ELEMENTARY EQUIVALENCE IN POSITIVE LOGIC VIA PRIME PRODUCTS

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ABSTRACT. We introduce *prime products* as a generalization of ultraproducts for positive logic. Prime products are shown to satisfy a version of Łoś's Theorem restricted to positive formulas, as well as the following variant of the Keisler Isomorphism Theorem: under the generalized continuum hypothesis, two models have the same positive theory if and only if they have isomorphic prime powers of ultrapowers.

1. INTRODUCTION

A map $f\colon M \to N$ between two structures M and N is said to be a *homomorphism* when for every atomic formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$,

$$M \vDash \varphi(a_1, \dots, a_n) \text{ implies } N \vDash \varphi(f(a_1), \dots, f(a_n)).$$

Positive model theory is the branch of model theory that deals with the formulas that are preserved by homomorphisms (see, e.g., [9, 10, 11, 14, 15, 16]). It is well known that these are precisely the *positive formulas*, that is, the formulas built from atomic formulas and $\bot$ using only $\exists$, $\wedge$, and $\vee$.

The Keisler-Shelah Isomorphism Theorem states that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers. This celebrated result was first proved by Keisler under the generalized continuum hypothesis (GCH) [5, Thm. 2.4]. This assumption was later shown to be redundant by Shelah [12, p. 244]. The aim of this paper is to prove a version of Keisler's original theorem in the context of positive model theory.

To this end, we say that two structures are *positively equivalent* when they have the same positive theory. In order to obtain a positive version of Keisler Isomorphism Theorem, we will introduce a generalization of the ultraproduct construction that captures positive equivalence. We term this construction a *prime product* because it is obtained by replacing the index set I typical of an ultraproduct by a poset $\mathbb{X}$ and the ultrafilter over I by a *prime* filter of the bounded distributive lattice of upsets of the poset $\mathbb{X}$. The case of traditional ultraproducts is then recovered by requiring the order of $\mathbb{X}$ to be the identity relation.

Prime products and positive formulas are connected by the natural incarnation of Łos Theorem in this context (Theorem 2.10). As a consequence, prime products preserve not only positive formulas, but also the universal closure of the implications between them, known as *basic h-inductive sentences* [11] (Proposition 2.13). This allows us to describe the classes of models of h-inductive theories as those closed under isomorphisms, prime products, and ultraroots (Corollary 2.16).

Our main result states that under GCH two structures have the same *positive theory* if and only if they have isomorphic *prime powers* of ultrapowers (Theorem 3.2). The same

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result holds without GCH, provided that prime powers are replaced by *prime products* in the statement (Theorem 3.10). Notably, the presence of ultrapowers cannot be removed from this theorems, as there exists positively equivalent structures without isomorphic prime powers (Example 3.5).

2. Prime products

A subset V of a poset $\mathbb{X} = \langle X; \leqslant \rangle$ is said to be an *upset* when for every $x, y \in X$,

$$\text{if } x \in V \text{ and } x \leqslant y, \text{ then } y \in V.$$

The *downsets* of $\mathbb{X}$ are defined dually. An upset V of $\mathbb{X}$ is *proper* when it differs from X, and it is *principal* when it coincides with

$$\uparrow x := \{y \in X \mid x \leqslant y\},$$

for some $x \in X$. When ordered under the inclusion relation, the family $\mathsf{Up}(\mathbb{X})$ of upsets of $\mathbb{X}$ forms a bounded distributive lattice

$$\mathsf{Up}(\mathbb{X}) := \langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \emptyset, X \rangle.$$

Definition 2.1. A *filter* over a poset $\mathbb{X}$ is a nonempty upset of the lattice $\mathsf{Up}(\mathbb{X})$ which, moreover, is closed under binary intersections. In this case, F is said to be *prime* when it is proper and for every $V, W \in \mathsf{Up}(\mathbb{X})$,

$$V \cup W \in F \text{ implies that either } V \in F \text{ or } W \in F.$$

Remark 2.2. Given a set X, we denote the poset whose universe is X and whose order is the identity relation by $\mathsf{id}(X)$. In this case, $\mathsf{Up}(\mathsf{id}(X)) = \mathcal{P}(X)$. Furthermore, the filters (resp. prime filters) over $\mathsf{id}(X)$ coincide with the filters (resp. ultrafilters) over the set X. ⊠

An *ordered system* (of structures) comprises a nonempty family $\{M_x \mid x \in X\}$ of similar structures indexed by a poset $\mathbb{X}$ and a family of homomorphisms $\{f_{xy} \colon M_x \to M_y \mid x, y \in X \text{ and } x \leqslant y\}$ such that f_{xx} is the identity map on M_x and for every $x, y, z \in X$,

$$x \leqslant y \leqslant z \text{ implies } f_{xz} = f_{yz} \circ f_{xy}.$$

A poset is said to be a *wellfounded forest* when its principal downsets are well ordered.

We will associate a new structure with every ordered system $\{M_x \mid x \in X\}$ indexed by a wellfounded forest $\mathbb{X}$ and every filter F over $\mathbb{X}$ as follows. First, for every $V \in F$ let

$$S_V := \{a \in \prod_{x \in V} M_x \mid \text{for every } y, z \in V,\ y \leqslant z \text{ implies } f_{yz}(a(y)) = a(z)\}$$

and consider the union

$$S_F := \bigcup_{V \in F} S_V.$$

Then for every $a \in S_F$ let V_a be the domain of a, that is,

$$V_a := \text{the unique } V \in F \text{ such that } a \in S_V.$$

It will often be convenient to restrict the sequence a to some $V \in F$ such that $V \subseteq V_a$ as follows:

$$a{\restriction}_V := \langle a(x) \mid x \in V \rangle.$$

Notice that from $a \in S_F$ it follows that $a_{\restriction_V} \in S_V$. Lastly, for every formula $\varphi(v_1, \dots, v_n)$ and $a_1, \dots, a_n \in S_F$ let

$$[\![\varphi(a_1, \dots, a_n)]\!] := \{x \in X \mid x \in V_{a_1} \cap \cdots \cap V_{a_n} \text{ and } M_x \vDash \varphi(a_1(x), \dots, a_n(x))\}.$$

We define an equivalence relation on S_F as follows: for every $a, b \in S_F$,

$$a \equiv_F b \text{ if and only if } [\![a = b]\!] \in F.$$

The proof of the following observation is a routine exercise.

Proposition 2.3. *Let g be a basic n-ary operation, R a basic n-ary relation, $a_1, \ldots, a_n, b_1, \ldots, b_n \in S_F$, and*

$$V_a := V_{a_1} \cap \cdots \cap V_{a_n} \text{ and } V_b := V_{b_1} \cap \cdots \cap V_{b_n}.$$

Then $g^{\prod_{x \in V_a} M_x}(a_1{\restriction}_{V_a}, \ldots, a_n{\restriction}_{V_a}), g^{\prod_{x \in V_b} M_x}(b_1{\restriction}_{V_b}, \ldots, b_n{\restriction}_{V_b}) \in S_F$. Moreover, if $a_m \equiv_F b_m$ for every $m \leqslant n$, then

$$[\![g^{\prod_{x \in V_a} M_x}(a_1{\restriction}_{V_a}, \ldots, a_n{\restriction}_{V_a}) = g^{\prod_{x \in V_b} M_x}(b_1{\restriction}_{V_b}, \ldots, b_n{\restriction}_{V_b})]\!] \in F$$

and

$$[\![R(a_1, \ldots, a_n)]\!] \in F \text{ if and only if } [\![R(b_1, \ldots, b_n)]\!] \in F.$$

In view of Proposition 2.3, the following structure is well defined:

Definition 2.4. The *filter product* $\prod_{x \in X} M_x / F$ is the structure with universe $S_F / {\equiv_F}$ where

(i) the basic n-ary operations g are defined as

$$g(a_1/{\equiv_F}, \ldots, a_n/{\equiv_F}) := g^{\prod_{x \in V_a} M_x}(a_1{\restriction}_{V_a}, \ldots, a_n{\restriction}_{V_a})/{\equiv_F}$$

for $V_a := V_{a_1} \cap \cdots \cap V_{a_n}$;

(ii) the basic n-ary relations R are defined as

$$\langle a_1/{\equiv_F}, \ldots, a_n/{\equiv_F} \rangle \in R \Longleftrightarrow [\![R(a_1, \ldots, a_n)]\!] \in F.$$

When each M_x is the same structure M, we say that M^X/F is a *filter power* of M.

Typical examples of filter products include reduced products.

Example 2.5 (Reduced products)**.** Recall from Remark 2.2 that the filters over a set X coincide with the filters over the poset $\mathsf{id}(X)$. We will show that the reduced product M of a family $Y = \{M_x \mid x \in X\}$ of similar structures induced by a filter F over X is isomorphic to the filter product of Y, viewed as an ordered system indexed by the wellfounded forest $\mathsf{id}(X)$, induced by the same filter F over $\mathsf{id}(X)$.

To this end, let $\mathsf{D}(Y, F)$ be the ordered system comprising the family of structures $\{\prod_{x \in V} M_x \mid V \in F\}$ indexed by the poset $\langle F; \supseteq \rangle$ and the canonical projections $f_{V,W} \colon \prod_{x \in V} M_x \to \prod_{x \in W} M_x$ for each $V, W \in F$ with $W \subseteq V$. As F is closed under binary intersections, $\mathsf{D}(Y, F)$ is a direct system. Furthermore, its direct limit coincides with the filter product of Y induced by F. This is because Y is indexed by $\mathsf{id}(X)$ and, therefore,

$$S_F = \bigcup \{ \prod_{x \in V} M_x \mid V \in F \}$$

and for every $a, b \in S_F$,

$$a \equiv_F b \text{ if and only if } f_{V_a, V_a \cap V_b}(a) = f_{V_b, V_a \cap V_b}(b).$$

As the direct limit of $\mathsf{D}(Y, F)$ is isomorphic to the reduced product of Y induced by F (see, e.g., [3, p. 109]), we conclude that so is the filter product of Y induced by F. ⊠

The following construction plays the role of an ultraproduct in positive model theory.

Definition 2.6. A filter product $\prod_{x \in X} M_x / F$ is said to be a *prime product* when F is prime. If, in addition, each M_x is the same structure M, we say that M^X/F is a *prime power* of M.

Example 2.7 (Ultraproduct)**.** Recall from Remark 2.2 that the ultrafilters over a set X coincide with the prime filters over the poset $\mathsf{id}(X)$. Therefore, Example 2.5 shows that the ultraproduct of a family $Y = \{M_x \mid x \in X\}$ of similar structures induced by an ultrafilter U over X is isomorphic to the prime product of Y, viewed as an ordered system indexed by $\mathsf{id}(X)$, induced by the prime filter U over $\mathsf{id}(X)$. ⊠

Let $\{M_x \mid x \in X\}$ be an ordered system indexed by a linearly ordered poset $\mathbb{X}$. It is easy to construct a nonempty well ordered subposet $\mathbb{Y}$ of $\mathbb{X}$ that is cofinal in $\mathbb{X}$, i.e., such that

$$\text{for every } x \in X \text{ there exists } y \in Y \text{ such that } x \leqslant y.$$

Furthermore, when viewed as an ordered system, $\{M_y \mid y \in Y\}$ has the same direct limit as $\{M_x \mid x \in X\}$. Because of this, when discussing ordered systems indexed by a linearly ordered poset $\mathbb{X}$, we will restrict our attention to the case where $\mathbb{X}$ is well ordered. Accordingly, by a *chain of structures* we understand an ordered system $\{M_x \mid x \in X\}$ indexed by a well ordered poset $\mathbb{X}$. A simple example of a prime product is the direct limit of a chain of structures, as we proceed to illustrate.

Example 2.8 (Limits of chains)**.** Let $\{M_x \mid x \in X\}$ be a chain of structures indexed by $\mathbb{X}$. Since $\mathbb{X}$ is nonempty and linearly ordered, $F := \{\uparrow x \mid x \in X\}$ is a filter over $\mathbb{X}$. Moreover, the direct limit of $\{M_x \mid x \in X\}$ is isomorphic to the prime product $\prod_{x \in X} M_x / F$. ⊠

For every class K of similar structures let

$\mathbb{P}_{\mathrm{R}}(\mathsf{K}) :=$ the class of structures isomorphic to a reduced product of members of K;

$\mathbb{P}_{\mathrm{U}}(\mathsf{K}) :=$ the class of structures isomorphic to a ultraproduct of members of K;

$\mathbb{P}_{\mathrm{F}}(\mathsf{K}) :=$ the class of structures isomorphic to a filter product of members of K;

$\mathbb{P}_{\mathrm{P}}(\mathsf{K}) :=$ the class of structures isomorphic to a prime product of members of K;

$\mathbb{L}_{\mathrm{C}}(\mathsf{K}) :=$ the class of structures isomorphic to the direct limit of a chain of members of K.

The proof of the following technical observation is contained in the Appendix.

Proposition 2.9. *For every class* K *of similar structures,*

$$\mathbb{P}_{\mathrm{R}}\mathbb{L}_{\mathrm{C}}(\mathsf{K}) \subseteq \mathbb{P}_{\mathrm{F}}(\mathsf{K}) \ \text{ and } \ \mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}(\mathsf{K}) \subseteq \mathbb{P}_{\mathrm{P}}(\mathsf{K}).$$

The importance of prime products derives from the following observation:

Positive Łos Theorem 2.10. *Let* $\prod_{x \in X} M_x / F$ *be a prime product. For every positive formula* $\varphi(v_1, \dots, v_n)$ *and* $a_1, \dots, a_n \in S_F$*,*

$$\prod_{x \in X} M_x / F \vDash \varphi(a_1/{\equiv_F}, \dots, a_n/{\equiv_F}) \ \text{ if and only if } \ [\![\varphi(a_1, \dots, a_n)]\!] \in F.$$

Consequently, a positive sentence holds in a structure M *if and only if it holds in some (equiv. every) prime power of* M*.*

Proof. We recall that positive formulas are preserved by homomorphisms. Therefore, the assumption that $a_1, \dots, a_n \in S_F$ guarantees that $[\![\psi(a_1, \dots, a_n)]\!]$ is an upset of $\mathbb{X}$, for every positive formula ψ. We will use this fact without further notice.

We reason by induction on the construction of φ. In the base case, φ is an atomic formula and the result holds by the definition of a prime product. The case where $\varphi = \psi_1 \wedge \psi_2$ follows from the inductive hypothesis and the fact that F is a filter over $\mathbb{X}$. The case where $\varphi = \psi_1 \vee \psi_2$ follows from the inductive hypothesis and the fact that F is prime. It only remains to consider the case where $\varphi = \exists w \psi(w, v_1, \ldots, v_n)$. By the induction hypothesis we have

$$\prod_{x\in X} M_x/F \vDash \varphi(a_1/{\equiv_F}, \ldots, a_n/{\equiv_F}) \ \text{ if and only if } \ \text{there exists } b \in S_F \text{ s.t. } [\![\psi(b, a_1, \ldots, a_n)]\!] \in F.$$

Therefore, it only remains to prove that

$$[\![\exists w \psi(w, a_1, \ldots, a_n)]\!] \in F \ \text{ if and only if } \ \text{there exists } b \in S_F \text{ s.t. } [\![\psi(b, a_1, \ldots, a_n)]\!] \in F.$$

For the sake of readability, we will write $E := [\![\exists w \psi(w, a_1, \ldots, a_n)]\!]$. Since E is an upset of $\mathbb{X}$ and $[\![\psi(b, a_1, \ldots, a_n)]\!] \subseteq E$ for every $b \in S_F$, the implication from right to left in the above display is straightforward. To prove the other implication, suppose that $E \in F$ and let Y be the set of minimal elements of E. For every $y \in Y \subseteq E$ there exists $b(y) \in M_y$ such that

$$M_y \vDash \psi(b(y), a_1(y), \ldots, a_n(y)). \tag{1}$$

As $\mathbb{X}$ is a wellfounded forest, for each $x \in E$ there exists a unique $y_x \in Y$ such that $y_x \leqslant x$. Therefore, we can define an element $b \in \prod_{x\in E} M_x$ as

$$b(x) := f_{y_x x}(b(y_x))$$

for each $x \in E$. As $E \in F$, we have $b \in S_E \subseteq S_F$. Furthermore, from Condition (1) and the fact that positive formulas are preserved by homomorphisms it follows

$$M_x \vDash \psi(b(x), a_1(x), \ldots, a_n(x)) \text{ for every } x \in E.$$

Consequently, $E \subseteq [\![\psi(b, a_1, \ldots, a_n)]\!]$. Since $E \in F$ and $[\![\psi(b, a_1, \ldots, a_n)]\!]$ is an upset of $\mathbb{X}$, we conclude that $[\![\psi(b, a_1, \ldots, a_n)]\!] \in F$ as desired. ⊠

Remark 2.11. The proof of the case of the existential quantifier in the Positive Łos Theorem reveals why prime products have been defined for systems of structures indexed by wellfounded forests (as opposed to arbitrary posets).

We remark that the assumption that forests are wellfounded is necessary, as shown by the following example. For each integer n, let M_n be the structure with universe $\mathbb{Z}$ whose language consists of a unary predicate P interpreted as $\{m \in \mathbb{Z} \mid m \leqslant n\}$. Let also $\{M_n \mid n \in \mathbb{Z}\}$ be the ordered system indexed by $\mathbb{Z}$ whose homomorphisms the identity function $i\colon \mathbb{Z} \to \mathbb{Z}$. Furthermore, consider the prime filter $F := \{\mathbb{Z}\}$ over $\mathbb{Z}$. Lastly, let $\prod_{n\in\mathbb{Z}} M_n/F$ be the structure obtained from these ingredients using the instructions in the definition of a filter product. We have

$$[\![\exists v P(v)]\!] = \mathbb{Z} \in F,$$

while the interpretation of P in $\prod_{n\in\mathbb{Z}} M_n/F$ is $\emptyset$ and, therefore, $\prod_{n\in\mathbb{Z}} M_n/F \nvDash \exists v P(v)$. ⊠

Since the notion of consequence is central to logic, the implications between positive formulas play also a fundamental role in positive model theory [11].

Definition 2.12. A formula φ is said to be

(i) *basic h-inductive* when there are two positive formulas ψ_1 and ψ_2 such that

$$\varphi = \forall v_1, \ldots, v_n(\psi_1 \to \psi_2);$$

(ii) *h-inductive* when it is a conjunction of basic h-inductive formulas.

A set of h-inductive sentences will be called an *h-inductive theory*.

As a consequence of the Positive Łos Theorem, we obtain the following:

Proposition 2.13. *H-inductive sentences persist in prime products: if φ is an h-inductive sentence and $\prod_{x\in X} M_x/F$ a prime product such that $M_x \vDash \varphi$ for every $x \in X$, then $\prod_{x\in X} M_x/F \vDash \varphi$.*

Proof. It suffices to prove the statement for the case where φ is basic h-inductive. Then there are two positive formulas $\psi_1(v_1, \ldots, v_n)$ and $\psi_2(v_1, \ldots, v_n)$ such that $\varphi = \forall v_1, \ldots, v_n (\psi_1 \to \psi_2)$. We will reason by contraposition. Suppose that $\prod_{x \in X} M_x / F \nvDash \varphi$. By the Positive Łos Theorem there are $a_1, \ldots, a_n \in S_F$ such that

$$\llbracket \psi_1(a_1, \ldots, a_n) \rrbracket \in F \text{ and } \llbracket \psi_2(a_1, \ldots, a_n) \rrbracket \notin F.$$

Since $\llbracket \psi_2(a_1, \ldots, a_n) \rrbracket$ is an upset of $\mathbb{X}$, this means that

$$\llbracket \psi_1(a_1, \ldots, a_n) \rrbracket \nsubseteq \llbracket \psi_2(a_1, \ldots, a_n) \rrbracket.$$

Consequently, there exists $x \in V_{a_1} \cap \cdots \cap V_{a_n}$ such that

$$M_x \vDash \psi_1(a_1(x), \ldots, a_n(x)) \wedge \neg \psi_2(a_1(x), \ldots, a_n(x)). \qquad \boxtimes$$

Definition 2.14. A class of similar structures is said to be *h-inductive* when it is closed under direct limits of chains of structures.

H-inductive theories and classes are related as follows (see, e.g., [11, p. 108]):

Theorem 2.15. *The class of models of an h-inductive theory is h-inductive. Conversely, every elementary h-inductive class is axiomatized by an h-inductive theory.*

Notably, h-inductive elementaty classes can also be characterized in terms of prime products.

Corollary 2.16. *A class of similar structures is elementary and h-inductive if and only if it is closed under isomorphisms, prime products, and ultraroots.*

Proof. We recall that a class of similar structures is elementary if and only if it is closed under isomorphisms, ultraproducts, and ultraroots [4, Thm 2.13]. Furthermore, such a class is h-inductive if and only if it is closed under direct limits of chains of structures. Since ultraproducts and direct limits of chains of structures are special cases of prime products (see Examples 2.5 and 2.8), it follows that every class of similar closed under isomorphisms, prime products, and ultraroots is h-inductive and elementary. Conversely, every h-inductive elementary class is closed under isomorphisms and ultraroots because it is elementary and under prime products by Proposition 2.13. $\boxtimes$

3. Positive equivalence

Definition 3.1. The *positive theory* of a structure M is the set of positive sentences valid in M. Two structures are *positively equivalent* when they have the same positive theory.

The Keisler Isomorphism Theorem states that, under the CGH, two structures are elementarily equivalent if and only if they have isomorphic ultrapowers [5, Thm. 2.4]. As shown by Shelah, the result holds also without GCH [12, p. 244]. The aim of this section is to establish the following:[1]

Positive Keisler Isomorphism Theorem 3.2. *Under* GCH*, two structures are positively equivalent if and only if they have isomorphic prime powers of ultrapowers.*

Before proving this result, we shall explain why it appears to be more complicated than the original Keisler Isomorphism Theorem. More precisely, the next examples show that

(i) A prime power of an ultrapower of a structure M need not be isomorphic to any filter power of M;
(ii) Two positively equivalent structures need not have isomorphic prime powers.

[1] It is an open problem whether one can dispense with GCH in the Positive Keisler Isomorphism Theorem too.

Example 3.3. Let $\mathbb{Q}$ be the poset of rational numbers with a constant for each element. Moreover, let $\mathbb{Q}_u$ be an ultrapower of $\mathbb{Q}$ containing an element m such that $q < m$ for every $q \in \mathbb{Q}$. Then consider the endomorphism $f \colon \mathbb{Q}_u \to \mathbb{Q}_u$ defined by the rule

$$f(p) = \begin{cases} m & \text{if } q < p \text{ for every } q \in \mathbb{Q}; \\ p & \text{otherwise.} \end{cases}$$

Lastly, let $\mathbb{Q}^*$ be the direct limit of the chain of structures

$$\mathbb{Q}_u \xrightarrow{f} \mathbb{Q}_u \xrightarrow{f} \mathbb{Q}_u \xrightarrow{f} \cdots$$

Recall from Example 2.8 that direct limits of chains of structures are prime powers. Therefore, $\mathbb{Q}^*$ is a prime power of an ultrapower of $\mathbb{Q}$ by construction. We will prove that $\mathbb{Q}^*$ is not isomorphic to any filter power of $\mathbb{Q}$. To this end, observe that $\mathbb{Q}^*$ has a greatest element, namely, the image of m in the direct limit. Therefore, it suffices to show that every nontrivial filter power of $\mathbb{Q}$ lacks a greatest element. Consider a nontrivial filter power $\mathbb{Q}^X/F$ of $\mathbb{Q}$ and let $a \in S_F$. We need to find $b \in S_F$ such that $a/{\equiv_F} < b/{\equiv_F}$. Let M be the set of minimal elements of V_a and for each $x \in M$ let $q_x \in \mathbb{Q}$ be such that $a(x) < q_x$. Since $\mathbb{X}$ is a wellfounded forest, for each $y \in V_a$ there exists a unique $x_y \in M$ such that $x_y \leqslant y$. Because of this, the unique element $b \in \prod_{y \in V_a} M_y$ defined for every $y \in V_a$ as $b(y) := f_{x_y y}(q_{x_y})$ belongs to S_{V_a}. Observe that the only endomorphism of $\mathbb{Q}$ is the identity function $i \colon \mathbb{Q} \to \mathbb{Q}$ because the language contains a constant for each element of $\mathbb{Q}$. Then for every $y \in V_a$,

$$a(y) = f_{x_y y}(a(x_y)) = i(a(x_y)) = a(x_y) < q_{x_y} = i(q_{x_y}) = f_{x_y y}(q_{x_y}) = b(y).$$

Therefore,

$$[\![a = b]\!] = \emptyset \notin F \quad \text{and} \quad [\![a \leqslant b]\!] = V_a \in F,$$

where $\emptyset \notin F$ because $\mathbb{Q}^X/F$ is nontrivial. By the definition of a filter power we conclude that $a/{\equiv_F} < b/{\equiv_F}$. ⊠

The next example relies on the following observation (see, e.g., [8, Thm. 3.1]):

Proposition 3.4. *A structure M satisfies the positive theory of a structure N if and only if there exist an ultrapower M_u of M and a homomorphism $f \colon N \to M_u$.*

Example 3.5. We will show that two positively equivalent structures need not have isomorphic prime powers. To this end, let $\mathbb{Q}$ be the structure defined in Example 3.3 and $\mathbb{Q}^+$ the structure obtained by adding a greatest element m to $\mathbb{Q}$. Clearly, $\mathbb{Q}$ is a substructure of $\mathbb{Q}^+$. Furthermore, $\mathbb{Q}^+$ is a substructure of the ultrapower $\mathbb{Q}_u$ of $\mathbb{Q}$ considered in Example 3.3. Therefore, $\mathbb{Q}$ and $\mathbb{Q}^+$ are positively equivalent by Proposition 3.4.

It only remains to prove that $\mathbb{Q}$ and $\mathbb{Q}^+$ do not have isomorphic prime powers. On the one hand, every filter power of $\mathbb{Q}$ induced by a proper filter (and, therefore, every prime power of $\mathbb{Q}$) lacks a greatest element, as shown in Example 3.3. On the other hand, every prime power $\mathbb{Q}^{+X}/F$ of $\mathbb{Q}^+$ has a greatest element, as we proceed to explain. Since the only endomorphism of $\mathbb{Q}^+$ is the identity function, the constant function $\hat{m} \colon X \to \mathbb{Q}^+$ with value m is an element of $S_X \subseteq S_F$. Furthermore, for every $a \in S_F$ we have $[\![a \leqslant \hat{m}]\!] = V_a \in F$. By the definition of a filter product, we conclude that $a/{\equiv_F} \leqslant \hat{m}/{\equiv_F}$. Hence, $\hat{m}/{\equiv_F}$ is the greatest element of $\mathbb{Q}^{+X}/F$ as desired. ⊠

The proof of the Positive Keisler Isomorphism Theorem relies on the next concept:

Definition 3.6. A structure M is said to be

(i) *positively κ-saturated* for a cardinal κ when for every $\vec{a} \in M^{\lambda}$ with $\lambda < \kappa$ and every set of positive formulas $p(x_1, \dots, x_n)$ with parameters in $\vec{a}$,

$$\text{if } p \text{ is finitely satisfiable in } M, \text{ then it is realized in } M;$$

(ii) *positively saturated* when it is positively $|M|$-saturated.

While every saturated model is positively saturated, the converse need not hold in general (for instance, when viewed as a poset, the extended real number line is positively saturated, but not saturated).

The proof of the next observation is a straightforward adaptation of the standard argument showing that saturated models are universal (see, e.g., [2, Thm. 5.1.14]).

Proposition 3.7. *Let M be a positively saturated structure. If M satisfies the positive theory of a structure N with $|N| \leqslant |M|$, there exists a homomorphism $f\colon N \to M$.*

We will also make use of the following result on classical saturation.

Theorem 3.8. *The following hold for a structure M:*

(i) *For every cardinal κ there exists a κ-saturated ultrapower of M;*
(ii) *Under GCH, if M is infinite, it has arbitrarily large saturated ultrapowers.*

Proof. Condition (i) follows from [6, Thm. 2.1] and [7, Thm. 3.2]. For Condition (ii), see the proof of [6, Cor. 2.3]. ⊠

Corollary 3.9. *For every cardinal κ, if M_u is an ultrapower of M and $f\colon N \to M_u$ a homomorphism, there exists a κ-saturated ultrapower M^* of M with a homomorphism $g\colon N \to M^*$.*

Proof. By Theorem 3.8(i) there exists a κ-saturated ultrapower M^* of M_u. As M_u is an ultrapower of M and ultrapowers of ultrapowers are still ultrapowers, we may assume that M^* is an ultrapower of M. Furthermore, as M_u embeds into M^*, we can view f as a homomorphism from N to M^*. ⊠

The Positive Keisler Isomorphism Theorem is a consequence of the next observation:

Theorem 3.10. *Two structures M_1 and M_2 are positively equivalent if and only if there exists*

$$N \in \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_1) \cap \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_2).$$

In addition, if each M_i is positively saturated and either finite or of size $\geqslant |L|$, we can take

$$N \in \mathbb{P}_{\mathrm{P}}(M_1) \cap \mathbb{P}_{\mathrm{P}}(M_2).$$

The next proof shows how to derive the Positive Keisler Isomorphism Theorem from the above result.

Proof. Consider two similar structures M_1 and M_2. If M_1 and M_2 have isomorphic prime powers of ultrapowers, then they are positively equivalent by the classical Łos Theorem and its positive version. Conversely, suppose that M_1 and M_2 are positively equivalent. By Theorem 3.8, under GCH each M_i has a saturated ultrapower M_i^*. Furthermore, by the same theorem M_i^* can be assumed to be either finite (if M_i is finite) or of size $\geqslant |L|$ (if M_i is infinite). Since M_1^* and M_2^* are also positively equivalent, we can apply Theorem 3.10 obtaining that there exists $N \in \mathbb{P}_{\mathrm{P}}(M_1) \cap \mathbb{P}_{\mathrm{P}}(M_2)$. ⊠

The rest of the paper is devoted to proving Theorem 3.10. To this end, we recall that a map $f\colon M \to N$ between two structures M and N is an *immersion* when for every positive formula $\varphi(v_1, \dots, v_n)$ and $a_1, \dots, a_n \in M$,

$$M \vDash \varphi(a_1, \dots, a_n) \text{ if and only if } N \vDash \varphi(f(a_1), \dots, f(a_n)).$$

Definition 3.11. A model M of a theory T is said to be *positively existentially closed* (*pec*, for short) when every homomorphism from M to a model of T is an immersion.

We rely on the following description of pec models:

Definition 3.12. Let $\varphi(\vec{v})$ be a positive formula and T an h-inductive theory. The *resultant* $\mathrm{Res}_T(\varphi)$ of φ over T is the set of positive formulas $\psi(\vec{v})$ such that $T \vdash \neg\exists\vec{v}(\varphi \wedge \psi)$.

Proposition 3.13 ([16, Lem. 14])**.** *A model M of an h-inductive theory T is pec if and only if for each positive formula $\varphi(\vec{v})$ and $\vec{a} \in M$ such that $M \nvDash \varphi(\vec{a})$ there exists $\psi \in \mathrm{Res}_T(\varphi)$ such that $M \vDash \psi(\vec{a})$.*

The next two results are instrumental in constructing pec models.

Proposition 3.14 ([16, Thm. 1, Lem. 12])**.** *The following holds for an h-inductive theory T:*

(i) *For every model M of T there exists a pec model N of T with a homomorphism $f\colon M \to N$;*
(ii) *The class of pec models of T is h-inductive.*

Proposition 3.15. *Let M be a pec model of an h-inductive theory T and $f\colon N \to M$ an immersion. Then, N is also a pec model of T.*

Proof. Since the theory T is h-inductive, from the assumption that f is an immersion and that $M \vDash T$ it follows that $N \vDash T$. To prove that N is pec, we will use Proposition 3.13. Consider a positive formula $\varphi(\vec{v})$ and $\vec{a} \in N$ such that $N \nvDash \varphi(\vec{a})$. Since f is an immersion and φ positive, we obtain $M \nvDash \varphi(f(\vec{a}))$. As M is a pec model of T, there exists $\psi(\vec{v}) \in \mathrm{Res}_T(\phi)$ such that $M \vDash \psi(f(\vec{a}))$. Since f is an immersion and ψ positive, this yields $N \vDash \psi(\vec{a})$ as desired. ⊠

Lastly, we will rely on the following criterion for elementary equivalence.

Proposition 3.16. *Two positively ω-saturated pec models of an h-inductive theory are elementarily equivalent if and only if they have the same positive theory.*

Proof. See the paragraph immediately after the proof of [11, Prop. 12]. ⊠

Positive equivalence is governed by the following concept.

Definition 3.17. Given a structure M with positive theory T, we let

$$\mathrm{Th}^+(M) := T \cup \{\neg\varphi\, (= \varphi \to \bot) \mid \varphi \text{ is a positive sentence and } M \nvDash \varphi\}.$$

Notice that $\mathrm{Th}^+(M)$ is an h-inductive theory. Furthermore, two structures M and N are positively equivalent if and only if $\mathrm{Th}^+(M) = \mathrm{Th}^+(N)$.

Proposition 3.18. *For every structure M there exists a positively ω-saturated pec model M_ω of $\mathrm{Th}^+(M)$ in $\mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M)$. In addition, if M is positively saturated and either finite or such that $|L| \leqslant |M|$, we can take $M_\omega \in \mathbb{L}_{\mathrm{C}}(M)$.*

Proof. We will define a chain of structures $\{M_n \mid n \in \omega\}$ such that $M_n \vDash \mathrm{Th}^+(M)$ for each $n \in \omega$. First let $M_0 := M$. Then suppose that the chain of structures $\{M_m \mid m \leqslant n\}$ has already been defined. Since M_n is a model of the h-inductive theory $\mathrm{Th}^+(M)$, by Proposition 3.14(i) there exists a pec model M_n^* of $\mathrm{Th}^+(M)$ with a homomorphism $g_n\colon M_n \to M_n^*$. Furthermore, as M_n^* is a model of $\mathrm{Th}^+(M)$, the structures M_n^* and M have the same positive theory. In particular, M satisfies the positive theory of M_n^*. Therefore, we can apply Proposition 3.4 obtaining an ultrapower M_{n+1} of M with a homomorphism $h_n\colon M_n^* \to M_{n+1}$. Since M is a model of $\mathrm{Th}^+(M)$, so is the ultrapower M_{n+1}. In addition, the ultrapower M_{n+1} can be assumed to be ω-saturated in virtue of Corollary 3.9. Then we let $\{M_m \mid m \leqslant n+1\}$ be the chain of structures obtained by extending $\{M_m \mid m \leqslant n\}$ with M_{n+1} and a homomorphism $f_{m\,n+1} \colon M_m \to M_{n+1}$ for each $m \leqslant n+1$ defined as follows:

$$f_{m\,n+1} \coloneqq \begin{cases} \text{the identity map } i \colon M_{n+1} \to M_{n+1} & \text{if } m = n+1; \\ h_n \circ g_n \circ f_{m\,n} & \text{otherwise.} \end{cases}$$

Lastly, let M_ω be the direct limit of the chain of structures $\{M_n \mid n \in \omega\}$. Since $\mathrm{Th}^+(M)$ is an h-inductive theory and $M_n \vDash \mathrm{Th}^+(M)$ for each $n \in \omega$, we can apply Theorem 2.15 obtaining that $M_\omega \vDash \mathrm{Th}^+(M)$.

We will prove that M_ω is a pec model of $\mathrm{Th}^+(M)$ that is positively ω-saturated and belongs to $\mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M)$. To this end, observe that M_ω is also the direct limit of

(i) a chain of structures whose members are of the form M_n^* for $n \in \omega$, and of
(ii) a chain of structures whose members are of the form M_{n+1} for $n \in \omega$.

On the one hand, each M_n^* is a pec model of $\mathrm{Th}^+(M)$ by construction. Therefore, from Condition (i) it follows that M_ω is the direct limit of a chain of pec models of $\mathrm{Th}^+(M)$. Since the theory $\mathrm{Th}^+(M)$ is h-inductive, from Proposition 3.14(ii) it follows that M_ω is a pec model of $\mathrm{Th}^+(M)$. On the other hand, each M_{n+1} is an ultrapower of M. Therefore, from Condition (ii) it follows $M_\omega \in \mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M)$.

It only remains to prove that M_ω is positively ω-saturated. Recall that M_ω is the direct limit of the chain of structures $\{M_n \mid n \in \omega\}$. For each $n \in \omega$ we will denote the canonical homomorphism from M_n to M_ω associated with the direct limit M_ω by $f_n \colon M_n \to M_\omega$. Then consider $a_1, \dots, a_n \in M_\omega$ and let $p(v_1, \dots, v_n, a_1, \dots, a_n)$ be a set of positive formulas that is finitely satisfiable in M_ω. Since M_ω is the direct limit of $\{M_n \mid n \in \omega\}$, there exist $m \in \omega$ and $\hat{a}_1, \dots, \hat{a}_n \in M_m$ such that $f_m(\hat{a}_1) = a_1, \dots, f_m(\hat{a}_n) = a_n$.

By the universal property of the direct limit we have

$$f_m = f_{m+1} \circ f_{m\,m+1} = f_{m+1} \circ h_m \circ g_m.$$

Thus,

$$\begin{aligned} p(v_1, \dots, v_n, a_1, \dots, a_n) &= p(v_1, \dots, v_n, f_m(\hat{a}_1), \dots, f_m(\hat{a}_n)) \\ &= p(v_1, \dots, v_n, f_{m+1}(h_m(g_m(\hat{a}_1))), \dots, f_{m+1}(h_m(g_m(\hat{a}_n)))). \end{aligned} \tag{2}$$

Since M_m^* is a pec model of $\mathrm{Th}^+(M)$ and M_ω a model of $\mathrm{Th}^+(M)$, the homomorphism $f_{m+1} \circ h_m \colon M_m^* \to M_\omega$ is an immersion. Consequently, from the assumption that the set $p(v_1, \dots, v_n, a_1, \dots, a_n)$ is finitely satisfiable in M_ω and the above display it follows that $p(v_1, \dots, v_n, g_m(\hat{a}_1), \dots, g_m(\hat{a}_n))$ is finitely satisfiable in M_m^*. As positive formulas are preserved by homomorphisms, the set $p(v_1, \dots, v_n, h_m(g_m(\hat{a}_1)), \dots, h_m(g_m(\hat{a}_n)))$ is finitely satisfiable in M_{m+1}. Since M_{m+1} is ω-saturated, there exist $b_1, \dots, b_n \in M_{m+1}$ such that

$$M_{m+1} \vDash p(b_1, \dots, b_n, h_m(g_m(\hat{a}_1)), \dots, h_m(g_m(\hat{a}_n))).$$

Since positive formulas are preserved by homomorphisms, from Condition (2) it follows that $M_\omega \vDash p(f_{m+1}(b_1), \dots, f_{m+1}(b_n), a_1, \dots, a_n)$. Hence, we conclude that M_ω is positively ω-saturated.

To prove the second part of the statement, suppose that M is positively saturated. If M is finite, we have $M_n \cong M$ for each $n \in \omega$ because M_n is an ultrapower of M. Therefore, the above construction yields $M_\omega \in \mathbb{L}_{\mathrm{C}}(M)$ as desired. Then we consider the case where M is infinite and $|L| \leqslant |M|$. Since M is a model of $\mathrm{Th}^+(M)$, by Proposition 3.14(i) there exists a pec model M^* of $\mathrm{Th}^+(M)$ with a homomorphism $g \colon M \to M^*$. As M is infinite and such that $|L| \leqslant |M|$, by the downward Löwenheim–Skolem Theorem there exists an elementary substructure N of M^* containing $g[M]$ such that $|N| \leqslant |M|$. Since N is an elementary substructure of M^* and M^* is a pec model of the h-inductive theory $\mathrm{Th}^+(M)$, we can apply Proposition 3.15 obtaining that N is also a pec model of $\mathrm{Th}^+(M)$. Therefore, we may assume without loss of generality that $M^* = N$ and, therefore, that $|M^*| \leqslant |M|$. As M^* is a model of $\mathrm{Th}^+(M)$, we know that M and M^* have the same positive theory. Together with $|M^*| \leqslant |M|$ and the assumption that M is positively saturated, this implies that there exists a homomorphism $h\colon M^* \to M$ by Proposition 3.7. Then we consider the endomorphism $f := h \circ g$ of M. Then let M_ω be the direct limit of the chain of strucutres

$$M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots$$

The argument detailed above shows that M_ω is an ω-saturated pec model of $\mathrm{Th}^+(M)$. Furthermore, $M_\omega \in \mathbb{L}_{\mathrm{C}}(M)$ by construction. ⊠

We are now ready to prove Theorem 3.10.

Proof. From the classical Łos Theorem and its positive version it follows that if there exists $N \in \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_1) \cap \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_2)$, then M_1 and M_2 are positively equivalent. Conversely, suppose that M_1 and M_2 are positively equivalent. Then let

$$T := \mathrm{Th}^+(M_1) = \mathrm{Th}^+(M_2).$$

By Proposition 3.18 there are

$$M_1^* \in \mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M_1) \text{ and } M_2^* \in \mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M_2)$$

positively ω-saturated pec models of T. Furthermore, M_1^* and M_2^* have the same positive theory by the classical Łos Theorem and its positive version. Consequently, M_1^* and M_2^* are elementarily equivalent by Proposition 3.16. In view of the Keisler-Shelah Isomorphism Theorem, there exists $N \in \mathbb{P}_{\mathrm{U}}(M_1^*) \cap \mathbb{P}_{\mathrm{U}}(M_2^*)$. Together with the above display, this yields

$$N \in \mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M_1) \cap \mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}\mathbb{P}_{\mathrm{U}}(M_2).$$

By Proposition 2.9 this simplifies to $N \in \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_1) \cap \mathbb{P}_{\mathrm{P}}\mathbb{P}_{\mathrm{U}}(M_2)$ as desired.

To prove the second part of the statement, suppose that M_1 and M_2 are positively saturated (in addition to positively equivalent) and that each M_i is either finite or of size $\geqslant |L|$. By Proposition 3.18 we can take

$$M_1^* \in \mathbb{L}_{\mathrm{C}}(M_1) \text{ and } M_2^* \in \mathbb{L}_{\mathrm{C}}(M_2)$$

and repeat the argument above obtaining that $N \in \mathbb{P}_{\mathrm{P}}(M_1) \cap \mathbb{P}_{\mathrm{P}}(M_2)$. ⊠

Remark 3.19. In practice, the assumption of GCH in the Positive Keisler Isomorphism Theorem can sometimes be dispensed with. For instance, this is the case for each pair M_1 and M_2 of positively equivalent structures for which the h-inductive theory

$$T := \mathrm{Th}^+(M_1) = \mathrm{Th}^+(M_2)$$

is *bounded*, i.e., the size of each pec model of T is $\leqslant \kappa$ for some cardinal κ [11]. To prove this, observe that by Theorem 3.8(i) each M_i has a κ-saturated ultrapower M_i^*. Moreover, by Proposition 3.14(i) there exists a pec model N_i of T with a homomorphism $g_i\colon M_i^* \to N_i$. The assumption that T is bounded guarantees that $|N_i| \leqslant \kappa$. Therefore, we can apply Proposition

3.7 obtaining a homomorphism $h_i \colon N_i \to M_i^*$. Then consider the endomorphism $f_i := h_i \circ g_i$ of M_i^*. Observe that the direct limit M_i^+ of the chain of structures

$$M_i^* \xrightarrow{f} M_i^* \xrightarrow{f} M_i^* \xrightarrow{f} \cdots$$

is a positively ω-saturated pec model of T. From Proposition 3.16 it follows that M_1^+ and M_2^+ are elementarily equivalent. Therefore, they have isomorphic ultrapowers by the Keisler-Shelah Isomorphism Thereom. Consequently, there exists $N \in \mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}(M_1^*) \cap \mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}(M_2^*)$. By Proposition 2.9 this simplifies to $N \in \mathbb{P}_{\mathrm{P}}(M_1^*) \cap \mathbb{P}_{\mathrm{P}}(M_2^*)$. Hence, we conclude that M_1 and M_2 have isomorphic prime powers of ultrapowers. ⊠

Example 3.20 (Passive structural completeness)**.** We close this paper with an application to algebraic logic. A quasivariety is said to be *passively structurally complete* when all its nontrivial members have the same positive theory [8]. From a logical standpoint, the interest of this notion is justified as follows: when a propositional logic $\vdash$ is algebraized by a quasivariety K in the sense of [1], then all the vacuously admissible rules of $\vdash$ are derivable in $\vdash$ if and only if K is passively structurally complete [13, Fact 2, p. 68]. Both the Positive Keisler Isomorphism Theorem and Theorem 3.10 yield immediate descriptions of passive structurally complete quasivarieties in terms of prime powers. ⊠

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Appendix

Proof of Proposition 2.9. We detail the proof of the inclusion $\mathbb{P}_{\mathrm{U}}\mathbb{L}_{\mathrm{C}}(\mathsf{K}) \subseteq \mathbb{P}_{\mathrm{P}}(\mathsf{K})$, since the proof of the other inclusion regarding reduced and filter products is analogous (in fact, simpler).

Let κ be a cardinal and for each $\alpha < \kappa$ let M_α be the direct limit of a chain of structures $\{M_x \mid x \in X_\alpha\}$ in K indexed by a well ordered poset $\mathbb{X}_\alpha$. Moreover, let U be a ultrafilter over κ. We need to prove that

$$\prod_{\alpha<\kappa} M_\alpha / U \in \mathbb{P}_{\mathrm{P}}(\mathsf{K}).$$

Without loss of generality, we may assume that the members of $\{\mathbb{X}_\alpha\}_{\alpha<\kappa}$ are pairwise disjoint. Then let $\mathbb{X}$ be the poset obtained as the disjoint union of $\{\mathbb{X}_\alpha\}_{\alpha<\kappa}$ and define

$$F := \{V \in \mathsf{Up}(\mathbb{X}) \mid \{\alpha < \kappa \mid X_\alpha \cap V \neq \emptyset\} \in U\}.$$

Claim 3.21. *The set F is a prime filter over $\mathbb{X}$.*

Proof of the Claim. Clearly, F is an upset of $\mathsf{Up}(\mathbb{X})$. Moreover, F is nonempty as it contains X. To prove that F is closed under binary intersections, consider $V, W \in F$. Then

$$\{\alpha < \kappa \mid X_\alpha \cap V \neq \emptyset\} \cap \{\alpha < \kappa \mid X_\alpha \cap W \neq \emptyset\} \in U. \tag{3}$$

We will show that

$$\{\alpha < \kappa \mid X_\alpha \cap V \neq \emptyset\} \cap \{\alpha < \kappa \mid X_\alpha \cap W \neq \emptyset\} \subseteq \{\alpha < \kappa \mid X_\alpha \cap V \cap W \neq \emptyset\}. \tag{4}$$

To this end, consider $\alpha < \kappa$ such that $X_\alpha \cap V \neq \emptyset$ and $X_\alpha \cap W \neq \emptyset$. Then there are $v \in X_\alpha \cap V$ and $w \in X_\alpha \cap W$. Since $\mathbb{X}_\alpha$ is linearly ordered, we may assume that $v \leqslant w$. Since $V \cap X_\alpha$ is an upset of $\mathbb{X}_\alpha$ (because V is an upset of $\mathbb{X}$), this implies $w \in X_\alpha \cap V$. Thus, $w \in X_\alpha \cap V \cap W$ and, therefore, $X_\alpha \cap V \cap W \neq \emptyset$. From Conditions (3) and (4) it follows that

$$\{\alpha < \kappa \mid X_\alpha \cap V \cap W \neq \emptyset\} \in U.$$

Thus, $V \cap W \in F$ as desired. We conclude that F is a filter over $\mathbb{X}$.

The definition of F guarantees that $\emptyset \notin F$. Therefore, F is proper. To prove that it is prime, consider $V, W \in \mathsf{Up}(\mathbb{X})$ such that $V \cup W \in F$. Then

$$\{\alpha < \kappa \mid X_\alpha \cap V \neq \emptyset\} \cup \{\alpha < \kappa \mid X_\alpha \cap W \neq \emptyset\} = \{\alpha < \kappa \mid X_\alpha \cap (V \cup W) \neq \emptyset\} \in U.$$

Since U is an ultrafilter over κ, it is also a prime filter over κ ordered under the identity relation (Remark 2.2). Therefore, from the above display it follows that

$$\text{either } \{\alpha < \kappa \mid X_\alpha \cap V \neq \emptyset\} \in U \text{ or } \{\alpha < \kappa \mid X_\alpha \cap W \neq \emptyset\} \in U.$$

This, in turn, implies that either V or W belongs to F. ⊠

Now, recall that $\mathbb{X}$ is the disjoint union of the well ordered posets $\mathbb{X}_\alpha$. Therefore, the union $\{M_x \mid x \in X\}$ of the ordered systems $\{M_x \mid x \in X_\alpha\}$ is a well-defined ordered system indexed by a wellfounded forest. Furthermore, F is a prime filter over $\mathbb{X}$ by the Claim. Consequently, we can form the associated prime product $\prod_{x \in X} M_x / F$.

In order to conclude the proof, it suffices to prove that

$$\prod_{\alpha < \kappa} M_\alpha / U \cong \prod_{x \in X} M_x / F.$$

To this end, observe that for every $\alpha < \kappa$ and $a \in M_\alpha$ there exist $z_a \in X_\alpha$ and $m_a \in M_{z_a}$ such that $f_{z_a}(m_a) = a$, where $f_{z_a} \colon M_{z_a} \to M_\alpha$ is the canonical homomorphism associated with the direct limit M_α. For every each $a \in \prod_{\alpha < \kappa} M_\alpha$, let

$$Y_a := \{x \in X \mid z_{a(\alpha)} \leqslant x \text{ for some } \alpha < \kappa\}.$$

Notice that each $x \in Y_a$ there exists exactly one $\alpha < \kappa$ such that $z_{a(\alpha)} \leqslant x$ because $\mathbb{X}$ is the disjoint union of the various $\mathbb{X}_\alpha$ and these are linearly ordered. We will denote this α by β_{ax}. Bearing this in mind, let $g(a)$ be the only element of $\prod_{x \in Y_a} M_x$ defined for every $x \in Y_a$ as

$$g(a)(x) := f_{z_{a(\beta_{ax})}x}(m_{a(\beta_{ax})}).$$

Claim 3.22. *For every $a \in \prod_{\alpha < \kappa} M_\alpha$ we have $g(a) \in S_F$ and $V_{g(a)} = Y_a$.*

Proof of the Claim. It suffices to show that $Y_a \in F$ and that for every $x, y \in Y_a$,

$$x \leqslant y \text{ implies } f_{xy}(g(a)(x)) = g(a)(y).$$

By definition Y_a is an upset of $\mathbb{X}$ which, moreover, is nondisjoint with every X_α because $z_{a(\alpha)} \in X_\alpha \cap Y_a$. Together with the definition of F, this yields $Y_a \in F$. Then consider $x, y \in Y_a$ such that $x \leqslant y$. From $z_{a(\beta_{ax})} \leqslant x \leqslant y$ and the fact that β_{ay} is the unique $\alpha < \kappa$ such that $z_{a(\alpha)} \leqslant y$ it follows $\beta_{ax} = \beta_{ay}$. Therefore, by the definition of $g(a)$ we obtain

$$g(a)(x) = f_{z_{a(\beta_{ax})}x}(m_{a(\beta_{ax})}) \ \text{ and } \ g(a)(y) = f_{z_{a(\beta_{ax})}y}(m_{a(\beta_{ax})}).$$

Furthermore, from $z_{a(\beta_{ax})} \leqslant x \leqslant y$ it follows $f_{z_{a(\beta_{ax})}y} = f_{xy} \circ f_{z_{a(\beta_{ax})}x}$. Together with the above display, this yields

$$f_{xy}(g(a)(x)) = f_{xy}(f_{z_{a(\beta_{ax})}x}(m_{a(\beta_{ax})})) = f_{z_{a(\beta_{ax})}y}(m_{a(\beta_{ax})}) = g(a)(y). \quad ⊠$$

Then we turn to prove the following:

Claim 3.23. *For every atomic formula $\varphi(v_1, \dots, v_n)$ and $a_1, \dots, a_n \in \prod_{\alpha<\kappa} M_\alpha$,*

$$\prod_{\alpha<\kappa} M_\alpha/U \vDash \varphi(a_1/{\equiv_U}, \dots, a_n/{\equiv_U}) \text{ if and only if } \prod_{x \in X} M_x/F \vDash \varphi(g(a_1)/{\equiv_F}, \dots, g(a_n)/{\equiv_F}). \tag{5}$$

Proof of the Claim. We begin by showing that

$$\begin{aligned}
&\{\alpha < \kappa \mid M_\alpha \vDash \varphi(a_1(\alpha), \dots, a_n(\alpha))\} \\
={}&\{\alpha < \kappa \mid M_\alpha \vDash \varphi(f_{z_{a_1(\alpha)}}(m_{a_1(\alpha)}), \dots, f_{z_{a_n(\alpha)}}(m_{a_n(\alpha)}))\} \\
={}&\{\alpha < \kappa \mid \text{there exists } x \geqslant z_{a_1(\alpha)}, \dots, z_{a_n(\alpha)} \text{ s.t. } M_x \vDash \varphi(f_{z_{a_1(\alpha)}x}(m_{a_1(\alpha)}), \dots, f_{z_{a_n(\alpha)}x}(m_{a_n(\alpha)}))\} \\
={}&\{\alpha < \kappa \mid \text{there exists } x \geqslant z_{a_1(\alpha)}, \dots, z_{a_n(\alpha)} \text{ s.t. } M_x \vDash \varphi(g(a_1)(x), \dots, g(a_n)(x))\} \\
={}&\{\alpha < \kappa \mid \text{there exists } x \in X_\alpha \cap V_{g(a_1)} \cap \dots \cap V_{g(a_n)} \text{ s.t. } M_x \vDash \varphi(g(a_1)(x), \dots, g(a_n)(x))\} \\
={}&\{\alpha < \kappa \mid X_\alpha \cap [\![\varphi(g(a_1), \dots, g(a_n))]\!] \neq \emptyset\}.
\end{aligned}$$

The first of the equalities above holds by the definition of $z_{a_i(\alpha)}$ and $m_{a_i(\alpha)}$, the second because M_α is the direct limit of the chain of structures $\{M_x \mid x \in X_\alpha\}$, the third by the definition of $g(a_i)$, and the fifth by the definition of $[\![\varphi(g(a_1), \dots, g(a_n))]\!]$. To prove the fourth, it suffices to show that

$$X_\alpha \cap V_{g(a_1)} \cap \dots \cap V_{g(a_n)} = \{x \in X \mid x \geqslant z_{a_1(\alpha)}, \dots, z_{a_n(\alpha)}\}.$$

The above equality, in turn, holds because $V_{g(a_i)} = Y_{a_i}$ by Claim 3.22 and $\mathbb{X}$ is the disjoint union of the various $\mathbb{X}_\beta$.

Observe that $[\![\varphi(g(a_1), \dots, g(a_n))]\!]$ is an upset of $\mathbb{X}$ (because positive formulas are preserved by homomorphisms). Therefore, from the above series of equalities and the definition of F it follows that

$$\{\alpha < \kappa \mid M_\alpha \vDash \varphi(a_1(\alpha), \dots, a_n(\alpha))\} \in U \text{ if and only if } [\![\varphi(g(a_1), \dots, g(a_n))]\!] \in F.$$

By the classical Łos Theorem and its positive version this yields the desired result. ⊠

In view of Claim 3.23, the map

$$\hat{g}\colon \prod_{\alpha<\kappa} M_\alpha/U \to \prod_{x \in X} M_x/F$$

defined by the rule $\hat{g}(a/{\equiv_U}) \coloneqq g(a)/{\equiv_F}$ is a well-defined embedding. Therefore, to prove that $\hat{g}$ is an isomorphism, it only remains to show that it is surjective.

To this end, consider $a \in S_F$. For each $\alpha < \kappa$ and $x \in X_\alpha \cap V_a$ let $f_x\colon M_x \to M_\alpha$ be the canonical homomorphism associated with the direct limit M_α. For each $\alpha < \kappa$ such that $X_\alpha \cap V_a \neq \emptyset$ there exists some $y_\alpha \in X_\alpha \cap V_a$ such that

$$z_{f_{y_\alpha}(a(y_\alpha))} \leqslant y_a \text{ and } a(y_\alpha) = f_{z_{f_{y_\alpha}(a(y_\alpha))}y_\alpha}(m_{f_{y_\alpha}(a(y_\alpha))}). \tag{6}$$

This is a consequence of the definition of the direct limit M_α and of the assumption that $X_\alpha \cap V_a$ is an upset of the linearly ordered poset $\mathbb{X}_\alpha$.

We define an element $b \in \prod_{\alpha<\kappa} M_\alpha$ as follows: for each $\alpha < \kappa$,

$$b(\alpha) \coloneqq \begin{cases} f_{y_\alpha}(a(y_\alpha)) & \text{if } X_\alpha \cap V_a \neq \emptyset; \\ \text{an arbitrary element of } M_\alpha & \text{otherwise.} \end{cases}$$

Our aim is to prove that $\hat{g}(b/{\equiv_U}) = a/{\equiv_F}$, i.e., $[\![g(b) = a]\!] \in F$. Since $g(b) \in S_F$, we know that $[\![g(b) = a]\!]$ is an upset of $\mathbb{X}$. Therefore, by the definition of F it suffices to show that

$$\{\alpha < \kappa \mid X_\alpha \cap [\![g(b) = a]\!] \neq \emptyset\} \in U.$$

As $a \in S_F$, we know that $V_a \in F$ and, therefore, that $\{\alpha < \kappa \mid X_\alpha \cap V_a \neq \emptyset\} \in U$. Therefore, our task reduces to that of proving the inclusion

$$\{\alpha < \kappa \mid X_\alpha \cap V_a \neq \emptyset\} \subseteq \{\alpha < \kappa \mid X_\alpha \cap [\![g(b) = a]\!] \neq \emptyset\}.$$

Accordingly, let $\alpha < \kappa$ be such that $X_\alpha \cap V_a \neq \emptyset$. Then $y_\alpha \in X_a \cap V_a$. Furthermore, $b(\alpha) = f_{y_\alpha}(a(y_\alpha))$ by the definition of b. Therefore, from Condition (6) it follows $z_{b(\alpha)} \leqslant y_\alpha$. By the definition of Y_b this amounts to $y_\alpha \in Y_b$. Since $Y_b = V_{g(b)}$ by Claim 3.22, we conclude that $y_\alpha \in V_{g(b)}$. Thus, $y_\alpha \in V_{g(b)} \cap V_a$. In addition, from the definition of $g(b)$ and Condition (6) it follows

$$g(b)(y_\alpha) = f_{z_{f_{y_\alpha}(a(y_\alpha))}\, y_\alpha}(m_{f_{y_\alpha}(a(y_\alpha))}) = a(y_\alpha).$$

Therefore, $y_\alpha \in [\![g(b) = a]\!]$. Hence, we conclude that $X_\alpha \cap [\![g(b) = a]\!] \neq \emptyset$ as desired. ⊠

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