

FINITE ELEMENT APPROXIMATION OF GENERAL SECOND ORDER HYPERBOLIC EQUATIONS

BY

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Summary

The vibration of elastic bodies and structures consisting of elastic bodies is an active research field in engineering. A mathematical model for such a vibrating system is a complex system of partial differential equations. In this dissertation only linear cases are considered and the variational form is of the same type as the variational form for the well known wave equation in each case. The generalized problem is to solve a second order differential equation for a function with values in a Hilbert space. This problem is referred to as the general second order hyperbolic equation in this dissertation.

Obviously numerical approximation of solutions is inevitable. The finite element method proved to be ideal. It is used for steady-state problems, eigenvalue problems and dynamic problems. This dissertation is a study of the state of the theory for dynamic problems: convergence of the finite element approximation and error estimates. The approach is to split the approximation problem. First the convergence for the semi-discrete problem is considered followed by the convergence of the fully discrete approximation to the solution of the semi-discrete problem. The focus is on the (continuous) Galerkin method. However one chapter is devoted to the application of the Trotter-Kato theorem and brief discussions on the mixed finite element method and discontinuous Galerkin method are included.

An in depth study was made of two articles and the relevant part of a PhD thesis. From a further five articles the useful parts (for this dissertation) were studied in depth. Five more articles were read but only to determine their contribution to the theory.

In this dissertation the theory for the Galerkin method is presented as a unit; the material from the sources are merged, duplication eliminated and similarities and differences noted. By standardizing the notation and expanding

proofs the presentation is (hopefully) more readable than the original publications.

Other notable contributions in this dissertation are the generalization of certain assumptions for wider application, weakening of some assumptions, the generalization of the main result in the 1976 article of Baker, application of Dupont's method (1973) to the Timoshenko beam with boundary damping and application of the Trotter-Kato theorem to the general second order hyperbolic equation.

The main conclusion is that the general results are not satisfactory. Apart from the application of the Trotter-Kato theorem, where error estimates are in general not available, the theory is not sufficiently general to cover all linear vibration models. It is also clear from examples that the assumptions in the theory are too restrictive.

The dissertation does not only provide an overview of the theory of the Galerkin method for dynamic problems but the presentation provides a basis for future research.

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Chapter 1

Model problems in variational form

1.1 Introduction

In the preface to the book of Strang and Fix [SF] we read: “The finite element method has been an astonishing success. It was created to solve the complicated equations of elasticity and structural mechanics, and for those problems it has essentially superseded the method of finite differences. Now other applications are rapidly developing. Whenever flexibility in the geometry is important - and the power of the computer is needed not only to *solve* a system of equations, but also to *formulate* and *assemble* the discrete approximation in the first place - the finite element method has something to contribute.”

In Chapter 7 of the same book: “So far we have discussed only steady-state problems - elliptic equations and eigenvalue problems. Galerkin’s principle is flexible enough to apply also to initial-value problems. They still have important advantages over finite differences, for general geometries and for problems which evolve comparatively slowly in time.”

In particular, we read in Section 7.3: “It is natural to experiment with finite elements also for hyperbolic problems.”

The book was published in 1973. A number of references can be found on numerical experiments with hyperbolic problems but nothing on convergence.

In Section 7.3 of [SF] a sketch of a convergence proof is given. We discuss this in Chapter 2.

1.1.1 The wave equation

As far as hyperbolic problems are concerned, we consider only mathematical models of vibration problems. The simplest example is the one-dimensional wave equation which we use for our introduction.

Problem 1 Given a positive constant α , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u \text{ in } (0, \ell) \text{ for } t > 0, \\ u(0, t) &= \partial_x u(\ell, t) = 0 \text{ for } t > 0, \\ u(x, 0) &= u_0(x) \text{ for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \text{ for } x \in (0, \ell).\end{aligned}$$

This problem is a mathematical model for longitudinal vibrations of a rod [I] or transverse vibrations of a string. It is necessary to prescribe the initial position and initial velocity of every point.

To apply the finite element method, the problem is written in variational form. Consider the term $\alpha \partial_x^2 u$ in the wave equation. We multiply the term by a function $\phi \in C^1[0, \ell]$ and integrate. Integration by parts yields

$$\int_0^\ell \alpha \partial_x^2 u(\cdot, t) \phi = - \int_0^\ell \alpha \partial_x u(\cdot, t) \phi' + [\alpha \partial_x u(\cdot, t) \phi]_0^\ell, \quad (1.1.1)$$

provided that $u(\cdot, t) \in C^2[0, \ell]$. Note that the continuity of $\partial_x u(\cdot, t)$ is essential.

We next introduce the so called test functions.

Test functions

$$T[0, \ell] = \{\phi \in C^1[0, \ell] \mid \phi(0) = 0\}.$$

From (1.1.1) we have that for each $\phi \in T[0, \ell]$

$$\int_0^\ell \alpha \partial_x^2 u(\cdot, t) \phi = - \int_0^\ell \alpha \partial_x u(\cdot, t) \phi' + \alpha \partial_x u(\ell, t) \phi(\ell). \quad (1.1.2)$$

Since $\partial_x u(\ell, t) = 0$ in Problem 1,

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = - \int_0^\ell \alpha \partial_x u(\cdot, t) \phi' \text{ for each } \phi \in T[0, \ell]. \quad (1.1.3)$$

1.1. INTRODUCTION

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Notation

$$b(u, v) = \int_0^\ell \alpha u' v' \quad \text{and} \quad (f, g) = \int_0^\ell f g.$$

Next we formulate the variational form of Problem 1.

Problem 1V Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$(\partial_t^2 u(\cdot, t), v) = -b(u(\cdot, t), v) \quad \text{for each } v \in T[0, \ell]$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

The original problem (Problem 1) and its variational form (Problem 1V) are equivalent in the sense that a solution of Problem 1 is a solution of Problem 1V and a solution of Problem 1V is a solution of Problem 1 if it is smooth enough.

In the finite element method we consider basis functions $\phi_1, \phi_2, \dots, \phi_n$ in $T[0, \ell]$ and denote the span of this set by S^h . The following problem is the finite element (Galerkin) approximation.

Problem 1V^h Find u^h such that for each $t > 0$, $u^h(\cdot, t) \in S^h$ and

$$(\partial_t^2 u^h(\cdot, t), v) = -b(u^h(\cdot, t), v) \quad \text{for each } v \in S^h,$$

while $u^h(\cdot, 0) = u_0^h$ and $\partial_t u^h(\cdot, 0) = u_1^h$ must be chosen appropriately.

In general u_0 and u_1 are not in S^h and hence $u_0^h \neq u_0$ and $u_1^h \neq u_1$. It is convenient to use the interpolants of u_0 and u_1 as initial conditions, but there are other possibilities. Since $u^h(\cdot, t) \in S^h$ for each t ,

$$u^h(x, t) = \sum_{j=1}^n u_j(t) \phi_j(x).$$

We find that u^h satisfy the variational equation in Problem 1V^h if and only if

$$\sum_{j=1}^n u_j''(t) (\phi_j, \phi_i) = - \sum_{j=1}^n u_j(t) b(\phi_j, \phi_i) \quad \text{for } i = 1, 2, \dots, n.$$

This problem is equivalent to a system of ordinary differential equations. Let

$$K_{ij} = b(\phi_j, \phi_i) \quad \text{and} \quad M_{ij} = (\phi_j, \phi_i),$$

then the vector function $\bar{u} = (u_1, u_2, \dots, u_n)$ satisfies an initial value problem for a system of ordinary differential equations.

Problem 1 ODE

$$M\bar{u}'' = -K\bar{u} \quad \text{with} \quad \bar{u}(0) = \bar{b} \quad \text{and} \quad \bar{u}'(0) = \bar{a},$$

where $u_0^h = \sum_{j=1}^n b_j \phi_j$ and $u_1^h = \sum_{j=1}^n a_j \phi_j$.

Problem 1 ODE is equivalent to Problem 1V^h (see Section 7.1).

To approximate the solution of Problem 1 ODE, a finite difference method is used. Consider for example:

$$\frac{1}{(\delta t)^2} M [\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] = \frac{\alpha}{4} K [\bar{u}_{k+1} + 2\bar{u}_k + \bar{u}_{k-1}].$$

$$\bar{u}_0 = \bar{b} \quad \text{and} \quad \bar{u}_1 - \bar{u}_{-1} = 2\delta t \bar{a}.$$

This is referred to as the fully discrete problem.

1.1.2 Engineering applications

As mentioned in [SF], the finite element method “... was created to solve the complicated equations of elasticity and structural mechanics, and for those problems it has essentially superseded the method of finite differences.” This remark referred to steady-state problems, but before the book appeared finite element experiments on vibration problems were conducted.

In the late sixties and early seventies many articles concerning the finite element method for second order hyperbolic equations appeared. Two important books (apart from [SF]) were [H] and [Z]. In the preface of [Z], published in 1977, we read: “Since the first volume (of [Z]) was written the number of research publications on the finite element method has been increasing almost exponentially. Close to 8000 references are recorded ” Chapter 20 of [Z] deals with “semi-discretization of field and dynamic problems” with 81 references – about half of them on vibration problems in engineering. In Chapter 21 the numerical treatment of the system of ordinary differential equations below is considered with 53 references, a number of them on vibration problems.

Significant for us, are the complex problems of elasto-dynamics and the vibration of structures. Early development of the finite element method for

structural dynamics was in the period 1969-1975, see [Wu03] for references. [Wu03]: “Structural dynamics is one of the most active and attractive research fields with a long history of development. Transient response plays an important role ...”

How to approximate the solution of the semi-discrete problem

$$Mx'' + Cx' + Kx = F,$$

was considered with mathematical rigor in [H], [Z] and other publications. As far as we could ascertain, the first results on convergence of the finite element approximation to the exact solution appeared in [SF, Chapter 7] and [D73]. But these results were for the wave equation and by that time engineers were already applying the methods to more complex vibration problems.

Our interest is mainly in the theory but before we start, a brief overview of the variety of applications is required. Damping may be introduced in the one-dimensional wave equation, see Section 1.2. The general wave equation is considered in Section 1.3, but it does not cover the vibration of beams and plates. Classical models for the vibration of beams and plates and improved theories are discussed in Section 1.4. In Section 1.5 elasto-dynamics is considered. Finally, in Section 1.6, we introduce structures consisting of more than one elastic body.

1.1.3 Theory

In this dissertation the aim is to investigate the state of the theory: Convergence of the finite element approximation and error estimates. Firstly we consider convergence of the solution of the semi-discrete problem to the solution of the variational problem and secondly the convergence of the solution of the fully discrete problem to the solution of the semi-discrete problem.

Concerning the convergence of the solution of the semi-discrete problem to the solution of the variational problem, we concentrate on the Galerkin method but devote one chapter to the application of semigroup theory. In the last chapter we mention other methods e.g. mixed finite elements and the discontinuous Galerkin method.

Any serious investigation of the Galerkin method must include [D73] and [B76]. Many articles refer to these two publications. On **MathSciNet** alone we find 34 citations on the 30th of June 2011 for [D73] (including 2 in 2011 and 3 in 2010) and 26 citations for [B76] (including 1 in 2011 and 2 in 2010).

It is also necessary to consider convergence of the solution of the fully discrete problem to the solution of the semi-discrete problem. This is done in Chapter 7 and error estimates are derived. It is then possible to combine errors and provide error estimates for the fully discrete approximation.

In this dissertation the aim was to examine what has been achieved and what more should be investigated.

The theory in [D73] and [B76] is for the n -dimensional wave equation. This is generalized in [OR], but the result is not sufficiently general to cover the complex problems mentioned in Subsection 1.1.2. So it is natural that these theories were adapted to other applications. Examples are the Timoshenko beam model in [S94] and [FXX99], the Reissner-Mindlin plate model in [Wu05] and linear elasto-dynamics in [Wu03]. An attempt to create a general theory of convergence for linear vibration problems was made in [ZVV99] and [Z-PhD].

Our objective is to determine to what extent the general approach - mentioned above - is applicable. In each of the articles mentioned above, we applied the theory in [Z-PhD] as far as possible and determined to what extent it was compatible.

In this chapter model problems are written in variational form. Once in variational form, all linear vibration problems have the same form. This suggests that a unified theory for convergence in the finite element method is possible.

1.2 The wave equation with damping

As mentioned in the introduction, realistic vibration problems can be extremely complicated. The one-dimensional wave equation is therefore useful to interpret the theory.

1.2.1 Model problems

We consider the wave equation with three types of damping. The damping term may be in the partial differential equation:

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$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u, \\ \partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u,\end{aligned}$$

with $\alpha > 0$ and $\gamma > 0$. The constant α represent physical properties of the bar (or string) and γ is the damping constant. The damping in the first equation is known as viscous damping which one may expect in the vibrating string model due to air resistance. The damping in the second equation is known as Kelvin-Voigt damping due to internal friction in the material of a vibrating bar.

Problem 2 Viscous damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u - \gamma \partial_t u + f \quad \text{in } (0, \ell) \quad \text{for } t \in \mathbb{R}, \\ u(0, t) &= \partial_x u(\ell, t) = 0 \quad \text{for } t \in \mathbb{R}, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

Problem 3 Kelvin-Voigt damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u + \gamma \partial_t \partial_x^2 u + f \quad \text{in } (0, \ell) \quad \text{for } t > 0, \\ u(0, t) &= \partial_x u(\ell, t) = 0 \quad \text{for } t > 0, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

Now consider the undamped wave equation but with damping at the boundary $x = \ell$:

$$\partial_x u(\ell, t) = -\gamma \partial_t u(\ell, t).$$

Problem 4 Boundary damping

Given positive constants α and γ and a function f , find u such that

$$\begin{aligned}\partial_t^2 u &= \alpha \partial_x^2 u + f \quad \text{in } (0, \ell) \quad \text{for } t > 0, \\ u(0, t) &= 0 \quad \text{for } t > 0, \\ \alpha \partial_x u(\ell, t) &= -\gamma \partial_t u(\ell, t) \quad \text{for } t > 0, \\ u(x, 0) &= u_0(x) \quad \text{for } x \in (0, \ell), \\ \partial_t u(x, 0) &= u_1(x) \quad \text{for } x \in (0, \ell).\end{aligned}$$

The term f in the wave equation represents a force per unit length.

1.2.2 Variational forms

The test functions for Problems 2 to 4 are the same as for Problem 1.

Now consider Problem 2. Multiply the equation

$$\partial_t^2 u = \alpha \partial_x^2 u - \gamma \partial_t u + f,$$

by a function $\phi \in T[0, \ell]$ and integrate. Using (1.1.2) we have

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t u(\cdot, t) \phi + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$. It is now possible to state the variational form of Problem 2.

Problem 2V Viscous damping

Find u such that for each t , $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t u(\cdot, t) \phi + \int_0^\ell f(\cdot, t) \phi$$

for each $\phi \in T[0, \ell]$ with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Next we consider Problem 3. Repeating the process above, we find that

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' + \gamma \int_0^\ell \partial_t \partial_x^2 u(\cdot, t) \phi + \int_0^\ell f(\cdot, t) \phi$$

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if $\phi \in T[0, \ell]$. Integration by parts on the damping term yields

$$\begin{aligned} \int_0^\ell \partial_t^2 u(\cdot, t) \phi &= -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot, t) \phi' \\ &\quad + (\alpha \partial_x u(\ell, t) + \gamma \partial_t \partial_x u(\ell, t)) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi, \end{aligned}$$

if $\phi \in T[0, \ell]$. It is given that $\partial_x u(\ell, t) = 0$ and it follows that $\partial_t \partial_x u(\ell, t) = 0$. We have the variational form of Problem 3.

Problem 3V Kelvin-Voigt damping

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \int_0^\ell \partial_t \partial_x u(\cdot, t) \phi' + \int_0^\ell f(\cdot, t) \phi,$$

for each $\phi \in T[0, \ell]$ with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Lastly, we consider Problem 4. From (1.1.2) it follows that

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' + \alpha \partial_x u(\ell, t) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi,$$

if $\phi \in T[0, \ell]$. Using the boundary condition at $x = \ell$ we arrive at the variational form of Problem 4.

Problem 4V Boundary damping

Find u such that for each $t > 0$, $u(\cdot, t) \in T[0, \ell]$ and

$$\int_0^\ell \partial_t^2 u(\cdot, t) \phi = -\alpha \int_0^\ell \partial_x u(\cdot, t) \phi' - \gamma \partial_t u(\ell, t) \phi(\ell) + \int_0^\ell f(\cdot, t) \phi,$$

for each $\phi \in T[0, \ell]$ with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

Undamped wave equation

If $\gamma = 0$ and $f = 0$ in Problem 2V, then it is the variational form of Problem 1 in the previous section.

Remark

In each of the above problems the natural boundary conditions are automatically satisfied. This is one of the advantages of the finite element method. Consider for example Problem 2. We showed that a solution of Problem 2 is a solution of Problem 2V. But Problem 2V has at most one solution, see Sections 2.1 and 2.2. Therefore a solution of Problem 2V will be a solution of Problem 2 if it is smooth enough.

1.2.3 General variational form

Using suitable notation, all three variational problems may be written in the same form. First, consider Problem 2V.

Notation

$$\begin{aligned} c(u, v) &= \int_0^\ell uv, \\ a(u, v) &= \gamma \int_0^\ell uv, \\ b(u, v) &= \alpha \int_0^\ell u'v'. \end{aligned}$$

The variational equation in Problem 2V may now be written in the form

$$c(\partial_t^2 u(\cdot, t), \phi) + a(\partial_t u(\cdot, t), \phi) + b(u(\cdot, t), \phi) = c(f(t), \phi), \quad (1.2.1)$$

where ϕ is a test function.

The variational equation in Problems 3V and 4V may also be written in the form above if we define the form a for damping appropriately:

$$\begin{aligned} a(u, v) &= \gamma \int_0^\ell u'v' \quad \text{for Problem 3V,} \\ a(u, v) &= \gamma u(\ell)v(\ell) \quad \text{for Problem 4V.} \end{aligned}$$

1.3 The n -dimensional wave equation

In 1973 Dupont considered “Galerkin methods for second order hyperbolic equations” [D73]. The problem was the “generalized” wave equation in a n -dimensional space. Baker [B76] also proved convergence of the Galerkin approximation for the wave equation in a n -dimensional space. They do not explain the practical application of the model problems.

1.3.1 Model problems

Consider the wave equation in a n -dimensional bounded domain denoted by Ω . If $n = 3$ it models the propagation of sound waves [W, Section 34],

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and in the two-dimensional case it models the vibration of a membrane [I]. Problem 5 below is as general as possible for theoretical purposes.

Problem 5

Let Σ be part of the boundary $\partial\Omega$ with unit outer normal vector n . We have different boundary conditions on Σ and $\partial\Omega - \Sigma$. Given functions f, g, u_0 and u_1 , find u defined on $\bar{\Omega} \times [0, T]$ such that

$$\begin{aligned}\rho \partial_t^2 u &= \nabla \cdot (A \nabla u) + f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega - \Sigma, \\ \beta_1 \partial_t u &= -A \nabla u \cdot n - \beta_2 u + g \quad \text{on } \Sigma,\end{aligned}$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.

The given parameters in the problem are the matrix of functions $A = (a_{ij})$ and the functions ρ, β_1 and β_2 .

Assumptions on the parameters

1. $a_{ij} \in C(\bar{\Omega}) \cap C^1(\Omega)$.
2. The matrix A is uniformly positive definite, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2,$$

for all $x \in \bar{\Omega}$ and all $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

3. There exist positive constants c_1 and c_2 such that $c_1 \leq \rho \leq c_2$, $c_1 \leq \beta_1$ and $c_1 \leq \beta_2$.
4. $\beta_i \in C(\partial\Omega)$ for $i = 1, 2$.

Problems B76 and D73 below are special cases of Problem 5.

Problem B76

Special case of Problem 5 with Σ empty.

Problem D73

Special case of Problem 5 with $\Sigma = \partial\Omega$ and $A = \alpha I$.

Remarks

1. Problem B76 is considered in [B76] but also in the first part of [D73].
2. Problem D73 is considered in [D73].
3. No explanation is given in [D73] for the boundary conditions on Σ . If $\beta_2 = g = 0$, then we have $\beta_1 \partial_t u = -A \nabla u \cdot n$ which represents linear boundary damping.

1.3.2 Variational form

Gauss's theorem (or divergence theorem)

If $F_i \in C^1(\bar{\Omega})$ for $i = 1, 2, 3$, then

$$\iiint_{\Omega} \nabla \cdot F \, dV = \iint_{\partial\Omega} F \cdot n \, dA.$$

where $\nabla \cdot F$ is the divergence of the vector F and n is the unit outward normal vector.

Proof See [?, p.340].

Proposition 1.3.1 *Green's formula*

If $u \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$, then

$$\iiint_{\Omega} -(\nabla \cdot (A \nabla u))v \, dV = \iiint_{\Omega} A \nabla u \cdot \nabla v \, dV - \iint_{\partial\Omega} v(A \nabla u) \cdot n \, dA.$$

Proof Since $\nabla \cdot ((A \nabla u)v) = (\nabla \cdot (A \nabla u))v + A \nabla u \cdot \nabla v$ we have that

$$\iiint_{\Omega} -(\nabla \cdot (A \nabla u))v \, dV = - \iiint_{\Omega} \nabla \cdot ((A \nabla u)v) \, dV + \iiint_{\Omega} A \nabla u \cdot \nabla v \, dV.$$

From Gauss's theorem it follows that

$$\iiint_{\Omega} -(\nabla \cdot (A \nabla u))v \, dV = - \iint_{\partial\Omega} v(A \nabla u) \cdot n \, dA.$$

The result follows. □

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If u is a solution of Problem 3, then for each $v \in C^1(\bar{\Omega})$

$$\begin{aligned} \iiint_{\Omega} \rho(\partial_t^2 u)v \, dV &= \iiint_{\Omega} (\nabla \cdot (A\nabla u))v \, dV + \iiint_{\Omega} f v \, dV \\ &= \iiint_{\Omega} A\nabla u \cdot \nabla v \, dV - \iint_{\partial\Omega} v(A\nabla u) \cdot n \, dA \\ &\quad + \iiint_{\Omega} f v \, dV. \end{aligned} \quad (1.3.1)$$

Test functions

$$T(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega - \Sigma\}.$$

From Equation (1.3.1) we have, if u is a solution of Problem 1, then for each $v \in T(\Omega)$

$$\begin{aligned} \iiint_{\Omega} \rho(\partial_t^2 u)v \, dV &= - \iint_{\Sigma} \beta_1(\partial_t u)v \, dS - \iiint_{\Omega} A\nabla u \cdot \nabla v \, dV \\ &\quad + \iint_{\Sigma} (-\beta_2 uv + gv) \, dS + \iiint_{\Omega} f v \, dV. \end{aligned}$$

Note that $\beta_1 \partial_t u = -A \nabla u \cdot n - \beta_2 u + g$ on Σ .

Bilinear forms for Problem 5

$$\begin{aligned} b(u, v) &= \iiint_{\Omega} A\nabla u \cdot \nabla v \, dV + \iint_{\Sigma} \beta_2 uv \, dS, \\ c(u, v) &= \iiint_{\Omega} \rho uv \, dV, \\ a(u, v) &= \iint_{\Sigma} \beta_1 uv \, dA. \end{aligned}$$

Notation

$$(f, g)_{\Omega} = \iiint_{\Omega} f g \, dV,$$

$$(f, g)_{\Sigma} = \iint_{\Sigma} f g \, dS.$$

Problem 5V

$T(\Omega)$ is the subset of $C^1(\bar{\Omega})$ that vanish on $\partial\Omega - \Sigma$. Find u such that for each $t > 0$, $u(\cdot, t) \in T(\Omega)$ and

$$c(\partial_t^2 u(\cdot, t), v) + a(\partial_t u(\cdot, t), v) + b(u(\cdot, t), v) = (f(\cdot, t), v)_\Omega + (g(\cdot, t), v)_\Sigma$$

for each $v \in T(\Omega)$.

Special cases of Problem 5V

Problem B76V

Special case of Problem 5V with $a = 0$ and $g = 0$.

Problem D73V

Special case of Problem 5V with $\Sigma = \partial\Omega$ and $A = \alpha I$.

Note that $T(\Omega) = C^1(\bar{\Omega})$ in [D73], but not in [B76].

1.4 Beam and plate models

1.4.1 Beam models

In 1921 Timoshenko introduced an improved theory for beams. It became known as the Timoshenko model and is generally considered to be an improvement on the Euler-Bernoulli and Rayleigh models. In fact, these models can be derived from the Timoshenko theory (see Chapter 4).

The Timoshenko theory is also interesting from a mathematical perspective. The mathematical model consists of a system of two (coupled) wave equations. Numerous mathematical papers have been published on the subject, even recently, see Chapter 4.

The Timoshenko model consists of two partial differential equations - one for the deflection w of the beam and the other for the angle ϕ due to the rotation of a cross section. The partial differential equations are derived from the equations of motion and the constitutive equations.

$$\begin{aligned}\rho A \partial_t^2 u &= \partial_x (AG \kappa^2 (\partial_x u - \phi)), \\ \rho I \partial_t^2 \phi &= AG \kappa^2 (\partial_x u - \phi) + \partial_x (EI \partial_x \phi).\end{aligned}$$

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In these equations ρ is the density, A the area of the cross section, I the area moment of inertia, E and G elastic constants and κ^2 the shear coefficient.

The variational form is derived in Chapter 4. At this point one should take note that the variational form is similar to that of the wave equation in Section 1.2.

Convergence for the finite element approximation was proved in 1994 in [S94].

1.4.2 Plate models

Classical plate theory was developed in the 19th century with contributions by Sophie Germain, Lagrange, G. R. Kirchhoff and Lord Kelvin (William Thompson), (see [Rei]). From 1945 to 1950 improvements to classical plate theory were made by E. Reissner, H. Hencky, Y. S. Uflyand and R. D. Mindlin (see [M51] for references). The improved model for vibration was introduced in [M51]. It became known as the *Reissner-Mindlin* model. Small transverse vibration of a plate is the subject in Chapter 5.

Where the Timoshenko beam model consists of two partial differential equations, there are three in the Reissner-Mindlin model. Nevertheless, the variational form is again similar to that of the wave equation. Actually from an abstract mathematical point of view, the variational forms are identical. The finite element approximation is considered in [Wu05].

1.5 Elasto-dynamics

Consider the case of so called small vibrations of a three-dimensional elastic body. The assumption is that the local displacements are small. The equation of motion is derived from the conservation law for momentum

$$\rho \partial_t^2 \mathbf{u} = \operatorname{div} \mathbf{T} + \rho \mathbf{b},$$

where $\mathbf{u}$ is the displacement, ρ the density, $\mathbf{T}$ the stress tensor and $\mathbf{b}$ the body force (density). It is possible to derive a system of partial differential equations using Hooke's law

$$\mathbf{T} = \frac{E}{1+\nu} \boldsymbol{\varepsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)} \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I},$$

where $\boldsymbol{\mathcal{E}}$ is the strain tensor, E Young's modulus and ν Poisson's ratio.

Linear vibration problem Suppose $\Omega \subset E_3$ is the reference configuration for a solid executing small vibrations. The boundary of Ω consists of two parts Σ and Γ . The problem is to find $\mathbf{u}$ such that

The equation of motion is satisfied in Ω ;

Hooke's law is satisfied in Ω ;

The specified displacement for $\mathbf{u}$ is satisfied on Σ ;

The specified traction $\mathbf{T}n$ is satisfied on Γ .

It is not necessary to consider the system of partial differential equations with corresponding boundary conditions as it is possible to derive the variational form directly from the equation of motion. The derivation is the same as for the Reissner-Mindlin plate and the details are given in Chapter 5.

The **variational form** of the vibration problem is to find a displacement $\mathbf{u}$ such that the boundary condition on Σ is satisfied and

$$\begin{aligned} & \int_{\Omega} \rho \partial_t^2 \mathbf{u} \cdot \mathbf{v} \, dV \\ = & \int_{\Omega} c \, \text{tr}(\boldsymbol{\mathcal{E}}V) + k \, \text{tr}(\boldsymbol{\mathcal{E}})\text{tr}(V) \, dV + \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} \, dV + \int_{\Gamma} \mathbf{T}n \cdot \mathbf{v} \, dS, \end{aligned}$$

for each $\mathbf{v}$ which is zero on Σ . The symbol tr denotes the trace of the tensor (matrix) and V is the Jacobi matrix for $\mathbf{v}$ (also denoted by $\nabla \mathbf{v}$).

If the bilinear form b is defined by

$$b(\mathbf{u}, \mathbf{v}) = \int_{\Omega} c \, \text{tr}(\boldsymbol{\mathcal{E}}V) + k \, \text{tr}(\boldsymbol{\mathcal{E}})\text{tr}(V) \, dV,$$

then the variational form of the problem is similar to that for the wave equation. The finite element approximation for this problem is considered in [Wu03] and [Wu06].

1.6 Hybrid models and structures

In the previous sections, we have seen that vibration models can become extremely complex. From a theoretical point of view matters get worse for systems where bodies interact. We present a few examples.

A hybrid model where a beam has complex boundary conditions due to a damping tip body attached to one end is considered in [AS02]. To apply the finite element method to compute an approximate solution is not too difficult but to obtain error estimates is a different matter.

A planar network of linked beams is considered in [LLS] together with other complex systems including a plate-beam system. The book is of a theoretical nature with a number of numerical examples, but there is no theory on the finite element method, except for references, for example [H].

In Section 5.6 we present a plate-beam model from a recent article [LVV09].

Finally, we mention a recent article [LH09] where the importance of vibration analysis of structures in engineering is stressed. The authors consider a system consisting of two plates.

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Chapter 2

General convergence results for the Galerkin method

2.1 Weak variational form of the one-dimensional wave equation

This section serves as an introduction to Chapter 2. Consider as an example Problem 2V in Section 1.2. We need to find a function u with $u(\cdot, t)$ a test function such that

$$c(\partial_t^2 u(\cdot, t), \phi) + a(\partial_t u(\cdot, t), \phi) + b(u(\cdot, t), \phi) = c(f(t), \phi), \quad (2.1.1)$$

for each test function ϕ . The initial conditions are $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$. The bilinear forms a , b and c are defined as

$$c(u, v) = \int_0^\ell uv, \quad a(u, v) = \gamma \int_0^\ell uv \quad \text{and} \quad b(u, v) = \alpha \int_0^\ell u'v'.$$

2.1.1 The weak formulation

It is important to note that a and c are defined for functions in $\mathcal{L}^2(0, \ell)$ and b is defined for functions $u \in H^1(0, \ell)$ (see Appendix A). But the partial derivatives $\partial_t u$ and $\partial_t^2 u$ do not make sense when a function may be changed arbitrarily on a set of measure zero. If we define a function w by $w(t) = u(\cdot, t)$,

then $w'(t)$ may be defined with respect to the norm of $\mathcal{L}^2(0, \ell)$:

$$\lim_{h \rightarrow 0} \|h^{-1}[w(t+h) - w(t)] - w'(t)\| = 0,$$

where $\|\cdot\|$ denotes the norm of $\mathcal{L}^2(0, \ell)$. Equation (2.1.1) may now be rewritten:

$$c(w''(t), \phi) + a(w'(t), \phi) + b(w(t), \phi) = c(f(t), \phi), \quad (2.1.2)$$

for each test function ϕ . The space of test functions $T[0, \ell]$ is defined in Section 1.2. by

$$T[0, \ell] = \{\phi \in C^1[0, \ell] \mid \phi(0) = 0\}.$$

Let $V(0, \ell)$ be the closure of $T[0, \ell]$ in $H^1(0, \ell)$. We need to show that Equation 2.1.2 holds for all $\phi \in V(0, \ell)$. For this the necessary estimates must be derived.

Recall that the inner product for $\mathcal{L}^2(0, \ell)$ is denoted by $(\cdot, \cdot)$ and the norm by $\|\cdot\|$. The norm of $H^1(0, \ell)$ is denoted by $\|\cdot\|_1$. It follows from Theorem A.4.4 that $\|u\| \leq \|u\|_1$ and $\|u'\| \leq \|u\|_1$ for $u \in H^1(0, \ell)$ (see Appendix A.4).

Now we consider the boundedness of the bilinear forms a , b and c . Using the Cauchy-Schwartz inequality we find that

$$|c(u, v)| = |(u, v)| \leq \|u\| \|v\| \leq \|u\|_1 \|v\|_1,$$

$$|a(u, v)| = \gamma |(u, v)| \leq \gamma \|u\|_1 \|v\|_1,$$

$$|b(u, v)| = \alpha |(u', v')| \leq \alpha \|u\|_1 \|v\|_1.$$

From the boundedness it follows that a , b and c are continuous on $H^1(0, \ell)$. Consider for example the bilinear form c :

$$|c(x, y) - c(x, z)| = |c(x, y - z)| \leq \|x\|_1 \|y - z\|_1.$$

For any $\phi \in V(0, \ell)$ there exists a sequence $\{\phi_n\} \subset T[0, \ell]$ such that $\|\phi_n - \phi\|_1 \rightarrow 0$ as $n \rightarrow \infty$. As a consequence we have for any $u \in H^1(0, \ell)$ that

$$c(u, \phi_n) \rightarrow c(u, \phi) \quad \text{as } n \rightarrow \infty.$$

The same is true for a and b . Consequently equation (2.1.2) holds for any $\phi \in V(0, \ell)$. It is now possible to formulate the weak variational form of Problem 2V.

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Problem 2W Find a function w with $w(t) \in V(0, \ell)$ such that

$$c(w''(t), \phi) + a(w'(t), \phi) + b(w(t), \phi) = c(f(t), \phi)$$

for each $\phi \in V(0, \ell)$. The initial conditions are $w(0) = u_0$ and $w'(0) = u_1$.

Consider the equivalence of Problem 2V and Problem 2W. We have shown that a solution of Problem 2V is a solution of Problem 2W, but it is not always true that a solution of Problem 2W is a solution of Problem 2V. The weak solution must satisfy so called regularity conditions and this is beyond the scope of this dissertation.

Consider necessary properties for w to be a possible solution. It is obviously necessary that $w'(t)$ and $w''(t)$ must exist in the sense defined above. Since **initial conditions** are prescribed for w and w' it is required that w and w' be continuous at zero - at least with respect to the norm of $\mathcal{L}^2(0, \ell)$.

Next we consider the **boundary conditions**. It was shown in Section 1.2 that the natural boundary condition $\partial_x u(\ell, t) = 0$ was automatically satisfied. The other boundary condition is forced by requiring that $u(\cdot, t) \in T[0, \ell]$ for each t . An appropriate condition in the Sobolev space $H^1(0, \ell)$ is required. It is possible to define a trace operator Γ_0 such that the boundary condition $u(0, t) = 0$ can be replaced by $\Gamma_0 u = 0$ when $u \in V(0, \ell)$. (See Appendix A.5).

We used Problem 2V to show how the weak variational form is derived. In the rest of this dissertation we consider only weak variational forms of problems in the theory. The connection between the variational form and the weak variational form is similar for all the problems.

The finite element method

We consider the finite element approximation of the weak variational form, e.g. Problem 2W. If the approximation converges, it does not imply that it converges to a classical solution of the partial differential equation. Consider the application of the finite element method to Problem 2W. A finite dimensional subspace S^h of $V(0, \ell)$ is constructed. Then the problem is to find a function u_h with $u_h(t) \in S^h$ such that

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = c(f(t), v), \quad (2.1.3)$$

for each $v \in S^h$. If the function u_h converges, it converges with respect to the so called energy norm (or the norm $\|\cdot\|_W$).

2.1.2 Existence of solutions

Proposition 2.1.1 *Suppose $f = 0$, $u_0 \in C^2[0, \ell]$ with $u_0(0) = u_0''(0) = u_0'(\ell) = 0$ and $u_1 \in C^1[0, \ell]$ with $u_1(0) = u_1'(\ell) = 0$. Then Problem 2 has a unique classical solution $u \in C^2([0, \ell] \times [0, \infty))$.*

Proof [W] Section 26.

Remark: If the second derivative of u_0 is only piecewise continuous or if $u_0''(0) \neq 0$, the second derivatives of u will have discontinuities.

It is not realistic to expect classical solutions in all cases. It is important to note that the existence of a solution is not a simple issue. It depends on the definition of a solution. Equally important is the fact that the rate of convergence depends on the type of solution as we show in the rest of this chapter. If a classical solution does not exist, it is possible to find a weak solution using the theory in Section 2.2.

Consider Problem 2W. The following results will be used to apply the existence theory and general convergence theory.

Proposition 2.1.2 *There exists a constant β such that $\|v\|^2 \leq \beta^2 b(v, v)$ for each $v \in V$.*

Proof For any v in V we have

$$b(v, v) = \alpha \int_0^\ell (v')^2 = \alpha \|v'\|^2.$$

From Proposition A.4.3 in the Appendix it follows that

$$\alpha \|v\|^2 \leq \alpha \ell^2 \|v'\|^2 = \ell^2 b(v, v).$$

□

Proposition 2.1.3 *The bilinear form b is an inner product on V .*

Proof The fact that the bilinear form b is symmetric follows directly from the definition of b . From Proposition 2.1.2 we see that b is nonnegative and $b(v, v) = 0$ implies that $v = 0$. □

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Definition We define the norm associated with the inner product b as

$$\|u\|_V = \sqrt{b(u, u)} \quad \text{for any } u \in V.$$

Remark

The norm $\|\cdot\|_V$ is referred to as the energy norm. Note that $\|\cdot\|_V = \|\cdot\|_E$ and both are used.

Proposition 2.1.4 *The norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(0, \ell)$ on V .*

Proof By definition $\|v\|_1^2 = \|v\|^2 + \|v'\|^2$, so clearly $\|v'\|^2 \leq \|v\|_1^2$. We also have that

$$\begin{aligned} \|v\|_V^2 = \alpha \|v'\|^2 &\geq \frac{1}{2} \alpha \|v'\|^2 + \frac{1}{2} \alpha \ell^{-2} \|v\|^2 \\ &\geq \frac{1}{2} \alpha \min\{1, \ell^{-2}\} \|v\|_1^2. \end{aligned}$$

Therefore there exist constants $K_1 > 0$ and $K_2 > 0$ such that

$$K_1 \|v\|_1^2 \leq \|v\|_V^2 \leq K_2 \|v\|_1^2,$$

and the result is proved. $\square$

Proposition 2.1.5 *V is dense in W .*

Proof From Theorem A.1.2 we have that $C_0^\infty[0, \ell]$ is dense in $\mathcal{L}^2(0, \ell) = W$. Since $C_0^\infty[0, \ell]$ is a subset of V , V is also dense in $\mathcal{L}^2(0, \ell)$. $\square$

2.2 Existence theory

2.2.1 General second order hyperbolic equation in variational form

In this section we consider the general linear vibration problem. Let X , W and V denote Hilbert spaces such that $V \subset W \subset X$.

Notation

X has inner product $(\cdot, \cdot)_X$ and norm $\|\cdot\|_X$;

W has inner product c and norm $\|\cdot\|_W$;

V has inner product b and norm $\|\cdot\|_V$.

Let J be a bounded or unbounded interval. J is either an open interval containing zero or it is of the form $[0, \tau)$ or $[0, \infty)$. Let Y be any Hilbert space and consider a function u on J with values in Y .

Notation

$C^k(J; Y)$: $u \in C^k(J; Y)$ if $u^{(k)}(t) \in Y$ for each $t \in J$.

The implication of $u'(t) \in Y$ is that the derivative exists with respect to the norm of Y .

Consider a bilinear form a defined on V and the following general problem.

Problem G

Given a function $f : J \rightarrow X$, find a function $u \in C(J; V)$ such that u' is continuous at 0 and for each $t \in J$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W,$$

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each } v \in V,$$

$$\text{while } u(0) = u_0, \quad u'(0) = u_1.$$

Recall that a **bilinear form** ϕ is called **bounded** on an inner product space Y if there exists a constant K such that for any u and v in Y

$$|\phi(u, v)| \leq K \|u\|_Y \|v\|_Y.$$

Assumptions

The following assumptions are made for the theory of this chapter.

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E1 V is dense in W ,

E2 There exists a constant C_b such that $\|v\|_W \leq C_b \|v\|_V$ for each $v \in V$,

E3 The bilinear form a is nonnegative, symmetric and bounded on V ,

E4 There exists a constant C_c such that $\|v\|_X \leq C_c \|v\|_W$ for each $v \in W$.

Theorem 2.2.1 *Suppose Assumptions E1, E2, E3 and E4 hold. If for $u_0 \in V$ and $u_1 \in V$, there exists some $y \in W$ such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each } v \in V, \quad (2.2.1)$$

then for each $f \in C^1([0, \tau]; X)$ there exists a unique solution

$$u \in C([0, \tau]; V) \cap C^1([0, \tau]; W) \cap C^1((0, \tau); V) \cap C^2((0, \tau); W),$$

for Problem G. If $f = 0$ then $u \in C^1([0, \infty); V) \cap C^2([0, \infty); W)$.

Proof [VV02]

A sharper result can be obtained for the undamped case or viscous type damping.

Definition

$$E_b = \{ x \in V \mid \text{there exists a } y \in W \text{ such that } c(y, v) = b(x, v) \text{ for all } v \in V \}.$$

Theorem 2.2.2 *Suppose Assumptions E1, E2, E3 and E4 hold and a is bounded with respect to the norm in W . Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W),$$

for Problem G for each $u_0 \in E_b$, $u_1 \in V$ and each $f \in C^1(J; X)$. If $f = 0$ then $u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W)$.

Proof Refer to [VV02]

2.2.2 The one-dimensional wave equation

Consider Problem 2W. For this problem the space $W = X = \mathcal{L}^2(0, \ell)$ and we have Assumption E4. The bilinear form b is an inner product for $V = V(0, \ell)$ by Proposition 2.1.2. Assumptions E1 and E2 hold due to Propositions 2.1.5 and 2.1.2.

Proposition 2.2.1 *If $f \in C^1(J, X)$, then Problem 2W has a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W),$$

for $u_0 \in E_b$ and $u_1 \in V$.

Proof The spaces $W = X = \mathcal{L}^2(0, \ell)$ and $V = V(0, \ell)$ satisfy E1, E2 and E4. It is obvious that a is symmetric and nonnegative.

Also $|a(u, v)| = \gamma |(u, v)| \leq \gamma \|u\| \|v\|$ from the Schwartz inequality. Thus a is a nonnegative, bounded bilinear form on W . The desired result follows from Theorem 2.2.2. $\square$

Sufficient conditions When do the initial conditions satisfy $u_0 \in E_b$ and $u_1 \in V$? Is it possible to derive sufficient conditions for this, which are also useful in applications?

For u_0 to be in E_b we need to find a function y in $\mathcal{L}^2(0, \ell)$ such that $c(y, v) = b(u_0, v)$ for any $v \in V$. Assume that $u_0 \in T(0, \ell) \cap C^2(0, \ell)$. For any $v \in T(0, \ell)$ integration by parts yields

$$\alpha(u'_0, v') = -\alpha(u''_0, v) + \alpha u'_0(\ell) v(\ell), \quad (2.2.2)$$

since $v(0) = 0$. If $u'_0(\ell) = 0$ then $b(u_0, v) = c(-\alpha u''_0, v)$ for any $v \in T[0, \ell]$. The bilinear form c is continuous on V and b is obviously continuous. Consequently (2.2.2) holds for any $v \in V$. This means that $u_0 \in E_b$. It is clear that if u_1 is a test function then $u_1 \in V$.

If $u_0 \in H^2(0, \ell) \cap V(0, \ell)$ and $\Gamma u'_0 = 0$, then u_0 may be approximated by a sequence $\{\phi_n\} \subset T(0, \ell) \cap C^2(0, \ell)$ with $\phi_n(\ell) = 0$. The approximation is in the sense that $\|\phi_n - u_0\|_2 \rightarrow 0$ and $\|\phi_n - u_0\|_V \rightarrow 0$.

We conclude that sufficient conditions on the initial values are that $u_0 \in V(0, \ell) \cap H^2(0, \ell)$, $\Gamma u'_0 = 0$ and $u_1 \in V(0, \ell)$.

Remarks

1. The sufficient conditions above should be compared with the sufficient conditions for a classical solution (Proposition 2.1.1).
2. By setting $\gamma = 0$ we have the undamped wave equation and Theorem 2.2.2 is still applicable. The sufficient conditions for the existence of a weak solution are given above.
3. How these results affect convergence of the Galerkin approximation is discussed in Section 3.2.
4. A consequence of Proposition 2.2.1 is that Problems 2 and 2V each have at most one solution.

2.3 Galerkin approximation

The Galerkin approximation was introduced in Section 1.1. In this section we consider the general case. Assumptions E1, E2, E3 and E4 are basic and assumed to hold for the general convergence theory.

Choose a set of basis functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ in V . Denote the span of these functions (called basis functions) by S^h .

Problem G^h

Given a function $f : J \rightarrow X$, find a function $u_h \in C(J, S^h)$ such that u'_h is continuous at 0 and for each $t \in J$

$$c(u''_h(t), v) + a(u'_h(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h,$$

$$\text{while } u_h(0) = u_0^h, \quad u'_h(0) = u_1^h.$$

The initial values u_0^h and u_1^h are elements of S^h as close as possible to u_0 and u_1 , see the next two sections.

Existence is no problem since f is continuous, and S^h is finite dimensional.

Theorem 2.3.1 *Suppose the bilinear form c is has the property that $u = 0$ if $c(u, u) = 0$ and $f \in C(J; S^h)$. Let J be an interval containing zero, then there exists a unique solution $u_h \in C^2(J; S^h)$ for Problem G^h for each u_0^h and u_1^h in S^h . If $f = 0$ then $u^h \in C^2((-\infty, \infty); S^h)$.*

Proof See Section 7.1.

Our concern is the difference between the solution u and the Galerkin approximation u_h . It is possible to estimate this error in terms of the projection error and errors for the initial conditions. This is called a projection method and was first used for parabolic problems, see [SF].

2.3.1 Discretization error

To find an estimate for the discretization error

$$\|u(t) - u_h(t)\|_E \quad \text{for } t \in J,$$

a projection is used. The projection operator P_h is defined by

$$b(u - P_h u, v) = 0 \quad \forall v \in S^h.$$

If no confusion is possible we will write P for P_h .

We also use the symbol P to denote the “projection” Pu of the solution u , i.e. $(Pu)(t) = Pu(t)$ for each $t \in J$.

The idea of the projection method is to split the errors. Let $e_h(t) = u(t) - u_h(t)$, $e(t) = Pu(t) - u_h(t)$ and $e_p(t) = u(t) - Pu(t)$. Then

$$e_h(t) = e_p(t) + e(t)$$

and

$$\|u(t) - u_h(t)\|_E \leq \|e_p(t)\|_E + \|e(t)\|_E. \quad (2.3.1)$$

Estimates for the norm of $e_p(t)$ are obtained from approximation theory in applications (see the next section). The problem is to find an estimate for $e(t)$, the difference between the projection of u and the Galerkin approximation. (The idea to split the error using the projection, was first used for parabolic problems - see [SF] for references.)

The convergence results that can be obtained, depend on the properties of the solution. But these may vary from problem to problem. The weakest assumption is the result of the existence theorem.

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Assumption A1

If u is a solution of Problem G, then

$$u \in C^1(J; V) \cap C^2(J; W).$$

Lemma 2.3.1 *If $u \in C^1(J, V)$, then $Pu \in C^1(J, V)$ and $(Pu)'(t) = Pu'(t)$.*

Proof As the projection operator P is a bounded linear operator with norm less than one, we have that $\|Px\|_E \leq \|x\|_E$. It follows that

$$\|(\delta t)^{-1} (Pu(t + \delta t) - Pu(t)) - Pu'(t)\|_E \leq \|(\delta t)^{-1} (u(t + \delta t) - u(t)) - u'(t)\|_E.$$

Since the term on the right converges to 0 as $\delta t \rightarrow 0$, $Pu \in C^1(J, V)$ and $(Pu)'(t) = Pu'(t)$. $\square$

Assumption A2

Assume that $u \in C(J, V)$ has the property that $(Pu) \in C^2(J, S^h)$.

Proposition 2.3.1 *If $u \in C^1(J, V) \cap C^2(J, W)$ and u satisfies Assumption A2, then*

$$e_p \in C^2(J, W).$$

Proof

$$e_p(t) = u(t) - (Pu)(t)$$

and since norms in S^h are equivalent, $Pu \in C^2(J, W)$. $\square$

Lemma 2.3.2 *If the solution u of Problem G satisfies Assumptions A1 and A2, then*

$$c(e_p''(t), v) + a(e_p'(t), v) + c(e''(t), v) + a(e'(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h.$$

Proof By subtracting the equation in Problem G^h ,

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = c(f(t), v),$$

from the equation in Problem G,

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = c(f(t), v),$$

we obtain the following identity.

For any $v \in S^h$,

$$c(u''(t) - u_h''(t), v) + a(u'(t) - u_h'(t), v) + b(u(t) - u_h(t), v) = 0. \quad (2.3.2)$$

Add and subtract $(Pu)''$, $(Pu)'$ and Pu respectively in the bilinear forms above.

Now we have that (2.3.2) can be written as

$$\begin{aligned} & c(u''(t) - (Pu)''(t), v) + a(u'(t) - (Pu)'(t), v) + b(u(t) - Pu(t), v) \\ & + c((Pu)''(t) - u_h''(t), v) + a((Pu)'(t) - u_h'(t), v) + b(Pu(t) - u_h(t), v) = 0, \end{aligned}$$

for all $v \in S^h$. Since P is the projection, we have

$$b(u(t) - Pu(t), v) = 0 \text{ for all } v \in S^h,$$

and the result follows. $\square$

Remark

We have used the notation from [SF], [ZVV99] and [Z-PhD]. The notation used in [D73] and [B76] is confusing, see Sections 3.3 and 3.4.

2.3.2 The general energy method

The following “energy” expression for e is convenient:

$$\begin{aligned} E(t) &= \frac{1}{2}c(e'(t), e'(t)) + \frac{1}{2}b(e(t), e(t)) \\ &= \frac{1}{2}\|e'(t)\|_W^2 + \frac{1}{2}\|e(t)\|_E^2. \end{aligned} \quad (2.3.3)$$

The energy method is due to [SF] and [D73]. The book as well as the article appeared in 1973. The problem in the book does not include damping ($a = 0$) whereas the problem in [D73] prescribe general boundary conditions including a damping term. However, the approach in [SF] encourages generalization.

Notation $J = [0, T]$.

Lemma 2.3.3 *If the solution u of Problem G satisfies Assumptions A1 and A2, then*

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)).$$

Proof Since $u^h \in C^2(J, S^h)$ and $Pu \in C^2(J, S^h)$, it follows that $e \in C^2(J, S^h)$. Also, since Pu and $u_h \in S^h$, $e(t) \in S^h$, it follows that $e'(t) \in S^h$. Choose $v = e'(t)$ in Lemma 2.3.2, then

$$c(e_p''(t), e'(t)) + c(e''(t), e'(t)) + a(e_p'(t), e'(t)) + a(e'(t), e'(t)) + b(e(t), e'(t)) = 0. \quad (2.3.4)$$

From the definition of the “energy” expression, and since b and c are inner products, we have

$$E'(t) = c(e''(t), e'(t)) + b(e(t), e'(t)).$$

Combining this with (2.3.4) gives us

$$E'(t) = -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)) - a(e'(t), e'(t)).$$

Since $a(e'(t), e'(t)) \geq 0$, we have that

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \quad \square$$

From this inequality the aim is to obtain an estimate for $E(t)$ in terms of $e_p'(t)$ and $e_p''(t)$. Since $\|e'(t)\|_W^2 + \|e(t)\|_E^2 = 2E(t)$, it will enable us to estimate the error $e_h(t)$. However, this is a major challenge and we will consider various options to obtain estimates for $e(t)$.

2.4 Modal damping

In [ZVV99] and [Z-PhD] the approach is based on [SF] (where only a sketch of a proof is given). The results apply to any problem of the form in Problem G provided that damping is modal. The bilinear form a is given by $a(u, v) = \mu b(u, v) + kc(u, v)$ where μ or k or both can be zero, i.e. it includes the undamped case. For this case general results are obtained but note that boundary damping is not modal. Boundary damping is considered in [D73], see Section 3.3.

The results in Sections 2.4 and 2.5 are based on [ZVV99] and [Z-PhD, Section 5.3]. There are modifications to facilitate comparison with [D73] and [B76]. Some of the assumptions are also generalized to make wider application of the results possible.

2.4.1 Fundamental estimate

Lemma 2.4.1 *If the solution u of Problem G satisfies Assumptions $A1$ and $A2$, then for any $t \in J$,*

$$\|e(t)\|_E + \|e'(t)\|_W \leq \|e(0)\|_E + \|e'(0)\|_W + \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) \, d\tau.$$

Proof Using Lemma 2.3.3 and substituting the bilinear form a we get

$$E'(t) \leq -c(e_p''(t), e'(t)) - \mu b(e_p'(t), e'(t)) - kc(e_p'(t), e'(t)). \quad (2.4.1)$$

Since $b(u(t) - Pu(t), v) = 0$ for all $v \in S^h$, we have that $b(u'(t) - (Pu)'(t), v) = 0$ for all $v \in S^h$. Therefore $\mu b(e_p'(t), e'(t)) = 0$ and (2.4.1) becomes

$$E'(t) = -c(e_p''(t), e'(t)) - kc(e_p'(t), e'(t)). \quad (2.4.2)$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |c(e_p''(t), e'(t))| &\leq \|e_p''(t)\|_W \|e'(t)\|_W, \\ |c(e_p'(t), e'(t))| &\leq \|e_p'(t)\|_W \|e'(t)\|_W, \end{aligned}$$

and therefore we obtain that

$$E'(t) \leq (\|e_p''(t)\|_W + k\|e_p'(t)\|_W) \|e'(t)\|_W. \quad (2.4.3)$$

From (2.3.3), $\|e'(t)\|_W \leq \sqrt{2E(t)}$. Thus

$$E'(t) \leq \sqrt{2E(t)} (\|e_p''(t)\|_W + k\|e_p'(t)\|_W).$$

Also,

$$\frac{d}{dt} \sqrt{E(t)} = \frac{1}{2} \frac{1}{\sqrt{E(t)}} E'(t),$$

and consequently

$$\frac{d}{dt} \sqrt{E(t)} \leq \frac{1}{\sqrt{2}} (\|e_p''(t)\|_W + k\|e_p'(t)\|_W).$$

Integrating the above on both sides yields

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \frac{1}{\sqrt{2}} \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) \, d\tau. \quad (2.4.4)$$

Since

$$E(0) = \frac{1}{2}\|e'(0)\|_W^2 + \frac{1}{2}\|e(0)\|_E^2,$$

we have that

$$\sqrt{2E(t)} \leq \sqrt{\|e'(0)\|_W^2 + \|e(0)\|_E^2} + \int_0^t (\|e_p''(\cdot)\|_W + k\|e_p'(\cdot)\|_W) \, d\tau.$$

Finally, $\|e(t)\|_E + \|e'(t)\|_W \leq \sqrt{2E(t)}$, again from (2.3.3), and the result follows. $\square$

2.4.2 Projection errors

To prove the convergence results, it is now necessary to consider the projection errors e_p' and e_p'' .

Assumption A3

There exists a subspace $H(V, k)$ of V , a positive constant $\hat{C}$ and a positive integer α such that for any $u \in H(V, k)$

$$\inf_{v \in S^h} \|u - v\|_E \leq \hat{C}h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

Remarks

1. Due to assumption A3,

$$\|e_p(t)\|_E \leq \hat{C}h^\alpha \|u(t)\|_{H(V, k)}. \quad (2.4.5)$$

2. Assumption A3 generalizes the assumptions in [D73] and [B76].
3. Compare A3 with A5 in the next section.

Note that differentiability with respect to the energy norm was required to prove that $(Pu)' = Pu'$.

Lemma 2.4.2 *If $u \in C^2(J, V)$, then $Pu \in C^2(J, V)$ and $(Pu)''(t) = Pu''(t)$.*

Proof This is proved in exactly the same way as Lemma 2.3.1. $\square$

Assumption A4

The solution u of Problem G satisfies

$$u \in C^2(J, V).$$

Remark

If Assumption A4 is satisfied, then $(Pu)''$ exists, but it is not a necessary condition. This may be considered an improvement on [Z-PhD] and [D73].

Lemma 2.4.3 *Assume that the solution u of Problem G satisfies Assumption A4. Then, for any $t \in J$,*

$$\begin{aligned} \|e(t)\|_E + \|e'(t)\|_W &\leq \|Pu_0 - u_0^h\|_E + \|Pu_1 - u_1^h\|_W \\ &\quad + \int_0^t (\|u'' - Pu''\|_W + k\|u' - Pu'\|_W) . \end{aligned}$$

Proof The result follows from Lemmas 2.3.1, 2.4.1 and 2.4.2. Note for example that

$$e'(0) = (Pu)'(0) - (u^h)'(0) = Pu'(0) - u_1^h = Pu_1 - u_1^h. \quad \square$$

2.4.3 Convergence

Theorem 2.4.1 *Suppose Assumptions E2 and A3 hold. Assume that the solution u satisfies Assumption A4 and $u(t)$, $u'(t)$ and $u''(t)$ are in $H(V, k)$. Then, for any $t \in J$,*

$$\begin{aligned} &\|u(t) - u_h(t)\|_E + \|u'(t) - u_h'(t)\|_W \\ &\leq \widehat{C}h^\alpha (\|u(t)\|_{H(V,k)} + C_b\|u'(t)\|_{H(V,k)}) + \|Pu_0 - u_0^h\|_E \\ &\quad + \|Pu_1 - u_1^h\|_W + C_b\widehat{C}h^\alpha \int_0^t (\|u''(\cdot)\|_{H(V,k)} + k\|u'(\cdot)\|_{H(V,k)}) . \end{aligned}$$

Proof We start by noting that

$$\begin{aligned} &\|u(t) - u_h(t)\|_E + \|u'(t) - u_h'(t)\|_W \\ &\leq \|e(t)\|_E + \|e_p(t)\|_E + \|e'(t)\|_W + \|e_p'(t)\|_W \\ &\leq \|e_p(t)\|_E + \|e_p'(t)\|_W + \|Pu_0 - u_0^h\|_E + \|Pu_1 - u_1^h\|_W \\ &\quad + \int_0^t (\|u'' - Pu''\|_W + k\|u' - Pu'\|_W) , \end{aligned} \tag{2.4.6}$$

by Lemma 2.4.3. The result follows from $\|u\|_W \leq C_b \|u\|_V$ (Assumption E2) and Assumption A3. $\square$

The error depends on u_0^h and u_1^h , but u_0^h and u_1^h are not given. In applications we must choose u_0^h and u_1^h . If we choose $Pu_0 = u_0^h$, for example, then $\|Pu_0 - u_0^h\|_E = 0$. In practice, it is impossible. Dupont [D73] suggested a numerical approximation for Pu_0 . A good choice is to use the interpolants of u_0 and u_1 , see the next section.

2.5 Interpolation

An approximation result such as in Assumption A3 is typically obtained from interpolation theory.

Assumption A5

There exists a subspace $H(V, k)$ of V , an interpolation operator Π and constants C_Π and α such that for $u \in H(V, k)$:

$$\|u - \Pi u\|_E \leq C_\Pi h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

Assumptions similar to A3 are often made in the literature, see e.g [D73] and [B76]. But P is a projection with respect to the inner product b , hence Assumption A3 follows from Assumption A5. In all our applications an estimate for e_p is obtained from Assumption A5 rather than Assumption A3.

Theorem 2.5.1 *Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume that Assumptions E2 and A5 hold, that the damping is modal and that the solution $u \in C^2(J, V)$ (Assumption A4). We also assume that $u(t)$, $u'(t)$ and $u''(t)$ are in $H(V, k) \cap V$ for $t \in J$. Then,*

$$\begin{aligned} & \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ & \leq C_\Pi h^\alpha (\|u(t)\|_{H(V, k)} + C_b \|u'(t)\|_{H(V, k)} + 2\|u_0\|_{H(V, k)} + 2C_b \|u_1\|_{H(V, k)} \\ & \quad + kC_b t \sup_{s \in J} \|u'(s)\|_{H(V, k)} + C_b t \sup_{s \in J} \|u''(s)\|_{H(V, k)}), \end{aligned}$$

for $t \in J$.

Proof We use the inequality (2.4.6) and Assumption A5 instead of A3. Next we find that

$$\begin{aligned}
\|Pu_0 - \Pi u_0\|_E &\leq \|Pu_0 - u_0\|_E + \|u_0 - \Pi u_0\|_E \\
&\leq 2C_\Pi h^\alpha \|u_0\|_{H(V,k)}; \\
\|Pu_1 - \Pi u_1\|_W &\leq C_b \|Pu_1 - \Pi u_1\|_E \\
&\leq C_b \|Pu_0 - u_0\|_E + C_b \|u_0 - \Pi u_0\|_E \\
&\leq 2C_b C_\Pi h^\alpha \|u_1\|_{H(V,k)}. \quad \square
\end{aligned}$$

Remarks

1. It is quite possible that the right hand side of the inequality is infinite if J is unbounded. The result is useful if applied on a bounded subinterval of J .
2. The assumption that $u(t) \in H(V, k)$ is necessary to obtain an error estimate involving h^α . Under less strict regularity conditions we can still show that the solution u_h of Problem G^h converges to the solution u of Problem G in the energy norm as h tends to zero.

Assumption A6

$H(V, k)$ is a dense subset of V .

Lemma 2.5.1 *Suppose Assumptions A5 and A6 hold. For any $\varepsilon > 0$ and any $u \in V$, there exists a $\delta > 0$, such that*

$$\|u - Pu\|_E < \varepsilon \quad \text{for } h < \delta.$$

Proof For any $u \in V$ there exists a $w \in H(V, k)$ such that $\|u - w\|_E \leq \varepsilon$. Consequently (since $\|P\| \leq 1$)

$$\begin{aligned}
\|Pu - u\|_E &\leq \|u - w\|_E + \|w - Pw\|_E + \|Pw - Pu\|_E \\
&\leq \varepsilon + C_\Pi h^\alpha \|w\|_{H(V,k)} + \varepsilon \\
&< 3\varepsilon \text{ for } h \text{ sufficiently small.} \quad \square
\end{aligned}$$

Theorem 2.5.2 *Suppose that Assumptions A5 and A6 hold. Assume that the solution $u \in C^2(J, V)$, (Assumption A4). If J_b is a bounded subinterval of J and*

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\|_W = 0,$$

then

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_E = 0 \text{ for } t \in J_b.$$

Proof From Lemma 2.4.3,

$$\begin{aligned} \|u(t) - u_h(t)\|_E &\leq \|e_p(t)\|_E + \|Pu_0 - u_0^h\|_E + \|Pu_1 - u_1^h\|_W \\ &\quad + \int_0^t (\|e_p''(s)\|_W + k\|e_p'(s)\|_W) ds. \end{aligned}$$

From Lemma 2.5.1 we know that for any $\varepsilon > 0$, $\|e_p(t)\|_E$, $\|e_p'(t)\|_W$ and $\|e_p''(t)\|_W$ are less than ε , provided that h is sufficiently small. $\square$

Remark

In most cases the conditions of the theorem will be satisfied if we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. This can only be understood when considering specific cases, see for example Subsection 3.2.2.

The main factor that determines the rate of convergence of the solution u_h of Problem G^h to the solution u of Problem G as h tends to zero, is the regularity of the weak solution u . However, all that can be guaranteed from the existence theory is Assumption A1. The regularity of u depends on the regularity of the initial values u_0 and u_1 . Rauch [R85] gave an example to show that the regularity of the solution is necessary to obtain optimal order convergence.

2.6 Inertia norm estimate

Assumption A4, $u \in C^2(J, V)$, is used in [D73] and [Z-PhD]. It is often not justified and then e_p'' cannot be estimated as in Sections 2.4 and 2.5. Baker [B76] derived error estimates for the undamped case using a weaker assumption. In this section we generalize the results of Baker [B76].

2.6.1 Fundamental estimate

Suppose that there is no damping, i.e. $a = 0$. The following lemma is a generalization of the estimate for $e = Pu - u_h$ in (3.15) in [B76]. We may use Lemma 2.3.2 with $a = 0$.

Lemma 2.6.1 *Suppose the solution u of Problem G satisfies Assumptions A1 and A2. If $c(u_1 - u_1^h, v) = 0$ for all $v \in S^h$, then for $t \in [0, T]$,*

$$\|e(t)\|_W \leq \sqrt{2}\|e(0)\|_W + 2\sqrt{T} \max_{t \in [0, T]} \|e'_p(t)\|_W.$$

Proof Using Lemma 2.3.2 with $a = 0$:

$$c(e''(t), v) + b(e(t), v) = -c(e''_p(t), v) \text{ for all } v \in S^h.$$

Now, if v is a function on $[0, T]$ with values in S^h , then

$$c(e''(t), v(t)) + b(e(t), v(t)) = -c(e''_p(t), v(t)). \quad (2.6.1)$$

Let $v'(t) = -e(t)$, then

$$\begin{aligned} \frac{d}{dt} [c(e, e) - b(v, v)] &= 2 c(e', e) - 2 b(v', v) \\ &= -2 c(e', v') + 2 b(e, v), \end{aligned}$$

and

$$\frac{d}{dt} c(e', v) = c(e'', v) + c(e', v').$$

Therefore we have that

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} c(e, e) - \frac{1}{2} b(v, v) + c(e', v) \right] &= b(e, v) + c(e'', v) \\ &= -c(e''_p, v) \text{ from (2.6.1).} \end{aligned}$$

Also, $\frac{d}{dt} c(e'_p, v) = c(e''_p, v) + c(e'_p, v')$, hence

$$\frac{d}{dt} \left[\frac{1}{2} c(e, e) - \frac{1}{2} b(v, v) + c(e', v) + c(e'_p, v) \right] = c(e'_p, v). \quad (2.6.2)$$

Choose $v(\tau) = 0$ for some $\tau \in (0, T)$, and integrate (2.6.2) between 0 and τ . We obtain

$$\begin{aligned} & - \left[\frac{1}{2} c(e(0), e(0)) - \frac{1}{2} b(v(0), v(0)) + c(e'(0), v(0)) + c(e'_p(0), v(0)) \right] \\ & + \frac{1}{2} c(e(\tau), e(\tau)) = \int_0^\tau c(e'_p, e), \end{aligned}$$

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and therefore, since $b(v(0), v(0)) \geq 0$, we have

$$\frac{1}{2} c(e(\tau), e(\tau)) \leq \frac{1}{2} c(e(0), e(0)) + c(e'(0) + e'_p(0), v(0)) + \int_0^\tau c(e'_p, e).$$

Recall that $e_h = e_p + e$. Since $c(u_1 - u_1^h, v(0)) = c(e'_h(0), v(0)) = 0$,

$$c(e(\tau), e(\tau)) \leq c(e(0), e(0)) + 2 \int_0^\tau c(e'_p, e). \quad (2.6.3)$$

We have the following estimate:

$$\begin{aligned} \left| \int_0^\tau c(e'_p, e) \right| &\leq \int_0^\tau |c(e'_p, e)| \\ &\leq \int_0^\tau \|e'_p\|_W \|e\|_W \\ &\leq \max \|e(t)\|_W \int_0^\tau \|e'_p\|_W \\ &\leq \frac{1}{4} [\max \|e(t)\|_W]^2 + \left(\int_0^\tau \|e'_p\|_W \right)^2, \end{aligned}$$

using Young's inequality (where $\max_{t \in [0, T]}$ is denoted by $\max$). Combining this with (2.6.3), we obtain

$$\|e(\tau)\|_W^2 \leq \|e(0)\|_W^2 + \frac{1}{2} \max \|e(t)\|_W^2 + 2T \max \|e'_p(t)\|_W^2.$$

Since this is true for all $\tau \in (0, T)$, we have that

$$\max \|e(t)\|_W^2 \leq \|e(0)\|_W^2 + \frac{1}{2} \max \|e(t)\|_W^2 + 2T \max \|e'_p(t)\|_W^2,$$

and hence

$$\frac{1}{2} \max \|e(t)\|_W^2 \leq \|e(0)\|_W^2 + 2T \max \|e'_p(t)\|_W^2.$$

It follows that

$$\begin{aligned} \|e(t)\|_W^2 &\leq 2\|e(0)\|_W^2 + 4\sqrt{2}\sqrt{T}\|e(0)\|_W\|e_p(t)\|_W + 4T \max \|e'_p(t)\|_W^2 \\ &\leq (\sqrt{2}\|e(0)\|_W + 2\sqrt{T} \max \|e'_p(t)\|_W)^2. \end{aligned}$$

□

2.6.2 Convergence

The error estimates follow readily from Lemma 2.6.1. Note that we use $(Pu)' = Pu'$ but not $(Pu)'' = Pu''$.

Theorem 2.6.1 *Suppose Assumption A3 holds. If the solution u of Problem G satisfies Assumptions A1 and A2 and $c(u_1 - u_1^h, v) = 0 \forall v \in S^h$, then*

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_b \hat{C} h^\alpha \|u(t)\|_{H(V,k)} + 2\sqrt{T} C_b \hat{C} h^\alpha \max \|u'(t)\|_{H(V,k)} \\ &\quad + \sqrt{2} C_b \hat{C} h^\alpha \|u_0\|_{H(V,k)} + \sqrt{2} \|u_0 - u_0^h\|_W, \end{aligned}$$

for each $t \in [0, T]$.

Proof Using Lemma 2.6.1, we have

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \|e_p(t)\|_W + \|e(t)\|_W \\ &\leq \|e_p(t)\|_W + \sqrt{2} \|e(0)\|_W + 2\sqrt{T} \max \|e'_p(t)\|_W \\ &\leq \|e_p(t)\|_W + 2\sqrt{T} \max \|e'_p(t)\|_W \\ &\quad + \sqrt{2} \|e_p(0)\|_W + \sqrt{2} \|e_h(0)\|_W. \end{aligned}$$

Now, $\|e'_p(t)\|_W = \|u'(t) - Pu'(t)\| \leq C_b \|u'(t) - Pu'(t)\|_E$ by Assumption E2 and Lemma 2.3.1. The result follows from Assumption A3. $\square$

Remark

Baker [B76] now simply chooses $u_0^h = Pu_0$, with the result that $\|u_0 - u_0^h\|_W \leq C_b \hat{C} h^\alpha \|u_0\|_{H(V,k)}$. This is in general not possible. However, in most cases the conditions of the theorem will be satisfied if we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. This can only be understood when considering specific cases as in Chapter 3, see e.g. Section 3.4.

2.7 Higher order inertia norm estimate

In this section we show that the Aubin-Nitsche trick, (due to [Au67] and [N69]) can be applied to find higher order estimates in the inertia norm for the discretization error for the case of **modal damping**. The presentation is based on [Z-PhD]. Recall the definition of the set E_b in Section 2.2.

Assumption A7

$E_b \subset H(V, k)$ and for any $u \in E_b$ we have $\|u\|_{H(V, k)} \leq \widehat{c}_b \|y\|_W$, where

$$b(u, v) = (y, v) \quad \forall v \in V.$$

Lemma 2.7.1 *Suppose that Assumptions A3 and A7 hold. There exists a constant $\widehat{C}$ such that, for all $u \in E_b$,*

$$\|u - Pu\|_W \leq \widehat{C}^2 h^{2\alpha} \widehat{c}_b \|u\|_{H(V, k)}.$$

Proof There exists a unique $y \in V$ such that

$$b(y, v) = (e_p, v) \quad \text{for all } v \in V. \quad (2.7.1)$$

By Assumption A7 we have $y \in E_b$ and there exists a $\widehat{c}_b$ such that

$$\|y\|_{H(V, k)} \leq \widehat{c}_b \|e_p\|_W. \quad (2.7.2)$$

Since P is a projection

$$b(e_p, v) = 0 \quad \text{for all } v \in S^h. \quad (2.7.3)$$

Let $v = e_p$ in (2.7.1) and $v = Py$ in (2.7.3), then

$$\|e_p\|_W^2 = b(y - Py, e_p) \leq \|y - Py\|_E \|e_p\|_E.$$

Using Assumption A3,

$$\|e_p\|_W^2 \leq \widehat{C} h^\alpha \|y\|_{H(V, k)} \|e_p\|_E.$$

From (2.7.2) we have

$$\|e_p\|_W^2 \leq \widehat{C} h^\alpha \widehat{c}_b \|e_p\|_E \|e_p\|_W.$$

It follows that

$$\|e_p\|_W \leq \widehat{C} h^\alpha \widehat{c}_b \|e_p\|_E \leq \widehat{C}^2 h^{2\alpha} \widehat{c}_b \|u\|_{H(V, k)}. \quad \square$$

Lemma 2.7.2 *Suppose Assumptions A3 and A7 hold. Assume that the solution u of Problem G satisfies Assumption A4 and both $u'(t)$ and $u''(t)$ in $H(V, k)$. Then, for any $t > 0$,*

$$\begin{aligned} \|e(t)\|_E + \|e'(t)\|_W &\leq \|Pu_0 - u_0^h\|_E + \|Pu_1 - u_1^h\|_W \\ &+ \widehat{C}^2 h^{2\alpha} \widehat{c}_b t \left(\max_{s \in [0, t]} \|u''(s)\|_{H(V, k)} + k \max_{s \in [0, t]} \|u'(s)\|_{H(V, k)} \right). \end{aligned}$$

Proof From Lemma 2.4.3, we have

$$\begin{aligned} \|e(t)\|_E + \|e'(t)\|_W &\leq \|Pu_0 - u_0^h\|_E + \|Pu_1 - u_1^h\|_W \\ &\quad + \int_0^t (\|u'' - Pu''\|_W + k\|u' - Pu'\|_W). \end{aligned}$$

The result follows from Lemma 2.7.1. $\square$

Now we show that the Aubin-Nitsche trick can be applied to obtain higher order inertia norm estimates for the discretization error.

Theorem 2.7.1 *Let $u_0^h = Pu_0$ and $u_1^h = Pu_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are all in $H(V, k)$ for all $t \geq 0$. Then,*

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \widehat{C}^2 h^{2\alpha} \widehat{C}_b \|u(t)\|_{H(V, k)} \\ &\quad + \widehat{C}^2 h^{2\alpha} \widehat{C}_b t \left(\max_{s \in [0, t]} \|u''(s)\|_{H(V, k)} + k \max_{s \in [0, t]} \|u'(s)\|_{H(V, k)} \right). \end{aligned}$$

Proof We have that

$$\|u(t) - u_h(t)\|_W = \|e_p(t)\|_W + \|e(t)\|_W \leq \|e_p(t)\|_W + \|e(t)\|_E + \|e'(t)\|_W.$$

The result follows from Lemma 2.7.1 and Lemma 2.7.2. $\square$

A useful result is also obtained if the Aubin-Nitsche trick is used only for the terms containing the integrals.

Theorem 2.7.2 *Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are in $H(V, k) \cap V$ for $t \geq 0$. Then,*

$$\begin{aligned} \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W &\leq 2C_\Pi h^\alpha \|u_0\|_{H(V, k)} + (\widehat{C}^2 h^{2\alpha} \widehat{C}_b + C_b C_\Pi h^\alpha) \|u_1\|_{H(V, k)} \\ &\quad + \widehat{C} h^\alpha \|u(t)\|_{H(V, k)} + \widehat{C}^2 h^{2\alpha} \widehat{C}_b \|u'(t)\|_{H(V, k)} \\ &\quad + \widehat{C}^2 h^{2\alpha} \widehat{C}_b t \left(\max_{s \in [0, t]} \|u''(s)\|_{H(V, k)} + k \max_{s \in [0, t]} \|u'(s)\|_{H(V, k)} \right), \end{aligned}$$

for $t \in [0, t]$.

Proof Use Lemma 2.7.2, Lemma 2.7.1 and Assumption A5. $\square$

Remark This result in [Z-PhD] is significant: It is advantageous to have an error estimate in the energy norm, while the terms containing t are dominated by $h^{2\alpha}$.

Chapter 3

Energy related methods for the wave equation

3.1 Existence of solutions for the wave equation

The weak variational form of the one-dimensional wave equation is considered in Section 2.1 and existence of solutions in Section 2.2. In this section we consider the two and three-dimensional cases.

3.1.1 Weak variational form for the n-dimensional wave equation

Consider Problem 5V. The space of test functions $T(\Omega)$ is defined in Section 1.3 by

$$T(\Omega) = \{ \phi \in C^1(\bar{\Omega}) \mid \phi = 0 \text{ on } \partial\Omega - \Sigma \}.$$

Let $V(\Omega)$ denote the closure of $T(\Omega)$ in $H^1(\Omega)$. The bilinear forms b and c are defined as in Section 1.3:

$$\begin{aligned} b(u, v) &= \iiint_{\Omega} A \nabla u \cdot \nabla v \, dV + \iint_{\Sigma} \beta_2 uv \, dS \text{ and} \\ c(u, v) &= \iiint_{\Omega} \rho uv \, dV, \end{aligned}$$

where $\partial_i u$ and $\partial_i v$ denote weak derivatives.

The trace operator Γ is defined in the Appendix A.5 and the bilinear form a is defined on $V(\Omega)$ by

$$a(u, v) = \iint_{\Sigma} \beta_1 \Gamma u \Gamma v \, dS.$$

Remarks

1. If $n = 2$, then Σ is a curve and the surface integral must be replaced by a line integral.
2. Dupont [D73] considered the case where n is any natural number.

Problem 5W

Find u such that for each $t > 0$, $u(t) \in V(\Omega)$, $\Gamma u'(t)$ exists, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) + b(u(t), v) + a(u'(t), v) = (f(t), v)_{\Omega} + (g(t), v)_{\Sigma} \quad \text{for each } v \in V(\Omega)$$

while $u(0) = u_0$, $u'(0) = u_1$.

Problem D73W

Special case of Problem 5W with $\Sigma = \partial\Omega$ and $A = \alpha I$.

Note that for this problem $V(\Omega) = H^1(\Omega)$.

Problem B76W

Special case of Problem 5W with Σ empty, $a = 0$ and $\beta_2 = 0$.

Note that for this problem $V(\Omega) = H_0^1(\Omega)$ and the term $(g, v)_{\Sigma} = 0$.

Remark

In the first part of [D73] the problem is actually the same as Problem B76W except that $A = \alpha I$. This is not significant since A is uniformly positive definite.

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3.1.2 Inner products and norms

The bilinear form c is an inner product for $X = \mathcal{L}^2(\Omega)$ and the corresponding norm is equivalent to the norm of $\mathcal{L}^2(\Omega)$. The space $\mathcal{L}^2(\Omega)$ with inner product c is denoted by W .

Assumption Poincare-Friedrichs inequality.

There exists a constant β such that $\|v\|^2 \leq \beta b(v, v)$ for each $v \in V$.

If $\partial\Omega - \Sigma$ has positive area (positive length for $n = 2$), then the assumption holds, see [Br, p.30]. Baker [B76] state this without proof or reference. It is not even mentioned in [D73].

If the Poincare-Friedrichs inequality is true, then b is positive definite on V with respect to the norm of $H^1(\Omega)$ and the bilinear form b is an inner product on V . (The fact that the bilinear form b is symmetric follows from the definition of b .)

Definition The norm corresponding to the inner product b is

$$\|v\|_V = \sqrt{b(v, v)} \text{ for any } v \in V.$$

The bilinear form b is bounded on $H^1(\Omega)$, hence the norm $\|\cdot\|_V$ is equivalent to the norm of $H^1(\Omega)$ on V .

Proposition 3.1.1 V is dense in $\mathcal{L}^2(\Omega)$.

Proof From Appendix A we have that $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$. Since $C_0^\infty(\Omega)$ is a subspace of V , V is also dense in $\mathcal{L}^2(\Omega)$. $\square$

3.1.3 Existence of a solution

We have that Assumptions E1, E2 and E4 are satisfied. For existence we also need Assumption E3.

Proposition 3.1.2 *The bilinear form a is nonnegative, symmetric and bounded on V (Assumption E3).*

Proof The bilinear form a is clearly symmetric. Since $a(u, u) = \iint_{\Sigma} \beta_1(\Gamma u)^2 dS$ and $\beta_1 \geq c_1 > 0$, it follows that a is nonnegative.

For any u and v in V , we have from the Schwartz inequality and Theorem A.5.3 in Appendix A that

$$\begin{aligned} |a(u, v)| &= \left| \iint_{\Sigma} \beta_1(\Gamma u \Gamma v) dS \right| \\ &\leq \max \beta_1 \|\Gamma u\|_{\Sigma} \|\Gamma v\|_{\Sigma} \\ &\leq \max \beta_1 K^2 \|u\|_1 \|v\|_1. \end{aligned}$$

From the equivalence of the norms $\|\cdot\|_V$ and $\|\cdot\|_1$ it follows that there exists a constant K_a such that

$$|a(u, v)| \leq K_a \|u\|_V \|v\|_V. \quad \square$$

Theorem 3.1.1 Suppose $f \in C^1(J, X)$ and $g = 0$. Problem 5W has a unique solution

$$u \in C^1(J; V) \cap C^2(J; W)$$

if for $u_0 \in V$ and $u_1 \in V$ there exists some $y \in W$ such that

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \text{for each } v \in V. \quad (3.1.1)$$

Proof Assumptions E1, E2, E3 and E4 hold. We use Theorem 2.2.1. All the Assumptions are satisfied and the initial condition is satisfied by (3.1.1). Therefore Problem 5W has a unique solution. $\square$

Remarks

1. Problem B76W is a special case of Problem 5W with $g = 0$, so existence is assured.
2. Existence of a solution for Problem D73 is uncertain. It is not considered in [D73].

3.2 Modal damping for the wave equation

In this section we apply the results of Chapter 2 to the wave equation.

3.2.1 The one-dimensional wave equation

The convergence results for modal damping can be applied to Problems 1 to 4. As an example we consider Problem 2. Existence of a solution is discussed in Subsection 2.2.2. But in this case we also know that a classical solution exists for suitable initial values. We may use Proposition 2.1.1 or Proposition 2.2.1.

Now consider the finite element approximation of Problem 2W. A finite dimensional subspace S^h of $V(0, \ell)$ is constructed, e.g. using piecewise linear basis functions.

Problem 2W^h

Find a function $u_h \in C(J, S^h)$ with u'_h continuous at 0 such that

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \quad (3.2.1)$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

We have to investigate whether the Assumptions in Chapter 2 hold for this problem in order to apply the theorems in Sections 2.4, 2.5 and 2.7. (The Assumptions are summarized in Appendix C for convenience.) Note that we do not use the results in Section 2.6 because we have damping.

Which theorem may be applied depends on the regularity of the solution and regularity theory for hyperbolic equations is beyond the scope of this dissertation. However, it is known that the smoothness of the solution depends on the given data. So, taking Problem 2 for example, the smoothness of the solution will depend on the smoothness of the functions f , u_0 and u_1 .

If f , u_0 and u_1 satisfy the conditions for Proposition 2.2.1, then a weak solution exists and Assumption A1 is a fact. (Sufficient conditions are given in Subsection 2.2.2.) However, we do not know if the solution u has a second derivative $u''(t) \in V(0, \ell)$ (Assumption A4) even for a classical solution. We need to assume that Assumption A4 holds.

For this problem we have that

$$H(V, k) = H^k(0, \ell) \cap V(0, \ell).$$

Instead of Assumption A5 we have a more specific estimate which depends on S^h and the interpolation operator Π . For the subspace S^h we consider only the well known piecewise linear basis functions and piecewise Hermite cubics, see e.g. [H], [OC] and [SF].

Interpolation (See Appendix B)

For piecewise linear basisfunctions (Corrolary B.1.2):

If $u \in H^k(0, \ell)$ for $k \geq 2$, then

$$\|\Pi u - u\|_1 \leq \hat{C}_L h |u|_2. \quad (3.2.2)$$

For piecewise Hermite cubics (Corrolary B.1.1):

If $u \in H^k(0, \ell)$ for $k = 2, 3$ or 4 , then

$$\|\Pi u - u\|_m \leq \hat{C}_C h^{k-m} |u|_k \quad \text{for } m = 0, 1, \dots, k. \quad (3.2.3)$$

If $u \in H^k(0, \ell)$ for $k > 4$, the value $k = 4$ must be used in equation (3.2.3).

Remarks

1. The semi-norm $|u|_j = \|u^{(j)}\|$ is defined in Appendix A.
2. The norm $\|\cdot\|_1$ is equivalent to the energy norm $\|\cdot\|_E$ for Problems 1 to 4.
3. For piecewise linear basis functions $m = 1$ and we have $u \in H^2(0, \ell)$.

Application of Theorem 2.5.1 with piecewise linear basis functions

Assumption A5 is now a fact, it is replaced by Equation (3.2.2). Suppose that $u(t)$, $u'(t)$ and $u''(t)$ are in $H^2(0, \ell)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ & \leq C_L h (|u(t)|_2 + C_b |u'(t)|_2 + 2|u_0|_2 + 2C_b |u_1|_2 \\ & \quad + \gamma C_b t \max_{s \in [0, t]} |u'(s)|_2 + C_b t \max_{s \in [0, t]} |u''(s)|_2), \end{aligned}$$

for $t \in [0, t)$. (Note that $k = \gamma$ for Problem 2.)

Application of Theorem 2.5.2 with piecewise linear basis functions

If we do not make the assumptions on u' and u'' , we may still use Theorem 2.5.2. We have Assumption A6 since $H^2(0, \ell)$ is dense in V . If we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\|_W = 0.$$

Consider for example the first limit.

$$\begin{aligned} \|Pu_0 - u_0^h\|_V &\leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\ &\leq 2\|u_0 - \Pi u_0\|_V \rightarrow 0 \text{ as } h \rightarrow 0 \text{ (Equation 3.2.2)}. \end{aligned}$$

Therefore, by Theorem 2.5.2,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_E = 0 \text{ for } t \in [0, \tau].$$

Note that it is not necessary to make any assumptions.

Application of Theorems 2.7.1 and 2.7.2

We need to prove that Assumption A7 is satisfied. For the rest the considerations are the same as for Theorem 2.5.1. If $u \in V$ is the solution of $b(u, v) = (f, v) \forall v \in V$ for a given $f \in \mathcal{L}^2(0, \ell)$. It is easy to prove for the one-dimensional case that $u \in H^2(0, \ell)$ and $\|u\|_2 \leq C\|f\|$. This implies that Assumption A7 is satisfied.

From Theorem 2.7.1:

Let $u_0^h = Pu_0$ and $u_1^h = Pu_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are all in H^2 for all $t \geq 0$. Then,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \widehat{C}^2 h^2 \widehat{c}_b |u(t)|_2 \\ &\quad + \widehat{C}^2 h^2 \widehat{c}_b t \left(\max_{s \in [0, t]} |u''(s)|_2 + k \max_{s \in [0, t]} |u'(s)|_2 \right). \end{aligned}$$

From Theorem 2.7.2:

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are in $H^2 \cap V$ for $t \geq 0$. Then,

$$\begin{aligned} &\|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ &\leq 2C_\Pi h |u_0|_2 + (\widehat{C}^2 h^2 \widehat{c}_b + C_b C_\Pi h) |u_1|_2 \\ &\quad + \widehat{C} h |u(t)|_2 + \widehat{C}^2 h^2 \widehat{c}_b |u'(t)|_2 \\ &\quad + \widehat{C}^2 h^2 \widehat{c}_b t \left(\max_{s \in [0, t]} |u''(s)|_2 + k \max_{s \in [0, t]} |u'(s)|_2 \right), \end{aligned}$$

for $t \in [0, t)$.

Remark

We see that the rate of convergence of the solution u_h of Problem $2W^h$ to the solution u of Problem $2W$ as h tends to zero, depends on the regularity of the weak solution u . The regularity of u depends on the regularity of the initial values u_0 and u_1 and the function f . [R85] gave an example to show that the regularity of the solution is necessary to obtain optimal order convergence.

Numerical integration

In Problem G^h we have the forcing term $(f(t), v)_X$. This is theoretically irrelevant as it is eliminated in Lemma 2.3.1. However, in practice it may be necessary to use some form of numerical integration.

It is often convenient to replace $(f(\cdot, t), v)_X$ by

$$(\Pi f(\cdot, t), v)_X = \sum_{i=1}^n f(x_i, t) \int_0^\ell \phi_i v.$$

Now we have a new problem, the Galerkin approximation changed.

Problem $2W^h$

Find a function $w_h \in C(J, S^h)$ with w'_h continuous at 0 such that

$$c(w_h''(t), v) + a(w_h''(t), v) + b(w_h(t), v) = \sum_{i=1}^n f(x_i, t)(\phi_i, v)_X, \quad (3.2.4)$$

for each $v \in S^h$, while $w_h(0) = w_0^h$ and $w'_h(0) = w_1^h$.

Remarks

1. Since u_h and w_h are not equal, there is an additional error, $u_h - w_h$, which should be investigated. However, we will not pursue this matter any further.
2. [Wu06] uses this method of replacing f with its interpolant.
3. Numerical integration is also required when the parameter α is not constant, but that is beyond scope of this dissertation.
4. The fully discrete problem is considered in Chapter 7.
5. Results of numerical experiments are given in Section 7.3.

3.2.2 The n-dimensional wave equation

Problem 5 is too general to consider using the theory in Chapter 2. The convergence results for modal damping can be applied to Problem B76 as the undamped case is a special case of modal damping. (Damping terms may even be included as long as the damping is modal.) Existence of a solution is proved in Section 3.1 (Theorem 3.1.1). (Problem D73 has boundary damping which is not modal, it is treated in Section 3.3.)

Now consider the finite element approximation of Problem B76W. A finite dimensional subspace S^h of $V(\Omega)$ is constructed, e.g. using piecewise linear basis functions on triangle elements or piecewise bilinear basis functions on quadrilateral elements.

Problem B76W^h

Find a function $u_h \in C(J, S^h)$ with u'_h continuous at 0 such that

$$c(u''_h(t), v) + b(u_h(t), v) = (f(t), v)_X, \quad (3.2.5)$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u'_h(0) = u_1^h$.

We need to investigate whether the Assumptions in Chapter 2 hold for this problem in order to apply the theorems in Sections 2.4, 2.5 and 2.7. (The Assumptions are summarized in Appendix C for convenience.) The application of the results in Section 2.6 is done in Section 3.4.

As mentioned in Subsection 3.2.1, the theorems that may be applied depend on the regularity of the solution which depends on the given data. So the smoothness of the solution will depend on the smoothness of the functions f , u_0 and u_1 .

If f , u_0 and u_1 satisfy the conditions for Theorem 3.1.1, then a weak solution exists and Assumption A1 is a fact. However, we do not know if the solution u has a second derivative $u''(t) \in V(\Omega)$ (Assumption A4). We need to assume that Assumption A4 holds to apply the results in Sections 2.4, 2.5 and 2.7.

For this problem we have that

$$H(V, k) = H^k(\Omega) \cap V(\Omega).$$

Instead of Assumption A5 we have a more specific estimate which depends on S^h and the interpolation operator Π . For simplicity we consider the two-dimensional wave equation on a rectangle and suppose we use triangle elements.

Interpolation (See Appendix B)

For piecewise linear basisfunctions on a triangle (Corrolary B.2.1):

If $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\|_1 \leq \hat{C}_T h |u|_2. \quad (3.2.6)$$

Remarks

1. The semi-norm $|u|_j$ is defined in Appendix B.
2. The norm $\|\cdot\|_1$ is equivalent to the energy norm $\|\cdot\|_E$ for Problem B76.
3. The estimate for piecewise bilinear basis functions on a rectangle is the same as for piecewise linear basis functions on a triangle.

Application of Theorem 2.5.1 with piecewise linear basis functions on triangle elements

Assumption A5 is now a fact, it is replaced by Equation (3.2.6). Suppose that $u(t)$, $u'(t)$ and $u''(t)$ are in $H^2(\Omega)$. Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Then,

$$\begin{aligned} & \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ & \leq C_L h (|u(t)|_2 + C_b |u'(t)|_2 + 2|u_0|_2 + 2C_b |u_1|_2 \\ & \quad + \gamma C_b t \max_{s \in [0, t]} |u'(s)|_2 + C_b t \max_{s \in [0, t]} |u''(s)|_2), \end{aligned}$$

for $t \in [0, t]$. (Note that $k = \gamma$ for Problem 2.)

Application of Theorem 2.5.2 with piecewise linear basis functions

If we do not make the assumptions on u' and u'' , we may still use Theorem 2.5.2. We have Assumption A6 since $H^2(\Omega)$ is dense in V . If we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, then

$$\lim_{h \rightarrow 0} \|Pu_0 - u_0^h\|_V = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \|Pu_1 - u_1^h\|_W = 0.$$

Consider for example the first limit.

$$\begin{aligned} \|Pu_0 - u_0^h\|_V & \leq \|Pu_0 - u_0\|_V + \|u_0 - \Pi u_0\|_V \\ & \leq 2\|u_0 - \Pi u_0\|_V \rightarrow 0 \text{ as } h \rightarrow 0 \text{ (Equation 3.2.6).} \end{aligned}$$

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Therefore, by Theorem 2.5.2,

$$\lim_{h \rightarrow 0} \|u(t) - u_h(t)\|_E = 0 \text{ for } t \in [0, \tau].$$

Note that no assumptions were made.

Application of Theorems 2.7.1 and 2.7.2

We need to prove that Assumption A7 is satisfied. In some cases we have that if $u \in V$ is the solution of $b(u, v) = (f, v) \forall v \in V$ for a given $f \in \mathcal{L}^2(\Omega)$, then $u \in H^2(\Omega)$ and $\|u\|_2 \leq C\|f\|$. This implies that Assumption A7 is satisfied. However, this regularity result for elliptic boundary value problems is not always true. For the rest the considerations are the same as for Theorem 2.5.1.

From Theorem 2.7.1:

Let $u_0^h = Pu_0$ and $u_1^h = Pu_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are all in H^2 for all $t \geq 0$. Then,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \hat{C}^2 h^2 \hat{c}_b |u(t)|_2 \\ &\quad + \hat{C}^2 h^2 \hat{c}_b t \left(\max_{s \in [0, t]} |u''(s)|_2 + k \max_{s \in [0, t]} |u'(s)|_2 \right). \end{aligned}$$

From Theorem 2.7.2:

Let $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. Assume that $u(t)$, $u'(t)$ and $u''(t)$ are in $H^2 \cap V$ for $t \geq 0$. Then,

$$\begin{aligned} &\|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ &\leq 2C_\Pi h |u_0|_2 + (\hat{C}^2 h^2 \hat{c}_b + C_b C_\Pi h) |u_1|_2 \\ &\quad + \hat{C} h |u(t)|_2 + \hat{C}^2 h^2 \hat{c}_b |u'(t)|_2 \\ &\quad + \hat{C}^2 h^2 \hat{c}_b t \left(\max_{s \in [0, t]} |u''(s)|_2 + k \max_{s \in [0, t]} |u'(s)|_2 \right), \end{aligned}$$

for $t \in [0, t)$.

As for the one-dimensional case the rate of convergence of the solution u_h of Problem B76W^h to the solution u of Problem B76W as h tends to zero, depends on the regularity of the weak solution u and this depends on the smoothness of the given data.

The application of Section 2.6 to Problem B76 is considered in Section 3.4.

Numerical integration

The forcing term $(f(\cdot, t), v)_X$ may be handled by the same methods as for the one-dimensional case (see Subsection 3.2.1).

If the parameters (i.e. the coefficients in the matrix A) are not constant, numerical integration is used (see e.g. [BD76] and [SPC03]). We do not consider numerical integration in this dissertation.

For the two-dimensional or three-dimensional wave equation we have the problem of a curved boundary. If the boundary of Ω does not consist of straight lines (two-dimensional case) or planes (three-dimensional case), then Ω is not the union of the elements. In [SF, Section 3.3] it is shown how to overcome this difficulty with isoparametric transformations.

Remark The fully discrete case is considered in Chapter 7.

3.3 The three-dimensional wave equation with boundary damping

We consider Problem D73W, (see Section 3.1). Boundary damping is not modal hence the results in Sections 2.4 to 2.7 may not be used as in Subsection 3.2.2. The most important contribution in [D73] is the method to deal with the damping term.

3.3.1 Fundamental estimate

The spaces V , W and X are defined in Section 3.1 and it is proved that Assumptions E1, E3 and E4 hold. Assuming that the Poincaré-Friedrichs inequality is valid, Assumption E2 follows.

We assume that Problem D73W has a unique solution satisfying Assumptions A1 and A2. Consequently we may use all the results in Section 2.3. Recall that the challenge is to estimate $e = Pu - u^h$. From Lemma 2.3.3, we have

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)). \quad (3.3.1)$$

In this case

$$a(u, v) = \int \int_{\Sigma} \beta_1 \Gamma u \Gamma v dS.$$

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Dupont uses the notation $\langle u, v \rangle$ and $|u| = \sqrt{\langle u, u \rangle}$. We use $(u, v)_\Sigma$ instead of $\langle u, v \rangle$ and $\|u\|_\Sigma$ instead of $|u|$ to avoid confusion.

There is a problem with the last term in Equation (3.3.1) since

$$a(e'_p(t), e'(t)) \leq K \|e'_p(t)\|_\Sigma \|e'(t)\|_\Sigma,$$

and $\|e'(t)\|_\Sigma$ is not bounded by $E(t)$. The next lemma from [D73] enables us to estimate this term. We need the following auxiliary result.

Proposition 3.3.1 *If $e_p \in C^2(J, V)$, then*

$$\int_0^t (e'_p, e')_\Sigma = (e'_p(t), e(t))_\Sigma - (e'_p(0), e(0))_\Sigma - \int_0^t (e''_p, e)_\Sigma.$$

Proof Since $\|u\|_\Sigma^2 = \iint_\Sigma \beta_1 (\Gamma u)^2 dS \leq K \|u\|_V$ for some constant K , it follows that

$$\Gamma e_p \in C^2(J, \mathcal{L}^2(\Sigma)) \quad \text{and} \quad (\Gamma e_p)'' = \Gamma e''_p.$$

On the other hand $e(t) = \sum_{i=1}^n e_i(t) \phi_i$, so

$$\Gamma e(t) = \sum_{i=1}^n e_i(t) \Gamma \phi_i \quad \text{and} \quad (\Gamma e)'' = \Gamma e''.$$

Using integration by parts we have

$$\int_0^t \Gamma e'_p \Gamma e' = \Gamma e'_p(t) \Gamma e(t) - \Gamma e'_p(0) \Gamma e(0) - \int_0^t \Gamma e''_p \Gamma e.$$

Now,

$$\begin{aligned} \int_0^t (e'_p(\tau), e'(\tau))_\Sigma d\tau &= \int_0^t \iint_\Sigma \beta_1 \Gamma e'_p(\tau) e'(\tau) dS d\tau \\ &= \iint_\Sigma \beta_1 \int_0^t \Gamma e'_p(\tau) e'(\tau) d\tau dS, \end{aligned}$$

by Fubini's theorem. The integral in the other term may also be interchanged and the result follows. $\square$

The proof of Lemma 3.3.1 is from [D73] using the notation of [Z-PhD] and [ZVV99]. The result above is used in [D73] with a different explanation, using results without sufficient references.

Lemma 3.3.1 *If the solution of Problem D73 satisfies Assumptions A1 and A4, then there exists a constant C such that for any $t \in [0, T]$*

$$\|e'(t)\|_W^2 + \|e(t)\|_E^2 \leq e^{CT} K_T, \quad \text{where}$$

$$K_T = \int_0^T \|e_p''\|_W^2 + 4K^2 \max \|e_p'(t)\|_E^2 + 2K \int_0^T \|e_p''(\cdot)\|_E^2 + \|e'(0)\|_W^2 + \|e(0)\|_E^2 + \|e_p'(0)\|_\Sigma^2 + \|e(0)\|_\Sigma^2.$$

Proof We have the following estimates from Equation (2.3.3):

$$\begin{aligned} |c(e_p''(t), e'(t))| &\leq \|e_p''(t)\|_W \|e'(t)\|_W \leq \frac{1}{2} \|e_p''(t)\|_W^2 + \frac{1}{2} \|e'(t)\|_W^2, \\ \|e'(t)\|_W^2 &\leq 2E(t). \end{aligned}$$

Using the above estimates in Equation (3.3.1) and integrating on both sides yields

$$E(t) \leq E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + \int_0^t E - \int_0^t (e_p', e')_\Sigma. \quad (3.3.2)$$

Using Proposition 3.3.1 we obtain

$$\begin{aligned} E(t) &\leq E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + \int_0^t E \\ &\quad - \left((e_p'(t), e(t))_\Sigma - (e_p'(0), e(0))_\Sigma - \int_0^t (e_p'', e)_\Sigma \right). \end{aligned}$$

We need estimates for the terms in brackets. We use Young's inequality (see Appendix E) and the definition of $E(t)$.

$$\begin{aligned} |(e_p'(t), e(t))_\Sigma| &\leq K \|e_p'(t)\|_\Sigma \|e(t)\|_\Sigma \\ &\leq K \|e_p'(t)\|_E \|e(t)\|_E \\ &\leq K\varepsilon^{-2} \|e_p'(t)\|_E^2 + K\varepsilon^2 \|e(t)\|_E^2 \\ &\leq K\varepsilon^{-2} \|e_p'(t)\|_E^2 + K\varepsilon^2 E(t), \end{aligned}$$

with ε arbitrary and K a generic constant. Also,

$$\begin{aligned} \left| \int_0^t (e_p'', e)_\Sigma \right| &\leq \int_0^t |(e_p'', e)_\Sigma| \\ &\leq K \int_0^t \|e_p''\|_E^2 + K \int_0^t \|e\|_E^2 \\ &\leq K \int_0^t \|e_p''\|_E^2 + K \int_0^t E. \end{aligned}$$

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Therefore we have that

$$\begin{aligned}
E(t) &\leq E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + K \int_0^t E + K\varepsilon^2 E(t) + K\varepsilon^{-2} \|e_p'(t)\|_E^2 \\
&\quad + (e_p'(0), e(0))_\Sigma + K \int_0^t \|e_p''(\cdot)\|_E^2 \\
&\leq E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + K \int_0^t E + \frac{1}{2} E(t) + 2K^2 \|e_p'(t)\|_E^2 \quad (3.3.3) \\
&\quad + (e_p'(0), e(0))_\Sigma + K \int_0^t \|e_p''(\cdot)\|_E^2,
\end{aligned}$$

by choosing ε such that $K\varepsilon^2 = \frac{1}{2}$.

Note that $(e_p'(0), e(0))_\Sigma \leq \frac{1}{2} \|e_p'(0)\|_\Sigma^2 + \frac{1}{2} \|e(0)\|_\Sigma^2$. Rewriting (3.3.3) we get

$$\begin{aligned}
\frac{1}{2} E(t) &\leq K \int_0^t E + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + 2K^2 \|e_p'(t)\|_E^2 \\
&\quad + K \int_0^t \|e_p''(\cdot)\|_E^2 + E(0) + \frac{1}{2} \|e_p'(0)\|_\Sigma^2 + \frac{1}{2} \|e(0)\|_\Sigma^2,
\end{aligned}$$

which can be written in the form needed for the Gronwall inequality:

$$\begin{aligned}
E(t) &\leq 2K \int_0^t E + K_T, \text{ where the constant} \\
K_T &= 2 \left[\frac{1}{2} \int_0^T \|e_p''\|_W^2 + 2K^2 \max \|e_p'(t)\|_E^2 + K \int_0^T \|e_p''(\cdot)\|_E^2 \right. \\
&\quad \left. + \frac{1}{2} \|e_p'(0)\|_\Sigma^2 + \frac{1}{2} \|e(0)\|_\Sigma^2 + \frac{1}{2} \|e_p'(0)\|_\Sigma^2 + \frac{1}{2} \|e(0)\|_\Sigma^2 \right].
\end{aligned}$$

By applying the Gronwall inequality (see Appendix E), we have

$E(t) \leq K_T e^{2KT}$. The result follows. $\square$

Lemma 3.3.1 corresponds to Lemma 3 in [D73]. For convenience we display the corresponding notation and Lemma 3 from the article.

Our notation	Dupont notation
S^h	M
u^h	U
Pu	W
$e_p = u - Pu$	$\eta = u - W$
$e = Pu - u_h$	$\zeta = W - U$

Lemma 3 [D73]

There exists a constant C such that

$$\begin{aligned} & \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^\infty(\mathcal{L}^2(\Omega))}^2 + \|\nabla \zeta\|_{L^\infty(\mathcal{L}^2(\Omega))}^2 + \left\| \alpha^{\frac{1}{2}} \frac{\partial \zeta}{\partial t} \right\|_{\mathcal{L}^2(\mathcal{L}^2(\partial\Omega))}^2 \\ & \leq C \left[\left\| \frac{\partial \zeta}{\partial t}(0) \right\|^2 + \|\nabla \zeta(0)\|^2 + \|\eta\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 \right. \\ & \quad + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^\infty(H^{-\frac{1}{2}}(\partial\Omega))}^2 + \left\| \frac{\partial^2 \eta}{\partial t^2} \right\|_{\mathcal{L}^2(H^{-\frac{1}{2}}(\partial\Omega))}^2 \\ & \quad \left. + \|\eta\|_{\mathcal{L}^2(\mathcal{L}^2(\Omega))}^2 + \left\| \frac{\partial^2 \eta}{\partial t^2} \right\|_{\mathcal{L}^2(\mathcal{L}^2(\Omega))}^2 \right]. \end{aligned}$$

3.3.2 Projection errors

We may now proceed as in Section 2.4. For boundary damping in the n -dimensional wave equation, Lemma 3.3.1 replaces Lemma 2.4.1 in the theory.

Lemma 3.3.2 *Assume that the solution of Problem G satisfies Assumption A_4 . Then there exists a constant C such that for any $t \in [0, T]$*

$$\|e'(t)\|_W^2 + \|e(t)\|_E^2 \leq e^{CT} K_T, \quad \text{where}$$

$$\begin{aligned} K_T = & \int_0^T \|u'' - Pu''\|_W^2 + 4K^2 \max \|u'(t) - Pu'(t)\|_E^2 + 2K \int_0^T \|u'' - Pu''\|_E^2 \\ & + \|Pu_1 - u_1^h\|_W + \|Pu_0 - u_0^h\|_E + \|u_1 - Pu_1\|_\Sigma^2 + \|Pu_0 - u_0^h\|_\Sigma^2. \end{aligned}$$

Proof The result follows from Lemmas 2.3.1, 2.4.2 and 3.3.1. $\square$

Lemma 3.3.3 *Suppose Assumptions $E2$ and $A3$ hold. Assume that the solution u of Problem G satisfies Assumption A_4 . Then there exists a constant C such that for any $t \in [0, T]$*

$$\|u(t) - u_h(t)\|_E^2 + \|u'(t) - u'_h(t)\|_W^2 \leq \|u(t) - Pu(t)\|_E^2 + \|u'(t) - Pu'(t)\|_W^2 + e^{CT} K_T,$$

where

$$\begin{aligned} K_T = & \int_0^T \|u'' - Pu''\|_W^2 + 4K^2 \max \|u'(t) - Pu'(t)\|_E^2 + 2K \int_0^T \|u'' - Pu''\|_E^2 \\ & + \|Pu_1 - u_1^h\|_W + \|Pu_0 - u_0^h\|_E + \|u_1 - Pu_1\|_\Sigma^2 + \|Pu_0 - u_0^h\|_\Sigma^2. \end{aligned}$$

Proof

$$\|u(t) - u_h(t)\|_E^2 + \|u'(t) - u'_h(t)\|_W^2 \leq \|e_p(t)\|_E^2 + \|e(t)\|_E^2 + \|e'_p(t)\|_W^2 + \|e'(t)\|_W^2$$

The result follows from Lemmas 3.3.2 and 2.3.1. $\square$

Theorem 3.3.1 *Suppose Assumptions E2 and A3 hold. Assume that the solution u of Problem G satisfies Assumption A4 and $u(t)$, $u'(t)$ and $u''(t)$ are in $H(V, k)$. Then there exists a C such that for any $t \in [0, T]$*

$$\begin{aligned} \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W &\leq \widehat{C}h^\alpha [\|u\|_{H(V, k)} + C_b\|u'\|_{H(V, k)}] \\ &+ e^{CT}\widehat{C}h^\alpha \left[C_b \int_0^t \|u''\|_{H(V, k)} + 2K \max \|u'\|_{H(V, k)} + \sqrt{2K} \int_0^t \|u''\|_{H(V, k)} \right] \\ &+ e^{CT}K_0, \end{aligned}$$

where $K_0 = \|Pu_1 - u_1^h\|_W + \|Pu_0 - u_0^h\|_E + C_s\|u_1 - Pu_1\|_1 + C_s\|Pu_0 - u_0^h\|_1$.

Proof We use the estimate in Lemma 3.3.3. To deal with the term on the right, we apply the estimates in Assumptions A3, E2 and Theorem A.5.4:

$$\begin{aligned} \|Pu - u\|_E &\leq \widehat{C}h^\alpha \|u\|_{H(V, k)}, \\ \|u\|_W &\leq C_b\|u\|_E, \\ \|u\|_\Sigma &\leq C_s\|u\|_1. \end{aligned}$$

The result is that

$$\begin{aligned} \|u(t) - u_h(t)\|_E^2 + \|u'(t) - u'_h(t)\|_W^2 &\leq \widehat{C}^2 h^{2\alpha} [\|u(t)\|_{H(V, k)}^2 + C_b^2 \|u'(t)\|_{H(V, k)}^2] \\ &+ K_e \int_0^t \widehat{C}^2 h^{2\alpha} \left[C_b^2 \int_0^t \|u''\|_{H(V, k)}^2 + 4K^2 \|u'(t)\|_{H(V, k)}^2 + 2K \int_0^t \|u''\|_{H(V, k)}^2 \right] \\ &+ K_0, \end{aligned}$$

where $K_0 = \|Pu_1 - u_1^h\|_W^2 + \|Pu_0 - u_0^h\|_E^2 + C_s\|u_1 - Pu_1\|_1^2 + C_s\|Pu_0 - u_0^h\|_1^2$. Since $(a + b)^2 \leq 2a^2 + 2b^2$ for positive a and b , the result follows. $\square$

Remarks

1. To apply the theory to the wave equation is the same as in Subsection 3.2.2. Note that Assumption A3 can be proved using interpolation.

2. The fully discrete case is considered in Chapter 7.
3. In Theorems 1 and 2 in [D73] it is **supposed** that

$$\|Pu_1 - u_1^h\| + \|Pu_0 - u_0^h\| \leq C_1 h^k.$$

Note that

$$\begin{aligned} \|Pu_0 - u_0^h\|_E &\leq \|Pu_0 - u_0\|_E + \|u_0 - u_0^h\| \\ &\leq \hat{C}h \|u_0\|_2, \end{aligned}$$

if we choose $u_0^h = \Pi u_0$.

3.3.3 Dupont's error estimates

For the wave equation $H(V, k) = H^k(\Omega) \cap V(\Omega)$ and for problem D73 the energy space $V(\Omega) = H^1(\Omega)$. So Assumption A3 may be formulated as $\|Pu - u\|_1 \leq Kh^{k-1}\|u\|_k$.

In [D73] Assumption (2.11) p.882 is used:

$$\inf_{\chi \in M_h} \|V - \chi\|_{H^s(\Omega)} \leq Ch^{\ell-s} \|V\|_{H^\ell(\Omega)}.$$

No reference is given for the estimate.

Theorem 2 [D73]

Let u be the solution to D73W, and let U be its Galerkin approximation. Suppose that $u, \frac{\partial u}{\partial t} \in L^\infty(H^k(\Omega))$ and that $\frac{\partial^2 u}{\partial t^2} \in \mathcal{L}^2(H^k(\Omega))$. Suppose that $U(0)$ and $\frac{\partial U}{\partial t}(0)$ are taken so that $\|(U - W)(0)\|_1 + \|\frac{\partial(U - W)}{\partial t}(0)\| \leq C_1 h^k$, with C_1 independent of h . Then there is a constant C independent of h such that

$$\left\| \frac{\partial}{\partial t}(U - u) \right\|_{L^\infty(\mathcal{L}^2(\Omega))} + \|U - u\|_{L^\infty(\mathcal{L}^2(\Omega))} \leq Ch^k.$$

Remarks

1. The notation $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$ used in the article may be misleading. Actually these derivatives are defined in Subsection 2.2.1.
2. Consider the assumption $\|(U - W)(0)\|_1 + \|\frac{\partial(U - W)}{\partial t}(0)\| \leq C_1 h^k$. It can be avoided by using interpolation as in Subsection 3.2.2.
3. Dupont's error estimates for the fully discrete case is considered in Section 7.4.

3.4 Inertia norm estimate for a weak solution

3.4.1 The error estimate

In this section we reconsider the convergence of the solution of Problem B76W^h to the solution of Problem B76W (see Subsection 3.2.2). If f, u_0 and u_1 satisfy the conditions for Theorem 3.1.1, then a weak solution exists and Assumption A1 is a fact. In this section we do not assume that Assumption A4 holds (as in Subsection 3.2.2) but we use the weaker Assumption A2.

We consider the application of the inertia norm estimate in Section 2.6. For convenience we state Theorem 2.6.1 again.

Theorem 3.4.1 *Suppose Assumption A3 holds. If the solution u of Problem G satisfies Assumptions A1 and A2 and $c(u_1 - u_1^h, v) = 0 \forall v \in S^h$, then*

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_b \hat{C} h^\alpha \|u(t)\|_{H(V,k)} + 2\sqrt{T} C_b \hat{C} h^\alpha \max \|u'(t)\|_{H(V,k)} \\ &\quad + \sqrt{2} C_b \hat{C} h^\alpha \|u_0\|_{H(V,k)} + \sqrt{2} \|u_0 - u_0^h\|_W, \end{aligned} \quad (3.4.1)$$

for each $t \in [0, T]$.

If we assume that $u(t)$ and $u'(t)$ are in $H^2(\Omega) \cap V(\Omega)$ and proceed as in Subsection 3.2.2, then the estimate (3.4.1) takes the form

$$\begin{aligned} \|u(t) - u_h(t)\| &\leq C_b \hat{C} h |u(t)|_2 + 2\sqrt{T} C_b \hat{C} h \max |u'(t)|_2 \\ &\quad + \sqrt{2} C_b \hat{C} h |u_0|_2 + \sqrt{2} \|u_0 - u_0^h\|. \end{aligned} \quad (3.4.2)$$

(As usual $\|\cdot\|$ denotes the norm of $\mathcal{L}^2(\Omega)$.)

To make the result useful, we need to estimate the last term in (3.4.2). If we choose $u_0^h = \Pi u_0$, then $\|u_0 - u_0^h\|_W \leq C_T h^2 |u_0|_2$, see Subsection 3.2.2.

3.4.2 Baker's error estimates

For convenience we display the corresponding notation and Theorem 3.1 from the article.

Our notation	Baker notation
S^h	S_h^r
u^h	U_h
Pu	w_h
$e_p = u - Pu$	$\eta = u - w_h$
$e = Pu - u_h$	$\psi = w_h - U_h$
$e_h = u - u_h$	$\ell = u - U_h$

Theorem 3.1 [B76]

Let u be the solution of B76W; then for each $h \in (0, 1]$, there exists a unique mapping

$$U_h \in \mathcal{L}^2(0, T; S_h^r(\Omega))$$

which satisfies

$$\left(\frac{\partial^2 U_h}{\partial t^2}(\cdot, t), v \right) + a(U_h(\cdot, t), v) = (f(\cdot, t), v)$$

for all $v \in S_h^r$, $t > 0$,

$$\begin{aligned} (U_h(\cdot, 0), v) &= (u_0, v) \quad \text{for all } v \in S_h^r(\Omega), \\ \left(\frac{\partial U_h}{\partial t}(\cdot, 0), v \right) &= (q_0, v) \quad \text{for all } v \in S_h^r(\Omega). \end{aligned}$$

Furthermore, if $u \in L^\infty(0, T; H^r(\Omega))$ and $\frac{\partial u}{\partial t} \in \mathcal{L}^2(0, T; H^r(\Omega))$, then there exists a constant $C = C(T)$ such that

$$\|u - U\|_{L^\infty} \leq Ch^r \left[\|u\|_{L^\infty} + \left\| \frac{\partial u}{\partial t} \right\|_{\mathcal{L}^2} \right].$$

To prove the result in the theorem the following assumption is made in (2.6) on p.566

$$\inf_{\chi \in S^h} \{ \|v - \chi\| + h \|v - \chi\|_1 \} \leq Ch^s \|v\|_s.$$

Implicit in the assumption is a combination of Assumptions A3 and A7. We discuss this in the next subsection.

Remarks

1. The notation $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$ used in the article may be misleading. Actually these derivatives are defined in Subsection 2.2.1.
2. The fully discrete case is considered in Chapter 7.

3.4.3 Higher order estimates

The value of Baker's theory lies in the fact that convergence is proved for a weak solution. It is therefore questionable whether one should try to obtain higher order estimates for the error. The Aubin-Nietsche trick requires spatial regularity which is unlikely. The best to hope for is that for piecewise linear basis functions on a triangle, if $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\| \leq \hat{C}_T h^2 |u|_2. \quad (3.4.3)$$

Even this is sometimes not possible as mentioned in Subsection 3.2.2 under the heading "Application of Theorems 2.7.1 and 2.7.2".

Another situation is questionable. Is it possible to have $u(t) \in H^r$ and $u'(t) \in H^r$ for $r \geq 2$ but $u \notin C^2([0, T], V)$?

Chapter 4

Vibration models for a Timoshenko beam

4.1 Equations of motion and constitutive equations

The Timoshenko model for a beam consists of two partial differential equations for the beam deflection w and the angle due to the rotation of a cross section, ϕ , as mentioned in Section 1.4. The partial differential equations are derived from the equations of motion and the constitutive equations.

Finite element approximations for the vibration of a Timoshenko beam are considered in [S94] and [FXX99]. The model problem in these articles contains a strange damping term. For realistic model problems we consider the equations of motion and constitutive equations.

4.1.1 Undamped vibrations

In this subsection we consider the case where there is no damping.

Equations of motion

$$\rho A \partial_t^2 w = \partial_x V + q, \quad (4.1.1)$$

$$\rho I \partial_t^2 \phi = V + \partial_x M. \quad (4.1.2)$$

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In these equations ρ denotes the density, A the area of the cross section, I the area moment of inertia, M the moment and V the shear force.

Constitutive equations

$$M = EI\partial_x\phi, \quad (4.1.3)$$

$$V = AG\kappa^2(\partial_x w - \phi), \quad (4.1.4)$$

where E and G are elastic constants and κ^2 the shear correction factor. For more detail, see e.g. [T, pp.337-338], [I, pp.337-338] and [N, pp.394-395].

Parameters The beam may be tapered and the cross-sectional area A and area moment I vary along the length of the beam.

The Euler-Bernoulli and Rayleigh models

Combining the equations of motion yields

$$\rho A \partial_t^2 w = \partial_x(\rho I \partial_t^2 \phi) - \partial_x^2 M.$$

If it is assumed that $\partial_x w = \phi$, then

$$\rho A \partial_t^2 w = \partial_x(\rho I \partial_t^2 \partial_x w) - \partial_x^2 M.$$

Using $M = EI\partial_x^2 w$, yields the model without shear due to Rayleigh (as in Section 1.4):

$$\rho A \partial_t^2 w = \partial_x(\rho I \partial_t^2 \partial_x w) - \partial_x^2(EI\partial_x^2 w).$$

The constitutive equation for the shear force V is thus redundant.

The term $\rho I \partial_t^2 \partial_x w$ in the equation above is referred to as the rotary inertia term. The Euler-Bernoulli model is a special case of the Rayleigh model where rotary inertia is ignored.

Dimensionless form

The length of the beam is denoted by ℓ . Set $\tau = \frac{t}{T}$, $\xi = \frac{x}{\ell}$,

$w^*(\xi, \tau) = \frac{w(x, t)}{\ell}$ and $\phi^*(\xi, \tau) = \phi(x, t)$, where T is to be specified. We introduce the dimensionless constants

$$\alpha = \frac{A\ell^2}{I}, \text{ and } \beta = \frac{AG\kappa^2\ell^2}{EI}.$$

For numerical computation it should be noted that α and β may be large, typically between 1000 and 10000.

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The dimensionless forms of the shear force and moment are

$$V^*(\xi, \tau) = \frac{V(x, t)}{AG\kappa^2}, \quad M^*(\xi, \tau) = \frac{M(x, t)}{AG\kappa^2\ell} \quad \text{and} \quad q^*(\xi, \tau) = \frac{q(x, t)\ell}{AG\kappa^2}.$$

A convenient choice for T is

$$T = \ell \sqrt{\frac{\rho}{G\kappa^2}}.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

$$\partial_t^2 w = \partial_x V + q, \tag{4.1.5}$$

$$\frac{1}{\alpha} \partial_t^2 \phi = V + \partial_x M, \tag{4.1.6}$$

$$M = \frac{1}{\beta} \partial_x \phi, \tag{4.1.7}$$

$$V = \partial_x w - \phi. \tag{4.1.8}$$

Homogenous boundary conditions

Various homogenous boundary conditions may be considered. The beam in [S94] and [FXX99] is clamped at both ends. The boundary conditions are

$$w(0, t) = \phi(0, t) = w(1, t) = \phi(1, t) = 0. \tag{4.1.9}$$

There are a number of other possibilities, for example clamped-free, then

$$w(0, t) = \phi(0, t) = 0 \quad \text{and} \\ M(1, t) = V(1, t) = 0.$$

The variational forms are virtually the same, see Section 4.3.

Problem T1

The mathematical model consists of the equations of motion (4.1.5) and (4.1.6), the constitutive equations (4.1.7) and (4.1.8) and the boundary conditions (4.1.9). The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_1$ and $\partial_t \phi(\cdot, 0) = \phi_1$.

4.1.2 Damping

For **viscous damping** Equation (4.1.1) changes to $\rho A \partial_t^2 w = \partial_x V - k \partial_t w$. The dimensionless form is

$$\partial_t^2 w = \partial_x V - \mu \partial_t w, \quad (4.1.10)$$

where $\mu = \frac{k\ell}{A\sqrt{\rho G \kappa^2}}$.

Problem T2

The mathematical model consists of the equations of motion (4.1.10) and (4.1.6), the constitutive equations (4.1.7) and (4.1.8) and the boundary conditions (4.1.9). The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_1$ and $\partial_t \phi(\cdot, 0) = \phi_1$.

The damping in [S94] and [FXX99] is unrealistic. To obtain the model, the second equation of motion (4.1.2) should change to

$$\rho I \partial_t^2 \phi = V + \partial_x M - k_1 \partial_t \phi. \quad (4.1.11)$$

The dimensionless form is

$$\frac{1}{\alpha} \partial_t^2 \phi = V + \partial_x M - \mu_1 \partial_t \phi, \quad (4.1.12)$$

where $\mu_1 = \frac{k_1}{\rho A \ell^2}$. This is physically unrealistic but we proceed in order to study the article.

Problem S94

The mathematical model consists of the equations of motion (4.1.10) and (4.1.12), the constitutive equations (4.1.7) and (4.1.8) and the boundary conditions (4.1.9). The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_1$ and $\partial_t \phi(\cdot, 0) = \phi_1$.

Remark

From a mathematical point of view Problems T1 and T2 are special cases of Problem S94: Let $\mu_1 = 0$ for Problem T2 and $\mu = \mu_1 = 0$ for Problem T1.

Constitutive equations for Kelvin-Voigt damping:

$$M = EI \partial_x \phi + \nu_1 I \partial_t \partial_x \phi, \quad (4.1.13)$$

$$V = AG \kappa^2 (\partial_x w - \phi) + \nu_2 A \partial_t (\partial_x w - \phi), \quad (4.1.14)$$

where ν_1 and ν_2 are damping parameters. To have modal damping in this model is also unrealistic. We will not consider this model.

4.2 Boundary damping or control

Now consider a cantilever beam with boundary damping. The length of the beam is denoted by ℓ . The beam is clamped at one end, thus

$$w(0, t) = \phi(0, t) = 0. \quad (4.2.1)$$

The boundary damping (or control) is modelled by (see e.g. [KR87])

$$V(\ell, t) = -k_1 \partial_t w(\ell, t), \quad (4.2.2)$$

$$M(\ell, t) = -k_2 \partial_t \phi(\ell, t). \quad (4.2.3)$$

Dimensionless form

The boundary conditions in dimensionless form are

$$w(0, t) = \phi(0, t) = 0, \quad (4.2.4)$$

$$V(1, t) = -\gamma_1 \partial_t w(1, t) \text{ and} \quad (4.2.5)$$

$$M(1, t) = -\gamma_2 \partial_t \phi(1, t), \quad (4.2.6)$$

where $\gamma_1 = \frac{k_1 \ell}{TAG\kappa^2}$ and $\gamma_2 = \frac{k_2}{AG\kappa^2 \ell T}$.

Problem TBD

The mathematical model consists of the equations of motion (4.1.5) and (4.1.6), the constitutive equations (4.1.7) and (4.1.8) and the boundary conditions (4.2.4), (4.2.5) and (4.2.6). The initial conditions are $w(\cdot, 0) = w_0$, $\phi(\cdot, 0) = \phi_0$, $\partial_t w(\cdot, 0) = w_d$ and $\partial_t \phi(\cdot, 0) = \phi_d$.

4.3 Variational form

4.3.1 Viscous type damping

Consider Problem S94. Multiplying (4.1.10) and (4.1.12) with arbitrary functions v and $\psi \in C^1[0, 1]$ respectively, integrating and using integration by

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parts yields

$$\int_0^1 \partial_t^2 w(\cdot, t) v + \mu \int_0^1 \partial_t w(\cdot, t) v \quad (4.3.1)$$

$$= - \int_0^1 V(\cdot, t) v' + [V(x, t) v(x)]_0^1 + \int_0^1 q v$$

$$\int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + \mu_1 \int_0^1 \partial_t \phi(\cdot, t) \psi \quad (4.3.2)$$

$$= \int_0^1 V(\cdot, t) \psi - \int_0^1 M(\cdot, t) \psi' + [M(x, t) \psi(x)]_0^1.$$

Test functions

A function v is a test function if $v \in C^1[0, 1]$ and $v(0) = v(1) = 0$. The space of these test functions is denoted by $T[0, 1]$.

Using the constitutive equations and the boundary conditions we have that for $v \in T[0, 1]$ and $\psi \in T[0, 1]$,

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi + \mu \int_0^1 \partial_t w(\cdot, t) v + \mu_1 \int_0^1 \partial_t \phi(\cdot, t) \psi \\ &= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) + \int_0^1 q v. \end{aligned} \quad (4.3.3)$$

In realistic models the damping is nonmodal. (For modal damping it is required that $\frac{\mu_1}{\mu} = \frac{1}{\alpha}$.)

Problem S94V

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T[0, 1]$, $\phi(\cdot, t) \in T[0, 1]$ and (4.3.3) holds for each pair $\langle v, \psi \rangle \in T[0, 1] \times T[0, 1]$.

Remark

Other boundary conditions may be accommodated by changing the set of test functions.

4.3.2 Boundary damping

Consider equations (4.3.1) and (4.3.2) with $\mu = \mu_1 = 0$.

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Test functions

A function v is a test function if $v \in C^1[0, 1]$ and $v(0) = 0$. The space of test functions is denoted by $T_B[0, 1]$.

Using the constitutive equations and the boundary conditions we have that for v and $\psi \in T_B[0, 1]$:

$$\begin{aligned} & \int_0^1 \partial_t^2 w(\cdot, t) v + \int_0^1 \frac{1}{\alpha} \partial_t^2 \phi(\cdot, t) \psi \\ &= - \int_0^1 \frac{1}{\beta} \partial_x \phi(\cdot, t) \psi' - \int_0^1 (\partial_x w(\cdot, t) - \phi(\cdot, t)) (v' - \psi) \\ & \quad - \gamma_1 \partial_t w(1, t) v(1) - \gamma_2 \partial_t \phi(1, t) \psi(1) + \int_0^1 q v. \end{aligned} \quad (4.3.4)$$

Problem TBDV

Find a pair of functions $\langle w, \phi \rangle$ such that $w(\cdot, t) \in T_B[0, 1]$, $\phi(\cdot, t) \in T_B[0, 1]$ and (4.3.4) holds for each pair $\langle v, \psi \rangle \in T_B[0, 1] \times T_B[0, 1]$.

4.4 Weak variational form

4.4.1 Viscous type damping

We start off as usual by considering the closure of the spaces of test functions. Let $V(0, 1)$ be the closure of $T[0, 1]$ in the Sobolev space $H^1(0, 1)$, then $V(0, 1)$ is a Hilbert space.

The following product spaces are necessary for the abstract problem.

Product spaces

$X = \mathcal{L}^2(0, 1) \times \mathcal{L}^2(0, 1)$. An element $y \in X$ is written as $y = \langle y_1, y_2 \rangle$.

$H^m = H^m(0, 1) \times H^m(0, 1)$.

$V = V(0, 1) \times V(0, 1)$.

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We define the following bilinear forms:

$$\begin{aligned} \text{For } u \text{ and } v \text{ in } H^1 \quad b(u, v) &= \int_0^1 \frac{1}{\beta} u'_2 v'_2 + \int_0^1 (u'_1 - u_2)(v'_1 - v_2), \\ \text{for } u \text{ and } v \text{ in } X \quad c(u, v) &= \int_0^1 u_1 v_1 + \int_0^1 \frac{1}{\alpha} u_2 v_2 \quad \text{and} \\ a(u, v) &= \mu \int_0^1 u_1 v_1 + \mu_1 \int_0^1 u_2 v_2. \end{aligned}$$

Remark Note that the bilinear forms are symmetric.

The inner product of $\mathcal{L}^2(0, 1)$ is denoted by $(\cdot, \cdot)$ and the norm by $\|\cdot\|$. A natural inner product for X is

$$(x, y)_X = (x_1, y_1) + (x_2, y_2).$$

We denote the corresponding norm by $\|\cdot\|_X$.

Proposition 4.4.1 *Let u and v be any functions in $V(0, 1)$. Then there exist positive constants K_i such that*

$$\begin{aligned} K_1 \|u'\|^2 &\leq \int_0^1 \frac{1}{\beta} (u')^2 \leq K_2 \|u'\|^2, \\ K_3 \|u\|^2 &\leq \int_0^1 \frac{1}{\alpha} u^2 \leq K_4 \|u\|^2. \end{aligned}$$

Proof This is trivial since α and β are constants or functions that are bounded above and below by positive constants. $\square$

Proposition 4.4.2 *The bilinear form c is an inner product for the space X .*

Proof The bilinear form c is clearly symmetric and $c(u, u) \geq \|u_1\|^2 + K_3 \|u_2\|^2$ by Proposition 4.4.1. $\square$

For the weak formulation we need the space W .

Definition Inertia space

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W .

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Notation

Let $f(t) = \langle q(\cdot, t), 0 \rangle$.

Problem S94W

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d \rangle$.

Before we discuss existence and convergence we need the following results.

Proposition 4.4.3 *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof It follows directly from Proposition 4.4.1 □

Proposition 4.4.4 *V is a dense subset of W .*

Proof $V(0, 1)$ is dense in $\mathcal{L}^2(0, 1)$, since $C_0^\infty(0, 1)$ is dense in $\mathcal{L}^2(0, 1)$. □

Proposition 4.4.5 *There exists a constant C_b such that*

$$b(u, u) \geq C_b \|u\|_W^2 \quad \text{for each } u \in V.$$

Proof By Proposition 4.4.1

$$c(u, u) \leq \|u_1\|^2 + K_4 \|u_2\|^2,$$

and from Appendix A4 $\|u_1\| \leq \|u'_1\|$ and $\|u_2\| \leq \|u'_2\|$ (since u_1 and u_2 are in $V(0, 1)$). Combining the inequalities we find that

$$\begin{aligned} c(u, u) \leq \|u'_1\|^2 + K_4 \|u'_2\|^2 &\leq \|u'_1 - u_2\|^2 + \|u_2\|^2 + K_4 \|u'_2\|^2 \\ &\leq \|u'_1 - u_2\|^2 + (K_4 + 1) \|u'_2\|^2. \end{aligned} \quad (4.4.1)$$

From Proposition 4.4.1 we have

$$b(u, u) \geq \|u'_1 - u_2\|^2 + K_1 \|u'_2\|^2. \quad (4.4.2)$$

The result follows from (4.4.1) and (4.4.2). □

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Corollary 4.4.1 *The bilinear form b is an inner product for V .*

Proposition 4.4.6

$$a(u, v) \leq k_a \|u\|_W \|v\|_W$$

Proof

$$\begin{aligned} |a(u, v)| &\leq |\mu(u_1, v_1)| + |\mu_1(u_2, v_2)| \leq \mu \|u_1\| \|v_1\| + \mu_1 \|u_2\| \|v_2\| \\ &\leq \max\{\mu, \mu_1\} \|u\|_X \|v\|_X. \end{aligned}$$

The result now follows from Proposition 4.4.3. $\square$

Definition The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_E$ is defined by $\|u\|_E = \sqrt{b(u, u)}$.

Note that Assumptions E1, E2, E3 and E4 have been established.

Theorem 4.4.1 *Let J be an interval containing zero and suppose $f \in C^1(J, V)$. Then there exists a unique solution*

$$u \in C^1(J; V) \cap C^2(J; W),$$

for Problem S94W for each $u_0 \in E_b$ and $u_1 \in V$.

Proof Assumptions E1, E2, E3 and E4 hold. We have that a is bounded with respect to the norm of W from Proposition 4.4.6. The result follows from Theorem 2.2.2. $\square$

4.4.2 Boundary damping

We start off by considering the closure of the spaces of test functions. Let $V_B(0, 1)$ be the closure of $T_B[0, 1]$ in the Sobolev space $H^1(0, 1)$, then $V_B(0, 1)$ is a Hilbert space. The product spaces X , W and H^m are defined in the previous subsection. The space V is now defined as $V_B(0, 1) \times V_B(0, 1)$.

The big difference between Problems S94 and TBD concerns the damping:

$$a(u, v) = \gamma_1 \Gamma_\ell u_1 \Gamma_\ell v_1 + \gamma_2 \Gamma_\ell u_2 \Gamma_\ell v_2.$$

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Note The trace operator Γ_ℓ is defined in Appendix A5.

Problem TBW

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) + a(u'(t), v) = (f(t), v)_X \text{ for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \phi_0 \rangle$ and $u'(0) = u_d = \langle w_d, \phi_d \rangle$.

Proposition 4.4.7 *There exists a constant K_a such that*

$$|a(u, v)| \leq K_a \|u\|_E \|v\|_E.$$

Proof

$$\begin{aligned} |a(u, v)| &\leq \gamma_1 |\Gamma u_1| |\Gamma v_1| + \gamma_2 |\Gamma u_2| |\Gamma v_2| \\ &\leq \gamma_1 \|u_1\|_1 \|v_1\|_1 + \gamma_2 \|u_2\|_1 \|v_2\|_1, \end{aligned} \quad (4.4.3)$$

using Theorem A.5.2 in the Appendix. As in the proof of Proposition 4.4.5, we have

$$\|u_1\|^2 \leq \|u'_1\|^2 \leq \|u'_1 - u_2\|^2 + \|u_2\|^2 \leq Kb(u, u),$$

and

$$\|u_2\|^2 \leq \|u'_2\|^2 \leq Kb(u, u).$$

It follows that

$$\|u_1\|_1 \leq \|u_1\| + \|u'_1\| \leq 2K\|u\|_V \quad \text{and} \quad \|u_2\|_1 \leq 2K\|u\|_V.$$

Combining these inequalities with Equation (4.4.3) yields the result. $\square$

The product spaces satisfy Assumptions E1, E2 and E4 and Assumption E3 holds from Proposition 4.4.7.

Theorem 4.4.2 *Suppose $f \in C^1(J, X)$. If $u_0 \in V$, $u_d \in V$ and there exists a $y \in W$ such that*

$$c(y, v) + b(u_0, v) + a(u_d, v) = (f(t), v)_X \text{ for each } v \in V,$$

then Problem TBW has a unique solution.

Proof It follows from Theorem 2.2.1. $\square$

4.5 Convergence for viscous type damping

Let S^h be a finite dimensional subspace of V .

Problem S94W^h

Find $u_h \in C(J, S^h)$ with u'_h continuous at 0 such that for each $t > 0$,

$$c(u''_h(t), v) + a(u'_h(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h,$$

while $u(0) = u_0^h = \langle w_0^h, \phi_0^h \rangle$ and $u'(0) = u_d^h = \langle w_d^h, \phi_d^h \rangle$.

Recall the notation e^h , e_p and e used in Sections 2.3 and 2.4. Lemma 2.4.1 is still valid. It is presented here for convenience as Lemma 4.5.1. A slight modification of the proof is required for this case. Note that Assumption A1 is implied by Theorem 4.4.1.

Lemma 4.5.1 *If the solution u of Problem G satisfies Assumption A2, then for any $t \in J$,*

$$\|e(t)\|_E + \|e'(t)\|_W \leq \|e(0)\|_E + \|e'(0)\|_W + \int_0^t (\|e''_p(\cdot)\|_W + k\|e'_p(\cdot)\|_W) \, d\tau.$$

Proof From Lemma 2.3.3 we have

$$E'(t) \leq -c(e''_p(t), e'(t)) - a(e'_p(t), e'(t)).$$

But $|a(u, v)| \leq K_a \|u\|_W \|v\|_W$, from Proposition 4.4.6. Therefore

$$E'(t) \leq (\|e''_p(t)\|_W + K\|e'_p(t)\|_W) \|e'(t)\|_W,$$

as in Equation (2.4.3) in the proof of Lemma 2.4.1. $\square$

It follows that all the results for modal damping in Section 2.4, 2.5 and 2.7 are valid for the Timoshenko model with viscous type damping.

The possibilities in applying the theory are the same as for the one-dimensional wave equation and will not be repeated here. However interpolation merits consideration. Note that $H(V, k) = H^k \cap V$ for the Timoshenko beam.

Interpolation on the product space V

Suppose we use piecewise linear basis functions.

Πu for $u \in H^k$ is defined by $\Pi u = \langle \Pi_L u_1, \Pi_L u_2 \rangle$ where Π_L is the interpolation operator for piecewise linear basis functions (see Appendix B.1).

Proposition 4.5.1 *There exists a constant K_b such that for each $u \in H^2 \cap V$*

$$\|\Pi u - u\|_E \leq K_b \widehat{C}_L h \|u\|_{H^2}.$$

Proof

$$\|\Pi u - u\|_{H^1}^2 = \|\Pi_L u_1 - u_1\|_1^2 + \|\Pi_L u_2 - u_2\|_1^2.$$

From Appendix B we have that

$$\|\Pi_L f - f\|_1 \leq \widehat{C}_L h |f|_2 \text{ for } f \in H^2(0, 1).$$

Therefore

$$\|\Pi u - u\|_{H^1}^2 \leq \widehat{C}_L^2 h^2 (|u_1|_2^2 + |u_2|_2^2) \leq \widehat{C}_L^2 h^2 \|u\|_{H^2}^2.$$

The result follows from Equation (4.4.2). $\square$

Lemma 4.5.2 *For $u \in H^2 \cap V$*

$$\|Pu - u\|_E \leq K_b \widehat{C}_L h \|u\|_{H^2}.$$

Proof $\|Pu - u\|_E \leq \|\Pi u - u\|_E \leq K_b \widehat{C}_L h \|u\|_{H^2}.$ $\square$

Making suitable assumptions, we may now apply the theory in Sections 2.5 and 2.7.

Remarks

1. Results on the regularity of weak solutions for the Timoshenko beam equations are not common knowledge and further study of the literature is required.
2. Due to the phenomenon of “locking”, the rate of convergence in practice is not as predicted by the theory. In [S94] the so called “mixed finite element method” is used as a remedy and references are made to other methods. However, for the Timoshenko beam locking can be avoided by using cubics (see [Br, p.302]).
3. The fully discrete case is considered in Chapter 7.

4.6 Convergence for boundary damping

We consider Problem TBW. Boundary damping is not modal hence the results in Section 2.4 to 2.7 do not apply. We adapt the method in [D73] to deal with the damping term.

4.6.1 Fundamental estimate

The spaces V , W and X are defined in Subsection 4.4.2 and it is proved that Assumptions E1, E2, E3 and E4 hold. Problem TBW has a unique solution satisfying Assumption A1 (Theorem 4.4.2).

Problem TBW^h (The Galerkin approximation)

Find $u_h(t) \in S^h$ such that for each $t > 0$,

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h,$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

Assuming that Assumption A2 holds, we may use all the results in Section 2.3 and from Lemma 2.3.3, we have

$$E'(t) \leq -c(e_p''(t), e'(t)) - a(e_p'(t), e'(t)), \quad (4.6.1)$$

where in this case

$$a(u, v) = \gamma_1 \Gamma_\ell u_1 \Gamma_\ell v_1 + \gamma_2 \Gamma_\ell u_2 \Gamma_\ell v_2.$$

Since $u \in C^1(J, V)$, $u'(t) \in C[0, 1]$ for each t and

$$(\Gamma_\ell u)' = \Gamma_\ell u' = \langle \partial_t u_1(1, t), \partial_t u_2(1, t) \rangle.$$

Therefore we have that

$$\begin{aligned} a(e_p'(t), e'(t)) &= \gamma_1 \Gamma_\ell e_{p,1}'(t) \Gamma_\ell e_1'(t) + \gamma_2 \Gamma_\ell e_{p,2}'(t) \Gamma_\ell e_2'(t) \\ &= \gamma_1 \partial_t e_{p,1}(1, t) \partial_t e_1(1, t) + \gamma_2 \partial_t e_{p,2}(1, t) \partial_t e_2(1, t). \end{aligned}$$

Equation (4.6.1) now becomes

$$E'(t) \leq -c(e_p''(t), e'(t)) - \gamma_1 \partial_t e_{p,1}(1, t) \partial_t e_1(1, t) - \gamma_2 \partial_t e_{p,2}(1, t) \partial_t e_2(1, t). \quad (4.6.2)$$

Note The derivatives $e'_p(t)$ and $e'(t)$ are defined in Subsection 2.2.1.

Recall that the challenge is to estimate $e = Pu - u^h$. The method used in the proof of the following lemma is similar to the method used in Lemma 3.3.1 and is due to [D73].

Lemma 4.6.1 *If the solution of Problem TBD satisfies Assumptions A1 and A2, then, for any $t \in [0, T]$*

$$\|e'(t)\|_W^2 + \|e(t)\|_E^2 \leq e^{3t} K_T, \quad \text{where}$$

$$K_T = \int_0^T \|e''_p\|_W^2 + 4K^2 \max \|e'_p(t)\|_E^2 + 4K^2 \int_0^T \|e''_p(\cdot)\|_E^2 \\ + 2E(0) - 2\gamma_1 \partial_t e_{p,1}(1, 0) e_1(1, 0) - 2\gamma_2 \partial_t e_{p,2}(1, 0) e_2(1, 0).$$

Proof We have the following estimates:

$$|c(e''_p(t), e'(t))| \leq \|e''_p(t)\|_W \|e'(t)\|_W \leq \frac{1}{2} \|e''_p(t)\|_W^2 + \frac{1}{2} \|e'(t)\|_W^2 \quad \text{and} \\ \|e'(t)\|_W^2 \leq 2E(t) \text{ from Equation (2.3.3).}$$

Using the estimates above in (4.6.2) and integrating on both sides yields

$$E(t) \leq E(0) + \frac{1}{2} \int_0^t \|e''_p\|_W^2 + \int_0^t E - \gamma_1 \int_0^t \partial_t e_{p,1}(1, \cdot) \partial_t e_1(1, \cdot) \\ - \gamma_2 \int_0^t \partial_t e_{p,2}(1, \cdot) \partial_t e_2(1, \cdot) \quad (4.6.3)$$

Applying integration by parts to the last two terms we get

$$\int_0^t \partial_t e_{p,1}(1, \cdot) \partial_t e_1(1, \cdot) = \partial_t e_{p,1}(1, t) e_1(1, t) - \partial_t e_{p,1}(1, 0) e_1(1, 0) \\ - \int_0^t \partial_t^2 e_{p,1}(1, \cdot) e_1(1, \cdot) \quad \text{and} \\ \int_0^t \partial_t e_{p,2}(1, \cdot) \partial_t e_2(1, \cdot) = \partial_t e_{p,2}(1, t) e_2(1, t) - \partial_t e_{p,2}(1, 0) e_2(1, 0) \\ - \int_0^t \partial_t^2 e_{p,2}(1, \cdot) e_2(1, \cdot).$$

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These can be substituted into equation (4.6.3) above to obtain:

$$\begin{aligned} E(t) \leq & E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + \int_0^t E + \gamma_1 \int_0^t \partial_t^2 e_{p,1}(1, \cdot) e_1(1, \cdot) \\ & + \gamma_2 \int_0^t \partial_t^2 e_{p,2}(1, \cdot) e_2(1, \cdot) - \gamma_1 (\partial_t e_{p,1}(1, t) e_1(1, t) - \partial_t e_{p,1}(1, 0) e_1(1, 0)) \\ & - \gamma_2 (\partial_t e_{p,2}(1, t) e_2(1, t) - \partial_t e_{p,2}(1, 0) e_2(1, 0)). \end{aligned}$$

Now we want to determine estimates for the terms on the right hand side.

$$\begin{aligned} |\partial_t e_{p,1}(1, t) e_1(1, t)| & \leq \|e_{p,1}'(t)\|_1 \|e_1(t)\|_1 \\ & \leq \varepsilon^2 \|e_1(t)\|_1^2 + \varepsilon^{-2} \|e_{p,1}'(t)\|_1^2, \end{aligned}$$

with ε arbitrary, using Young's inequality (see Appendix E). A similar estimate holds for $|\partial_t e_{p,2}(1, t) e_2(1, t)|$ and we obtain

$$\begin{aligned} \gamma_1 |\partial_t e_{p,1}(1, t) e_1(1, t)| + \gamma_2 |e_{p,2}'(1, t) e_2(1, t)| & \leq K \varepsilon^2 \|e(t)\|_{H^1}^2 + K \varepsilon^{-2} \|e_p'(t)\|_{H^1}^2 \\ & \leq K \varepsilon^2 \|e(t)\|_E^2 + K \varepsilon^{-2} \|e_p'(t)\|_E^2 \\ & \leq K \varepsilon^2 E(t) + K \varepsilon^{-2} \|e_p'(t)\|_E^2, \end{aligned}$$

where K is a generic constant that depends on the bilinear forms a and b .

Thus, we also have that

$$\begin{aligned} \gamma_1 \int_0^t |\partial_t^2 e_{p,1}(1, \cdot) e_1(1, \cdot)| + \gamma_2 \int_0^t |\partial_t^2 e_{p,2}(1, \cdot) e_2(1, \cdot)| \\ \leq K \varepsilon^{-2} \int_0^t \|e_p''(\cdot)\|_E^2 + K \varepsilon^2 \int_0^t E. \end{aligned}$$

Therefore we have that

$$\begin{aligned} E(t) \leq & E(0) + \frac{1}{2} \int_0^t \|e_p''\|_W^2 + (1 + K \varepsilon^2) \int_0^t E + K \varepsilon^2 E(t) \\ & + K \varepsilon^{-2} \|e_p'(t)\|_E^2 + K \varepsilon^{-2} \int_0^t \|e_p''(\cdot)\|_E^2 \\ & - \gamma_1 \partial_t e_{p,1}(1, 0) e_1(1, 0) - \gamma_2 \partial_t e_{p,2}(1, 0) e_2(1, 0). \end{aligned}$$

Choosing ε such that $K\varepsilon^2 = \frac{1}{2}$ and rewriting the inequality we get

$$\begin{aligned} \frac{1}{2}E(t) &\leq \frac{3}{2} \int_0^t E + \frac{1}{2} \int_0^t \|e_p''\|_W^2 \\ &\quad + 2K^2 \|e_p'(t)\|_E^2 + 2K^2 \int_0^t \|e_p''(\cdot)\|_E^2 \\ &\quad + E(0) - \gamma_1 \partial_t e_{p,1}(1,0) e_1(1,0) - \gamma_2 \partial_t e_{p,2}(1,0) e_2(1,0). \end{aligned}$$

This can be written in the form needed for the Gronwall inequality:

$$\begin{aligned} E(t) &\leq 3 \int_0^t E + K_T, \quad \text{where} \\ K_T &= \int_0^T \|e_p''\|_W^2 + 4K^2 \max \|e_p''(t)\|_E^2 + 4K^2 \int_0^T \|e_p''(\cdot)\|_E^2 \\ &\quad + 2E(0) - 2\gamma_1 \partial_t e_{p,1}(1,0) e_1(1,0) - 2\gamma_2 \partial_t e_{p,2}(1,0) e_2(1,0). \end{aligned}$$

The result follows by applying the Gronwall inequality, see Appendix E. $\square$

4.6.2 Projection errors

Theorem 4.6.1 *Assume that the solution u of Problem TBD satisfies Assumption A4. Then, for any $t \in [0, T]$*

$$\|u(t) - u_h(t)\|_E^2 + \|u'(t) - u_h'(t)\|_W^2 \leq \|u(t) - Pu(t)\|_E^2 + \|u'(t) - Pu'(t)\|_W^2 + e^{3t} K_T,$$

where

$$\begin{aligned} K_T &= \int_0^T \|e_p''\|_W^2 + 4K^2 \max \|e_p''(t)\|_E^2 + 4K^2 \int_0^T \|e_p''(\cdot)\|_E^2 \\ &\quad + 2E(0) - 2\gamma_1 \partial_t e_{p,1}(1,0) e_1(1,0) - 2\gamma_2 \partial_t e_{p,2}(1,0) e_2(1,0). \end{aligned}$$

Proof

$$\|u(t) - u_h(t)\|_E^2 + \|u'(t) - u_h'(t)\|_W^2 \leq \|e_p(t)\|_E^2 + \|e(t)\|_E^2 + \|e_p'(t)\|_W^2 + \|e'(t)\|_W^2.$$

The result follows from Lemmas 4.6.1 and 2.3.1. $\square$

We use Assumption A5 in modified form.

Assumption A5T

For a given interpolation operator Π there exists a positive integer k such that for each $u \in H^k$

$$\|\Pi u - u\|_{H^1} \leq \widehat{C} h^{k-1} \|u\|_{H^k}.$$

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Theorem 4.6.2 *Suppose Assumptions E2 and A3 hold. Assume that the solution u of Problem TBD satisfies Assumption A4 and both $u'(t)$ and $u''(t)$ are in H^k . Then, for any $t \in [0, T]$*

$$\begin{aligned} & \|u(t) - u_h(t)\|_E + \|u'(t) - u'_h(t)\|_W \\ & \leq K_b \widehat{C} h^{(k-1)} [\|u\|_{H^k} + C_b \|u'\|_{H^k}] \\ & \quad + e^{\sqrt{3}t} K_b \widehat{C} h^{(k-1)} \left[C_b \int_0^T \|u''\|_{H^k} + 2K \max \|u'\|_{H^k} + \int_0^T 2K \|u''\|_{H^k} \right] \\ & \quad + e^{\sqrt{3}t} K_0, \end{aligned}$$

where

$$K_0 = \|Pu_d - u_d^h\|_W + \|Pu_0 - u_0^h\|_E + K_a \|u_d - Pu_d\|_E \|Pu_0 - u_0^h\|_E.$$

Proof From Theorem 4.6.1, and by using the following estimates

$$\begin{aligned} \|Pu - u\|_{H^1} & \leq \widehat{C} h^{k-1} \|u\|_{H^k} \quad (\text{from Assumption A5T}), \\ \|u\|_W & \leq C_b \|u\|_E \quad (\text{Assumption E2}) \text{ and} \\ |a(u, v)| & \leq K_a \|u\|_E \|v\|_E \quad (\text{Proposition 4.4.7}), \end{aligned}$$

we have that

$$\begin{aligned} & \|u(t) - u_h(t)\|_E^2 + \|u'(t) - u'_h(t)\|_W^2 \\ & \leq K_b^2 \widehat{C}^2 h^{2(k-1)} [\|u\|_{H^k}^2 + C_b^2 \|u'\|_{H^k}^2] \\ & \quad + e^{3t} K_b^2 \widehat{C}^2 h^{2(k-1)} \left[C_b^2 \int_0^t \|u''\|_{H^k}^2 + 4K^2 \max \|u'\|_{H^k}^2 + 2K \int_0^T \|u''\|_{H^k}^2 \right] \\ & \quad + e^{3t} K_0, \end{aligned}$$

where

$$K_0 = \|e'(0)\|_W^2 + \|e(0)\|_E^2 + K_a \|e'_p(0)\|_E \|e(0)\|_E.$$

Since $(a + b)^2 \leq 2a^2 + 2b^2$ for positive a and b , the result follows. $\square$

Interpolation on the product space $\mathbf{V}$

The definition of interpolation on the product space is given in Section 4.5. For piecewise linear basis functions we have Assumption A5T with $k = 2$:

$$\|\Pi u - u\|_{H^1} \leq \widehat{C} h \|u\|_{H^2}.$$

Other basis functions, for example Hermite cubics, can also be used.

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When we use $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$ we get

$$\begin{aligned}\|Pu_0 - u_0^h\|_E &\leq \|Pu_0 - u_0\|_E + \|u_0 - u_0^h\|_E \\ &= \|Pu_0 - u_0\|_E + \|u_0 - \Pi u_0\|_E \\ &\leq 2K_b \hat{C}h \|u_0\|_{H^2},\end{aligned}$$

and

$$\|Pu_d - u_d^h\|_E \leq 2K_b \hat{C}h \|u_d\|_{H^2}.$$

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Chapter 5

Vibration of a Reissner-Mindlin plate

5.1 Reissner-Mindlin plate theory

5.1.1 Equations of motion

We consider small transverse vibration of a thin plate with thickness h and density ρ . The reference configuration for the plate is a domain Ω in the plane. The transverse displacement of $\mathbf{x}$ at time t is denoted by $w(\mathbf{x}, t)$. The angle between a “material line” and a perpendicular to the plane is $\psi(\mathbf{x}, t)$ and the angle between the projection of the material line in the plane and the unit vector $\mathbf{e}_1$ is $\phi(\mathbf{x}, t)$ (see [Rei, Sec 3.2, Sec 3.5]). For a linear model $\boldsymbol{\psi}$ is approximated by

$$\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T = [\psi \cos \phi \ \psi \sin \phi]^T.$$

Then the equations of motion (see [M51] and [Rei, p.152]) are given by

$$\rho h \partial_t^2 w = \operatorname{div} \mathbf{Q} + q, \quad (5.1.1)$$

$$\rho I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} \mathbf{M} - \mathbf{Q}, \quad (5.1.2)$$

where $I = \frac{h^3}{12}$ is the length moment of inertia. $\mathbf{Q}$ represents a force density, $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a moment density and q an external load on the plate.

5.1.2 Constitutive equations

We restrict our attention to the linear theory. The following assumptions are made for small curvature and small partial derivatives (see [Rei, p.61] and [M51]).

$$\mathbf{Q} = \kappa^2 G h (\nabla w + \boldsymbol{\psi}), \quad (5.1.3)$$

where G is the shear modulus and κ^2 a correction factor.

$$M = \frac{1}{2} D \begin{bmatrix} 2(\partial_1 \psi_1 + \nu \partial_2 \psi_2) & (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) \\ (1 - \nu)(\partial_1 \psi_2 + \partial_2 \psi_1) & 2(\partial_2 \psi_2 + \nu \partial_1 \psi_1) \end{bmatrix}. \quad (5.1.4)$$

D is a measure of stiffness of the plate and is given by

$$D = \frac{EI}{1 - \nu^2},$$

where E is Young's modulus and ν Poisson's ratio.

The correction factor κ^2 is chosen in such a way that the solution of the plate model compares well with the solution of the three-dimensional model. The value of κ^2 depends on Poisson's ratio ν and ranges almost linearly from 0.76 to 0.91 if ν increases from 0 to 0.5 (see [M51]). Also mentioned in this reference is that Reissner used $\kappa^2 = \frac{5}{6}$.

The equations of motion and the constitutive equations above are known as the **Reissner-Mindlin** plate model. The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [Rei, p.152] and [M51]). In our approach these partial differential equations are not used.

Remark

For classical plate theory, $\boldsymbol{\psi}$ is replaced by $-\nabla w$ and the constitutive equation for $\mathbf{Q}$ is no longer necessary. This is sometimes referred to as the Kirchhoff plate model.

5.1.3 Dimensionless form

We introduce the dimensionless variables

$$\tau = \frac{t}{t_0}, \quad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell},$$

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where ℓ is a suitable length and t_0 must still be specified.

The dimensionless variables, with $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, are

$$\begin{aligned} w^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{\ell}\right) w(\mathbf{x}, t), \quad \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) = \boldsymbol{\psi}(\mathbf{x}, t), \\ \mathbf{Q}^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{\ell G \kappa^2}\right) \mathbf{Q}(\mathbf{x}, t), \quad M^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell^2 G \kappa^2}\right) M(\mathbf{x}, t) \\ \text{and } q^*(\boldsymbol{\xi}, \tau) &= \left(\frac{1}{G \kappa^2}\right) q(\mathbf{x}, t). \end{aligned}$$

The dimensionless constants that are used are given by

$$h^* = \frac{h}{\ell}, \quad I^* = \frac{(h^*)^3}{12} \quad \text{and} \quad \beta = \frac{\ell^3 G \kappa^2}{EI}.$$

We choose $t_0 = \ell \sqrt{\frac{\rho}{G \kappa^2}}$ and use the original notation for the corresponding dimensionless quantities. The equations of motion and constitutive equations in dimensionless form are presented below (using the original notation).

Reissner-Mindlin plate model

$$h \partial_t^2 w = \operatorname{div} \mathbf{Q} + q, \quad (5.1.5)$$

$$I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \mathbf{Q}, \quad (5.1.6)$$

$$\mathbf{Q} = h(\nabla w + \boldsymbol{\psi}), \quad (5.1.7)$$

$$M = \frac{1}{2\beta(1-\nu^2)} \begin{bmatrix} 2(\partial_1 \psi_1 + \nu \partial_2 \psi_2) & (1-\nu)(\partial_1 \psi_2 + \partial_2 \psi_1) \\ (1-\nu)(\partial_1 \psi_2 + \partial_2 \psi_1) & 2(\partial_2 \psi_2 + \nu \partial_1 \psi_1) \end{bmatrix}. \quad (5.1.8)$$

5.1.4 Model problem

Problem Wu05

First, in the domain Ω , we have Equations of motion (5.1.5) and (5.1.6) together with the constitutive equations (5.1.7) and (5.1.8). The boundary conditions on $\partial\Omega$ are conventional homogeneous boundary conditions, $w = \psi_1 = \psi_2 = 0$ where the plate is clamped. The initial conditions are

$$\begin{aligned} w(\cdot, 0) &= w_0, \quad \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_0, \\ \partial_t w(\cdot, 0) &= w_1 \quad \text{and} \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1. \end{aligned}$$

This model problem is considered in [Wu05] and [Wu06].

5.2 Variational form

Using integration by parts, we obtain the following Green formulas: For any scalar valued function v

$$\iint_{\Omega} (\operatorname{div} \mathbf{Q})v \, dA = - \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA + \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n})v \, ds. \quad (5.2.1)$$

For any vector valued function $\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T$,

$$\iint_{\Omega} \operatorname{div} M \cdot \boldsymbol{\phi} \, dA = - \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA + \int_{\partial\Omega} M\mathbf{n} \cdot \boldsymbol{\phi} \, ds. \quad (5.2.2)$$

where $\Phi = \begin{bmatrix} \partial_1\phi_1 & \partial_2\phi_1 \\ \partial_1\phi_2 & \partial_2\phi_2 \end{bmatrix}$ and “tr” denotes the trace of the matrix. Note that we used the symmetry of M .

Multiply equation (5.1.5) by an arbitrary scalar valued function v and integrate. Using (5.2.1), we find that

$$h \iint_{\Omega} \partial_t^2 w v \, dA + \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA - \int_{\partial\Omega} (\mathbf{Q} \cdot \mathbf{n})v \, ds = \iint_{\Omega} qv \, dA. \quad (5.2.3)$$

From equation (5.1.6) we have

$$I \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} = \operatorname{div} M \cdot \boldsymbol{\phi} - \mathbf{Q} \cdot \boldsymbol{\phi},$$

where $\boldsymbol{\phi}$ is an arbitrary vector valued function. Using (5.2.2) we find that

$$\begin{aligned} I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_{\Omega} \operatorname{tr}(M\Phi) \, dA - \int_{\partial\Omega} M\mathbf{n} \cdot \boldsymbol{\phi} \, ds \\ + \iint_{\Omega} \mathbf{Q} \cdot \boldsymbol{\phi} \, dA = 0. \end{aligned} \quad (5.2.4)$$

Test functions

Choose two spaces of test functions $T_1(\Omega)$ and $T_2(\Omega)$:

$$T_1(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega\},$$

$$T_2(\Omega) = \{\boldsymbol{\phi} = [\phi_1 \ \phi_2]^T \mid \phi_1, \phi_2 \in C^1(\bar{\Omega}), \ \boldsymbol{\phi} = 0 \text{ on } \partial\Omega\}.$$

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From 5.2.3

$$h \iint_{\Omega} \partial_t^2 w v \, dA + \iint_{\Omega} \mathbf{Q} \cdot \nabla v \, dA = \iint_{\Omega} q v \, dA, \quad (5.2.5)$$

for each $v \in T_1(\Omega)$.

From 5.2.4

$$I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + \iint_{\Omega} \text{tr}(M\Phi) \, dA + \iint_{\Omega} \mathbf{Q} \cdot \boldsymbol{\phi} \, dA = 0, \quad (5.2.6)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

We define a bilinear form b_B by

$$\begin{aligned} b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) &= \iint_{\Omega} \text{tr}(M\Phi) \, dA \\ &= \frac{1}{\beta(1-\nu^2)} \iint_{\Omega} ((\partial_1 \psi_1 + \nu \partial_2 \psi_2) \partial_1 \phi_1 + (\partial_2 \psi_2 + \nu \partial_1 \psi_1) \partial_2 \phi_2) \, dA \\ &\quad + \frac{1}{2\beta(1+\nu)} \iint_{\Omega} (\partial_1 \psi_2 + \partial_2 \psi_1) (\partial_1 \phi_2 + \partial_2 \phi_1) \, dA, \end{aligned}$$

for each $\boldsymbol{\psi}, \boldsymbol{\phi}$ in $H^1(\Omega)^2$.

Finally, the second variational equation is given by

$$I \iint_{\Omega} \partial_t^2 \boldsymbol{\psi} \cdot \boldsymbol{\phi} \, dA + b_B(\boldsymbol{\psi}, \boldsymbol{\phi}) + h \iint_{\Omega} (\nabla w + \boldsymbol{\psi}) \cdot \boldsymbol{\phi} \, dA = 0, \quad (5.2.7)$$

for each $\boldsymbol{\phi} \in T_2(\Omega)$.

Problem Wu05V

Find w and $\boldsymbol{\psi}$ such that, for $t > 0$, $w(\cdot, t) \in T_1(\Omega)$, $\boldsymbol{\psi}(\cdot, t) \in T_2(\Omega)$ and Equations (5.2.5) and (5.2.7) hold for each $v \in T_1(\Omega)$ and each $\boldsymbol{\phi} \in T_2(\Omega)$. The initial conditions are

$$w(\cdot, 0) = w_0, \quad \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_0,$$

$$\partial_t w(\cdot, 0) = w_1 \quad \text{and} \quad \partial_t \boldsymbol{\psi}(\cdot, 0) = \boldsymbol{\psi}_1.$$

The variational form above is used for computational purposes, but for theoretical purposes we consider the weak form of the variational problem in the next section.

5.3 Weak variational form

To see that this problem is of the same form as Problem G, add Equations (5.2.5) and (5.2.7).

$$\begin{aligned} & h \iint_{\Omega} \partial_t^2 w v \, dA + I \iint_{\Omega} \partial_t^2 \psi \cdot \phi \, dA \\ & = -b_B(\psi, \phi) - h \iint_{\Omega} (\nabla w + \psi)(\nabla v + \phi) \, dA + \iint_{\Omega} qv \, dA \end{aligned}$$

This is of the form $c(\partial_t^2 u, z) = -b(u, z) + (q, v)_X$. To make the formulation precise, we introduce product spaces.

Product spaces

$$\begin{aligned} X &= \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)^2 \\ H^k &= H^k(\Omega) \times H^k(\Omega)^2 \\ T &= T_1(\Omega) \times T_2(\Omega) \end{aligned}$$

We are now ready to consider the weak variational form. We define V as the closure of T in H^1 .

Bilinear forms For u and v in V , define

$$\begin{aligned} c(u, v) &= h(u_1, v_1)_{\Omega} + I(u_2, v_2)_{0,2}^{\Omega} \text{ and} \\ b(u, v) &= b_B(u_2, v_2) + h(\nabla u_1 + u_2, \nabla v_1 + v_2)_{0,2}^{\Omega}. \end{aligned}$$

The norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$. A natural inner product for X is

$$(x, y)_X = (x_1, y_1)_0 + (x_2, y_2)_{0,2}.$$

(See Appendix A3 on Sobolev spaces of vector valued functions.) We denote the corresponding norm by $\|\cdot\|_X$.

Proposition 5.3.1 *Let u and v be any elements of X . Then there exist positive constants K_i such that*

$$K_1 \|u\|_X \leq c(u, u) \leq K_2 \|u\|_X^2.$$

Proof This is trivial since $c(u, u) = h\|u_1\|_{\Omega}^2 + I(\|u_2\|_{0,2}^{\Omega})^2$. □

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Proposition 5.3.2 *The bilinear form c is an inner product for the space X .*

Proof The bilinear form c is clearly symmetric and $c(u, u) \geq K_1 \|u\|_X^2$ by Proposition 5.3.1. $\square$

Definition Inertia space:

The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u, u)}$. We refer to the vector space X equipped with the norm $\|\cdot\|_W$ as the space W .

Let $f(t) = \langle q(\cdot, t), \mathbf{0} \rangle$.

Problem Wu05W

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each } v \in V,$$

while $u(0) = u_0 = \langle w_0, \psi_0 \rangle$ and $u'(0) = u_1 = \langle w_1, \psi_1 \rangle$.

Before we discuss existence and convergence we need the following results.

Proposition 5.3.3 *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof It follows directly from Proposition 5.3.1. $\square$

Proposition 5.3.4 *V is a dense subset of W .*

Proof $T_1(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ and $T_2(\Omega)$ is dense in $\mathcal{L}^2(\Omega)^2$. Consequently T is dense in X . $\square$

Theorem 5.3.1 *There exist a constant K_b such that*

$$|b(u, v)| \leq K_b \|u\|_{H^1} \|v\|_{H^1},$$

for each $u \in V$.

Proof The terms in $b_B(u_2, v_2)$ are all of the form $\iint_{\Omega} \partial_i u_{2,i} \partial_j v_{2,j} dA$. Therefore

$$|b_B(u_2, v_2)| \leq K \|u_2\|_{1,2} \|v_2\|_{1,2}.$$

For the other term in b we have

$$|(\nabla u_1 + u_2, \nabla v_1 + v_2)_{0,2}| \leq \|\nabla u_1 + u_2\|_{0,2} \|\nabla v_1 + v_2\|_{0,2}.$$

But

$$\begin{aligned} \|\nabla u_1 + u_2\|_{0,2}^2 &\leq \|u_1\|_1^2 + \|u_2\|_{0,2}^2 \\ &\leq 2\|u_1\|_1^2 + 2\|u_2\|_{0,2}^2 \leq 2\|u\|_{H^1}^2. \end{aligned} \quad \square$$

Next we have Korn's lemma.

Theorem 5.3.2 *There exists a constant C_b such that*

$$b_B(u, u) \geq C_b \|u\|_{1,2}^2 \text{ for each } u \in V.$$

Proof See [Br, p.289].

Corollary 5.3.1 *The bilinear form b is an inner product for V .*

Definition Energy space

The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_E$ is defined by $\|u\|_E = \sqrt{b(u, u)}$.

Corollary 5.3.2 *The norms $\|\cdot\|_E$ and $\|\cdot\|_{H^1}$ are equivalent on V .*

We have shown that Assumptions E1, E2 and E4 hold. (Since the vibration is undamped, E3 is not relevant.)

Theorem 5.3.3 *Suppose $q \in C^1(J, X)$. Let J be an interval containing zero, then there exists a unique solution*

$$u \in C^1((-\infty, \infty); V) \cap C^2((-\infty, \infty); W),$$

for Problem Wu05W for each $u_0 \in E_b$ and $u_1 \in V$.

Proof It follows from Theorem 2.2.2. $\square$

5.4 Convergence

Let S^h be a finite dimensional subspace of V .

Problem Wu05W^h

Find $u_h \in C(J, S^h)$ such that u'_h is continuous at 0 and for each $t \in J$

$$c(u''_h(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h,$$

while $u(0) = u_0^h = \langle w_0^h, \psi_0^h \rangle$ and $u'(0) = u_1^h = \langle w_1^h, \psi_1^h \rangle$.

It follows that all the results for modal damping in Section 2.4, 2.5 and 2.7 are valid. Since no damping is present, the result in Section 2.6 may also be used. It remains to derive the interpolation error.

Interpolation

In [Wu05] the domain Ω is a rectangle. We must define an interpolation operator on $H^k = H^k(\Omega) \times H^k(\Omega)^2$. To illustrate how this is done, we use piecewise bilinear basis functions on rectangular elements. Πu for $u \in H^k$ is defined by: $\Pi u = \langle \Pi_b u_1, \Pi_B u_2 \rangle$ where Π_b and Π_B are defined in Appendix B.2 and B.3 respectively.

Proposition 5.4.1 *There exists a $\hat{C} > 0$ such that*

$$\|\Pi u - u\|_{H^1} \leq \hat{C}h \|u\|_{H^2} \quad \text{for each } u \in H^2.$$

Proof From the Appendix we have that

$$\|\Pi_b f - f\|_{1,\Omega} \leq \hat{C}h |f|_{2,\Omega} \quad \text{for } f \in H^2(\Omega) \quad \text{and}$$

$$\|\Pi_B f - f\|_{1,2} \leq \hat{C}h |f|_{2,2} \quad \text{for } f \in H^2(\Omega)^2.$$

Therefore, from the definition of Π , we have

$$\begin{aligned} \|\Pi u - u\|_{H^1}^2 &= \|\Pi_b u_1 - u_1\|_1^2 + \|\Pi_B u_2 - u_2\|_{1,2}^2 \\ &\leq \hat{C}^2 h^2 |u_1|_{2,\Omega}^2 + \hat{C}^2 h^2 |u_2|_{2,2}^2. \end{aligned}$$

The result follows. □

Lemma 5.4.1 *For $u \in H^2$, $\|Pu - u\|_E \leq \hat{C}h \|u\|_2$.*

Proof We use Theorem 5.3.3 and Proposition 5.4.1.

$$\|Pu - u\|_E \leq \sqrt{K_b} \|\Pi u - u\|_{H^1} \leq \sqrt{K_b} \hat{C} h \|u\|_{H^2}. \quad \square$$

Remarks

1. Theoretically convergence follows from Lemma 5.4.1 but in practice locking may be a problem as observed in the numerical experiments in [Wu05]. However, with piecewise bicubic basis functions satisfactory results are obtained (see [LVV09]).
2. The fully discrete case is considered in Chapter 7.

5.5 The approach taken by Wu

From the introduction to the article [Wu05], we conclude that this is the first result on convergence of the finite element method for the vibration of a Reissner-Mindlin plate. Wu formulate it as follows: “The static analysis is the foundation of the dynamic analysis. On the other hand, the a priori estimates for hyperbolic problems are developed in [D73] and [B76]. The method is extended to elasto-dynamic problems by explicit finite elements [Wu03].”

In Section 2 of [Wu05] the plate vibration problem (Problem Wu05) is formulated and the weak variational form is stated (but not derived). This is Equation (2.6). Estimates and the existence theorem with references to other articles are given.

The fully discrete problem is formulated in Section 3 and the explicit scheme explained.

In Section 4 the projection is defined and the decomposition of the errors introduced as in Section 2.3 of this dissertation. The notation in [Wu05] is the author’s own and differs from [D73] and [B76]. (The fact that all these papers use different notation makes comparison a time consuming task.) Since the semi-discrete problem is not considered in [Wu05], we postpone further discussion to Section 7.6.

5.6 A plate-beam system

In this section we present the plate-beam system in [LVV09].

Consider small transverse vibration of a thin rectangular plate supported by identical beams at two opposing sides and rigidly supported at the remaining sides. The beams are supported at their endpoints. Assume furthermore the case of free vibration, i.e. $q = 0$.

The reference configuration for the plate is the rectangle Ω , where $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq b$. The plate is rigidly supported at $x_1 = 0$ and $x_1 = a$. These sections of the boundary of Ω are denoted by Σ_0 and Σ_1 respectively. The plate is supported by beams at $x_2 = 0$ and $x_2 = b$ and these sections are denoted by Γ_0 and Γ_1 respectively.

Equations of motion

In the mathematical model the Reissner-Mindlin plate theory and the Timoshenko beam theory are used. On the rectangle Ω , the equations of motion (5.1.1) and (5.1.2) are satisfied and on Γ_0 and Γ_1 , the two sets of equations of motion are given by adapted versions of (4.1.1) and (4.1.2):

$$\rho_b A \partial_t^2 w = \partial_x V + P, \quad (5.6.1)$$

$$\rho_b I \partial_t^2 \phi = V + \partial_x M + L. \quad (5.6.2)$$

In the equations above L represents a moment density transmitted from the plate to the beam and P a force density transmitted from the plate to the beam.

Boundary conditions on Σ_0 and Σ_1

On these sections of the boundary, the conventional homogeneous boundary conditions for a rigidly supported plate are used, i.e.

$$w = 0, \quad \psi_2 = 0 \quad \text{and} \quad M \mathbf{n} \cdot \mathbf{n} = 0, \quad (5.6.3)$$

where $\mathbf{n}$ is the unit exterior normal (see [Rei, p.66]).

Interface conditions on Γ_0 and Γ_1

On Γ_0 and Γ_1 the interaction between the plate and the beams is considered.

$$w_b(x_1, t) = w(x_1, 0, t) \text{ on } \Gamma_0, \quad w_b(x_1, t) = w(x_1, a, t) \text{ on } \Gamma_1, \quad (5.6.4)$$

$$\phi_b(x_1, t) = -\psi_1(x_1, 0, t) \text{ on } \Gamma_0, \quad \phi_b(x_1, t) = -\psi_1(x_1, a, t) \text{ on } \Gamma_1. \quad (5.6.5)$$

The interface conditions for the force densities and moment densities on Γ_0 and Γ_1 are given by

$$\mathbf{Q} \cdot \mathbf{n} = -P, \quad (5.6.6)$$

$$M\mathbf{n} \cdot \boldsymbol{\tau} = L, \quad (5.6.7)$$

$$M\mathbf{n} \cdot \mathbf{n} = 0, \quad (5.6.8)$$

where $\boldsymbol{\tau}$ is the unit tangent oriented in such a way that Ω is on the left hand side of $\boldsymbol{\tau}$. For a detailed explanation of the moments $M\mathbf{n} \cdot \mathbf{n}$ and $M\mathbf{n} \cdot \boldsymbol{\tau}$, see [Rei, p.66].

Conditions at the endpoints of Γ_0 and Γ_1

At the endpoints of Γ_0 and Γ_1 we have the obvious boundary conditions for the beams, namely

$$w_b = 0 \quad \text{and} \quad M_b = 0. \quad (5.6.9)$$

The mathematical model

The vibration problem for the plate-beam system is given by the following equations.

Equations of motion for the plate: (5.1.1) and (5.1.2) on Ω .

Constitutive equations for the plate: (5.1.3) and (5.1.4) on Ω .

Equations of motion for the beams: (5.6.1) and (5.6.2) on Γ_0 and Γ_1 .

Constitutive equations for the beams: (4.1.3) and (4.1.4) on Γ_0 and Γ_1 .

Interface conditions: (5.6.4) to (5.6.8) on Γ_0 and Γ_1 .

Boundary conditions: (5.6.3) on Σ_0 and Σ_1 .

Endpoint conditions: (5.6.9) at the endpoints of Γ_0 and Γ_1 .

Chapter 6

Application of the Trotter-Kato theorem

In this chapter a completely different approach is considered and some preparation is necessary. In Section 6.1 we consider first order abstract differential equations and we show in Section 6.2 that Problem G is equivalent to such a differential equation.

6.1 Abstract linear differential equations and semigroups

In this section a summary of basic definitions and theorems from semi-group theory is presented for the convenience of the reader.

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Recall that the set of bounded linear operators (or linear transformations) on H is itself a linear space, denoted by $B(H)$.

Remark

A special case is the Euclidean space E_n . Recall that the linear operators on E_n are bounded.

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Theorem 6.1.1 *For any $A \in B(H)$ and any $b \in H$, $e^{tA}b$ is a differentiable function and it is the unique solution of*

$$x' = Ax \quad \text{with} \quad x(0) = b.$$

Proof Refer to [P, pp.2-3].

Abstract initial value problem

Suppose L is a linear operator with $\mathcal{D}(L) \subset H$ and $\mathcal{R}(L) \subset H$. Given a function f on some interval J and $\mathcal{R}(f) \subset H$, find w such that for each t , $w(t) \in \mathcal{D}(L)$ and

$$w' = Lw + f \quad \text{with} \quad w(0) = b.$$

Definition Semigroup

A family of bounded linear operators $S(t)$ defined for $t \geq 0$ is referred to as a semigroup if it has the following properties:

For positive t and s , $S(t+s) = S(t)S(s)$ and $S(0) = I$.

Remark

Note that S is a function that maps $[0, \infty)$ into $B(H)$. We will use the notation $S(\cdot)$.

Instead of defining a C_0 semigroup and its infinitesimal generator, we present the following theorem.

Theorem 6.1.2 *Let $S(\cdot)$ be a C_0 semigroup and let L be its infinitesimal generator. Then*

(a) *For any $b \in \mathcal{D}(L)$ and each $t > 0$,*

$$S(t)b \in \mathcal{D}(L) \quad \text{and} \quad LS(t)b = S(t)Lb.$$

(b) *For any $b \in \mathcal{D}(L)$,*

$$\frac{d}{dt} S(t)b = LS(t)b.$$

(c) *For any $b \in \mathcal{D}(L)$,*

$$S(t)b - S(s)b = \int_s^t S(\tau)Lb \, d\tau = \int_s^t LS(\tau)b \, d\tau.$$

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Corollary 6.1.1 *For any $b \in \mathcal{D}(L)$, $S(\cdot)b$ is a solution of the initial value problem $x' = Lx$ with $x(0) = b$.*

Proof [P, p.5]

Definition C_0 semigroup of contractions

If $S(\cdot)$ is a C_0 semigroup and

$$\|S(t)\| \leq 1 \quad \text{for } t \in [0, \infty),$$

then $S(\cdot)$ is called a C_0 semigroup of contractions.

Theorem 6.1.3 *If $S(\cdot)$ is a C_0 semigroup of contractions, and L its infinitesimal generator, then $(\lambda I - L)^{-1} \in B(H)$ for each $\lambda > 0$ and $\|(\lambda I - L)^{-1}\| \leq \lambda^{-1}$.*

Proof [P, p.12].

Dissipative operators

Definition Dissipative operator

A linear operator L is dissipative if for every $x \in \mathcal{D}(L)$, $\langle Lx, x \rangle \leq 0$.

Theorem 6.1.4 *A dissipative linear operator L is the infinitesimal generator of a C_0 semigroup of contractions if the following conditions are satisfied.*

- (a) $\mathcal{D}(L)$ is dense in H ,
- (b) For each $\lambda > 0$, $\mathcal{R}(\lambda I - L) = H$.

Proof [P, p.14].

6.2 The general second order hyperbolic problem as a first order system

In this section we write Problem G as an initial value problem for a first order differential equation in a Hilbert space. Recall that

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W is a Hilbert space with inner product c and norm $\|\cdot\|_W$;

V is a Hilbert space with inner product b and norm $\|\cdot\|_V$;

V is a subspace of W .

A summary of properties of these spaces is given in Appendix C.

Let $H = V \times W$ with inner product

$$(x, y)_H = b(x_1, y_1) + c(x_2, y_2),$$

for all x and $y \in H$. For $x \in H$ denote x and its components by $x = \langle x_1, x_2 \rangle$.

The idea is to define a linear operator with certain properties that will enable us to apply semigroup theory. For convenience we present results from [VV02]. An expanded version of the paper can be found in Chapter 4 of the dissertation by M de Villiers [D-MSc].

Proposition 6.2.1 *Suppose $\lambda \geq 0$. For each $y \in H$ there exists a unique $x \in H$ such that*

$$\begin{aligned} \lambda x_1 - x_2 &= y_1 \quad \text{and} \\ b(x_1, v) + a(x_2, v) + \lambda c(x_2, v) &= c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

Remarks

1. $x_2 \in V$ in Proposition 6.2.1.
2. The result remains valid in the undamped case (i.e. when $a = 0$).

Definition The mapping Λ

For $\lambda = 0$, the mapping Λ is defined on H by $\Lambda y = -x$ where $-x_2 = y_1$ and $x_1 \in V$ such that

$$b(x_1, v) + a(x_2, v) = c(y_2, v) \quad \text{for each } v \in V.$$

Remark

If x is in the range of Λ , then x_1 and x_2 are in V .

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Proposition 6.2.2 Λ is a bounded linear operator.

Proposition 6.2.3 The nullspace of Λ is trivial.

Notation

Denote the range of the operator Λ by $\mathcal{R}(\Lambda)$ and its domain by $\mathcal{D}(\Lambda)$.

Proposition 6.2.4 $\mathcal{R}(\Lambda)$ is dense in H .

The null space of Λ is trivial by Proposition 6.2.3, hence Λ is one-to-one.

Definition Operator A

Let $\mathcal{D}(A) = \mathcal{R}(\Lambda)$ and $A = \Lambda^{-1}$.

Lemma 6.2.1 A is a closed linear operator.

Lemma 6.2.2 The domain of A is dense in H .

Lemma 6.2.3 For any $x \in \mathcal{D}(A)$, $Ax = y$ if and only if $x_2 = y_1$ and

$$b(x_1, v) + a(x_2, v) = -c(y_2, v) \quad \text{for each } v \in V.$$

Corollary 6.2.1 $x \in \mathcal{D}(A)$ if and only if $x_2 \in V$ and there exists a $y \in W$ such that

$$b(x_1, v) + a(x_2, v) = c(y, v) \quad \text{for each } v \in V.$$

Lemma 6.2.4 For any $\lambda \geq 0$, $\mathcal{R}(\lambda - A) = H$.

Lemma 6.2.5 $(Ax, y)_H = b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2)$ for each x and $y \in \mathcal{D}(A)$.

Corollary 6.2.2 $(Ax, x)_H = -a(x_2, x_2)$ for $x \in \mathcal{D}(A)$.

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Remark

Corollary 6.2.2 implies that A is dissipative.

Problem AH

Given a function $F : [0, \tau) \rightarrow H$, find $U \in C([0, \tau); H)$ such that for each $t \in (0, \tau)$, $U(t) \in \mathcal{D}(A)$, $U'(t) \in H$ and

$$\begin{aligned} U'(t) &= AU(t) + F(t), \\ U(0) &= U_0. \end{aligned}$$

The following results enable us to write Problem G as an initial value problem for a first order system.

Proposition 6.2.5 Suppose $J = [0, \tau)$ and $F(t) = \langle 0, f(t) \rangle$ for each $t \in J$. If

$$u \in C([0, \tau); V) \cap C^1([0, \tau); W)$$

is a solution of Problem G then $U = \langle u, u' \rangle$ is a solution of Problem AH with $U_0 = \langle u_0, u_1 \rangle$.

If

$$u \in C([0, \tau); V) \cap C^1([0, \tau); W) \cap C^1((0, \tau); V) \cap C^2((0, \tau); W),$$

then

$$U \in C([0, \tau); H) \cap C^1((0, \tau); H).$$

Proposition 6.2.6 Suppose $J = [0, \tau)$ and $F(t) = \langle 0, f(t) \rangle$ for each $t \in J$. If U is a solution of Problem AH with $U(0) = \langle u_0, u_1 \rangle$, then the first component $u = U_1$ of U is a solution of Problem G. If $U \in C^1((0, \tau); H)$ then

$$u \in C([0, \tau); V) \cap C^1([0, \tau); W) \cap C^1((0, \tau); V) \cap C^2((0, \tau); W).$$

Remark This result is true even if $a = 0$.

Lemma 6.2.6 The linear operator A defined in this section is the infinitesimal generator of a C_0 semigroup of contractions in H .

Corollary 6.2.3 If $U_0 \in \mathcal{D}(A)$ and $F \in C^1([0, \tau); H)$ then Problem ADE 1 has a unique solution. If $F = 0$ then this solution is in $C^1([0, \infty); H)$.

Remark

If $F \neq 0$ then $U \in C([0, \tau); H) \cap C^1((0, \tau); H)$.

6.3 Approximation

In this section the results of Section 6.2 are adapted for a finite dimensional subspace. The objective is to find an approximation for the linear operator Λ .

We consider Problem G and assume that the Assumptions E1 to E4 hold. We also use Assumptions A5 and A6.

Proposition 6.3.1 *For any $\lambda \geq 0$ and any given $y \in H$ there exists a unique $x^h \in S^h \times V$ such that*

$$\begin{aligned} y_1 &= \lambda x_1^h - x_2^h \quad \text{and} \\ b(x_1^h, v) + a(x_2^h, v) + \lambda c(x_2^h, v) &= c(y_2, v) \quad \text{for any } v \in S^h. \end{aligned}$$

Proof Let $v \in S^h$ be arbitrary. Consider the problem of solving for x_1^h in

$$b(x_1^h, v) + \lambda a(x_1^h, v) + \lambda^2 c(x_1^h, v) = a(y_1, v) + c(\lambda y_1 + y_2, v). \quad (6.3.1)$$

Set $F(v) = a(y_1, v) + c(\lambda y_1 + y_2, v)$ for each $v \in V$. Since $\|v\|_W < \beta \|v\|_V$ for any $v \in V$ and a is a bounded bilinear form on V

$$\begin{aligned} |F(v)| &\leq |a(y_1, v)| + |c(\lambda y_1 + y_2, v)| \\ &\leq K \|y_1\|_V \|v\|_V + (\lambda \|y_1\|_W + \|y_2\|_W) \|v\|_W \\ &\leq K \|y_1\|_V \|v\|_V + \lambda \beta^2 \|y_1\|_V \|v\|_V + \beta \|y_2\|_W \|v\|_V \\ &\leq C (\|y_1\|_V + \|y_2\|_W) \|v\|_V, \end{aligned}$$

where $C = \max\{K + \lambda\beta^2; \beta\}$. So F is a bounded linear functional on $V \supset S^h$.

Since $b + \lambda a + \lambda^2 c$ defines an inner product on S^h , it follows from Riesz's Theorem that there exist a unique $x_1^h \in S^h$ such that (6.3.1) is satisfied.

Let $x_2^h = \lambda x_1^h - y_1$ and the result follows. $\square$

Lemma 6.3.1 *Suppose $\lambda \geq 0$ and $y \in H$. If $x \in H$ is such that*

$$\begin{aligned} \lambda x_1 - x_2 &= y_1 \quad \text{and} \\ b(x_1, v) + a(x_2, v) + \lambda c(x_2, v) &= c(y_2, v) \quad \text{for each } v \in V, \end{aligned}$$

and $x^h \in S^h \times V$ is such that

$$\begin{aligned} \lambda x_1^h - x_2^h &= y_1 \quad \text{and} \\ b(x_1^h, v) + a(x_2^h, v) + \lambda c(x_2^h, v) &= c(y_2, v) \quad \text{for any } v \in S^h, \end{aligned}$$

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then

$$\lim_{h \rightarrow 0} \|x - x^h\|_H = 0.$$

Proof Let $v \in S^h$ be arbitrary. Consider the problem of solving for x_1 in

$$b(x_1, v) + \lambda a(x_1, v) + \lambda^2 c(x_1, v) = a(y_1, v) + c(\lambda y_1 + y_2, v),$$

and x_1^h in

$$b(x_1^h, v) + \lambda a(x_1^h, v) + \lambda^2 c(x_1^h, v) = a(y_1, v) + c(\lambda y_1 + y_2, v).$$

It is clear that

$$b(x_1 - x_1^h, v) + \lambda a(x_1 - x_1^h, v) + \lambda^2 c(x_1 - x_1^h, v) = 0$$

for any $v \in S^h$, and it is also clear that $b + \lambda a + \lambda^2 c$ is an inner product for V . But then x_1^h is the projection of x with respect to this inner product. Due to assumptions E2 and E3 the error in the corresponding norm is bounded by $K\|x_1 - x_1^h\|_E$ for some constant K . Using Assumptions A5 and A6 it follows that $\|x_1 - x_1^h\|_E \rightarrow 0$ as $h \rightarrow 0$. Consequently

$$\|x_2 - x_2^h\|_W = \lambda\|x_1 - x_1^h\|_W \rightarrow 0. \quad \square$$

Definition The mapping Λ_h

For $\lambda = 0$, the mapping Λ_h is defined on H by $\Lambda_h y = -x^h$, where $-x_2^h = y_1$ and $x_1^h \in V$ such that

$$b(x_1^h, v) + a(x_2^h, v) = c(y_2, v) \quad \text{for each } v \in S^h.$$

Proposition 6.3.2 Λ_h is a bounded linear operator.

Proof Clearly

$$\|x_1^h\|_E^2 = b(x_1^h, x_1^h) \leq C\|x_2^h\|_E\|x_1^h\|_E + C_b\|y_2\|_W\|x_1^h\|_E,$$

using Assumption E2. It follows that

$$\|x_1^h\|_W \leq C\|y_1\|_E + C_b^2\|y_2\|_W.$$

The fact that $x_2^h = -y_1$, yields the result. $\square$

Lemma 6.3.2 *For any $y \in H$ and any $\lambda > 0$, there exists a unique $x \in H$ such that*

$$x = \Lambda(y - \lambda x),$$

a unique $x^h \in S^h \times V$ such that

$$x^h = \Lambda_h(y - \lambda x^h)$$

and

$$\lim_{h \rightarrow 0} \|x^h - x\|_H = 0.$$

Proof The existence of x and x^h follows from Propositions 6.2.1 and 6.3.1. Lemma 6.3.1 provides the convergence result. $\square$

6.4 Application of the Trotter-Kato theorem

We consider Problem G with the same assumptions as in the previous section. From Lemma 6.2.6 we have that the linear operator A is the infinitesimal generator of a C_0 semigroup, say $T(\cdot)$. In this section the objective is to approximate T by a semigroup on the subspace $V^h = S^h \times S^h$.

Obviously Λ_h is not invertible but we consider its restriction Λ_{V^h} to V^h .

Proposition 6.4.1 *The nullspace of Λ_{V^h} is trivial.*

Proof If $x^h = 0$, then $c(y_2, v) = 0$ for each $v \in S^h$, and hence $y_2 = 0$. Also, $y_1 = x_2^h = 0$. $\square$

We denote the inverse of Λ_{V^h} by L_h .

Since S^h is finite dimensional, e^{tL_h} is defined (and bounded) and $e^{tL_h}b$ is the unique solution of

$$x' = L^h x \quad \text{with} \quad x(0) = b$$

for any $b \in V^h$.

Let Q_h denote the projection of H onto V^h with respect to the inner product of H . Then $T_h(\cdot)$, defined by

$$T_h(t) = e^{tL_h}Q_h,$$

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is a C_0 semigroup. The infinitesimal generator of $T_h(\cdot)$ is $A_h = L_h Q_h$. But L_h is dissipative since $(L_h x, x)_H = -a(x, x) \leq 0$. Consequently

$$\|T_h(t)\| = \|e^{tL_h} P_h\| \leq \|e^{tL_h}\| \cdot \|P_h\| \leq 1,$$

which means that $T_h(\cdot)$ is a contraction semigroup.

The following result is a simplified version of the Trotter approximation theorem, Theorem 4.2 p.85 in the book of Pazy [P].

Trotter approximation theorem

Let L be the infinitesimal generator of a contraction semigroup $S(\cdot)$ and $\{L_n\}$ be a sequence of infinitesimal generators of contraction semigroups $\{S_n(\cdot)\}$, then the following statements are equivalent.

- (a) For every $x \in H$ and $\lambda > 0$, $(\lambda I - L_n)^{-1}x \rightarrow (\lambda I - L)^{-1}x$ as $n \rightarrow \infty$.
- (b) For every $x \in H$ and $t \geq 0$, $S_n(t)x \rightarrow S(t)x$ as $n \rightarrow \infty$.

Moreover, the convergence in (b) is uniform on bounded intervals. □

Now consider a sequence $\{h_n\}$ where $h_n \rightarrow 0$ as $n \rightarrow \infty$. For each $n \in N$, we let $V_n = V_{h_n}$, $Q_n = Q_{h_n}$, $A_n = A_{h_n}$ and $T_n = T_{h_n}$. We intend to prove that the semigroup T is approximated by the sequence T_n using the Trotter approximation theorem. Now A is the infinitesimal generator of T and A_n is the infinitesimal generator of T_n . To prove the required result, we will show that for $\lambda > 0$,

$$(\lambda I - A_n)^{-1}y \rightarrow (\lambda I - A)^{-1}y \quad \text{as } n \rightarrow \infty,$$

for any $y \in H$.

We adapt the proof of Theorem 2.1 p.53 in [IKS91]. Our case is simpler since we have dissipative operators and contraction semigroups.

Remarks

1. We consider Problem AH defined in Section 6.2.
2. In [IKS91] σ is defined by a general linear operator whereas in our case $\sigma(x, y) = (Ax, y) = b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2)$.

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Lemma 6.4.1 *For any $y \in H$ and any $\lambda > 0$,*

$$(\lambda - A_n)^{-1}y \rightarrow (\lambda - A)^{-1}y \quad \text{as } n \rightarrow \infty.$$

Proof For any n let

$$x_n = (\lambda - A_n)^{-1}y.$$

This is true if and only if $-A_n x_n = y - \lambda x_n$, which is the same as

$$Q_n x_n = \lambda_n(y - \lambda x_n).$$

But $Q_n x_n \in V_n$ and hence

$$Q_n x_n = \Lambda_n(y - \lambda Q_n x_n).$$

The result follows from Lemma 6.3.2. □

Corollary 6.4.1 *For any $y \in H$*

$$T_n(t)y \rightarrow T(t)y \quad \text{as } n \rightarrow \infty,$$

and the convergence is uniform on bounded intervals.

Remarks

1. The theory in [IKS91] as adapted in this chapter can be applied to any problem that is of the format of Problem G. The details will vary from case to case as in Chapter 3.
2. The disadvantage of the method is that no results on the order of convergence are obtained and it does not provide error estimates.

6.5 Other applications

In [IKS91] a general approach to delay systems is considered. The authors apply the theory to size structured population models. (In this dissertation it is shown that the method can be applied to Problem G.)

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In [IK98] another formulation of the Trotter-Kato theorem is presented, which provides a framework for approximation of solutions of partial differential equations. An application to the wave equation with boundary damping is given in Subsection 4.1.

The wave equation with boundary damping is also considered in [F01]. The semigroup that generates the solution is exponentially stable, i.e. $\|T(t)\|_H \leq Me^{-wt}$ for $t \geq 0$. However, this is not the case for the standard Galerkin approximation, see [F01, p.4553]. Fabiano shows that the Hilbert space may be renormed and uses a new approximation scheme that is uniformly exponentially stable. The theory is based on Theorem 2.1 in [IKS91].

Chapter 7

Fully discrete approximation

7.1 A system of ordinary differential equations

Consider Problems G and G^h defined in Sections 2.1 and 2.3. In this chapter we consider the discretization of Problem G^h . Recall that S^h is the span of the set of basis functions $\{\phi_1, \phi_2, \dots, \phi_n\}$.

Problem G^h

Find u_h such that for each $t > 0$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h,$$

while $u_h(0) = d^h$ and $u_h'(0) = v^h$. (In this chapter it is convenient to use d^h and v^h for initial conditions instead of u_0^h and u_1^h .)

This problem is equivalent to a system of ordinary differential equations as we prove below. Since $u^h(t) \in S^h$ for each t , it is of the form $u^h(x, t) = \sum_{j=1}^n u_j(t)\phi_j(x)$. For any function $v(x, t) = \sum_{j=1}^n v_j(t)\phi_j(x)$ it is a fact that $\lim_{t \rightarrow \tau} v(x, t) = \sum_{j=1}^n w_j\phi_j(x)$ if and only if $\lim_{t \rightarrow \tau} v_j(t) = w_j$ for each j . The norm used to define $\lim_{t \rightarrow \tau} v(x, t)$ does not matter since S^h is finite dimensional. Consequently u^h is differentiable if and only if u_j is differentiable for each j and

$$\frac{d}{dt} \left(\sum_{j=1}^n u_j(t)\phi_j(x) \right) = \sum_{j=1}^n u_j'(t)\phi_j(x).$$

The same is true for the second order derivative of u_h .

We find that u^h satisfies the variational equation in Problem G^h if and only if

$$c(u_h''(t), \phi_i) + a(u_h'(t), \phi_i) + b(u_h(t), \phi_i) = (f(t), \phi_i), \quad (7.1.1)$$

for $i = 1, 2, \dots, n$.

Due to the fact that u_h may be differentiated componentwise, u_h satisfies Equation (7.1.1) if and only if

$$\begin{aligned} \sum_{j=1}^n u_j''(t)c(\phi_j, \phi_i) + \sum_{j=1}^n u_j'(t)a(\phi_j, \phi_i) + \sum_{j=1}^n u_j(t)b(\phi_j, \phi_i) \\ = (f(t), \phi_i)_X \quad \text{for } i = 1, 2, \dots, n. \end{aligned} \quad (7.1.2)$$

Let

$$K_{ij} = b(\phi_j, \phi_i), \quad C_{ij} = a(\phi_j, \phi_i), \quad M_{ij} = (\phi_j, \phi_i) \quad \text{and} \quad F_i(t) = (f(t), \phi_i)_X,$$

then u^h satisfies Equation (7.1.1) if and only if

$$\sum_{j=1}^n M_{ij}u_j'' + \sum_{j=1}^n C_{ij}u_j' + \sum_{j=1}^n K_{ij}u_j = F_i(t) \quad \text{for } i = 1, 2, \dots, n.$$

Notation

The vector valued function with components u_i are denoted by $\bar{u}$, i.e.

$$\bar{u} = (u_1, u_2, \dots, u_n) = [u_1 \ u_2 \ \dots \ u_n]^t.$$

We also introduce the vectors $\bar{d} = (d_1, d_2, \dots, d_n)$ and $\bar{v} = (v_1, v_2, \dots, v_n)$ where $d^h = \sum_{j=1}^n d_j \phi_j$ and $v^h = \sum_{j=1}^n v_j \phi_j$.

Problem G-ODE

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F(t) \quad \text{with} \quad \bar{u}(0) = \bar{d} \quad \text{and} \quad \bar{u}'(0) = \bar{v}.$$

We have proved the following result:

Proposition 7.1.1 *Suppose M , K , C , F , $\bar{d}$ and $\bar{v}$ are defined as above. Then, the function u^h is a solution of Problem G^h if and only if the function $\bar{u}$ is a solution of Problem G-ODE.*

Proposition 7.1.2 *Suppose the bilinear form c has the property that $u = 0$ if $c(u, u) = 0$. If $F \in C(J)$, then Problem G-ODE has a unique solution for each pair of vectors $\bar{d}$ and $\bar{v}$.*

Proof Due to the assumption on the bilinear form c , $M\bar{x} \cdot \bar{x} = 0$ only if $\bar{x} = 0$. This implies that M is invertible and the differential equation in Problem G-ODE may be written in the form

$$\bar{u}'' + M^{-1}C\bar{u}' + M^{-1}K\bar{u} = M^{-1}F(t).$$

It follows from the theory of linear differential equations that the initial value problem where $\bar{u}(0) = \bar{d}$ and $\bar{u}'(0) = \bar{v}$ has a unique solution.

Proof of Theorem 2.3.1

Since S^h is finite dimensional, $f \in C(J; S^h)$ if and only if $F \in C(J)$. It follows that Problem G-ODE has a unique solution (by Proposition 7.1.2) and hence Problem G^h has a unique solution (by Proposition 7.1.1).

Remark

For existence the continuity of F is sufficient but for $\bar{u}$ to have derivatives of order $2 + k$, we need to assume that F has derivatives of order k .

7.2 Finite differences

A finite difference method is used to approximate the solution of the system. Suppose an approximation for the solution is required on the time interval $[0, \tau]$. The interval is divided into N steps resulting in a step length $\delta t = \tau/N$. We denote the approximation of $\bar{u}(t_k)$ by $\bar{u}_k$.

Any finite difference algorithm will be implicit due to the matrix M . Consider for example central differences.

$$(\delta t)^{-2}M[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] + (2\delta t)^{-1}C[\bar{u}_{k+1} - \bar{u}_{k-1}] + K\bar{u}_k = F(t_k).$$

In early experiments it was found that the matrix M could be replaced by a diagonal matrix without a significant loss of accuracy and enhanced stability (see [SF]). The process is referred to as mass lumping. With a diagonal matrix in the place of the matrix M , the algorithm is effectively explicit. However the method is only conditionally stable, see Section 7.5. It is natural

to search for methods with as “high a degree of stability” as possible, even unconditional stability.

A general approach to finite differences for second order hyperbolic systems are the so called Newmark schemes. Many standard algorithms are special cases, e.g. central differences. Chapter 9 in the book of Hughes [H] is devoted to various “algorithms for hyperbolic and parabolic-hyperbolic problems”.

One special case of a Newmark scheme is the average acceleration scheme. It is an implicit method which is unconditionally stable and the accuracy is of order 2.

The average acceleration method

$$(\delta t)^{-2} M[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] + (2\delta t)^{-1} C[\bar{u}_{k+1} - \bar{u}_{k-1}] + \frac{1}{4} K[\bar{u}_{k+1} + 2\bar{u}_k + \bar{u}_{k-1}] = \frac{1}{4} [F(t_{k+1}) + 2F(t_k) + F(t_{k-1})],$$

with $\bar{u}_0 = \bar{d}$ and $\bar{u}_1 - \bar{u}_{-1} = 2\delta t \bar{v}$.

This algorithm is used in the next section where we report on a few numerical experiments and convergence is considered in Section 7.4.

7.3 Application to the one-dimensional wave equation

This dissertation is aimed at establishing the state of convergence theory for Galerkin’s method. However, in view of the assumptions that are rather restrictive, it is of interest to consider some examples.

In this section we present numerical results for the wave equation. Consider the initial value problem

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u, \\ u(0, t) &= \partial_x u(1, t) = 0, \end{aligned}$$

with initial conditions $u(x, 0) = u_0(x)$ and $\partial_t u(x, 0) = 0$.

The idea is to see how the smoothness of the initial value u_0 affects the rate of convergence.

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For the following examples we look at the results at time $t = 4$ (after one oscillation) or $t = 8$, at the right endpoint ($x = 1$). Note that we have the actual error and not the relative difference. In the experiments the ratio $r = \frac{\delta t}{h^2}$ is fixed at $r = \frac{1}{2}$.

Notation We denote the error at $x = 1$ for n elements by E_n .

Example 1

$$u_0(x) = 10 \sin\left(\frac{\pi}{2}x\right)$$

This initial value satisfies the conditions for Proposition 2.1.1, therefore we have a unique classical solution. Actually we need the theory only for uniqueness since the solution is obviously $u(x, t) = 10 \sin(\frac{\pi}{2}x) \cos(\frac{\pi}{2}t)$. Furthermore, $u \in C^\infty([0, 1] \times (-\infty, \infty))$, hence all the assumptions are satisfied.

The results are given in Table 1.

Table 1

Number of elements (n)	E_n for $t = 4$	E_n for $t = 8$
8	0.0005018738	0.0020074448
16	0.0000317184	0.0001268733
32	0.0000019879	0.0000079515
64	0.0000001243	0.0000004974
128	0.0000000066	0.0000000323

We have that $\frac{E_n}{E_{2n}} \approx 16$, but $\frac{E_n}{E_{2n}} = 4$ according to the theory.

Example 2

$$u_0(x) = 10 \sin\left(\frac{\pi}{2}x\right) + 10 \sin\left(\frac{5\pi}{2}x\right)$$

This initial value also results in a classical solution and the same remarks as in Example 1 are valid. Note that the energy norm for this initial value is greater than for the one in Example 1 and it features in the estimates.

The results are in Table 2.

Table 2

Number of elements (n)	E_n for $t = 4$	E_n for $t = 8$
8	6.947995	18.138448
16	0.494096	1.927563
32	0.031090	0.124168
64	0.001943	0.007773
128	0.000121	0.000486

Again $\frac{E_n}{E_{2n}} \approx 16$.

Example 3

$$u_0(x) = 3x - x^3$$

All the conditions for Proposition 2.1.1 are satisfied by this initial condition, and therefore we have a unique classical solution, but not infinitely differentiable.

The results are shown in Table 3.

Table 3

n	E_n
8	0.0051815
16	0.0013041
32	0.0003265
64	0.0000816
128	0.0000204

For this initial value $\frac{E_n}{E_{2n}} \approx 4$.

Example 4

$$u_0(x) = x(2 - x)$$

In this case the initial value does not satisfy the conditions for Proposition 2.1.1 (since $u_0''(0) \neq 0$), and therefore we do not have a classical solution. However from d'Alembert's method it is clear that the solution is almost a classical solution. The initial value does satisfy the sufficient conditions for Proposition 2.2.1 however, so we have a weak solution.

The results are shown in Table 4.

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Table 4

n	E_n
8	0.0005455
16	0.0009644
32	0.0001476
64	0.0000106
128	0.0000014

For this initial value $\frac{E_{16}}{E_{32}} = 6.5$, $\frac{E_{32}}{E_{64}} = 14$ and $\frac{E_{64}}{E_{128}} = 7.6$.

Example 5

$$u_0(x) = x$$

This initial value does not even satisfy the boundary condition at $x = 1$. However, it is possible to calculate a mild solution using d'Alembert's method. The results are shown in Table 5.

Table 5

n	E_n
8	0.09941
16	0.06183
32	0.04047
64	0.02581
128	0.01599

Here we have $1.53 \leq \frac{E_n}{E_{2n}} \leq 1.61$.

Remarks on order of convergence

1. For Example 1 and 2 the order of convergence is h^4 , this is better than the order that the theory predicts.
2. The order of convergence for Example 3 is h^2 .
3. Note that even though the error is smaller for Example 4 than for Example 2, the order of convergence for Example 2 is better than the order for Example 4. This is because of the shape of the initial value in Example 2 - more elements are needed to approximate this shape than the one in Example 4.
4. The order of convergence for Example 5 is less than h .

7.4 Convergence for the average acceleration method

7.4.1 Finite differences for the system of ordinary differential equations

In Chapters 2 to 6 we considered the convergence of the solution of the semi-discrete problem (Problem G^h) to the solution of the general second order problem (Problem G). But what is actually calculated, is the solution of the fully discrete problem. So we need also consider convergence of the solution of the fully discrete problem to the solution of the semi-discrete problem.

Recall that the time interval $[0, \tau]$ is divided into N steps with in a step length $\delta t = \tau/N$ and that we denote the approximation of $\bar{u}(t_k)$ by $\bar{u}_k$. As far as possible we consider a general fully discrete method.

Problem G-ODE-D

Suppose ρ_0 and ρ_1 are positive numbers such that $\rho_0 + 2\rho_1 = 1$. Find a sequence $\{\bar{u}_k\} \subset R_n$ such that for each k ,

$$\begin{aligned} & \frac{1}{(\delta t)^2} M [\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] + \frac{1}{2\delta t} C [\bar{u}_{k+1} - \bar{u}_{k-1}] \\ & + K [\rho_1 \bar{u}_{k+1} + \rho_0 \bar{u}_k + \rho_1 \bar{u}_{k-1}] = \rho_1 F(t_{k+1}) + \rho_0 F(t_k) + \rho_1 F(t_{k-1}), \end{aligned}$$

with $\bar{u}_0 = \bar{d}$ and $\bar{u}_1 - \bar{u}_{-1} = 2\delta t \bar{v}$.

Remark

For the average acceleration method $\rho_0 = 1/2$ and $\rho_1 = 1/4$.

Proposition 7.4.1 *Problem G-ODE-D has a unique solution for any pair of vectors $\bar{d}$ and $\bar{v}$.*

Proof Consider

$$\begin{aligned} & M [\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] + \frac{\delta t}{2} C [\bar{u}_{k+1} - \bar{u}_{k-1}] \\ & + \delta t^2 K [\rho_1 \bar{u}_{k+1} + \rho_0 \bar{u}_k + \rho_1 \bar{u}_{k-1}] = \delta t^2 [\rho_1 F(t_{k+1}) + \rho_0 F(t_k) + \rho_1 F(t_{k-1})], \end{aligned}$$

with $\bar{u}_0 = \bar{d}$ and $\bar{u}_1 - \bar{u}_{-1} = 2\delta t \bar{v}$. Consequently

$$\begin{aligned} & [2M + 2\rho_1 \delta t^2 K] \bar{u}_1 = 2M \bar{d} + 2\delta t M \bar{v} - \delta t^2 C \bar{v} - \rho_0 \delta t^2 K \bar{d} + 2\rho_1 \delta t^3 K \bar{v} \\ & + \delta t^2 [\rho_1 F(t_{k+1}) + \rho_0 F(t_k) + \rho_1 F(t_{k-1})]. \end{aligned}$$

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The matrix $2M + 2\rho_1\delta t^2 K$ is invertible since

$$[2M + 2\rho_1\delta t^2 K]\bar{x} \cdot \bar{x} = 2M\bar{x} \cdot \bar{x} + 2\rho_1\delta t^2 K\bar{x} \cdot \bar{x} > 0,$$

for $\bar{x} \neq \bar{0}$. (In other notation $X^t[2M + 2\rho_1\delta t^2 K]X > 0$ for $X \neq 0$.)

This implies that $\bar{u}_1$ is uniquely determined. Now consider $\bar{u}_{k+1}$:

$$\begin{aligned} [M + \frac{\delta t}{2} C + \delta t^2 \rho_1 K] \bar{u}_{k+1} &= 2M\bar{u}_k - M\bar{u}_{k-1} + \frac{\delta t}{2} C \bar{u}_{k-1} \\ &- \delta t^2 K [\rho_0 \bar{u}_k + \rho_1 \bar{u}_{k-1}] + \delta t^2 [\rho_1 F(t_{k+1}) + \rho_1 F(t_k) + \rho_0 F(t_{k-1})]. \end{aligned}$$

Again,

$$[M + \frac{\delta t}{2} C + \delta t^2 \rho_1 K]\bar{x} \cdot \bar{x} > 0,$$

for $\bar{x} \neq \bar{0}$ and the inverse of $[M + \frac{\delta t}{2} C + \delta t^2 \rho_1 K]$ exists. Therefore $\bar{u}_{k+1}$ is uniquely determined. $\square$

Convergence for the system of ordinary differential equations

As mentioned before, Chapter 9 in [H] is devoted to various algorithms including Newmark schemes. We find the following statement in Section 9.1: “Again, stability plus consistency implies convergence.” A rather general approach to Newmark schemes is followed leaving a number of proofs as exercises. In Exercises 2 and 3 on p.495 the central difference average acceleration algorithm is derived from the general Newmark scheme. Actually these are examples, leaving only elementary calculations to the reader. It is also stated in Section 9.1 that the central difference average acceleration algorithm is unconditionally stable. No proof is given but three references are given.

7.4.2 The error

Our aim is to prove convergence of the solution of the fully discrete problem to the solution of the the general second order problem and derive error estimates. Consider the error

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h].$$

Estimates for the error $u(t_k) - u_h(t_k)$ were obtained in Chapters 2 to 5 and estimates for the error $\bar{u}(t_k) - \bar{u}_k$ are available in [H, Chapter 9]. But what

is required, is an estimate for $\|u_h(t_k) - u_k^h\|_W$ and there is no indication in [H, Chapter 9] how to obtain this. The natural approach is to consider the fully discrete problem in variational form as in [D73].

Problem G^h -D

Find a sequence $\{u_k^h\} \subset S^h$ such that for each k ,

$$\begin{aligned} & c(\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + a((2\delta t)^{-1}[u_{k+1}^h - u_{k-1}^h], v) \\ & + b(\rho_1 u_{k+1}^h + \rho_0 u_k^h + \rho_1 u_{k-1}^h, v) = (\rho_1 f(t_{k+1}) + \rho_0 f(t_k) + \rho_1 f(t_{k-1}), v)_X, \end{aligned}$$

for each $v \in S^h$ while $u_0^h = d^h$ and $u_1^h - u_{-1}^h = 2\delta t v^h$.

The algorithm for the wave equation without damping is the scheme (3.2) [D73, p.886] and (3.10) [D73, p.888] for boundary damping. Problem G^h -D above is the general version. This approach was followed by others e.g. [B76] and [Z-PhD]. However, in [Z-PhD] an estimate for $u_h(t_k) - u_k^h$ is obtained while in [D73] (and others) the error $u(t_k) - u_k^h$ is estimated directly. In this section we follow the approach in [Z-PhD] and discuss the “direct approach” in Section 7.5.

It is convenient to introduce the following transformation. For any $\bar{x} \in R_n$, let

$$T_h \bar{x} = \sum_{i=1}^n x_i \phi_i \in S^h.$$

The mapping T_h is obviously a linear bijection.

Proposition 7.4.2 *The sequence $\{\bar{u}_k\}$ is a solution of Problem G -ODE-D if and only if $\{u_k^h\} = \{T_h \bar{x}\}$ is a solution of Problem G^h -D.*

Proof The proof is similar to the proof of Proposition 1 in Section 7.1 (only easier). $\square$

Corollary 7.4.1 *Problem G^h -D has a unique solution for any pair of functions d^h and v^h in S^h .*

Remark

We consider Problem G^h -D for theory but Problem G -ODE-D is used for computation.

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The objective is to prove that $\|u_h(t_k) - u_k^h\|_W \leq K_1 \delta t^2$ for some constant K_1 . Then, since $\|u(t_k) - u_h(t_k)\|_W \leq K_2 h^\kappa$ for some constant K_2 and some positive integer κ , the estimate below follows.

Error estimate

$$\begin{aligned} \|u(t_k) - u_k^h\|_W &\leq \|u(t_k) - u_h(t_k)\|_W + \|u_h(t_k) - u_k^h\|_W \\ &\leq K_2 h^\kappa + K_1 \delta t^2. \end{aligned}$$

7.4.3 The local error

If $f \in C^2(J)$, then $u_h \in C^4(J)$ and optimal error estimates can be obtained. The first step is to substitute the solution of problem G^h into Problem G^h -D and estimate the truncation error. The procedure, using Taylor expansions, is standard. Note that $\rho_0 = \frac{1}{4}$ and $\rho_1 = \frac{1}{2}$ for the average acceleration method.

Proposition 7.4.3 *If $u_h \in C^4(J)$ is the solution of Problem G^h , then*

$$\begin{aligned} 2c(u_h(\delta t), v) + \frac{\delta t^2}{2} b(u_h(\delta t), v) &= 2c(d^h, v) - \frac{\delta t^2}{2} b(d^h, v) \\ - 2\delta t c(v^h, v) - \delta t^2 a(v^h, v) + \frac{\delta t^3}{2} b(v^h, v) &+ \frac{\delta t^2}{2} (f(\delta t) + f(0), v)_X \\ + c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \leq \max_J \|u_h^{(4)}(t)\|_W \delta t^3$ and $\|r_2^h\|_E \leq \max_J \|u_h^{(4)}(t)\|_E \delta t^3$.

Proposition 7.4.4 *If $u_h \in C^4(J)$ is the solution of Problem G^h , then*

$$\begin{aligned} &c(u_h(t + \delta t) - 2u_h(t) + u_h(t - \delta t), v) \\ &+ \frac{\delta t}{2} a(u_h(t + \delta t) - u_h(t - \delta t), v) \\ &+ \frac{\delta t^2}{4} b(u_h(t + \delta t) + 2u_h(t) + u_h(t - \delta t), v) \\ &= \frac{\delta t^2}{4} (f(t + \delta t) + 2f(t) + f(t - \delta t), v)_X \\ &+ c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \leq \max_J \|u_h^{(4)}(t)\|_W \delta t^3$ and $\|r_2^h\|_E \leq \max_J \|u_h^{(4)}(t)\|_E \delta t^3$.

Lemma 7.4.1 *If $u_h \in C^4(J)$ is the solution of Problem G^h and $\{u_k^h\}$ the solution of Problem G^h -D, then*

$$\|u_h(\delta t) - u_1^h\|_W \leq \|e_1^h\|_W + \delta t \|e_2^h\|_E \|u_h(\delta t) - u_1^h\|_E \|u_h(\delta t) - u_1^h\|_W^{-1}.$$

Proof Recall that

$$\begin{aligned} & c(u_1^h - 2u_0^h + u_{-1}^h, v) + \frac{\delta t}{2} a(u_1^h - u_{-1}^h, v) \\ & + \frac{\delta t^2}{4} b(u_1^h + 2u_0^h + u_{-1}^h, v) = \frac{\delta t^2}{4} (f(t_1) + 2f(t_0) + f(t_{-1}), v)_X, \end{aligned}$$

for each $v \in S^h$, while $u_0^h = d^h$, $u_1^h - u_{-1}^h = 2\delta t v^h$ and $f(t_{-1}) = f(t_1)$. It follows that

$$\begin{aligned} & c(2u_1^h, v) + \frac{\delta t^2}{2} b(u_1^h, v) = 2c(d^h, v) - \frac{\delta t^2}{2} b(d^h, v) \\ & + 2\delta t c(v^h, v) - \delta t^2 a(v^h, v) + \frac{\delta t^3}{2} b(v^h, v) + \frac{\delta t^2}{2} (f(t_1) + f(t_0), v)_X, \end{aligned}$$

for each $v \in S^h$. By Proposition 7.4.3

$$\begin{aligned} & 2c(u_h(\delta t), v) + \frac{\delta t^2}{2} b(u_h(\delta t), v) = 2c(d^h, v) - \frac{\delta t^2}{2} b(d^h, v) \\ & - 2\delta t c(v^h, v) - \delta t^2 a(v^h, v) + \frac{\delta t^3}{2} b(v^h, v) + \frac{\delta t^2}{2} (f(\delta t) + f(0), v)_X \\ & + c(r_1^h, v) + a(r_2^h, v), \end{aligned}$$

for each $v \in S^h$, where $\|r_1^h\|_W \leq K\delta t^3$ and $\|r_2^h\|_E \leq K\delta t^3$. Consequently

$$2c(u_h(\delta t) - u_1^h, v) + \frac{\delta t^2}{2} b(u_h(\delta t) - u_1^h, v) = c(r_1^h, v) + a(r_2^h, v),$$

for each $v \in S^h$. Therefore

$$\begin{aligned} \|u_h(\delta t) - u_1^h\|_W^2 & \leq \frac{1}{2} |c(e_1^h, u_h(\delta t) - u_1^h)| + \frac{\delta t}{2} |a(e_2^h, u_h(\delta t) - u_1^h)| \\ & \leq \|e_1^h\|_W \|u_h(\delta t) - u_1^h\|_W + \delta t \|e_2^h\|_E \|u_h(\delta t) - u_1^h\|_E \end{aligned}$$

and the desired estimate follows. $\square$

Corollary 7.4.2 *For δt sufficiently small,*

$$\|u_h(\delta t) - u_1^h\|_W \leq K \max_j \|u_h^{(4)}(t)\|_W \delta t^3.$$

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Lemma 7.4.2 Suppose $u_h \in C^4(J)$ is the solution of Problem G^h and $u^+ = u^+(u_h, t, \delta t)$ is defined by

$$\begin{aligned} & c(u^+ - 2u_h(t) + u_h(t - \delta t), v) \\ & + \frac{\delta t}{2} a(u^+ - u_h(t - \delta t), v) \\ & + \delta t^2 b(\rho_1 u^+ + \rho_0 u_h(t) + \rho_1 u_h(t - \delta t), v) \\ & = \delta t^2 (\rho_1 f(t + \delta t) + \rho_0 f(t) + \rho_1 f(t - \delta t), v)_X, \end{aligned}$$

for each $v \in S^h$. If $\Delta u = u_h(t + \delta t) - u^+$, then

$$\|\Delta u\|_W \leq \|e_1^h\|_W + \delta t \|e_2^h\|_E \|\Delta u\|_E \|\Delta u\|_W^{-1}.$$

Proof Using the result of Proposition 7.4.4, we have

$$c(\Delta u, v) + \frac{\delta t}{2} a(\Delta u, v) + \rho_1 \delta t^2 b(\Delta u, v) = c(e_1^h, v) + a(e_2^h, v),$$

for each $v \in S^h$. Therefore

$$\begin{aligned} \|\Delta u\|_W^2 & \leq |c(e_1^h, \Delta u)| + \delta t |a(e_2^h, \Delta u)| \\ & \leq \|e_1^h\|_W \|\Delta u\|_W + \delta t \|e_2^h\|_E \|\Delta u\|_E. \end{aligned}$$

This yields the desired estimate. $\square$

Corollary 7.4.3 For δt sufficiently small,

$$\|\Delta u\|_W \leq K \max_j \|u_h^{(4)}(t)\|_W \delta t^3.$$

7.4.4 Stability and convergence

To obtain an estimate for the global error, we need a stability result. Following [Z-PhD], we prove that the error growth is bounded by some constant times the number of time steps. It then follows from the estimate for the local error that

$$\|u_h(t_k) - u_k^h\|_W \leq K_1 \delta t^2.$$

Suppose $\{u_k^h\}$ and $\{w_k^h\}$ are solutions of Problem G^h -D with different initial values. Let $e_k = u_k^h - w_k^h$, then e_k is also a solution of Problem G^h -D but with $f = 0$:

$$\begin{aligned} & c(e_{k+1} - 2e_k + e_{k-1}, v) + \frac{\delta t}{2} a(e_{k+1} - e_{k-1}, v) \\ & + \delta t^2 b(\rho_1 e_{k+1} + \rho_0 e_k + \rho_1 e_{k-1}, v) = 0, \end{aligned} \tag{7.4.1}$$

for each $v \in S^h$.

Following [Z-PhD] we assume modal damping. It is then possible to consider each mode individually using eigenvectors.

Eigenvalue problem

The vector w^h is an eigenvector with corresponding eigenvalue λ if

$$b(w^h, v) = \lambda c(w^h, v) \text{ for each } v \in S^h. \quad (7.4.2)$$

Let $w^h = T_h \bar{w}$, then w^h is an eigenvector with corresponding eigenvalue λ if and only if $K\bar{w} = \lambda M\bar{w}$. Since K and M are symmetric, there exist n mutually orthogonal eigenvectors for the matrix eigenvalue problem and it follows that there are n linearly independent eigenvectors for (7.4.2). But eigenvectors corresponding to different eigenvalues are orthogonal with respect to the inner product c . Therefore we may assume the existence of a set of orthonormal eigenvectors $\{w_1^h, w_2^h, \dots, w_n^h\}$ which forms a basis for S^h .

Consider a solution $\{e_k\}$ of (7.4.1). For $k = 1$ to N ,

$$e_k = \sum_{j=1}^n r_{k,j} w_j^h \text{ for } j = 1, 2, \dots, n.$$

If we substitute for e_k in (7.4.1), we find that

$$\begin{aligned} & \sum_{j=1}^n (r_{k+1,j} - 2r_{k,j} + r_{k-1,j}) c(w_j^h, v) \\ & + \frac{\delta t}{2} \sum_{j=1}^n (r_{k+1,j} - r_{k-1,j}) a(w_j^h, v) \\ & + \delta t^2 \sum_{j=1}^n (\rho_1 r_{k+1,j} + \rho_0 r_{k,j} + \rho_1 r_{k-1,j}) b(w_j^h, v) = 0, \end{aligned}$$

for each $v \in S^h$. But for each $v \in S^h$ we have

$$\begin{aligned} b(w_j^h, v) &= \lambda_j c(w_j^h, v), \\ a(w_j^h, v) &= \mu b(w_j^h, v) + k c(w_j^h, v) = (\mu \lambda_j + k) c(w_j^h, v). \end{aligned}$$

Consequently, for each mode,

$$\begin{aligned} & (r_{k+1} - 2r_k + r_{k-1}) + (\mu \lambda + k) \frac{\delta t}{2} (r_{k+1} - r_{k-1}) \\ & + \lambda \delta t^2 (\rho_1 r_{k+1} + \rho_0 r_k + \rho_1 r_{k-1}) = 0, \end{aligned}$$

with the subscript j dropped.

Consider the equation

$$\alpha r_{k+1} + \beta r_k + \gamma r_{k-1} = 0,$$

where

$$\begin{aligned}\alpha &= 1 + (\mu\lambda + k)\frac{\delta t}{2} + \lambda\delta t^2\rho_1, \\ \beta &= -2 + \lambda\delta t^2\rho_0, \\ \gamma &= 1 - (\mu\lambda + k)\frac{\delta t}{2} + \lambda\delta t^2\rho_1.\end{aligned}$$

It has a solution $r_k = r^k$ where

$$\alpha r^2 + \beta r + \gamma = 0.$$

If $\rho_0 \leq 2\rho_1$, it follows by direct calculation that $|r| \leq 1$ for real or imaginary roots. Consequently r_k is bounded by $|r_1|$ and $|r_0|$. Therefore we have the following stability result.

Lemma 7.4.3 *If $\rho_0 \leq 2\rho_1$, then there exists a $K > 0$ such that*

$$\|e_k\|_W \leq K(\|e_1\|_W + \|e_0\|_W).$$

Remarks

1. For the average acceleration method $\rho_0 = \frac{1}{2}$ and $\rho_1 = \frac{1}{4}$. Therefore the method is unconditionally stable.
2. A different approach is followed in [D73] and [B76], see the next section.
3. Recall that the problem without damping is a special case of modal damping.

7.5 Direct convergence result for the fully discrete approximation

In [D73] Dupont does not estimate $\|u_h(t_k) - u_k^h\|_W$, but $\|u(t_k) - u_k^h\|_W$ directly. The disadvantage of this approach is that he assumes the existence of a 4-th order time derivative for the exact solution. This is a very restrictive

assumption as one can see with the one-dimensional wave equation using d'Alembert's method. On the other hand, it is quite reasonable to assume that the solution u_h of Problem G^h has a derivative of order four (as was done in Section 7.4). Note that only the wave equation is considered in [D73] where the cases with no damping and boundary damping are considered separately.

For the undamped case Scheme (3.2) in [D73, p.886] is used:

$$\begin{aligned} & (\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + \frac{1}{4} b(\theta u_{k+1}^h + (1 - 2\theta)u_k^h + \theta u_{k-1}^h, v) \\ &= \frac{1}{4} (\theta f(t_{k+1}) + (1 - 2\theta)f(t_k) + \theta f(t_{k-1}), v) \end{aligned}$$

for each $v \in S^h$. Note that $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$.

We find the following paragraph at the bottom of p.886: “This is a second order correct in Δt approximation to (2.5). The reason for the choice $\theta = 1/4$ is that for $\theta \geq 1/4$ the analogues of (3.2) are stable independently of Δt and S^h , and the time truncation is minimized over this class by taking $\theta = 1/4$.” No references are given.

For boundary damping Scheme (3.10) in [D73, p.888] with $\theta = 1/4$, is used:

$$\begin{aligned} & (\delta t^{-2}[u_{k+1}^h - 2u_k^h + u_{k-1}^h], v) + (2\delta t)^{-1} \langle u_{k+1}^h - u_{k-1}^h, v \rangle \\ &+ \frac{1}{4} b(\theta u_{k+1}^h + (1 - 2\theta)u_k^h + \theta u_{k-1}^h, v) \\ &= \frac{1}{4} (\theta f(t_{k+1}) + (1 - 2\theta)f(t_k) + \theta f(t_{k-1}), v) \\ &+ \frac{1}{4} \langle (\theta g(t_{k+1}) + (1 - 2\theta)g(t_k) + \theta g(t_{k-1})), v \rangle, \end{aligned}$$

for each $v \in S^h$. Stability is not even mentioned.

The convergence results are Theorem 3 in [D73, p.888] for the undamped case and Theorem 4 in [D73, p.889] for boundary damping. It needs to be emphasized that in both cases the existence of a 4-th order time derivative for the exact solution is assumed.

The book [OR] appeared in 1976. Section 9.6 deals with “Hyperbolic equations of second order.” However, only the undamped case is considered. For the fully discrete approximation central differences is used. The authors also consider the direct approach as in [D73]. In Remark 1 on p.415 we read: “The techniques used here represent adaptations of Dupont [D73] to an explicit fully discrete scheme ...”. The convergence result suffers from the

same limitation as the results in [D73]: “ ... following Dupont [D73], we assume that the (exact) solution is such that $\partial^4 u / \partial t^4 \in L^2(L^2(\Omega))$.”

Following [D73], Baker in [B76] also derive a direct convergence result and with the same limitation. Recall that the convergence result for the semi-discrete problem in [B76] has the advantage that less regularity is assumed. But this advantage is lost when convergence for the fully discrete case is treated. Stability is not mentioned but it is implicit in Lemma 4.2. on p.570. The proof starts on p.571 and continues to the middle of p.575. But the problem in [B76] is for the wave equation without damping which may be treated the same as modal damping, and the assumption regarding a 4-th order time derivative for the exact solution is clearly not necessary.

The idea to estimate $\|u(t_k) - u_k^h\|_W$ directly (as in [D73]) is used in other articles, e.g.[S94], [FXX99], [Wu03], [Wu05] and [Wu06]. In all these papers the existence of a fourth order time derivative for the exact solution is assumed.

7.6 An explicit method for large systems

As mentioned in Section 7.2, any finite element method tends to be implicit due to the presence of the mass matrix M , but that it was changed to a diagonal matrix by so called mass lumping. The idea was based on intuition and was without mathematical justification. However, the numerical results were good and stability improved (as mentioned before). In stead of

$$\delta t^{-2} M[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] = K\bar{u}_k + F(t_k),$$

the system

$$\delta t^{-2} D[\bar{u}_{k+1} - 2\bar{u}_k + \bar{u}_{k-1}] = K\bar{u}_k + F(t_k)$$

is considered. The matrix M is replaced by the diagonal matrix D . The system is now essentially explicit since it is a trivial matter to compute D^{-1} . This explicit central difference method is of course not unconditionally stable and it is necessary to consider stability criteria. However, it has advantages for large systems that arise in engineering applications.

In the introduction to [Wu06] the importance of the explicit method is stressed. “The explicit finite element method has been extensively developed for the transient dynamic analysis to meet the increasing demand of engineering application.” As mentioned above, a diagonal mass matrix is needed

for an explicit method. In Section 3 of the paper, piecewise linear and bilinear basis functions are considered for one-, two- and three-dimensional cases. The equivalent diagonal mass matrices are derived and the errors estimated in Lemma 3.3. Using this result it is proved in Theorem 4.1 that the rate of convergence is the same as for using consistent mass matrices.

The procedure followed by Wu is best explained in [Wu03]. In the abstract it is formulated: “The explicit finite element method for transient dynamics of linear elasticity is formulated by using Galerkin method for space and the central difference method for time.” The fully discrete problem is formulated as follows.

$$\begin{aligned}\bar{u}_n &= \bar{u}_{n-1} + \delta_t \bar{v}_{n-\frac{1}{2}}, \\ F_n &= K \bar{u}_n + F(t_n), \\ \bar{a}_n &= D^{-1} F_n, \\ \bar{v}_{n+\frac{1}{2}} &= \bar{v}_{n-\frac{1}{2}} + \delta_t \bar{a}_n.\end{aligned}$$

The same procedure is used in [Wu05] for the Reissner-Mindlin plate.

To prove convergence the error $e^h = u(t_n) - u_n^h$ is split similar to the semi-discrete case as

$$e^h = e_p + e = \langle \bar{w}, \bar{\beta} \rangle + \langle \hat{w}, \hat{\beta} \rangle.$$

The term on the right is the notation of [Wu05]. Similar to the semi-discrete case the challenge is to estimate e . This is done in Theorem 4.1 [Wu05]. Halfway through the proof a stability assumption (4.11) is introduced. The disadvantage of Wu’s approach to the theory is discussed in the previous section.

Chapter 8

Conclusion

8.1 Galerkin method

Despite the decision to focus on Galerkin's method for this dissertation, the project turned out to be rather ambitious and it was not possible to cover the material envisaged. Numerical integration is important for reasons already mentioned but is not discussed in this dissertation. The articles [BD76], [S02] and [SPC03] were mentioned but a detailed study could be a report on its own. However, enough results have been studied to conclude that the theory is not complete. It may be stated that

1. Except for the theory in [IKS91], the general results are not suitable to cover all linear vibration models;
2. Assumptions in the theory are too restrictive.

The contribution of this dissertation to the theory may be summarized as follows.

1. The proofs in Sections 2.3, 2.4 and 2.5 are from [Z-PhD]. The auxiliary results are formulated in a different way to make the presentation more readable. Also, the assumptions are numbered for easy reference.
2. The notation for errors is standardized in this dissertation to make comparison between different articles easier.

3. The similarities and differences between [Z-PhD, Section 5.3] and the semi-discrete case in [D73] are explained.
4. A generalized version of the assumptions in [D73] and [B76] are formulated in Section 2.4. This makes it possible to consider vibrating structures and not only the wave equation.
5. It is shown in Section 2.6 that the convergence result of [B76] for the time continuous approximation can be generalized. It holds for any linear vibration problem without damping.
6. In the general theory it is often necessary to assume that $(Pu)''$ exists. A sufficient condition is $u \in C^2([0, T], V)$ but it is not necessary and this last assumption is too restrictive. In this regard the experiments in Section 7.3 are illuminating. Further investigation is necessary.
7. In Chapter 6, Theorem 2.1 in [IKS91] is modified and it is shown that it may be applied to the general linear vibration problem.
8. In Chapter 4, the method used in [D73] is applied to the Timoshenko beam with boundary damping.
9. The presentation of the theory in this dissertation provides a basis for future research. It is clear where better theoretical results are desired.

8.2 Mixed finite element methods

To illustrate the basic idea behind the mixed finite element method, we consider the vibration of a membrane. The first equation of motion for a plate is also valid for a membrane; it is Equation (5.1.1):

$$\rho h \partial_t^2 w = \operatorname{div} \mathbf{Q} + q.$$

The constitutive equation for a tightly stretched membrane is $\mathbf{Q} = k \nabla w$ (see [I, Section 6.6]).

For the mixed finite element method a different variational form is used. The constitutive equation is not substituted into the variational equation:

$$(\rho h \partial_t^2 w, v) = (\operatorname{div} \mathbf{Q}, v) + (q, v) \quad (8.2.1)$$

(for each test function v). Instead an additional variational equation

$$(\mathbf{Q}, z)_{0,2} = -(w, \operatorname{div} z), \quad (8.2.2)$$

is used for each $z \in H(\Omega, \operatorname{div})$. This space is the subspace of $\mathcal{L}^2(\Omega)^2$ consisting of vector valued functions z with $\operatorname{div} z \in \mathcal{L}^2(\Omega)$. In the mixed finite element method the aim is to find a pair $\langle w, z \rangle$ that satisfies 8.2.1 and 8.2.2.

In [CDW90] the authors consider the undamped n -dimensional wave equation. It is the problem described above but not restricted to $n = 2$. The advantages of the method are mentioned briefly: displacements and stresses are approximated simultaneously and a higher order approximation of the stresses is obtained.

In [CDW96] Problem D73 without boundary forcing is considered. It may be seen as a follow up article of [CDW90]. Boundary damping (or feedback control) is introduced and error estimates are derived for continuous and fully discrete approximations.

The mixed finite element method is also used for the vibration of a Timoshenko beam in [S94] and [FXX99] (see Chapter 4). In both articles the authors state that their reason for using this method is to overcome locking. The locking effect becomes severe when the dimensionless thickness $d \ll 1$. (Alternatively this happens when the parameter α , defined in section 4.1.1 is extremely large.) In [S94] it is stated that a reduced integration approach is equivalent to the mixed finite element method. Reference to the work of Arnold is given.

8.3 Discontinuous Galerkin method

We briefly considered a recent paper (2006) on the discontinuous Galerkin method. In this article [GSS06]:

“The symmetric interior penalty discontinuous Galerkin finite element method is presented for the numerical discretization of the second-order wave equation.”

In the introduction the authors discuss difficulties encountered with the continuous Galerkin method.

“To avoid these difficulties, we consider instead discontinuous Galerkin (DG) methods. Based on discontinuous finite element spaces, these methods easily handle elements of various types and shapes, irregular nonmatching grids,

and even locally varying polynomial order; thus, they are ideally suited for hp-adaptivity. Here continuity is weakly enforced across mesh interfaces by adding suitable bilinear forms, so-called numerical fluxes, to standard variational formulations. These fluxes are easily included within an existing conforming finite element code.”

Only the undamped wave equation (Problem B76) is considered. Also, error estimates are derived only for the semi-discrete problem - based on the method in [B76]. For the fully discrete problem, the authors refer to [H].

It should be noted that the continuous Galerkin method is still used and in [LH09] the theory is based on [D73] and [B76]. In this dissertation we focussed on the continuous Galerkin method and cannot compare it to the discontinuous Galerkin method.

Appendix A

Sobolev spaces

A.1 The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $\mathbb{R}^n$ and denote its closure by $\bar{\Omega}$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω .

Theorem A.1.1 *The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with inner product*

$$(f, g) = \int_{\Omega} f g = \int_{\Omega} f g \, d\mu,$$

where μ is the n -dimensional Lebesgue measure.

Proof See [Ru, Theorem 3.11, p.69].

Notation Unless otherwise stated the norm of $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|$.

Definition Set $S_f = \text{closure}\{x \in \Omega \mid f(x) \neq 0\}$. Then $C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) \mid S_f \subset \Omega\}$.

Remark

If $f \in C_0^\infty(\Omega)$, then the distance between S_f and the boundary of Ω is positive.

Theorem A.1.2 $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

Proof See [Ad, Theorem 2.13, p.28].

A.2 Sobolev spaces

The one-dimensional case

Suppose Ω is a bounded open interval. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition For f, g in $H^m(\Omega)$,

$$[f, g]_m = (f^{(m)}, g^{(m)}) \quad \text{and} \\ |f|_m = \sqrt{[f, f]_m} \quad \text{for } m = 0, 1, \dots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

Definition Inner product

For f and g in $H^m(\Omega)$, $(f, g)_m = \sum_{k=0}^m (f^{(k)}, g^{(k)})$ for $m = 0, 1, \dots$

The bilinear form $(\cdot, \cdot)_m$ has all the properties of an inner product.

Definition Norm

For f in $H^m(\Omega)$, $\|f\|_m = \sqrt{(f, f)_m}$ for $m = 0, 1, \dots$

The higher dimensional case

Suppose Ω is an open subset of $\mathbb{R}^n$. The Sobolev spaces $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Notation Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, then $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Remark $|\alpha|$ denotes the order of the derivative.

Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^2$.

Definition For f, g in $H^m(\Omega)$,

$$[f, g]_m = \sum_{|\alpha|=m} \partial^\alpha f \partial^\alpha g \quad \text{and} \\ |f|_m = \sqrt{[f, f]_m}.$$

Definition The inner product for $H^m(\Omega)$ is defined by

$$(f, g)_m = \sum_{k=0}^m [f, g]_k \quad \text{for } m = 0, 1, \dots$$

Definition The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f, g)_m} \quad \text{for } m = 0, 1, \dots$$

Sobolev spaces of vector valued functions

Definition

$$u \in \mathcal{L}^2(\Omega)^2 \quad \text{if } u_i \in \mathcal{L}^2(\Omega) \quad \text{for } i = 1, 2.$$

$$u \in \mathcal{L}^2(\Gamma)^2 \quad \text{if } u_i \in \mathcal{L}^2(\Gamma) \quad \text{for } i = 1, 2.$$

$$u \in H^k(\Omega)^2 \quad \text{if } u_i \in H^k(\Omega) \quad \text{for } i = 1, 2.$$

$$[u, v]_{m,2} = [u_1, v_1]_m + [u_2, v_2]_m \quad \text{for } u \in H^m(\Omega)^2 \quad \text{and } v \in H^m(\Omega)^2.$$

$$|u|_{m,2} = \sqrt{[u, u]_{m,2}} \quad \text{for } u \in \mathcal{L}^2(\Omega)^2.$$

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|_{m,2}^\Omega$ and $\|\cdot\|_{m,2}^\Gamma$.

Definition The inner product for $H^m(\Omega)^2$ is defined by

$$(f, g)_{m,2} = \sum_{k=0}^m [f, g]_{k,2} \quad \text{for } m = 0, 1, \dots$$

Definition The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f, g)_{m,2}} \quad \text{for } m = 0, 1, \dots$$

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

A.3 Fundamental properties of Sobolev spaces

Assumption Suppose Ω is a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

Remark

It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Notation $H^0(\Omega) = \mathcal{L}^2(\Omega)$.

Theorem A.3.1 *The space $H^m(\Omega)$ is complete.*

Proof See [Ad, Theorem 3.2, p.45].

Theorem A.3.2 *$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.*

Proof See [OR, Theorem 2.10, p.53].

Remark

A function in $H^m(\Omega)$ can be approximated by a function in $C^m(\bar{\Omega})$: if $u \in H^m(\Omega)$ then for any $\varepsilon > 0$ there exists a $\phi \in C^m(\bar{\Omega})$ such that $\|u - \phi\|_m < \varepsilon$.

Theorem A.3.3 *Sobolev's lemma*

Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > m + n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \leq \|u\|_p \quad \text{for } |\alpha| \leq m.$$

Proof See [OR, Theorem 3.10, p.80].

Remark One-dimensional case

If $n = 1$ in Theorem 3, then $u \in C^{p-1}(\bar{\Omega})$ and $\|u^{(k)}\|_{\sup} \leq \|u\|_p$ for $k \leq m-1$.

A.4 Inequalities

The one-dimensional case

Proposition A.4.1 *Consider any $u \in C^1[0, \ell]$. For any two points x and y in $[0, \ell]$,*

$$|u(x)| \leq \sqrt{\ell} \|u'\| + |u(y)|.$$

Proof For any f and $g \in \mathcal{L}^2(\Omega)$ we have the Cauchy-Schwartz inequality

$$\left(\int_y^x fg \right)^2 \leq \left(\int_y^x f^2 \right) \left(\int_y^x g^2 \right)$$

for $x > y$. By choosing $g = 1$ we find that

$$\left(\int_y^x f \right)^2 \leq \left(\int_y^x f^2 \right) (x - y) \leq \ell \|f\|^2. \quad (\text{A.4.1})$$

Hence $\left| \int_y^x f \right| \leq \sqrt{\ell} \|f\|$ for each $f \in \mathcal{L}^2(0, \ell)$. Since $u(x) - u(y) = \int_y^x u'$,

$$\begin{aligned} |u(x)| &\leq \left| \int_y^x u' \right| + |u(y)| \\ &\leq \sqrt{\ell} \|u'\| + |u(y)|, \end{aligned}$$

from Equation (A.4.1). □

Proposition A.4.2 For any $u \in C^1[0, \ell]$ with a zero in $[0, \ell]$ we have

$$\|u\|_{\sup} \leq \sqrt{\ell} \|u'\|.$$

Proof Suppose $u(y) = 0$, then $|u(x)| \leq \sqrt{\ell} \|u'\|$ by Proposition A.4.1. The result follows from the fact that $\sqrt{\ell} \|u'\|$ is an upper bound for $|u|$. □

Proposition A.4.3 For any $u \in C^1[0, \ell]$ with a zero in $[0, \ell]$ we have

$$\|u\| \leq \ell \|u'\|.$$

Proof We use Proposition A.4.2,

$$\|u\|^2 = \int_0^\ell (u(x))^2 dx \leq \ell \|u\|_{\sup}^2 \leq \ell^2 \|u'\|^2. \quad \square$$

Proposition A.4.4 Let $T[0, \ell] = \{u \in C^1[0, \ell] \mid u(0) = 0\}$. Denote the closure of $T[0, \ell]$ in $H^1(0, \ell)$ by $V(0, \ell)$. For any $u \in V(0, \ell)$,

$$\|u\| \leq \ell \|u'\|.$$

Proof There exists a sequence $\{u_n\} \in C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$. Since $|\|u_n\|_1 - \|u\|_1| \leq \|u_n - u\|_1$, $\|u_n\|_1 \rightarrow \|u\|_1$ as $n \rightarrow \infty$. Consequently $\|u_n\| \rightarrow \|u\|$ and $\|u'_n\| \rightarrow \|u'\|$ as $n \rightarrow \infty$.

The result follows from Proposition A.4.3 by taking the limits. $\square$

Proposition A.4.5 *For any $u \in C^1[0, \ell]$ there exists a constant $K_\ell > 0$ such that*

$$|u(\ell)| \leq K_\ell \|u\|_1.$$

Proof If u has a zero in $[0, \ell]$, then $|u(\ell)| \leq \sqrt{\ell} \|u'\|$ by Proposition A.4.1. If u does not have a zero, suppose $u > 0$ on $[0, \ell]$ and let m be the minimum of u on $[0, \ell]$. Let $w(x) = u(x) - m$, then

$$|u(\ell)| \leq |w(\ell)| + m \leq \sqrt{\ell} \|w'\| + m, \quad (\text{A.4.2})$$

by Proposition A.4.2. But $w' = u'$ and $\|u\|^2 = \int_0^\ell u^2 \geq m^2 \ell$.

Therefore, from Equation (A.4.2),

$$|u(\ell)| \leq \sqrt{\ell} \|u'\| + \frac{1}{\sqrt{\ell}} \|u\|$$

which implies the result. If $u < 0$, then $-u > 0$ and we have the result. $\square$

A.5 Trace

The one-dimensional case

From Theorem A.3.2 we have that for each $u \in H^1(0, \ell)$, there exists a sequence $\{u_n\} \subset C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

Theorem A.5.1 *Suppose $u \in H^1(0, \ell)$. Then, there exists a unique real number u_ℓ such that for any sequence $\{u_n\} \subset C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} u_n(\ell) = u_\ell$.*

Proof Let $\{u_n\}$ be any sequence in $C^1[0, \ell]$ such that $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$. The sequence $\{u_n\}$ is Cauchy in $H^1(0, \ell)$ so by Proposition A.4.5

$\{u_n(\ell)\}$ is Cauchy in $\mathbb{R}$. Thus $\lim_{n \rightarrow \infty} u_n(\ell)$ exists. This limit is independent of the sequence $\{u_n\}$. Indeed, if $\{v_n\}$ is another sequence in $C^1[0, \ell]$ such that $\|v_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$ then

$$\|v_n - u_n\|_1 \leq \|v_n - u\|_1 + \|u - u_n\|_1 \rightarrow 0,$$

as $n \rightarrow \infty$. So by Proposition 2.4.4 we have that $|v_n(\ell) - u_n(\ell)| \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Definition Trace operator Γ_ℓ

For each $u \in H^1(0, \ell)$ let

$$\Gamma_\ell u = u_\ell.$$

Theorem A.5.2 *The mapping Γ_ℓ is linear and bounded. In fact*

$$|\Gamma_\ell u| \leq K_\ell \|u\|_1.$$

Proof The linearity follows from the properties of limits. We have that,

$$\frac{|u_n(\ell)|}{\|u_n\|_1} \leq K_\ell \text{ for all } n,$$

and by considering the limits we have that

$$|\Gamma_\ell u| \leq K_\ell \|u\|_1.$$

This proves that Γ_ℓ is bounded. $\square$

Remark

A trace operator Γ_0 can be defined similarly at zero.

The two-dimensional and three-dimensional cases

Suppose Ω is an open convex subset of $\mathbb{R}_n$ with a piecewise smooth boundary $\partial\Omega$ and $\Sigma \subset \partial\Omega$ is connected. It is then possible to define $\mathcal{L}^2(\Sigma)$.

Theorem A.5.3 *Suppose $u \in H^1(\Omega)$. Then there exists a unique $u_\Sigma \in \mathcal{L}^2(\Sigma)$ such that for any sequence $\{u_n\} \subset C^1(\bar{\Omega})$ where $\|u_n - u\|_1 \rightarrow 0$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} u_n = u_\Sigma$.*

Proof [OR, pp.141-142]

Definition The trace operator Γ
For each $u \in H^1(\Omega)$ let

$$\Gamma u = u_\Sigma.$$

Theorem A.5.4 *The mapping Γ is linear and bounded. In fact*

$$\|\Gamma u\|_\Sigma \leq C_s \|u\|_1.$$

Proof See [OR, pp.141-142]

Appendix B

Interpolation

The results in this appendix are simplified versions of the theory in [OR, Chapter 6], [OC, Chapter 4] and [SF, Chapter 5].

B.1 The one dimensional case

The interval $I = [0, \ell]$ is partitioned into n elements referred to as a finite element mesh. Using this mesh a finite dimensional space S^h is defined. An interpolation operator Π is a transformation from $H^k(I)$ to S^h .

It is first necessary to define interpolation on an element.

Definition Interpolation on the element $I_e = [a, b]$.

Suppose there are n basis functions that are nonzero on the element I_e , denoted by ψ_i . Let $(x_1, x_2, \dots, x_n)$ be an ordered n -tuple of points in I_e – not necessarily distinct. Suppose $(\beta_1, \beta_2, \dots, \beta_n)$ is an ordered n -tuple of nonnegative integers – each less than or equal to m . For a function in $u \in C^m(I_e)$ we define the interpolant Π_e by

$$\Pi_e u = \sum_{i=1}^n D^{\beta_i} u(x_i) \psi_i.$$

Examples

1. Hermite cubic basis functions:

There are four basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = x_3 = a$, $x_2 = x_4 = b$, $\beta_1 = \beta_2 = 0$ and $\beta_3 = \beta_4 = 1$.

$$\begin{aligned}\Pi_e u &= \sum_{i=1}^4 D^{\beta_i} u(x_i) \psi_i \\ &= u(a) \psi_1 + u(b) \psi_2 + u'(a) \psi_3 + u'(b) \psi_4.\end{aligned}$$

2. Piecewise linear basis functions:

There are two basis functions ψ_i that are nonzero on $[a, b]$. Let $x_1 = a$, $x_2 = b$ and $\beta_1 = \beta_2 = 0$.

$$\Pi_e u = \sum_{i=1}^2 D^{\beta_i} u(x_i) \psi_i = u(a) \psi_1 + u(b) \psi_2.$$

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element I_e is denoted by $[f]_{I_e}$.

Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{I_e} = \Pi_e[u]_{I_e} \quad \text{for each element } I_e.$$

Notation

$\mathcal{P}_j(I_e)$: the set of polynomials on I_e of degree at most j is denoted by $\mathcal{P}_j(I_e)$.

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e .

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e .

Remark

If $k \geq s(\Pi_e) + 1$, then the interpolation operator Π_e is defined on $H^k(I_e)$.

Recall that $|\cdot|_k$ denotes the seminorm of order k , i.e.

$$|u|_k = \|u^{(k)}\|.$$

(See Appendix A.)

The interpolation error

Theorem B.1.1 below is formulated as a special case of a general result. This result may be found in [SF, p.144], [OC, p.76] and [OR, p.279].

We will use $\widehat{C}$ to denote a generic constant.

Theorem B.1.1 *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1,$$

for the interpolation operator Π . Then there exists a constant $\widehat{C}$ such that for any $u \in H^k(I)$ we have

$$|\Pi u - u|_{m,I} \leq \widehat{C} h^{k-m} |u|_{k,I} \quad \text{for } m = 0, 1, \dots, k.$$

The interpolation operator is denoted by Π_L for piecewise linear basis functions and by Π_c for Hermite cubics.

Corollary B.1.1 *Hermite cubic basis functions.*

There exists a constant $\widehat{C}_c$ such that if $u \in H^k(I)$ for

a) $2 \leq k \leq 4$, then

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{k-m} |u|_k, \quad m = 0, 1, \dots, k.$$

b) $k > 4$, then

$$\|u - \Pi_c u\|_m \leq \widehat{C}_c h^{4-m} |u|_4, \quad m = 0, 1, \dots, 4.$$

Proof It is clear that $s(\Pi_c) = 1$ and it can be shown that $r(\Pi_c) = 3$. Consequently Theorem B.1.1 is applicable with $k = 2, 3$ or 4 . $\square$

Corollary B.1.2 *Piecewise linear basis functions*

There exists a constant $\widehat{C}_L$ such that if $u \in H^k(0, \ell)$ for $k \geq 2$, then

$$\|\Pi_L u - u\|_1 \leq \widehat{C}_L h |u|_2.$$

Proof It is clear that $s(\Pi_L) = 1$ and it can be shown that $r(\Pi_L) = 1$. Consequently Theorem B.1.1 is applicable with $k = 2$. $\square$

B.2 The two-dimensional case

Recall that the domain Ω is partitioned into n elements. An interpolation operator Π is a transformation from $H^k(\Omega)$ to S^h , the finite dimensional subspace.

It is first necessary to define interpolation on an element.

Definition Interpolation on the element Ω_e .

Suppose there are n basis functions that are nonzero on the element Ω_e , denoted by ψ_i . Let $(x_1, x_2, \dots, x_n)$ be an ordered n -tuple of points in Ω_e – not necessarily distinct. Suppose $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is an ordered n -tuple of multi-indices with $|\alpha_i| \leq m$. For a function in $u \in C^m(\Omega_e)$ we define the interpolant $\Pi_e u$ by

$$\Pi_e u = \sum_{i=1}^n \partial^{\alpha_i} u(x_i) \psi_i.$$

Examples

1. For piecewise linear basis functions on the triangle Ω_e :

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3,$$

where a, b and c are the vertices of the triangle.

2. For piecewise bilinear basis functions on a quadrilateral Ω_e :

$$\Pi_e w = w(a)\psi_1 + w(b)\psi_2 + w(c)\psi_3 + w(d)\psi_4,$$

where a, b, c and d are the vertices of Ω_e .

3. For Hermite cubic basis functions on a triangle Ω_e :

There are 10 basis functions which are not zero on Ω_e . If a, b and c are the vertices of Ω_e and d the centroid, let $x_i = a$ for $i = 1, 2, 3$, $x_i = b$ for $i = 4, 5, 6$, $x_i = c$ for $i = 7, 8, 9$ and $x_{10} = d$. The multi-indices are $\alpha_i = (0, 0)$ for $i = 1, 4, 7, 10$, $\alpha_i = (1, 0)$ for $i = 2, 5, 8$ and $\alpha_i = (0, 1)$ for $i = 3, 6, 9$.

The interpolation operator Π must be defined in such a way that Π_e is the restriction of Π to an element. This is stated precisely in the next definition. For the definition we need the following notation: the restriction of a function f to the element Ω_e is denoted by $[f]_{\Omega_e}$.

Definition. Interpolation

The interpolation operator Π is defined by

$$[\Pi u]_{\Omega_e} = \Pi_e[u]_{\Omega_e} \quad \text{for each element } \Omega_e.$$

Notation

$\mathcal{P}_j(\Omega_e)$: the set of polynomials on Ω_e of degree at most j .

$r(\Pi_e)$: the highest degree of polynomials left invariant by Π_e .

$s(\Pi_e)$: the integer $s(\Pi_e)$ is the highest order derivative used in the definition of Π_e .

Recall that for a two-dimensional domain Ω , $|\cdot|_k$ denotes the seminorm of order k and

$$|u|_k^2 = \sum_{i+j=k} \|\partial_1^i \partial_2^j u\|^2.$$

(See Appendix A.)

Theorem B.2.1 below is formulated as a special case of a general result. This result may be found in [SF, p.144], [OC, p.76] and [OR, p.279]. In the theorem, $h = \max h_e$, where h_e is the diameter of the element Ω_e .

Theorem B.2.1 *Suppose there exists an integer k such that for each element*

$$s(\Pi_e) + 2 \leq k \leq r(\Pi_e) + 1,$$

for the interpolation operator Π . Then there exists a constant $\hat{C}$ such that for any $u \in H^k(\Omega)$ we have

$$|\Pi u - u|_{m,\Omega} \leq \hat{C} h^{k-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, \dots, k.$$

Remark

The constant $\hat{C}$ depends on the shape of the elements in the finite element mesh.

Corollary B.2.1 *Piecewise linear basis functions on triangle elements.*

The interpolation operator is denoted by Π_Δ . If $k \geq 2$, then there exists a constant $\hat{C}$ such that for any $u \in H^k(\Omega)$ we have

$$|\Pi_\Delta u - u|_{m,\Omega} \leq \hat{C} h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

Corollary B.2.2 *Piecewise bilinear basis functions on rectangle elements. The interpolation operator is denoted by Π_b . If $k \geq 2$, then there exists a constant $\hat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_b u - u|_{m,\Omega} \leq \hat{C} h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

Corollary B.2.3 *Hermite cubic basis functions. The interpolation operator is denoted by Π_c . If $k \geq 2$, then there exists a constant $\hat{C}$ such that for any $u \in H^k(\Omega)$ we have*

$$|\Pi_c u - u|_{m,\Omega} \leq \hat{C} h^{2-m} |u|_{k,\Omega} \quad \text{for } m = 0, 1, 2.$$

B.3 Vector-valued functions

If an interpolation operator Π is defined on $H^k(\Omega)$ we may define one on $H^k(\Omega)^2$. For $u = \langle u_1, u_2 \rangle \in H^k(\Omega)^2$, we define

$$\Pi_2 u = \langle \Pi u_1, \Pi u_2 \rangle.$$

The **seminorm** of order k for $H^k(\Omega)^2$ is denoted by $|\cdot|_{k,2}$ and

$$|u|_{k,2}^2 = |u_1|_k^2 + |u_2|_k^2.$$

(See Appendix A.)

Lemma B.3.1 *If $\|\Pi v - v\|_m \leq \hat{C} h^{k-m} |v|_k$ for $v \in H^k(\Omega)$, then*

$$\|u - \Pi_2 u\|_{m,2} \leq \hat{C} h^{k-m} |u|_{k,2} \quad \text{for } u \in H^k(\Omega)^2.$$

Proof

$$|u - \Pi_2 u|_{m,2}^2 = |u_1 - \Pi u_1|_m^2 + |u_2 - \Pi u_2|_m^2.$$

□

For piecewise bilinear basis functions on rectangles, let $\Pi_B u = \langle \Pi_b u_1, \Pi_b u_2 \rangle$.

Corollary B.3.1 *If $k \geq 2$, then there exists a constant $\hat{C}$ such that, for all $u \in H^k(\Omega)^2$*

$$|u - \Pi_B u|_{m,2} \leq \hat{C} h |u|_{2,2} \quad \text{for } m = 0, 1, 2.$$

Proof The result follows from Corollary B.2.2 and Lemma B.3.1. □

Appendix C

Assumptions in Chapter 2

Assumption A1 (Subsection 2.3.1)

If u is a solution of Problem G, then

$$u \in C^1(J; V) \cap C^2(J; W).$$

Assumption A2 (Subsection 2.3.1)

Assume that $u \in C(J, V)$ has the property that $(Pu) \in C^2(J)$.

Assumption A3 (Subsection 2.4.2)

There exists a subspace $H(V, k)$ of V , a positive constant $\hat{C}$ and a positive integer α such that for any $u \in H(V, k)$

$$\inf_{v \in S^h} \|u - v\|_E \leq \hat{C} h^\alpha \|u\|_{H(V, k)},$$

where $\|\cdot\|_{H(V, k)}$ is a norm or semi-norm associated with $H(V, k)$.

Assumption A4 (Subsection 2.4.2)

The solution u of Problem G satisfies

$$u \in C^2(J, V).$$

Assumption A5 (Section 2.5)

There exists a subspace $H(V, k)$ of V , an interpolation operator Π and constants C_Π and α such that for $u \in H(V, k)$:

$$\|u - \Pi u\|_E \leq C_\Pi h^\alpha \|u\|_{H(V, k)}.$$

Assumption A6 (Section 2.5)

$H(V, k)$ is a dense subset of V .

Assumption A7 (Section 2.7)

$E_b \subset H(V, k)$ and for any $u \in E_b$ we have $\|u\|_{H(V, k)} \leq \hat{c}_b \|y\|_W$ where

$$b(u, v) = (y, v) \quad \forall v \in V.$$

Appendix D

Summary of abstract spaces

The properties of the spaces V , W and X are of critical importance in the theory. For convenience we present a summary here.

Note that $V \subset W \subset X$.

Space	Inner product	Norm
Energy space V	$b(\cdot, \cdot)$	Energy norm $\ \cdot\ _E$
Inertia space W	$c(\cdot, \cdot)$	Inertia norm $\ \cdot\ _W$
X	$(\cdot, \cdot)_X$	$\ \cdot\ _X$

Estimates

There exist constants C_b and C_c such that

$$\begin{aligned} \|u\|_W &\leq C_b \|u\|_E \text{ for all } u \in V, \\ \|u\|_X &\leq C_c \|u\|_W \text{ for all } u \in W. \end{aligned}$$

Topological property

V is dense in W with respect to the inertia norm.

Appendix E

Gronwall and Young's inequalities

Proposition E.0.1 *If there exists a constant $c > 0$ such that $\phi' \leq c\phi$, then*

$$\phi(t) \leq e^{ct} \phi(0), \quad \text{for } t > 0.$$

Proof

$$\frac{d}{dt}(\phi(t) e^{-ct}) = e^{-ct} (\phi'(t) - c\phi(t)) \leq 0.$$

Therefore $\phi(t) e^{-ct} - \phi(0) \leq 0$ and the result follows. $\square$

Lemma E.0.2 *Gronwall's inequality*

Suppose $\psi \in C^1[0, T]$. If there exist positive constants c and K such that

$$\psi(t) \leq c \int_0^t \psi + K,$$

then

$$\psi(t) \leq K e^{ct}, \quad \text{for each } t \in [0, T].$$

Proof Let $\phi(t) = c \int_0^t \psi + K$, then $\phi' = c\psi$ and $\phi(0) = K$. Since $\phi' \leq c\phi$, we have

$$\phi(t) \leq e^{ct} K \quad \text{from Proposition E.0.1.}$$

The result follows from the fact that $\psi \leq \phi$. $\square$

The following simple inequality is very useful.

Lemma E.0.3 *Young's inequality*

$$ab \leq \frac{1}{2} (\varepsilon^2 a^2 + \varepsilon^{-2} b^2).$$

Proof $(\varepsilon a - \varepsilon^{-1} b)^2 \geq 0$ implies that

$$\varepsilon^2 a^2 + \varepsilon^{-2} b^2 \geq 2ab.$$

□

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