

Traveling wave solution of the Kuramoto-Sivashinsky equation: A computational study

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Abstract. This work considers the numerical solution of the Kuramoto-Sivashinsky equation using the fractional time splitting method. We will investigate the numerical behavior of two categories of the traveling wave solutions documented in the literature (Hooper & Grimshaw (1998)), namely: the regular shocks and the oscillatory shocks. We will also illustrate the ability of the scheme to produce convergent chaotic solutions.

Keywords: Kuramoto-Sivashinsky equation, traveling wave solution, time-step splitting.

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INTRODUCTION

The one dimensional Kuramoto-Sivashinsky (K-S) equation

$$u_t + uu_x + u_{xx} + u_{xxx} = 0, \quad (1)$$

has attracted considerable attention in the study of nonlinear dynamics of partial differential equations. It was derived in various physical contexts such as: propagation of combustion fronts in gas, surface waves in film of a viscous fluid - to name just a few. Equation (1) also represents a class of equations capable of pattern formation [4], and is also a good example of a model for chaos, [5, 7].

In the setting where the initial data is periodic with zero average, the K-S equation has several interesting properties, see for example [6, 1]. These include the preservation of the periodicity for all t , bounds for the mean energy, and bounds for the first and second derivatives. In this work, we investigate, numerically, the traveling wave solutions of the K-S equation derived by [3]. We investigate the stability of the traveling wave solutions by comparing them to the solution of the full equation obtained using the time step splitting algorithm. In particular, for the numerical solution of the full K-S equation, we split equation (1) into the hyperbolic (inviscid Burgers) equation and the diffusion equation, respectively, as follows

$$u_t + f(u)_x = 0, \quad (2)$$

and

$$u_t + \phi(u)_{xx} = 0, \quad (3)$$

where $f(u) = u^2/2$ and $\phi(u) = u + u_{xx}$. From a numerical point of view, the discretised forms for equations (2) and (3) are handled as follows

$$\begin{cases} \frac{v^{n*} - v^n}{\Delta t} + A_1(v^n) = 0, \\ \frac{v^{n+1} - v^{n*}}{\Delta t} + A_2(v^{n*}, v^{n+1}) = 0, \end{cases} \quad (4)$$

where A_1 and A_2 are the discretisations for the nonlinear (convection) operator and the linear diffusion operator respectively. Here the index n refers to the current time step and $n+1$ the next time step. The asterisk denotes an intermediate step. The advantage of solving the equations (2) and (3) separately is that we avoid the restrictive stability requirements of the the fourth order derivative. In particular, here we solve the nonlinear convection equation

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using different schemes separately, while the linear equation is solved using the traditional Crank-Nicolson scheme. For the convection term we will explore the use of the following second order schemes: Godunov scheme, non-staggered central difference scheme (NSTG), semi discrete scheme (SemiD), fully discrete scheme (FullD), fully implicit scheme (FUIM), and the Crank Nicolson scheme (CNS). The choice for the scheme to handle the convection part of the problem was based on the need: (a) to eliminate any spurious oscillations, (b) for a scheme that possesses an appropriate form of consistency with the weak form of the conservation form - among other things.

TRAVELING WAVE SOLUTION

Using the transformation $u(x,t) = u(z)$ where $z = x - st$ and s is the wave speed, the traveling wave solution of the K-S equation satisfies, [3],

$$u''' = c + su - \frac{1}{2}u^2 - u', \quad (5)$$

where the prime denotes the derivative with respect to z . The wave speed s and the constant of integration c are determined by the far field solutions as

$$s = \frac{u_l + u_r}{2}, \quad c = -\frac{u_l u_r}{2}, \quad (6)$$

where $u \rightarrow u_l$ as $z \rightarrow -\infty$ and $u \rightarrow u_r$ as $z \rightarrow +\infty$. Note, the wave speed can also be found via the Rankine-Hugoniot condition to be

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}. \quad (7)$$

Here equation (5) is a third order nonlinear boundary value problem whose numerical solution can be explored in several ways. For example [2] suggested the use of B-spline functions. In our work, a second order finite difference scheme is used to discretise the equation and the resulting system of non-linear algebraic equations are solved using the Newton method.

NUMERICAL RESULTS

In this section we present numerical solution of the full K-S equation in (1) using the time splitting scheme. We start in the immediate section by checking the stability of the traveling wave solution. This is illustrated for the regular and the oscillatory shock by using the traveling wave solution as the initial condition. We conclude this section by illustrating the capability of the scheme to produce convergent chaotic solutions.

Traveling wave

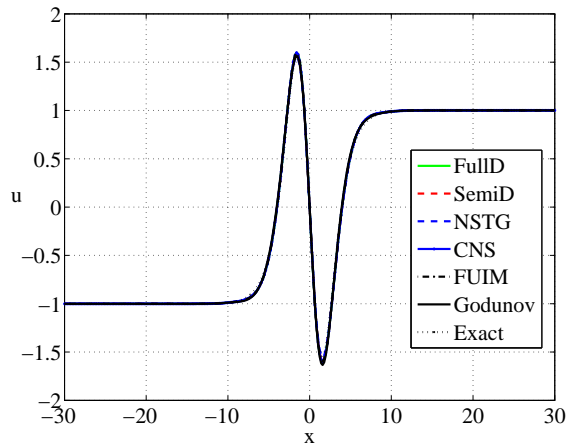
Hooper and Grimshaw (1988) classified the solutions of (5) based on the shock development as either regular shocks, solitary waves or oscillatory shocks. Here we will explore in detail the regular and the oscillatory shocks. In the next figures we compare the numerical solution of the full equation and the traveling wave solution. Fig. 1 compares the regular shock profile and Fig. 2 compares the oscillatory shock profile. Note that, from here forthwith, the labels in figures denote the scheme used to hand the convection part of the equation.

The results in Figs. 1 and 2 illustrate the capabilities of the chosen schemes. In addition to the presented figures, we perform a grid refinement study for regular shock initial conditions. The results, not shown here, confirmed a second order convergence in space for all the schemes.

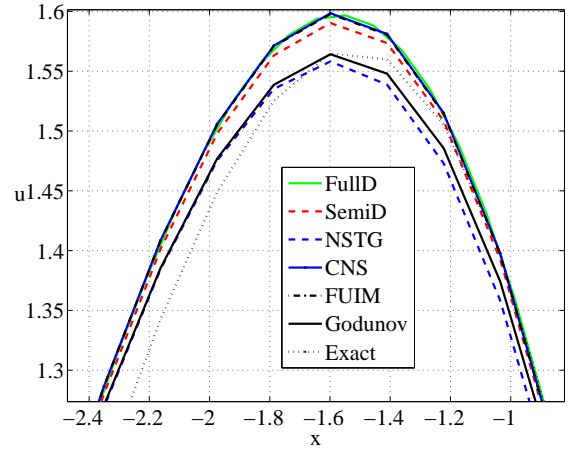
Chaotic solution

In this section we test the capabilities of our approach to simulate chaotic behavior. We note that a chaotic solution is observed when equation (1) is solved on a domain with periodic boundary conditions. Here we use the initial condition

$$u(x,0) = \cos\left(\frac{x}{16}\right) \left(1 + \sin\left(\frac{x}{16}\right)\right), \quad (8)$$

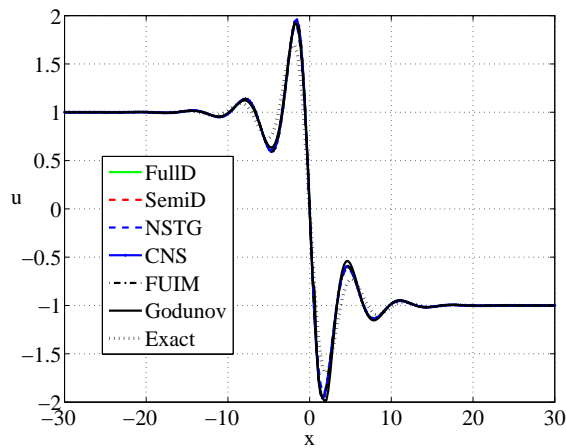


(a) All schemes with the regular shock.

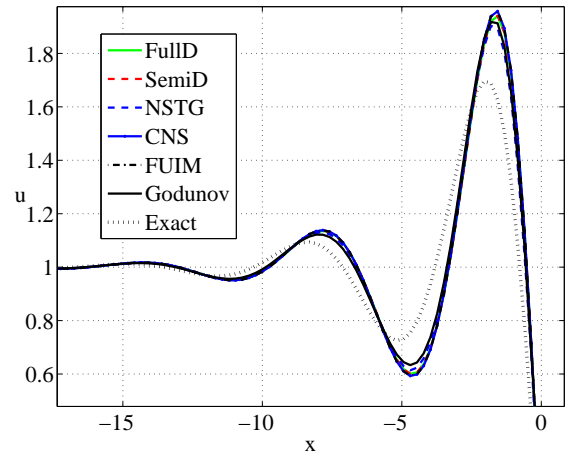


(b) A zoom-in of the solution in (a).

FIGURE 1. Comparison of the regular shock profile with the numerical solutions.



(a) All schemes with the oscillatory shock.



(b) A zoom-in of the solution in (a).

FIGURE 2. Comparison of the oscillatory shock profile with the numerical solutions.

consistent with the work of [8]. In Fig. 3 we present simulations run for 300 grid cells and time $T = 150$ where the contour lines show regions of equal peak. The figure illustrates the chaotic nature of the solution and we highlight that, for results not shown here, the results numerically converged. That is, the same profile was maintained for grid cells above 300.

CONCLUSIONS AND FUTURE WORK

In this work we developed a time splitting numerical method for the K-S equation. We highlight that all the considered schemes performed within the same range of accuracy except for the Godunov scheme. In particular, for the Godunov scheme one had to play around with the CFL number to produce the presented simulations. We also note the discrepancy in the traveling wave solution and the numerical solution in Fig. 2. As highlighted in [3], we are not certain whether this is linked to the chaotic behavior of the full equation with respect to the traveling wave.

In all the cases presented, a first order Euler scheme was used for the time stepping procedure. In future we intend to use higher order time differencing schemes such as the fourth order Runge-Kutta method.

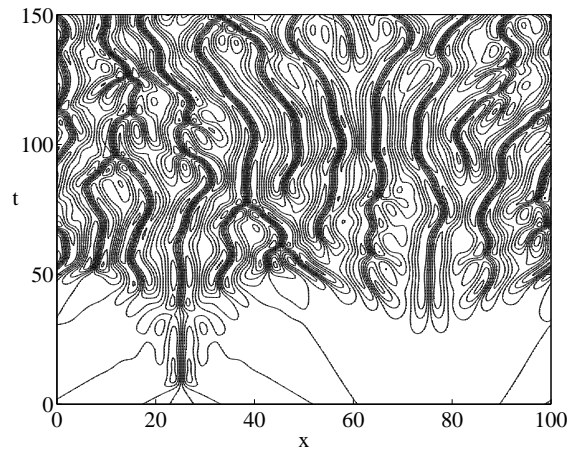


FIGURE 3. The chaotic solution of the K-S equation.

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