i}{\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{kr}{(1-\rho)} + \frac{kr}{\rho} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{\lambda_1}{\rho^2} L_{n_1-r-1}^r(x_1) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_1-r}}{\lambda_1} \prod_{i=2}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad \vdots \\
&\quad - \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{\lambda_k}{\rho^2} L_{n_k-r-1}^r(x_k) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_k-r}}{\lambda_k} \prod_{i=1}^{k-1} \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&= \left[\frac{\left(\sum_{i=1}^k n_i - k\right)}{\rho} - \frac{k}{(1-\rho)} + \sum_{i=1}^k \frac{2}{(1-\rho)\rho} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] \\
&\times f^w(0, \dots, 0) - \left[\sum_{i=1}^k \frac{\lambda_i}{(1-\rho)\rho} \frac{(n_i+1)}{\lambda_i} \frac{L_{n_i-1}^1(x_i)}{L_{n_i-1}^2(x_i)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{\left(\sum_{i=1}^k n_i - k\right)}{\rho} - \frac{k}{(1-\rho)} + \sum_{i=1}^k \frac{1}{(1-\rho)\rho} \right] \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&- \frac{\lambda_{k+1}}{2!} \left[\sum_{i=1}^k \frac{\lambda_i}{(1-\rho)\rho} \frac{n_i}{\lambda_i} \frac{L_{n_i-1}^0(x_i)}{L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{\sum_{i=1}^k n_i}{\rho} - \frac{2k}{(1-\rho)\rho} \right] \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&- \frac{k\lambda_{k+1}}{(1-\rho)\rho} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{(r+1)!} \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&- \frac{\lambda_1}{(1-\rho)\rho} \sum_{r=1, n_1 \neq 1}^{\min(n_1-1, (n_2, \dots, n_k))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_1-r-1}^r(x_1) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_1-r}}{\lambda_1} \prod_{i=2}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\vdots \\
&- \frac{\lambda_k}{(1-\rho)\rho} \sum_{r=1, n_k \neq 1}^{\min(n_k-1, (n_1, \dots, n_{k-1}))} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_k-r-1}^r(x_k) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_k-r}}{\lambda_k} \prod_{i=1}^{k-1} \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] f^w(n_1, \dots, n_k) - \frac{\sum_{i=1}^k \lambda_i g_i}{(1-\rho)\rho} - \frac{k\lambda_{k+1} g_{k+1}}{(1-\rho)\rho}.
\end{aligned}$$

Second derivative with respect to λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.37), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2} \\
&= \left[\frac{e^{-\lambda_j}}{(1 - e^{-\lambda_j})^2} - \frac{1}{\lambda_j^2} \right] f^w(0, \dots, 0) + \frac{1}{f^w(0, \dots, 0)} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 \\
&= \frac{1}{(1 - e^{-\lambda_j})} \left[1 + \frac{e^{-\lambda_j}}{(1 - e^{-\lambda_j})} - 1 \right] f^w(0, \dots, 0) \\
&\quad - \frac{f^w(0, \dots, 0)}{\lambda_j^2} + \frac{1}{f^w(0, \dots, 0)} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 \\
&= \frac{1}{(1 - e^{-\lambda_j})^2} f^w(0, \dots, 0) - \frac{1}{(1 - e^{-\lambda_j})} f^w(0, \dots, 0) \\
&\quad - \frac{f^w(0, \dots, 0)}{\lambda_j^2} + \frac{1}{f^w(0, \dots, 0)} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 \\
&= \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{f^w(0, \dots, 0)}{\lambda_j}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (S.4.0.33) and (S.4.0.34) and using (A.12), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&= \left[\frac{(1 - \rho)(n_j + 1)}{\rho} \frac{L_{n_j - 2}^2(x_j)}{\lambda_j L_{n_j - 1}^2(x_j)} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad + \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} h_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2} - \frac{1}{f^w(0, \dots, 0)^2} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 + \frac{1}{\lambda_j^2} \right] \\
&\quad \times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \left[\frac{(1 - \rho)(n_j + 1)}{\rho} \frac{1}{\lambda_j} \left[L_{n_j - 1}^2(x_j) - L_{n_j - 1}^1(x_j) \right] \frac{1}{L_{n_j - 1}^2(x_j)} \right] \\
&\quad \times f^w(n_a, \dots, n_b, 0, \dots, 0) - \left[-\frac{1}{\lambda_j} + 1 \right] h_j \\
&\quad + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&= \frac{(1-\rho)(n_j+1)}{\rho \lambda_j} f^w(n_a, \dots, n_b, 0, \dots, 0) - \frac{1}{\rho} h_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} + 1 \right] h_j \\
&+ \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] h_j \\
&= h'_{j,j} + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] h_j.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (S.4.0.37), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j^2} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&- \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)^2} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 \\
&+ \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2} \\
&= \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (4.19), (S.4.0.26) and (S.4.0.27) and using (A.10) and (A.12), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} \\
&= \left[\frac{(1-\rho)(n_j+1)}{\rho} \frac{L_{n_j-2}^2(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j^2 L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho) n_j L_{n_j-2}^1(x_j)}{\rho \lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} \frac{1}{\lambda_j} L_{n_j-r-1}^r(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-2}^{r+1}(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} g_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j^2} - \frac{1}{f^w(0, \dots, 0)^2} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \right)^2 + \frac{1}{\lambda_j^2} \right] \\
&\quad \times f^w(n_1, \dots, n_k)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} \\
&= \left[\frac{(1-\rho)(n_j+1)}{\rho \lambda_j} \left[L_{n_j-1}^2(x_j) - L_{n_j-1}^1(x_j) \right] \frac{1}{L_{n_j-1}^2(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j^2 L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho) n_j}{\rho \lambda_j} \left[L_{n_j-1}^1(x_j) - L_{n_j-1}^0(x_j) \right] \frac{1}{L_{n_j-1}^1(x_j)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{1}{\lambda_j} \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} \left[L_{n_j-r-1}^{r+1}(x_j) - L_{n_j-r-1}^r(x_j) \right] \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad - \left[-\frac{1}{\lambda_j} + 1 \right] g_j + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j^2} \\
&= \frac{(n_j + 1)}{\lambda_j} \left[\frac{1}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} + \frac{(1-\rho)}{\rho} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{(1-\rho) n_j}{\rho \lambda_j} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-1}^{r+1}(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\rho} g_j + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j \\
&+ \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j \\
&= g'_{j,j} + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} + 1 \right] g_j.
\end{aligned}$$

Second derivative with respect to λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Similarly to (S.4.0.40), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, \dots, 0)}{\partial^2 \lambda_{k+1}} \\
&= \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_{k+1}} + 1 \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&- \frac{f^w(0, \dots, 0)}{\lambda_{k+1}}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. Substituting from (S.4.0.38), it follows that

$$\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial^2 \lambda_{k+1}} = \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_{k+1}^2}.$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (S.4.0.28) and (S.4.0.29), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} \\
&= -\frac{1}{\lambda_{k+1}^2} \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{r \lambda_{k+1}^{r-1}}{(r+1)!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} g_{k+1} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_{k+1}^2} - \frac{1}{f^w(0, \dots, 0)^2} \left(\frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \right)^2 + \frac{1}{\lambda_{k+1}^2} \right] \\
&\times f^w(n_1, \dots, n_k) \\
&= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\lambda_{k+1}} \left[\frac{1}{\lambda_{k+1}} \prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\lambda_{k+1}} \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] \\
&\times f^w(0, \dots, 0) - \left[\frac{1}{\lambda_{k+1}} + 1 \right] g_{k+1} \\
&+ \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_{k+1}} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}^2} \\
&= \sum_{r=1}^{\min(n_1, \dots, n_k)} \frac{\lambda_{k+1}^{r-1}}{r!} \left[\prod_{i=1}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{\lambda_{k+1}} \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] g_{k+1} \\
&+ \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_{k+1}} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} + 1 \right] g_{k+1} \\
&= g'_{k+1, k+1} + \left[\frac{2}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{2}{\lambda_j} + 1 \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&- \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} + 1 \right] g_{k+1}.
\end{aligned}$$

Second derivative with respect to λ_j and λ_ℓ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.4.0.37), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \left[\frac{1}{\lambda_j} - \frac{1}{(1-e^{-\lambda_j})} \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j, \ell \in \{a, \dots, b\}$. Substituting from (S.4.0.33), (S.4.0.37) and (S.4.0.41), it

follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{f^w(n_a, \dots, n_b, 0, \dots, 0)} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} h_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \right] \\
&\times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{1}{f^w(n_a, \dots, n_b, 0, \dots, 0)} h_j h_\ell \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] h_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] h_\ell \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \\
&\times \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] f^w(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$, $\ell \notin \{a, \dots, b\}$. Substituting from (S.4.0.33), (S.4.0.37) and (S.4.0.41), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{f^w(n_a, \dots, n_b, 0, \dots, 0)} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} h_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \right] \\
&\times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j, \ell \notin \{a, \dots, b\}$. Substituting from (S.4.0.37) and (S.4.0.41), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell} \\
&\quad - \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \\
&\quad + \frac{f^w(n_a, \dots, n_b, 0, \dots, 0)}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_\ell}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.4.0.26), (S.4.0.30) and (S.4.0.41) and using (S.3.0.1) and (S.3.0.2), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{\lambda_\ell} \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\frac{(1-\rho) L_{n_\ell-2}^3(x_\ell)}{\rho L_{n_\ell-1}^2(x_\ell)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{(1-\rho) L_{n_\ell-2}^2(x_\ell)}{\rho L_{n_\ell-1}^1(x_\ell)} \right] \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\lambda_\ell} \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)}{\rho} L_{n_\ell-r-1}^r(x_\ell) \right] \left[\frac{(1-\rho)^{2r} \rho^{n_j+n_\ell-2r}}{\lambda_j \lambda_\ell} \right. \\
&\times \left. \prod_{i=1, i \neq j, \ell}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} g_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_\ell} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} \right] \\
&\times f^w(n_1, \dots, n_k)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \frac{1}{\lambda_\ell} \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \\
&\times \left[\frac{(1-\rho)}{\rho} \left[\frac{(n_\ell + 1) \rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^1(x_\ell) - \frac{2\rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^2(x_\ell) \right] \frac{1}{L_{n_\ell-1}^2(x_\ell)} \right] \\
&\times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \\
&\times \left[\frac{(1-\rho)}{\rho} \left[\frac{n_\ell \rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^0(x_\ell) - \frac{\rho}{\lambda_\ell (1-\rho)} L_{n_\ell-1}^1(x_\ell) \right] \frac{1}{L_{n_\ell-1}^1(x_\ell)} \right] \\
&\times \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&- \frac{1}{\lambda_\ell} \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell) \right] \\
&\times \left[\frac{(1-\rho)^{2r} \rho^{n_j+n_\ell-2r}}{\lambda_j \lambda_\ell} \prod_{i=1, i \neq j, \ell}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] \\
&\times f^w(0, \dots, 0) + \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} g_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= -\frac{1}{\lambda_\ell} \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{1}{\lambda_\ell} \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad - \frac{1}{\lambda_\ell} \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\quad \times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\frac{(n_\ell + 1) L_{n_\ell-1}^1(x_\ell)}{\lambda_\ell L_{n_\ell-1}^2(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{n_\ell L_{n_\ell-1}^0(x_\ell)}{\lambda_\ell L_{n_\ell-1}^1(x_\ell)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_j+n_\ell-2r}}{\lambda_j \lambda_\ell} \prod_{i=1, i \neq j, \ell}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} g_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_\ell} \\
&= \left[\frac{(n_j + 1) L_{n_j-1}^1(x_j)}{\lambda_j L_{n_j-1}^2(x_j)} \right] \left[\frac{(n_\ell + 1) L_{n_\ell-1}^1(x_\ell)}{\lambda_\ell L_{n_\ell-1}^2(x_\ell)} \right] \\
&\quad \times \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i + 1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \frac{\lambda_{k+1}}{2!} \left[\frac{n_j L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\frac{n_\ell L_{n_\ell-1}^0(x_\ell)}{\lambda_\ell L_{n_\ell-1}^1(x_\ell)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&\quad + \sum_{r=1, n_j, n_\ell \neq 1}^{\min(n_j-1, n_\ell-1, (n_1, \dots, n_k) \neq n_j, n_\ell)} \frac{\lambda_{k+1}^{r+1}}{(r+2)!} \left[\frac{(1-\rho)^2}{\rho^2} L_{n_j-r-1}^r(x_j) L_{n_\ell-r-1}^r(x_\ell) \right] \\
&\quad \times \left[\frac{(1-\rho)^{2r} \rho^{n_j+n_\ell-2r}}{\lambda_j \lambda_\ell} \prod_{i=1, i \neq j, \ell}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_\ell} - \frac{1}{\lambda_\ell} \right] g_j \\
&\quad + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell} \\
&= g'_{j,\ell} + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_\ell}.
\end{aligned}$$

Second derivative with respect to λ_j and λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.37), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \left[\frac{1}{\lambda_j} - \frac{1}{(1-e^{-\lambda_j})} \right] \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. Substituting from (S.4.0.33), (S.4.0.37) and (S.4.0.42), it

follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \frac{1}{f^w(n_a, \dots, n_b, 0, \dots, 0)} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} h_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \right] \\
&\times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. Substituting from (S.4.0.37) and (S.4.0.42), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \right] \\
&\times f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&= \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.4.0.26), (S.4.0.31) and (S.4.0.42), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \frac{1}{2!} \left[\frac{n_i L_{n_j-1}^0(x_j)}{\lambda_j L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{(r+1) \lambda_{k+1}^r}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} g_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda_j \partial \lambda_{k+1}} - \frac{1}{f^w(0, \dots, 0)^2} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} \right] \\
&\times f^w(n_1, \dots, n_k) \\
&= \frac{1}{2!} \left[\frac{n_i L_{n_j-1}^0(x_j)}{\lambda_j L_{n_i-1}^1(x_i)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&- \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{(r+2)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} g_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_j \partial \lambda_{k+1}} \\
&= \left[\frac{n_i}{\lambda_j} \frac{L_{n_j-1}^0(x_j)}{L_{n_j-1}^1(x_j)} \right] \left[\prod_{i=1}^k \frac{(1-\rho) \rho^{n_i-1}}{n_i} L_{n_i-1}^1(x_i) \right] f^w(0, \dots, 0) \\
&+ \frac{1}{\lambda_{k+1}} \left[\frac{(n_j+1)}{\lambda_j} \frac{L_{n_j-1}^1(x_j)}{L_{n_j-1}^2(x_j)} \right] \left[\prod_{i=1}^k \frac{\lambda_i (1-\rho) \rho^{n_i-1}}{n_i (n_i+1)} L_{n_i-1}^2(x_i) \right] f^w(0, \dots, 0) \\
&+ \sum_{r=1, n_j \neq 1}^{\min(n_j-1, (n_1, \dots, n_k) \neq n_j)} \frac{\lambda_{k+1}^r}{(r+1)!} \left[\frac{(1-\rho)}{\rho} L_{n_j-r-1}^r(x_j) \right] \\
&\times \left[\frac{(1-\rho)^r \rho^{n_j-r}}{\lambda_j} \prod_{i=1, i \neq j}^k \frac{(1-\rho)^r \rho^{n_i-r}}{\lambda_i} \left(L_{n_i-r}^{r-1}(x_i) - \binom{n_i-1}{n_i-r} \right) \right] f^w(0, \dots, 0) \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] g_j \\
&+ \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} \\
&= g'_{j,k+1} + \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_j} - \frac{1}{\lambda_j} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}}.
\end{aligned}$$

Second derivative with respect to ρ and λ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho \partial \lambda_j} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \in \{a, \dots, b\}$. From (S.4.0.33) and (S.4.0.34), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} \\
&- \frac{h_j}{(1-\rho)\rho} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b$, $c \leq d$ and let $j \notin \{a, \dots, b\}$. From (S.4.0.35), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_j} \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_j} - \frac{\sum_{i=a}^b \lambda_i h'_{i,j}}{(1-\rho)\rho}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$ and let $j \in \{1, \dots, k\}$. From (S.4.0.27), (S.4.0.30) and (S.4.0.32), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_j} \\ &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_j} - \frac{g_j}{(1-\rho)\rho} \\ & \quad - \frac{\sum_{i=1}^k \lambda_i g'_{i,j}}{(1-\rho)\rho} - \frac{k \lambda_{k+1} g'_{k+1,j}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ and λ_{k+1}

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. From (S.4.0.36), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho \partial \lambda_{k+1}} \\ &= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{\sum_{i=a}^b \lambda_i h'_{i,k+1}}{(1-\rho)\rho}. \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. From (S.4.0.29), (S.4.0.31) and (S.4.0.32), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho \partial \lambda_{k+1}} \\ &= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1}} - \frac{k g_{k+1}}{(1-\rho)\rho} \\ & \quad - \frac{\sum_{i=1}^k \lambda_i g'_{i,k+1}}{(1-\rho)\rho} - \frac{k \lambda_{k+1} g'_{k+1}}{(1-\rho)\rho}. \end{aligned}$$

Second derivative with respect to ρ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0)}{\partial \rho^2} = 0.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{n_a, \dots, n_b, 0, \dots, 0\}$, $\{n_a, \dots, n_b\} \in \{1, 2, \dots\}$, $a \leq b, c \leq d$. Substituting from (S.4.0.33), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho^2} \\
&= \left[\frac{\sum_{i=a}^b n_i}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&\quad - \left[\frac{\sum_{i=a}^b n_i}{\rho^2} + \frac{(b-a+1)}{(1-\rho)\rho^2} - \frac{(b-a+1)}{(1-\rho)^2\rho} \right] f^w(n_a, \dots, n_b, 0, \dots, 0) \\
&\quad + \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \sum_{i=a}^b \lambda_i h_i \\
&\quad - \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&= \left[\frac{\sum_{i=a}^b n_i - 1}{\rho} + \frac{(b-a+1)}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \\
&\quad - \frac{\sum_{i=a}^b \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \frac{\partial f^w(n_a, \dots, n_b, 0, \dots, 0)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=a}^b n_i}{(1-\rho)\rho} f^w(n_a, \dots, n_b, 0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{1, 2, \dots\}$. Substituting from (S.4.0.26) and (S.4.0.28), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \rho^2} \\
&= \left[\frac{\sum_{i=1}^k n_i}{\rho} + \frac{2k}{(1-\rho)\rho} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \left[\frac{\sum_{i=1}^k n_i}{\rho^2} + \frac{2k}{(1-\rho)\rho^2} - \frac{2k}{(1-\rho)^2\rho} \right] f^w(n_1, \dots, n_k) \\
&\quad + \left[\frac{1}{(1-\rho)\rho^2} - \frac{1}{(1-\rho)^2\rho} \right] \left[\sum_{i=1}^k \lambda_i g_i + k\lambda_{k+1} g_{k+1} \right] \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} \right. \\
&\quad \quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&= \left[\frac{\sum_{i=1}^k n_i - 1}{\rho} + \frac{2k}{(1-\rho)\rho} + \frac{1}{(1-\rho)} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \\
&\quad - \frac{\sum_{i=1}^k \lambda_i}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_i \partial \rho} \right. \\
&\quad \quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_i} - \frac{1}{\lambda_i} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{k\lambda_{k+1}}{(1-\rho)\rho} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda_{k+1} \partial \rho} \right. \\
&\quad \quad \left. - \left[\frac{1}{f^w(0, \dots, 0)} \frac{\partial f^w(0, \dots, 0)}{\partial \lambda_{k+1}} - \frac{1}{\lambda_{k+1}} \right] \frac{\partial f^w(n_1, \dots, n_k)}{\partial \rho} \right] \\
&\quad - \frac{\sum_{i=1}^k n_i}{(1-\rho)\rho} f^w(n_1, \dots, n_k).
\end{aligned}$$

Chapter 5

Type II bivariate Pólya-Aeppli distributions

In this supplementary material for Chapter 5, it is shown how the MLEs for the Type II bivariate Pólya-Aeppli distributions, referred to in Chapter 5, were obtained.

5.3 Type II bivariate Pólya-Aeppli distributions

5.3.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 5.3.5 for the BPA_{II} distribution. The Newton-Raphson algorithm uses the equation in (5.13) for $t \geq 1$ and the MoM estimates in Subsection 5.3.4 are used as initial values of the parameters $\lambda^{(0)}$, $\theta_1^{(0)}$ and $\theta_2^{(0)}$ for the BPA_{II} distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t + 1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F in (5.11) and the Hessian matrix in (5.12), such that the Newton-Raphson algorithm in (5.13) can be utilized, the first and second derivatives for each element of the *pmf* in (5.9) need to be calculated with respect to each of the parameters: λ , θ_1 and θ_2 .

The results of the calculation of the first and second derivatives of the *pmf* in (5.9) for the BPA_{II} distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f(0, 0)$ and $f(n_1, n_2)$ where $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$ with respect to each of the parameters: λ , θ_1 and θ_2 are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f(n_1, n_2)}{\partial \lambda} - \left(\frac{1}{\lambda} - 1\right) f(n_1, n_2) \\ &= \frac{\lambda\theta^2}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 1}^2(x) f(0, 0), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}. \end{aligned} \tag{S.5.0.1}$$

$$\begin{aligned} g' &= \frac{\partial g}{\partial \lambda} \\ &= \frac{\partial^2 f(n_1, n_2)}{\partial \lambda^2} - \left(\frac{1}{\lambda} - 1\right) \frac{\partial f(n_1, n_2)}{\partial \lambda} + \frac{f(n_1, n_2)}{\lambda^2} \\ &= \frac{\lambda\theta^3}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 2}^3(x) f(0, 0) + \left(\frac{1}{\lambda} - (1 - \theta)\right) g, \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}. \end{aligned} \tag{S.5.0.2}$$

It then follows that the first and second derivatives of the pmf in (5.9) are

First derivative with respect to λ

$$\begin{aligned} \frac{\partial f(0, 0)}{\partial \lambda} &= -(1 - \theta) f(0, 0) \\ \frac{\partial f(n_1, n_2)}{\partial \lambda} &= g + \left(\frac{1}{\lambda} - 1\right) f(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}. \end{aligned} \tag{S.5.0.3}$$

First derivative with respect to θ_1

$$\begin{aligned} \frac{\partial f(0, 0)}{\partial \theta_1} &= -\lambda f(0, 0) \\ \frac{\partial f(n_1, n_2)}{\partial \theta_1} &= -\frac{\lambda}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) f(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}. \end{aligned} \tag{S.5.0.4}$$

First derivative with respect to θ_2

$$\begin{aligned}
\frac{\partial f(0,0)}{\partial \theta_2} &= -\lambda f(0,0) \\
\frac{\partial f(n_1, n_2)}{\partial \theta_2} &= -\frac{\lambda}{\theta} g + \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f(n_1, n_2), \\
\{n_1, n_2\} &\in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.
\end{aligned} \tag{S.5.0.5}$$

Second derivative with respect to λ

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \lambda^2} &= (1-\theta)^2 f(0,0) \\
\frac{\partial^2 f(n_1, n_2)}{\partial \lambda^2} &= g' + \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \lambda} - \frac{f(n_1, n_2)}{\lambda^2}, \\
\{n_1, n_2\} &\in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.
\end{aligned}$$

Second derivative with respect to θ_1 and λ

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \theta_1 \partial \lambda} &= (\lambda(1-\theta) - 1) f(0,0) \\
\frac{\partial^2 f(n_1, n_2)}{\partial \theta_1 \partial \lambda} &= -\frac{\lambda}{\theta} g' + \left(\frac{n_1}{\theta_1} - \frac{2}{\theta} \right) g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{1}{\lambda} - 1 \right) f(n_1, n_2), \\
\{n_1, n_2\} &\in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.
\end{aligned} \tag{S.5.0.6}$$

Second derivative with respect to θ_2 and λ

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial \theta_2 \partial \lambda} &= (\lambda(1-\theta) - 1) f(0,0) \\
\frac{\partial^2 f(n_1, n_2)}{\partial \theta_2 \partial \lambda} &= -\frac{\lambda}{\theta} g' + \left(\frac{n_2}{\theta_2} - \frac{2}{\theta} \right) g + \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) \left(\frac{1}{\lambda} - 1 \right) f(n_1, n_2), \\
\{n_1, n_2\} &\in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.
\end{aligned} \tag{S.5.0.7}$$

Second derivative with respect to θ_1

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \theta_1^2} &= \lambda^2 f(0,0) \\ \frac{\partial^2 f(n_1, n_2)}{\partial \theta_1^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{(\lambda+1)}{\theta} \right) g \\ &\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right)^2 f(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{\lambda}{\theta} \frac{1}{(1-e^{-\lambda})} \right) g \\ &\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f^w(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\ &\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.\end{aligned}$$

Second derivative with respect to θ_2

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \theta_2^2} &= \lambda^2 f(0,0) \\ \frac{\partial^2 f(n_1, n_2)}{\partial \theta_2^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_2}{\theta_2} - \frac{(\lambda+1)}{\theta} \right) g \\ &\quad + \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right)^2 f(n_1, n_2) - \left(\frac{n_2}{\theta_2^2} + \frac{1}{\theta^2} \right) f(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.\end{aligned}$$

Second derivative with respect to θ_1 and θ_2

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial \theta_1 \partial \theta_2} &= \lambda^2 f(0,0) \\ \frac{\partial^2 f(n_1, n_2)}{\partial \theta_1 \partial \theta_2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{(\lambda+1)}{\theta} \right) g \\ &\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f(n_1, n_2) - \frac{f(n_1, n_2)}{\theta^2}, \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}.\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (5.9) is given here.

First derivative with respect to λ

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (5.9), it follows that

$$\begin{aligned} & \frac{\partial f(0, 0)}{\partial \lambda} \\ &= -(1 - \theta) e^{-\lambda(1-\theta)} \\ &= -(1 - \theta) f(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (5.9), (S.5.0.1) and (S.5.0.3) and using (A.12), it follows that

$$\begin{aligned} & \frac{\partial f(n_1, n_2)}{\partial \lambda} \\ &= \frac{\lambda \theta^2}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 2}^2(x) f(0, 0) \\ & \quad + \frac{f(n_1, n_2)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda} + \frac{f(n_1, n_2)}{\lambda} \\ &= \frac{\lambda \theta^2}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} \left[L_{n_1 + n_2 - 1}^2(x) - L_{n_1 + n_2 - 1}^1(x) \right] f(0, 0) \\ & \quad - (1 - \theta) f(n_1, n_2) + \frac{f(n_1, n_2)}{\lambda} \\ &= \frac{\lambda \theta^2}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 1}^2(x) f(0, 0) + \left(\frac{1}{\lambda} - 1 \right) f(n_1, n_2) \\ &= g + \left(\frac{1}{\lambda} - 1 \right) f(n_1, n_2). \end{aligned}$$

First derivative with respect to θ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (5.9), it follows that

$$\begin{aligned} & \frac{\partial f(0, 0)}{\partial \theta_1} \\ &= -\lambda e^{-\lambda(1-\theta)} \\ &= -\lambda f(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (5.9), (S.5.0.1) and (S.5.0.4) and using (A.12), it follows that

$$\begin{aligned}
& \frac{\partial f(n_1, n_2)}{\partial \theta_1} \\
&= \frac{\lambda \theta}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \left(\frac{n_1}{\theta_1}\right) \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-1}^1(x) f(0, 0) \\
&\quad - \frac{\lambda^2 \theta}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-2}^2(x) f(0, 0) \\
&\quad + \frac{f(n_1, n_2)}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \theta_1} - \frac{f(n_1, n_2)}{\theta} \\
&= -\frac{\lambda^2 \theta}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} \left[L_{n_1+n_2-1}^2(x) - L_{n_1+n_2-1}^1(x) \right] f(0, 0) \\
&\quad + \frac{n_1}{\theta_1} f(n_1, n_2) - \lambda f(n_1, n_2) - \frac{f(n_1, n_2)}{\theta} \\
&= -\frac{\lambda^2 \theta}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-1}^2(x) f(0, 0) + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) f(n_1, n_2) \\
&= -\frac{\lambda}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) f(n_1, n_2).
\end{aligned}$$

The first derivative with respect to θ_2 follows similarly.

Second derivative with respect to λ

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.5.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(0, 0)}{\partial \lambda^2} \\
&= -(1 - \theta) \frac{\partial f(0, 0)}{\partial \lambda} \\
&= (1 - \theta)^2 f(0, 0).
\end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (S.5.0.2) and (S.5.0.3), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \lambda^2} \\ &= \frac{\lambda \theta^3}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 2}^3(x) f(0, 0) + \left[\frac{1}{f(0, 0)} \frac{\partial f(0, 0)}{\partial \lambda} + \frac{1}{\lambda} \right] g \\ & \quad + \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \lambda} - \frac{f(n_1, n_2)}{\lambda^2} \\ &= \frac{\lambda \theta^3}{(n_1 + n_2)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 2}^3(x) f(0, 0) + \left(\frac{1}{\lambda} - (1 - \theta) \right) g \\ & \quad + \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \lambda} - \frac{f(n_1, n_2)}{\lambda^2} \\ &= g' + \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \lambda} - \frac{f(n_1, n_2)}{\lambda^2}. \end{aligned}$$

Second derivative with respect to θ_1 and λ

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.5.0.3), it follows that

$$\begin{aligned} & \frac{\partial^2 f(0, 0)}{\partial \theta_1 \partial \lambda} \\ &= -f(0, 0) - \lambda \frac{\partial f(0, 0)}{\partial \lambda} \\ &= (\lambda(1 - \theta) - 1) f(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (S.5.0.3), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \theta_1 \partial \lambda} \\ &= -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f(n_1, n_2)}{\partial \lambda} \\ &= -\frac{\lambda}{\theta} g' + \left(\frac{n_1}{\theta_1} - \frac{2}{\theta} \right) g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{1}{\lambda} - 1 \right) f(n_1, n_2). \end{aligned}$$

The second derivative with respect to θ_2 and λ follows similarly.

Second derivative with respect to θ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.5.0.4), it follows that

$$\begin{aligned} & \frac{\partial^2 f(0, 0)}{\partial \theta_1^2} \\ &= -\lambda \frac{\partial f(0, 0)}{\partial \theta_1} \\ &= \lambda^2 f(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (S.5.0.4) and (S.5.0.6), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \theta_1^2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda \partial \theta_1} - \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \theta_1} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f(n_1, n_2)}{\partial \theta_1} - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f(n_1, n_2) \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{(\lambda+1)}{\theta} \right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right)^2 f(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f(n_1, n_2). \end{aligned}$$

The second derivative with respect to θ_2 follows similarly.

Second derivative with respect to θ_1 and θ_2

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.5.0.5), it follows that

$$\begin{aligned} & \frac{\partial^2 f(0, 0)}{\partial \theta_1 \partial \theta_2} \\ &= -\lambda \frac{\partial f(0, 0)}{\partial \theta_2} \\ &= \lambda^2 f(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$. Substituting from (S.5.0.5) and (S.5.0.7), it follows that

$$\begin{aligned} & \frac{\partial^2 f(n_1, n_2)}{\partial \theta_1 \partial \theta_2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f(n_1, n_2)}{\partial \lambda \partial \theta_2} - \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f(n_1, n_2)}{\partial \theta_2} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f(n_1, n_2)}{\partial \theta_2} - \frac{f(n_1, n_2)}{\theta^2} \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{(\lambda+1)}{\theta} \right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f(n_1, n_2) - \frac{f(n_1, n_2)}{\theta^2}. \end{aligned}$$

5.4 Weighted Type II bivariate Pólya-Aeppli distribution Case I

5.4.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 5.4.5 for the WBPA_{II}⁽¹⁾ distribution. The Newton-Raphson algorithm uses the equation

in (5.13) for $t \geq 1$ and where the MoM estimates in Subsection 5.4.4 are used as initial values of the parameters $\lambda^{(0)}, \theta_1^{(0)}$ and $\theta_2^{(0)}$ for the $WBPA_{II}^{(1)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (5.14) need to be calculated with respect to each of the parameters: λ, θ_1 and θ_2 .

The results of the calculation of the first and second derivatives of the *pmf* for the $WBPA_{II}^{(1)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(n_1, n_2)$ where $\{n_1, n_2\} \in \{0, 1, \dots\}$ with respect to each of the parameters: λ, θ_1 and θ_2 are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda} + f^w(n_1, n_2) \\ &= \theta^2 \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2}^1(x) e^{-\lambda(1-\theta)}, \\ &\{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned} \quad (S.5.0.8)$$

$$\begin{aligned} g' &= \frac{\partial g}{\partial \lambda} \\ &= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} + \frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\ &= \theta^3 \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 1}^2(x) e^{-\lambda(1-\theta)} \\ &\quad - (1 - \theta) g, \\ &\{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned} \quad (S.5.0.9)$$

It then follows that the first and second derivatives of the *pmf* in (5.14) are

First derivative with respect to λ

$$\begin{aligned} \frac{\partial f^w(n_1, n_2)}{\partial \lambda} &= g - f^w(n_1, n_2), \\ &\{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned} \quad (S.5.0.10)$$

First derivative with respect to θ_1

$$\frac{\partial f^w(n_1, n_2)}{\partial \theta_1} = \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2) - \frac{\lambda}{\theta} g, \quad (S.5.0.11)$$

$$\{n_1, n_2\} \in \{0, 1, \dots\}.$$

First derivative with respect to θ_2

$$\frac{\partial f^w(n_1, n_2)}{\partial \theta_2} = \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f^w(n_1, n_2) - \frac{\lambda}{\theta} g, \quad (S.5.0.12)$$

$$\{n_1, n_2\} \in \{0, 1, \dots\}.$$

Second derivative with respect to λ

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} = g' - g + f^w(n_1, n_2),$$

$$\{n_1, n_2\} \in \{0, 1, \dots\}.$$

Second derivative with respect to θ_1 and λ

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \lambda} = -\frac{\lambda}{\theta} g' + \left(\frac{n_1}{\theta_1} - \frac{2}{\theta} \right) g - \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2), \quad (S.5.0.13)$$

$$\{n_1, n_2\} \in \{0, 1, \dots\}.$$

Second derivative with respect to θ_2 and λ

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_2 \partial \lambda} = -\frac{\lambda}{\theta} g' + \left(\frac{n_2}{\theta_2} - \frac{2}{\theta} \right) g - \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f^w(n_1, n_2), \quad (S.5.0.14)$$

$$\{n_1, n_2\} \in \{0, 1, \dots\}.$$

Second derivative with respect to θ_1

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{(\lambda+2)}{\theta} \right) g \\ &\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f^w(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned}$$

Second derivative with respect to θ_2

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_2^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_2}{\theta_2} - \frac{(\lambda+2)}{\theta} \right) g \\ &\quad + \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right)^2 f^w(n_1, n_2) - \left(\frac{n_2}{\theta_2^2} + \frac{1}{\theta^2} \right) f^w(n_1, n_2), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned}$$

Second derivative with respect to θ_1 and θ_2

$$\begin{aligned} \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \theta_2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{(\lambda+2)}{\theta} \right) g \\ &\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f^w(n_1, n_2) - \frac{f^w(n_1, n_2)}{\theta^2}, \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (5.14) is given here.

First derivative with respect to λ

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (5.14) and (S.5.0.8) and using (A.13), it follows that

$$\begin{aligned} &\frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\ &= \theta^2 \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-1}^1(x) e^{-\lambda(1-\theta)} - (1-\theta) f^w(n_1, n_2) \\ &= \theta^2 \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} [L_{n_1+n_2}^1(x) - L_{n_1+n_2}^0(x)] e^{-\lambda(1-\theta)} - (1-\theta) f^w(n_1, n_2) \\ &= \theta^2 \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2}^1(x) e^{-\lambda(1-\theta)} - f^w(n_1, n_2) \\ &= g - f^w(n_1, n_2). \end{aligned}$$

First derivative with respect to θ_1

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (5.14) and (S.5.0.8) and using (A.13), it follows that

$$\begin{aligned}
 & \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} \\
 &= \theta \binom{n_1 + n_2}{n_2} \left(\frac{n_1}{\theta_1} \right) \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2}^0(x) e^{-\lambda(1-\theta)} \\
 &\quad - \lambda \theta \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-1}^1(x) e^{-\lambda(1-\theta)} - \left(\lambda + \frac{1}{\theta} \right) f^w(n_1, n_2) \\
 &= -\lambda \theta \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} [L_{n_1+n_2}^1(x) - L_{n_1+n_2}^0(x)] e^{-\lambda(1-\theta)} \\
 &\quad + \left(\frac{n_1}{\theta_1} - \lambda - \frac{1}{\theta} \right) f^w(n_1, n_2) \\
 &= -\lambda \theta \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2}^1(x) e^{-\lambda(1-\theta)} + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2) \\
 &= \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2) - \frac{\lambda}{\theta} g.
 \end{aligned}$$

The first derivative with respect to θ_2 follows similarly.

Second derivative with respect to λ

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (S.5.0.9) and (S.5.0.10), it follows that

$$\begin{aligned}
 & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} \\
 &= \theta^3 \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-1}^2(x) e^{-\lambda(1-\theta)} - \frac{\partial f^w(n_1, n_2)}{\partial \lambda} - (1-\theta) g \\
 &= g' - \frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\
 &= g' - g + f^w(n_1, n_2).
 \end{aligned}$$

Second derivative with respect to θ_1 and λ

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (S.5.0.10), it follows that

$$\begin{aligned}
 & \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \lambda} \\
 &= \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda} - \frac{1}{\theta} g - \frac{\lambda}{\theta} g' \\
 &= -\frac{\lambda}{\theta} g' + \left(\frac{n_1}{\theta_1} - \frac{2}{\theta} \right) g - \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2).
 \end{aligned}$$

The second derivative with respect to θ_2 and λ follows similarly.

Second derivative with respect to θ_1

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (S.5.0.11) and (S.5.0.13), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda \partial \theta_1} + \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f^w(n_1, n_2) \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{(\lambda+2)}{\theta} \right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2} \right) f^w(n_1, n_2). \end{aligned}$$

The second derivative with respect to θ_2 follows similarly.

Second derivative with respect to θ_1 and θ_2

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$. Substituting from (S.5.0.12) and (S.5.0.14), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \theta_2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda \partial \theta_2} + \frac{\partial f^w(n_1, n_2)}{\partial \theta_2} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_2} - \frac{f^w(n_1, n_2)}{\theta^2} \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{(\lambda+2)}{\theta} \right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta} \right) f^w(n_1, n_2) - \frac{f^w(n_1, n_2)}{\theta^2}. \end{aligned}$$

5.5 Weighted Type II bivariate Pólya-Aeppli distribution Case II

5.5.5 Maximum likelihood estimation

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 5.5.5 for the $WBPA_{II}^{(2)}$ distribution. The Newton-Raphson algorithm uses the equation in (5.13) for $t \geq 1$ and where the MoM estimates in Subsection 5.5.4 are used as initial values of the parameters $\lambda^{(0)}$, $\theta_1^{(0)}$ and $\theta_2^{(0)}$ for the $WBPA_{II}^{(2)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (5.15) need to be calculated with respect to each of the parameters: λ , θ_1 and θ_2 .

The results of the calculation of the first and second derivatives of the *pmf* for the $WBPA_{II}^{(2)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(0,0)$, $f^w(1,0)$, $f^w(0,1)$ and $f(n_1, n_2)$ where $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$ with respect to each of the parameters: λ , θ_1 and θ_2 are written in a shorter form to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta$ and let $L_n^\alpha(x)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f^w(n_1, n_2)}{\partial \lambda} - \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) f^w(n_1, n_2) \\ &= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})(n_1 + n_2)(n_1 + n_2 - 1)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 2}^3(x), \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\ &\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}. \end{aligned} \quad (\text{S.5.0.15})$$

$$\begin{aligned} g' &= \frac{\partial g}{\partial \lambda} \\ &= \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} - \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\ &\quad + \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, n_2) \\ &= \frac{\lambda^2 \theta^3 e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})(n_1 + n_2)(n_1 + n_2 - 1)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 3}^4(x) \\ &\quad + \left(\frac{2}{\lambda} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} - (1 - \theta) \right) g, \\ &\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\ &\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}. \end{aligned} \quad (\text{S.5.0.16})$$

It then follows that the first and second derivatives of the *pmf* in (5.15) are

First derivative with respect to λ

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \lambda} &= \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f^w(0,0) + \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial f^w(1,0)}{\partial \lambda} &= \theta \theta_1 f^w(0,0) + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f^w(1,0) \\
\frac{\partial f^w(0,1)}{\partial \lambda} &= \theta \theta_2 f^w(0,0) + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f^w(0,1) \\
\frac{\partial f^w(n_1, n_2)}{\partial \lambda} &= g + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, n_2), \\
&\{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned} \tag{S.5.0.17}$$

First derivative with respect to θ_1

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \theta_1} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0,0) - \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial f^w(1,0)}{\partial \theta_1} &= \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda \right) f^w(1,0) - \lambda \theta_1 f^w(0,0) \\
\frac{\partial f^w(0,1)}{\partial \theta_1} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0,1) - \lambda \theta_2 f^w(0,0) \\
\frac{\partial f^w(n_1, n_2)}{\partial \theta_1} &= -\frac{\lambda}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2), \\
&\{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned} \tag{S.5.0.18}$$

First derivative with respect to θ_2

$$\begin{aligned}
\frac{\partial f^w(0,0)}{\partial \theta_1} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0,0) - \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial f^w(1,0)}{\partial \theta_1} &= \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda \right) f^w(1,0) - \lambda \theta_1 f^w(0,0) \\
\frac{\partial f^w(0,1)}{\partial \theta_1} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0,1) - \lambda \theta_2 f^w(0,0) \\
\frac{\partial f^w(n_1, n_2)}{\partial \theta_1} &= -\frac{\lambda}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) f^w(n_1, n_2), \\
&\{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned} \tag{S.5.0.19}$$

Second derivative with respect to λ

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \lambda^2} &= \frac{(\theta-1)}{(1-e^{-\lambda})} f^w(0,0) + \left(\theta - \frac{2}{(1-e^{-\lambda})} \right) \frac{\partial f^w(0,0)}{\partial \lambda} \\
\frac{\partial^2 f^w(1,0)}{\partial \lambda^2} &= \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} f^w(1,0) + \theta \theta_1 \frac{\partial f^w(0,0)}{\partial \lambda} \\
&\quad + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(1,0)}{\partial \lambda} \\
\frac{\partial^2 f^w(0,1)}{\partial \lambda^2} &= \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} f^w(0,1) + \theta \theta_2 \frac{\partial f^w(0,0)}{\partial \lambda} \\
&\quad + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(0,1)}{\partial \lambda} \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} &= g' + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\
&\quad - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} \right) f^w(n_1, n_2), \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned}$$

Second derivative with respect to θ_1 and λ

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0,0)}{\partial \lambda} - f^w(0,0) \\
&\quad + \frac{1}{\theta} \left(\frac{\lambda}{(1-e^{-\lambda})} - 1 \right) \left(\frac{e^{-\lambda}}{(1-e^{-\lambda})} \right) \\
\frac{\partial^2 f^w(1,0)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda \right) \frac{\partial f^w(1,0)}{\partial \lambda} - f^w(1,0) \\
&\quad - \lambda \theta_1 \frac{\partial f^w(0,0)}{\partial \lambda} - \theta_1 f^w(0,0) \\
\frac{\partial^2 f^w(0,1)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0,1)}{\partial \lambda} - f^w(0,1) \\
&\quad - \lambda \theta_2 \frac{\partial f^w(0,0)}{\partial \lambda} - \theta_2 f^w(0,0) \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \lambda} &= -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda}, \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned} \tag{S.5.0.20}$$

Second derivative with respect to θ_2 and λ

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,0)}{\partial \lambda} - f^w(0,0) \\
&\quad + \frac{1}{\theta} \left(\frac{\lambda}{1-e^{-\lambda}} - 1\right) \left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right) \\
\frac{\partial^2 f^w(1,0)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1,0)}{\partial \lambda} - f^w(1,0) \\
&\quad - \lambda \theta_1 \frac{\partial f^w(0,0)}{\partial \lambda} - \theta_1 f^w(0,0) \\
\frac{\partial^2 f^w(0,1)}{\partial \theta_1 \partial \lambda} &= \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,1)}{\partial \lambda} - f^w(0,1) \\
&\quad - \lambda \theta_2 \frac{\partial f^w(0,0)}{\partial \lambda} - \theta_2 f^w(0,0) \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \lambda} &= -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda}, \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned} \tag{S.5.0.21}$$

Second derivative with respect to θ_1

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \theta_1^2} &= \frac{1}{\theta^2} f^w(0,0) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,0)}{\partial \theta_1} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{1-e^{-\lambda}} \\
\frac{\partial^2 f^w(1,0)}{\partial \theta_1^2} &= \left(\frac{1}{\theta^2} - \frac{1}{\theta_1^2}\right) f^w(1,0) + \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1,0)}{\partial \theta_1} - \lambda \theta_1 \frac{\partial f^w(0,0)}{\partial \theta_1} - \lambda f^w(0,0) \\
\frac{\partial^2 f^w(0,1)}{\partial \theta_1^2} &= \frac{1}{\theta^2} f^w(0,1) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,1)}{\partial \theta_1} - \lambda \theta_2 \frac{\partial f^w(0,0)}{\partial \theta_1} \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{\lambda}{\theta} \frac{1}{1-e^{-\lambda}}\right) g \\
&\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2}\right) f^w(n_1, n_2), \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned}$$

Second derivative with respect to θ_2

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \theta_1^2} &= \frac{1}{\theta^2} f^w(0,0) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,0)}{\partial \theta_1} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial^2 f^w(1,0)}{\partial \theta_1^2} &= \left(\frac{1}{\theta^2} - \frac{1}{\theta_1^2}\right) f^w(1,0) + \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1,0)}{\partial \theta_1} - \lambda \theta_1 \frac{\partial f^w(0,0)}{\partial \theta_1} - \lambda f^w(0,0) \\
\frac{\partial^2 f^w(0,1)}{\partial \theta_1^2} &= \frac{1}{\theta^2} f^w(0,1) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,1)}{\partial \theta_1} - \lambda \theta_2 \frac{\partial f^w(0,0)}{\partial \theta_1} \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{\lambda}{\theta} \frac{1}{(1-e^{-\lambda})}\right) g \\
&\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2}\right) f^w(n_1, n_2), \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned}$$

Second derivative with respect to θ_1 and θ_2

$$\begin{aligned}
\frac{\partial^2 f^w(0,0)}{\partial \theta_1 \partial \theta_2} &= \frac{f^w(0,0)}{\theta^2} + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,0)}{\partial \theta_2} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial^2 f^w(1,0)}{\partial \theta_1 \partial \theta_2} &= \frac{f^w(1,0)}{\theta^2} + \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1,0)}{\partial \theta_2} - \lambda \theta_1 \frac{\partial f^w(0,0)}{\partial \theta_2} \\
\frac{\partial^2 f^w(0,1)}{\partial \theta_1 \partial \theta_2} &= \frac{f^w(0,1)}{\theta^2} + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0,1)}{\partial \theta_2} - \lambda \theta_2 \frac{\partial f^w(0,0)}{\partial \theta_2} - \lambda f^w(0,0) \\
\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \theta_2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{\lambda}{\theta} \frac{1}{(1-e^{-\lambda})}\right) g \\
&\quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta}\right) f^w(n_1, n_2) - \frac{f^w(n_1, n_2)}{\theta^2}, \\
&\quad \{n_1, n_2\} \in \{0, 1, \dots\}, \{n_1, n_2\} \neq \{0, 0\}, \\
&\quad \{n_1, n_2\} \neq \{1, 0\}, \{n_1, n_2\} \neq \{0, 1\}.
\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (5.15) is given here.

First derivative with respect to λ

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (5.15), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, 0)}{\partial \lambda} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta e^{\lambda\theta}}{\theta} - \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \frac{(e^{\lambda\theta} - 1)}{\theta} - f^w(0, 0) \\ &= \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) f^w(0, 0) + \frac{e^{-\lambda}}{(1 - e^{-\lambda})}. \end{aligned}$$

Let $\{n_1, n_2\} = \{1, 0\}$. Substituting from (5.15), it follows that

$$\begin{aligned} & \frac{\partial f^w(1, 0)}{\partial \lambda} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_1}{\theta} (\theta e^{\lambda\theta}) + \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_1}{\theta} [\theta(\lambda\theta - 1)e^{\lambda\theta}] \\ & \quad - \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \frac{\theta_1}{\theta} [(\lambda\theta - 1)e^{\lambda\theta} + 1] - f^w(1, 0) \\ &= \lambda\theta^2\theta_1 f^w(0, 0) - \frac{1}{(1 - e^{-\lambda})} f^w(1, 0) + \lambda\theta\theta_1 \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \\ &= \theta\theta_1 f^w(0, 0) + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) f^w(1, 0). \end{aligned}$$

Let $\{n_1, n_2\} = \{0, 1\}$. It follows that

$$\frac{\partial f^w(0, 1)}{\partial \lambda} = \theta\theta_2 f^w(0, 0) + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) f^w(0, 1).$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. Substituting from (5.15) and (S.5.0.15) and using (A.14), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, n_2)}{\partial \lambda} \\
&= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-3}^3(x) \\
&\quad - \frac{\lambda^2 \theta e^{-\lambda(1-\theta)} e^{-\lambda}}{(1-e^{-\lambda})^2 (n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-2}^2(x) \\
&\quad + \frac{2}{\lambda} f^w(n_1, n_2) - (1-\theta) f^w(n_1, n_2) \\
&= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} \left[L_{n_1+n_2-2}^3(x) - L_{n_1+n_2-2}^2(x) \right] \\
&\quad + \left(\frac{2}{\lambda} - (1-\theta) - \frac{e^{-\lambda}}{(1-e^{-\lambda})} \right) f^w(n_1, n_2) \\
&= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-2}^3(x) \\
&\quad + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, n_2) \\
&= g + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, n_2).
\end{aligned}$$

First derivative with respect to θ_1

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (5.15), it follows that

$$\begin{aligned}
& \frac{\partial f^w(0, 0)}{\partial \theta_1} \\
&= \frac{f^w(0, 0)}{\theta} - \frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\lambda e^{\lambda\theta}}{\theta} \\
&= \left(\frac{1}{\theta} - \lambda \right) f^w(0, 0) - \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1-e^{-\lambda})}.
\end{aligned}$$

Let $\{n_1, n_2\} = \{1, 0\}$. Substituting from (5.15), it follows that

$$\begin{aligned}
& \frac{\partial f^w(1, 0)}{\partial \theta_1} \\
&= -\frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_1}{\theta} (\lambda e^{\lambda\theta}) - \frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_1}{\theta} [\lambda(\lambda\theta-1)e^{\lambda\theta}] + \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right) f^w(1, 0) \\
&= \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda \right) f^w(1, 0) - \lambda \theta_1 f^w(0, 0).
\end{aligned}$$

Let $\{n_1, n_2\} = \{0, 1\}$. Substituting from (5.15), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, 1)}{\partial \theta_1} \\ &= -\frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_2}{\theta} (\lambda e^{\lambda\theta}) - \frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_2}{\theta} [\lambda(\lambda\theta - 1)e^{\lambda\theta}] + \frac{f^w(0, 1)}{\theta} \\ &= \left(\frac{1}{\theta} - \lambda\right) f^w(0, 1) - \lambda\theta_2 f^w(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. Substituting from (5.15) and (S.5.0.15) and using (A.14), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \frac{n_1}{\theta_1} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-2}^2(x) \\ & \quad - \frac{\lambda^3 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-3}^3(x) \\ & \quad - \left(\lambda + \frac{1}{\theta}\right) f^w(n_1, n_2) \\ &= -\frac{\lambda^3 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} \left[L_{n_1+n_2-2}^3(x) - L_{n_1+n_2-2}^2(x) \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta} - \lambda\right) f^w(n_1, n_2) \\ &= -\frac{\lambda^3 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})(n_1+n_2)(n_1+n_2-1)} \binom{n_1+n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1+n_2-2}^3(x) \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) f^w(n_1, n_2) \\ &= -\frac{\lambda}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) f^w(n_1, n_2). \end{aligned}$$

The first derivative with respect to θ_2 follows similarly.

Second derivative with respect to λ

Let $\{n_1, n_2\} = \{0, 0\}$. Substituting from (S.5.0.17), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, 0)}{\partial \lambda^2} \\ &= \left(\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}\right) f^w(0, 0) + \left(\theta - \frac{1}{(1-e^{-\lambda})}\right) \frac{\partial f^w(0, 0)}{\partial \lambda} - \frac{e^{-\lambda}}{(1-e^{-\lambda})} - \frac{(e^{-\lambda})^2}{(1-e^{-\lambda})^2} \\ &= \frac{(\theta-1)}{(1-e^{-\lambda})} f^w(0, 0) + \left(\theta - \frac{2}{(1-e^{-\lambda})}\right) \frac{\partial f^w(0, 0)}{\partial \lambda}. \end{aligned}$$

Let $\{n_1, n_2\} = \{1, 0\}$. It follows that

$$\frac{\partial^2 f^w(1, 0)}{\partial \lambda^2} = \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} f^w(1, 0) + \theta \theta_1 \frac{\partial f^w(0, 0)}{\partial \lambda} + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(1, 0)}{\partial \lambda}.$$

Let $\{n_1, n_2\} = \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(0, 1)}{\partial \lambda^2} = \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} f^w(0, 1) + \theta \theta_2 \frac{\partial f^w(0, 0)}{\partial \lambda} + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(0, 1)}{\partial \lambda}.$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. Substituting from (S.5.0.16), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda^2} \\ &= \frac{\lambda^2 \theta^3 e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})(n_1 + n_2)(n_1 + n_2 - 1)} \binom{n_1 + n_2}{n_2} \theta_1^{n_1} \theta_2^{n_2} L_{n_1 + n_2 - 3}^4(x) \\ &+ \left(\frac{2}{\lambda} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} - (1 - \theta) \right) g \\ &+ \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda} - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, n_2) \\ &= g' + \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda} - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, n_2) \end{aligned}$$

Second derivative with respect to θ_1 and λ

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, 0)}{\partial \theta_1 \partial \lambda} \\ &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, 0)}{\partial \lambda} - f^w(0, 0) \\ &\quad - \frac{1}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} + \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} + \frac{\lambda}{\theta} \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \\ &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, 0)}{\partial \lambda} - f^w(0, 0) + \frac{1}{\theta} \left(\frac{\lambda}{(1 - e^{-\lambda})} - 1 \right) \left(\frac{e^{-\lambda}}{(1 - e^{-\lambda})} \right). \end{aligned}$$

Let $\{n_1, n_2\} = \{1, 0\}$. It follows that

$$\frac{\partial^2 f^w(1, 0)}{\partial \theta_1 \partial \lambda} = \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda \right) \frac{\partial f^w(1, 0)}{\partial \lambda} - f^w(1, 0) - \lambda \theta_1 \frac{\partial f^w(0, 0)}{\partial \lambda} - \theta_1 f^w(0, 0).$$

Let $\{n_1, n_2\} = \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(0, 1)}{\partial \theta_1 \partial \lambda} = \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0, 1)}{\partial \lambda} - f^w(0, 1) - \lambda \theta_2 \frac{\partial f^w(0, 0)}{\partial \lambda} - \theta_2 f^w(0, 0).$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \lambda} = -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \frac{\partial f^w(n_1, n_2)}{\partial \lambda}.$$

The second derivative with respect to θ_2 and λ follows similarly.

Second derivative with respect to θ_1

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \theta_1^2} = \frac{1}{\theta^2} f^w(0, 0) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0, 0)}{\partial \theta_1} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

Let $\{n_1, n_2\} = \{1, 0\}$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(1, 0)}{\partial \theta_1^2} \\ &= \left(\frac{1}{\theta^2} - \frac{1}{\theta_1^2}\right) f^w(1, 0) + \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1, 0)}{\partial \theta_1} \\ & \quad - \lambda \theta_1 \frac{\partial f^w(0, 0)}{\partial \theta_1} - \lambda f^w(0, 0). \end{aligned}$$

Let $\{n_1, n_2\} = \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(0, 1)}{\partial \theta_1^2} = \frac{1}{\theta^2} f^w(0, 1) + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0, 1)}{\partial \theta_1} - \lambda \theta_2 \frac{\partial f^w(0, 0)}{\partial \theta_1}.$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. Substituting from (S.5.0.18) and (S.5.0.20), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1^2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda \partial \theta_1} - \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})}\right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_1} - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2}\right) f^w(n_1, n_2) \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_1}{\theta_1} - \frac{\lambda}{\theta} \frac{1}{(1 - e^{-\lambda})}\right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right)^2 f^w(n_1, n_2) - \left(\frac{n_1}{\theta_1^2} + \frac{1}{\theta^2}\right) f^w(n_1, n_2). \end{aligned}$$

The second derivative with respect to θ_2 follows similarly.

Second derivative with respect to θ_1 and θ_2

Let $\{n_1, n_2\} = \{0, 0\}$. It follows that

$$\frac{\partial^2 f^w(0, 0)}{\partial \theta_1 \partial \theta_2} = \frac{f^w(0, 0)}{\theta^2} + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0, 0)}{\partial \theta_2} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

Let $\{n_1, n_2\} = \{1, 0\}$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(1, 0)}{\partial \theta_1 \partial \theta_2} \\ &= \frac{f^w(1, 0)}{\theta^2} + \left(\frac{1}{\theta} + \frac{1}{\theta_1} - \lambda\right) \frac{\partial f^w(1, 0)}{\partial \theta_2} - \lambda \theta_1 \frac{\partial f^w(0, 0)}{\partial \theta_2}. \end{aligned}$$

Let $\{n_1, n_2\} = \{0, 1\}$. It follows that

$$\frac{\partial^2 f^w(0, 1)}{\partial \theta_1 \partial \theta_2} = \frac{f^w(0, 1)}{\theta^2} + \left(\frac{1}{\theta} - \lambda\right) \frac{\partial f^w(0, 1)}{\partial \theta_2} - \lambda \theta_2 \frac{\partial f^w(0, 0)}{\partial \theta_2} - \lambda f^w(0, 0).$$

Let $\{n_1, n_2\} \in \{0, 1, \dots\}$, $\{n_1, n_2\} \neq \{0, 0\}$, $\{n_1, n_2\} \neq \{1, 0\}$, $\{n_1, n_2\} \neq \{0, 1\}$. Substituting from (S.5.0.19) and (S.5.0.21), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, n_2)}{\partial \theta_1 \partial \theta_2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, n_2)}{\partial \lambda \partial \theta_2} - \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})}\right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_2} \right] \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \frac{\partial f^w(n_1, n_2)}{\partial \theta_2} - \frac{f^w(n_1, n_2)}{\theta^2} \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_1}{\theta_1} + \frac{n_2}{\theta_2} - \frac{\lambda}{\theta} \frac{1}{(1 - e^{-\lambda})}\right) g \\ & \quad + \left(\frac{n_1}{\theta_1} - \frac{1}{\theta}\right) \left(\frac{n_2}{\theta_2} - \frac{1}{\theta}\right) f^w(n_1, n_2) - \frac{f^w(n_1, n_2)}{\theta^2}. \end{aligned}$$

Chapter 6

Multivariate extension of Type II bivariate Pólya-Aeppli distributions

This supplementary material for Chapter 6 provides the derivations and calculations for the following results: the *pgfs*, moments, and moment-based functions in Section 6.2; the *pmfs* and selected results for the conditional distributions and expected values; and the MoM estimators and MLEs for the BPA_{II} , $WBPA_{II}^{(1)}$, and $WBPA_{II}^{(2)}$ distributions, as discussed in Sections 6.3, 6.4, and 6.5, respectively.

6.2 Properties

6.2.1 Probability generating functions

Theorem 6.2.1. *Let $(N_1, \dots, N_k) \sim MPA_{II}(\lambda, \theta_1, \dots, \theta_k)$. From Minkova and Balakrishnan [2], the joint *pgf* of the MPA_{II} distribution is*

$$\psi_{N_1, \dots, N_k}(s_1, \dots, s_k) = e^{-\lambda \left(1 - \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)}\right)},$$

where $0 < \theta_i < 1$, $\theta = 1 - \sum_{i=1}^k \theta_i \neq 0$ and $i \in \{1, \dots, k\}$.

Proof. Using the *pgf* of the Poisson random variable $Y \sim \text{Poi}(\lambda)$ in Table 2.1 and the *pgf* of the multivariate geometric distribution, it follows from (A.39) that the joint *pgf* of $(N_1, \dots, N_k) \sim MPA_{II}(\lambda, \theta_1, \dots, \theta_k)$ is

$$\psi_{N_1, \dots, N_k}(s_1, \dots, s_k) = \psi_2(\psi_1(s_1, \dots, s_k)) = e^{-\lambda(1 - \psi_1(s_1, \dots, s_k))} = e^{-\lambda \left(1 - \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)}\right)}.$$

□

Theorem 6.2.2. *Let $(N_1^w, \dots, N_k^w) \sim WMPA_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$. The joint *pgf* of the $WMPA_{II}^{(1)}$ distribution is*

$$\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) = \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)} e^{-\lambda \left(1 - \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)}\right)},$$

where $0 < \theta_i < 1$, $\theta = 1 - \sum_{i=1}^k \theta_i \neq 0$ and $i \in \{1, \dots, k\}$.

Proof. Using the *pgf* of the weighted Poisson random variable $Y^w \sim \text{WPoi}^{(1)}(\lambda)$ in Table 2.1 and the *pgf* of the multivariate geometric distribution, it follows from (A.39) and similar to Theorem 6.2.1, that the joint *pgf* of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$ is

$$\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) = \psi_1(s_1, \dots, s_k) e^{-\lambda(1 - \psi_1(s_1, \dots, s_k))} = \frac{\theta}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)} e^{-\lambda \left(1 - \frac{\theta}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}\right)}.$$

□

Theorem 6.2.3. Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$. The joint *pgf* of the $\text{WMPA}_{II}^{(2)}$ distribution is

$$\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}{\theta} \left(e^{\frac{\lambda \theta}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}} - 1 \right),$$

where $0 < \theta_i < 1$, $\theta = 1 - \sum_{i=1}^k \theta_i \neq 0$ and $i \in \{1, \dots, k\}$.

Proof. Using the *pgf* of the weighted Poisson random variable $Y^w \sim \text{WPoi}^{(2)}(\lambda)$ in Table 2.1 and the *pgf* of the multivariate geometric distribution, it follows from (A.39) in the same way as in Theorem 6.2.2, that the joint *pgf* of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$ is

$$\begin{aligned} \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\left(e^{\lambda \psi_1(s_1, \dots, s_k)} - 1\right)}{\psi_1(s_1, \dots, s_k)} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}{\theta} \left(e^{\frac{\lambda \theta}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}} - 1 \right). \end{aligned}$$

□

6.2.2 Moments and moment-based functions

The moments of the multivariate distributions can be obtained in a similar way to the bivariate case presented in Section 5.2.2. This involves calculating the first, second, and joint partial derivatives of the *pgfs* in (6.1), (6.2), and (6.3) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}$, $i \neq j$.

For $(N_1, \dots, N_k) \sim \text{MPA}_{II}(\lambda, \theta_1, \dots, \theta_k)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (6.1) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}, i \neq j$ are

$$\begin{aligned}\frac{\partial \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i} &= \frac{\lambda \theta_i}{\theta} \psi_1(s_1, \dots, s_k)^2 \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ \frac{\partial \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i \partial s_j} &= \frac{\lambda \theta_i \theta_j}{\theta^2} \psi_1(s_1, \dots, s_k)^3 [2 + \lambda \psi_1(s_1, \dots, s_k)] \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial s_i^2} &= \frac{\lambda \theta_i^2}{\theta^2} \psi_1(s_1, \dots, s_k)^3 [2 + \lambda \psi_1(s_1, \dots, s_k)] \psi_{N_1, \dots, N_k}(s_1, \dots, s_k).\end{aligned}$$

For $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (4.2) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}, i \neq j$ are

$$\begin{aligned}\frac{\partial \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i} &= \frac{\theta_i}{\theta} \psi_1(s_1, \dots, s_k) [1 + \lambda \psi_1(s_1, \dots, s_k)] \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i \partial s_j} &= \frac{\theta_i \theta_j}{\theta^2} \psi_1(s_1, \dots, s_k)^2 [2 + 4\lambda \psi_1(s_1, \dots, s_k) + \lambda^2 \psi_1(s_1, \dots, s_k)^2] \\ &\quad \times \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i^2} &= \frac{\theta_i^2}{\theta^2} \psi_1(s_1, \dots, s_k)^2 [2 + 4\lambda \psi_1(s_1, \dots, s_k) + \lambda^2 \psi_1(s_1, \dots, s_k)^2] \\ &\quad \times \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)\end{aligned}$$

and for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$ the derivatives of $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ in (4.3) with respect to s_i and s_j , where $i, j \in \{1, \dots, k\}, i \neq j$ are

$$\begin{aligned}\frac{\partial \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i} &= \left[\frac{\lambda \psi_1(s_1, \dots, s_k)}{(1 - e^{-\lambda \psi_1(s_1, \dots, s_k)})} - 1 \right] \frac{\theta_i}{\theta} \psi_1(s_1, \dots, s_k) \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i \partial s_j} &= \frac{\lambda^2}{(1 - e^{-\lambda \psi_1(s_1, \dots, s_k)})} \frac{\theta_i \theta_j}{\theta^2} \psi_1(s_1, \dots, s_k)^4 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ \frac{\partial^2 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_i^2} &= \frac{\lambda^2}{(1 - e^{-\lambda \psi_1(s_1, \dots, s_k)})} \frac{\theta_i^2}{\theta^2} \psi_1(s_1, \dots, s_k)^4 \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k).\end{aligned}$$

6.3 Type I multivariate Pólya-Aeppli distribution

6.3.1 Joint probability mass function

Theorem 6.3.1. *The pmf of $(N_1, \dots, N_k) \sim \text{MPA}_{II}(\lambda, \theta_1, \dots, \theta_k)$ is*

$$\begin{aligned}f(0, \dots, 0) &= e^{-\lambda(1-\theta)} \\ f(n_1, \dots, n_k) &= \lambda \theta \left(\sum_{i=1}^k n_i - 1 \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^1(x) f(0, \dots, 0), \\ &\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\},\end{aligned}$$

where $\theta = 1 - \sum_{i=1}^k \theta_i$, $x = -\lambda\theta$ and $L_n^\alpha(x)$ is the Laguerre polynomial in (A.3).

Proof. For the first part of the proof, $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows from (A.17) and (6.1) that

$$f(0, \dots, 0) = \psi_{N_1, \dots, N_k}(0, \dots, 0) = e^{-\lambda(1-\theta)}.$$

The second part of the proof is derived for the case where $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$. Let $x = -\lambda\theta$ and $z_i = \theta_i s_i$ for $i \in \{1, \dots, k\}$, then the pgf in (6.1) can be written as

$$\begin{aligned} & \psi_{N_1, \dots, N_k}(s_1, \dots, s_k) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} f(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\ &= e^{-\lambda(1-\theta)} e^{\left(\frac{x \sum_{i=1}^k z_i}{\sum_{i=1}^k z_i - 1}\right)}. \end{aligned}$$

Differentiating $\psi_{N_1, \dots, N_k}(s_1, \dots, s_k)$ with respect to z_1 and using results (A.1) and (A.2), it can be seen that

$$\begin{aligned} & \frac{\partial \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial z_1} \\ &= \frac{\lambda\theta}{\left(1 - \sum_{i=1}^k z_i\right)^2} e^{\left(\frac{x \sum_{i=1}^k z_i}{\sum_{i=1}^k z_i - 1}\right)} f(0, \dots, 0) \\ &= \lambda\theta \sum_{n_1=0}^{\infty} L_{n_1}^1(x) \left(\sum_{i=1}^k z_i\right)^{n_1} f(0, \dots, 0) \\ &= \lambda\theta \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_1} \left(\prod_{i=1}^{k-1} \binom{n_i}{n_{i+1}}\right) L_{n_1}^1(x) z_1^{n_1-n_2} \cdots z_{k-1}^{n_{k-1}-n_k} z_k^{n_k} f(0, \dots, 0) \\ &= \lambda\theta \sum_{n_1, \dots, n_k=0}^{\infty} \left(\prod_{j=1}^{k-1} \binom{\sum_{i=j}^k n_i}{\sum_{i=j+1}^k n_i}\right) L_{\sum_{i=1}^k n_i}^1(x) z_1^{n_1} \cdots z_k^{n_k} f(0, \dots, 0) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} \left[\lambda\theta \frac{\left(\sum_{i=1}^k n_i\right)!}{\left(\prod_{i=1}^k n_i!\right)} L_{\sum_{i=1}^k n_i}^1(x) f(0, \dots, 0) \right] z_1^{n_1} \cdots z_k^{n_k} \\ &= \sum_{n_1=1}^{\infty} \sum_{n_2, \dots, n_k=0}^{\infty} \left[\lambda\theta \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{(n_1 - 1)! \left(\prod_{i=2}^k n_i!\right)} L_{\sum_{i=1}^k n_i - 1}^1(x) f(0, \dots, 0) \right] z_1^{n_1-1} z_2^{n_2} \cdots z_k^{n_k}. \end{aligned}$$

Furthermore, from (A.16),

$$\frac{\partial \psi_{N_1, \dots, N_k}(s_1, \dots, s_k)}{\partial z_1} = \sum_{n_1=1}^{\infty} \sum_{n_2, \dots, n_k=0}^{\infty} n_1 \frac{f(n_1, \dots, n_k)}{\left(\prod_{i=1}^k \theta_i^{n_i}\right)} z_1^{n_1-1} z_2^{n_2} \cdots z_k^{n_k}.$$

Similar results can be derived by differentiating with respect to z_i for $i \in \{2, \dots, k\}$. From these results it follows that for $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ where $i \in \{2, \dots, k\}$ that

$$f(n_1, \dots, n_k) = \lambda \theta \left(\sum_{i=1}^k n_i - 1 \right)! \left(\prod_{i=1}^k \frac{\theta^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^1(x) f(0, \dots, 0).$$

□

6.3.4 Method of moments estimates

Let $(N_1, \dots, N_k) \sim \text{MPA}_{II}(\lambda, \theta_1, \dots, \theta_k)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$. Furthermore, for $i, j \in \{1, \dots, k\}, i \neq j$, let

$$\bar{n}_i = \frac{1}{m} \sum_{\ell=1}^m n_{i\ell}$$

$$s_{ij} = \frac{1}{(m-1)} \sum_{\ell=1}^m (n_{i\ell} - \bar{n}_i)(n_{j\ell} - \bar{n}_j).$$

The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$ and $\text{Cov}(N_i^w, N_j^w)$, where $i, j \in \{1, \dots, k\}$ as presented in Table 6.1 in Section 6.2.2.

Therefore, if $\hat{\lambda}$ is the MoM estimator for λ , then

$$\frac{\hat{\lambda} \hat{\theta}_i}{\left(1 - \sum_{i=1}^k \hat{\theta}_i\right)} = \bar{n}_i$$

$$\sum_{i=1}^k \hat{\theta}_i = \frac{\bar{n}_i - \hat{\lambda} \hat{\theta}_i}{\bar{n}_i},$$

and an MoM estimator for $\theta_i, i \in \{1, \dots, k\}$ is

$$\frac{\hat{\lambda} \left(\sum_{i=1}^k \hat{\theta}_i\right)}{\left(1 - \sum_{i=1}^k \hat{\theta}_i\right)} = \sum_{i=1}^k \bar{n}_i$$

$$\frac{\bar{n}_i - \hat{\lambda} \hat{\theta}_i}{\hat{\theta}_i} = \sum_{i=1}^k \bar{n}_i$$

$$\hat{\theta}_i = \frac{\bar{n}_i}{\hat{\lambda} + \sum_{i=1}^k \bar{n}_i}.$$

In order to obtain $\hat{\lambda}$, the sample moment s_{ij} is equated with the population moment $\text{Cov}(N_i^w, N_j^w)$. It follows that

$$\begin{aligned} \frac{1}{2} \sum_{i,j=1,i \neq j}^k \frac{2\hat{\lambda}\hat{\theta}_i\hat{\theta}_j}{\left(1 - \sum_{i=1}^k \hat{\theta}_i\right)\left(1 - \sum_{j=1}^k \hat{\theta}_j\right)} &= \frac{1}{2} \sum_{i,j=1,i \neq j}^k s_{ij} \\ \frac{2}{\hat{\lambda}} \sum_{i,j=1,i \neq j}^k \bar{n}_i\bar{n}_j &= \sum_{i,j=1,i \neq j}^k s_{ij} \\ \hat{\lambda} &= 2 \sum_{i,j=1,i \neq j}^k \frac{\bar{n}_i\bar{n}_j}{s_{ij}}. \end{aligned}$$

6.3.5 Maximum likelihood estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 6.3.5 for the WMPA_{II} distribution. The Newton-Raphson algorithm uses the equation in (6.9) for $t \geq 1$ and the MoM estimates in Subsection 6.3.4 are used as initial values of the parameters $\lambda^{(0)}, \theta_1^{(0)}, \dots, \theta_k^{(0)}$ for the MPA_{II} distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{\text{th}}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F in (6.7) and the Hessian matrix in (6.8), such that the Newton-Raphson algorithm in (6.9) can be utilized, the first and second derivatives for each element of the *pmf* in (6.5) need to be calculated with respect to each of the parameters: λ and $\theta_1, \dots, \theta_k$.

The results of the calculation of the first and second derivatives of the *pmf* in (6.5) for the WMPA_{II} distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f(0, \dots, 0)$ and $f(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ with respect to each of the parameters: $\lambda, \theta_1, \dots, \theta_k$ are written in a shorter form in order to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta$, $i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k) \\ &= \lambda\theta^2 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 1}^2(x) f(0, \dots, 0), \quad (\text{S.6.0.1}) \\ &\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}. \end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial \lambda} = g' &= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} - \left(\frac{1}{\lambda} - 1\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} + \frac{f^w(n_1, \dots, n_k)}{\lambda^2} \\
&= \lambda \theta^3 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 2}^3(x) f(0, \dots, 0) \\
&\quad + \left(\frac{1}{\lambda} - (1 - \theta)\right) g, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned} \tag{S.6.0.2}$$

It then follows that the first and second derivatives of the pmf in (6.5) are

First derivative with respect to λ

$$\begin{aligned}
\frac{\partial f(0, \dots, 0)}{\partial \lambda} &= -(1 - \theta) f(0, \dots, 0) \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} &= g + \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned} \tag{S.6.0.3}$$

First derivative with respect to θ_j

$$\begin{aligned}
\frac{\partial f(0, \dots, 0)}{\partial \theta_j} &= -\lambda f(0, \dots, 0) \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} &= -\frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}, j \in \{1, \dots, k\}.
\end{aligned} \tag{S.6.0.4}$$

Second derivative with respect to λ

$$\begin{aligned}
\frac{\partial^2 f(0, \dots, 0)}{\partial \lambda^2} &= (1 - \theta)^2 f(0, \dots, 0) \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} &= g' + \left(\frac{1}{\lambda} - 1\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \frac{f^w(n_1, \dots, n_k)}{\lambda^2}, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned}$$

Second derivative with respect to θ_j and λ

$$\begin{aligned}\frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j \partial \lambda} &= (\lambda(1-\theta) - 1) f(0, \dots, 0) \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} &= -\frac{\lambda}{\theta} g' + \left(\frac{n_j}{\theta_j} - \frac{2}{\theta}\right) g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k), \\ &\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}, j \in \{1, \dots, k\}.\end{aligned}\tag{S.6.0.5}$$

Second derivative with respect to θ_j

$$\begin{aligned}\frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j^2} &= \lambda^2 f(0, \dots, 0) \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{(\lambda+1)}{\theta}\right) g \\ &\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2}\right) f^w(n_1, \dots, n_k), \\ &\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}, j \in \{1, \dots, k\}.\end{aligned}$$

Second derivative with respect to θ_j and θ_ℓ

$$\begin{aligned}\frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j \partial \theta_\ell} &= \lambda^2 f(0, \dots, 0) \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left[\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{(\lambda+1)}{\theta}\right] g \\ &\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta}\right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}, \\ &\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}, j, \ell \in \{1, \dots, k\}.\end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (6.5) is given here.

First derivative with respect to λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (6.5), it follows that

$$\begin{aligned}\frac{\partial f(0, \dots, 0)}{\partial \lambda} &= -(1-\theta) e^{-\lambda(1-\theta)} \\ &= -(1-\theta) f(0, \dots, 0).\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$. Substituting from (6.5), (S.6.0.1) and (S.6.0.3) and using (A.12), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\
&= \lambda \theta^2 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 2}^2(x) f(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda} + \frac{f^w(n_1, \dots, n_k)}{\lambda} \\
&= \lambda \theta^2 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) \left[L_{\sum_{i=1}^k n_i - 1}^2(x) - L_{\sum_{i=1}^k n_i - 1}^1(x)\right] f(0, \dots, 0) \\
&\quad - (1 - \theta) f^w(n_1, \dots, n_k) + \frac{f^w(n_1, \dots, n_k)}{\lambda} \\
&= \lambda \theta^2 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 1}^2(x) f(0, \dots, 0) \\
&\quad + \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k) \\
&= g + \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k).
\end{aligned}$$

First derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (6.5), it follows that

$$\begin{aligned}
& \frac{\partial f(0, \dots, 0)}{\partial \theta_j} \\
&= -\lambda e^{-\lambda(1-\theta)} \\
&= -\lambda f(0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (6.5), (S.6.0.1) and (S.6.0.4) and using (A.12), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \\
&= \lambda \theta \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\frac{n_j}{\theta_j} \prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 1}^1(x) f(0, \dots, 0) \\
&\quad - \lambda^2 \theta \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 2}^2(x) f(0, \dots, 0) \\
&\quad + \frac{f^w(n_1, \dots, n_k)}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \theta_j} - \frac{f^w(n_1, \dots, n_k)}{\theta} \\
&= -\lambda^2 \theta \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) \left[L_{\sum_{i=1}^k n_i - 1}^2(x) - L_{\sum_{i=1}^k n_i - 1}^1(x)\right] f(0, \dots, 0) \\
&\quad + \frac{n_j}{\theta_j} f^w(n_1, \dots, n_k) - \lambda f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta} \\
&= -\lambda^2 \theta \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 1}^2(x) f(0, \dots, 0) \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) f^w(n_1, \dots, n_k) \\
&= -\frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (S.6.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(0, \dots, 0)}{\partial \lambda^2} \\
&= -(1 - \theta) \frac{\partial f(0, \dots, 0)}{\partial \lambda} \\
&= (1 - \theta)^2 f(0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$. Substituting from (S.6.0.2) and (S.6.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} \\
&= \lambda \theta^3 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 2}^3(x) f(0, \dots, 0) \\
&+ \left[\frac{1}{f(0, \dots, 0)} \frac{\partial f(0, \dots, 0)}{\partial \lambda} + \frac{1}{\lambda} \right] g \\
&+ \left(\frac{1}{\lambda} - 1\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \frac{f^w(n_1, \dots, n_k)}{\lambda^2} \\
&= \lambda \theta^3 \frac{\left(\sum_{i=1}^k n_i - 1\right)!}{\left(\prod_{i=1}^k n_i!\right)} \left(\prod_{i=1}^k \theta_i^{n_i}\right) L_{\sum_{i=1}^k n_i - 2}^3(x) f(0, \dots, 0) \\
&+ \left(\frac{1}{\lambda} - (1 - \theta)\right) g + \left(\frac{1}{\lambda} - 1\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \frac{f^w(n_1, \dots, n_k)}{\lambda^2} \\
&= g' + \left(\frac{1}{\lambda} - 1\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \frac{f^w(n_1, \dots, n_k)}{\lambda^2}.
\end{aligned}$$

Second derivative with respect to θ_j and λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j \partial \lambda} \\
&= -f(0, \dots, 0) - \lambda \frac{\partial f(0, \dots, 0)}{\partial \lambda} \\
&= (\lambda(1 - \theta) - 1) f(0, \dots, 0).
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.3), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} \\
&= -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\
&= -\frac{\lambda}{\theta} g' + \left(\frac{n_j}{\theta_j} - \frac{2}{\theta}\right) g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta}\right) \left(\frac{1}{\lambda} - 1\right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.4), it follows that

$$\begin{aligned} & \frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j^2} \\ &= -\lambda \frac{\partial f(0, \dots, 0)}{\partial \theta_j} \\ &= \lambda^2 f(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.4) and (S.6.0.5), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} \\ &= \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k) \\ & \quad - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_j} - \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \right] - \frac{\lambda}{\theta^2} g \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{(\lambda+1)}{\theta} \right) g \\ & \quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to θ_j and θ_ℓ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.6.0.4), it follows that

$$\begin{aligned} & \frac{\partial^2 f(0, \dots, 0)}{\partial \theta_j \partial \theta_\ell} \\ &= -\lambda \frac{\partial f(0, \dots, 0)}{\partial \theta_\ell} \\ &= \lambda^2 f(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.6.0.4) and (S.6.0.5), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} \\ &= \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} - \frac{\lambda}{\theta^2} g \\ & \quad - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_\ell} - \left(\frac{1}{\lambda} - 1 \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} \right] - \frac{f^w(n_1, \dots, n_k)}{\theta^2} \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left[\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{(\lambda + 1)}{\theta} \right] g \\ & \quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}. \end{aligned}$$

6.4 Weighted Type II multivariate Pólya-Aeppli distribution Case I

6.4.1 Joint probability mass function

Theorem 6.4.1. *The pmf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$ is*

$$\begin{aligned} f^w(n_1, \dots, n_k) &= \theta \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i}^0(x) e^{-\lambda(1-\theta)}, \\ & \quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \end{aligned}$$

where $\theta = 1 - \sum_{i=1}^k \theta_i$, $x = -\lambda\theta$ and $L_n^\alpha(x)$ is the Laguerre polynomial in (A.3).

Proof. Suppose $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and let $x = -\lambda\theta$ and $z_i = \theta_i s_i$ for $i \in \{1, \dots, k\}$. Using results (A.1) and (A.2), the pgf in (6.2) can be written as

$$\begin{aligned}
& \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\
&= \frac{\theta}{\left(1 - \sum_{i=1}^k z_i\right)} e^{\left(\frac{x \sum_{i=1}^k z_i}{\sum_{i=1}^k z_i - 1}\right)} e^{-\lambda(1-\theta)} \\
&= \theta \sum_{n_1=0}^{\infty} L_{n_1}^0(x) \left(\sum_{i=1}^k z_i\right)^{n_1} e^{-\lambda(1-\theta)} \\
&= \theta \sum_{n_1=n_2}^{\infty} \sum_{n_2=n_3}^{\infty} \cdots \sum_{n_k=0}^{\infty} \left(\prod_{i=1}^{k-1} \binom{n_i}{n_{i+1}}\right) L_{n_1}^0(x) z_1^{n_1-n_2} \cdots z_{k-1}^{n_{k-1}-n_k} z_k^{n_k} e^{-\lambda(1-\theta)} \\
&= \theta \sum_{n_1, \dots, n_k=0}^{\infty} \left(\prod_{j=1}^{k-1} \binom{\sum_{i=j}^k n_i}{\sum_{i=j+1}^k n_i}\right) L_{\sum_{i=1}^k n_i}^0(x) z_1^{n_1} \cdots z_k^{n_k} e^{-\lambda(1-\theta)} \\
&= \theta \sum_{n_1, \dots, n_k=0}^{\infty} \frac{\left(\sum_{i=1}^k n_i\right)!}{\left(\prod_{i=1}^k n_i!\right)} L_{\sum_{i=1}^k n_i}^0(x) z_1^{n_1} \cdots z_k^{n_k} e^{-\lambda(1-\theta)} \\
&= \sum_{n_1, \dots, n_k=0}^{\infty} \left[\theta \left(\sum_{i=1}^k n_i\right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!}\right) L_{\sum_{i=1}^k n_i}^0(x) e^{-\lambda(1-\theta)} \right] s_1^{n_1} \cdots s_k^{n_k}.
\end{aligned}$$

Furthermore, from (A.16),

$$\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) = \sum_{n_1, \dots, n_k=0}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k}.$$

The result for the proof follows from this. □

6.4.3 Conditional distributions and expected values

Theorem 6.4.4. *The pgf of N_2^w, \dots, N_k^w conditioned on N_1^w for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$ is*

$$\begin{aligned}
\psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) &= \left(\frac{1 - \sum_{i=2}^k \theta_i}{1 - \sum_{i=2}^k \theta_i s_i}\right)^{n_1+1} \\
&\quad \frac{\sum_{m=0}^{\infty} \binom{m+n_1}{m} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i s_i\right)^m}}{\sum_{m=0}^{\infty} \binom{m+n_1}{m} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i\right)^m}}, \\
& n_1 \in \{0, 1, \dots\},
\end{aligned}$$

where $\theta = 1 - \sum_{i=1}^k \theta_i$.

Proof. Using (A.1) the pgf in (6.2) can be written in the following form

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \psi_1(s_1, \dots, s_k)^{m+1} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m+1}}{m!} \frac{1}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m+1}}. \end{aligned}$$

It then follows using (A.37) for $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ that

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}^{(n_1, \dots, n_k)}(s_1, \dots, s_k) \\ &= \frac{\partial^{n_1 + \dots + n_k} \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_1^{n_1} \dots \partial s_k^{n_k}} \\ &= e^{-\lambda} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m+1}}{m!} \frac{(m+1)(m+2) \dots (m+n_1)(m+n_1+1) \dots (m + \sum_{i=1}^k n_i)}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m + \sum_{i=1}^k n_i + 1}} \\ &= e^{-\lambda} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m+1}}{m!} \frac{\left(m + \sum_{i=1}^k n_i\right)!}{m! \left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m + \sum_{i=1}^k n_i + 1}}. \end{aligned}$$

Therefore, from (A.36), the pgf of N_2^w, \dots, N_k^w conditioned on N_1^w is

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) \\ &= \frac{\psi_{N_1^w, \dots, N_k^w}^{(n_1, 0, \dots, 0)}(0, s_2, \dots, s_k)}{\psi_{N_1^w, \dots, N_k^w}^{(n_1, 0, \dots, 0)}(0, 1, \dots, 1)} \\ &= \frac{e^{-\lambda} \theta_1^{n_1} \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m+1}}{m!} \frac{(m+n_1)!}{m! \left(1 - \sum_{i=2}^k \theta_i s_i\right)^{m+n_1+1}}}{e^{-\lambda} \theta_1^{n_1} \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m+1}}{m!} \frac{(m+n_1)!}{m! \left(1 - \sum_{i=2}^k \theta_i\right)^{m+n_1+1}}} \\ &= \left(\frac{1 - \sum_{i=2}^k \theta_i}{1 - \sum_{i=2}^k \theta_i s_i} \right)^{n_1+1} \frac{\sum_{m=0}^{\infty} \binom{m+n_1}{m} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i s_i\right)^m}}{\sum_{m=0}^{\infty} \binom{m+n_1}{m} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i\right)^m}}. \end{aligned}$$

□

6.4.4 Method of moments estimates

Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(1)}(\lambda, \theta_1, \dots, \theta_k)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$.

Let \bar{n}_i and s_{ij} be the sample moments given in (6.6), where $i, j \in \{1, \dots, k\}$, $i \neq j$. The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$ and $\text{Cov}(N_i^w, N_j^w)$, as presented in Table 6.1 in Section 6.2.2.

Therefore, if $\hat{\lambda}$ is the MoM estimator for λ , then

$$\frac{\hat{\theta}_i (1 + \hat{\lambda})}{(1 - \sum_{i=1}^k \hat{\theta}_i)} = \bar{n}_i$$

$$\sum_{i=1}^k \hat{\theta}_i = \frac{\bar{n}_i - \hat{\theta}_i (1 + \hat{\lambda})}{\bar{n}_i}$$

and an MoM estimator for θ_i , $i \in \{1, \dots, k\}$ is

$$\frac{(\sum_{i=1}^k \hat{\theta}_i) (1 + \hat{\lambda})}{(1 - \sum_{i=1}^k \hat{\theta}_i)} = \sum_{i=1}^k \bar{n}_i$$

$$\frac{\bar{n}_i - \hat{\theta}_i (1 + \hat{\lambda})}{\hat{\theta}_i} = \sum_{i=1}^k \bar{n}_i$$

$$\hat{\theta}_i = \frac{\bar{n}_i}{\hat{\lambda} + \sum_{i=1}^k \bar{n}_i + 1}.$$

In order to obtain $\hat{\lambda}$, the sample moment s_{ij} is equated with the population moment $\text{Cov}(N_i^w, N_j^w)$. It follows that

$$\frac{1}{2} \sum_{i=1, j=1, i \neq j}^k \frac{(2\lambda + 1) \theta_i \theta_j}{(1 - \sum_{i=1}^k \hat{\theta}_i) (1 - \sum_{i=1}^k \hat{\theta}_j)} = \frac{1}{2} \sum_{i=1, j=1, i \neq j}^k s_{ij}$$

$$\frac{(2\lambda + 1)}{(\hat{\lambda} + 1)^2} \sum_{i=1, j=1, i \neq j}^k \bar{n}_i \bar{n}_j = \sum_{i=1, j=1, i \neq j}^k s_{ij}$$

$$(\hat{\lambda} + 1)^2 \sum_{i=1, j=1, i \neq j}^k s_{ij} = (\hat{\lambda}^2 + 2\lambda + 1 - \hat{\lambda}^2) \sum_{i=1, j=1, i \neq j}^k \bar{n}_i \bar{n}_j$$

$$\frac{(\hat{\lambda} + 1)^2}{\hat{\lambda}^2} = \sum_{i=1, j=1, i \neq j}^k \frac{\bar{n}_i \bar{n}_j}{(\bar{n}_i \bar{n}_j - s_{ij})}$$

$$\hat{\lambda} = \frac{1}{\pm \sqrt{\sum_{i=1, j=1, i \neq j}^k \frac{\bar{n}_i \bar{n}_j}{(\bar{n}_i \bar{n}_j - s_{ij})}} - 1}$$

$$\hat{\lambda} = \frac{1}{\sqrt{\sum_{i=1, j=1, i \neq j}^k \frac{\bar{n}_i \bar{n}_j}{(\bar{n}_i \bar{n}_j - s_{ij})}} - 1}.$$

6.4.5 Maximum likelihood estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 6.4.5 for the $WMPA_{II}^{(1)}$ distribution. The Newton-Raphson algorithm uses the equation in (6.9) for $t \geq 1$ and the MoM estimates in Subsection 6.4.4) are used as initial values of the parameters $\lambda^{(0)}, \theta_1^{(0)}, \dots, \theta_k^{(0)}$ for the $WMPA_{II}^{(1)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (6.10) need to be calculated with respect to each of the parameters: λ and $\theta_1, \dots, \theta_k$.

The results of the calculation of the first and second derivatives of the *pmf* for the $WMPA_{II}^{(1)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ with respect to each of the parameters: $\lambda, \theta_1, \dots, \theta_k$ are written in a shorter form in order to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta, i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} + f^w(n_1, \dots, n_k) \\ &= \theta^2 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i}^1(x) e^{-\lambda(1-\theta)}, \\ &\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}. \end{aligned} \tag{S.6.0.6}$$

$$\begin{aligned} g' &= \frac{\partial g}{\partial \lambda} \\ &= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\ &= \theta^3 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^2(x) e^{-\lambda(1-\theta)} - (1-\theta)g, \\ &\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}. \end{aligned} \tag{S.6.0.7}$$

It then follows that the first and second derivatives of the *pmf* in (6.10) are

First derivative with respect to λ

$$\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} = g - f^w(n_1, \dots, n_k),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}.$$
(S.6.0.8)

First derivative with respect to θ_j

$$\frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} = -\frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, j \in \{1, \dots, k\}.$$
(S.6.0.9)

Second derivative with respect to λ

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} = g' - g + f^w(n_1, \dots, n_k),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}.$$

Second derivative with respect to θ_j and λ

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} = -\frac{\lambda}{\theta} g' + \left(\frac{n_j}{\theta_j} - \frac{2}{\theta} \right) g - \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, j \in \{1, \dots, k\}.$$
(S.6.0.10)

Second derivative with respect to θ_j

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} = \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{(\lambda+2)}{\theta} \right) g$$

$$+ \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, j \in \{1, \dots, k\}.$$

Second derivative with respect to θ_j and θ_ℓ

$$\begin{aligned} \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{(\lambda+2)}{\theta} \right) g \\ &\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}, \\ \{n_1, \dots, n_k\} &\in \{0, 1, \dots\}, j, \ell \in \{1, \dots, k\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the *pmf* in (6.10) is given here.

First derivative with respect to λ

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$. Substituting from (6.10) and (S.6.0.6) and using (A.13), it follows that

$$\begin{aligned} &\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\ &= \theta^2 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^1(x) e^{-\lambda(1-\theta)} - (1-\theta) f^w(n_1, \dots, n_k) \\ &= \theta^2 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) \left[L_{\sum_{i=1}^k n_i}^1(x) - L_{\sum_{i=1}^k n_i}^0(x) \right] e^{-\lambda(1-\theta)} \\ &\quad - (1-\theta) f^w(n_1, \dots, n_k) \\ &= \theta^2 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i}^1(x) e^{-\lambda(1-\theta)} - f^w(n_1, \dots, n_k) \\ &= g - f^w(n_1, \dots, n_k). \end{aligned}$$

First derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and let $j = \{1, \dots, k\}$. Substituting from (6.10) and (S.6.0.6) and using (A.13), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \\
&= \theta \left(\sum_{i=1}^k n_i \right)! \left(\frac{n_j}{\theta_j} \prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i}^0(x) e^{-\lambda(1-\theta)} \\
&\quad - \lambda \theta \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^1(x) e^{-\lambda(1-\theta)} - \left(\lambda + \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) \\
&= -\lambda \theta \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) \left[L_{\sum_{i=1}^k n_i}^1(x) - L_{\sum_{i=1}^k n_i}^0(x) \right] e^{-\lambda(1-\theta)} \\
&\quad + \left(\frac{n_j}{\theta_j} - \lambda - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) \\
&= -\lambda \theta \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i}^1(x) e^{-\lambda(1-\theta)} + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) \\
&= -\frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to λ

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$. Substituting from (S.6.0.7) and (S.6.0.8), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} \\
&= \theta^3 \left(\sum_{i=1}^k n_i \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - 1}^2(x) e^{-\lambda(1-\theta)} - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - (1-\theta) g \\
&= g' - \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\
&= g' - g + f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to θ_j and λ

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.8), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} \\
&= \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \frac{1}{\theta} g - \frac{\lambda}{\theta} g' \\
&= -\frac{\lambda}{\theta} g' + \left(\frac{n_j}{\theta_j} - \frac{2}{\theta} \right) g - \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and let $j \in \{1, \dots, k\}$. Substituting from (S.6.0.9) and (S.6.0.10), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} \\
&= \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k) \\
&\quad - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_j} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \right] - \frac{\lambda}{\theta^2} g \\
&= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{(\lambda+2)}{\theta} \right) g \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to θ_j and θ_ℓ

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.6.0.9) and (S.6.0.10), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} \\
&= \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} - \frac{f^w(n_1, \dots, n_k)}{\theta^2} \\
&\quad - \frac{\lambda}{\theta} \left(\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_\ell} + \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} \right) - \frac{\lambda}{\theta^2} g \\
&= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{(\lambda+2)}{\theta} \right) g \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}.
\end{aligned}$$

6.5 Weighted Type II multivariate Pólya-Aeppli distribution Case II

6.5.1 Joint probability mass function

Theorem 6.5.1. The pmf of $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$ is

$$f^w(0, \dots, 0) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{(e^{\lambda\theta} - 1)}{\theta}$$

$$f^w(n_a, \dots, n_b, \dots, n_c) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_b}{\theta} [(\lambda\theta - 1)e^{\lambda\theta} + 1],$$

$$\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} = \{0, \dots, 1, \dots, 0\},$$

$$a \leq b \leq c, a \neq c$$

$$f^w(n_a, \dots, n_b, n_c, \dots, n_d) = \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \left(\sum_{i=a}^b n_i - (b - a + 1) \right)! \left(\prod_{i=a}^b \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=a}^b n_i - (b-a+1)}^2(x)$$

$$\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{1, \dots, 1, 0, \dots, 0\},$$

$$a < b \leq c \leq d$$

$$f^w(n_1, \dots, n_k) = \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \left(\sum_{i=1}^k n_i - k \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - k}^2(x),$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\},$$

$$\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} \neq \{1, \dots, 1, 0, \dots, 0\},$$

$$a < b \leq c \leq d,$$

where $\theta = 1 - \sum_{i=1}^k \theta_i$, $x = -\lambda\theta$ and $L_n^\alpha(x)$ is the Laguerre polynomial in (A.3).

Proof. For the first part of the proof $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. It follows from (A.17) and (6.3) that

$$f^w(0, \dots, 0) = \psi_{N_1^w, \dots, N_k^w}(0, \dots, 0) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{(e^{\lambda\theta} - 1)}{\theta}.$$

The second part of the proof is derived for the case where one of the $\{n_1, \dots, n_k\}$ is equal to one and the rest are zero. Therefore, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} = \{0, \dots, 1, \dots, 0\}$ and $a \leq b \leq c, a \neq c$. The pgf in (6.3) can be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ & \equiv \psi_{N_a^w, \dots, N_b^w, \dots, N_c^w}(s_a, \dots, s_b, \dots, s_c) \\ & = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{(1 - \sum_{i=a}^c \theta_i s_i)}{\theta} \left[\sum_{m=0}^{\infty} \left(\frac{\lambda\theta}{1 - \sum_{i=a}^c \theta_i s_i} \right)^m \frac{1}{m!} - 1 \right] \\ & = \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=1}^{\infty} \frac{\lambda^m \theta^{m-1}}{m! (1 - \sum_{i=a}^c \theta_i s_i)^{m-1}}. \end{aligned}$$

It then follows from (A.18) that

$$\begin{aligned} & \psi_{N_a^w, \dots, N_b^w, \dots, N_c^w}^{(0, \dots, 1, \dots, 0)}(s_a, \dots, s_b, \dots, s_c) \\ &= \frac{\partial \psi_{N_a^w, \dots, N_b^w, \dots, N_c^w}(s_a, \dots, s_b, \dots, s_c)}{\partial s_b} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=1}^{\infty} \frac{\theta_b (m-1) \lambda^m \theta^{m-1}}{m! (1 - \sum_{i=a}^c \theta_i s_i)^m} \end{aligned}$$

and

$$\begin{aligned} & f^w(n_a, \dots, n_b, \dots, n_c) \\ &= \psi_{N_a^w, \dots, N_b^w, \dots, N_c^w}^{(0, \dots, 1, \dots, 0)}(s_a, \dots, s_b, \dots, s_c) \Big|_{s_a=0, \dots, s_b=0, \dots, s_c=0} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=1}^{\infty} \frac{\theta_b (m-1) \lambda^m \theta^{m-1}}{m!} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_b}{\theta} \left[\sum_{m=1}^{\infty} \frac{\lambda^m \theta^m}{(m-1)!} - \sum_{m=1}^{\infty} \frac{\lambda^m \theta^m}{m!} \right] \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_b}{\theta} \left[\sum_{m=0}^{\infty} \frac{\lambda^{m+1} \theta^{m+1}}{m!} - \sum_{m=0}^{\infty} \frac{\lambda^m \theta^m}{m!} + 1 \right] \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_b}{\theta} \left[(\lambda\theta - 1) \sum_{m=0}^{\infty} \frac{\lambda^m \theta^m}{m!} + 1 \right] \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_b}{\theta} [(\lambda\theta - 1) e^{\lambda\theta} + 1]. \end{aligned}$$

The third part of the proof is derived for the case where more than one of the $\{n_1, \dots, n_k\}$ are equal to one and the rest are zero, or where all of the $\{n_1, \dots, n_k\}$ are equal to one. Therefore $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{1, \dots, 1, 0, \dots, 0\}$ and $a < b \leq c \leq d$. Let $x = -\lambda\theta$ and $z_i = \theta_i s_i$, then the pgf in (6.3) can then be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ & \equiv \psi_{N_a^w, \dots, N_b^w, N_c^w, \dots, N_d^w}(s_a, \dots, s_b, s_c, \dots, s_d) \\ &= \sum_{n_a, \dots, n_b=0}^{\infty} \sum_{n_c, \dots, n_d=0}^{\infty} f^w(n_a, \dots, n_b, n_c, \dots, n_d) s_a^{n_a} \dots s_b^{n_b} s_c^{n_c} \dots s_d^{n_d} \\ &= \frac{(1 - \sum_{i=a}^b \theta_i s_i - \sum_{i=c}^d \theta_i s_i)}{\theta (1 - e^{-\lambda})} \left[e^{-\lambda \left(1 - \frac{\theta}{(1 - \sum_{i=a}^b \theta_i s_i - \sum_{i=c}^d \theta_i s_i)}\right)} - e^{-\lambda} \right] \\ &= \frac{(1 - \sum_{i=a}^b \theta_i s_i - \sum_{i=c}^d \theta_i s_i)}{\theta (1 - e^{-\lambda})} \left[e^{-\lambda \left(1 - \theta + \theta - \frac{\theta}{(1 - \sum_{i=a}^b \theta_i s_i - \sum_{i=c}^d \theta_i s_i)}\right)} - e^{-\lambda} \right] \\ &= \frac{(1 - \sum_{i=a}^b z_i - \sum_{i=c}^d z_i)}{\theta (1 - e^{-\lambda})} \left[e^{-\lambda(1-\theta)} e^{\left(\frac{x(\sum_{i=a}^b z_i + \sum_{i=c}^d z_i)}{(\sum_{i=a}^b z_i + \sum_{i=c}^d z_i - 1)}\right)} - e^{-\lambda} \right]. \end{aligned}$$

Finding the derivative of $\psi_{N_a^w, \dots, N_b^w, N_c^w, \dots, N_d^w}(s_a, \dots, s_b, s_c, \dots, s_d)$ with respect to z_a, \dots, z_b and using results (A.1) and (A.2), it can be seen that

$$\begin{aligned} & \frac{\partial^{b-a+1} \psi_{N_a^w, \dots, N_b^w, N_c^w, \dots, N_d^w}(s_a, \dots, s_b, s_c, \dots, s_d)}{\partial z_a \cdots \partial z_b} \\ &= \left(\frac{\lambda^2 \theta}{(1-e^{-\lambda}) \left(1 - \sum_{i=a}^b z_i - \sum_{i=c}^d z_i\right)^3} \right) e^{-\lambda(1-\theta)} e^{\left(\frac{x(\sum_{i=a}^b z_i + \sum_{i=c}^d z_i)}{(\sum_{i=a}^b z_i + \sum_{i=c}^d z_i - 1)}\right)} \\ &= \sum_{n_a, \dots, n_b=0}^{\infty} \sum_{n_c, \dots, n_d=0}^{\infty} \left[\frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \frac{(\sum_{i=a}^b n_i + \sum_{i=c}^d n_i)!}{(\prod_{i=a}^b n_i!) (\prod_{i=c}^d n_i!)} L_{\sum_{i=a}^b n_i + \sum_{i=c}^d n_i}^2(x) \right] \\ & \quad \times z_a^{n_a} \cdots z_b^{n_b} \cdots z_c^{n_c} \cdots z_d^{n_d} \\ &= \sum_{n_a, \dots, n_b=1}^{\infty} \sum_{n_c, \dots, n_d=0}^{\infty} \left[\frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \frac{(\sum_{i=a}^b n_i + \sum_{i=c}^d n_i - (b-a+1))!}{(\prod_{i=a}^b (n_i-1)!) (\prod_{i=c}^d n_i!)} \right] \\ & \quad \times L_{\sum_{i=a}^b n_i + \sum_{i=c}^d n_i - (b-a+1)}^2(x) \left] z_a^{n_a-1} \cdots z_b^{n_b-1} \cdots z_c^{n_c} \cdots z_d^{n_d} \end{aligned}$$

and from (A.16)

$$\begin{aligned} & \frac{\partial^{b-a+1} \psi_{N_a^w, \dots, N_b^w, N_c^w, \dots, N_d^w}(s_a, \dots, s_b, s_c, \dots, s_d)}{\partial z_a \cdots \partial z_b} \\ &= \sum_{n_a, \dots, n_b=1}^{\infty} \sum_{n_c, \dots, n_d=0}^{\infty} \left(\prod_{i=a}^b n_i \right) \frac{f^w(n_a, \dots, n_b, n_c, \dots, n_d)}{(\prod_{i=a}^b \theta_i^{n_i}) (\prod_{i=c}^d \theta_i^{n_i})} z_a^{n_a-1} \cdots z_b^{n_b-1} \cdots z_c^{n_c} \cdots z_d^{n_d}. \end{aligned}$$

From these results, it follows that for $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} = \{1, \dots, 1, 0, \dots, 0\}$ and $a < b \leq c \leq d$ that

$$\begin{aligned} & f^w(n_a, \dots, n_b, n_c, \dots, n_d) \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \left(\sum_{i=a}^b n_i + \sum_{i=c}^d n_i - (b-a+1) \right)! \left(\prod_{i=a}^b \frac{\theta_i^{n_i}}{n_i!} \right) \left(\prod_{i=c}^d \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=a}^b n_i + \sum_{i=c}^d n_i - (b-a+1)}^2(x) \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \left(\sum_{i=a}^b n_i - (b-a+1) \right)! \left(\prod_{i=a}^b \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=a}^b n_i - (b-a+1)}^2(x) \end{aligned}$$

In the case where $\{n_1, \dots, n_k\} = \{1, \dots, 1\}$, then $k = b = c = d$ and $a = 1$, such that

$$f^w(n_1, \dots, n_k) = \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \left(\sum_{i=1}^k n_i - k \right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!} \right) L_{\sum_{i=1}^k n_i - k}^2(x).$$

The fourth and final part of the proof is derived for the case where $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, and $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} \neq \{1, \dots, 1, 0, \dots, 0\}$ and

$a < b \leq c \leq d$. Let $x = -\lambda\theta$ and $z_i = \theta_i s_i$, then the pgf in (6.3) can then be written as

$$\begin{aligned} & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} f^w(n_1, \dots, n_k) s_1^{n_1} \cdots s_k^{n_k} \\ &= \frac{(1 - \sum_{i=1}^k \theta_i s_i)}{\theta(1 - e^{-\lambda})} \left[e^{-\lambda \left(1 - \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)}\right)} - e^{-\lambda} \right] \\ &= \frac{(1 - \sum_{i=1}^k \theta_i s_i)}{\theta(1 - e^{-\lambda})} \left[e^{-\lambda \left(1 - \theta + \theta - \frac{\theta}{(1 - \sum_{i=1}^k \theta_i s_i)}\right)} - e^{-\lambda} \right] \\ &= \frac{(1 - \sum_{i=1}^k z_i)}{\theta(1 - e^{-\lambda})} \left[e^{-\lambda(1-\theta)} e^{\left(\frac{x(\sum_{i=1}^k z_i)}{(\sum_{i=1}^k z_i - 1)}\right)} - e^{-\lambda} \right]. \end{aligned}$$

Finding the k^{th} derivative of $\psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)$ with respect to z_1 and using results (A.1) and (A.2), it can be seen that

$$\begin{aligned} & \frac{\partial^k \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial z_1^k} \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda}) \left(1 - \sum_{i=1}^k z_i\right)^3} e^{\left(\frac{x(\sum_{i=1}^k z_i)}{\sum_{i=1}^k z_i - 1}\right)} \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \sum_{n_1=0}^{\infty} L_{n_1}^2(x) \left(\sum_{i=1}^k z_i\right)^{n_1} \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \sum_{n_1=n_2}^{\infty} \sum_{n_2=n_3}^{\infty} \cdots \sum_{n_k=0}^{\infty} \left[\prod_{i=1}^{k-1} \binom{n_i}{n_{i+1}} \right] L_{n_1}^2(x) z_1^{n_1 - n_2} \cdots z_{k-1}^{n_{k-1} - n_k} z_k^{n_k} \\ &= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \sum_{n_1, \dots, n_k=0}^{\infty} \left[\prod_{i=1}^{k-1} \binom{\sum_{m=i}^k n_m}{\sum_{m=i+1}^k n_m} \right] L_{\sum_{i=1}^k n_i}^2(x) z_1^{n_1} \cdots z_k^{n_k} \\ &= \sum_{n_1, \dots, n_k=0}^{\infty} \left[\frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \frac{(\sum_{i=1}^k n_i)!}{(\prod_{i=1}^k n_i!)} L_{\sum_{i=1}^k n_i}^2(x) \right] z_1^{n_1} \cdots z_k^{n_k}. \end{aligned}$$

If $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} \neq \{1, \dots, 1, 0, \dots, 0\}$ and $a < b \leq c \leq d$ then

$$\begin{aligned} & \frac{\partial^k \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial z_1^k} \\ &= \sum_{n_1=k}^{\infty} \sum_{n_2, \dots, n_k=0}^{\infty} \left[\frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1 - e^{-\lambda})} \frac{(\sum_{i=1}^k n_i - k)!}{(n_1 - k)! (\prod_{i=2}^k n_i!)} L_{\sum_{i=1}^k n_i - k}^2(x) \right] z_1^{n_1 - k} z_2^{n_2} \cdots z_k^{n_k} \end{aligned}$$

and from (A.16)

$$\frac{\partial^k \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial z_1^k} = \sum_{n_1=k}^{\infty} \sum_{n_2, \dots, n_k=0}^{\infty} \frac{n_1!}{(n_1-k)!} \frac{f^w(n_1, \dots, n_k)}{\left(\prod_{i=1}^k \theta_i^{n_i}\right)} z_1^{n_1-k} z_2^{n_2} \dots z_k^{n_k}.$$

Similar results can be derived by differentiating with respect to z_i for $i \in \{2, \dots, k\}$. From these results it follows that for $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, and $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c, \dots, n_d\} \neq \{1, \dots, 1, 0, \dots, 0\}$ and $a < b \leq c \leq d$ that

$$f^w(n_1, \dots, n_k) = \frac{\lambda^2 \theta e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \left(\sum_{i=1}^k n_i - k\right)! \left(\prod_{i=1}^k \frac{\theta_i^{n_i}}{n_i!}\right) L_{\sum_{i=1}^k n_i - k}^2(x).$$

□

6.5.3 Conditional distributions and expected values

Theorem 6.5.4. *The pgf of N_2^w, \dots, N_k^w conditioned on N_1^w for $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$ is*

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1^w=0)}(s_2, \dots, s_k) \\ &= \left(\frac{1 - \sum_{i=2}^k \theta_i s_i}{1 - \sum_{i=2}^k \theta_i}\right) \frac{\left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}} - 1\right)}{\left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}} - 1\right)} \\ & \psi_{N_2^w, \dots, N_k^w | (N_1^w=1)}(s_2, \dots, s_k) \\ &= \frac{\frac{(1 - \theta_1 - \sum_{i=2}^k \theta_i s_i)}{\theta} \left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}} - 1\right) + \frac{\lambda \theta_1}{(1 - \sum_{i=2}^k \theta_i s_i)} e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}}}{\frac{(1 - \sum_{i=1}^k \theta_i)}{\theta} \left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}} - 1\right) + \frac{\lambda \theta_1}{(1 - \sum_{i=2}^k \theta_i)} e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}}} \\ & \psi_{N_2^w, \dots, N_k^w | (N_1^w=n_1)}(s_2, \dots, s_k) \\ &= \left(\frac{1 - \sum_{i=2}^k \theta_i}{1 - \sum_{i=2}^k \theta_i s_i}\right)^{n_1-1} \frac{\sum_{m=2}^{\infty} \binom{m+n_1-2}{m-2} \frac{\lambda^m \theta^m}{m!} \frac{1}{(1 - \sum_{i=2}^k \theta_i s_i)^m}}{\sum_{m=2}^{\infty} \binom{m+n_1-2}{m-2} \frac{\lambda^m \theta^m}{m!} \frac{1}{(1 - \sum_{i=2}^k \theta_i)^m}}, \end{aligned}$$

$$n_1 \in \{2, 3, \dots\},$$

where $\theta = 1 - \sum_{i=1}^k \theta_i$.

Proof. For the first part of the proof $n_1 = 0$. It follows from (A.36) that the conditional *pgf* of N_2^w, \dots, N_k^w given N_1^w is

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1^w=0)}(s_2, \dots, s_k) \\ &= \frac{\psi_{N_1^w, \dots, N_k^w}^{(0, \dots, 0)}(0, s_2, \dots, s_k)}{\psi_{N_1^w, \dots, N_k^w}^{(0, \dots, 0)}(0, 1, \dots, 1)} \\ &= \left(\frac{1 - \sum_{i=2}^k \theta_i s_i}{1 - \sum_{i=2}^k \theta_i} \right) \frac{\left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}} - 1 \right)}{\left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}} - 1 \right)}. \end{aligned}$$

For the second part of the proof, $n_1 = 1$ and by calculating the first derivative of the *pgf* in (6.3) with respect to s_1

$$\begin{aligned} \frac{\partial \psi_{N_1^w}(s_1, \dots, s_k)}{\partial s_1} &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{(1 - \theta_1 - \sum_{i=2}^k \theta_i s_i)}{\theta} \left(e^{\frac{\lambda \theta}{(1 - \sum_{i=1}^k \theta_i s_i)}} - 1 \right) \\ &\quad + \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\lambda \theta_1}{(1 - \sum_{i=1}^k \theta_i s_i)} e^{\frac{\lambda \theta}{(1 - \sum_{i=1}^k \theta_i s_i)}} \end{aligned}$$

it follows from (A.36) that the conditional *pgf* of N_2^w, \dots, N_k^w given N_1^w is

$$\begin{aligned} & \psi_{N_2^w, \dots, N_k^w | (N_1^w=1)}(s_2, \dots, s_k) \\ &= \frac{\psi_{N_1^w, \dots, N_k^w}^{(1, 0, \dots, 0)}(0, s_2, \dots, s_k)}{\psi_{N_1^w, \dots, N_k^w}^{(1, 0, \dots, 0)}(0, 1, \dots, 1)} \\ &= \frac{\left(\frac{1 - \theta_1 - \sum_{i=2}^k \theta_i s_i}{\theta} \right) \left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}} - 1 \right) + \frac{\lambda \theta_1}{(1 - \sum_{i=2}^k \theta_i s_i)} e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i s_i)}}}{\frac{(1 - \sum_{i=1}^k \theta_i)}{\theta} \left(e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}} - 1 \right) + \frac{\lambda \theta_1}{(1 - \sum_{i=2}^k \theta_i)} e^{\frac{\lambda \theta}{(1 - \sum_{i=2}^k \theta_i)}}}. \end{aligned}$$

For the third and final part of the proof $n_1 \in \{2, 3, \dots\}$. The *pgf* in (6.3) can be written in the following form

$$\begin{aligned}
 & \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k) \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \psi_1(s_1, \dots, s_k)^{m-1} - \psi_1(s_1, \dots, s_k)^{-1} \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=0}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{1}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m-1}} - \frac{\left(1 - \sum_{i=1}^k \theta_i s_i\right)}{\theta} \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \sum_{m=1}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{1}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m-1}}.
 \end{aligned}$$

It then follows that for $\{n_1, \dots, n_k\} \in \{2, 3, \dots\}$

$$\begin{aligned}
 & \psi_{N_1^w, \dots, N_k^w}^{(n_1, \dots, n_k)}(s_1, \dots, s_k) \\
 &= \frac{\partial^{n_1 + \dots + n_k} \psi_{N_1^w, \dots, N_k^w}(s_1, \dots, s_k)}{\partial s_1^{n_1} \dots \partial s_k^{n_k}} \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \sum_{m=1}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{(m-1)(m) \dots (m+n_1-2) \dots \left(m + \sum_{i=1}^k n_i - 2\right)}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m + \sum_{i=1}^k n_i - 1}} \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \sum_{m=2}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{(m-1)(m) \dots (m+n_1-2) \dots \left(m + \sum_{i=1}^k n_i - 2\right)}{\left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m + \sum_{i=1}^k n_i - 1}} \\
 &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \sum_{m=2}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{\left(m + \sum_{i=1}^k n_i - 2\right)!}{(m-2)! \left(1 - \sum_{i=1}^k \theta_i s_i\right)^{m + \sum_{i=1}^k n_i - 1}}.
 \end{aligned}$$

Therefore, from (A.36) the conditional *pgf* of N_2^w, \dots, N_k^w given N_1^w is

$$\begin{aligned}
& \psi_{N_2^w, \dots, N_k^w | (N_1^w = n_1)}(s_2, \dots, s_k) \\
&= \frac{\psi_{N_1^w, \dots, N_k^w}^{(n_1, 0, \dots, 0)}(0, s_2, \dots, s_k)}{\psi_{N_1^w, \dots, N_k^w}^{(n_1, 0, \dots, 0)}(0, 1, \dots, 1)} \\
&= \frac{e^{-\lambda} \theta_1^{n_1} \sum_{m=2}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{(m+n_1-2)!}{(m-2)! \left(1 - \sum_{i=2}^k \theta_i s_i\right)^{m+n_1-1}}}{\frac{e^{-\lambda}}{(1-e^{-\lambda})} \theta_1^{n_1} \sum_{m=2}^{\infty} \frac{\lambda^m \theta^{m-1}}{m!} \frac{(m+n_1-2)!}{(m-2)! \left(1 - \sum_{i=2}^k \theta_i\right)^{m+n_1-1}}} \\
&= \left(\frac{1 - \sum_{i=2}^k \theta_i}{1 - \sum_{i=2}^k \theta_i s_i} \right)^{n_1-1} \frac{\sum_{m=2}^{\infty} \binom{m+n_1-2}{m-2} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i s_i\right)^m}}{\sum_{m=2}^{\infty} \binom{m+n_1-2}{m-2} \frac{\lambda^m \theta^m}{m!} \frac{1}{\left(1 - \sum_{i=2}^k \theta_i\right)^m}}.
\end{aligned}$$

□

6.5.4 Method of moments estimates

Let $(N_1^w, \dots, N_k^w) \sim \text{WMPA}_{II}^{(2)}(\lambda, \theta_1, \dots, \theta_k)$ and consider a random sample of size m from this distribution. The observed values are then $(n_{1\ell}, \dots, n_{k\ell})$ where $\ell \in \{1, \dots, m\}$.

Let \bar{n}_i and s_{ij} , where $i, j \in \{1, \dots, k\}$, be the sample moments given in (6.8). The MoM estimators can be derived using these sample moments along with the corresponding population moments for $E(N_i^w)$ and $\text{Cov}(N_i^w, N_j^w)$, as presented in Table 6.1 in Section 6.2.2.

Therefore, if $\hat{\lambda}$ is the MoM estimator for λ , then

$$\begin{aligned}
& \left(\frac{\hat{\lambda}}{(1-e^{-\hat{\lambda}})} - 1 \right) \frac{\hat{\theta}_i}{\left(1 - \sum_{i=1}^k \hat{\theta}_i\right)} = \bar{n}_i \\
& 1 - \left(\frac{\hat{\lambda}}{(1-e^{-\hat{\lambda}})} - 1 \right) \frac{\hat{\theta}_i}{\bar{n}_i} = \sum_{i=1}^k \hat{\theta}_i
\end{aligned}$$

and an MoM estimator for $\theta_i, i \in \{1, \dots, k\}$ is

$$\begin{aligned} \frac{\bar{n}_i (1 - \sum_{i=1}^k \hat{\theta}_i)}{\hat{\theta}_i} \frac{\sum_{i=1}^k \hat{\theta}_i}{(1 - \sum_{i=1}^k \hat{\theta}_i)} &= \sum_{i=1}^k \bar{n}_i \\ \frac{\bar{n}_i}{\hat{\theta}_i} - \left(\frac{\hat{\lambda}}{(1 - e^{-\hat{\lambda}})} - 1 \right) &= \sum_{i=1}^k \bar{n}_i \\ \hat{\theta}_i &= \frac{\bar{n}_i}{\sum_{i=1}^k \bar{n}_i + \frac{\hat{\lambda}}{(1 - e^{-\hat{\lambda}})} - 1}. \end{aligned}$$

As in Section 5.5.4, an explicit expression for the MoM estimator of λ is unavailable, therefore, the Newton-Raphson method is employed to estimate λ (Press et al. [3]). This involves iteratively refining the estimate by evaluating the function and its derivative at each step, updating λ until convergence is achieved.

Therefore, the Newton-Raphson algorithm can be employed to derive the MoM estimator for λ , such that

$$\hat{\lambda}^{(t+1)} = \hat{\lambda}^{(t)} - \frac{f(\hat{\lambda}^{(t)})}{f'(\hat{\lambda}^{(t)})}$$

for $t \geq 1$.

In order to obtain $\hat{\lambda}$, the sample moments $s_{ij} + \bar{n}_i \bar{n}_j$ is equated with the population moments $E(N_i^w N_j^w) = \text{Cov}(N_i^w, N_j^w) + E(N_i^w) E(N_j^w)$. It follows that

$$\begin{aligned} \frac{1}{2} \sum_{i=1, j=1, i \neq j}^k \frac{\hat{\lambda}^2 \hat{\theta}_i \hat{\theta}_j}{(1 - e^{-\hat{\lambda}}) \hat{\theta}^2} &= \frac{1}{2} \sum_{i=1, j=1, i \neq j}^k (\bar{s}_{ij} + \bar{n}_i \bar{n}_j) \\ \sum_{i=1, j=1, i \neq j}^k \frac{\hat{\lambda}^2 (1 - e^{-\hat{\lambda}}) \bar{n}_i \bar{n}_j}{(\hat{\lambda} - 1 + e^{-\hat{\lambda}})^2} &= \sum_{i=1, j=1, i \neq j}^k (\bar{s}_{ij} + \bar{n}_i \bar{n}_j). \end{aligned}$$

The expressions for the function $f(\hat{\lambda})$ and its derivative $f'(\hat{\lambda})$ are

$$\begin{aligned} f(\hat{\lambda}) &= \sum_{i=1, j=1, i \neq j}^k \frac{\hat{\lambda}^2 (1 - e^{-\hat{\lambda}}) \bar{n}_i \bar{n}_j}{(\hat{\lambda} - 1 + e^{-\hat{\lambda}})^2} - \bar{s}_{ij} - \bar{n}_i \bar{n}_j \\ f'(\hat{\lambda}) &= \sum_{i=1, j=1, i \neq j}^k \frac{2\hat{\lambda} (1 - e^{-\hat{\lambda}}) \bar{n}_i \bar{n}_j + \hat{\lambda}^2 e^{-\hat{\lambda}} \bar{n}_i \bar{n}_j - 2\hat{\lambda}^2 (1 - e^{-\hat{\lambda}})^2 \bar{n}_i \bar{n}_j}{(\hat{\lambda} - 1 + e^{-\hat{\lambda}})^2}. \end{aligned}$$

The MoM estimator from the MPA_{II} distribution in Section 6.3.4 is used as an initial value for $\lambda^{(0)}$. In cases where the MoM estimator in Section 5.3.4 is unavailable, a grid search method, as

suggested by Woodford and Phillips [4], can be used as an alternative to determine the starting value. However, this approach will not be discussed further in this thesis.

The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate value is reached or until a specified maximum number of iterations is reached.

6.5.5 Maximum likelihood estimates

The results derived in this section are used in the Newton-Raphson algorithm to obtain the MLEs in Subsection 6.5.5 for the $WMPA_{II}^{(2)}$ distribution. The Newton-Raphson algorithm uses the equation in (6.9) for $t \geq 1$ and the MoM estimates in Subsection 6.5.4 are used as initial values of the parameters $\lambda^{(0)}, \theta_1^{(0)}, \dots, \theta_k^{(0)}$ for the $WMPA_{II}^{(2)}$ distribution. The iterative process is repeated until a given tolerance level ϵ between the t^{th} and $(t+1)^{th}$ iterate values is reached or until a specified maximum number of iterations is reached.

In order to calculate the gradient vector ∇F and the Hessian matrix, such that the Newton-Raphson algorithm can be utilized, the first and second derivatives for each element of the *pmf* in (6.11) need to be calculated with respect to each of the parameters: λ and $\theta_1, \dots, \theta_k$.

The results of the calculation of the first and second derivatives of the *pmf* for the $WMPA_{II}^{(2)}$ distribution are given here. Following this, the calculations for how each result was obtained will be given.

The first and second derivatives of $f^w(0, \dots, 0)$, $f^w(0, \dots, 1)$, and $f^w(n_1, \dots, n_k)$ where $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 1\}$ with respect to each of the parameters $\lambda, \theta_1, \dots, \theta_k$ are written in a shorter form in order to mathematically and programmatically simplify the expressions.

Results of the derivatives

Let $x = -\lambda\theta, i, j, \ell \in \{1, \dots, k\}$ and let $L_{n_i}^\alpha(x_i)$ be the Laguerre polynomial in (A.3). Then let

$$\begin{aligned} g &= \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) f^w(n_1, \dots, n_k) \\ &= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)!}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k}^3(x), \end{aligned} \quad (S.6.0.11)$$

$$\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b,$$

$$\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.$$

$$\begin{aligned}
g' &= \frac{\partial g}{\partial \lambda} \\
&= \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} - \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\
&\quad + \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} \right) f^w(n_1, \dots, n_k) \\
&= \frac{\lambda^2 \theta^3 e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)! \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k - 1}^4(x)}{(1-e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} L_{\sum_{i=1}^k n_i - k - 1}^4(x) \\
&\quad + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} + \theta \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda}, \\
&\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned} \tag{S.6.0.12}$$

It then follows that the first and second derivatives of the pmf in (6.11) are;

First derivative with respect to λ

$$\begin{aligned}
\frac{\partial f^w(0, \dots, 0)}{\partial \lambda} &= \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f^w(0, \dots, 0) + \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\
\frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} &= \theta \theta_j f^w(0, \dots, 0) + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f^w(0, \dots, 0, n_c), \\
&\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \tag{S.6.0.13} \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} &= g + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, \dots, n_k), \\
&\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned}$$

First derivative with respect to θ_j

$$\begin{aligned}
\frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0, \dots, 0) - \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \\
\frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} &= \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) f^w(0, \dots, 0, n_c) - \lambda \theta_j f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \\
\frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} &= \left(\frac{1}{\theta} - \lambda \right) f^w(0, \dots, 0, n_c) - \lambda \theta_\ell f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j \neq c, \ell = c \\
\frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} &= -\frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned} \tag{S.6.0.14}$$

Second derivative with respect to λ

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda^2} &= \frac{(\theta - 1)}{(1 - e^{-\lambda})} f^w(0, \dots, 0) + \left(\theta - \frac{2}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} \\
\frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} f^w(0, \dots, 0, n_c) + \theta \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} \\
&\quad + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda}, \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} &= g' + \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\
&\quad - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned}$$

Second derivative with respect to θ_j and λ

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j \partial \lambda} &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - f^w(0, \dots, 0) \\
&\quad + \frac{1}{\theta} \left(\frac{\lambda}{(1 - e^{-\lambda})} - 1 \right) \left(\frac{e^{-\lambda}}{(1 - e^{-\lambda})} \right) \\
\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \lambda} &= \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} - f^w(0, \dots, 0, n_c) \\
&\quad - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - \theta_j f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \\
\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \lambda} &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} - f^w(0, \dots, 0, n_c) \\
&\quad - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - \theta_\ell f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j \neq c, \ell = c \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} &= -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda}, \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned} \tag{S.6.0.15}$$

Second derivative with respect to θ_j

$$\begin{aligned}
\frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j^2} &= \frac{1}{\theta^2} f^w(0, \dots, 0) + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \\
\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j^2} &= \left(\frac{1}{\theta^2} - \frac{1}{\theta_j^2} \right) f^w(0, \dots, 0, n_c) + \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} \\
&\quad - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} - \lambda f^w(0, \dots, 0), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \\
\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j^2} &= \frac{f^w(0, \dots, 0, n_c)}{\theta^2} + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j}, \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j \neq c, \ell = c \\
\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{\lambda}{\theta} \frac{1}{(1 - e^{-\lambda})} \right) g \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k), \\
&\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\
&\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}.
\end{aligned}$$

Second derivative with respect to θ_j and θ_ℓ

$$\begin{aligned} \frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j \partial \theta_\ell} &= \frac{f^w(0, \dots, 0)}{\theta^2} + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \\ \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \theta_\ell} &= \frac{f^w(0, \dots, 0, n_c)}{\theta^2} + \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_\ell} - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell}, \\ &\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j = c \\ \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \theta_\ell} &= \frac{f^w(0, \dots, 0, n_c)}{\theta^2} + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_\ell} \\ &\quad - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell} - \lambda f^w(0, \dots, 0), \\ &\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}, a \leq b, j \neq c, \ell = c \\ \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{\lambda}{\theta} \frac{1}{(1 - e^{-\lambda})} \right) g \\ &\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}, \\ &\quad \{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} \neq \{0, \dots, 0, 1\}, a \leq b, \\ &\quad \{n_1, \dots, n_k\} \in \{0, 1, \dots\}, \{n_1, \dots, n_k\} \neq \{0, \dots, 0\}. \end{aligned}$$

Calculation of the derivatives

The calculation of the first and second derivatives of the pmf in (6.11) is given here.

First derivative with respect to λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (6.11), it follows that

$$\begin{aligned} &\frac{\partial f^w(0, \dots, 0)}{\partial \lambda} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta e^{\lambda \theta}}{\theta} - \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \frac{(e^{\lambda \theta} - 1)}{\theta} - f^w(0, \dots, 0) \\ &= \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) f^w(0, \dots, 0) + \frac{e^{-\lambda}}{(1 - e^{-\lambda})}. \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. Substituting from (6.11), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} \\ &= \frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_j}{\theta} (\theta e^{\lambda\theta}) + \frac{e^{-\lambda}}{(1-e^{-\lambda})} \frac{\theta_j}{\theta} [\theta(\lambda\theta - 1)e^{\lambda\theta}] \\ & \quad - \frac{(e^{-\lambda})^2}{(1-e^{-\lambda})^2} \frac{\theta_j}{\theta} [(\lambda\theta - 1)e^{\lambda\theta} + 1] - f(0, \dots, 0, 1) \\ &= \lambda\theta^2 \theta_j f^w(0, \dots, 0) - \frac{1}{(1-e^{-\lambda})} f(0, \dots, 0, 1) + \lambda\theta \theta_j \frac{e^{-\lambda}}{(1-e^{-\lambda})} \\ &= \theta \theta_j f^w(0, \dots, 0) + \left(\theta - \frac{1}{(1-e^{-\lambda})} \right) f(0, \dots, 0, 1). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$. Substituting from (6.11) and (S.6.0.11) and using (A.14), it follows that

$$\begin{aligned} & \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\ &= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \frac{(\sum_{i=1}^k n_i - k)!}{(\prod_{i=1}^k n_i!)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k - 1}^3(x) \\ & \quad - \frac{\lambda^2 \theta e^{-\lambda(1-\theta)} e^{-\lambda}}{(1-e^{-\lambda})^2} \frac{(\sum_{i=1}^k n_i - k)!}{(\prod_{i=1}^k n_i!)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k}^2(x) \\ & \quad + \frac{2}{\lambda} f^w(n_1, \dots, n_k) - (1-\theta) f^w(n_1, \dots, n_k) \\ &= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \frac{(\sum_{i=1}^k n_i - k)!}{(\prod_{i=1}^k n_i!)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \left[L_{\sum_{i=1}^k n_i - k}^3(x) - L_{\sum_{i=1}^k n_i - k}^2(x) \right] \\ & \quad + \left(\frac{2}{\lambda} - (1-\theta) - \frac{e^{-\lambda}}{(1-e^{-\lambda})} \right) f^w(n_1, \dots, n_k) \\ &= \frac{\lambda^2 \theta^2 e^{-\lambda(1-\theta)}}{(1-e^{-\lambda})} \frac{(\sum_{i=1}^k n_i - k)!}{(\prod_{i=1}^k n_i!)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k}^3(x) \\ & \quad + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, \dots, n_k) \\ &= g + \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) f^w(n_1, \dots, n_k). \end{aligned}$$

First derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. Substituting from (6.11), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} \\ &= \frac{f^w(0, \dots, 0)}{\theta} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\lambda e^{\lambda\theta}}{\theta} \\ &= \left(\frac{1}{\theta} - \lambda\right) f^w(0, \dots, 0) - \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})}. \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. Substituting from (6.11), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} \\ &= -\frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_j}{\theta} (\lambda e^{\lambda\theta}) - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_j}{\theta} [\lambda(\lambda\theta - 1) e^{\lambda\theta}] \\ & \quad + \left(\frac{1}{\theta} + \frac{1}{\theta_j}\right) f(0, \dots, 0, 1) \\ &= \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda\right) f(0, \dots, 0, 1) - \lambda \theta_j f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j \neq c$, $\ell = c$. Substituting from (6.11), it follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \theta_j} \\ &= -\frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_\ell}{\theta} (\lambda e^{\lambda\theta}) - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} \frac{\theta_\ell}{\theta} [\lambda(\lambda\theta - 1) e^{\lambda\theta}] + \frac{f(0, \dots, 0, 1)}{\theta} \\ &= \left(\frac{1}{\theta} - \lambda\right) f(0, \dots, 0, 1) - \lambda \theta_\ell f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$ and let $j \in \{1, \dots, k\}$. Substituting from (6.11) and (S.6.0.11)

and using (A.14), it follows that

$$\begin{aligned}
& \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \\
&= \frac{\lambda^2 \theta e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)!}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \left(\frac{n_j}{\theta_j} \prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k}^2(x) \\
&\quad - \frac{\lambda^3 \theta e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)!}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k - 1}^3(x) \\
&\quad - \left(\lambda + \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) \\
&= - \frac{\lambda^3 \theta e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)!}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) \left[L_{\sum_{i=1}^k n_i - k}^3(x) - L_{\sum_{i=1}^k n_i - k}^2(x) \right] \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} - \lambda \right) f^w(n_1, \dots, n_k) \\
&= - \frac{\lambda^3 \theta e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)!}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k}^3(x) \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) \\
&= - \frac{\lambda}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k).
\end{aligned}$$

Second derivative with respect to λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$. Substituting from (S.6.0.13), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(0, \dots, 0)}{\partial \lambda^2} \\
&= \left(\frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(0, \dots, 0) + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} \\
&\quad - \frac{e^{-\lambda}}{(1 - e^{-\lambda})} - \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \\
&= \frac{(\theta - 1)}{(1 - e^{-\lambda})} f^w(0, \dots, 0) + \left(\theta - \frac{2}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda}.
\end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. It follows that

$$\begin{aligned} & \frac{\partial f^w(0, \dots, 0, n_c)}{\partial \lambda} \\ &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} f^w(0, \dots, 1) + \theta \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} \\ & \quad + \left(\theta - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \lambda}. \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$. Substituting from (S.6.0.12), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda^2} \\ &= \frac{\lambda^2 \theta^3 e^{-\lambda(1-\theta)} \left(\sum_{i=1}^k n_i - k \right)! \left(\prod_{i=1}^k \theta_i^{n_i} \right) L_{\sum_{i=1}^k n_i - k - 1}^4(x)}{(1 - e^{-\lambda}) \left(\prod_{i=1}^k n_i! \right)} \\ & \quad + \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} + \theta \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} \\ & \quad + \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, \dots, n_k) \\ &= g' + \left(\frac{2}{\lambda} - \frac{1}{(1 - e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda} - \left(\frac{2}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \right) f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to θ_j and λ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j \partial \lambda} \\ &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - f^w(0, \dots, 0) \\ & \quad - \frac{1}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} + \frac{\lambda}{\theta} \frac{e^{-\lambda}}{(1 - e^{-\lambda})} + \frac{\lambda}{\theta} \frac{(e^{-\lambda})^2}{(1 - e^{-\lambda})^2} \\ &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - f^w(0, \dots, 0) + \frac{1}{\theta} \left(\frac{\lambda}{(1 - e^{-\lambda})} - 1 \right) \left(\frac{e^{-\lambda}}{(1 - e^{-\lambda})} \right). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \lambda} \\ &= \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \lambda} - f(0, \dots, 0, 1) \\ & \quad - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - \theta_j f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j \neq c$, $\ell = c$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \lambda} \\ &= \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \lambda} - f(0, \dots, 0, 1) \\ & \quad - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \lambda} - \theta_\ell f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$ and let $j \in \{1, \dots, k\}$. It follows that

$$\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \lambda} = -\frac{\lambda}{\theta} g' - \frac{1}{\theta} g + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \lambda}.$$

Second derivative with respect to θ_j

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j \in \{1, \dots, k\}$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j^2} = \frac{1}{\theta^2} f^w(0, \dots, 0) + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1 - e^{-\lambda})}.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j^2} \\ &= \left(\frac{1}{\theta^2} - \frac{1}{\theta_j^2} \right) f(0, \dots, 0, 1) + \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \theta_j} \\ & \quad - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j} - \lambda f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j \neq c$, $\ell = c$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j^2} = \frac{1}{\theta^2} f(0, \dots, 0, 1) + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \theta_j} - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \theta_j}.$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$. Substituting from (S.6.0.14) and (S.6.0.15), it follows that

$$\begin{aligned} & \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j^2} \\ &= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_j} - \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} \right] \\ & \quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_j} - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k) \\ &= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{2n_j}{\theta_j} - \frac{\lambda}{\theta} \frac{1}{(1-e^{-\lambda})} \right) g \\ & \quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right)^2 f^w(n_1, \dots, n_k) - \left(\frac{n_j}{\theta_j^2} + \frac{1}{\theta^2} \right) f^w(n_1, \dots, n_k). \end{aligned}$$

Second derivative with respect to θ_j and θ_ℓ

Let $\{n_1, \dots, n_k\} = \{0, \dots, 0\}$ and let $j, \ell \in \{1, \dots, k\}$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0)}{\partial \theta_j \partial \theta_\ell} = \frac{f^w(0, \dots, 0)}{\theta^2} + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell} - \frac{\lambda}{\theta^2} \frac{e^{-\lambda}}{(1-e^{-\lambda})}.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j = c$. It follows that

$$\frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \theta_\ell} = \frac{f(0, \dots, 0, 1)}{\theta^2} + \left(\frac{1}{\theta} + \frac{1}{\theta_j} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \theta_\ell} - \lambda \theta_j \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell}.$$

Let $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, n_c\} = \{0, \dots, 0, 1\}$, $a \leq b$, $j \neq c$, $\ell = c$. It follows that

$$\begin{aligned} & \frac{\partial^2 f^w(0, \dots, 0, n_c)}{\partial \theta_j \partial \theta_\ell} \\ &= \frac{f(0, \dots, 0, 1)}{\theta^2} + \left(\frac{1}{\theta} - \lambda \right) \frac{\partial f(0, \dots, 0, 1)}{\partial \theta_\ell} \\ & \quad - \lambda \theta_\ell \frac{\partial f^w(0, \dots, 0)}{\partial \theta_\ell} - \lambda f^w(0, \dots, 0). \end{aligned}$$

Let $\{n_1, \dots, n_k\} \in \{0, 1, \dots\}$, $\{n_1, \dots, n_k\} \neq \{0, \dots, 0\}$, $\{n_1, \dots, n_k\} \equiv \{n_a, \dots, n_b, \dots, n_c\} \neq \{0, \dots, 1, \dots, 0\}$, $a \leq b \leq c$, $a \neq c$ and let $j, \ell \in \{1, \dots, k\}$. Substituting from (S.6.0.14) and

(S.6.0.15), it follows that

$$\begin{aligned}
& \frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \theta_j \partial \theta_\ell} \\
&= -\frac{\lambda}{\theta^2} g - \frac{\lambda}{\theta} \left[\frac{\partial^2 f^w(n_1, \dots, n_k)}{\partial \lambda \partial \theta_\ell} - \left(\frac{2}{\lambda} - \frac{1}{(1-e^{-\lambda})} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} \right] \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \frac{\partial f^w(n_1, \dots, n_k)}{\partial \theta_\ell} - \frac{f^w(n_1, \dots, n_k)}{\theta^2} \\
&= \frac{\lambda^2}{\theta^2} g' - \frac{\lambda}{\theta} \left(\frac{n_j}{\theta_j} + \frac{n_\ell}{\theta_\ell} - \frac{\lambda}{\theta} \frac{1}{(1-e^{-\lambda})} \right) g \\
&\quad + \left(\frac{n_j}{\theta_j} - \frac{1}{\theta} \right) \left(\frac{n_\ell}{\theta_\ell} - \frac{1}{\theta} \right) f^w(n_1, \dots, n_k) - \frac{f^w(n_1, \dots, n_k)}{\theta^2}.
\end{aligned}$$

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