Cayley graphs of given degree and diameters 3, 4 and 5

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Abstract

Let $C_{d,k}$ be the largest number of vertices in a Cayley graph of degree d and diameter k. We show that $C_{d,3} \geq \frac{3}{16}(d-3)^3$ and $C_{d,5} \geq 25(\frac{d-7}{4})^5$ for any $d \geq 8$, and $C_{d,4} \geq 32(\frac{d-8}{5})^4$ for any $d \geq 10$. For sufficiently large d our graphs are the largest known Cayley graphs of degree d and diameters 3, 4 and 5.

Keywords: Cayley graph, degree, diameter

A Cayley graph $C(G, X)$ is specified by a group G and a unit-free generating set X for this group such that $X = X^{-1}$. The vertices of $C(G, X)$ are the elements of G and there is an edge between two vertices u and v in $C(G, X)$ if and only if there is a generator $a \in X$ such that $v = ua$.

The degree-diameter problem for Cayley graphs is to determine the largest number of vertices in Cayley graphs of given degree and diameter. Let $C_{d,k}$ be the largest order of a Cayley graph of degree d and diameter k. The number of vertices in a graph of maximum degree d and diameter k can not exceed the Moore bound $M_{d,k} = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$. In [1] and [2] Bannai and Ito improved the upper bound and showed that for any $d, k \geq 3$ there are no graphs of order greater than $M_{d,k} - 2$, therefore $C_{d,k} \leq M_{d,k} - 2$ for such d and k. Since the Moore graphs of diameter 2 and degree 3 or 7, and the potential Moore graph(s) of diameter 2 and degree 57 are non-Cayley (see [3]), Cayley graphs of order equal to the Moore bound exist only in the trivial cases when $d = 2$ or $k = 1$.

We focus on constructions of Cayley graphs of small diameter. The case $k = 2$ and $d \to \infty$ has been widely studied. The best known construction of diameter 2 was recently found by Šiagiová and Širáň [8] who showed that $C_{d,2} \geq d^2 - 6\sqrt{2}d^{\frac{3}{2}}$ for an infinite set of degrees d. Macbeth et. al. [7] presented large Cayley graphs giving the bound $C_{d,k} \geq k(\frac{d-3}{3})^k$ for any diameter $k \geq 3$ and degree $d \geq 5$. Let us also mention the Faber-Moore-Chen graphs [4] of odd degree $d \geq 5$, diameter k, such that $3 \leq k \leq \frac{d+1}{2}$, and order

$(\frac{d+3}{2})!/(\frac{d+3}{2}-k)!$. These graphs are vertex-transitive and in [7] it is proved that for any $k \geq 4$ and sufficiently large d the Faber-Moore-Chen graphs are not Cayley. Large Cayley graphs of given degree d and diameter k, where both d and k are small, were obtained by use of computers, see [5] and [6].

For diameters 3, 4 and 5 we present Cayley graphs which yield the bounds $C_{d,3} \geq \frac{3}{16}(d-3)^3$ and $C_{d,5} \geq 25(\frac{d-7}{4})^5$ for any $d \geq 8$, and $C_{d,4} \geq 32(\frac{d-8}{5})^4$ for any $d \geq 10$, improving thus the corresponding bounds of [7] for large d. Particularly for diameter 3 we improve the lower bound considerably. It can be easily checked that the graphs of Faber, Moore and Chen are smaller than our graphs for diameter 3 and large degree, and they are larger than our graphs for diameters 4 and 5. However, for $k = 4$ and $d \geq 21$, and for $k = 5$ and $d \geq 23$, the Faber-Moore-Chen graphs are non-Cayley. To the best of our knowledge, for sufficiently large d there is no construction of Cayley graphs of degree d and diameter 3, 4 or 5 of order greater than the order of our graphs.

Now we describe the groups G which we use to produce large Cayley graphs. Let H be a group of order $m \geq 2$ with unit element e. We denote by H^k the product $H \times H \times \ldots \times H$, where H appears k times. Let α be the automorphism of the group H^k which shifts coordinates by one to the right, that is, $\alpha(x_1, x_2, \ldots, x_k) = (x_k, x_1, x_2, \ldots, x_{k-1})$. The cyclic group of order p will be denoted by Z_p.

We study the semidirect products $G = H^k \rtimes Z_p$, where p is a multiple of k, with multiplication given by

$$(x, y)(x', y') = (x\alpha^y(x'), y + y'), \qquad (1)$$

where α^y is the composition of α with itself y times, $x, x' \in H^k$ and $y, y' \in Z_p$. Elements of G will be written in the form $(x_1, x_2, \ldots, x_k; y)$, where $x_1, x_2, \ldots, x_k \in H$ and $y \in Z_p$.

We consider generating sets X which consist of classes of elements of the form $(x_1, x_2, \ldots, x_k; y)$ where x_i, $1 \leq i \leq k$, is either e or g for any $g \in H$. In the case of diameters 3 and 5 we found generating sets for large Cayley graphs using four such classes, whereas for diameter 4 we needed five classes. In our search for relatively small generating sets (to result in large Cayley graphs in terms of their degree) it proved efficient to consider generating sets containing $(k+1)$-tuples as above with at most two non-identity entries among the first k coordinates; increasing this number did not yield better graphs.

We now state and prove our main result.

Theorem 1.
(i) $C_{d,3} \geq \frac{3}{16}d^3$ for $d \geq 8$ such that d is a multiple of 4.
(ii) $C_{d,4} \geq 32(\frac{d}{5})^4$ for $d \geq 10$ such that d is a multiple of 5.
(iii) $C_{d,5} \geq 25(\frac{d}{4})^5$ for $d \geq 8$ such that d is a multiple of 4.

Proof. We use the group G with multiplication (1) defined earlier.
(i) Let $G = H^3 \rtimes Z_{12}$ and $X = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$ where $a_g = (g, g, e; 1)$, $\bar{a}_{g'} = (g', e, g'; -1)$, $b_h = (h, e, e; 8)$ and $\bar{b}_{h'} = (e, h', e; 4)$. Since $a_g^{-1} = \bar{a}_{g^{-1}}$ and $b_h^{-1} = \bar{b}_{h^{-1}}$, we have $X = X^{-1}$. The Cayley graph $C(G, X)$ is of degree $d = |X| = 4m$, $m \geq 2$ and order $|G| = 12m^3 = 12(\frac{d}{4})^3 = \frac{3}{16}d^3$.

We show that the diameter of $C(G, X)$ is at most 3, which is equivalent to showing that each element of G can be obtained as a product of at most 3 generators of X. For any $x_1, x_2, x_3 \in H$ we have

$(x_1, x_2, x_3; 0) = (x_1, e, e; 8)(x_3, e, e; 8)(x_2, e, e; 8)$,
$(x_1, x_2, x_3; 1) = (x_1x_3^{-1}, e, e; 8)(x_3, x_3, e; 1)(e, x_2, e; 4)$,
$(x_1, x_2, x_3; 2) = (x_1, e, x_1; -1)(x_2, e, x_2; -1)(e, x_2^{-1}x_1^{-1}x_3, e; 4)$,
$(x_1, x_2, x_3; 3) = (x_1x_2^{-1}, e, e; 8)(x_3, e, e; 8)(x_2, e, x_2; -1)$,
$(x_1, x_2, x_3; 4) = (x_3, e, x_3; -1)(e, x_3^{-1}x_1x_2^{-1}, e; 4)(x_2, x_2, e; 1)$,
$(x_1, x_2, x_3; 5) = (x_1, e, e; 8)(x_3x_2^{-1}, e, e; 8)(x_2, x_2, e; 1)$,
$(x_1, x_2, x_3; 6) = (x_2x_3^{-1}, x_2x_3^{-1}, e; 1)(x_3, x_3, e; 1)(e, x_3x_2^{-1}x_1, e; 4)$.

It is easy to see that if $(x_1, x_2, x_3; y) = abc$, where $a, b, c \in X$ and $0 \leq y \leq 6$, then

$$(x^{-1}_{y \pmod 3)+1}, x^{-1}_{y+1 \pmod 3)+1}, x^{-1}_{y+2 \pmod 3)+1}; -y) = c^{-1}b^{-1}a^{-1}.$$

Note that the diameter of $C(G, X)$ cannot be smaller than 3, because the order is greater than the Moore bound for diameter 2.

(ii) Let $G' = H^4 \rtimes Z_{32}$ and $X' = \{a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'}, c_j \mid \text{for any } g, g', h, h', j \in H\}$, where $a_g = (g, e, e, e; 1)$, $\bar{a}_{g'} = (e, e, e, g'; -1)$, $b_h = (e, h, h, e; 7)$, $\bar{b}_{h'} = (e, e, h', h'; -7)$ and $c_j = (e, j, e, e; 16)$. Clearly, $X = X^{-1}$ since $a_g^{-1} = \bar{a}_{g^{-1}}$, $b_h^{-1} = \bar{b}_{h^{-1}}$ and $c_j^{-1} = c_{j^{-1}}$. The Cayley graph $C(G', X')$ has degree $d = |X'| = 5m$ and order $|G'| = 32m^4 = 32(\frac{d}{5})^4$.

We show that every element of G' can be expressed as a product of 4 generators of X'. For any $x_1, x_2, x_3, x_4 \in H$ we have

$(x_1,x_2,x_3,x_4;0)=b_{x_2}a_{x_4x_3^{-1}x_2}\bar{b}_{x_2^{-1}x_3}\bar{a}_{x_1},$
$(x_1,x_2,x_3,x_4;1)=\bar{b}_{x_4x_1^{-1}}c_{x_1x_4^{-1}x_3}\bar{b}_{x_1}\bar{a}_{x_2},$
$(x_1,x_2,x_3,x_4;2)=a_{x_1}c_{x_3}a_{x_2}c_{x_4},$
$(x_1,x_2,x_3,x_4;3)=a_{x_1x_2^{-1}x_4^{-1}}c_{x_3}\bar{b}_{x_4}\bar{b}_{x_2},$
$(x_1,x_2,x_3,x_4;4)=a_{x_1}a_{x_2}a_{x_3}a_{x_4},$
$(x_1,x_2,x_3,x_4;5)=c_{x_2x_4x_1^{-1}x_3^{-1}}b_{x_3}b_{x_1x_4^{-1}}b_{x_4},$
$(x_1,x_2,x_3,x_4;6)=\bar{a}_{x_4x_1^{-1}}\bar{a}_{x_3}b_{x_1}a_{x_2},$
$(x_1,x_2,x_3,x_4;7)=c_{x_2}\bar{b}_{x_3}\bar{a}_{x_1}\bar{a}_{x_3^{-1}x_4},$
$(x_1,x_2,x_3,x_4;8)=c_{x_2}a_{x_1}c_{x_3x_4^{-1}}b_{x_4},$
$(x_1,x_2,x_3,x_4;9)=c_{x_2x_3^{-1}}\bar{a}_{x_4}\bar{b}_{x_3}a_{x_1},$
$(x_1,x_2,x_3,x_4;10)=b_{x_3}a_{x_4}a_{x_1}a_{x_3^{-1}x_2},$
$(x_1,x_2,x_3,x_4;11)=c_{x_2}a_{x_1x_4^{-1}}\bar{b}_{x_4}a_{x_3},$
$(x_1,x_2,x_3,x_4;12)=b_{x_2}\bar{a}_{x_2^{-1}x_3}b_{x_4}\bar{a}_{x_4^{-1}x_1},$
$(x_1,x_2,x_3,x_4;13)=\bar{a}_{x_4}c_{x_1}\bar{a}_{x_3}\bar{a}_{x_2},$
$(x_1,x_2,x_3,x_4;14)=a_{x_1x_2^{-1}}b_{x_3}\bar{a}_{x_3^{-1}x_4}b_{x_2},$
$(x_1,x_2,x_3,x_4;15)=c_{x_2x_3^{-1}x_4}\bar{b}_{x_4}\bar{a}_{x_1}b_{x_4^{-1}x_3},$
$(x_1,x_2,x_3,x_4;16)=a_{x_1x_3x_4^{-1}}a_{x_2}b_{x_4x_3^{-1}}b_{x_3}.$

Elements of G' with the last coordinate y, where $17\leq y\leq 31$, can be obtained as inverses of the above ones, therefore the diameter of $C(G',X')$ is at most 4. It is easy to show that, for example, no element of G' with the last coordinate 4 can be obtained as a product of at most 3 elements of X', hence the diameter of $C(G',X')$ cannot be smaller than 4.

(iii) Let $G''=H^5\rtimes Z_{25}$ and $X''=\{a_g,\bar{a}_{g'},b_h,\bar{b}_{h'}\mid \text{for any } g,g',h,h'\in H\}$ where $a_g=(g,e,e,e,e;1)$, $\bar{a}_{g'}=(e,e,e,e,g';-1)$, $b_h=(h,e,e,h,e;-4)$ and $\bar{b}_{h'}=(e,e,h',e,h';4)$. The Cayley graph $C(G'',X'')$ is of degree $d=|X''|=4m$ and order $|G''|=25m^5=25(\frac{d}{4})^5$.

In order to prove that the diameter of $C(G'',X'')$ is at most 5, it suffices to show that any element $(x_1,x_2,x_3,x_4,x_5;y)$ of G'', where $0\leq y\leq 12$, can be obtained as a product of 5 generators of X''. It can be checked that

$(x_1,x_2,x_3,x_4,x_5;0)=a_{x_1}a_{x_2}a_{x_3x_5^{-1}}a_{x_4}b_{x_5},$
$(x_1,x_2,x_3,x_4,x_5;1)=\bar{b}_{x_3}\bar{a}_{x_4x_2^{-1}}b_{x_2}a_{x_3^{-1}x_5}a_{x_1},$
$(x_1,x_2,x_3,x_4,x_5;2)=b_{x_4x_2^{-1}}\bar{a}_{x_2x_4^{-1}x_1x_3^{-1}}\bar{a}_{x_5}\bar{b}_{x_2}\bar{b}_{x_3},$
$(x_1,x_2,x_3,x_4,x_5;3)=a_{x_1}b_{x_5}a_{x_3}a_{x_4x_2^{-1}x_5}\bar{b}_{x_5^{-1}x_2},$
$(x_1,x_2,x_3,x_4,x_5;4)=\bar{b}_{x_5}\bar{b}_{x_2}\bar{a}_{x_5^{-1}x_3x_1^{-1}}b_{x_1}a_{x_2^{-1}x_4},$

$$
\begin{aligned}
(x_1,x_2,x_3,x_4,x_5;5) &= a_{x_1}a_{x_2}a_{x_3}a_{x_4}a_{x_5},\\
(x_1,x_2,x_3,x_4,x_5;6) &= \bar{b}_{x_3}\bar{b}_{x_2}a_{x_2^{-1}x_4x_1^{-1}}a_{x_3^{-1}x_5}b_{x_1},\\
(x_1,x_2,x_3,x_4,x_5;7) &= a_{x_1x_2x_4^{-1}}\bar{b}_{x_4x_2^{-1}}\bar{a}_{x_5}\bar{b}_{x_2}\bar{a}_{x_3},\\
(x_1,x_2,x_3,x_4,x_5;8) &= \bar{a}_{x_5x_2^{-1}x_4^{-1}x_1x_3^{-1}}b_{x_3x_1^{-1}x_4}b_{x_4}b_{x_2}b_{x_4{}^{-1}x_1},\\
(x_1,x_2,x_3,x_4,x_5;9) &= \bar{a}_{x_5}\bar{b}_{x_2}\bar{b}_{x_1}a_{x_1^{-1}x_3}a_{x_2^{-1}x_4},\\
(x_1,x_2,x_3,x_4,x_5;10) &= \bar{b}_{x_5}\bar{b}_{x_4}\bar{b}_{x_5^{-1}x_3}\bar{a}_{x_4^{-1}x_2}\bar{a}_{x_3^{-1}x_5x_1},\\
(x_1,x_2,x_3,x_4,x_5;11) &= \bar{b}_{x_3}\bar{b}_{x_2}a_{x_2^{-1}x_4}a_{x_3^{-1}x_5}a_{x_1},\\
(x_1,x_2,x_3,x_4,x_5;12) &= a_{x_1x_2x_4^{-1}}\bar{b}_{x_4x_2^{-1}}\bar{b}_{x_5}\bar{b}_{x_2}\bar{a}_{x_5^{-1}x_3}.
\end{aligned}
$$

Hence, $C_{d,5} \geq 25(\frac{d}{4})^5$ for any $d \geq 8$ such that d is a multiple of 4. $\square$

By adding new elements to the generating sets, we get Cayley graphs of any degree $d \geq 10$ if $k = 4$, and $d \geq 8$ if $k = 3$ or 5.

Theorem 2.
(i) $C_{d,3} \geq \frac{3}{16}(d-3)^3$ for any $d \geq 8$.
(ii) $C_{d,4} \geq 32(\frac{d-8}{5})^4$ for any $d \geq 10$.
(iii) $C_{d,5} \geq 25(\frac{d-7}{4})^5$ for any $d \geq 8$.

Proof. We use the notation of the proof of Theorem 1.

(i) By Theorem 1, $C_{d,3} \geq \frac{3}{16}d^3$ for $d = 4m$, where $m \geq 2$. Let u be an element of G such that $u \notin X$ and $u \neq u^{-1}$. Let $X_1 = X \cup \{v\}$, $X_2 = X \cup \{u, u^{-1}\}$ and $X_3 = X \cup \{u, u^{-1}, v\}$ where $v = (e, e, e, 6)$. Then the Cayley graph $C(G, X_i)$ has degree $d = |X_i| = 4m + i$, diameter at most $k = 3$ and order $|G| = 12m^3 = \frac{3}{16}(d-i)^3$, where $i = 1, 2, 3$. Hence $C_{d,3} \geq \frac{3}{16}(d-3)^3$ for any $d \geq 8$.

(ii) We know that $C_{d,4} \geq 32(\frac{d}{5})^4$ for $d = 5m$, $m \geq 2$. Let u_i, $i = 1, 2, 3, 4$, be non-involuntary elements of G' such that $u_i \notin X'$. Let $X'_i = X' \cup \{u_1, \ldots, u_i, u_1^{-1}, \ldots, u_i^{-1}\}$ where $i = 1, 2, 3, 4$. The Cayley graph $C(G', X'_i)$ has degree $d = |X'_i| = 5m + 2i$ and order $|G'| = 32m^4 = 32(\frac{d-2i}{5})^4$. It remains to show that $C_{d,4} \geq 32(\frac{d-8}{5})^4$ for $d = 11$ and 13. However, if $m = 2$, then G' has an involution which is not in X'. By using this involution, we can obtain Cayley graphs of order $32(\frac{10}{5})^4 = 2^9$ and degree 11 or 13.

(iii) Let $m \geq 2$ be even. Then the order of H is even and G'' must contain an involution, say v, other than the identity. Let $X''_0 = X'' \cup \{v\}$ and $X''_i = X'' \cup \{v, u_1, \ldots, u_i, u_1^{-1}, \ldots, u_i^{-1}\}$ for $i = 1, 2, 3$, where $u_i \neq u_i^{-1}, u_i \notin X''$. The graph $C(G'', X''_i)$, $i = 0, 1, 2, 3$, is of degree $|X''_i| = 4m + 2i + 1$, diameter at most $k = 5$ and order $|G''| = 25m^5 = 25(\frac{d-2i-1}{4})^5$.

Since for any $m \geq 2$ we can obtain a Cayley graph of degree $4m+2$ and order $25(\frac{d-2}{4})^5$ by adjoining u_1 and u_1^{-1} to X'', we have $C_{d,5} \geq 25(\frac{d-7}{4})^5$ for any $d \geq 8$ as desired. □

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