Existence Theory for Linear Vibration Models of Elastic Bodies

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Summary

The vibration of elastic bodies or systems of elastic bodies are modeled by partial differential equations or systems of partial differential equations. In this dissertation we consider the wave equation (in one and three dimensions), different partial differential equations for the vibration of beams and other complex structures. Various forms of damping are included in the models.

It is well known that solutions of partial differential equations are often not possible to determine explicitly. To further complicate matters, solutions often do not exist if it is required that the partial differential equation be satisfied in every point of the relevant domain. (An example is provided in the introduction.) In such cases a solution may exist if the definition of a solution is "relaxed" and it turns out that satisfactory numerical results can be obtained. However, an alternative definition of a solution must be unambiguous and the solutions must be useful in applications. The weak variational form for vibration problems meet these requirements.

In this dissertation we consider only linear vibration problems. The aim is to provide a general framework for existence theory. This framework is based on an article by Van Rensburg and Van der Merwe in 2002 where an existence theory for a general linear vibration problem in variational form, is presented.

In this dissertation the pre-knowledge for the existence theory (Sobolev spaces and semigroup theory) is provided and a detailed version of the existence theory is provided. More auxiliary results are also included to clarify the exposition. Great care was taken in the formulation of results that link the existence results to semigroup theory.

It is shown through examples that all linear vibration problems have the same variational form. (A proof is not possible but the examples are convincing.) The general theory is then applied to the model problems in weak variational form. Six of the problems are standard (but not trivial), e.g. the three-dimensional wave equation with boundary damping. However, two of the problems are from recent articles and are complex mathematical models.

Contents

Chapter 1

Linear vibration problems in variational form

1.1 Introduction

Mathematical models for the vibration of elastic bodies consist of partial differential equations or systems of partial differential equations. It is well known that solutions of partial differential equations are often not possible to determine explicitly and numerical methods are then used to approximate the solution. Preferably one should know whether a solution exist in such a case. However, the existence of solutions for partial differential equations is not a clear cut issue as we show in the examples below.

We use the **wave equation** to illustrate as it is the simplest second order hyperbolic partial differential equation. Here it is possible to construct solutions with different orders of smoothness. An elegant way to determine precise conditions for existence of solutions is provided by d'Alembert's method, see [W].

Example 1 The undamped wave equation

$$\begin{aligned}
\partial_t^2\, u &= c^2 \partial_x^2\, u \quad \text{in} \quad (0,\ell), \\
u(0,t) &= \partial_x u(\ell,t) = 0, \\
u(x,0) &= u_0(x) \quad \text{for} \quad x \in (0,\ell), \\
\partial_t u(x,0) &= 0 \quad \text{for} \quad x \in (0,\ell)
\end{aligned}$$

Solution

It is easy to see that the function

$$u(x,t) = \frac{1}{2}g(x+ct) + \frac{1}{2}g(x-ct) \tag{1.1.1}$$

is a solution of the partial differential equation if g is defined on the real line and twice differentiable. It is also clear that the initial conditions are satisfied if g is an extension of u_0.

However, for u to satisfy the boundary conditions, u_0 must satisfy the boundary conditions and the extension g of u_0 cannot be arbitrary. For convenience consider the case $\ell = c = 1$. Let

$$g_1(x) = u_0(2-x) \ \text{ for } \ 1 < x \le 2.$$

Next let

$$g_2(x) = -g_1(-x) \ \text{ for } \ -2 \le x < 0.$$

For any integer n and any $x \in (-2,2)$ let $g(x+4n) = g_2(x)$. The result is that $g(x) = -g(-x)$ and $g(x) = g(2-x)$ for any x in $(-\infty,\infty)$. Consequently

$$\begin{aligned} u(0,t) &= \frac{1}{2}g(ct) + \frac{1}{2}g(-ct) = 0, \\ \partial_x u(1,t) &= \frac{1}{2}g'(1+ct) + \frac{1}{2}g'(1-ct) = 0. \end{aligned}$$

It is necessary to investigate the differentiability of g. If $u_0 \in C^1[0,1]$ and $u_0(0) = u_0'(1) = 0$, it is not difficult to see that the right (hand) and left (hand) derivatives $D^+g(n)$ and $D^-g(n)$ are equal for any integer n.
If $u_0 \in C^2[0,1]$, then g'' will exist on any open interval $\big(2n, 2(n+1)\big)$ with n an integer.

For a classical solution of the problem to exist, the conditions are that
$u_0 \in C^2[0,1]$, $u_0(0) = u_0'(1) = 0$ and $u_0''(0) = 0$. Note that the condition $u_0''(0) = 0$ is necessary for g'' to exist at the even integers. Consider for example $x = 0$. Since $g_2(x) = -u_0(-x)$, it follows that $g_2''(x) = -u_0''(-x)$ and hence

$$\lim_{x\to 0-} g_2''(x) = -\lim_{x\to 0+} u_0''(-x).$$

Example 2

Consider Example 1 where $u_0(x) = x(2-x)$. We see that $u_0''(x) = -2$ on $[0,2]$ and $g_2''(x) = 2$ on $[-2,0]$. It follows that $g''(0)$ does not exist. We have the same situation at each even integer and this means that $\partial_t^2\, u$ and $\partial_x^2\, u$ do not exist along the lines $x+t=2n$ and $x-t=2n$ for each integer n.

Does the fact that $\partial_t^2\, u$ and $\partial_x^2\, u$ do not exist along the lines $x+t=2n$ and $x-t=2n$ mean that (1.1.1) is not a valid solution of the problem in this case? There is a need to define what is meant by a solution. The conditions imposed on a solution will then determine whether the “solution” is acceptable.

Definition Classical solution

A function u is a classical solution of the problem in Example 1 if

$$u \in C^1\big([0,\ell]\times[0,\infty)\big),$$

$\partial_t^2 u$ and $\partial_x^2 u$ exist in $(0,\ell)\times(0,\infty)$, and u satisfies the partial differential equation, the boundary conditions and the initial conditions.

Remark The problem in Example 2 does not have a classical solution.

The next case is worse.

Example 3

Consider Example 1 with $u_0(x)=x^2$. The result is that $g_2'(1)$ does not exist and consequently the problem does not have a classical solution.

However, a definition that requires too much from a function to be a solution may be impractical.

Alternative definition of a solution

The definition above may be modified as follows: for each t a finite number of points may exist where the second order derivatives of the solution do not exist.

A motivation for the alternative definition is provided by the finite element method (FEM). The first step when using this method to approximate a solution of a partial differential equation, is to multiply the partial differential equation by a so called test function and integrate:

$$\int_0^\ell \big(\partial_t^2 u(x,t)\big)v(x)\ dx = \int_0^\ell \big(\partial_x^2 u(x,t)\big)v(x)\ dx.$$

Now, if $\partial_t^2 u(x,t)$ and $\partial_x^2 u(x,t)$ are continuous on $(0,\ell)$ except for a finite number of points, then the integrals exist and the equation above makes sense. The next step is to use integration by parts to derive a variational form of the problem. This is done in the next section where we reconsider Examples 2 and 3. The existence of so called weak solutions are treated in Section 5.2.

It should be clear that the existence of a solution is not a clear cut issue. Also, consider the fact that (in general) numerical methods are used to approximate solutions of partial differential equations. If we have convergence, it does not mean that the approximation converges to a classical solution of the partial differential equation (see Section 5.1).

There can be no doubt that existence theory is important. The aim of this dissertation is to present an existence theory for linear vibration problems: an extended version of the paper [VV] with a variety of applications.

1.2 The wave equation

1.2.1 Model problems

As mentioned in the introduction, the wave equation is the simplest example of a linear vibration problem for an elastic body and it can be found in most textbooks on partial differential equations, such as [EP]. The partial differential equation is a mathematical model for longitudinal vibrations of a bar or transverse vibrations of a string.

We consider the wave equation with three types of damping. First consider the cases where the damping term is in the partial differential equation.

$$\begin{aligned} \partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u, \\ \partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u, \end{aligned}$$

with $\alpha > 0$ and $\gamma > 0$. The constant α represent physical properties of the bar (or string) and γ is the damping constant. The damping in the first equation is known as viscous damping which one may expect in the vibrating string model due to air resistance. The damping in the second equation is known as Kelvin-Voigt damping due to internal friction in the material of a vibrating bar (see Subsection 1.2.4).

The vibration of the bar or string is also determined by the action at the endpoints. Suppose the length is ℓ and consider the boundary conditions

$$u(0,t) = \partial_x u(\ell,t) = 0.$$

This means that the endpoint at $x = 0$ is fixed and the other endpoint is free.

Now consider the undamped wave equation but with damping at the boundary $x = \ell$:

$$\partial_x u(\ell,t) = -\gamma \partial_t u(\ell,t).$$

We proceed to formulate three model problems that are typical examples of the general linear vibration problem. It is necessary to prescribe the initial position and initial velocity of every point.

Problem 1 Viscous damping

Given a positive constants α and γ, find u such that

$$\begin{aligned}
\partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u \quad \text{in } (0,\ell) \ \text{ for } t \in \mathbb{R}, \\
u(0,t) &= \partial_x u(\ell,t) = 0 \ \text{ for } t \in \mathbb{R}, \\
u(x,0) &= u_0(x) \ \text{ for } \ x \in (0,\ell), \\
\partial_t u(x,\, 0) &= u_1(x) \ \text{for } \ x \in (0,\ell).
\end{aligned}$$

Problem 2 Kelvin-Voigt damping

Given a positive constants α and γ, find u such that

$$\begin{aligned}
\partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u \ \text{ in } (0,\ell) \ \text{ for } t > 0, \\
u(0,t) &= \partial_x u(\ell,t) = 0 \ \text{ for } t > 0, \\
u(x,0) &= u_0(x) \ \text{ for } \ x \in (0,\ell), \\
\partial_t u(x,\, 0) &= u_1(x) \ \text{ for } \ x \in (0,\ell).
\end{aligned}$$

Problem 3 Boundary damping

Given a positive constants α and γ, find u such that

$$\begin{aligned}
\partial_t^2 u &= \alpha \partial_x^2 u \ \text{ in } (0,\ell) \ \text{ for } t > 0, \\
u(0,t) &= 0 \ \text{ for } t > 0, \\
\alpha \partial_x u(\ell,t) &= -\gamma \partial_t u(\ell,t) \ \text{ for } t > 0, \\
u(x,0) &= u_0(x) \ \text{ for } \ x \in (0,\ell), \\
\partial_t u(x,\, 0) &= u_1(x) \ \text{ for } \ x \in (0,\ell).
\end{aligned}$$

Remark The general existence theory is presented in Chapter 4 and the terms weak and strong damping are defined there. The damping in Problems 1 and 2 are examples of weak and strong damping respectively. The boundary damping in Problem 3 is an example of damping that can not be classified as either weak or strong. The existence theory is applied to the three problems in Section 5.2.

1.2.2 Variational form of the model problems

The notation $C^k[0,\ell]$ is well known. A function in $C^k[0,\ell]$ has derivatives up to order k and they are continuous on the closed interval $[0,\ell]$. For the derivation of the variational form of a model problem this notation is not always suitable and the following alternative notation is introduced.

Notation

$C^k_*[0,\ell] = \{v \in C^{k-1}[0,\ell] \mid v^{(k)} \text{ is integrable.}\}$

Consider the term $\alpha\partial_x^2 u$ in the wave equation. We multiply the term by a function $\phi \in C^1_*[0,\ell]$ and integrate. Integration by parts yields

$$\alpha \int_0^\ell \partial_x^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' + [\alpha\partial_x u(\cdot,t)\phi]_0^\ell, \tag{1.2.1}$$

provided that $u(\cdot,t) \in C^2_*(0,\ell)$. Note that the continuity of $\partial_x u(\cdot,t)$ is essential.

We now introduce the so called test functions – the same for Problems 1 to 3.

Test functions

$$T[0,\ell] = \{\phi \in C^1_*[0,\ell] \mid \phi(0) = 0\}.$$

From (1.2.1) we have that for each $\phi \in T[0,\ell]$

$$\alpha \int_0^\ell \partial_x^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' + \alpha\partial_x u(\ell,t)\phi(\ell) \tag{1.2.2}$$

Now consider Problem 1. Multiply the equation

$$\partial_t^2 u = \alpha\partial_x^2 u - \gamma\partial_t u$$

by a function $\phi \in T[0,\ell]$ and integrate. Using (1.2.2) we have

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t u(\cdot,t)\phi + \alpha \partial_x u(\ell,t)\phi(\ell) \qquad (1.2.3)$$

for each $\phi \in T[0,\ell]$. Given the boundary condition $\partial_x u(\ell,t) = 0$ it follows that

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t u(\cdot,t)\phi. \qquad (1.2.4)$$

It is now possible to give the variational form of Problem 1.

Problem 1V Viscous damping

Find u such that for each t, $u(\cdot,t) \in T[0,\ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi \;=\; -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t u(\cdot,t)\phi$$

for each $\phi \in T[0,\ell]$ with $u(\cdot,0) = u_0$ and $\partial_t u(\cdot,0) = u_1$.

Remark If u is a solution of Problem 1 (in the sense of the alternative definition), then u satisfies Equation (1.2.4). Recall that the continuity of $\partial_x u(\cdot,t)$ is essential.

Next we consider Problem 2. Repeating the process above, we find that

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi \;=\; -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' + \gamma \int_0^\ell \partial_t \partial_x^2 u(\cdot,t)\phi + \alpha \partial_x u(\ell,t)\phi(\ell)$$

if $\phi \in T[0,\ell]$. Integration by parts on the damping term yields

$$\begin{aligned}\int_0^\ell \partial_t^2 u(\cdot,t)\phi \;&=\; -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot,t)\phi' \\ &\quad+\; (\alpha \partial_x u(\ell,t) + \gamma \partial_t \partial_x u(\ell,t))\phi(\ell) \qquad (1.2.5)\end{aligned}$$

if $\phi \in T[0,\ell]$. It is given that $\partial_x u(\ell,t) = 0$ and it follows that $\partial_t \partial_x u(\ell,t) = 0$. We have the variational form of Problem 2.

Problem 2V Kelvin-Voigt damping

Find u such that for each $t > 0$, $u(\cdot,t) \in T[0,\ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi \;=\; -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot,t)\phi'$$

for each $\phi \in T[0,\ell]$ with $u(\cdot,0) = u_0$ and $\partial_t u(\cdot,0) = u_1$.

Lastly, we consider Problem 3. From (1.2.2) it follows that

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' + \alpha\partial_x u(\ell,t)\phi(\ell) \tag{1.2.6}$$

if $\phi \in T[0,\ell]$. Using the boundary condition at $x = \ell$ we arrive at the variational form of Problem 3.

Problem 3V Boundary damping

Find u such that for each $t > 0$, $u(\cdot,t) \in T[0,\ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma\partial_t u(\ell,t)\phi(\ell)$$

for each $\phi \in T[0,\ell]$ with $u(\cdot,0) = u_0$ and $\partial_t u(\cdot,0) = u_1$.

Undamped wave equation

If $\gamma = 0$ in Problem 1V, then it is the variational form of Example 1 in the previous section. We consider the function u given by Equation (1.1.1). If $g' \in C(-\infty,\infty)$ and g'' exists except for isolated points, then u is a solution of Problem 1V. For Example 2 - a special case of Example 1 - a solution of the variational problem exists. However, for Example 3 (another special case of Example 1) the derivation of the variational form above is not valid since $\partial_x u(\cdot,t)$ is not continuous ($g' \notin C(-\infty,\infty)$).

1.2.3 Equivalence of problems

It is necessary to consider the equivalence of the model problems and their respective variational forms. The following result has been derived.

Theorem 1

A solution of Problem 1 or 2 or 3 is a solution of Problem 1V or 2V or 3V.

To prove that the solutions of the variational problems are solutions of the original problems, we introduce a subset of the test functions and an auxiliary result.

Notation

$$T_0[0,\ell] = \{\phi \in C_*^1[0,\ell] \mid \phi(0) = \phi(\ell) = 0\}.$$

Proposition 1

Suppose f is integrable on $(0,\ell)$. If $\int_0^\ell f\phi = 0$ for each $\phi \in T_0[0,\ell]$, then $\int_0^\ell f\phi = 0$ for each ϕ that is integrable on $(0,\ell)$.

If, in addition $f \in C[0,\ell]$, then $f = 0$.

Proof See Section 2.1.

Theorem 2

If u is a solution of Problem 1V and $u(\cdot,t) \in C^2_*[0,\ell]$ for each t, then u is a solution of Problem 1.

Proof From (1.2.2) we have that

$$\alpha \int_0^\ell \partial_x^2 u(\cdot,t)\phi = -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' + \alpha \partial_x u(\ell,t)\phi(\ell)$$

for each $\phi \in T[0,\ell]$. If u is a solution of Problem 1V, it follows that

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi = \alpha \int_0^\ell \partial_x^2 u(\cdot,t)\phi - \gamma \int_0^\ell \partial_t u(\cdot,t)\phi - \alpha \partial_x u(\ell,t)\phi(\ell)$$

for each $\phi \in T[0,\ell]$. Therefore

$$\int_0^\ell (\partial_t^2 u(\cdot,t) - \alpha \partial_x^2 u(\cdot,t) + \gamma \partial_t u(\cdot,t))\phi = 0$$

for each $\phi \in T_0[0,\ell]$. But from Proposition 1 we know that this is also the case for for each integrable function ϕ. We conclude that

$$\int_0^\ell (\partial_t^2 u(\cdot,t) - \alpha \partial_x^2 u(\cdot,t) + \gamma \partial_t u(\cdot,t))^2 = 0$$

and hence

$$\partial_t^2 u - \alpha \partial_x^2 u + \gamma \partial_t u = 0 \quad \text{a.e.}$$

Thus we have proved that u is a solution of the partial differential equation. But we know more: Equation (1.2.3) is true for each $\phi \in T[0,\ell]$. This fact together with

$$\int_0^\ell \partial_t^2 u(\cdot,t)\phi \;=\; -\alpha \int_0^\ell \partial_x u(\cdot,t)\phi' - \gamma \int_0^\ell \partial_t u(\cdot,t)\phi$$

for each $\phi \in T[0,\ell]$, imply that $\partial_x u(\ell,t)\phi(\ell) = 0$ for each $\phi \in T[0,\ell]$. Since $\phi(\ell)$ is arbitrary, it follows that $\partial_x u(\ell,t) = 0$ which completes the proof. $\square$

We consider Problem 3V next due to complications associated with Problem 2V.

Theorem 3

If u is a solution of Problem 3V and $u(\cdot,t) \in C^2_*[0,\ell]$ for $t > 0$, then u is a solution of Problem 3.

Proof The first part of the proof is the same as for Theorem 2 and we find that u is a solution of the partial differential equation. Consequently u satisfies (1.2.6). Using the fact that u is also a solution of Problem 3V, we have for each $\phi \in T[0,\ell]$ that

$$\partial_x u(\ell,t)\phi(\ell) = -\gamma\partial_t u(\ell,t))\phi(\ell)$$

and since $\phi(\ell)$ is arbitrary, it follows that $\partial_x u(\ell,t) = -\gamma\partial_t u(\ell,t)$. $\square$

Theorem 4

If u is a solution of Problem 2V, $u(\cdot,t) \in C^2_*[0,\ell]$ for $t \geq 0$ and $u_0'(\ell) = 0$, then u is a solution of Problem 2.

Proof The first part of the proof is the same as for Theorem 2 and we find that u is a solution of the partial differential equation. As a consequence u satisfies (1.2.5). Since u is also a solution of Problem 2V,

$$(\alpha\partial_x u(\ell,t) + \gamma\partial_t\partial_x u(\ell,t))\phi(\ell) = 0$$

for each $\phi \in T[0,\ell]$. Since $\phi(\ell)$ is arbitrary, it follows that

$$\alpha\partial_x u(\ell,t) + \gamma\partial_t\partial_x u(\ell,t) = 0.$$

From this we may only conclude that $\partial_x u(\ell,t) = K\exp(-(\alpha/\gamma)t)$. But $\partial_x u$ is continuous and $\partial_x u(\ell,0) = u_0'(\ell) = 0$, which implies that $K = 0$. Therefore $\partial_x u(\ell,t) = 0$ and the proof is complete. $\square$

Remark What if $u_0'(\ell) \neq 0$? Then Problem 2V is equivalent to another problem. This problem is the same as Problem 2 except for the fact that one boundary condition changes to

$$\alpha\partial_x u(\ell,t) + \gamma\partial_t\partial_x u(\ell,t) = 0.$$

This seems like a paradox since Problem 2V is well posed, see the next subsection. However, we cannot expect $\partial_x u_0(\ell, t)$ to be continuous if $u_0'(\ell) \neq 0$. It is also instructive to consider the longitudinal vibrations in an elastic bar as we do in the next subsection.

1.2.4 Longitudinal vibrations in a bar

The partial differential equation in Problem 2 models longitudinal vibrations in an elastic bar. Consider a prismatic bar with density ρ, length ℓ and cross-sectional area A. The equation of motion

$$\rho A \partial_t^2 u = \partial_x F$$

follows from Newton's second law (see e.g. [EP]). Hooke's law in its simplest form provides the constitutive equation

$$F = EA\partial_x u.$$

Combining the two equations yields the wave equation $\partial_t^2 u = c^2 \partial_x^2 u$, where $c^2 = E/\rho$. In Problem 2 we have the fixed-free case with boundary conditions $u(0) = 0$ and $\partial_x u(\ell, t) = 0$. The second boundary condition follows from the fact that $F(\ell, t) = 0$.

However, if we consider the model with Kelvin-Voigt damping, then

$$F = EA\partial_x u + \gamma \partial_t \partial_x u.$$

Combining this with the equation of motion yields the partial differential equation in Problem 2. If the bar is free at $x = \ell$, then the boundary condition is $EA\partial_x u(\ell, t) + \gamma \partial_t \partial_x u(\ell, t) = 0$ since $F(\ell, t) = 0$.

1.2.5 General variational form

Using suitable notation, all three variational problems may be written in the same form. First, consider Problem 1V.

Notation

$$\begin{aligned} c(u,v) &= \int_0^\ell uv, \\ a(u,v) &= \gamma\int_0^\ell uv, \\ b(u,v) &= \alpha\int_0^\ell u'v'. \end{aligned}$$

The variational equation in Problem 1V may now be written in the form

$$c(\partial_t^2 u(\cdot,t),\phi) + a(\partial_t u(\cdot,t),\phi) + b(u(\cdot,t),\phi) = 0 \tag{1.2.7}$$

where ϕ is a test function.

The variational equation in Problems 2V and 3V may also be written in the form above if we define the form a for damping appropriately:

$$\begin{aligned} a(u,v) &= \gamma\int_0^\ell u'v' \quad \text{for Problem 2V}, \\ a(u,v) &= \gamma u(\ell)v(\ell) \quad \text{for Problem 3V}. \end{aligned}$$

Note that the initial conditions must be specified for a well posed problem.

Energy

The mechanical energy (kinetic energy plus elastic potential energy) of the system is (not in standard units)

$$E(t) = \frac{1}{2}\,c(\partial_t u(\cdot,t),\partial_t u(\cdot,t)) + \frac{1}{2}\,b(u(\cdot,t),u(\cdot,t)).$$

Using the symmetry of b and c and assuming that $\partial_t^2 u$ is continuous on $[0,\ell]\times[0,T]$ (see remark below), we have

$$\begin{aligned} \frac{d}{dt}c(\partial_t u,\partial_t u) &= 2c(\partial_t^2 u,\partial_t u) \quad \text{and} \\ \frac{d}{dt}b(u,u) &= 2b(\partial_t u,u). \end{aligned}$$

Note that we simplified the notation. It follows that

$$E'(t) = c(\partial_t^2 u,\partial_t u) + b(u,\partial_t u) = -a(\partial_t u,\partial_t u).$$

It is obvious that $a(u,u)$ is nonnegative for all three models, hence $E'(t) \leq 0$. This result is to be expected from Physics.

Remark The assumption of continuity is sufficient to justify "differentiation under the integral sign" (see [ApA, p219]). In Section 4.6 we prove that $E'(t) \leq 0$ without assuming the continuity of $\partial_t^2 u$.

Uniqueness

Consider for example Problem 1V and suppose u and $\tilde{u}$ are solutions. Let $w = u - \tilde{u}$, then w is also a solution of Problem 1V but with initial values $w(x,0) = \partial_t w(x,0) = 0$. The result is that $E(0) = 0$. Since $E(t) \leq E(0)$, we have $E(t) = 0$ for $t > 0$. But $b(w,w)$ and $c(\partial_t w, \partial_t w)$ are both nonnegative and hence $b(w,w) = 0$. This implies that $\partial_x w = 0$ with the result that $w(x,t)$ is constant for fixed t. The boundary condition $w(0,t) = 0$ now yields the fact that w is identically zero hence $u - \tilde{u} = 0$ a.e. This proves uniqueness. The same is true for of Problems 2V and 3V.

It is important to note that certain values of u and its derivatives are **not** uniquely determined, e.g. $\partial_x u(\ell, t)$.

1.3 The Euler-Bernoulli beam

1.3.1 The model problem

The simplest mathematical model for a beam is the so called Euler-Bernoulli theory which can be found in most books on partial differential equations, for instance [EP]. (It is a special case of the Timoshenko beam, see Chapter 7.) Consider the transverse vibration of a beam with density ρ, length ℓ and cross-sectional area A. It is convenient to denote the moment and shear force by M and V respectively.

The Euler-Bernoulli model for a beam consists of two equations of motion for the beam deflection w given by

$$\rho A \partial_t^2 w = \partial_x V - k\partial_t w, \tag{1.3.1}$$

$$0 = \partial_x M + V. \tag{1.3.2}$$

The term $-k\partial_t w$ models a force density due to the resistance of a medium where k denotes the damping constant or parameter. This is referred to as viscous damping and is an example of weak damping (see Subsection 1.2.1).

The following constitutive equation for the moment is used to derive the partial differential equation:

$$M = EI\partial_x^2 w, \tag{1.3.3}$$

where I is the area moment of inertia and the constant E is referred to as Young's modulus. The reader is referred to [I, p 337-8] and [N, p 392-5].

We consider a cantilever beam: the left endpoint of the beam is clamped but the other endpoint is free. This means that $V(\ell,t) = M(\ell,t) = 0$ and the boundary conditions are

$$w(0,t) = \partial_x w(0,t) = \partial_x^2 w(\ell,t) = \partial_x^3 w(\ell,t) = 0.$$

Problem 4

Given w_0 and w_1, find w such that

$$\begin{aligned}
\rho A\partial_t^2\, w &= -EI\partial_x^4\, w - k\partial_t\, w \ \text{ in } \ (0,\ell) \ \text{ for } \ t \in \mathbb{R}, \\
w(0,t) &= \partial_x w(0,t) = 0 \text{ for } \ t \in \mathbb{R}, \\
\partial_x^2 w(\ell,t) &= \partial_x^3 w(\ell,t) = 0 \text{ for } \ t \in \mathbb{R}, \\
w(x,0) &= w_0 \text{ for } \ x \in (0,\ell), \\
\partial_t w(x,\,0) &= w_1 \text{ for } \ x \in (0,\ell).
\end{aligned}$$

1.3.2 Variational form

Multiply Equation (1.3.1) with a function $v \in C_*^2[0,\ell]$ and do integration by parts twice using also (1.3.2). We find that

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v \\
= \ & -\int_0^\ell M(\cdot,t)v'' - k\int_0^\ell (\partial_t w)v + \big[V(x,t)v(x)\big]_0^\ell + \big[M(x,t)v'(x)\big]_0^\ell.
\end{aligned}$$

Since $V(\ell,t) = M(\ell,t) = 0$ it follows that

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v \\
= \ & -\int_0^\ell M(\cdot,t)v'' - k\int_0^\ell (\partial_t w)v + V(0,t)v(0) + M(0,t)v'(0).
\end{aligned}$$

Test functions

$$T[0,\ell] = \{v \in C^2_*[0,\ell] \mid v(0) = v'(0) = 0\}.$$

For each $v \in T[0,\ell]$

$$\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v = -\int_0^\ell M(\cdot,t)v'' - k\int_0^\ell (\partial_t w)v.$$

Using Equation (1.3.3) it follows that

$$\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v = -\int_0^\ell EI\partial_x^2 w(\cdot,t)v'' - k\int_0^\ell (\partial_t w)v$$

for each $v \in T[0,\ell]$.

Notation

$$\begin{aligned} b(u,v) &= \int_0^\ell EIu''v'' \\ a(u,v) &= \int_0^\ell kuv \\ \text{and} \quad c(u,v) &= \int_0^\ell \rho Auv. \end{aligned}$$

Problem 4V

Find w such that for each t, $w(\cdot,t) \in T[0,\ell]$ and

$$c(\partial_t^2 w(\cdot,t),v) + a(\partial_t w(\cdot,t),v) + b(w(\cdot,t),v) = 0 \quad \text{for each} \quad v \in T[0,\ell]$$

while $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$.

Theorem

A solution of Problem 4 is a solution of Problem 4V. If w is a solution of Problem 4V and $w(\cdot,t) \in C^4_*[0,\ell]$ for each t, then w is a solution of Problem 4.

Proof The first part has been proved and the second part of the proof is similar to the proof of Theorem 1.2.2 (i.e. Theorem 2 in Section 1.2).

1.4 The Raleigh beam

1.4.1 The model problem

The Raleigh beam model is the same as the Euler Bernoulli model but a rotary inertia term is included here. It is also a special case of the Timoshenko beam, see Chapter 7.

The Rayleigh model for a beam consists of two equations of motion for the beam deflection w and the angle $\partial_x w$ due to the rotation of a cross section:

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x V + Q, && (1.4.1) \\ \rho I \partial_t^2 \partial_x w &= V + \partial_x M. && (1.4.2) \end{aligned}$$

This is an example with no damping, but a distributed load Q is applied to the beam. We consider a cantilever beam where the left endpoint of the beam is clamped. The boundary conditions are $V(\ell,t) = M(\ell,t) = 0$ which means that

$$\begin{aligned} & w(0,t) = \partial_x w(0,t) = \partial_x^2 w(\ell,t) = 0 \\ \text{and } \; & \partial_x^3 w(\ell,t) = E^{-1}\rho \; \partial_t^2 \partial_x w(\ell,t). \end{aligned}$$

Problem 5

Given w_0 and w_1, find w such that

$$\begin{aligned} \rho A \partial_t^2 \, w &= -\rho I \partial_t^2 \partial_x^2 w - EI\partial_x^4 \, w + Q \;\text{ in }\; (0,\ell) \;\text{ for }\; t \in \mathbb{R}, \\ w(0,t) &= \partial_x w(0,t) = 0 \text{ for } \; t \in \mathbb{R}, \\ \partial_x^2 w(\ell,t) &= 0 \text{ for } \; t \in \mathbb{R}, \\ \partial_x^3 w(\ell,t) &= E^{-1}\rho \; \partial_t^2 \partial_x w(\ell,t) \text{ for } \; t \in \mathbb{R}, \\ w(x,0) &= w_0 \text{ for } \; x \in (0,\ell), \\ \partial_t w(x,\, 0) &= w_1 \text{ for } \; x \in (0,\ell). \end{aligned}$$

1.4.2 Variational form

Multiply (1.4.1) by $v \in C_*^2[0,\ell]$ and integrate:

$$\int_0^\ell \rho A \partial_t^2 w(\cdot,t) v = \int_0^\ell \partial_x V(\cdot,t) v + \int_0^\ell Q(\cdot,t) v.$$

Integration by parts yields that

$$\int_0^\ell \partial_x V(\cdot,t)v = -\int_0^\ell V(\cdot,t)v' + \big[V(x,t)v(x)\big]_0^\ell$$

hence

$$\begin{aligned}&\int_0^\ell \rho A\partial_t^2 w(\cdot,t)v\\ =\ &-\int_0^\ell V(\cdot,t)v' + \big[V(x,t)v(x)\big]_0^\ell + \int_0^\ell Q(\cdot,t)v. \end{aligned}\tag{1.4.3}$$

Now multiply (1.4.2) by v' and integrate

$$\int_0^\ell \rho I\partial_x\partial_t^2 w(\cdot,t)v' = \int_0^\ell V(\cdot,t)v' + \int_0^\ell \partial_x M(\cdot,t)v'.$$

Integration by parts yields

$$\int_0^\ell \partial_x M(\cdot,t)v' = -\int_0^\ell M(\cdot,t)v'' + \big[M(x,t)v'(x)\big]_0^\ell$$

and we find that

$$\begin{aligned}&\int_0^\ell \rho I\partial_x\partial_t^2 w(\cdot,t)v'\\ =\ &\int_0^\ell V(\cdot,t)v' - \int_0^\ell M(\cdot,t)v'' + \big[M(x,t)v'(x)\big]_0^\ell. \end{aligned}\tag{1.4.4}$$

By the addition of (1.4.3) and (1.4.4) it follows that

$$\begin{aligned}&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v + \int_0^\ell \rho I(\partial_x\partial_t^2 w(\cdot,t))v'\\ =\ &-\int_0^\ell M(\cdot,t)v'' + [M(x,t)v'(x)]_0^\ell + [V(x,t)v(x)]_0^\ell + \int_0^\ell Q(\cdot,t)v.\end{aligned}$$

Some boundary terms vanish due to the fact that $V(\ell,t) = M(\ell,t) = 0$. The others are eliminated by a proper choice of test function.

Test functions

$$T[0,\ell] = \{v \in C_*^2[0,\ell] \mid v(0) = v'(0) = 0\}.$$

Notation

$$\begin{aligned} (f,g) &= \int_0^\ell fg, \\ b(u,v) &= \int_0^\ell EIu''v'' \\ \text{and} \quad c(u,v) &= \int_0^\ell \rho Auv + \int_0^\ell \rho Iu'v'. \end{aligned}$$

Problem 5V

Find w such that for each t, $w(\cdot,t) \in T[0,\ell]$ and

$$c(\partial_t^2 w(\cdot,t),v) + b(w(\cdot,t),v) = \big(Q(t),v\big) \quad \text{for each} \quad v \in T[0,\ell]$$

while $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$.

Theorem

A solution of Problem 5 is a solution of Problem 5V. If w is a solution of Problem 5V and $w(\cdot,t) \in C_*^4[0,\ell]$ for each t, then w is a solution of Problem 5.

Proof The first part of the theorem is proved above. The proof of the second part is similar to the proof of Theorem 1.2.2.

1.5 The three-dimensional wave equation

1.5.1 Model problem

We consider the wave equation in a three-dimensional cube denoted by Ω. It models the propagation of sound waves in a room [W, Section 34]. On one "wall" of the boundary of Ω we have boundary damping and zero boundary conditions on the rest.

Problem 6

Suppose

$$\Omega = \{x \in \mathbb{R}^3 \mid 0 < x_i < d\}.$$

Let Σ be the part of the boundary $\partial\Omega$ where $x_1 = d$ and let $\partial\Omega - \Sigma$ be the rest of the boundary. Given the positive constants α and γ, find u such that

$$\begin{aligned} \partial_t^2 u &= \alpha \nabla^2 u \quad \text{in} \;\; \Omega, \\ u &= 0 \qquad \text{on} \;\; \partial\Omega - \Sigma, \\ \gamma \partial_t u &= -\alpha\, \partial_1 u \quad \text{on} \;\; \Sigma, \end{aligned}$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Notation $\partial_i u = \dfrac{\partial u}{\partial x_i}$

1.5.2 Variational form

Notation Suppose the function $u \in C(\bar{\Omega})$ and $p \in \partial\Omega$. A directional derivative for u can be considered in the direction of a vector e if $p + \lambda\, e \in \Omega$ for λ in some interval $[0, \tau]$. If the directional derivatives of u exists in all such possible directions and are continuous, then we say $\nabla u \in C(\bar{\Omega})$ or $u \in C^1(\bar{\Omega})$.

> $u \in C^1_*(\bar{\Omega})$ if u is in $C(\bar{\Omega})$ and its partial derivatives are integrable on Ω.
>
> $u \in C^2_*(\bar{\Omega})$ if u is in $C^1(\bar{\Omega})$ and its second order partial derivatives are integrable on Ω.

Gauss's theorem

If $F_i \in C^1_*(\Omega)$ for $i = 1, 2, 3$, then

$$\iiint_\Omega \nabla \cdot F \, dV = \iint_{\partial\Omega} F \cdot n \, dA.$$

where $\nabla \cdot F$ is the divergence of the vector F and n is the unit outward normal vector.

Proof See [ApA, p340].

Gauss's theorem is often referred to as the divergence theorem.

Proposition 1 Green's formula

If $u \in C^2_*(\bar{\Omega})$ and $v \in C^1_*(\bar{\Omega})$, then

$$\iiint_\Omega -(\nabla^2 u)v \; dV = \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV - \iint_{\partial\Omega} v(\nabla u) \cdot n \; dA.$$

Proof Since $\nabla \cdot (v\nabla u) = v\nabla^2 u + \nabla u \cdot \nabla v$ we have that

$$\iiint_\Omega -(\nabla^2 u)v \; dV = -\iiint_\Omega \nabla \cdot (v\nabla u) \; dV + \iiint_\Omega \nabla u \cdot \nabla v \; dV.$$

From Gauss's theorem it follows that

$$\iiint_\Omega -(\nabla^2 u)v \; dV = -\iint_{\partial\Omega} v\nabla u \cdot n \; dA + \iiint_\Omega \nabla u \cdot \nabla v \; dV.$$

The result follows from the fact that

$$\sum_{i=1}^{3} \partial_i u \partial_i v = \nabla u \cdot \nabla v.$$

$\square$

If u is a solution of Problem 6, then for each $v \in T(\Omega)$

$$\begin{aligned} \iiint_\Omega (\partial_t^2 u)v \; dV &= \alpha \iiint_\Omega (\nabla^2 u)v \; dV \\ &= \alpha \iint_\Sigma v(\nabla u \cdot n) \; dA - \alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV. \qquad (1.5.1) \end{aligned}$$

Note that $\nabla u \cdot n = \partial_1 u$ and $\alpha \; \partial_1 u = -\gamma \; \partial_t u$ on Σ.

Test functions

$$T(\Omega) = \{ v \in C^1_*(\Omega) \mid v = 0 \text{ on } \partial\Omega - \Sigma \}.$$

From Equation (1.5.1) we have

$$\iiint_\Omega (\partial_t^2 u)v \; dV = -\gamma \iint_\Sigma (\partial_t u)v \; dA - \alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV.$$

Notation

$$\begin{aligned} b(u,v) &= \alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV, \\ c(u,v) &= \iiint_\Omega uv \; dV, \\ a(u,v) &= \gamma \iint_\Sigma u \; v \; dA. \end{aligned}$$

Problem 6V

Find u such that for each $t > 0$, $u(\cdot,t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot,t), v) + a(\partial_t u(\cdot,t), v) + b(u(\cdot,t), v) = 0 \quad \text{for each} \quad v \in T(\Omega).$$

1.5.3 Equivalence of problems

Definition $T_0(\Omega) = \{\phi \in C_*^1(\Omega) \mid \phi = 0 \text{ on } \partial\Omega\}$

Proposition 2

Suppose f is integrable on Ω.

If $\iiint_\Omega f\phi \; dV = 0$ for each $\phi \in T_0(\Omega)$, then $\iiint_\Omega f\phi \; dV = 0$ for each ϕ that is integrable on Ω.

If, in addition $f \in C(\bar{\Omega})$, then $f = 0$.

Proof See Section 2.1.

Theorem

A solution of Problem 6 is a solution of Problem 6V. If u is a solution of Problem 6V and $u(\cdot,t) \in C_*^2(\bar{\Omega})$ for each t, then u is a solution of Problem 6.

Proof The first part of the theorem is proved in the previous subsection.

If u is a solution of Problem 6V, we have that for each $v \in T_0(\Omega)$

$$\iiint_\Omega (\partial_t^2 u) v \; dV = -\alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV.$$

The notation $u(\cdot,t)$ is suppressed when convenient. From Green's formula it follows that

$$\begin{aligned} & -\alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV \\ = \; & \alpha \iiint_\Omega (\nabla^2 u) v \; dV - \alpha \iint_{\partial\Omega} v(\nabla u)\cdot n \; dA \end{aligned}$$

For each $v \in T_0(\Omega)$ it follows that

$$\iiint_\Omega (\partial_t^2 u - \alpha \; \nabla^2 u) v \; dV = 0.$$

But from Proposition 2 we know that this is also the case for for each integrable function v. We conclude that

$$\iiint_\Omega (\partial_t^2 u - \alpha \; \nabla^2 u)^2 dV = 0$$

and hence

$$\partial_t^2 u - \alpha \; \nabla^2 u = 0 \quad \text{a.e. on} \quad \Omega.$$

Thus we have proved that u is a solution of the partial differential equation.

Since u is a solution of the partial differential equation, u satisfies (1.5.1). But u is also a solution of Problem 6V:

$$\begin{aligned} & \iiint_\Omega (\partial_t^2 u) v \; dV \\ = \; & -\gamma \iint_\Sigma v(\partial_t u) \; dA - \alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \;\; dV \end{aligned} \tag{1.5.2}$$

for each $v \in T(\Omega)$.

By comparing (1.5.1) and (1.5.2) we have that for each $v \in T(\Omega)$

$$\alpha \iint_\Sigma v(\nabla u \cdot n) \; dA = -\gamma \iint_\Sigma v(\partial_t u) \; dA. \tag{1.5.3}$$

So for an arbitrary test function v we have that

$$\iint_\Sigma (\alpha \; \partial_1 u + \gamma \; \partial_t u) v \; dA = 0.$$

Note that $\partial_1 u \in T(\Omega)$ and $\partial_t u \in T(\Omega)$ and we have that

$$\iint_{\Sigma} \left(\alpha\ \partial_1 u + \gamma\ \partial_t u\right)^2\, dA = 0.$$

Since $\alpha\ \partial_1 u + \gamma\ \partial_t u$ is continuous, $\alpha\ \partial_1 u + \gamma\ \partial_t u = 0$ on Σ. This shows that the boundary conditions are satisfied and the proof is complete. ☐

1.6 The general linear vibration problem

1.6.1 General variational form

There are a variety of problems in Chapter 1. However, when considering the variational forms of these problems, we see that all may be written in the same form and it is therefore possible to formulate a **general linear vibration problem**.

Suppose Ω is a bounded open subset of $\mathbb{R}^n$ for $n = 1,\ 2$ or 3. (If $n = 1$, then $\Omega = (0, \ell)$.) Let $T(\Omega)$ denote the space of test functions, then $a(u, v)$, $b(u, v)$ and $c(u, v)$ are defined when u and v are in $T(\Omega)$. The form c is actually defined on a larger set of functions and this is also possible for a and b.

The precise definitions of a, b and c are not of importance now but their general properties are. First of all a, b and c are **bilinear**: for any functions u, v and w and any constant k we have

$$\begin{aligned} c(ku, v) &= kc(u, v) = c(u, kv), \\ c(u + v, w) &= c(u, w) + c(v, w), \\ c(u, v + w) &= c(u, v) + c(u, w). \end{aligned}$$

This is easy to verify for each example considered thus far. The same is also true for a and b. Secondly a, b and c are **symmetric**: for any functions u and v it is true that

$$c(u, v) = c(v, u)$$

and the same holds for a and b. Finally the corresponding quadratic forms are **nonnegative**:

$$c(u, u) \geq 0 \quad \text{for each } u$$

and again, the same is true for a and b.

For the **general linear vibration problem** we assume that the forms a, b and c are bilinear and symmetric and that the corresponding quadratic forms are nonnegative.

Suppose J is an interval and let $D = \Omega \times J$. Suppose also that u is defined on D, then $u(\cdot, t)$ is defined on Ω. The problems in this chapter are all of the form

$$c(\partial_t^2 u(\cdot,t), v) + a(\partial_t u(\cdot,t), v) + b(u(\cdot,t), v) = \int_\Omega f(\cdot,t)v \tag{1.6.1}$$

for each $v \in T(\Omega)$.

Remark Equation (1.6.1) may be used to calculate approximations for solutions. This is one variation of the finite element method.

Energy

As in Subsection 1.2.5 the mechanical energy (kinetic energy plus elastic potential energy) of the system is defined as

$$E(t) = \frac{1}{2}\, c(\partial_t u, \partial_t u) + \frac{1}{2}\, b(u, u)$$

and we have

$$E'(t) = c(\partial_t^2 u, \partial_t u) + b(u, \partial_t u) = -a(\partial_t u, \partial_t u).$$

If $a(u, u)$ is nonnegative, then $E'(t) \le 0$. A proof is provided in Section 4.6.

Uniqueness Equation (1.6.1) with given initial conditions $u(\cdot, 0)$ and $\partial_t u(\cdot, 0)$ has at most one solution. The proof is the same as in Subsection 1.2.5.

1.6.2 Existence theory

Although the variational form of a problem is a weaker form of the original model problem, it is still not suitable for the application of general methods, especially functional analysis and measure theory.

The problem is the classical pointwise definition of derivatives. In the modern theory of partial differential equations this is replaced by weak derivatives. These derivatives may be viewed as describing average behaviour rather than providing detailed information at each point. Weak derivatives and Sobolev spaces are treated in Chapter 2.

Consider for example Problem 1V. The bilinear forms a and c are actually defined for functions in $\mathcal{L}^2(0,\ell)$ (defined in Section 2.1) and b for functions in the Sobolev space $H^1(0,\ell)$ (defined in Section 2.2). For all the other variational problems we have a similar situation.

The problems in this chapter are all of the form (1.6.1), as discussed in the previous subsection. But $\partial_t u(\cdot,t)$ and $\partial_t^2 u(\cdot,t)$ do not make sense if $u(\cdot,t)$ may be changed arbitrarily on a set of measure zero. It is therefore necessary to replace the time partial derivatives: the rate of change should be measured in an average sense. In the one-dimensional case we consider v to be the derivative of u if

$$\int_0^\ell \left(h^{-1}[u(x,t+h)-u(x,t)]-v(x,t)\right)^2\, dx \to 0 \quad \text{as} \quad h \to 0.$$

The discussion above is made precise in Sections 3.6 and 5.1.

Equation (1.6.1) with weak derivatives and “average time derivatives” is referred to as a weak variational form. The weak variational form is introduced rigorously in Section 5.1. It is a natural choice for existence theory and solutions are referred to as weak solutions. To investigate the smoothness of weak solutions is referred to as regularity theory and is beyond the scope of this dissertation.

The general existence theory is presented in Chapter 4 where the general variational form is linked to semigroup theory. Application of the general theory to the model problems in this chapter is presented in Chapter 5.

Conditions

It is impossible to specify precise conditions for the existence of weak solutions at this point. However, it may be mentioned that it depends on the relation between the test functions $T(\Omega)$ and $\mathcal{L}^2(\Omega)$ (defined in Section 2.1) and the properties of the bilinear forms a, b and c.

It is required that $T(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$: For any $u \in \mathcal{L}^2(\Omega)$ and any $e > 0$ there must exists a test function ϕ such that $\int_\Omega (u-\phi)^2 < \epsilon$.

The conditions on the bilinear forms are as follows.

If $c(\phi,\phi)=0$ then $\phi=0$,

There exists a constant $K>0$ such that $b(\phi,\phi)\geq Kc(\phi,\phi)$ for any $\phi\in T(\Omega)$ and

There exists a constant $K^*>0$ such that $a(\phi,\psi)^2\leq K^*b(\phi,\phi)b(\psi,\psi)$ for any pair ϕ and ψ in $T(\Omega)$.

1.6.3 Parameters

The parameters in a model problem are not always constant. We chose constants for simplicity since the focus is on existence theory. Consider for example the **Euler Bernoulli** model in Section 1.3. Suppose the beam is not prismatic but tapered, then the cross-sectional area A and the area moment I are not constant but vary along the length of the beam.

From the equations of motion (1.3.1) and (1.3.2) we have

$$\rho A\partial_t^2 w=\partial_x V=-\partial_x^2 M.$$

Combining this with the constitutive equation $M=EI\partial_x^2 w$ yields the partial differential equation which we do not use. Instead we derived the variational form directly from the equations above.

As in Subsection 1.3.2 we have for each $v\in T[0,\ell]$

$$\begin{aligned}&\int_0^\ell \rho A(x)\ \partial_t^2 u(x,t)v(x)\ dx\\ =&-\int_0^\ell M(x,t)\ v''(x)\ dx-\int_0^\ell k(x)\partial_t u(x,t)v(x)\ dx\\ =&-\int_0^\ell EI(x)\ \partial_x^2 u(x,t)v''(x)\ dx-\int_0^\ell k(x)\partial_t u(x,t)v(x)\ dx.\end{aligned}$$

The fact that A and I are not constant does not create any problems.

The bilinear forms are defined as follows

$$\begin{aligned}b(u,v)&=\int_0^\ell EI(x)u''(x)v''(x)\ dx\\ a(u,v)&=\int_0^\ell k(x)u(x)v(x)\ dx\\ \text{and}\quad c(u,v)&=\int_0^\ell \rho A(x)u(x)v(x)\ dx.\end{aligned}$$

The functions A, I and k can of course not be arbitrary. It is necessary that they are nonnegative, bounded, continuous functions with a positive minimum. This will ensure that a, b and c have the required properties.

Chapter 2

Sobolev Spaces

2.1 The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$ and denote its closure by $\bar{\Omega}$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω.

Theorem 1

The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with inner product

$$(f,g) = \int_\Omega fg = \int_\Omega fg\ d\mu$$

where μ is the n-dimensional Lebesgue measure.

Proof See [Ru, Theorem 3.11, p 69].

Notation Unless otherwise stated the norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$.

Definition Set $A_f = \{\ x \in \Omega \bigm| f(x) \neq 0\ \}$ and let S_f be the closure of A_f in Ω. Define $C_0^\infty(\Omega) = \{\ f \in C^\infty(\Omega) \bigm| S_f \subset \Omega\ \}$.

Remark If $f \in C_0^\infty(\Omega)$, then the distance between S_f and the boundary of Ω is positive.

Theorem 2

$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

Proof See [Ad, Theorem 2.13, p 28].

Proof of Proposition 1.2.1

Suppose f is integrable on $(0, \ell)$ and $\int_0^\ell f\phi = 0$ for each $\phi \in T_0[0, \ell]$. Since $T_0[0, \ell]$ contains C_0^∞, $T_0[0, \ell]$ is dense in $\mathcal{L}^2(0, \ell)$. From the continuity of the inner product it follows that $\int_0^\ell f\phi = 0$ for each ϕ that is integrable on $(0, \ell)$. So $\int_0^\ell f^2 = 0$ and we have that f equals zero almost everywhere.

Suppose $f \in C[0, \ell]$ and $f(x) \neq 0$). If $f(x) > 0$ then $f > 0$ on some interval $(x - \delta, x + \delta)$ by continuity. This contradiction proves that $f = 0$. $\square$

Proof of Proposition 1.5.2

The proof is almost identical to the proof of Proposition 1.2.1.

2.2 Sobolev spaces

The one-dimensional case

Suppose Ω is a bounded open interval. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition Inner product

For f and g in $H^m(\Omega)$, $(f, g)_m = \sum_{k=0}^m (f^{(k)}, g^{(k)})$ for $m = 0, 1, \ldots$

The bilinear form $(\cdot, \cdot)_m$ has all the properties of an inner product.

Definition Norm

For f in $H^m(\Omega)$, $\|f\|_m = \sqrt{(f, f)_m}$ for $m = 0, 1, \ldots$

The higher dimensional case

Suppose Ω is an open subset of $\mathbb{R}^n$. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Notation Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$, then $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 \cdots \alpha_n$.

Remark $|\alpha|$ denotes the order of the derivative.

Definition Inner product

For f and g in $H^m(\Omega)$,

$$(f,g)_m = \sum_{|\alpha|\leq m} (\partial^\alpha f, \partial^\alpha g) \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

The bilinear form $(\cdot,\cdot)_m$ has all the properties of an inner product.

Definition Norm

The norm for $H^m(\Omega)$ is defined by $\|f\|_m = \sqrt{(f,f)_m}$ for $m = 0,\, 1,\, \ldots$

2.3 Fundamental properties of Sobolev spaces

Assumption Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

Remark It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$.

Theorem 1

The space $H^m(\Omega)$ is complete.

Proof See [Ad, Theorem 3.2, p 45].

Theorem 2

$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.

Proof See [OR, Theorem 2.10, p 53].

Remark A function in $H^m(\Omega)$ can be approximated by a function in $C^m(\bar{\Omega})$: if $u \in H^m(\Omega)$ then for any $\varepsilon > 0$ there exists a $\phi \in C^m(\bar{\Omega})$ such that $\|u - \phi\|_m < \varepsilon$.

Theorem 3 Sobolev's lemma

Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > m + n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \leq \|u\|_p \quad \text{for} \quad |\alpha| \leq m.$$

Proof See [OR, Theorem 3.10, p 80].

Remark One-dimensional case

If $n = 1$ in Theorem 3, then $u \in C^{p-1}(\bar{\Omega})$ and $\|u^{(k)}\|_{\sup} \leq \|u\|_p$ for $k \leq m-1$.

2.4 Inequalities

The one-dimensional case

Recall the notation $C_*^m[0,\ell]$ that was defined in Section 1.2.2.

Proposition 1

Consider any $u \in C_*^1[0,\ell]$. For any two points x and y in $[0,\ell]$,

$$|u(x)| \leq \sqrt{\ell}\|u'\| + |u(y)|.$$

Proof Assuming that $x > y$ (without loss of generality), we have

$$u(x) = \int_y^x u' + u(y).$$

For any f and $g \in \mathcal{L}^2(\Omega)$ we have the Cauchy-Schwartz inequality

$$\left(\int_y^x fg\right)^2 \leq \left(\int_y^x f^2\right)\left(\int_y^x g^2\right)$$

and by choosing $g = 1$ we find that

$$(\int_y^x f)^2 \leq (\int_y^x f^2)(x-y) \leq \ell\|f\|^2.$$

Hence $\left|\int_y^x f\right| \leq \sqrt{\ell}\|f\|$ for each $f \in \mathcal{L}^2(0,\ell)$. This inequality is also true for u' since u' is integrable by assumption, so for $u \in C_*^1[0,\ell]$ it follows that

$$\begin{aligned} |u(x)| &\leq \left|\int_y^x u'\right| + |u(y)| \\ &\leq \sqrt{\ell}\|u'\| + |u(y)|. \end{aligned}$$

□

Proposition 2

For any $u \in C^1_*[0,\ell]$ with a zero in $[0,\ell]$ we have $\|u\|_{\text{sup}} \leq \sqrt{\ell}\|u'\|$.

Proof Suppose $u(y) = 0$, then $|u(x)| \leq \sqrt{\ell}\|u'\|$ by Proposition 1. Since x is arbitrary, $\sqrt{\ell}\|u'\|$ is an upper bound for for $|u|$, but $\|u\|_{\text{sup}}$ is the least upper bound for $|u|$. It follows that $\|u\|_{\text{sup}} \leq \sqrt{\ell}\|u'\|$. □

Proposition 3

For any $u \in C^1_*[0,\ell]$ with a zero in $[0,\ell]$ we have $\|u\| \leq \ell\|u'\|$.

Proof

$$\|u\|^2 = \int_0^\ell \big(u(x)\big)^2 dx \leq \|u\|^2_{\text{sup}} \int_0^\ell \, dx \leq \ell\|u\|^2_{\text{sup}}.$$

From Proposition 2 we have that

$$\|u\| \leq \sqrt{\ell}\|u\|_{\text{sup}} \leq \ell\|u'\|.$$

□

Lemma

Let

$$C_Z[0,\ell] = \{u \in C^1_*[0,\ell] \bigm| u(x) = 0 \ \text{ for some } \ x \in [0,\ell]\}.$$

Denote the closure of $C_Z[0,\ell]$ in $H^1(0,\ell)$ by $\overline{C_Z}$. For any $u \in \overline{C_Z}$,

$$\|u\| \leq \ell\|u'\|. \tag{2.4.1}$$

Proof For any $u \in \overline{C_Z}$, there exist a sequence $\{u_n\} \in C_Z[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

Since $\big|\|u_n\| - \|u\|\big| \leq \|u_n - u\| \leq \|u_n - u\|_1$ we find that $\|u_n\| \to \|u\|$ as $n \to \infty$. Likewise $\|u'_n\| \to \|u'\|$ as $n \to \infty$.

$u_n \in C_Z[0,\ell]$ so from Proposition 3 we have $\|u_n\| \leq \ell\|u'_n\|$ for all n. From this it follows that

$$\frac{\|u_n\|}{\|u'_n\|} \leq \ell$$

and by considering limits we find that $\|u\| \leq \ell\|u'\|$. □

Proposition 4

For any $u \in C^1_*[0,\ell]$ there exists a constant $K_\ell > 0$ such that

$$|u(\ell)| \leq K_\ell \|u\|_1.$$

Proof Let $g(x) = x$. Since $g(0) = 0$

$$\ell u(\ell) = \int_0^\ell (gu)' = \int_0^\ell (g'u + gu')\,.$$

By making use of the Schwartz inequality

$$\ell|u(\ell)| = \left|\int_0^\ell (g'u + gu')\right| \leq \|g'\|\|u\| + \|g\|\|u'\|.$$

Since $\|g'\| = \sqrt{\ell}$ and $\|g\| = \sqrt{\frac{\ell^3}{3}}$, we have

$$\ell|u(\ell)| \leq \sqrt{\ell}\|u\| + \sqrt{\frac{\ell^3}{3}}\|u'\|.$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, it follows that

$$|u(\ell)|^2 \leq 2\big(\ell^{-1}\|u'\|^2 + \frac{\ell}{3}\|u\|^2\big) \leq K\|u\|_1^2$$

where $K = 2\max\{\ell^{-1}, \frac{\ell}{3}\}$. □

The three-dimensional case

The following result is referred to as the Poincare-Friedrichs inequality or Friedrichs's inequality or Poincare's inequality. We consider only the situation as encountered in Problem 6V in Section 5.5.

Proposition 5

Suppose

$$\Omega = \{x \in \mathbb{R}^3 \;\big|\; 0 < x_i < d\}.$$

Let Σ be the part of the boundary where $x_1 = d$ and $\partial\Omega - \Sigma$ the rest of the boundary. Recall that the set of test functions is defined as

$$T(\Omega) = \{u \in C^1_*(\bar{\Omega}) \;\big|\; u = 0 \text{ on } \partial\Omega - \Sigma\}.$$

There exists a constant k such that, for each $u \in T(\Omega)$,

$$\|u\|^2 \leq k \sum_{i=1}^3 \|\partial_i u\|^2.$$

Proof Consider $u \in T(\Omega)$. From Proposition 3 we have

$$\begin{aligned} & \int_0^d [u(x_1,x_2,x_3)]^2 \, dx_1 \\ \leq \; & d^2 \int_0^d [\partial_1 u(x_1,x_2,x_3)]^2 \, dx_1 \end{aligned}$$

for any $x_2 \in [0,d]$ and any $x_3 \in [0,d]$. Clearly

$$\begin{aligned} \iiint_\Omega u^2 & \leq d^2 \iiint_\Omega (\partial_1 u)^2 \\ & \leq d^2 \iiint_\Omega \sum_{i=1}^3 (\partial_i u)^2 \end{aligned}$$

□

Remark The result in Proposition 5 is true for a more general set Ω. See e.g. [Br, p 30].

2.5 Trace

The one-dimensional case

From Theorem 2.3.2 we have that for each $u \in H^1(0,\ell)$, there exists a sequence $\{u_n\} \subset C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

Theorem 1

For each $u \in H^1(0,\ell)$ let

$$\Gamma u = \lim_{n\to\infty} u_n(\ell)$$

where $\{u_n\}$ is a sequence in $C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

The mapping Γ is linear and bounded. In fact $|\Gamma u| \leq K_\ell \, \|u\|_1$.

Proof Let $\{u_n\}$ be any sequence in $C^1[0,\ell]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

The sequence $\{u_n\}$ is Cauchy in $H^1(0,\ell)$ so by Proposition 2.4.4 $\{u_n(\ell)\}$ is Cauchy in $\mathbb{R}$. Thus $\lim\limits_{n\to\infty} u_n(\ell)$ exists. Moreover this limit is independent of of the choice of the sequence $\{u_n\}$. Indeed, if $\{v_n\}$ is another sequence in $C^1[0,\ell]$ such that $\|v_n - u\|_1 \to\ 0$ as $n \to \infty$ then

$$\|v_n - u_n\|_1 \leq \|v_n - u\|_1 + \|u - u_n\|_1 \to\ 0$$

as $n \to \infty$. So by Proposition 2.4.4 we have that $\big|v_n(\ell) - u_n(\ell)\big| \to\ 0$ as $n \to \infty$.

The linearity follows from the properties of limits.

Finally,

$$\frac{|\ u_n(\ell)|}{\|u_n\|_1} \leq K_\ell \ \text{ for all } \ n$$

and by considering the limits we have that

$$|\ \Gamma u| \leq K_\ell \|u\|_1.$$

This proves that Γ is bounded. □

Remark A trace operator can be defined similarly at zero.

The three-dimensional case

Suppose

$$\Omega = \{x \in \mathbb{R}^3 \ \big|\ 0 < x_i < d\}.$$

Let Σ be the part of the boundary where $x_1 = d$ and $\partial\Omega - \Sigma$ the rest of the boundary.

Definition Trace operator Γ in three dimensions

For $u \in C(\bar{\Omega})$, the function Γu is the restriction of the function u to Σ.

Proposition

$\|\Gamma u\|_\Sigma \leq K \|u\|_1^\Omega$ for each $u \in C^1(\bar{\Omega})$.

Proof Consider any pair (x_2, x_3) such that $0 \leq x_2 \leq d$ and $0 \leq x_3 \leq d$. From Proposition 2.4.4 we have

$$\big[u(d_1, x_2, x_3)\big]^2 \leq K_1 \int_0^d [u(x_1, x_2, x_3)]^2\, dx_1 + K_1 \int_0^d [\partial_1 u(x_1, x_2, x_3)]^2\, dx_1$$

Hence

$$\begin{aligned} \iint_\Sigma (\Gamma\ u)^2 &\leq K_1 \iiint_\Omega \big(u^2 + (\partial_1 u)^2\big) \\ &= K_1 \big(\|u\|^2 + \|\partial_1 u\|^2\big) \\ &\leq K_1 \|u\|_1^2. \end{aligned}$$

□

Theorem 2

The trace operator Γ can be extended to a bounded linear operator which maps $H^1(\Omega)$ into $\mathcal{L}^2(\Sigma)$ and

$$\|\Gamma u\|_\Sigma \leq K\|u\|_1^\Omega.$$

Proof From Theorem 2.3.2 we have that for each $u \in H^1(\Omega)$, there exists a sequence $\{u_n\} \subset C^1(\bar{\Omega})$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

For each $u \in H^1(\Omega)$ define

$$\Gamma u = \lim_{n\to\infty} \Gamma u_n.$$

This mapping is linear (from the properties of limits).

The sequence $\{u_n\}$ is Cauchy in $H^1(\Omega)$ so by the previous proposition $\{\Gamma u_n\}$ is Cauchy in $\mathcal{L}^2(\Sigma)$. Thus $\lim\limits_{n\to\infty} \Gamma u_n$ exists. Moreover this limit is independent of of the choice of the sequence $\{u_n\}$. Indeed, if $\{v_n\}$ is another sequence in $C^1(\bar{\Omega})$ such that $\|v_n - u\|_1 \to 0$ as $n \to \infty$ then

$$\|v_n - u_n\|_1 \leq \|v_n - u\|_1 + \|u - u_n\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

From the previous proposition $\|\Gamma v_n - \Gamma u_n\|_\Sigma \leq K\|v_n - u_n\|_1 \to 0$ as $n \to \infty$. Finally,

$$\frac{\|\Gamma u_n\|_\Sigma}{\|u_n\|_1} \leq K \quad \text{for all} \quad n$$

and by considering the limits we have that

$$\| \ \Gamma u\| \leq K\|u\|_1.$$

This proves that Γ is bounded. □

Remark This result is a special case of results in [OR, p 141-142].

Chapter 3

Semigroups and differential equations

The existence theory in Chapter 4 is based on semigroup theory and so called abstract differential equations.

3.1 Integration and differentiation

To analyze a differential equation of a function with values in a general Banach space, we need the definition of a derivative. Let J be any of the following intervals: (a, b), $[a, b)$ or $[a, \infty)$. Let Y be any Banach space and consider a function u on J with values in Y.

Definition Derivative

Let t be any interior point of J. Suppose there exists a $v \in Y$ such that

$$\lim_{h \to 0} \left\| h^{-1}\big(u(t+h) - u(t)\big) - v \right\|_Y = 0,$$

then v is the derivative of u at t. We write $u'(t)$ for the derivative and $u'(t) \in Y$ to show that the derivative exists with respect to the norm of Y. The derivative (function) u' and the second order derivative u'' are defined in the usual way.

Suppose there exists a $v \in Y$ such that

$$\lim_{h \to 0+} \left\| h^{-1}\big(u(t+h) - u(t)\big) - v \right\|_Y = 0,$$

then v is the right (hand) derivative of u at t and we write $D^+u(t)$.

Notation $C^k\big(J;Y\big)$

$u \in C^k\big(J;Y\big)$ if $u^{(k)}(t) \in Y$ for each $t \in J$.

In the book [Pa] it is assumed that the reader is familiar with integration of functions with values in a Banach space. For completeness we define the integral and derive certain properties. In a Hilbert space Riesz's theorem is used. Let H be a real Hilbert space and denote its inner product by $\langle \cdot,\cdot \rangle$ and the corresponding norm by $\| \cdot \|$.

Riesz's Theorem

For any bounded linear functional f defined on H, there exists a unique $u \in H$ such that

$$\langle u, x\rangle = f(x) \quad \text{for each } x \in H.$$

Proof See [Kr, Theorem 3.8-1, p 189].

The remainder of this section is from [VDS].

Proposition 1

If $g \in C([a,b];H)$, then there exists a unique $y \in H$ such that

$$\langle y, w\rangle = \int_a^b \langle g, w\rangle \quad \text{for each} \quad w \in H.$$

Proof The real valued function $\langle g(\cdot), w\rangle$ is continuous. Define the functional f by

$$f(w) = \int_a^b \langle g, w\rangle \quad \text{for each} \quad w \in H.$$

Clearly f is linear and

$$|f(w)| \leq \|w\| \int_a^b \|g(\cdot)\| \quad \text{for each} \quad w \in H,$$

i.e. there exists a constant K such that $|f(w)| \leq K\|w\|$ for each $w \in H$. By Riesz's theorem, there exists a unique $y \in H$ such that $\langle y, w\rangle = f(w)$ for each $w \in H$. $\square$

Definition If $\langle y, w\rangle = \int_a^b \langle g, w\rangle$ for each $w \in H$, then $y = \int_a^b g$.

Remark We have the useful formula

$$\langle \int_a^b g, w\rangle = \int_a^b \langle g, w\rangle \quad \text{for each} \quad w \in H.$$

□

Definition If $b > a$, then $\int_b^a g = -\int_a^b g$ and $\int_a^a g = 0 \in H$.

Remark In this case the formula above is still valid.

Proposition 2

$$\| \int_a^b u\| \leq | \int_a^b \|u\| \, |.$$

Proof First suppose $b > a$ and let $v = \int_a^b u$. Then

$$\| \int_a^b u\|^2 = \int_a^b \langle u(\cdot), v\rangle \leq \int_a^b \|u(\cdot)\| \, \|v\|.$$

If $b < a$, then

$$\| \int_a^b u\| = \| \int_b^a u\| \leq \int_b^a \|u\| = | \int_a^b \|u\| \, |.$$

□

Next we prove the fundamental theorem of calculus for Hilbert space valued functions.

Theorem

Suppose the function g is integrable on $[a, b]$ and continuous at $c \in (a, b)$. If f is defined by $f(t) = \int_a^t g$, then $f'(c) \in H$ and $f'(c) = g(c)$.

Proof Note that $f(t+h) - f(t) = \int_t^{t+h} g$. It is therefore sufficient to prove that

$$\lim_{h\to 0} h^{-1} \int_t^{t+h} g = g(t).$$

Consider

$$\begin{aligned} \|h^{-1} \int_t^{t+h} g - g(t)\| &\leq \int_t^{t+h} \|h^{-1}[g - g(t)]\| \\ &\leq |h^{-1}| \int_t^{t+h} \|g - g(t)\|. \end{aligned}$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|g(s) - g(t)\| < \epsilon$ for $|s - t| < \delta$. It follows that

$$\|h^{-1}\int_t^{t+h} g - g(t)\| \leq |h^{-1}|\epsilon|h| = \epsilon.$$

$\square$

3.2 Semigroups

In the remainder of this chapter we consider a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. Let A be a linear operator that maps H into itself and let $B(H)$ denote the space consisting of all bounded linear operators on H.

Remark Pazy [Pa] present semigroup theory for Banach spaces but Hilbert spaces are sufficient for our purpose.

Definition Semigroup

A family of bounded linear operators $T(t)$ defined for $t \geq 0$ is referred to as a semigroup if it has the following properties:
for positive t and s, $T(t+s) = T(t)T(s)$ and $T(0) = I$.

Remark T is a function that maps $[0,\infty)$ into $B(H)$. We will use the notation $T(\cdot)$.

Definition Strongly continuous semigroups

A semigroup $T(\cdot)$ is called strongly continuous if

$$\lim_{h\to 0+} T(h)x = x \quad \text{for each} \quad x \in H.$$

Remarks

1. The definition does not imply that $T(\cdot)$ is right continuous at 0 with respect to the operator norm.
2. A strongly continuous semigroup is also referred to as a semigroup of class C_0 or a C_0 semigroup.

Definition Infinitesimal generator

Consider a semigroup $T(\cdot)$. For $h > 0$ the linear operator A_h is defined by

$$A_h x = h^{-1}(T(h) - I)x.$$

Let

$$\mathcal{D}(A) = \{x \in H \mid \lim_{h \to 0+} A_h x \quad \text{exists}\}$$

and define the operator A with domain $\mathcal{D}(A)$ by

$$Ax = \lim_{h \to 0+} A_h x \quad \text{for} \quad x \in \mathcal{D}(A).$$

The operator A is called the infinitesimal generator of $T(\cdot)$.

Remarks

1. If $x \in \mathcal{D}(A)$, then $\lim_{h \to 0+} \|A_h x - Ax\| = 0$.
2. The set $\mathcal{D}(A)$ in the definition above is a subspace of H and dense in H, see [Pa, Corollary 2.5, p 5].

Definition C_0 semigroup of contractions

If $T(\cdot)$ is a C_0 semigroup and $\|T(t)\| \leq 1 \quad \text{for} \quad t \in [0, \infty)$, then $T(\cdot)$ is called a C_0 semigroup of contractions.

Notation It is convenient to denote λI simply by λ.

Theorem 1 Hille-Yosida

A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions on H if and only if the following conditions are satisfied: A is closed, $\mathcal{D}(A)$ is dense in H, for each $\lambda > 0$, $(\lambda - A)^{-1} \in B(H)$ and $\|(\lambda - A)^{-1}\| \leq \lambda^{-1}$.

Proof See [Pa, Theorem 3.1, p 8]

Definition Dissipative operator

A linear operator A is dissipative if for every $x \in \mathcal{D}(A)$, $\langle Ax, x \rangle \leq 0$.

Theorem 2

Let A be a linear operator with a dense domain in H. If A is dissipative and for $\lambda > 0$ the range $R\big(\lambda - A\big) = H$, then A is the infinitesimal generator of a C_0 semigroup of contractions on H.

Proof

$$\langle(\lambda - A)x, x\rangle = \lambda\|x\|^2 - \langle Ax, x\rangle \geq \lambda\|x\|^2.$$

By making use of the Schwartz inequality

$$\big|\langle(\lambda - A)x, x\rangle\big| \leq \|(\lambda - A)x\|\|x\|.$$

Combining these two inequalities we find that

$$\lambda\|x\| \leq \|(\lambda - A)x\|.$$

It follows that if $(\lambda - A)x = 0$ then $x = 0$ so the operator $(\lambda - A)$ has a trivial nullspace. Therefore, for $\lambda > 0$,

$$\|(\lambda - A)^{-1}y\| \leq \lambda^{-1}\|y\|.$$

We conclude that $(\lambda - A)^{-1}$ is bounded and $\|(\lambda - A)^{-1}\| \leq \lambda^{-1}$. Since $(\lambda - A)^{-1}$ is bounded, $(\lambda - A)$ is closed and hence A is closed. Note that all the conditions of the Hille-Yosida theorem are satisfied. $\square$

3.3 Abstract differential equations

In this section we consider the initial value problem

$$\begin{aligned} u'(t) &= Au(t) + f(t) \quad \text{for} \quad t > 0, && (3.3.1)\\ u(0) &= b. && (3.3.2) \end{aligned}$$

We use the same definition for a solution as Pazy [Pa, p 105].

Definition Solution

A function u is said to be a solution of the initial value problem above if it satisfies (3.3.1) and (3.3.2) and for each $t > 0$, $u(t) \in \mathcal{D}(A)$ and

$$u \in C([0, \tau); H) \cap C^1((0, \tau); H).$$

Assumption For simplicity we assume that the linear operator A is dissipative and in all our applications this is the case.

Theorem 1

The initial value problem (3.3.1)-(3.3.2) has at most one solution.

Proof Firstly it is shown that the initial value problem

$$u'(t) = Au(t) \quad \text{for} \quad t > 0 \quad \text{with} \quad u(0) = 0$$

has only a trivial solution. Assume that u is a solution of this initial value problem, then

$$\begin{aligned} \frac{d}{dt}\|u(t)\|^2 &= \frac{d}{dt}\langle u(t), u(t)\rangle \\ &= 2\langle u(t), u'(t)\rangle \\ &= 2\langle u(t), Au(t)\rangle \quad \text{(since } u \text{ is a solution)} \\ &\leq 0 \quad \text{(since } A \text{ is dissipative).} \end{aligned}$$

Thus $\|u(t)\|^2$ is a nonincreasing function with $u(0) = 0$ which implies that $\|u(t)\|^2 = 0$ for all $t > 0$.

Now assume that the initial value problem (3.3.1)-(3.3.2) has more than one solution; say u and $\bar{u}$ are solutions. A is a linear operator hence

$$u'(t) - \bar{u}'(t) = A\big(u(t) - \bar{u}(t)\big) \quad \text{for all} \quad t > 0.$$

Since $u'(0) - \bar{u}'(0) = 0$, this initial value problem can have only the zero solution by the result just proved. This means that $u(t) - \bar{u}(t) = 0$ for all $t > 0$ which shows that $u = \bar{u}$. ☐

Remark Pazy proves uniqueness without assuming that A is dissipative.

Proposition

Suppose $T(\cdot)$ is a C_0 semigroup generated by the linear operator A. For a function $f \in C([0, \tau); H)$, if the initial value problem (3.3.1)-(3.3.2) has a solution, it is given by

$$u(t) = T(t)b + \int_0^t T(t-s)f(s)ds.$$

Proof See [Pa, Corollary 2.2, p 106]

Remark In general, the continuity of the function f is not sufficient to ensure the solvability of the initial value problem (3.3.1)-(3.3.2). An example is given by Pazy [Pa, p 106].

Theorem 2

Suppose A is the infinitesimal generator of a C_0 semigroup, then the abstract initial value problem (3.3.1)-(3.3.2) has a solution for each $b \in \mathcal{D}(A)$ and $f \in C^1([0,\tau); H)$. If $f = 0$ then the solution is in $C^1([0,\infty); H)$.

Proof See [Pa, Corollary 2.5, p 107 and Theorem 1.3, p 102].

Remark If u is a solution we have from the definition of a solution that u' is continuous on $(0,\tau)$. If $f = 0$ then $u' \in C([0,\infty); H)$.

3.4 Strongly continuous groups

Definition Group

If $T(\cdot)$ is a semigroup and $T(t+s) = T(t)T(s)$ for t and s real numbers, then $T(\cdot)$ is called a group.

Definition Strongly continuous groups

A group $T(\cdot)$ is called strongly continuous if

$$\lim_{h\to 0} T(h)x = x \quad \text{for each} \quad x \in H.$$

Definition Infinitesimal generator

Consider a group $T(\cdot)$. Let

$$\mathcal{D}(A) = \{x \in H \mid \lim_{h\to 0} A_h x \quad \text{exists}\}$$

and define the operator A with domain $\mathcal{D}(A)$ by

$$Ax = \lim_{h\to 0} A_h x \quad \text{for} \quad x \in \mathcal{D}(A).$$

The operator A is called the infinitesimal generator of $T(\cdot)$.

Theorem 1

Let A be a closed linear operator with $\mathcal{D}(A)$ dense in H. If for every real λ such that $|\lambda| > k$, $(\lambda - A)^{-1} \in B(H)$ and $\|\big((\lambda - A)^{-1}\big)^n\| \leq (|\lambda| - k)^{-n}$ for $n = 1, 2...$ then A is the infinitesimal generator of a C_0 group.

Proof See [Pa, Theorem 6.3, p 23].

Remark Consider the differential equation (3.3.1) on an open interval J containing zero. The definition of a **solution** in Section 3.3 is still applicable provided that the interval $(0, \tau)$ is replaced by the open interval J. An existence result for the case were a group is involved, is not proved in [Pa]. However, Theorem 2 of the previous section may be used to prove the following result.

Theorem 2

Let A be the infinitesimal generator of a C_0 group and $f \in C^1(\bar{J}, H)$. Then the initial value problem (3.3.1)-(3.3.2) has a unique solution $u \in C^1(J, H)$ for each $b \in \mathcal{D}(A)$. If $f = 0$, then $u \in C^1((-\infty, \infty), H)$.

Proof Suppose that $\bar{J} = [a, b]$ and define a function g on $[0, b - a]$ by $g(t) = f(t + a)$. Since $f \in C^1([a, b], H)$, $g \in C^1([0, b - a], H)$. By Theorem 2 of the previous section there exists a function $y \in C^1((0, b - a), H)$ such that

$$y' = Ay + g \quad \text{on} \quad (0, b - a).$$

Define a function w on J by $w(t) = y(t - a)$, then $w \in C^1(J, H)$ and
$w' = Aw + f$.

Since b and $w(0)$ are in $\mathcal{D}(A)$, the function $T(\cdot)(b - w(0))$ is a solution of the homogeneous differential equation. Consequently $u = T(\cdot)(b - w(0)) + w$ is a solution of the nonhomogeneous differential equation and since $u(0) = b$, u is the solution of the initial value problem (3.3.1)-(3.3.2). □

3.5 Analytic semigroups

3.5.1 Semigroups on a complex Hilbert space

A real Hilbert H space may be imbedded in a complex Hilbert space

$$\widetilde{H} = \{x + iy \mid x \in H, y \in H\}.$$

Note that if $\alpha + i\ \beta$ is a complex number, then

$$(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i\beta x + i\alpha y.$$

This construction can be made rigorous, see [Sch, p153].

The inner product for $\tilde{H}$ is defined as

$$\langle x + iy, u + iv\rangle_{\tilde{H}} = \langle x, u\rangle_H + \langle y, v\rangle_H + i\langle y, u\rangle_H - i\langle x, v\rangle_H,$$

Any linear operator L on H may be extended to $\tilde{L}$:

$$\begin{aligned} D(\tilde{L}) &= \{x + iy \mid x \in \mathcal{D}(L), y \in \mathcal{D}(L)\}, \\ \tilde{L}(x + iy) &= Lx + iLy. \end{aligned}$$

In the rest of this section we consider an arbitrary complex Hilbert space X with inner product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X$. The induced operator norm is denoted by $\|\cdot\|$, i.e $\|A\|$ for $A \in B(X)$.

Definitions Resolvent set. Resolvent operator.

Let A be a linear operator with domain and range contained in X. The resolvent set $\rho(A)$ of A is the set of complex numbers λ such that $(\lambda - A)^{-1}$ is a bounded linear operator defined on X.
The linear operator $R(\lambda, A) = (\lambda - A)^{-1}$ is referred to as the resolvent operator of A or simply the resolvent of A.

Theorem 1

The resolvent set $\rho(A)$ is an open subset of the complex plane $\mathbb{C}$.

Proof See [Sch, Theorem 10.1, p 304]

Lemma

Let A be a closed operator on a complex Banach space. If λ is a boundary point of $\rho(A)$ and $\{\lambda_n\}$ is a sequence of points in $\rho(A)$ converging to λ, then

$$\|(A-\lambda_n)^{-1}\| \to \infty.$$

Proof See [Sch, Lemma 10.4, p 307].

3.5.2 Differentiable semigroups

Definition Uniformly bounded semigroup

A semigroup $T(\cdot)$ is called uniformly bounded if there exists a constant $C > 0$ such that $\|T(t)\| \leq C$ for $t \geq 0$.

Definition Differentiability

A semigroup $T(\cdot)$ is called differentiable if the function $T(\cdot)x$ is differentiable on $(0, \infty)$ for every $x \in X$.

Remark We do not consider the case where the function $T(\cdot)x$ is differentiable on $[0, \infty)$ for every $x \in X$ since this is only possible when the infinitesimal generator of the semigroup is bounded.

Notation For any $\phi \in (0, \pi/2)$, let

$$\begin{aligned} \Delta_\phi &= \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \phi\} \quad \text{and} \\ \Sigma_\phi^+ &= \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \pi/2 + \phi\} \cup \{0\}. \end{aligned}$$

Definition Analytic semigroup

Let $T(z)$ be a bounded linear operator for each $z \in \Delta_\phi$. The function $T(\cdot)$ is an analytic semigroup in Δ_ϕ if

1. $T(\cdot)$ is analytic in Δ_ϕ;
2. $T(0) = I$ and
$$\lim_{z \to 0,\ z \in \Delta_\phi} T(z)x = x \quad \text{for each} \quad x \in H;$$
3. for z_1 and z_2 in Δ_ϕ, $T(z_1 + z_2) = T(z_1)T(z_2)$.

Definition Operator of Type (ϕ, M)

A linear operator A is of Type (ϕ, M) if there exist a $\phi \in (0, \pi/2)$ and $M > 0$ such that $\rho(A)$ contains Σ_ϕ^+ and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \quad \text{for} \quad \lambda \in \Sigma_\phi^+, \ \lambda \neq 0.$$

Remark The terminology "Type (ϕ, M)" is due to [F].

Theorem 2

If the linear operator A is densely defined on H and of Type (ϕ, M), then it is the infinitesimal generator of a uniformly bounded C_0 semigroup.

Proof See [Pa, Theorem 1.7.7, p 30]

Theorem 3

Suppose $T(\cdot)$ is a uniformly bounded C_0 semigroup and let A be its infinitesimal generator. If A is of Type (ϕ, M) and $0 \in \rho(A)$ then $T(\cdot)$ is differentiable and an analytic semigroup in Δ_ϕ.

Proof The result is part of Theorem 5.2 in [Pa] on p 61. The required result is the fact that Statement (c) implies Statements (a) and (d). $\square$

Theorems 2 and 3 above provide a sufficient condition for an operator to be the infinitesimal generator of a uniformly bounded analytic semigroup. In a Hilbert space X a sufficient condition for the infinitesimal generator can be formulated in terms of the numerical range of the form $\langle Ax, x\rangle_X$. We present the result in [VV].

In our presentation we elaborate to make the proof more readable. The next well known result is also included for this purpose.

Proposition 1

Let A be a subset of the complex plane. Suppose p and q are are complex numbers with $p \in A$ and $q \notin A$. If ℓ is the line segment joining p and q then ℓ contains a boundary point of A.

Theorem 4 [VV, Theorem 5]

The linear operator A is the infinitesimal generator of a uniformly bounded analytic (holomorphic) semigroup in a complex Hilbert space X if $\mathcal{D}(A)$ is dense in X, $0 \in \rho(A)$ and there exists a constant $C_A > 0$ such that

$$C_A Re\langle Ax, x\rangle_X + \left|Im\langle Ax, x\rangle_X\right| \leq 0 \text{ for each } x \in \mathcal{D}(A). \tag{3.5.1}$$

Proof Due to Theorem 3 we need to show that there exist $0 < \delta < \pi/2$ and $M > 0$ such that

$$\Sigma_\delta^+ \subset \rho(A) \text{ and}$$
$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \text{ for all } \lambda \in \Sigma_\delta^+, \ \lambda \neq 0.$$

The first step is to establish the inequality (3.5.2) below. Choose $\theta = \arctan C_A$. Then $0 < \theta < \pi/2$ and $-\langle Ax, x\rangle_X \in \Delta_\theta$ for all $x \in \mathcal{D}(A)$. Note that $Re\langle Ax, x\rangle \leq 0$ from assumption (3.5.1).

For any complex number λ and $x \in \mathcal{D}(A)$ consider

$$\big\langle(\lambda - A)x, x\big\rangle_X = \lambda\|x\|_X^2 - \langle Ax, x\rangle_X.$$

This is of the form $r\lambda - \mu$ with $-\mu \in \Delta_\theta$ and $r = \|x\|_X^2 > 0$. (The result is trivial if $x = 0$).

Choose $\delta > 0$ such that $\theta + \delta < \pi/2$. Applying the cosine rule in the complex plane yields that

$$|r\lambda - \mu|^2 = r^2|\lambda|^2 - 2r|\lambda||\mu|\cos(\theta_\lambda - \theta_\mu) + |\mu|^2$$

with $\theta_\lambda = \arg\lambda$ and $\theta_\mu = \arg\mu$.

Note that for $\lambda \in \Sigma_\delta^+$ we have that $|\theta_\lambda - \theta_\mu| \geq \pi/2 - \theta - \delta > 0$, hence $\cos(\theta_\lambda - \theta_\mu) \leq \cos(\pi/2 - \delta - \theta) < 1$. Denoting this upper bound by α and using the inequality $-2ab \geq -a^2 - b^2$, we find that

$$\begin{aligned} |r\lambda - \mu|^2 &\geq r^2|\lambda|^2 - 2\alpha r|\lambda||\mu| + |\mu|^2 \\ &\geq (1 - \alpha^2)r^2|\lambda|^2. \end{aligned}$$

This implies the existence of a constant $C > 0$ such that

$$\left|\big\langle(\lambda - A)x, x\big\rangle_X\right| \geq C|\lambda|\|x\|_X^2 \tag{3.5.2}$$

for all $x \in \mathcal{D}(A)$ and all $\lambda \in \Sigma_\delta^+$.

For $\lambda \in \rho(A) \cap \Sigma_\delta^+$, $(\lambda - A)^{-1} \in B(X)$ and applying the Schwartz inequality yields that

$$\|(\lambda - A)^{-1}\|_X \leq C^{-1}|\lambda|^{-1}.$$

As $\rho(A)$ is open and $0 \in \rho(A)$ there exists a ball $B_\varepsilon = \{\lambda \mid |\lambda| < \varepsilon\}$ contained in $\rho(A)$. If $\lambda \in \rho(A) \cap \left(\Sigma_\delta^+ - B_\varepsilon\right)$ then

$$\|(\lambda I - A)^{-1}\|_X \leq C^{-1}\varepsilon^{-1}.$$

Since A is closed, we conclude from the lemma above that no point in $\Sigma_\delta^+ - B_\varepsilon$ can be a boundary point of $\rho(A)$ and hence that $\Sigma_\delta^+ \subset \rho(A)$ by Proposition 1. We have proved that A is of type (ϕ, M) and the result follows from Theorem 3. ☐

The condition in Theorem 3 for A to be the infinitesimal generator of an analytic semigroup is not necessary. Consider the following formal calculation:

$$\frac{d}{dt}\{e^{\alpha t}T(t)x\} = \alpha e^{\alpha t}T(t)x + e^{\alpha t}AT(t)x = (\alpha + A)e^{\alpha t}T(t)x.$$

If this can be justified, then it means that $e^{\alpha t}T(t)x$ is differentiable when $T(t)x$ is differentiable. (It is referred to as the "exponential shift trick" in [VV].) Justification for the calculation above is now provided.

Proposition 2

Suppose A is the infinitesimal generator of an analytic semigroup and let $L = \alpha + A$. Then L is the infinitesimal generator of an analytic semigroup.

Proof Suppose $T(\cdot)$ be the analytic semigroup generated by A and let $S(t) = e^{\alpha t}T(t)$. We prove that $S(\cdot)$ is an analytic semigroup generated by L. For any $x \in H$

$$\begin{aligned} \|h^{-1}(S(h)x - x) - Lx\|_X &= \|h^{-1}(e^{\alpha h}T(h)x - x) - (\alpha + A)x\|_X \\ &\leq \|h^{-1}(e^{\alpha h}T(h)x - e^{\alpha h}x) - Ax\|_X \\ &\quad + \|h^{-1}(e^{\alpha h}x - x) - \alpha x\|_X. \end{aligned}$$

Clearly

$$\lim_{h \to 0+} \|h^{-1}(e^{\alpha h} - 1)x - \alpha x\|_X = 0.$$

On the other hand

$$\begin{aligned} & \|h^{-1}(e^{\alpha h}T(h)x - e^{\alpha h}x) - Ax\|_X \\ \leq\ & |e^{\alpha h}|\ \|h^{-1}(T(h)x - x) - Ax\|_X + |e^{\alpha h} - 1|\ \|Ax\|_X \end{aligned}$$

and both terms on the right tend to zero as h $\to$ 0+. ☐

3.5.3 Application to the initial value problem

Definition Locally Hölder continuous function

Let J be an interval and $\alpha \in (0,1]$. A function $f : J \to H$ is Hölder continuous on J with exponent α if there exists a $L > 0$ such that

$$\|f(t) - f(s)\| \leq L|t-s|^{\alpha} \quad \text{for } t, s \in J.$$

The function f is locally Hölder continuous on J (with exponent α) if every $t \in J$ has a neighbourhood in which f is Hölder continuous (with exponent α).

Remark If a function $f : J \to H$ is locally Lipschitz it is locally Hölder continuous with exponent one and if $f \in C^1(J, H)$ it is locally Lipschitz.

Theorem 4

Let A be the infinitesimal generator of an analytic semigroup. If the function f is locally Holder continuous on $(0, \tau)$, then the initial value problem (3.3.1)-(3.3.2) has a unique solution u for each $b \in H$. If $f = 0$, then $u \in C^{\infty}\big((0,\infty), H\big)$.

Proof [Pa, Corollary 3.3, p113]

3.6 Derivatives and partial derivatives in function spaces

The results in this section are necessary to apply the theory of this chapter to partial differential equations. In applications the Hilbert space is some function space for example $\mathcal{L}^2(\Omega)$.

Consider the general variational equation (Equation (1.6.1)) in Section 1.6. Suppose Ω is a bounded open subset of $\mathbb{R}^n$ for $n = 1,\ 2$ or 3. The function u is defined on $D = \Omega \times J$ where J is some interval. Let $w : J \to \mathcal{L}^2(\Omega)$ such that $w(t) = u(\cdot, t)$ for each t. Our aim is to show that $w'(t) = \partial_t u(\cdot, t)$. The inner product in $\mathcal{L}^2(\Omega)$ is denoted by $(\cdot, \cdot)$.

The material of this section is from [VDS].

Proposition

Suppose J is a bounded interval and let $D = \Omega \times J$. Consider $u \in \mathcal{L}^2(D)$ and define v by $v(x,t) = \int_a^t u(x,\tau)\, d\tau$ for each $(x,t) \in D$. If $w(t) = u(\cdot,t)$ and $y(t) = v(\cdot,t)$, then $y(t) = \int_a^t w$.

Proof Let $g \in C(\overline{\Omega})$ be arbitrary. Since D is bounded, it follows that $u \in \mathcal{L}^1(D)$. This implies that gu is integrable on D with respect to the Lebesgue measure for $\mathbb{R}^4$. By Fubini's theorem [Roy, p269],

$$\int_a^t \int_0^\ell g(x)u(x,\tau)\, dx d\tau = \int_0^\ell \int_a^t g(x)u(x,\tau)\, d\tau dx.$$

Now,

$$\begin{aligned}\int_a^t \int_0^\ell g(x)u(x,\tau)\, dx d\tau &= \int_0^\ell g(x) \int_a^t u(x,\tau)\, d\tau dx,\\ &= \int_0^\ell g(x)v(x,t)\, dx,\\ &= (g, y(t)).\end{aligned}$$

We have also that

$$\int_a^t \int_0^\ell g(x)u(x,\tau)\, dx d\tau = \int_a^t (g, w(\cdot)).$$

Since $C(\overline{\Omega})$ is dense in $\mathcal{L}^2(\Omega)$,

$$(z, y(t)) = \int_a^t (z, w(\cdot)) \quad \text{for each } z \in \mathcal{L}^2(\Omega)$$

and the result follows. □

Theorem 1

Suppose J is an interval and let $D = \Omega \times J$. Suppose also that v is defined on D and $\partial_t v(\cdot,t) \in \mathcal{L}^2(\Omega)$ for each $t \in J$. Let $y(t) = v(\cdot,t)$ for each t, then $y'(t) \in \mathcal{L}^2(\Omega)$ and for each t and

$$y'(t) = \partial_t v(\cdot,t) \quad \text{a.e. in } \Omega.$$

Proof Let $u = \partial_t v$, then

$$v(x,t) = \int_a^t u(x,\tau)\, d\tau \quad \text{for each } (x,t) \in D.$$

If $w(t) = u(\cdot,t)$, then $y(t) = \int_a^t w$ by the proposition. But then we may apply the fundamental theorem (Section 3.1) and it follows that $y'(t) = w(t)$. □

Corollary

Suppose that v is defined on D and $\partial_t^2 v$ exists in D. Let $y(t) = v(\cdot, t)$ for each t, then $y''(t) \in \mathcal{L}^2(\Omega)$ and for each t,

$$y''(t) = \partial_t^2 v(\cdot, t) \quad \text{a.e. in } \Omega.$$

Assumptions for the set Ω

If $n = 1$, then $\Omega = (0, \ell)$ and if $n = 2$ or 3 then Ω is convex.

Notation

Suppose J is an interval and let $D = \Omega \times J$ and $\widetilde{D} = \overline{\Omega} \times J$.

In the next theorem we use Sobolev's lemma (see Theorem 2.3.3). We also use the usual maximum norm $\| \cdot \|_\infty$ for vectors in $\mathbb{R}^n$.

Theorem 2

Suppose $u \in C^m(J; H^k(\Omega)$ where m and k are integers, $m \geq 0$ and $k \geq 1$. For the case $n = 2$ or 3, assume that $k \geq 2$ and Ω is convex. Then

(a) there exists a unique function $\tilde{u} \in C(\widetilde{D})$ such that

$$\|\tilde{u}(\cdot, t) - u(t)\|_k = 0;$$

(b) for $m \geq 1$, $\partial_t^m \tilde{u}$ exists in $\widetilde{D}$ and for each t,

$$\|\partial_t^m \tilde{u}(\cdot, t) - w^{(m)}(t)\|_k = 0.$$

Proof (a) By Sobolev's lemma there exists a unique $\bar{u}(t) \in C(\overline{\Omega})$ for each t such that $\|\bar{u}(t) - u(t)\|_k = 0$. Note that $\bar{u} \in C(J; C(\overline{\Omega}))$. Consequently $\bar{u}(t)(x)$ is uniquely determined for each $(x, t) \in \widetilde{D}$. Let $\tilde{u}(x, t) = \bar{u}(t)(x)$ for each $(x, t) \in \widetilde{D}$, then $\tilde{u}$ is the desired function. We must prove that it is continuous.

Consider $(x, t) \in \widetilde{D}$. For any $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$\|\bar{u}(s) - \bar{u}(t)\|_{\text{sup}} < \epsilon \quad \text{for} \quad |s - t| < \delta_1.$$

Since $\bar{u}(t) \in C(\overline{\Omega})$ there exists a $\delta_2 > 0$ such that

$$|\bar{u}(t)(x) - \bar{u}(t)(y)| < \epsilon \quad \text{for} \quad \|x - y\|_\infty < \delta_2.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, then

$$|\tilde{u}(t,x) - \tilde{u}(s,y)| < 2\epsilon$$

for any pair (y,s) such that $\|(x,t)-(y,s)\|_\infty < \delta$.

(b) By Sobolev's lemma

$$\lim_{h\to 0} \|h^{-1}[u(t+h)-u(t)] - u'(t)\|_{\text{sup}} = 0.$$

For each $x \in [0,\ell]$

$$|h^{-1}[\tilde{u}(x,t+h) - \tilde{u}(x,t)] - u'(t)(x)| \to 0 \quad \text{as} \quad h \to 0.$$

□

Chapter 4

Existence

This chapter is devoted entirely to the article [VV]. The proofs are given in greater detail to render them more readable. Additional propositions are also formulated for this purpose.

4.1 General linear vibration problems

In this section we consider the general linear vibration problem. Let W and V denote vector spaces with the following properties:

W is a Hilbert space with inner product c and norm $\|\cdot\|_W$;

V is a Hilbert space with inner product b and norm $\|\cdot\|_V$;

V is a subspace of W.

Let J be a bounded or unbounded interval. J is either an open interval containing zero or it is of the form $[0,\tau)$ or $[0,\infty)$.

For any Banach space Y the spaces $C^k\big(J;Y\big)$ are defined in Chapter 3. Recall that the implication of $u'(t) \in Y$ is that the derivative exists with respect to the norm of Y, as discussed in Section 3.1. Consider a bilinear form a defined on V and the following general problem:

Problem G

Given a function $f : J \to W$ find a function $u \in C(J; V)$ such that u' is continuous at 0 and for each $t \in J$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W,$$

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = c\big(f(t), v\big) \quad \text{for each } v \in V$$

$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Recall that a **bilinear form** ϕ is called **bounded** on an inner product space Y if there exists a constant K such that for any u and v in Y

$$|\phi(u, v)| \leq K\|u\|_Y\|v\|_Y.$$

Assumptions The following assumptions are made for the existence results of this chapter.

A1 V is dense in W,

A2 There exists a constant β such that $\|v\|_W \leq \beta\|v\|_V$ for each $v \in V$,

A3 The bilinear form a is nonnegative, symmetric and bounded on V.

The following theorems are the main results of this chapter and will be used in subsequent chapters to prove the existence results for different types of problems. The proofs of these theorems are the aim of the current chapter.

Theorem 1

Suppose Assumptions **A1**, **A2** and **A3** hold. If for $u_0 \in V$ and $u_1 \in V$, there exists some $y \in W$ such that

$$b\big(u_0, v\big) + a\big(u_1, v\big) = c\big(y, v\big) \quad \text{for each } v \in V \tag{4.1.1}$$

then for each $f \in C^1\big([0, \tau); W\big)$ there exists a unique solution

$$u \in C\big([0, \tau); V\big) \cap C^1\big([0, \tau); W\big) \cap C^1\big((0, \tau); V\big) \cap C^2\big((0, \tau); W\big)$$

for Problem G. If $f = 0$ then $u \in C^1\big([0, \infty); V\big) \cap C^2\big([0, \infty); W\big)$.

Proof See Section 4.3.

Definition

$E_b = \{\, x \in V \,\big|\, \text{there exists a } y \in W \text{ such that } c(y,v) = b(x,v) \text{ for all } v \in V \,\}.$

Definition Weak damping

In general the bilinear form a is only defined on V. When a is defined on W and bounded with respect to the norm of W, it is referred to as weak damping.

Theorem 2 Weak damping

Suppose Assumptions **A1**, **A2** and **A3** hold and a is bounded with respect to the norm in W. Let J be an interval containing zero, then there exists a unique solution

$$u \in C^1\big(J;V\big) \cap C^2\big(J;W\big)$$

for Problem G for each $u_0 \in E_b$, $u_1 \in V$ and each $f \in C^1\big(J;W\big)$. If $f = 0$ then $u \in C^1\big((-\infty,\infty);V\big) \cap C^2\big((-\infty,\infty);W\big)$.

Proof See Section 4.4.

Remark Theorem 2 is still applicable if the bilinear form a is 0.

Recall that a is called positive definite on V if $a(u,u) \geq K\|u\|_V^2$ for any $u \in V$.

Definition Strong damping

When a is positive definite on V the damping is referred to as strong damping.

Theorem 3 Strong damping

Suppose Assumptions **A1**, **A2** and **A3** hold and a is positive definite on V. Let $f : [0,\tau] \to W$ be Hölder continuous. Then there exists a unique solution

$$u \in C\big([0,\tau);V\big) \cap C^1\big([0,\tau);W\big) \cap C^2\big((0,\tau);W\big)$$

for Problem G for any $u_0 \in V$, $u_1 \in W$. If $f = 0$ then

$$u \in C\big([0,\infty);V\big) \cap C^1\big([0,\infty);W\big) \cap C^\infty\big((0,\infty);V\big).$$

Proof See Section 4.5.

4.2 Abstract differential equation

The aim is to write Problem G as an initial value problem for a first order differential equation in a Hilbert space. To motivate the method, Problem G is written as a first order system:

$$\begin{aligned} u'(t) &= w(t) \quad \text{and} \\ c(w'(t),v) + a(w(t),v) + b(u(t),v) &= c(f(t),v) \quad \text{for each } v \in V. \end{aligned}$$

As we will show, this system can then be written as a first order abstract differential equation of the form (3.3.1)-(3.3.2). Let $H = V \times W$ with inner product

$$(x,y)_H = b(x_1,y_1) + c(x_2,y_2)$$

for all x and $y \in H$. For $x \in H$ denote x and its components by $x = \langle x_1, x_2 \rangle$.

The idea is to define a linear operator with certain properties that will enable us to apply the existence theory in Chapter 3 to this abstract differential equation.

Proposition 1

Suppose $\lambda \geq 0$. For each $y \in H$ there exists a unique $x \in H$ such that

$$\begin{aligned} \lambda x_1 - x_2 &= y_1 \quad \text{and} \\ b(x_1,v) + a(x_2,v) + \lambda c(x_2,v) &= c(y_2,v) \quad \text{for each } v \in V. \end{aligned}$$

Proof Let $v \in V$ be arbitrary. Consider the problem of solving for x_1 in

$$b(x_1,v) + \lambda a(x_1,v) + \lambda^2 c(x_1,v) = a(y_1,v) + c(\lambda y_1 + y_2, v). \tag{4.2.1}$$

Set $F(v) = a(y_1,v) + c(\lambda y_1 + y_2, v)$ for each $v \in V$. Since $\|v\|_W < \beta \|v\|_V$ for any $v \in V$ and a is a bounded bilinear form on V

$$\begin{aligned} |F(v)| &\leq |\, a(y_1,v)| + |\, c(\lambda y_1 + y_2, v)| \\ &\leq K\|y_1\|_V\|v\|_V + \big(\lambda\|y_1\|_W + \|y_2\|_W\big)\|v\|_W \\ &\leq K\|y_1\|_V\|v\|_V + \lambda\beta^2\|y_1\|_V\|v\|_V + \beta\|y_2\|_W\|v\|_V \\ &\leq C\big(\|y_1\|_V + \|y_2\|_W\big)\|v\|_V \end{aligned}$$

where $C = \max\{K + \lambda\beta^2; \beta\}$. So F is a bounded linear functional on V.

Since $b + \lambda a + \lambda^2 c$ defines an inner product on V, it follows from Riesz's Theorem that there exist a unique $x_1 \in V$ such that (4.2.1) is satisfied.

Let $x_2 = \lambda x_1 - y_1$ and the result is proved. $\square$

Remark $x_2 \in V$ in Proposition 1.

Note that $b + \lambda a + \lambda^2 c$ defines an inner product on V even if $a = 0$ hence the result remains valid in the undamped case.

Definition The mapping Λ

For $\lambda = 0$, the mapping Λ is defined on H by $\Lambda y = -x$ where $-x_2 = y_1$ and $x_1 \in V$ such that

$$b(x_1, v) + a(x_2, v) = c(y_2, v) \quad \text{for each } v \in V.$$

Remark If x is in the range of Λ, then x_1 and x_2 are in V.

Proposition 2

Λ is a bounded linear operator.

Proof (Linearity) Let α be any nonzero real number and let v be any member of V. Consider $-x = \Lambda(\alpha y)$. The definition of the operator Λ implies that

$$-x_2 = \alpha y_1 \quad \text{and} \quad b(x_1, v) = c(\alpha y_2, v) - a(x_2, v).$$

Also consider $\Lambda(y) = -w$. From the definition of the operator we have that

$$-w_2 = y_1 \quad \text{and} \quad b(w_1, v) = c(y_2, v) - a(w_2, v).$$

Since $x_2 = \alpha w_2$, $b(\alpha w_1, v) = \alpha c(y_2, v) - a(x_2, v) = b(x_1, v)$ for each $v \in V$. Since b is an inner product defined on V, this shows that $w_1 = x_1$. Consequently

$$\Lambda(\alpha y) = x = \alpha w = \alpha \Lambda(y).$$

Now consider $-x = \Lambda(y + z)$. By definition $-x_2 = y_1 + z_1$ and

$$b(x_1, v) = c(y_2 + z_2, v) - a(x_2, v) \quad \text{for any} \quad v \in V.$$

Also consider $-\tilde{x} = \Lambda(y)$ and $-\tilde{w} = \Lambda(z)$. By definition $-\tilde{x}_2 - \tilde{w}_2 = y_1 + z_1$ and

$$b(\tilde{x}_1 + \tilde{w}_1, v) = c(y_2 + z_2, v) - a(\tilde{x}_2 + \tilde{w}_2, v)$$

for any $v \in V$. It is clear that $-x_2 = -\tilde{x}_2 - \tilde{w}_2$. Since $b(\tilde{x}_1 + \tilde{w}_1, v) = b(x_1, v)$ and b is an inner product on V we find that $x_1 = \tilde{x}_1 + \tilde{w}_1$.

This shows that $\Lambda(y + z) = \Lambda(y) + \Lambda(z)$. □

Proof (Boundedness) We use the following facts: the inner products for V and W are b and c and a is bounded on V.

Consider any $y \in H$ and suppose $\Lambda y = -x$. By the definition of the operator Λ

$$\begin{aligned} -x_2 &= y_1 \quad \text{and} && (4.2.2) \\ b(x_1, v) + a(x_2, v) &= c(y_2, v) \quad \text{for each } v \in V. && (4.2.3) \end{aligned}$$

By Assumption **A2**, $\|x_2\|_W \leq \beta \|x_2\|_V$ and from (4.2.2)

$$\|x_2\|_W \leq \beta \|y_1\|_V. \tag{4.2.4}$$

It follows from (4.2.3) that

$$\begin{aligned} \|x_1\|_V^2 &= b(x_1, x_1) \\ &\leq |\, a(x_2, x_1)| + |\, c(y_2, x_1)| \\ &\leq |\, a(x_2, x_1)| + \beta \|y_2\|_W \, \|x_1\|_V \\ &\leq K \|x_2\|_V \, \|x_1\|_V + \beta \|y_2\|_W \, \|x_1\|_V \\ &= K \|y_1\|_V \, \|x_1\|_V + \beta \|y_2\|_W \, \|x_1\|_V. \end{aligned}$$

Consequently

$$\|x_1\|_V \leq K\big(\|y_1\|_V + \beta \|y_2\|_W\,\big). \tag{4.2.5}$$

Combine (4.2.4) and (4.2.5) and apply the inequality $2\alpha\eta \leq \alpha^2 + \eta^2$. It follows that there exists a constant C such that

$$\|x_1\|_V^2 + \|x_2\|_W^2 \leq C\big(\|y_1\|_V^2 + \|y_2\|_W^2\big) = C\|y\|_H^2.$$

We have that

$$\|\Lambda y\|_H^2 = \|x\|_H^2 = \|x_1\|_V^2 + \|x_2\|_W^2 \leq C\|y\|_H^2$$

and it follows that Λ is bounded. □

Proposition 3

The nullspace of Λ is trivial.

Proof Assume that for some $y \in H$, $\Lambda y = 0$. From the definition of Λ it follows that $y_1 = 0$ and

$$c(y_2, v) = 0 \quad \text{for each } v \in V. \tag{4.2.6}$$

Take any $w \in W$. Since V is dense in W, there exist a sequence $\{v_n\} \in V$ such that $\|v_n - w\|_W \to 0$ as $n \to \infty$.

Since c is an inner product on on W we have from the Schwartz inequality that

$$|\ c(y_2, w - v_n)| \leq \|y_2\|_W \|w - v_n\|_W.$$

From the continuity of c it follows that

$$c(y_2, v_n) \to c(y_2, w) \quad \text{as} \quad n \to \infty.$$

It follows from (4.2.6) that $c(y_2, v_n) = 0$ for each v_n therefore

$$c(y_2, w) = 0 \ \text{ for any } w \in W.$$

For $y_2 \in W$, $c(y_2, y_2) = 0$ which shows that $y_2 = 0$. It follows that the null space of is trivial since y=0. ☐

Notation Denote the range of the operator Λ by $\mathcal{R}(\Lambda)$ and its closure by $\overline{\mathcal{R}(\Lambda)}$. Similarly the domain of the operator is denoted by $\mathcal{D}(\Lambda)$.

Proposition 4

$\mathcal{R}(\Lambda)$ is dense in H.

Proof Suppose that $\mathcal{R}(\Lambda)$ is not dense in H. Let $x \in H$ but not in $Y = \overline{\mathcal{R}(\Lambda)}$. Then there exists a $y \in Y$, $y \neq x$ such that $x - y$ is orthogonal to Y, since Y is a closed subspace of H (see [Kr, Lemma 3.3-2, p 145]).

Let $w = x - y$ then $w \neq 0$ and $(w, z)_H = 0$ for all $z \in Y$, i.e.

$$(w, z)_H = 0 \quad \text{for each } z \in \overline{\mathcal{R}(\Lambda)}. \tag{4.2.7}$$

This will also be true for any $z \in \mathcal{R}(\Lambda)$. Consider any $y \in H$. Since $\Lambda y = x \in \mathcal{R}(\Lambda)$,

$$b(w_1, x_1) + c(w_2, x_2) = (w, x)_H = 0. \tag{4.2.8}$$

Now suppose $y = \langle w_1, w_1 + w_2 \rangle$. Then, by the definition of Λ, $x_2 = -w_1$ and

$$b(x_1, v) + a(x_2, v) = c(w_1 + w_2, v) \quad \text{for each } v \in V.$$

Rewriting this equation we find that for any $v \in V$

$$b(x_1, v) - c(w_2, v) = c(w_1, v) - a(x_2, v) \quad \text{for each } v \in V.$$

Since $w_1 = -x_2$ and $w_1 \in V$,

$$b(x_1, w_1) + c(w_2, x_2) = c(w_1, w_1) + a(w_1, w_1).$$

Using (4.2.8) and the fact that a is nonnegative, we have $c(w_1, w_1) = 0$. So w_1 must be zero.

But (4.2.8) holds for any $x \in \mathcal{R}(\Lambda)$. Therefore $c(w_2, y_1) = 0$ for any $y_1 \in V$. Since V is dense in W, $c(w_2, z) = 0$ for any $z \in W$. This shows that $w_2 = 0$.

But $w \neq 0$ by assumption and this contradiction proves the result. □

The operator A and its properties

The null space of Λ is trivial by Proposition 3, hence Λ is one-to-one.

Definition Operator A

Let $\mathcal{D}(A) = \mathcal{R}(\Lambda)$ and $A = \Lambda^{-1}$.

Lemma 1

A is a closed linear operator.

Proof Since the linear operator Λ is defined on all of H and Λ is bounded by Proposition 2 it follows from the definition of a closed operator that Λ is a closed operator. From the definition of a closed operator it is obvious that a linear operator is closed if and only if its inverse is closed. □

Lemma 2

The domain of A is dense in H.

Proof Since $\mathcal{D}(A) = \mathcal{R}(\Lambda)$ and $\mathcal{R}(\Lambda)$ is dense in H from Proposition 4, the result follows. □

Lemma 3

For any $x \in \mathcal{D}(A)$, $Ax = y$ if and only if $x_2 = y_1$ and

$$b(x_1, v) + a(x_2, v) = -c(y_2, v) \quad \text{for each } v \in V.$$

Proof This follows from the definitions of the operators Λ and A. □

Corollary 1 $x \in \mathcal{D}(A)$ if and only if $x_2 \in V$ and there exists a $y \in W$ such that

$$b(x_1, v) + a(x_2, v) = c(y, v) \quad \text{for each } v \in V.$$

Lemma 4

For any $\lambda \geq 0$, $\mathcal{R}(\lambda - A) = H$.

Proof For any $\lambda \geq 0$ and any given $y \in H$ there exists a unique $x \in H$ such that $y_1 = \lambda x_1 - x_2$ and $b(x_1, v) + a(x_2, v) + \lambda c(x_2, v) = c(y_2, v)$ for any $v \in V$ from Proposition 1. Thus

$$b(x_1, v) + a(x_2, v) = c(y_2 - \lambda x_2, v) \quad \text{for each } v \in V$$

and $-x_2 = y_1 - \lambda x_1$. From the definition of Λ we find that $\Lambda(y - \lambda x) = -x$ which means that $(\lambda - A)x = y$. □

Lemma 5

$(Ax, y)_H = b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2)$ for each x and $y \in \mathcal{D}(A)$.

Proof Let $Ax = w$. For any $v \in V$ it follows from Lemma 3 that $x_2 = w_1$ and

$$c(w_2, v) = -b(x_1, v) - a(x_2, v).$$

By the definition of the inner product of H,

$$\begin{aligned} (Ax, y)_H &= b(w_1, y_1) + c(w_2, y_2) \\ &= b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2). \end{aligned}$$

□

Corollary 2 $(Ax, x)_H = -a(x_2, x_2)$ for $x \in \mathcal{D}(A)$.

Remark Corollary 2 implies that A is dissipative.

4.3 General existence result

We use the notation of Section 4.2. Consider the following initial value problem:

Problem ADE 1

Given a function $F : [0, \tau) \to H$, find $U \in C([0, \tau); H)$ such that for each $t \in (0, \tau)$, $U(t) \in \mathcal{D}(A)$, $U'(t) \in H$ and

$$\begin{aligned} U'(t) &= AU(t) + F(t), \\ U(0) &= U_0. \end{aligned}$$

The following results enable us to write Problem G as an initial value problem for a first order system.

Proposition 1

Suppose $J = [0, \tau)$ and $F(t) = \langle 0, f(t) \rangle$ for each $t \in J$. If

$$u \in C([0, \tau); V) \cap C^1([0, \tau); W)$$

is a solution of Problem G then $U = \langle u, u' \rangle$ is a solution of Problem ADE 1 with $U_0 = \langle u_0, u_1 \rangle$.

If

$$u \in C\big([0, \tau); V\big) \cap C^1\big([0, \tau); W\big) \cap C^1\big((0, \tau); V\big) \cap C^2\big((0, \tau); W\big)$$

then

$$U \in C\big([0, \tau); H\big) \cap C^1\big((0, \tau); H\big).$$

Proof Assume that u is a solution of Problem G, then

$$b(u(t), v) + a(u'(t), v) = c(f(t) - u''(t), v) \quad \text{for all } v \in V.$$

Set $U(t) = \langle u(t), u'(t) \rangle$. Since $f(t) - u''(t) \in W$ for any $t \in (0, \tau)$, Corollary 4.2.1 implies that $U(t) \in \mathcal{D}(A)$. Also $U'(t) = \langle u'(t), u''(t) \rangle \in H$.

Let $Y_1(t) = U_2(t)$ and $Y_2(t) = u''(t) - f(t)$. It follows that

$$b(U_1(t), v) + a(U_2(t), v) = -c(Y_2(t), v) \quad \text{for all } v \in V.$$

From Lemma 4.2.3 we have that $AU(t) = Y(t)$ and

$$\begin{aligned} AU(t) &= \langle U_1'(t), U_2'(t) - f(t) \rangle \\ &= \langle U_1'(t), U_2'(t) \rangle - \langle 0, f(t) \rangle \\ &= U'(t) - F(t). \end{aligned}$$

Since $u \in C\big([0,\tau);V\big)$ and $u' \in C([0,\tau);W)$ it follows that $U \in C([0,\tau);H)$.

Finally, $U(0) = \langle u(0), u'(0)\rangle = \langle u_0, u_1\rangle$. This shows that U is a solution of Problem ADE 1.

Moreover, if $u \in C^1\big((0,\tau);V\big)$ and $u' \in C^1\big((0,\tau);W\big)$ then $U \in C^1\big((0,\tau);H\big)$. □

Proposition 2

Suppose $J = [0,\tau)$ and $F(t) = \langle 0, f(t)\rangle$ for each $t \in J$. If U is a solution of Problem ADE 1 with $U(0) = \langle u_0, u_1\rangle$, then the first component $u = U_1$ of U is a solution of Problem G. If $U \in C^1\big((0,\tau);H\big)$ then

$$u \in C\big([0,\tau);V\big) \cap C^1\big([0,\tau);W\big) \cap C^1\big((0,\tau);V\big) \cap C^2\big((0,\tau);W\big).$$

Proof Assume that U is a solution of Problem ADE 1. This means that $U(t) \in \mathcal{D}(A)$ and $AU(t) = U'(t) - F(t)$ where $F(t) = \langle 0, f(t)\rangle$. It follows that

$$AU(t) = \langle U_1'(t), U_2'(t) - f(t)\rangle.$$

Lemma 4.2.3 implies that $U_2(t) = U_1'(t)$ and

$$b(U_1(t), v) + a(U_2(t), v) = -c(U_2'(t) - f(t), v) \quad \text{for all } v \in V.$$

Rewriting this equation we find that

$$c(U_2'(t), v) + a(U_2(t), v) + b(U_1(t), v) = c(f(t), v) \quad \text{for all } v \in V. \quad (4.3.1)$$

Let $u(t) = U_1(t)$ so $u'(t) = U_1'(t) = U_2(t)$ and $u''(t) = U_2'(t)$. For $t \in (0,\tau)$, $U(t) \in \mathcal{D}(A)$ so $u(t) \in V$. Since $U'(t) \in H$ it follows that $u'(t) \in V$ and that $u''(t) \in W$.

From (4.3.1) it follows that

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = c(f(t), v) \quad \text{for all } v \in V.$$

Since $U \in C\big([0,\tau);H\big)$ we find that $u \in C\big([0,\tau);V\big) \cap C^1\big([0,\tau);W\big)$.

The initial conditions are satisfied since

$$\langle u(0), u'(0)\rangle = U(0) = \langle u_0, u_1\rangle.$$

We conclude that the first component of U is indeed a solution of Problem G.

Moreover, if $U \in C^1\big((0,\tau);H\big)$ then $\langle u', u''\rangle = U' \in C\big((0,\tau);H\big)$. This implies that $u' \in C\big((0,\tau);V\big)$ and $u'' \in C\big((0,\tau);W\big)$. $\square$

Remark This result is true even if $a = 0$.

Lemma

A is the infinitesimal generator of of a C_0 semigroup of contractions in H.

Proof For $\lambda \geq 0$ we have that $\mathcal{R}(\lambda - A) = H$ from Lemma 4.2.4. Since a is nonnegative, Corollary 4.2.2 implies that $(Ax, x)_H \leq 0$ for all $x \in \mathcal{D}(A)$, so A is dissipative.

From Lemma 4.2.2 we find that $\mathcal{D}(A)$ is dense in H. We can apply Theorem 3.2.2 and it follows that A is the infinitesimal generator of of a C_0 semigroup of contractions in H. $\square$

Corollary

If $U_0 \in \mathcal{D}(A)$ and $F \in C^1([0,\tau);H)$ then Problem ADE 1 has a unique solution. If $F = 0$ then this solution is in $C^1([0,\infty);H)$.

Proof Apply Theorem 3.3.2 $\square$

Remark If $F \neq 0$ then $U \in C([0,\tau);H) \cap C^1((0,\tau);H)$ - see the definition of a solution in Section 3.3.

Proof of Theorem 4.1.1.

Note that $\langle u_0, u_1\rangle \in \mathcal{D}(A)$ by Corollary 4.2.1 and $f \in C^1([0,\tau);W)$ implies that $F = \langle 0, f\rangle \in C^1([0,\tau);H)$.

It follows from the corollary above that Problem ADE 1 has a unique solution U. From Proposition 2 the first component of U is a solution of Problem G and

$$u \in C\big([0,\tau);V\big) \cap C^1\big([0,\tau);W\big) \cap C^1\big((0,\tau);V\big) \cap C^2\big((0,\tau);W\big).$$

If $F = 0$ then $U \in C^1([0,\infty);H)$ which means that

$$u \in C^1([0,\infty);V) \cap C^2((0,\infty);W).$$

$\square$

4.4 Weak damping

In this section Theorem 4.1.2 is proved. The definition of weak damping is given in Section 4.1. All the results in this section are true for the case $a = 0$ (no damping).

Let J be an open interval containing 0 as in Section 3.4.

Problem ADE 2

Given a function $F : J \to H$, find $U \in C(J;H)$ such that for each $t \in J$, $U(t) \in \mathcal{D}(A)$, $U'(t) \in H$ and

$$\begin{aligned} U'(t) &= AU(t) + F(t), \\ U(0) &= U_0. \end{aligned}$$

We consider the connection between Problem G and Problem ADE 2.

Proposition 1

Suppose $F(t) = \langle 0, f(t) \rangle$ for each $t \in J$. If $u \in C(J;V) \cap C^1(J;W)$ is a solution of Problem G, then $U = \langle u, u' \rangle$ is a solution of Problem ADE 2 with $U_0 = \langle u_0, u_1 \rangle$.

If $u \in C^1\big(J;V\big) \cap C^2\big(J;W\big)$ then $U \in C^1\big(J;H\big)$.

Proof The proof is similar to the proof of Proposition 4.3.1.

Proposition 2

Suppose $F(t) = \langle 0, f(t) \rangle$ for each $t \in J$. If U is a solution of Problem ADE 2 with $U_0 = \langle u_0, u_1 \rangle$, then the first component $u = U_1$ of U is a solution of Problem G. If $U \in C^1\big(J;H\big)$ then $u \in C^1\big(J;V\big) \cap C^2\big(J;W\big)$.

Proof The proof is similar to the proof of Proposition 4.3.2.

Lemma 1

If a is nonnegative and bounded with respect to the norm of W, then A is the infinitesimal generator of a C_0 group.

Proof Since a is bounded on W, there exist a $K > 0$ such that

$$|\, a(u,u)| \leq K \|u\|_W^2 \quad \text{for each} \quad u \in W.$$

Let λ be any real number such that $|\lambda| > K$. For any $u \in V$

$$\lambda a(u,u) + \lambda^2 c(u,u) \geq |\lambda|\big(|\lambda| - K\big)c(u,u).$$

For $|\lambda| > K$, the form $b + \lambda a + \lambda^2 c$ defines an inner product on V. As in the proof of Lemma 4.2.4 it then follows that $\mathcal{R}\big(\lambda I - A\big) = H$ for all real λ with $|\lambda| > K$.

Next we prove that $(\lambda - A)^{-1}$ is bounded for $|\lambda| > K$. For any x in $\mathcal{D}(A)$

$$|\ (Ax,x)_H| = |\ a(x_2,x_2)| \leq K\|x_2\|_W^2 \leq K\|x\|_H^2.$$

Consequently

$$\begin{aligned} |\lambda|\ \|x\|_H^2 &= \big|\big((\lambda - A)x, x\big)_H + (Ax,x)_H\big| \\ &\leq \|(\lambda I - A)x\|_H\|x\|_H + K\|x\|_H^2. \end{aligned}$$

Therefore

$$\|(\lambda - A)x\|_H \geq \big(|\lambda| - K\big)\|x\|_H$$

which implies that

$$\|(\lambda - A)^{-1}y\|_H \leq \big(|\lambda| - K\big)^{-1}\|y\|_H \ \text{ for each } \ y \in H.$$

Hence for any real λ with $|\lambda| > K$ it follows that $(\lambda - A)^{-1} \in B(H)$. From the definition of the operator norm we have that

$$\|(\lambda - A)^{-1}\| \leq \big(|\lambda| - K\big)^{-1}.$$

It follows that

$$\|\big((\lambda I - A)^{-1}\big)^n\| \leq \big(|\lambda - K|\big)^{-n} \ \text{ for each } \ n \in \mathbb{N}.$$

From Lemmas 4.2.1 and 4.2.2 we have that $\mathcal{D}(A)$ is dense in H and A is a closed linear operator. The result follows from Theorem 3.4.1. ☐

Corollary If $U_0 \in \mathcal{D}(A)$ and $F \in C^1\big(J;H\big)$ then Problem ADE 2 has a unique solution. If $F = 0$ then this solution is in $C^1\big((-\infty,\infty);H\big)$.

Proof Apply Theorem 3.4.2. ☐

If a is nonnegative and bounded with respect to the norm of W then the domain of A is characterized by the following result:

Lemma 2

$\mathcal{D}(A) = E_b \times V$

Proof Let $x \in E_b \times V$. From the definition of E_b there exists an f such that $b(x_1, v) = c(f, v)$ for all v in V.

Consider $F(w) = -c(f, w) - a(x_2, w)$ for any $w \in W$.

$$\begin{aligned} |F(w)| &\leq | \, c(f,w)| + | \, a(x_2, w)| \\ &\leq \|f\|_W \|w\|_W + K\|x_2\|_W \|w\|_W \\ &= \big(\|f\|_W + K\|x_2\|_W\big)\|w\|_W \end{aligned}$$

so F is a bounded linear functional on W. By Riesz's theorem there exists a unique $y_2 \in W$ such that $-c(f, w) - a(x_2, w) = c(y_2, w) \quad$ for each $\quad w \in W$. But this equality will also hold for any $w \in V$ so

$$-b(x_1, w) - a(x_2, w) = c(y_2, w) \quad \text{for each} \quad w \in V.$$

By setting $y_1 = x_2$ it follows from Lemma 4.2.3 that $Ax = y$.

Now assume that $x \in \mathcal{D}(A)$ and that $Ax = y$. From Lemma 4.2.3 it follows that $x_2 = y_1$ (so $x_2 \in V$) and $b(x_1, v) + a(x_2, v) = -c(y_2, v)$ for all $v \in V$.

Let $F(w) = -a(x_2, w) - c(y_2, w)$ for all $w \in W$ then

$$|F(w)| \leq \big(K\|x_2\|_W + \|y_2\|_W\big)\|w\|_W$$

so F is a bounded linear functional on W. By Riesz's theorem there exists an $f \in W$ such that $c(f, w) = F(w)$ for any $w \in W$.

This equality will also hold for $w \in V$ so

$$\begin{aligned} c(f, w) &= -a(x_2, w) - c(y_2, w) \quad \text{for all} \quad w \in V \\ &= b(x_1, w) \end{aligned}$$

which shows that $x_1 \in E_b$. □

Proof of Theorem 4.1.2.

Since $u_0 \in E_b$ and $u_1 \in V$, Lemma 2 yields that $\langle u_0, u_1 \rangle \in D(A)$. For $F(t) = \langle 0, f(t) \rangle$, $f \in C^1(J; W)$ implies that $F \in C^1(J; H)$. By setting $U_0 = \langle u_0, u_1 \rangle$, the corollary implies that Problem ADE 2 has a unique solution. From Proposition 2 it follows that the first component $u = U_1$ of this solution is a solution of Problem G. If $f = 0$, then $F = 0$ and it follows from the corollary above that

$$u \in C^1\big((-\infty, \infty); V\big) \cap C^2\big((-\infty, \infty); W\big).$$

□

4.5 Strong damping

Consider Problem ADE 1. If Assumptions **A1**, **A2** and **A3** hold, we know that the operator A is the infinitesimal generator of a C_0 semigroup and Problem ADE 1 has a unique solution for $U_0 \in \mathcal{D}(A)$ and $F \in C^1([0,\tau), W)$. In this section we show that both these conditions may be relaxed if a is positive definite on V.

Assumption There exists a constant c_a such that $a(u,u) \geq c_a\|u\|_V^2$ for each $u \in V$.

Recall that uniqueness is established by Theorem 3.3.1.

A real Hilbert H space may be imbedded in a complex Hilbert space as discussed in Subsection 3.5.1. For convenience the main facts are restated

$$\widetilde{H} = \{x + iy \mid x \in H, y \in H\}.$$

The inner product for $\widetilde{H}$ is given by

$$\langle x + iy, u + iv\rangle_{\widetilde{H}} = \langle x, u\rangle_H + \langle y, v\rangle_H + i\langle y, u\rangle_H - i\langle x, v\rangle_H.$$

The linear operator A (in Problem ADE 1) may extend to $\widetilde{A}$:

$$\begin{aligned} D(\widetilde{A}) &= \{x + iy \mid x \in D(A), y \in D(A)\}, \\ \widetilde{A}(x + iy) &= Ax + iAy. \end{aligned}$$

Our aim is to show that the operator $T = \widetilde{A} - I$ is of Type (ϕ, M) (see Section 3.5).

Proposition 1

For any $w = x + iy \in D(\widetilde{A})$

$$\begin{aligned} Re(\widetilde{A}w, w)_{\widetilde{H}} &= (Ax, x)_H + (Ay, y)_H, \\ Im(\widetilde{A}w, w)_{\widetilde{H}} &= (Ay, x)_H - (Ax, y)_H. \end{aligned}$$

Proof

$$(\widetilde{A}(x + iy), x + iy)_{\widetilde{H}} = (Ax, x)_H + (Ay, y)_H + i(Ay, x)_H - i(Ax, y)_H.$$

$\square$

Proposition 2

There exists a constant c_a such that for any $w = x + iy \in D(\widetilde{A})$ we have

$$Re(\widetilde{A}w, w)_{\widetilde{H}} \leq -c_a(\|x_2\|_V^2 + \|y_2\|_V^2).$$

Proof

$$Re(\widetilde{A}w, w)_{\widetilde{H}} = (Ax, x)_H + (Ay, y)_H$$

Using Corollary 4.2.2

$$Re(\widetilde{A}w, w)_{\widetilde{H}} = -a(x_2, x_2) - a(y_2, y_2).$$

Now use the assumption $a(u, u) \geq c_a\|u\|_V^2$. □

Proposition 3

For any $w = x + iy \in D(\widetilde{A})$ we have

$$|Im(\widetilde{A}w, w)|_{\widetilde{H}} \leq \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2.$$

Proof

$$\begin{aligned} Im(\widetilde{A}w, w)_{\widetilde{H}} &= (Ay, x)_H - (Ax, y)_H \\ &= b(y_2, x_1) - b(y_1, x_2) - a(x_2, y_2) \\ &- (b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2)) \\ &= 2b(x_1, y_2) - 2b(x_2, y_1), \end{aligned}$$

by Lemma 5. Now

$$2b(u, v) \leq 2\|u\|_V\|v\|_V \leq \|u\|_V^2 + \|v\|_V^2$$

for any u and v in V and we have the desired result. □

Proposition 4

There exists a constant K such that

$$K\ Re(Tw, w)_{\widetilde{H}} + |Im(Tw, w)|_{\widetilde{H}} \leq 0 \quad \text{for any } w \in D(\widetilde{A}).$$

Proof For any $w = x + iy \in D(\widetilde{A})$

$$\begin{aligned} Re(\widetilde{A}w, w)_{\widetilde{H}} &\leq -c_a(\|x_2\|_V^2 + \|y_2\|_V^2) \quad \text{and} \\ |Im(\widetilde{A}w, w)_{\widetilde{H}}| &\leq \|x_1\|_V^2 + \|x_2\|_V^2 + \|y_1\|_V^2 + \|y_2\|_V^2. \end{aligned}$$

Since $(Tw, w)_{\widetilde{H}} = (\widetilde{A}w, w)_{\widetilde{H}} - \|w\|^2_{\widetilde{H}}$, we have

$$\begin{aligned} Re(Tw, w)_{\widetilde{H}} &\leq -c_a(\|x_2\|^2_V + \|y_2\|^2_V) \\ &\quad -(\|x_1\|^2_V + \|x_2\|^2_W + \|y_1\|^2_V + \|y_2\|^2_W) \\ &\leq -c_a(\|x_2\|^2_V + \|y_2\|^2_V) \\ &\quad -(\|x_1\|^2_V + \|y_1\|^2_V) \ \text{ and} \end{aligned}$$

$$|Im(Tw, w)|_{\widetilde{H}} \leq \|x_1\|^2_V + \|x_2\|^2_V + \|y_1\|^2_V + \|y_2\|^2_V.$$

Then, for any $K > 0$

$$\begin{aligned} &K \ Re(Tw, w)_{\widetilde{H}} + |Im(Tw, w)|_{\widetilde{H}} \\ \leq\ &(1 - Kc_a)(\|x_2\|^2_V + \|y_2\|^2_V) + (1 - K)(\|x_1\|^2_V + K\|y_1\|^2_V). \end{aligned}$$

If we choose $K > \max\{1, c_a^{-1}\}$, we have the desired result. □

Proposition 5

The operator T is the infinitesimal generator of an analytic semigroup.

Proof $\mathcal{D}(A)$ is dense in H (Lemma 4.2.2) and it follows that $\mathcal{D}(\widetilde{A})$ is dense in $\widetilde{H}$. Therefore $\mathcal{D}(T)$ is dense in $\widetilde{H}$.

The range $\mathcal{R}(\lambda - A) = H$ for $\lambda \geq 0$ (Lemma 4.2.4) and it follows that $\mathcal{R}(\lambda - \widetilde{A}) = \widetilde{H}$. Therefore $\mathcal{R}(T) = \widetilde{\widetilde{H}}$.

Next we have

$$Re(\widetilde{A}w, w)_{\widetilde{H}} \leq 0 \ \text{ for any } w \in D(\widetilde{A}),$$

hence

$$Re(Tw, w)_{\widetilde{H}} \geq \|w\|^2_{\widetilde{H}} \ \text{ for any } w \in D(T).$$

Using the Cauchy-Schwartz inequality it follows that T^{-1} is bounded and hence $0 \in \rho(T)$.

By Proposition 4 the operator T satisfies all the conditions of Theorem 3.5.4 and it follows that T is the infinitesimal generator of an analytic semigroup.

□

Theorem 1

The operator $\widetilde{A}$ is the infinitesimal generator of an analytic semigroup.

Proof The result is a direct consequence of Proposition 3.5.2. □

Theorem 2

If $F : [0,\tau) \to H$ is locally Hölder continuous on $(0,\tau)$, then Problem ADE 1 has a unique solution U for each $U_0 \in H$. If $F = 0$, then $U \in C^\infty\big((0,\infty); H\big)$.

Proof The initial value problem $Y' = \tilde{A}Y + F$ with $Y(0) = U_0$ has a unique solution and $Y \in C^\infty\big((0,\infty); \tilde{H}\big)$ if $F = 0$ by Theorem 3.5.4. But $Y(t) = U(t) + iW(t)$ with $U(t) \in H$. Since $F(t) \in H$ for each $t \in [0,\tau)$ we have $U' = AU + F$. Finally $Y(0) = U_0 \in H$ and it follows that $U(0) = U_0$.

□

Proof of Theorem 4.1.3 If $u_0 \in V$ and $u_1 \in W$, then $\langle u_0, u_1\rangle \in H$ and F is locally Hölder continuous with respect to the norm $\|\cdot\|_H$ if f is locally Hölder continuous with respect to $\|\cdot\|_W$. Therefore Problem ADE 1 has a unique solution U. By Proposition 4.3.2, the first component $u = U_1$ of U is a solution of Problem G. Proposition 4.3.2 also yields the desired properties for U. □

4.6 Energy

In this section we generalize the energy inequality first derived in Subsection 1.2.5. We consider solutions of the general variational equation in Problem G Section 4.1 and use the same notation. First we derive a general product rule.

Proposition

Suppose J is an interval and u and v are function defined on J with values in a Hilbert spaces H. Let β be a bounded bilinear form on H and set $\phi(t) = \beta(u(t), v(t))$ for each $t \in J$. If u and v are differentiable in H, then ϕ is differentiable and

$$\phi'(t) = \beta(u'(t), v(t)) + \beta(u(t), v'(t)).$$

Proof

$$\begin{aligned} & \beta(u(t+h), v(t+h)) - \beta(u(t), v(t)) \\ = \; & \beta(u(t+h), v(t+h)) - \beta(u(t), v(t+h)) + \beta(u(t), v(t+h)) - \beta(u(t), v(t)). \end{aligned}$$

Now

$$\begin{aligned}
& |h^{-1}[\beta(u(t+h),v(t+h)) - \beta(u(t),v(t+h))] - \beta(u'(t),v(t))| \\
= \; & |\beta(h^{-1}[(u(t+h) - u(t)] - u'(t), v(t+h)) \\
& +\beta(u'(t), v(t+h) - v(t))| \\
\leq \; & |\beta(h^{-1}[(u(t+h) - u(t)] - u'(t), v(t))| \\
& +|\beta(u'(t), v(t+h) - v(t))| \\
= \; & \|h^{-1}[(u(t+h) - u(t)] - u'(t)\| \; \|v(t+h)\| \\
& +\|u'(t)\| \; \|v(t+h) - v(t)\| \to 0 \text{ as } h \to 0.
\end{aligned}$$

Therefore

$$\lim_{h\to 0} \; h^{-1}[\beta(u(t+h),v(t+h)) - \beta(u(t),v(t+h)) = \beta(u'(t),v(t)).$$

By the same reasoning

$$\lim_{h\to 0} \; h^{-1}[\beta(u(t),v(t+h)) - \beta(u(t),v(t)) = \beta(u(t),v'(t))$$

and the proof is complete. □

Corollary

If u' and v' are differentiable in H, and $\phi(t) = \beta(u'(t),v'(t))$ for each $t \in J$, then

$$\phi'(t) = \beta(u''(t),v'(t)) + \beta(u'(t),v''(t)).$$

In the theorem below we use the notation of Section 4.1.

Theorem

Suppose Assumptions **A2** and **A3** hold and for each $t \in J$

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = 0$$

for each $v \in V$. If

$$E(t) = \frac{1}{2}\, c(u'(t),u'(t)) + \frac{1}{2}\, b(u(t),u(t)).$$

for each $t \in J$, then $E' \leq 0$.

Proof

$$E'(t) = c(u''(t),u'(t)) + b(u(t),u'(t)) = -a(u'(t),u'(t)) \leq 0.$$

□

Corollary

Problem G has at most one solution.

Proof The proof is the same as in Subsection 1.2.5.

Chapter 5

Routine applications

The existence theory for the general linear vibration problem is presented in Chapter 4. In this chapter we apply the theory to the model problems considered in Chapter 1. As mentioned in Subsection 1.6.2, the theory cannot be applied to the variational form but it can be applied to the so called weak variational form. In Section 5.1 we introduce the weak variational form.

5.1 Weak variational form

This section serves as an introduction to Chapter 5. Consider as an example Problem 1V in Section 1.2. We need to find a function u with $u(\cdot,t)$ a test function such that

$$c(\partial_t^2 u(\cdot,t),\phi) + a(\partial_t u(\cdot,t),\phi) + b(u(\cdot,t),\phi) = 0 \tag{5.1.1}$$

for each test function ϕ. The initial conditions are $u(\cdot,0) = u_0$ and $\partial_t u(\cdot,0) = u_1$. The bilinear forms a, b and c are defined as

$$c(u,v) = \int_0^\ell uv, \quad a(u,v) = \gamma \int_0^\ell uv \ \text{ and } \ b(u,v) = \alpha \int_0^\ell u'v'.$$

(Equation (5.1.1) is the same as Equation (1.2.7).)

It is important to note that a and c are defined for functions in $\mathcal{L}^2(0,\ell)$ (Section 2.1) and b is defined for functions $u \in H^1(0,\ell)$ (Section 2.2). But the partial derivatives $\partial_t u$ and $\partial_t^2 u$ do not make sense when a function may be changed arbitrarily on a set of measure zero, as mentioned in Section 1.6.

If we define a function w by $w(t) = u(\cdot, t)$, then $w'(t)$ may be defined with respect to the norm of $\mathcal{L}^2(0, \ell)$. (Recall the definition of a derivative of a function with the respect to the norm of some space Y as discussed in Section 3.1.) If $\partial_t u(\cdot, t) \in \mathcal{L}^2(0, \ell)$ and $\partial_t^2 u(\cdot, t) \in \mathcal{L}^2(0, \ell)$, we may apply Theorem 3.6.1. and find that $w'(t) = \partial_t u(\cdot, t)$ and $w''(t) = \partial_t^2 u(\cdot, t)$. Equation (5.1.1) may now be rewritten:

$$c(w''(t), \phi) + a(w'(t), \phi) + b(w(t), \phi) = 0 \tag{5.1.2}$$

for each test function ϕ.

Let $V(0, \ell)$ be the closure of $T[0, \ell]$ in $H^1(0, \ell)$. Two Hilbert spaces V and W are used to formulate Problem G in Section 4.1. To apply the theory we set $W = \mathcal{L}^2(0, \ell)$ and $V = V(0, \ell)$. We need to show that Equation 5.1.2 holds for all $\phi \in V(0, \ell)$.

Recall that the inner product for $\mathcal{L}^2(0, \ell)$ is denoted by $(\cdot, \cdot)$ and the norm by $\| \cdot \|$. The norm of $H^1(0, \ell)$ is denoted by $\| \cdot \|_1$. We have $\|u\| \leq \|u\|_1$ and $\|u'\| \leq \|u\|_1$ for $u \in H^1(0, \ell)$ (see Section 2.2).

Next we consider the boundedness of the bilinear forms a, b and c. Using the Cauchy-Schwarts inequality

$$|c(u, v)| = |(u, v)| \leq \|u\| \|v\| \leq \|u\|_1 \|v\|_1.$$

Next

$$|a(u, v)| = \gamma |(u, v)| \leq \gamma \|u\|_1 \|v\|_1$$

and

$$|b(u, v)| = \alpha |(u', v')| \leq \alpha \|u\|_1 \|v\|_1.$$

For any $\phi \in V(0, \ell)$ there exists a sequence $\{\phi_n\} \subset T[0, \ell]$ such that $\|\phi_n - \phi\|_1 \to 0$ as $n \to \infty$. As a consequence we have for any $x \in H^1(0, \ell)$:

$$b\big(x, \phi_n\big) \to b\big(x, \phi\big), \quad a\big(x, \phi_n\big) \to a\big(x, \phi\big)$$
$$\text{and} \quad c\big(x, \phi_n\big) \to c\big(x, \phi\big) \quad \text{as} \quad n \to \infty.$$

To see this, consider for example

$$|c(x, \phi) - c(x, \phi_n)| = |c(x, \phi - \phi_n)| \leq \|x\|_1 \|\phi - \phi_n\|_1.$$

It is now possible to formulate the weak variational form of Problem 1V.

Problem 1W Find a function w with $w(t) \in V(0,\ell)$ such that

$$c(w''(t),\phi) + a(w'(t),\phi) + b(w(t),\phi) = 0$$

for each $\phi \in V$. The initial conditions are $w(0) = u_0$ and $w'(0) = u_1$.

Consider the equivalence of Problem 1V and Problem 1W. We have shown that a solution of Problem 1V is a solution of Problem 1W, but due to the assumptions in Theorem 3.6.2 it is not always true that a solution of Problem 1W is a solution of Problem 1V. The weak solution must satisfy so called regularity conditions and as mentioned before, this is beyond the scope of this dissertation.

Consider necessary properties for w to be a possible solution. It is obviously necessary that $w''(t) \in \mathcal{L}^2(0,\ell)$. Since **initial conditions** are prescribed for w and w' it is required that w and w' be continuous at zero - at least with respect to the norm of $\mathcal{L}^2(0,\ell)$.

What about the **boundary conditions**? It is shown in Section 1.2 that the natural boundary condition $\partial_x u(\ell,t) = 0$ is automatically satisfied. The other boundary condition is forced by requiring that $u(\cdot,t) \in T[0,\ell]$ for each t. An appropriate condition in the Sobolev space $H^1(0,\ell)$ is required. Recall that $V(0,\ell)$ is the closure of $T[0,\ell]$ in $H^1(0,\ell)$. It is possible to define a trace operator Γ_0 such that $\Gamma_0 u = u(0)$ when u is continuous. (See the remark in Section 2.5). For $u \in H^1(0,\ell)$

$$\Gamma_0 u = \lim_{n\to\infty} u_n(0) \quad \text{where} \quad u_n \in C^1[0,\ell].$$

If $u \in V(0,\ell)$, then $u_n \in T[0,\ell]$ for each n and hence $\Gamma_0 u = 0$. If $w(t) \in V(0,\ell)$, then $w(t)(0) = 0$ in the sense that $\Gamma_0 w(t) = 0$.

We used Problem 1V to show how the weak variational form is derived. In the rest of this chapter we consider only weak variational forms of problems and the existence of weak solutions. The connection between the variational form and the weak variational form is similar for all the problems.

The finite element method

As mentioned in the introduction, solutions of partial differential equations are in general approximated using a numerical method. If the approximation converges, it does not mean that it converges to a classical solution of the partial differential equation. Consider the application of the finite element method to Problem 1V (or rather Problem 1W). A finite dimensional subspace S^h of $V(0,\ell)$ is constructed. Then the problem is to find a function u_h

with $u_h(t) \in S^h$ such that

$$c(u_h''(t), v) + a(u_h''(t), v) + b(u_h(t), v) = 0 \tag{5.1.3}$$

for each $v \in S^h$. If the function u_h converges, it converges with respect to the so called energy norm (see the next section) and the convergence is dependent on the initial conditions. In this particular case the energy norm is equivalent to the norm of $H^1(0, \ell)$. The reader is referred to [SF].

5.2 The wave equation

5.2.1 Application of the existence theory

Consider Problem 1W as discussed in Section 5.1. To apply the theorems in Section 4.1 we need to define the spaces V and W and verify that Assumptions **A1**, **A2** and **A3** are valid. Let W be the set $\mathcal{L}^2(0, \ell)$ (as in Section 5.1) and consider the bilinear form c. Note that

$$c(u, v) = \int_0^\ell uv,$$

the usual inner product for $\mathcal{L}^2(0, \ell)$.

The space of test functions $T[0, \ell]$ is defined in Section 1.2. by

$$T[0, \ell] = \{\phi \in C_*^1[0, \ell] \; \big| \phi(0) = 0 \}.$$

Recall the definitions of $V(0, \ell)$ and the bilinear form b in the previous section. In this application, define $V = V(0, \ell)$. It will be shown that b is an inner product for V.

Proposition 1

There exists a constant β such that $\|v\|^2 \leq \beta^2 \, b(v, v)$ for each $v \in V$.

Proof For any v in V we have $c(v, v) = (v, v) = \|v\|^2$ and

$$b(v, v) = \alpha \int_0^\ell (v')^2 = \alpha \|v'\|^2$$

for $\alpha > 0$. From the lemma in Section 2.4 it follows that

$$\alpha \|v\|^2 \leq \alpha \ell^2 \|v'\|^2 = \ell^2 b(v, v).$$

So $\beta^2 = \ell^2/\alpha$ and the result is proved. □

Proposition 2

The bilinear form b is an inner product on V.

Proof The fact that the bilinear form b is symmetric follows directly from the definition of b. From Proposition 1 we have the estimate

$$\beta\, b(v,v) \geq \|v\|^2 \geq 0$$

so it is clear that b is nonnegative. If $b(v,v) = 0$ then $\|v\|^2 = 0$, which means that $v = 0$. □

Definition We define the norm associated with the inner product b as

$$\|u\|_V = \sqrt{b(u,u)} \quad \text{for any} \quad u \in V.$$

Remarks

1. Note that $\|v\|_V^2 = \alpha\|v'\|^2$ for $v \in V$ and from Proposition 1 we have that
$$\|v\| \leq \beta\|v\|_V \quad \text{for any} \quad v \in V. \tag{5.2.1}$$
2. The bilinear form c is continuous on V.

Proposition 3

The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(0,\ell)$ on V.

Proof By definition $\|v\|_1^2 = \|v\|^2 + \|v'\|^2$, so clearly $\|v'\|^2 \leq \|v\|_1^2$. We also have that

$$\begin{aligned} \|v\|_V^2 = \alpha\|v'\|^2 &\geq \frac{1}{2}\alpha\|v'\|^2 + \frac{1}{2}\alpha\, \ell^{-2}\|v\|^2 \\ &\geq \frac{1}{2}\alpha \min\{1, \ell^{-2}\}\|v\|_1^2. \end{aligned}$$

Therefore there exists constants $K_1 > 0$ and $K_2 > 0$ such that

$$K_1\|v\|_1^2 \leq \|v\|_V^2 \leq K_2\|v\|_1^2$$

and the result is proved. □

Remark The norm $\|\cdot\|_V$ is referred to as the energy norm.

Proposition 4

V is dense in W.

Proof From Theorem 2.1.3 we have that $C_0^\infty[0,\ell]$ is dense in $\mathcal{L}^2(0,\ell) = W$. Since $C_0^\infty[0,\ell]$ is a subset of V, V is also dense in $\mathcal{L}^2(0,\ell)$. □

We have verified that Assumptions **A1** and **A2** hold. The damping term is different for Problems 1, 2 and 3 and each problem is treated in a separate subsection.

5.2.2 Weak damping

Consider Problem 1W and recall that the bilinear form a is defined on W by

$$a(u,v) = \gamma \int_0^\ell uv.$$

The definition of E_b is given in Section 4.1:

$E_b = \{\, x \in V \mid \text{there exists a } y \in W \text{ such that } c(y,v) = b(x,v) \text{ for all } v \in V \,\}.$

Theorem 1 Viscous Damping

Problem 1W has a unique solution

$$u \in C^1\big((-\infty,\infty);V\big) \cap C^2\big((-\infty,\infty);W\big)$$

for $u_0 \in E_b$ and $u_1 \in V$.

Proof It is obvious that a is symmetric. For any u and $v \in V$

$$a(v,v) = \gamma(v,v) \geq 0 \quad \text{since} \quad \gamma \geq 0.$$

Also $|a(u,v)| = \gamma|(u,v)| \leq \gamma\|u\|\|v\|$ from the Schwartz inequality. Thus a is a nonnegative, bounded bilinear form on W. Using (5.2.1) we conclude that a is bounded on V (Assumption **A3**). The desired result follows from Theorem 4.1.2. □

Sufficient conditions When do the initial conditions satisfy $u_0 \in E_b$ and $u_1 \in V$? Is it possible to derive sufficient conditions for this, which are also useful in applications?

For u_0 to be in E_b we need to find a function y in $\mathcal{L}^2(0,\ell)$ such that $c(y,v) = b(u_0,v)$ for any $v \in V$. Assume that $u_0 \in T[0,\ell] \cap C^2_*[0,\ell]$. For any $v \in T[0,\ell]$ integration by parts yields

$$\alpha(u_0', v') = -\alpha(u_0'', v) + \alpha u_0'(\ell)v(\ell) \tag{5.2.2}$$

since $v(0) = 0$. If $u_0'(\ell) = 0$ then $b(u_0,v) = c(-\alpha u_0'', v)$ for any $v \in T[0,\ell]$. The bilinear form c is continuous on V (Remark 2 after Proposition 2) and b is obviously continuous. Consequently (5.2.2) holds for any $v \in V$. This means that $u_0 \in E_b$. It is clear that if u_1 is a test function then $u_1 \in V$.

We conclude that the sufficient conditions on the initial conditions are that

$$u_0 \in T[0,\ell] \cap C^2_*[0,\ell], \quad u_0'(\ell) = 0$$

$$\text{and} \quad u_1 \in T[0,\ell].$$

Undamped wave equation

It is now possible to consider the examples in Section 1.1. By setting $\gamma = 0$ we have the undamped wave equation and Theorem 1 is still applicable. The sufficient conditions for the existence of weak solutions are given above. For Example 2, $u_0(x) = x(2-x)$ and $u_1 = 0$. Clearly $u_0 \in T[0,\ell] \cap C^2[0,\ell]$ and $u_0'(\ell) = 0$. Recall that $u_1 = 0$ and hence trivially in $T[0,\ell]$.

However, for Example 3 we have that $u_0 \in T[0,\ell]$ but $u_0'(\ell) \neq 0$. The result is that $u_0 \notin E_b$ so we cannot prove that a weak solution exists.

5.2.3 Strong damping

We now consider the weak variational form of Problem 2. For Kelvin-Voigt damping the bilinear a form is defined on V by

$$a(u,v) = \gamma \int_0^\ell u'v'.$$

Problem 2W

Find a function u such that for each $t \in J$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W \quad \text{and}$$
$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = 0 \quad \text{for each } v \in V$$
$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_1.$$

Theorem 2 Kelvin-Voigt Damping

For $u_0 \in V$, $u_1 \in W$ and $\gamma > 0$, Problem 2W has a unique solution

$$u \in C\big([0,\infty); V\big) \cap C^1\big([0,\infty); W\big) \cap C^\infty\big((0,\infty); V\big).$$

Proof It is obvious that a is symmetric. For any u and v in V

$$a(u,u) = \gamma\big(u', u'\big) = \gamma\|u'\|^2$$

so it is clear that a is nonnegative. Since b is an inner product on V it follows that

$$a(u,v) = \gamma\alpha^{-1}b(u',v') \leq \gamma\alpha^{-1}\|u\|_V\|v\|_V$$

so Assumption **A3** holds. Since

$$a(u,u) = \gamma\alpha^{-1}b(u',u') = \gamma\alpha^{-1}\|u\|_V^2,$$

a is positive definite on V and the result follows by applying Theorem 4.1.3.

$\square$

Sufficient conditions The sufficient conditions in this case are simply that u_0 must be a test function and that u_1^2 must be integrable.

Application of Theorem 4.1.1.

Theorem 4.1.1 can also be applied to Problem 2W to prove an existence result. If the initial conditions satisfy Condition (4.1.1), then Problem 2W has a unique solution

$$u \in C^1\big([0,\infty); V\big) \cap C^2\big([0,\infty); W\big).$$

For u_0 and u_1 in $T[0,\ell] \cap C^2_*[0,\ell]$ and any $v \in T[0,\ell]$ we find from (5.2.2) that

$$b(u_0, v) = (-\alpha\; u_0'', v) + \alpha u_0'(\ell)v(\ell).$$

Integration by parts yields

$$a(u_1, v) = \gamma(u_1', v') = -\gamma(u_1'', v) + \gamma u_1'(\ell)v(\ell).$$

Thus for any $v \in T[0, \ell]$

$$b(u_0, v) + a(u_1, v) = (-\alpha u_0'' - \gamma u_1'', v) + \big(\alpha u_0'(\ell) + \gamma u_1'(\ell)\big)v(\ell).$$

If $\alpha u_0'(\ell) + \gamma u_1'(\ell) = 0$ then

$$b(u_0, v) + a(u_1, v) = (-\alpha u_0'' - \gamma u_1'', v)$$

for any $v \in T[0, \ell]$. Since a is continuous on V this is also true for any $v \in V$ and hence Condition (4.1.1) is satisfied.

We conclude that sufficient conditions on the initial conditions in this case are that u_0 and u_1 are in $T[0, \ell] \cap C_*^2[0, \ell]$ and $\alpha u_0'(\ell) + \gamma u_1'(\ell) = 0$.

Comparison of existence results

To apply different existence theorems, different sufficient conditions are needed.

When applying Theorem 4.1.3 it is sufficient that u_0 is a test function and u_1 is integrable. The sufficient conditions when applying Theorem 4.1.1 are stronger: it is required that u_0 and u_1 are test functions and in $C_*^2[0, \ell]$. It is also required that $\alpha u_0'(\ell) + \gamma u_1'(\ell) = 0$.

By applying different existence theorems, we obtain solutions with different continuity properties. An application of Theorem 4.1.1 yields the existence of a solution u where

$$u \in C^1\big([0, \infty); V\big) \cap C^2\big([0, \infty); W\big).$$

However, an application of Theorem 4.1.3 yields

$$u \in C\big([0, \infty); V\big) \cap C^1\big([0, \infty); W\big) \cap C^\infty\big((0, \infty); V\big).$$

5.2.4 Boundary damping

For boundary damping the bilinear form a is defined on V by

$$a(u, v) = \gamma u(\ell)v(\ell).$$

Problem 3W

Find a function u such that for each $t > 0$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W \quad \text{and}$$

$$c\big(u''(t), v\big) + a\big(u'(t), v\big) + b\big(u(t), v\big) = 0 \quad \text{for each } v \in V$$

$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_1.$$

Theorem 3 Boundary Damping

Problem 3W has a unique solution

$$u \in C^1\big([0,\infty); V\big) \cap C^2\big([0,\infty); W\big)$$

if $u_0 \in V$, $u_1 \in V$ and there exists some $y \in W$ such that

$$b\big(u_0, v\big) + a\big(u_1, v\big) = c\big(y, v\big) \quad \text{for each } v \in V.$$

Proof The symmetry of the bilinear form a is obvious from the definition and it is nonnegative since $a\big(u, u\big) = \gamma(\Gamma u)^2 \geq 0$.

For any u and v in V, we have from Theorem 2.5.1 that

$$\gamma|\Gamma u||\Gamma v| \leq \gamma K_\ell^2 \|u\|_1 \|v\|_1 = \tilde{K}\|u\|_1\|v\|_1.$$

We have from Proposition 3 that the norms $\|u\|_V$ and $\|u\|_1$ are equivalent and it follows that a is bounded on V. Consequently a has the properties required in Assumption **A3**. By assumption u_0 and u_1 satisfy Condition (4.1.1) and the result follows from Theorem 4.1.1. $\square$

Sufficient conditions To apply this result to a specified problem, one would have to know whether the given initial conditions satisfy Condition (4.1.1).

Assume that $u_0 \in T[0,\ell] \cap C_*^2[0,\ell]$. Then integration by parts yields

$$b(u_0, v) = \alpha(u_0', v') = (-\alpha\ u_0'', v) + \alpha u_0'(\ell)v(\ell)$$

for any $v \in T[0,\ell]$. We find that

$$b(u_0, v) + a(u_1, v) = c(-\alpha u_0'', v) + \alpha u_0'(\ell)v(\ell) + \gamma u_1'(\ell)v(\ell)$$

for any $v \in T[0,\ell]$. If the initial conditions are chosen such that the pair u_0 and u_1 satisfies

$$\alpha u_0'(\ell) + \gamma u_1'(\ell) = 0$$

then

$$b(u_0, v) + a(u_1, v) = c(-\alpha u_0'', v) \text{ for any } v \in T[0, \ell].$$

Since a, b and c are continuous with respect to the norm of V, this will also be true for any $v \in V$. Sufficient conditions on the initial conditions are therefore

$$u_0 \in T[0, \ell] \cap C_*^2[0, \ell], \quad u_1 \in T[0, \ell]$$

$$\text{and } \ \alpha\, u_0'(\ell) + \gamma\, u_1'(\ell) = 0.$$

5.3 The Euler-Bernoulli beam

This model and the bilinear forms a, b and c were discussed in Section 1.3 where the space of test functions is defined as

$$T[0, \ell] = \{v \in C_*^2[0, \ell] \mid v(0) = v'(0) = 0\}.$$

Definition Let W be the set of function $\mathcal{L}^2(0, \ell)$ and let V be defined as the closure of the test functions in $H^2(0, \ell)$.

The weak variational form of Problem 4V is Problem 4W.

Problem 4W

Find u such that for each t

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W \text{ and}$$

$$c(u''(\cdot, t), v) + a(u'(\cdot, t), v) + b(u(\cdot, t), v) = 0 \text{ for each } v \in V$$

while $u(0) = u_0$ and $u'(0) = u_1$.

Comparing Problems 1W and 4W, we see that they are essentially the same. The only difference is the fact that the space V is defined differently for these two problems. As a consequence the existence theory is the same.

Proposition 1

There exists a constant $K > 0$ such that $c(v, v) \leq K b(v, v)$ for each $v \in V$.

Proof For any v in V

$$\begin{aligned} c(v,v) &= \int_0^\ell \rho A v^2 \quad \text{and} && (5.3.1) \\ b(v,v) &= \int_0^\ell EI(v'')^2 && (5.3.2) \end{aligned}$$

We have from the lemma in Section 2.4 that $\|v\| \leq \ell^2 \|v''\|$. Combining this inequality with (5.3.1) and (5.3.2) we find that

$$c(v,v) \leq \ell^4 \big(EI\big)^{-1} \rho A \; b(v,v)$$

and the result is proved. □

The remainder of this section's proofs are the same as the proofs in Subsections 5.2.1 and 5.2.2.

Proposition 2

The bilinear form b is an inner product on V.

Definition The norm associated with the inner product b is defined as

$$\|u\|_V = \sqrt{b(u,u)} \quad \text{for any} \quad u \in V.$$

Proposition 3

The bilinear form c is an inner product on W.

Definition The norm associated with the inner product c is defined as

$$\|u\|_W = \sqrt{c(u,u)} \quad \text{for any} \quad u \in W.$$

Remark From the definitions of these norms and from Proposition 1 it follows that there exists a $K > 0$ such that

$$\|v\|_W \leq K \; \|v\|_V \quad \text{for any} \quad v \in V. \qquad (5.3.3)$$

Proposition 4

V dense in W.

It is therefore clear that Assumptions **A1** and **A2** hold. It is clear that the bilinear form a is symmetric and nonnegative.

Proposition 5

The bilinear form a is bounded on W.

Proof

$$\begin{aligned} a(u,v) &= k(u,v) = k(\rho A)^{-1}c(u,v) \\ &\leq k(\rho A)^{-1}\|u\|_W\|v\|_W \end{aligned}$$

☐

From (5.3.3) it follows that a is also bounded on V. We may now apply Theorem 4.2.2 to obtain the following existence result.

Theorem

For each $u_0 \in E_b$ and $u_1 \in V$, there exists a unique solution

$$u \in C^1\big((-\infty,\infty);V\big) \cap C^2\big((-\infty,\infty);W\big)$$

for Problem 4W.

Sufficient conditions

If $u_0 \in T[0,\ell] \cap C^4_*[0,\ell]$ then integration by parts yields

$$\begin{aligned} EI(u_0'',v'') &= -EI(u_0^{(3)},v') + EI[u_0''v']_0^\ell \\ &= EI(u_0^{(4)},v) - EI[u_0^{(3)}v]_0^\ell + EI[u_0''v']_0^\ell. \end{aligned}$$

For any v a test function, $v(0) = v'(0) = 0$ and

$$b(u_0,v) = EI(u_0^{(4)},v) - EI\ u_0^{(3)}(\ell)v(\ell) + EI\ u_0''(\ell)v'(\ell).$$

If $u_0^{(3)}(\ell) = 0$ and $u_0''(\ell) = 0$ then for any test function v

$$b(u_0,v) = EI(u_0^{(4)},v) \tag{5.3.4}$$

Since b is an inner product on V and the inner product of $\mathcal{L}^2$ is bounded by $\|\cdot\|_V$, this will also be true for any $v \in V$. This shows that $u_0 \in E_b$. The sufficient conditions are therefore

$$\begin{gathered} u_0 \in T[0,\ell] \cap C^4_*[0,\ell], \quad u_0^{(3)}(\ell) = 0 = u_0''(\ell) \\ \text{and} \ \ u_1 \in T[0,\ell]. \end{gathered}$$

Remark The boundary conditions on u_0 implies that $M(\ell,0) = V(\ell,0) = 0$.

5.4 The Raleigh beam

The definitions of the space of test functions, $T[0,\ell]$, and the bilinear form b are the same as in the previous section. The difference is with regards to the bilinear form c which is defined by

$$c(u,v) = \int_0^\ell \rho A u v + \int_0^\ell \rho I u' v'.$$

We still need to define the spaces W and V.

Definitions Let W be the closure of the the test functions in $H^1(0,\ell)$ and define V as the closure of $T[0,\ell]$ in $H^2(0,\ell)$.

Remark Note that $V \subset W \subset \mathcal{L}^2(0,\ell)$ and that the bilinear forms b and c are defined on V and W respectively.

Problem 5W

Let J be an open interval containing zero. Given $Q \in C^1\big(J; \mathcal{L}^2(0,\ell)\big)$, find u such that for each $t \in J$, $u(t) \in V$, $u''(t) \in W$ and

$$c(u''(t),v) + b(u(t),v) = \big(Q(t),v\big) \quad \text{for each} \quad v \in V$$

while $u(0) = u_0$ and $u'(0) = u_1$.

As in the previous section, b defines an inner product for V (see Proposition 5.3.2).

Definition The norm associated with b is defined as

$$\|u\|_V = \sqrt{b(u,u)} \quad \text{for any} \quad u \in V.$$

Proposition 1

The bilinear form c defines an inner product for W.

Proof From the definition of the bilinear form the symmetry of c is obvious and c is nonnegative since ρ, A and I are nonnegative.

If $c(u,u) = 0$ for some $u \in W$, then $\rho A(u,u) + \rho I(u',u') = 0$. Both the terms on the left are nonnegative which means that $\rho A(u,u) = 0$ and consequently $u = 0$. $\square$

Definition The norm associated with c is defined as

$$\|u\|_W = \sqrt{c(u,u)} \ \text{ for any } \ u \in W.$$

Proposition 2

$\|\cdot\|_W$ is equivalent to $\|\cdot\|_1$.

Proof This follows from the fact that $\|u\|_W^2 = \rho A\|u\|^2 + \rho I\|u'\|^2$. □

Remark Note that $\|u\|_V^2 = b(u,u) = EI\|u''\|^2$ as in the previous section.

Proposition 3

There exists constants β_1 and β_2 such that

$$\|v\| \leq \beta_1\|v\|_W \leq \beta_2\|v\|_V$$

for each $v \in V$.

Proof Consider any $v \in V$. From the lemma in Section 2.4 we have that $\|v\|^2 \leq \ell^2\|v'\|^2 \leq \ell^4\|v''\|^2$ since $v(0) = v'(0) = 0$. So

$$\|v\|_W^2 = \rho A\|v\|^2 + \rho I\|v'\|^2 \geq \left(\rho A + \ell^{-2}\rho I\right)\|v\|^2$$

and the first inequality is proved.

Also we have that

$$\|v\|_W^2 \leq \left(\rho A\ell^4 + \rho I\ell^2\right)\|v''\|^2.$$

Since $\|v\|_V^2 = EI\|v''\|^2$ it follows that

$$EI\|v\|_W^2 \leq \left(\rho A\ell^4 + \rho I\ell^2\right)\|v\|_V^2$$

and the second inequality is proved. □

Proposition 4

V is dense in W and W is dense in $\mathcal{L}^2(0,\ell)$.

Proof Since $T[0,\ell]$ is by definition dense in W with respect to the norm of $H^1(0,\ell)$ and $\|\cdot\|_1$ is equivalent to $\|\cdot\|_W$, it follows that $T[0,\ell]$ is dense in W. So V is dense in W. Since W contains C_0^∞ which is dense in $\mathcal{L}^2(0,\ell)$ it follows that W is dense in $\mathcal{L}^2(0,\ell)$. □

Note that Assumptions **A1** and **A2** are satisfied and **A3** is not relevant. However, the function Q does not satisfy the assumption in Theorem 4.1.2. The following propositions are necessary to rectify the situation.

Proposition 5

(a) For each $y \in \mathcal{L}^2(0,\ell)$ there exists a unique $z \in W$ such that

$$c(z,v) = (y,v) \quad \text{for each} \quad v \in W.$$

(b) If the mapping T is defined by $z = Ty$ then T is a bounded linear operator, i.e. there exists a β such that

$$\|Ty\|_W \le \beta\, \|y\| \quad \text{for all} \quad y \in \mathcal{L}^2(0,\ell).$$

Proof of (a) For any $y \in \mathcal{L}^2(0,\ell)$ and any $v \in W$, define $\phi(v) = (y,v)$.

It follows from the Schwartz inequality and Proposition 3 that

$$|\phi(v)| = |(y,v)| \le \|y\|\|v\| \le \beta_1\|y\|\|v\|_W.$$

Therefore ϕ is a bounded linear functional on W. From Riesz's Theorem there exists a unique $z \in W$ such that

$$c(z,v) = \phi(v) = (y,v) \quad \text{for each} \quad v \in W. \tag{5.4.1}$$

Consequently

$$\|z\|_W^2 = c(z,z) = (y,z) \le \|y\|\|z\| \le \beta\|y\|\|z\|_W.$$

Proof of (b) It is clear that T is linear and $\|z\|_W = \|Ty\|_W \le \beta\|y\|$. □

Proposition 6

For each $f \in C(J;\mathcal{L}^2(0,\ell))$ there exists a unique $g \in C(J;W)$ such that $c(g(t),v) = (f(t),v)$ for each $v \in W$ and any $t \in J$.

Proof Define $g(t) = Tf(t)$ for any $t \in J$, i.e. $c(g(t),v) = (f(t),v)$ for any $v \in V$. Since $f \in C(J;\mathcal{L}^2(0,\ell))$ we have that $\|f(t) - f(s)\| < \varepsilon$ if $|t-s| < \delta$. Since T is linear and bounded it follows that

$$\|g(t) - g(s)\|_W = \|T(f(t) - f(s))\|_W \le \beta\|f(t) - f(s)\|.$$

Hence $g \in C(J; W)$ and the proof is complete. $\square$

Proposition 7

If $f \in C^1(J; \mathcal{L}^2(0, \ell))$ then $g' = Tf'$ and $g \in C^1(J; W)$.

Proof For any $t \in J$

$$\begin{aligned} \|h^{-1}(g(t+h) - g(t)) - Tf'(t)\|_W &= \|h^{-1}(Tf(t+h) - Tf(t)) - Tf'(t)\|_W \\ &\le \beta\|h^{-1}(f(t+h) - f(t)) - f'(t)\| \end{aligned}$$

and $\|h^{-1}(f(t+h) - f(t)) - f'(t)\| \longrightarrow 0$ as $h \to 0$.

So the derivative $g' = Tf'$ of g exists. The continuity of g' follows from Proposition 6. $\square$

Theorem

If $Q \in C^1\big(J; \mathcal{L}^2(0, \ell)\big)$ then Problem 5W has a unique solution

$$u \in C^1\big(J; V\big) \cap C^2\big(J; W\big)$$

for each $u_0 \in E_b$ and $u_1 \in V$.

Proof Let $g(t) = Tf(t)$ i.e. $c(g(t), v) = (Q(t), v)$ for all $v \in V$. From Proposition 7 we have that $g \in C^1\big(J; W\big)$ since $Q \in C^1\big(J; \mathcal{L}^2(0, \ell)\big)$. The existence result now follows from Theorem 4.1.2. $\square$

Sufficient conditions

Suppose $u_0 \in T(0, \ell) \cap C^3_*[0, \ell]$. For any test function v

$$b(u_0, v) = -EI(u_0''', v') + EI\ u_0''(\ell)\ v'(\ell)$$

since $v'(0) = 0$. If $u_0''(\ell) = 0$ then

$$b(u_0, v) = -EI(u_0''', v')$$

for any test function v. This is also true for $v \in V$ since V is the closure of the test functions in $H^2(0, \ell)$ and b is an inner product on V.

Define the functional ϕ by $\phi(v) = EI(u_0''', v')$. This is clearly linear. By making use of the Schwartz inequality, it follows that

$$|\phi(v)| \le EI\|u_0'''\|\ \|v'\|.$$

From the definition of $\| \cdot \|_W$, it is easy to see that $\sqrt{\rho I}\|v'\| < \|v\|_W$ and hence

$$|\phi(v)| \leq \left(EI\sqrt{\rho I}^{-1}\|u_0'''\|\right)\|v\|_W.$$

We have a bounded linear function on W and by Riesz's Theorem it follows that there exist a $y \in W$ such that

$$\phi(v) = c(y, v)$$

for any $v \in W$. Therefore u_0 is in E_b.

We conclude that $u_0 \in T(0,\ell) \cap C_*^3[0,\ell]$, with $u_0''(\ell) = 0$ and $u_1 \in T[0,\ell]$ are sufficient conditions.

5.5 The three-dimensional wave equation

Consider Problem 6V. The space of test functions $T(\Omega)$ is defined in Section 1.5. by

$$T(\Omega) = \{\phi \in C_*^1(\Omega) \mid \phi = 0 \ \text{ on } \ \partial\Omega - \Sigma\ \}.$$

Let $V(\Omega)$ denote the closure of $T(\Omega)$ in $H^1(\Omega)$.

The trace operator Γ is defined in Section 2.5 and the bilinear form a is defined on $V(\Omega)$ by

$$a(u, v) = \gamma \iint_\Sigma \Gamma u\ \Gamma v\ dA.$$

Problem 6W

Find u such that for each $t > 0$, $u(t) \in V(\Omega)$, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) = -b(u(t), v) - a(u'(t), v) \quad \text{for each} \ \ v \in V(\Omega)$$

while $u(0) = u_0\,, \quad u'(0) = u_1$.

It is important to realize that this problem is similar to Problem 3W in Section 5.2.4, but here we are considering three dimensions.

Definition Let $W = \mathcal{L}^2(\Omega)$.

The bilinear form c is the usual inner product of $\mathcal{L}^2(\Omega)$.

Next consider the bilinear form b which is defined on $V(\Omega)$ (see Section 1.5) by

$$b(u,v) = \alpha \iiint_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v \; dV$$

where $\partial_i u$ and $\partial_i v$ denote weak derivatives. It will be shown that b is an inner product for V.

Proposition 1

There exists a constant β such that $\|v\|^2 \leq \beta \; b(v,v)$ for each $v \in V$.

Proof For any v in V we have $c(v,v) = (v,v) = \|v\|^2$ and

$$b(v,v) = \alpha \iiint_\Omega \sum_{i=1}^{3} (\partial_i v)^2 \; dV = \alpha \sum_{i=1}^{3} \|\partial_i v\|^2.$$

From Proposition 2.4.5 it follows that for any test function v

$$\alpha \|v\|^2 \leq \alpha \; k \sum_{i=1}^{3} \|\partial_i v\|^2 = k \; b(v,v).$$

Since $V(\Omega)$ is the closure of the test functions in $H^1(\Omega)$, this inequality is also true for any $v \in V(\Omega)$.

This shows that $\beta = \alpha^{-1} k$ and the result is proved. □

Proposition 2

The bilinear form b is an inner product on V.

Proof The fact that the bilinear form b is symmetric follows from the definition of b. Since

$$b(v,v) = \alpha \sum_{i=1}^{3} \|\partial_i v\|^2$$

for any $v \in V$, it is also clear that b is nonnegative.

If $b(v,v) = 0$ we have from Proposition 1 that $\|v\|^2 \leq \beta \; b(v,v) = 0$ which means that $v = 0$. □

Definition The norm corresponding to the inner product b is

$$\|v\|_V = \sqrt{b(v,v)} \quad \text{for any} \quad v \in V.$$

Remark From Proposition 1 we have that

$$\|v\| \leq \beta \|v\|_V \quad \text{for any} \quad v \in V. \tag{5.5.1}$$

Proposition 3

The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(0,\ell)$ on V.

Proof For any $v \in V$ we have

$$\|v\|_1^2 = \|v\|^2 + \sum_{i=1}^{3} \|\partial_i v\|^2 \tag{5.5.2}$$

so it is clear that

$$\|v\|_V^2 = \alpha \sum_{i=1}^{3} \|\partial_i v\|^2 \leq \alpha \|v\|_1^2.$$

From (5.5.1) and (5.5.2) we have that

$$\alpha \|v\|_1^2 \leq \alpha\beta^2 \|v\|_V^2 + \|v\|_V^2$$

and it follows that

$$\|v\|_1^2 \leq (\beta^2 + \alpha^{-1}) \|v\|_V^2. \tag{5.5.3}$$

Hence there exists a constant $K_1 > 0$ such that

$$K_1 \|v\|_1^2 \leq \|v\|_V^2 \leq \alpha \|v\|_1^2.$$

□

Proposition 4

V is dense in $\mathcal{L}^2(\Omega)$.

Proof From Theorem 2.1.3 we have that $C_0^\infty(\bar{\Omega})$ is dense in $\mathcal{L}^2(\Omega)$. Since $C_0^\infty(\bar{\Omega})$ is a subspace of V, V is also dense in $\mathcal{L}^2(\Omega)$. □

Theorem

Problem 6W has a unique solution

$$u \in C^1\big([0,\infty);V\big) \cap C^2\big([0,\infty);W\big)$$

if for $u_0 \in V$ and $u_1 \in V$ there exists some $y \in W$ such that

$$b\big(u_0, v\big) + a\big(u_1, v\big) = c\big(y, v\big) \quad \text{for each } v \in V. \tag{5.5.4}$$

Proof We use Theorem 4.1.1. The bilinear form a is clearly symmetric. Since $a(u,u) = \gamma(\Gamma u)^2 \geq 0$ it follows that a is nonnegative.

For any u and v in V, we have from the Schwartz inequality and Theorem 2.5.2 that

$$\begin{aligned} |\ a(u,v)| &= \gamma \ \Big| \iint_\Sigma (\Gamma u \ \Gamma v) \ dA \Big| \\ &\leq \gamma \ \|\Gamma u\|_\Sigma \ \|\Gamma v\|_\Sigma \\ &\leq \gamma K^2 \ \|u\|_1 \ \|v\|_1. \end{aligned}$$

From (5.5.3) it follows that

$$|\ a(u,v)| \leq \gamma K^2(\beta^2 + \alpha^{-1}) \ \|u\|_V \ \|v\|_V$$

hence a has all the properties required in Assumption **A3**.

Condition (4.1.1) is satisfied by assumption. Therefore Problem 6W has a unique solution from Theorem 4.1.1. ☐

Sufficient conditions To apply this result to a specified problem, one would have to know whether the given initial conditions satisfy the requirements $u_0 \in V$, $u_1 \in V$ and there exist a $y \in W$ such that Equation (5.5.4) is satisfied.

Assume that $u_0 \in T(\Omega) \cap C^2_*(\Omega)$. By Green's formula

$$b(u_0, v) = \alpha \iiint_\Omega -(\nabla^2 u_0)v \ dV + \alpha \iint_\Sigma (\nabla u_0 \cdot n)v \ dA.$$

If $u_1 \in T(\Omega)$ then

$$a(u_1, v) = \gamma \iint_\Sigma u_1 v \ dA.$$

We find that

$$\begin{aligned} & b(u_0,v) + a(u_1,v) \\ = \ & \alpha \iiint_\Omega -(\nabla^2 u_0)v \ dV + \alpha \iint_\Sigma (\nabla u_0 \cdot n)v \ dA + \gamma \iint_\Sigma u_1 v \ dA \\ = \ & -c(\alpha(\nabla^2 u_0), v) + \iint_\Sigma \alpha \ (\nabla u_0 \cdot n)v \ dA + \iint_\Sigma \gamma \ u_1 v \ dA \qquad (5.5.5) \end{aligned}$$

for any $v \in T(\Omega)$. If $\alpha \ \partial_1 u_0 + \gamma \ u_1 = 0$ on Σ then

$$\iint_\Sigma \alpha \ (\nabla u_0 \cdot n)v \ dA + \iint_\Sigma \gamma \ u_1 v \ dA = 0$$

(since $n = (1, 0, 0)$ on Σ) and

$$b(u_0, v) + a(u_1, v) = -c(\alpha(\nabla^2 u_0), v) \text{ for any } v \in T(\Omega). \tag{5.5.6}$$

From (5.5.1) it follows that c is bounded with respect to $\| \cdot \|_V$. Since a satisfy Assumption **A3**, Equation (5.5.4) will be true for any $v \in V$ and consequently Condition (4.1.1) will be satisfied.

We conclude that the sufficient conditions are that u_1 be a test function, $u_0 \in T(\Omega) \cap C^2_*(\Omega)$ and $\alpha\ \partial_1 u_0 + \gamma\ u_1 = 0$ on Σ.

Chapter 6

Beam with a damping tip body

In the article [AS] the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. They establish the existence of a unique solution for the model problem. The authors state that "the problem contains more complicated boundary conditions" than problems "treated previously". This is due to the fact that the model is more realistic – as we explain below.

In this article the Euler-Bernoulli theory for a beam is used with Kelvin-Voigt damping. The beam is clamped at one end and a rigid body is attached to the other end. It is an interesting model for more than one reason. The dynamics of the rigid body is treated in a realistic way unlike other papers where a particle model is used for the body. The fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account. The damping mechanism is also explained unlike other papers where vibration models with boundary damping are considered (see Chapter 7).

In this chapter we also prove the existence of a unique solution, using the results from Chapter 4. The approach differs from that in [AS] and there are some differences regarding assumptions and properties of the solution.

6.1 Model problem

The Euler-Bernoulli model for the transverse vibration of a beam is presented in Section 1.3. Recall that it is derived from

$$\rho A \partial_t^2 w \quad = \quad \partial_x V + f, \tag{6.1.1}$$

and

$$0 = \partial_x M + V. \tag{6.1.2}$$

The usual constitutive equation is $M = EI\partial_x^2 w$, but due to Kelvin-Voigt damping this changes. In [AS] they use

$$M = EI\partial_x^2 w + \lambda \partial_t \partial_x^2 w, \tag{6.1.3}$$

where λ denotes the damping parameter. The partial differential equation (which we do not use) is

$$\rho A \partial_t^2 w \quad = \quad -EI\partial_x^4 w + \lambda \partial_t \partial_x^4 w + f.$$

The left endpoint of the beam is clamped and the boundary conditions are the usual (see Section 1.3):

$$w(0,t) = \partial_x w(0,t) = 0. \tag{6.1.4}$$

The interface conditions at the other endpoint are determined by the interaction between the beam and the rigid body. This is explained in [AS] in some detail. It is necessary to consider the equations of motion for the rigid body carefully when deriving the boundary conditions.

The position of the center of mass of the tip body relative to the endpoint of the beam is

$$d\cos\theta\ \mathbf{i} + d\sin\theta\ \mathbf{j}$$

where θ is the angle of the neutral plane with the horizontal. Therefore the velocity $\mathbf{v}_C$ and acceleration $\mathbf{a}_C$ of the center of mass are given by

$$\begin{aligned}
\mathbf{v}_C &= \partial_t w(\ell,t)\ \mathbf{j} - d\dot\theta\sin\theta\ \mathbf{i} + d\dot\theta\cos\theta\ \mathbf{j},\\
\mathbf{a}_C &= \partial_t^2 w(\ell,t)\ \mathbf{j} - d\ddot\theta\sin\theta\ \mathbf{i} + d\ddot\theta\cos\theta\ \mathbf{j} - d\dot\theta^2\cos\theta\ \mathbf{i} - d\dot\theta^2\sin\theta\ \mathbf{j},
\end{aligned}$$

where ℓ denotes the length of the beam. For the linear approximation it is assumed that the term $\dot\theta^2\sin\theta\ \mathbf{j}$ may be neglected, $\dot\theta \approx \partial_t\partial_x w(\ell,t)$, $\ddot\theta \approx \partial_t^2\partial_x w(\ell,t)$ and $\cos\theta \approx 1$. Using these approximations, we have the following expressions for the vertical components of the velocity and acceleration:

$$\partial_t w(\ell,t) + d\partial_t\partial_x w(\ell,t) \quad\text{and}\quad \partial_t^2 w(\ell,t) + d\partial_t^2\partial_x w(\ell,t).$$

In [AS] the term $d\partial_t\partial_x w(\ell,t)$ in the expressions for the vertical component of the velocity is neglected. As shown in [DV], this should not be done. In the next section we discuss the decay of energy for the system and it is also evident from energy considerations that the additional term is necessary.

An external force $f_B(t)$ may act on the rigid body, e.g. gravity. Using Newton's second law for the motion of the center of mass we have

$$\begin{aligned}
& m\partial_t^2 w(\ell,t) + md\ \partial_t^2\partial_x w(\ell,t)\\
= {}& -V(\ell,t) + f_B(t) - \gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t). \qquad (6.1.5)
\end{aligned}$$

Taking moments about the center of mass we have

$$J\partial_t^2\partial_x w(\ell,t) = -M(\ell,t) + dV(\ell,t) - d\gamma^*\partial_t\partial_x w(\ell,t). \qquad (6.1.6)$$

Following [AS] we combine Equations (6.1.5) and (6.1.6) and find that

$$\begin{aligned}
& md\ \partial_t^2 w(\ell,t) + (J+md^2)\partial_t^2\partial_x w(\ell,t)\\
= {}& -M(\ell,t) - \gamma d\partial_t w(\ell,t) - (\gamma d+\gamma^*)d\partial_t\partial_x w(\ell,t) + df_B(t). \qquad (6.1.7)
\end{aligned}$$

It is convenient to rewrite Equations (6.1.5) and (6.1.7):

$$\begin{aligned}
V(\ell,t) &= -m\partial_t^2 w(\ell,t) - md\ \partial_t^2\partial_x w(\ell,t)\\
&\quad -\gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t) + f_B(t), \qquad (6.1.8)\\
M(\ell,t) &= -md\ \partial_t^2 w(\ell,t) - (J+md^2)\partial_t^2\partial_x w(\ell,t)\\
&\quad -\gamma d\partial_t w(\ell,t) - (\gamma d+\gamma^*)d\partial_t\partial_x w(\ell,t) + df_B(t). \qquad (6.1.9)
\end{aligned}$$

Problem 7

The mathematical model consists of the Equations of motion (6.1.1) and (6.1.2) and the Constitutive equation (6.1.3) for the beam, the Boundary conditions (6.1.4) and the Interface conditions (6.1.8) and (6.1.9). The initial conditions are $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$.

6.2 Variational form

Multiply Equation (6.1.1) by an arbitrary smooth function v and integrate. Using also Equation (6.1.2), integration by parts yields

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v \\
= \; & -\int_0^\ell V(\cdot,t)v' + [V(x,t)v(x)]_0^\ell + \int_0^\ell f(\cdot,t)v \\
= \; & -\int_0^\ell M(\cdot,t)v'' + [V(x,t)v(x)]_0^\ell + [M(x,t)v'(x)]_0^\ell + \int_0^\ell f(\cdot,t)v
\end{aligned}$$

Test functions

The space of test functions is defined by

$$T[0,\ell] = \{v \in C_*^2[0,\ell] \mid v(0) = v'(0) = 0\}.$$

It follows that

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v \\
= \; & -\int_0^\ell M(\cdot,t)v'' + V(\ell,t)v(\ell) + M(\ell,t)v'(\ell) + \int_0^\ell f(\cdot,t)v
\end{aligned}$$

for each $v \in T[0,\ell]$.

We now use the constitutive equation $M = EI\partial_x^2 w + \lambda\partial_t\partial_x^2 w$ and the initial conditions (6.1.8) and (6.1.9) to derive the variational form of the model problem.

It is convenient to introduce the following bilinear forms

$$\begin{aligned}
\bar{b}(u,v) &= \int_0^\ell EIu''v'' \\
\bar{c}(u,v) &= \int_0^\ell \rho Auv + mu(\ell)v(\ell) + mdu'(\ell)v(\ell) + mdu(\ell)v'(\ell) \\
&\quad +(J+md^2)u'(\ell)v'(\ell) \\
\bar{a}(u,v) &= \int_0^\ell \lambda u''v'' + \gamma u(\ell)v(\ell) + \gamma du'(\ell)v(\ell) + \gamma du(\ell)v'(\ell) \\
&\quad +(\gamma d + \gamma^*)du'(\ell)v'(\ell).
\end{aligned}$$

Remark Note that the bilinear forms $\bar{a}$, $\bar{b}$ and $\bar{c}$ are symmetric. The additional term $\gamma du'(\ell)v(\ell)$ in the definition of $\bar{a}$ is necessary for symmetry.

Variational form of the model problem

Problem 7V

Find w such that for each $t>0$, $w(\cdot,t)\in T(0,\ell)$ and

$$\begin{aligned}
&\bar{c}(\partial_t^2 w(\cdot,t),v) + \bar{a}(\partial_t w(\cdot,t),v) + \bar{b}(w(\cdot,t),v) \\
= \; &(f(\cdot,t),v) + f_B(t)v(\ell) + df_B(t)v'(\ell)
\end{aligned}$$

for each $v\in T(0,\ell)$ with $w(\cdot,0)=w_0$ and $\partial_t w(\cdot,0)=w_1$.

Mechanical energy

The mechanical energy (kinetic energy plus elastic potential energy) of the system is

$$E(t) = \frac{1}{2}\,\bar{c}(\partial_t w(\cdot,t),\partial_t w(\cdot,t)) + \frac{1}{2}\,\bar{b}(w(\cdot,t),w(\cdot,t)).$$

Using the symmetry of $\bar{b}$ and $\bar{c}$ and assuming that w is sufficiently smooth, we have

$$E'(t) = \bar{c}(\partial_t^2 w,\partial_t w) + \bar{b}(w,\partial_t w) = -\bar{a}(\partial_t w,\partial_t w).$$

It is obvious that $\bar{b}(u,u)$ is nonnegative and not difficult to show that $\bar{a}(u,u)$

and $\bar{c}(u,u)$ are nonnegative.

$$\begin{aligned}\bar{c}(u,u) &= \int_0^\ell \rho A u^2 + m[u(\ell)]^2 + 2mdu'(\ell)u(\ell) + (J+md^2)[u'(\ell)]^2 \\ &= \int_0^\ell \rho A u^2 + m[u(\ell)+du'(\ell)]^2 + J[u'(\ell)]^2 \geq 0. \\ \bar{a}(u,u) &= \int_0^\ell \lambda(u'')^2 + \gamma[u(\ell)]^2 + 2\gamma du(\ell)u'(\ell) + (\gamma d+\gamma^*)d[u'(\ell)]^2 \\ &= \int_0^\ell \lambda(u'')^2 + \gamma[u(\ell)+du'(\ell)]^2 + \gamma^* d[u'(\ell)]^2 \geq 0.\end{aligned}$$

As a result $E'(t) \leq 0$. This result is to be expected from Physics. The fact that $\bar{a}$ is symmetric and $\bar{a}(u,u)$ nonnegative, is due to the additional term.

6.3 Weak variational form

Let $V(0,\ell)$ be the closure of $T[0,\ell]$ in $H^2(0,\ell)$.

The following product spaces are necessary for the abstract problem.

Product spaces

$X = \mathcal{L}^2(0,\ell) \times \mathbb{R} \times \mathbb{R}$. In this chapter an element $y \in X$ is written as $y = \langle y_1, y_2, y_3 \rangle$.

$H^m = H^m(0,\ell) \times \mathbb{R} \times \mathbb{R}$

$V_P = V(0,\ell) \times \mathbb{R} \times \mathbb{R}$

$V = \{v \in V_P \bigm| v_2 = \Gamma v_1,\ v_3 = \Gamma v_1' \}$

An obvious inner product for X is

$$(u,v)_X = \int_0^\ell u_1 v_1 + u_2 v_2 + u_3 v_3.$$

We denote the corresponding norm by $\|\cdot\|_X$.

Definition Bilinear forms

For u and v in X

$$c(u,v) = \int_0^\ell \rho A u_1 v_1 + m u_2 v_2 + md(u_3 v_2 + u_2 v_3) + (J+md^2) u_3 v_3;$$

and for u and v in H^2

$$\begin{aligned} b(u,v) &= \int_0^\ell EIu_1''v_1''; \\ a(u,v) &= \int_0^\ell \lambda u_1''v_1'' + \gamma u_2 v_2 + \gamma d(u_3 v_2 + u_2 v_3) + (\gamma d + \gamma^*) d u_3 v_3. \end{aligned}$$

Remarks

1. The bilinear forms a, b and c are symmetric.
2. For u and v in V, $a(u,v) = \bar{a}(u_1, v_1)$, $b(u,v) = \bar{b}(u_1, v_1)$ and $c(u,v) = \bar{c}(u_1, v_1)$.
3. It is essential to use product spaces since the bilinear form c is not defined on $\mathcal{L}^2(0,\ell)$.

For the weak variational form of the model problem we need to show that c is an inner product.

Proposition 1

The bilinear form c is an inner product for the space X.

Proof It follows directly from

$$0 = c(u,u) = \rho A \|u_1\|^2 + m(u_2 + du_3)^2 + Ju_3^2$$

that $u = 0$. $\square$

Definition

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W.

We proceed to determine a weak variational form of the model problem. Let $\tilde{f}(t) = \langle f(\cdot,t), f_B(t), df_B(t)\rangle$, $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3}\rangle$ and $u_1 = \langle w_1, \tilde{u}_{1,2}, \tilde{u}_{1,3}\rangle$. Note that $\tilde{u}_{1,2}$ and $\tilde{u}_{1,3}$ may be arbitrary.

Remark It is natural to think that $\tilde{u}_{0,2} = \Gamma u_0$, etc. are the correct initial conditions. This is discussed in Section 6.5.

Problem 7W

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{f}(t), v)_X$$

for each $v \in V$ with $u(0) = u_0$ and $u'(0) = u_1$.

Proposition 2

The space V is a dense subset of W.

Proof Consider any $y \in W$. Since $C_0^\infty(0, \ell)$ is dense in $\mathcal{L}^2(0, \ell)$ there exists a sequence $\{\phi_n\} \subset C_0^\infty(0, \ell)$ such that $\|\phi_n - y_1\| \to 0$.

It is not difficult to construct sequences $\{\eta_n\}$ and $\{\zeta_n\}$ in $H^2(0, \ell)$ with the following properties

$$\begin{aligned} \Gamma\eta_n &= 1, \quad \Gamma\eta_n' = 0 \quad \text{and} \quad \|\eta_n\| \to 0, \\ \Gamma\zeta_n &= 0, \quad \Gamma\zeta_n' = 1 \quad \text{and} \quad \|\eta_n\| \to 0. \end{aligned}$$

Now, let $v_n = \phi_n + y_2\eta_n + y_3\zeta_n$, then $v_n \in V(0, \ell)$, $\Gamma v_n = y_2$ and $\Gamma v_n' = y_3$.

Consequently $u_n = \langle v_n, v_n(\ell), v_n'(\ell)\rangle \in V$ and $\|u_n - y\|_W \to 0$. $\square$

6.4 Estimates

For each $u \in V(0, \ell)$ we have from the lemma in Section 2.4 that

$$\|u\| \leq \ell\|u'\| \leq \ell^2\|u''\|. \tag{6.4.1}$$

The boundedness of Γ is also required.

Proposition 1

$$|\Gamma u| \leq \sqrt{\ell}\|u'\| \text{ for each } u \in V(0, \ell). \tag{6.4.2}$$

Proof From Proposition 2.4.1, we have $|u(\ell)| \leq \sqrt{\ell}\|u'\|$ when $u(0) = 0$. As in the proof of Proposition 2.4.4 it follows that $|\Gamma u| \leq \sqrt{\ell}\|u'\|$ for each $u \in V(0, \ell)$. $\square$

Proposition 2

There exists a constant K such that

$$b(u,u) \;\geq\; K\|u_1\|_2^2 \;\text{ for each } u \in V.$$

Proof We use (6.4.1).

$$\begin{aligned}
\|u_1\|_2^2 &= \|u_1\|^2 + \|u_1'\|^2 + \|u_1''\|^2 \\
&\leq \|u_1''\|^2 \; (\ell^2 + \ell + 1) \\
&\leq K_\ell \|u''\|^2. \\
&= \frac{K_\ell}{EI}\, b(u,u).
\end{aligned}$$

$\square$

Corollary The bilinear form b is an inner product for V.

Proof Clearly $b(u,u) = 0$ implies that $u_1 = 0$ and therefore $u_2 = \Gamma u_1 = 0$ and $u_3 = \Gamma u_1' = 0$. $\square$

Definition The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

Proposition 3

There exists a constant K such that

$$b(u,u) \;\geq\; Kc(u,u) \;\text{ for each } u \in V.$$

Proof We use (6.4.1) and (6.4.2).

$$\begin{aligned}
c(u,u) &= \int_0^\ell \rho A u_1^2 + m(u_2 + du_3)^2 + Ju_3^2 \\
&= \int_0^\ell \rho A u_1^2 + m(\Gamma u_1 + d\Gamma u_1')^2 + J(\Gamma u_1')^2 \\
&= \rho A \|u_1\|^2 + m(\Gamma u_1 + d\Gamma u_1')^2 + J(\Gamma u_1')^2 \\
&\leq \rho A \ell^4 \|u_1''\|^2 + 2m(\Gamma u_1)^2 + 2md^2(\Gamma u_1')^2 + J(\Gamma u_1')^2 \\
&\leq \|u_1''\|^2 \; (\rho A \ell^4 + 2m\ell^3 + 2md^2\ell + J\ell) \\
&\leq K_\ell \|u_1''\|^2. \\
&= \frac{K_\ell}{EI}\, b(u,u).
\end{aligned}$$

$\square$

Proposition 4

There exists a constant K such that

$$|a(u,v)| \quad \leq \quad K\|u\|_V \; \|v\|_V.$$

for each u and v in V.

Proof We use (6.4.1) and (6.4.2).

$$\begin{aligned}
|a(u,v)| \quad &\leq \quad \lambda\|u_1''\| \; \|v_1''\| + \gamma|u_2v_2| + \gamma d|u_3v_2| + \gamma d|u_2v_3| \\
&\quad\quad +(\gamma d+\gamma^*)d|u_3v_3| \\
&= \quad \lambda\|u_1''\| \; \|v_1''\| + \gamma|\Gamma u_1\Gamma v_1| + \gamma d|\Gamma u_1'\Gamma v_1| + \gamma d|\Gamma u_1\Gamma v_1'| \\
&\quad\quad +(\gamma d+\gamma^*)d|\Gamma u_1'\Gamma v_1'| \\
&\leq \quad \|u_1''\| \; \|v_1''\| \; (\lambda + \gamma\ell^3 + 2\gamma d\ell^2 + (\gamma d+\gamma^*)d\ell \;).
\end{aligned}$$

$\square$

The result above is true for $\lambda \geq 0$. If $\lambda > 0$, the bilinear form a is positive definite on V and this has implications for existence results.

Proposition 5

$$a(u,u) \geq \frac{\lambda}{EI}\|u\|_V^2 \;\; \text{for} \;\; u \in V.$$

Proof

$$a(u,u) \geq \lambda\|u_1''\|^2 = \frac{\lambda}{EI}\|u\|_V^2.$$

$\square$

Remark [AS] consider only the case $\lambda > 0$.

Proposition 6

There exists a constant K such that

$$c(u,u) \quad \geq \quad K(u,u)_X \;\; \text{for each } u \in X.$$

Proof It is sufficient to show that there exists a constant K_1 such that $m(u_2+du_3)^2+Ju_3^2 \geq K_1(u_2^2+u_3^2)$ for each $(u_2,u_3) \in \mathbb{R}^2$. Using the inequality $2\alpha\beta \leq \frac{1}{\varepsilon}\alpha^2 + \varepsilon\beta^2$, we have

$$m(u_2+du_3)^2 \geq m(1-\epsilon)u_2^2 + m(1-\frac{1}{\epsilon})d^2u_3^2.$$

By the definition of the moment of inertia of a rigid body, $J \geq \beta m d^2$ for some $\beta > 0$. Now choose $0 < \epsilon < 1$ and $\epsilon < (1+\beta)^{-1}$ and the result follows.

□

6.5 Existence

Theorem 4.1.1 is applied to the case where $\lambda = 0$ and Theorem 4.1.3 to the case where $\lambda > 0$. Note that the Assumptions **A1**, **A2** and **A3** are satisfied due to Propositions 6.3.2, 6.4.3 and 6.4.4. In the formulation of the theorems we denote $t \to f(\cdot, t)$ by f_1.

Theorem 1

Suppose $\lambda = 0$ and

(a) $f_1 \in C^1\big([0,\tau), \mathcal{L}^2(0,\ell)\big)$ and $f_B \in C^1\big([0,\tau), \mathbb{R}\big)$,

(b) $u_0 \in V$, $u_1 \in V$ and there exists a $y \in W$ such that

$$b(u_0, v) + a(u_1, v) = c(y, v) \text{ for each } v \in V.$$

Then Problem 7W has a unique solution

$$u \in C\big([0,\tau), V\big) \cap C^1\big([0,\tau), W\big) \cap C^1\big((0,\tau), V\big) \cap C^2\big((0,\tau), W\big).$$

Proof Clearly $\tilde{f} \in C^1([0,\tau), X)$. Due to Proposition 6.4.6, it follows that $\tilde{f} \in C^1([0,\tau), W)$. The result now follows from Theorem 4.1.1. □

Remark To have $u_0 \in V$ and $u_1 \in V$ it is necessary that $\tilde{u}_{0,2} = \Gamma u_0$, $\tilde{u}_{0,3} = \Gamma u_0'$, $\tilde{u}_{1,2} = \Gamma u_1$ and $\tilde{u}_{1,3} = \Gamma u_1'$.

Theorem 2

Suppose $\lambda > 0$ and

(a) f_1 is locally Hölder continuous on $[0,\tau)$ with respect to the norm $\|\cdot\|$ and f_B is locally Hölder continuous on $[0,\tau)$,

(b) $u_0 \in V$ and $u_1 \in W$.

Then Problem 7W has a unique solution

$$u \in C\big([0,\tau),V\big) \cap C^1\big([0,\tau),W\big) \cap C^2\big((0,\tau),W\big).$$

If $f_1 = f_B = 0$, then $u \in C^\infty\big((0,\infty),W\big)$.

Proof Clearly $\tilde{f}$ is locally Hölder continuous on $[0,\tau)$ with respect to the norm $\|\cdot\|_X$. Due to Proposition 6.4.5, it follows that $\tilde{f}$ is locally Hölder continuous on $[0,\tau)$ with respect to the norm $\|\cdot\|_W$. The result now follows from Theorem 4.1.3. $\square$

Remark To have $u_0 \in V$ it is necessary that $\tilde{u}_{0,2} = \Gamma u_0$ and $\tilde{u}_{0,3} = \Gamma u_0'$. However $\tilde{u}_{1,2}$ and $\tilde{u}_{1,3}$ may be chosen in an arbitrary way.

Sufficient conditions for the case $\lambda > 0$

If $w_0 \in T[0,\ell]$, $w_1 \in \mathcal{L}^2(0,\ell)$, $\partial_t f$ is continuous and f_B' is continuous, then we have sufficient conditions for existence.

Sufficient conditions for the case $\lambda = 0$

We want sufficient conditions for Condition (4.1.1).

Using integration by parts twice we have that

$$\begin{aligned}
\bar{b}(w_0,v) &= \int_0^\ell EIw_0''\, v'' \\
&= -\int_0^\ell EIw_0'''\, v' + [EIw_0''(x)\, v'(x)]_0^\ell \\
&= \int_0^\ell EIw_0^{(4)}\, v - [EIw_0'''(x)\, v(x)]_0^\ell + [EIw_0''(x)\, v'(x)]_0^\ell \\
&= \int_0^\ell EIw_0^{(4)}\, v - EIw_0'''(\ell)\, v(\ell) + EIw_0''(\ell)\, v'(\ell)
\end{aligned}$$

for each $v \in T[0,\ell]$. From the definition of the bilinear form a we have that

$$\begin{aligned}
\bar{a}(w_1,v) &= \gamma w_1(\ell)v(\ell) + \gamma d w_1'(\ell)v(\ell) + \gamma d w_1(\ell)v'(\ell) \\
&\quad +(\gamma d + \gamma^*) d w_1'(\ell)v'(\ell).
\end{aligned}$$

Suppose that

$$\begin{aligned}
-EIw_0'''(\ell) &= m\,(\rho A)^{-1} EIw_0^{(4)}(\ell) + md\,(\rho A)^{-1} EI\, w_0^{(5)}(\ell) \\
&\quad -\gamma w_1(\ell) - \gamma d w_1'(\ell) \qquad (6.5.1)
\end{aligned}$$

and

$$\begin{aligned} EIw_0''(\ell) &= md\ (\rho A)^{-1}EI\ w_0^{(4)}(\ell) + (J + md^2)\ (\rho A)^{-1}EI\ w_0^{(5)}(\ell) \\ &\quad -\gamma d w_1(\ell) - (\gamma d + \gamma^*) d w_1'(\ell), \end{aligned} \tag{6.5.2}$$

then for each $v \in T[0, \ell]$ we have that

$$\begin{aligned} \bar{b}(w_0, v) + \bar{a}(w_1, v) &= \int_0^\ell EIw_0^{(4)}\ v + m\ (\rho A)^{-1}EIw_0^{(4)}(\ell)\ v(\ell) - md\ (\rho A)^{-1}EI\ w_0^{(5)}(\ell)\ v(\ell) \\ &\quad +md\ (\rho A)^{-1}EI\ w_0^{(4)}(\ell)\ v'(\ell) - (J + md^2)\ (\rho A)^{-1}EI\ w_0^{(5)}(\ell)\ v'(\ell) \\ &= \bar{c}\big((\rho A)^{-1}EI\ w_0^{(4)}, v\big). \end{aligned}$$

Since $\bar{a}$, $\bar{b}$ and $\bar{c}$ are continuous, we have that

$$\bar{b}(w_0, v) + \bar{a}(w_1, v) = \bar{c}\big((\rho A)^{-1}EI\ w_0^{(4)}, v\big) \quad \text{for any} \quad v \in V.$$

We conclude that the sufficient conditions are that $w_0 \in C^{(5)}[0, \ell] \cap T[0, \ell]$ satisfying Equations (6.5.1) and (6.5.2), $w_1 \in T[0, \ell]$, $\partial_t f$ and f_B' continuous.

Chapter 7

The Timoshenko model for a beam

7.1 Introduction

In 1921 Timoshenko introduced an improved theory for beams. It became known as the Timoshenko model and is generally considered to be an improvement on the Euler-Bernoulli and Rayleigh models. In Section 7.2 we show that the Rayleigh and Euler-Bernoulli models can be derived from Timoshenko model by making additional assumptions.

The Timoshenko theory is remarkable in that we have a one-dimensional theory that compares well with higher dimensional theories, see [LVV] and [SP].

The Timoshenko theory is also interesting from a mathematical perspective. The mathematical model consists of a system of two (coupled) wave equations (see Section 7.2). Numerous papers have been published on the subject, even recently. We mention only [KimR], [LVV], [Shu] and [SP].

In the article [Kim R] the Timoshenko theory for a beam is used with boundary damping. The beam is clamped at one end and a "control device" at the other end (it amounts to boundary damping). They establish the existence of a semigroup directly.

In the article [Shu] a cantilever Timoshenko beam with boundary damping is also considered. The main concern of the author is the completeness of the sequence of eigenfunctions but the existence of a semigroup is also proved.

In this chapter we prove the existence of a unique solution for the model problem in [KimR], using Theorem 4.1.1. This means that the approach here is completely different from that in [KimR] and [Shu].

7.2 The model problem

The Timoshenko model for a beam consists of two partial differential equations for the beam deflection w and the angle ϕ due to the rotation of a cross section:

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x V, & (7.2.1) \\ \rho I \partial_t^2 \phi &= V + \partial_x M. & (7.2.2) \end{aligned}$$

In these equations ρ denotes the density, A the area of the cross section and I the area moment of inertia.

The following constitutive equations for the moment M and the shear force V are used to derive the partial differential equations and to model some of the boundary conditions.

$$\begin{aligned} M &= EI\partial_x \phi, & (7.2.3) \\ V &= AG\kappa^2(\partial_x w - \phi), & (7.2.4) \end{aligned}$$

where E and G are elastic constants and κ^2 the shear coefficient. We refer the reader to [T, p 337-8], [I, p 337-8] and [N, p 392-5]. The equations of motion and the constitutive equations yield a system of partial differential equations which we do not use.

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x\big(AG\kappa^2(\partial_x w - \phi)\big), \\ \rho I \partial_t^2 \phi &= AG\kappa^2(\partial_x w - \phi) + \partial_x(EI\partial_x \phi). \end{aligned}$$

We consider a cantilever beam with boundary damping as in [KimR]. The length of the beam is denoted by ℓ. The beam is clamped at one end and we have that

$$w(0,t) = \phi(0,t) = 0. \qquad (7.2.5)$$

The boundary damping is modelled by

$$\begin{aligned} V(\ell,t) &= -\gamma \partial_t w(\ell,t), & (7.2.6) \\ M(\ell,t) &= -\gamma^* \partial_t \phi(\ell,t). & (7.2.7) \end{aligned}$$

Problem 8

The mathematical model consists of the Equations of motion (7.2.1) and (7.2.2), the Constitutive equations (7.2.3) and (7.2.4) and the Boundary conditions (7.2.5), (7.2.6) and (7.2.7). The initial conditions are $w(\cdot,0) = w_0$, $\phi(\cdot,0) = \phi_0$, $\partial_t w(\cdot,0) = w_1$ and $\partial_t\phi(\cdot,0) = \phi_1$.

Parameters We assume that the beam may be tapered and the cross-sectional area A and area moment I vary along the length of the beam.

The Euler-Bernoulli and Rayleigh models

Combining the equations of motion yields

$$\rho A\partial_t^2 w = \partial_x(\rho I\partial_t^2\phi) - \partial_x^2 M.$$

If it is assumed that $\partial_x w = \phi$, then

$$\rho A\partial_t^2 w = \partial_x(\rho I\partial_t^2\partial_x w) - \partial_x^2 M.$$

Using $M = EI\partial_x^2 w$, yields the model without shear due to Rayleigh (as in Section 1.4):

$$\rho A\partial_t^2 w = \partial_x(\rho I\partial_t^2\partial_x w) - \partial_x^2(EI\partial_x^2 w).$$

The constitutive equation for the shear force V is now redundant.

The term $\rho I\partial_t^2\partial_x w$ in the equation above is referred to as the rotary inertia term. The Euler-Bernoulli model is a special case of the Rayleigh model where rotary inertia is ignored.

7.3 Variational form

Multiply Equation (7.2.1) by an arbitrary function $v \in C_*^1[0,\ell]$ and integrate. Using integration by parts yields

$$\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v = -\int_0^\ell V(\cdot,t)v' + [V(x,t)v(x)]_0^\ell.$$

Multiply Equation (7.2.2) by an arbitrary function $\psi \in C_*^1[0,\ell]$ and integrate. Using integration by parts yields

$$\int_0^\ell \rho I(\partial_t^2\phi(\cdot,t))\psi = -\int_0^\ell M(\cdot,t)\psi' + \int_0^\ell V(\cdot,t)\psi + [M(x,t)\psi(x)]_0^\ell.$$

Test functions

A function v is a test function if $v \in C^1_*[0,\ell]$ and $v(0) = 0$. The space of test functions is denoted by $T[0,\ell]$.

It follows that for v and ψ in $T[0,\ell]$ we have

$$\begin{aligned}
\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v &= -\int_0^\ell V(\cdot,t)v' + V(\ell,t)v(\ell),\\
\int_0^\ell \rho I(\partial_t^2 \phi(\cdot,t))\psi &= -\int_0^\ell M(\cdot,t)\psi' + \int_0^\ell V(\cdot,t)\psi + M(\ell,t)\psi(\ell).
\end{aligned}$$

Adding the equations above yields

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v + \int_0^\ell \rho I(\partial_t^2 \phi(\cdot,t))\psi\\
= &-\int_0^\ell M(\cdot,t)\psi' - \int_0^\ell V(\cdot,t)(v'-\psi) + V(\ell,t)v(\ell) + M(\ell,t)\psi(\ell).
\end{aligned}$$

We now use the Constitutive equations (7.2.3) and (7.2.4) and the Boundary conditions (7.2.5),(7.2.6) and (7.2.7) and obtain

$$\begin{aligned}
&\int_0^\ell \rho A(\partial_t^2 w(\cdot,t))v + \int_0^\ell \rho I(\partial_t^2 \phi(\cdot,t))\psi\\
= &-\int_0^\ell EI\partial_x\phi(\cdot,t)\psi' - \int_0^\ell AG\kappa^2(\partial_x w(\cdot,t) - \phi(\cdot,t))(v'-\psi)\\
&-\gamma\partial_t w(\ell,t)v(\ell) - \gamma^*\partial_t\phi(\ell,t)\psi(\ell). \qquad (7.3.1)
\end{aligned}$$

It is now possible to present the variational form of the model problem.

Problem 8V

Find a pair of functions $\langle w, \phi\rangle$ such that for each $t > 0$, $w(\cdot,t)$ and $\phi(\cdot,t)$ are in $T(0,\ell)$ and Equation (7.3.1) is satisfied for each pair $\langle v, \psi\rangle$ such that v and ψ are in $T(0,\ell)$. The initial conditions are $w(\cdot,0) = w_0$, $\phi(\cdot,0) = \phi_0$, $\partial_t w(\cdot,0) = w_1$ and $\partial_t\phi(\cdot,0) = \phi_1$.

Theorem

If $\langle w, \phi\rangle$ is a solution of Problem 8, then it is a solution of Problem 8V. If $\langle w, \phi\rangle$ is a solution of Problem 8V and $w(\cdot,t) \in C^2_*[0,\ell]$ and $\phi(\cdot,t) \in C^2[0,\ell]$, then $\langle w, \phi\rangle$ is a solution of Problem 8.

Proof First part of the theorem has been proved. The second part is similar to the proof of Theorem 1.2.1. ☐

Remark Problem 8V may be used to compute an approximation for the solution by the finite element method.

7.4 Weak variational form

Consider Problem 8V. We start of as usual by considering the closure of the space of test functions. Let $V(0,\ell)$ be the closure of $T[0,\ell]$ in $H^1(0,\ell)$, then $V(0,\ell)$ is a Hilbert space (being a closed subspace of a Hilbert space).

We require the trace operator which is denoted by Γ (see Section 2.5).

The following product spaces are necessary for the abstract problem.

Product spaces

$X = \mathcal{L}^2(0,\ell) \times \mathcal{L}^2(0,\ell)$. An element $y \in X$ is written as $y = \langle y_1, y_2 \rangle$.

$H^m = H^m(0,\ell) \times H^m(0,\ell)$.

$V = V(0,\ell) \times V(0,\ell)$.

It is convenient to introduce the following bilinear forms:

$$\begin{array}{llcl}
\text{For } u \text{ and } v \text{ in } V & b(u,v) & = & \displaystyle\int_0^\ell EIu_2'v_2' + \int_0^\ell AG\kappa^2(u_1' - u_2)(v_1' - v_2), \\
 & a(u,v) & = & \gamma\Gamma u_1\Gamma v_1 + \gamma^*\Gamma u_2\Gamma v_2 \ \text{ and} \\
\text{for } u \text{ and } v \text{ in } X & c(u,v) & = & \displaystyle\int_0^\ell \rho A u_1 v_1 + \int_0^\ell \rho I u_2 v_2.
\end{array}$$

Remark Note that the bilinear forms a, b and c are symmetric.

Proposition 1

Let u and v be any functions in $V(0,\ell)$. Then there exist positive constants K_i such that

$$\begin{aligned}
K_1\|u'\|^2 &\leq \int_0^\ell EI(u')^2 \leq K_2\|u'\|^2,\\
K_3\|u'-v\|^2 &\leq \int_0^\ell AG\kappa^2(u'-v)^2 \leq K_4\|u'-v\|^2,\\
K_5\|u\|^2 &\leq \int_0^\ell \rho Au^2 \leq K_6\|u\|^2,\\
K_7\|u\|^2 &\leq \int_0^\ell \rho Iu^2 \leq K_8\|u\|^2.
\end{aligned}$$

Proof It follows directly from the assumption that the functions A and I are bounded above and below by positive constants. □

Proposition 2

The bilinear form c is an inner product for the space X.

Proof It is clear that c is a symmetric bilinear form and it follows directly from Proposition 1 that $u = 0$ when $c(u,u) = 0$. □

Definition

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W.

Proposition 3

The space V is a dense subset of W.

Proof Since $C_0^\infty(0,\ell)$ is dense in $\mathcal{L}^2(0,\ell)$ it follows that $V(0,\ell)$ dense in $\mathcal{L}^2(0,\ell)$ and we have $V(0,\ell)\times V(0,\ell)$ dense in $\mathcal{L}^2(0,\ell)\times\mathcal{L}^2(0,\ell)$. □

Proposition 4

There exists a constant K_b such that

$$b(u,u) \quad \geq \quad K_b\|u\|_W^2 \quad \text{for each } u \in V.$$

Proof From the lemma in Section 2.4, we have

$$\|u_1\| \leq \ell\|u_1'\| \quad \text{and} \quad \|u_2\| \leq \ell\|u_2'\|. \tag{7.4.1}$$

Now (using Proposition 1)

$$\begin{aligned} c(u,u) &= \int_0^\ell \rho A u_1^2 + \int_0^\ell \rho I u_2^2 \\ &\leq K_6\|u_1\|^2 + K_8\|u_2\|^2. \end{aligned}$$

But (using (7.4.1))

$$\begin{aligned} \|u_1\|^2 \leq \ell^2\|u_1'\|^2 &\leq \ell^2\|u_1' - u_2\|^2 + \ell^2\|u_2\|^2 \\ &\leq \ell^2\|u_1' - u_2\|^2 + \ell^4\|u_2'\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} c(u,u) &\leq K_6(\ell^2\|u_1' - u_2\|^2 + \ell^4\|u_2'\|^2) + K_8\ell^2\|u_2'\|^2 \\ &= K_6\ell^2\|u_1' - u_2\|^2 + (K_6\ell^4 + K_8\ell^2)\|u_2'\|^2. \qquad (7.4.2) \end{aligned}$$

On the other hand

$$\begin{aligned} b(u,u) &= \int_0^\ell EI(u_2')^2 + \int_0^\ell AG\kappa^2(u_1' - u_2)^2 \\ &\geq K_1\|u_2'\|^2 + K_3\|u_1' - u_2\|^2, \qquad (7.4.3) \end{aligned}$$

by Proposition 1. Combining (7.4.2) and (7.4.3) yields the result. ☐

Corollary

There exists a constant $K_b^* > 0$ such that

$$b(u,u) \geq K_b^*\left(\|u_1'\|^2 + \|u_2'\|^2\right).$$

Proposition 5

The bilinear form b is an inner product for V.

Proof It is clear that b is a symmetric bilinear form. If $b(u,u) = 0$, it follows from Proposition 4 that $c(u,u) = 0$ and hence $u = 0$ (by Proposition 2). ☐

Definition

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

Problem 8W

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = 0 \quad \text{for each} \quad v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_1 = \langle w_1, \phi_1 \rangle$.

7.5 Existence

Proposition

There exists a constant K such that

$$|a(u, v)| \;\leq\; K\|u\|_V \; \|v\|_V.$$

for each u and v in V.

Proof

$$\begin{aligned} |a(u,v)| &\leq \gamma|\Gamma u_1| \; |\Gamma v_1| + \gamma^*|\Gamma u_2| \; |\Gamma v_2| \\ &\leq \gamma K_\ell^2\|u_1\|_1 \; \|v_1\|_1 + \gamma^* K_\ell^2\|u_2\|_1 \; \|v_2\|_1, \end{aligned}$$

using Proposition 2.4.4. Clearly there exists a constant $K_d > 0$ such that

$$|a(u,v)| \leq K_d\big(\|u_1\|_1^2 + \|v_1\|_1^2 + \|u_2\|_1^2 + \|v_2\|_1^2\big). \tag{7.5.1}$$

From Proposition 7.4.1 it follows that

$$c(u,u) \geq K_5\|u_1\|^2 + K_7\|u_2\|^2$$

and from Proposition 7.4.4 we have that

$$b(u,u) \geq K_b K_5\|u_1\|^2 + K_b K_7\|u_2\|^2. \tag{7.5.2}$$

The corollary in the previous section implies that

$$b(u,u) \geq K_b^*\big(\|u_1'\|^2 + \|u_2'\|^2\big). \tag{7.5.3}$$

The desired result now follows from (7.5.1), (7.5.2) and (7.5.3). ☐

We apply Theorem 4.1.1 to establish existence of solutions. Note that the Assumption **A1** is satisfied due to Proposition 7.4.3 and Assumption **A2** is satisfied due to Propositions 7.4.4 and 7.4.5. From the definition of the bilinear form a it is clear that a is symmetric and nonnegative. Due to the above proposition, a has all the properties required in Assumption **A3**.

Theorem If $u_0 \in V$, $u_1 \in V$ and there exists a $y \in W$ such that

$$b(u_0, v) + a(u_1, v) = c(y, v) \text{ for each } v \in V,$$

then Problem 8W has a unique solution

$$u \in C\big([0,\tau), V\big) \cap C^1\big([0,\tau), W\big) \cap C^1\big((0,\tau), V\big) \cap C^2\big((0,\tau), W\big).$$

Proof The result follows from Theorem 4.1.1. ☐

Sufficient conditions

We need to find conditions on w_0 and ϕ_0 such that Condition 4.1.1 is satisfied. Let v and ψ be arbitrary test functions and denote $\langle v, \psi\rangle$ by z. Using integration by parts, we obtain

$$\begin{aligned} b(u_0, z) &= \int_0^\ell EI\phi_0\psi + \int_0^\ell \kappa^2 AG(w_0' - \phi_0)(v' - \psi) \\ &= -\int_0^\ell EI\phi_0''\psi + EI\phi_0'(\ell)\psi(\ell) - \int_0^\ell \kappa^2 AG(w_0' - \phi_0)\psi \\ &\quad - \int_0^\ell \kappa^2 AG(w_0' - \phi_0)'v + \kappa^2 AG\big(w_0'(\ell) - \phi_0(\ell)\big)v(\ell). \end{aligned}$$

Recall that

$$a(u_1, z) = \gamma w_1(\ell)v(\ell) + \gamma^*\phi_1(\ell)\psi(\ell).$$

Suppose

$$\kappa^2 AG(w_0'(\ell) - \phi_0(\ell)) = -\gamma w_1(\ell) \text{ and} \tag{7.5.4}$$

$$EI\phi_0(\ell) = -\gamma^*\phi_1(\ell). \tag{7.5.5}$$

Using the boundary conditions on w_0, w_1, ϕ_0 and ϕ_1, we obtain for any $z \in T[0,\ell] \times T[0,\ell]$ that

$$\begin{aligned} b(u_0, z) + a(u_1, z) &= -\int_0^\ell \kappa^2 AG(w_0' - \phi_0)'v - \int_0^\ell \big[EI\phi_0'' - \kappa^2 AG(w_0' - \phi_0)\big]\psi \\ &= c(y, z), \end{aligned}$$

where

$$\begin{aligned} y_1 &= \kappa^2 AG(\rho A)^{-1}(w_0' - \phi_0)' \in \mathcal{L}^2(0,\ell) \;\text{ and} \\ y_2 &= (\rho I)^{-1}\big[EI\phi_0'' - \kappa^2 AG(w_0' - \phi_0)\big] \in \mathcal{L}^2(0,\ell). \end{aligned}$$

Since the bilinear forms a, b and c are continuous with respect to the norm of V, this result will also be true for any $z \in V$.

We conclude that the sufficient conditions are that the pair w_0 and ϕ_0 is in $T[0,\ell] \cap C^2_*[0,\ell]$ and Equations (7.5.4) and (7.5.5) are satisfied.

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