Risk-based optimal portfolio of an insurance firm with regime switching and noisy memory

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Abstract. In this paper, we consider a risk-based optimal investment problem of an insurance firm in a regime-switching jump diffusion model with noisy memory. Using the model uncertainty modeling, we formulate the investment problem as a zero-sum, stochastic differential delay game between the insurance firm and the market, with a convex risk measure of the terminal surplus and the Brownian delay surplus over a period $[T-\varrho, T]$, where $\varrho \geq 0$. Then, by the BSDE approach, the game problem is solved. Finally, we derive analytical solutions of the game problem, for a particular case of a quadratic penalty function and a numerical example is considered.

1. Introduction

Stochastic delay equations are equations whose coefficients depend also on past history of the solution. They appear naturally in Economics, Life Science, Finance, Engineering, Biology, etc. In Mathematical Finance, the basic assumption of the evolution price processes is that they are Markovian. In reality, these processes possess some memory which cannot be neglected. Stochastic delay control problems have received much interest in recent times and these are solved by different methods. For instance, when the state process depends on the discrete and average delay, Elsanoni *et. al.* [18] studied an optimal harvesting problem using the dynamic programming approach. On the other hand, a maximum principle approach was used to solve stochastic optimal control systems with delay. See e.g., Øksendal and Sulem [29], Pamen [28]. When the problem allows a noisy memory, i.e., a delay modeled by a Brownian motion, Dahl *et. al* [8] proposed a maximum principle approach with Malliavin derivatives to solve their problem. For detailed information

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on the theory of stochastic delay differential equations (SDDE) and their applications to stochastic control problems, see, e.g., Baños *et. al.* [2], Kuang [24], Mohammed [27] and references therein.

In this paper, we consider an insurance firm risk-based optimal investment problem with noisy memory. Noisy memory refers to incorporating past observations or experiences that are subject to random fluctuations or noise into the investment decision-making process. In the insurance industry, it is important to consider noisy memory due to several reasons, for instance, it allows the insurance firm to account for the inherent uncertainties and fluctuations in the market. This can help in avoiding potential losses and preserving the financial stability of the insurer. Furthermore, it enables the insurance firm to capture important market trends and patterns that may not be immediately apparent. Some examples illustrating the relevance of incorporating noisy memory for an insurance firm include: utilizing past claim data, which may be subject to noise, historical market data with noise to identify long-term investment opportunities, and information from external sources, such as, news and changes of laws.

The financial market model set-up is composed by one risk-free asset and one risky asset whose price dynamics are described by a hidden Markov regime-switching jump diffusion process. The jump-diffusion models represent a valuable extension of the diffusion models for modeling the asset prices [29]. They capture some sudden changes in the market such as the existence of high-frequency data, volatility clusters and regime switching. It is important to note that in the Markov regime-switching diffusion models, we can have random coefficients possibly with jumps, even if the return process is a diffusion one. In this paper, we consider a jump diffusion model, which incorporates jumps in the asset price as well as in the model coefficients, i.e., a Markov regime-switching jump-diffusion model. Furthermore, we consider the Markov chain to represent different modes of the economic environment such as, political situations, natural catastrophes or law. Such kind of models have been considered for option pricing of the contingent claim, see for example, Elliott *et. al* [14], Siu [36] and references therein. For stochastic optimal control problems, we mention the works by Bäuerle and Rieder [3], Meng and Siu [26]. In these works a portfolio asset allocation and a risk-based asset allocation of a Markov-modulated jump process model has been considered and solved via the dynamic programming approach. We also mention a recent work by Pamen and Momeya [30], where a maximum principle approach has been applied to an optimization problem described by a Markov-modulated regime switching jump-diffusion model.

In this paper, we assume that the company receives premiums at the constant rate and pays the aggregate claims modeled by a hidden Markov-modulated pure jump process. We assume the existence of capital inflow or outflow from the insurer's current wealth, where the amount of the capital is proportional to the past performance of the insurer's wealth. Then, the surplus process is governed by a stochastic delay differential equation with the delay, which may be random. Therefore we find it reasonable to consider also a delay modeled by Brownian motion. In literature, a mean-variance problem of an insurance firm was considered, but the wealth process is given by a diffusion model with distributed delay, solved via the maximum principle approach (Shen and Zeng [33]). Chunxiang and

Li [5] extended this mean-variance problem of an insurance firm to the Heston stochastic volatility case and solved using dynamic programming approach. For thorough discussion on different types of delay, we refer to Baños *et. al.* [2], Section 2.2.

We adopt a convex risk measure first introduced by Frittelli and Gianin [20] and Föllmer and Schied [19]. This generalizes the concept of coherent risk measure first introduced by Artzner *et. al.* [1], since it includes the nonlinear dependence of the risk of the portfolio due to the liquidity risks. Moreover, it relaxes a sub-additive and positive homogeneous properties of the coherent risk measures and substitute these by a convex property.

When the risky share price is described by a diffusion process and without delay, such kind of risk-based optimization problems of an insurance firm have been widely studied and reported in literature, see e.g., Elliott and Siu [16, 17], Siu [34, 35], Peng and Hu [31]. For a jump-diffusion case, we refer to Mataramvura and Øksendal [25]. For hidden Markov modulated jump diffusion European option pricing problem, we refer to [36].

To solve our optimization problem, we first transform the unobservable Markov regime-switching problem into one with complete observation by using the so-called optimal filter-based estimate for the Markov chain. For interested readers, we refer to Elliott *et. al.* [15], Elliott and Siu [17], Cohen and Elliott [6] and Kallianpur [22]. Then we formulate a convex risk measure described by a terminal surplus process as well as the dynamics of the noisy memory surplus over a period $[T-\varrho, T]$, $\varrho \geq 0$ of the insurance firm to measure the risks. The main objective of the insurance firm is to select the optimal investment strategy so as to minimize the risk. This is a two-player zero-sum stochastic delayed differential game problem. Using delayed backward stochastic differential equations (BSDE) with a jump approach, we solve this game problem by an application of a comparison principle for BSDE with jumps. Our modeling framework follows that in Elliott and Siu [16], later extended to the regime switching case by Peng and Hu [31].

Our paper differs from the existing literature in two aspects: First, the model in our paper is driven by a general jump-diffusion process with regime switching and partial observable processes for an insurance firm. Secondly, we assume the capital inflows/outflows which are subject noisy and distributed delay.

The rest of the paper is organized as follows: In Section 2, we introduce the dynamic of state process described by SDDE in the Hidden Markov regime switching jump-diffusion market. In Section 3, we use the filtering theory to turn the model into one with complete observation. We also derive the optimal Markov chain. Section 4, is devoted to the formulation of our risk-based optimization problem as a zero-sum stochastic delayed differential game problem, which is then solved in Section 5. Finally, in Section 6, we derive the explicit solutions for a particular case of a quadratic penalty function and we give an example to show how one can apply these results in a concrete situation.

2. Model formulation

Suppose we have an insurance firm investing in a finite investment period $[0, T]$, with $T < \infty$. Consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is a filtration; $\mathcal{F}$ is a σ-field on the sample space Ω; $\mathbb{P}$ is a probability measure on the measurable space

$(\Omega, \mathcal{F})$. Let $\{\Lambda(t);\ t \in [0,T]\}$ be a continuous time finite state hidden Markov chain defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite state space $\mathcal{S} = \{e_1, e_2, \dots, e_D\} \subset \mathbb{R}^D$, $e_j = (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^D$, where $D \in \mathbb{N}$ is the number of states of the chain, and the jth component of e_n is the Kronecker delta δ_{nj}, for each $n, j = 1, 2, \dots, D$. $\{\Lambda(t);\ t \in [0,T]\}$ describes the evolution of the unobserved state of the model parameters in the financial market over time, i.e., a process which collects factors that are relevant for the model, such as, political situations, laws or natural catastrophes (see, e.g. Bauerle and Rieder [3], Elliott and Siu [17]). The main property of the Markov chain $\Lambda(t)$ with the canonical state space $\mathcal{S}$ is that, any nonlinear function of Λ, is linear in Λ, i.e., $\varphi(\Lambda(t)) = \langle \varphi, \Lambda(t) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^D$. For detailed information, see, for instance, Elliott *et. al.* [15].

To describe the probability law of the chain Λ under the probability measure $\mathbb{P}$, we define a family of intensity matrices $A(t) := \{a_{ji}(t);\ t \in [0,T]\}$, where $a_{ji}(t)$ is the instantaneous transition intensity of the chain Λ from state e_i to state e_j at time $t \in [0,T]$. Then it was proved in Elliott *et. al.* [15], that Λ admits the following semi-martingale dynamics:

$$\Lambda(t) = \Lambda(0) + \int_0^t A(s)\Lambda(s-)ds + \Phi(t)\,,$$

where Φ is an $\mathbb{R}^D$-valued martingale with respect to the natural filtration generated by Λ, that is, $\{\mathcal{F}_t^\Lambda\}_{0 \le t \le T}$. And $\mathcal{F}_t^\Lambda$ is the $\mathbb{P}$-augmentation of the σ-field generated by the history of the markov chain Λ up to and including time t.

To describe the dynamics of the financial market, we consider a Brownian motion $\{W(t)\}_{t \in [0,T]}$ and a Markov regime-switching Poisson random measure, with the dual predictable projection ν_Λ defined by

$$\nu_\Lambda(dt, dz) = \sum_{j=1}^D \langle \Lambda(t-), e_j \rangle \varepsilon_j(t) \nu_j(dz) dt\,,$$

where ν_j is the conditional Levy measure of the random jump size and ε_j is the intensity rate when the Markov chain Λ is in state e_j at time t. The compensated Markov regime-switching Poisson random measure $\tilde{N}_\Lambda(dt, dz) := N(dt, dz) - \nu_\Lambda(dz)dt$. We suppose that the processes W and N are independent under the probability measure $\mathbb{P}$.

Let $\{\mathcal{F}_t\}_{0 \le t \le T}$ be the $\mathbb{P}$-augmentation of natural filtration generated by W and N respectively. We consider a financial market consisting of one risk-free asset $(B(t))_{0 \le t \le T}$ and one risky asset $(S(t))_{0 \le t \le T}$. Their respective prices are given by the following regime-switching stochastic differential equations (SDE):

$$\begin{aligned} dB(t) &= r(t)B(t)dt\,, \quad B(0) = 1\,, \\ dS(t) &= S(t)\Big[\alpha^\Lambda(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \Big(e^z - 1\Big) N(dt, dz)\Big] \\ &= S(t)\Big[\Big(\alpha^\Lambda(t) + \sum_{j=1}^D \int_{\mathbb{R}} \Big(e^z - 1\Big) \langle \Lambda(t-), e_j \rangle \varepsilon_j(t) \nu_j(dz)\Big)dt \end{aligned} \tag{2.1}$$

$$+\beta(t)dW(t) + \int_{\mathbb{R}} \Big(e^z - 1\Big)\tilde{N}_\Lambda(dt, dz)\Big]\,, \tag{2.2}$$

with initial value $S(0) = s > 0$. We suppose that the instantaneous interest rate $r(t) > 0$ is a deterministic function, the appreciation rate $\alpha^\Lambda(t)$ is modulated by the Markov chain Λ, as follows:

$$\alpha^\Lambda(t) \;\;:=\;\; \langle \alpha(t), \Lambda(t)\rangle = \sum_{j=1}^{D} \alpha_j(t)\langle \Lambda(t), e_j\rangle\,,$$

α_j represents the appreciation rate, when the Markov chain is in state e_j of the economy. We suppose that $\alpha(t)$ is $\mathbb{R}^D$-valued $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable and uniformly bounded processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Otherwise, the volatility rate $\beta(t) > 0$ is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted uniformly bounded process. Note that we may consider a Markov modulated volatility process, however it would lead in a complicated, if not possible filtering issue in the following section. As was pointed out by Siu [36] and references therein, the other reason is that, the volatility can be determined from a price path of the risky share, i.e., the volatility is observable.

We now model the liabilities by a Markov regime-switching pure jump process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We follow the modeling framework of Elliott and Siu [17], Siu [35], Pamen and Momeya [30].

Consider a real valued pure jump process $Z := \{Z(t);\ t \in [0, T]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Z denotes the aggregate amount of the claims up to time t. Then, we can write Z as

$$Z(t) = \sum_{0<s\le t} \Delta Z(s)\,; \quad Z(0) = 0, \;\; \mathbb{P}-\text{a.s.}, \;\; \text{t} \in [0, \text{T}]\,,$$

where $\Delta Z(s) := Z(s) - Z(s-)$, for each $s \in [0, T]$, represents the jump size of Z at time s.

Suppose that the state space of the claim size $\mathcal{Z}$ is $(0, \infty)$. Consider a random measure $N^0(\cdot, \cdot)$ defined on a measurable space $([0, T] \times \mathcal{Z}, \mathcal{B}([0, T]) \times \mathcal{B}(\mathcal{Z}))$, which selects the random claim arrivals and size $z := Z(s) - Z(s-)$, at time s, where $\mathcal{B}(\mathcal{Z})$ stands for the Borel σ-field generated by the open subsets of $\mathcal{Z}$. The aggregate insurance claim process Z can be written as

$$Z = \int_0^t \int_0^\infty z N^0(ds, dz); \quad t \in [0, T]\,.$$

Define, for each $t \in [0, T]$,

$$M(t) := \int_0^t \int_0^\infty N^0(ds, dz); \quad t \in [0, T]\,.$$

$M(t)$ counts the number of claim arrivals up to time t. Suppose that under $\mathbb{P}$, $M := \{M(t),\, t \in [0, T]\}$ is a conditional Poisson process on $(\Omega, \mathcal{F}, \mathbb{P})$, given the information about the realized path of the chain, with intensity $\lambda^\Lambda(t)$ modulated by the Markov chain given by

$$\lambda^{\Lambda}(t) := \langle \lambda(t), \Lambda(t) \rangle = \sum_{j=1}^{D} \lambda_j \langle \Lambda(t), e_j \rangle \,,$$

where $\lambda_j(t)$ is the jth entry of the vector $\lambda(t)$ and represents the intensity rate of M when the Markov chain is in the state space e_j at time t.

Let $f_j(z)$, $j = 1, \ldots, D$ be the probability density function of the claim size $z = Z(s) - Z(s-)$, when $\Lambda(s-) = e_j$. Then the Markov regime-switching compensator of the random measure $N^0(\cdot, \cdot)$ under $\mathbb{P}$, is given by

$$\nu^0_{\Lambda}(ds, dz) := \sum_{j=1}^{D} \langle \Lambda(s-), e_j \rangle \lambda_j(s) f_j(dz) ds \,.$$

Therefore, a compensated version of the random measure is given by

$$\tilde{N}^0_{\Lambda}(ds, dz) = N^0(ds, dz) - \nu^0_{\Lambda}(ds, dz) \,.$$

We suppose that $\tilde{N}^0_{\Lambda}$ is independent of W and $\tilde{N}_{\Lambda}$.

Let $p(t)$ be the premium rate at time t. We suppose that the premium rate process $\{p(t),\, t \in [0,T]\}$ is $\{\mathcal{F}^Z_t\}_{t\in[0,T]}$-progressively measurable and uniformly bounded process on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values on $(0, \infty)$. $\{\mathcal{F}^Z_t,\, t \in [0,T]\}$ denotes the $\mathbb{P}$-augmentation os the natural filtration generated by the history of the insurance claim process Z. Let $R := \{R(t),\, t \in [0,T]\}$ be the insurance risk process of the insurance company without investment. Then, $R(t)$ is given by

$$\begin{aligned} R(t) &:= r_0 + \int_0^t p(s) ds - Z(t) \\ &= r_0 + \int_0^t p(s) ds - \int_0^t \int_0^{\infty} z N^0(ds, dz) \,. \end{aligned}$$

Here r_0 is the initial surplus of the insurance firm.

Let $\pi(t)$ be the amount of the money invested in the risky asset at time t. We denote the surplus process of the insurance firm by $\{X(t),\, t \in [0,T]\}$, then we formulate the surplus process with delay, which is caused by the capital inflow/outflow function from the insurance firm's current wealth. We suppose that the capital inflow/outflow function is given by

$$\varphi(t, X(t), \bar{Y}(t), U(t)) = (\vartheta(t) + \xi) X(t) - \vartheta(t) \bar{Y}(t) - \xi U(t) \,,$$

where $\vartheta(t) \geq 0$ is uniformly bounded function of t, $\xi \geq 0$ is a constant and

$$Y(t) = \int_{t-\varrho}^{t} e^{\zeta(s-t)} X(s) dW_1(s) \,; \qquad \bar{Y}(t) = \frac{Y(t)}{\int_{t-\varrho}^{t} e^{\zeta(s-t)} ds} \,; \quad U(t) = X(t - \varrho) \,.$$

Here, Y, $\bar{Y}$, U denote respectively the integrated, average and pointwise delayed information of the wealth process in the interval $[t-\varrho, t]$. $\zeta \geq 0$ is the average parameter and $\varrho \geq 0$ the delay parameter. $W_1 := \{W_1(t),\, t \in [0,T]\}$ is a standard Brownian motion which is supposed to be independent of W, $\tilde{N}^0_{\Lambda}$ and $\tilde{N}_{\Lambda}$ under the probability measure $\mathbb{P}$.

The parameters ϑ and ξ represent the weights proportional to the past performance of $X-\bar{Y}$ and $X-U$, respectively. A good performance ($\varphi>0$), may bring to the insurance firm more wealth, so that he can pay part of the wealth to the policyholders. Otherwise, a bad performance ($\varphi<0$) may forces the insurance firm to use the reserve or look for further capital in the market to cover the losses in order to achieve the final performance.

Remark. According to the definition of our capital inflow/outflow function, we take a noisy memory into account, thus generalizing the inflow/outflow function considered in Shen and Zeng [33]. To the best of our knowledge, this kind of noisy delay has just been applied in a stochastic control problem recently by Dahl *et. al.* [8] using a maximum principle techniques with Malliavin derivatives. Unlike in Dahl *et. al.* [8], we suppose that the noisy delay is derived by an independent Brownian motion. We believe that this assumption is more realistic since the delay of the information may not be caused by the same source of randomness as the one driving the stock price. Furthermore, when the delay is driven by the same noisy with the asset price, the filtering theory we apply in the next section, fails to turn the model into one with complete observations, as the dynamics of $Y(t)$ would still be dependent on some hidden parameters. Under derivative pricing, such kind of delays have been applied to consider some stochastic volatility models, but with the delay driven by independent Poisson process, see, e.g., Swishchuk [37].

Note that we can write the noisy memory information Y in a differential form by

$$\begin{aligned} dY(t) &= -\zeta Y(t)dt + X(t)dW_1(t) - e^{-\zeta\varrho}X(t-\varrho)dW_1(t-\varrho) \\ &= -\zeta Y(t)dt + X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]})dW_1(t) \quad t\in[-\varrho,T]\,, \end{aligned} \tag{2.3}$$

where χ_A denotes the characteristic function defined in a set A.

Then, the surplus process of the insurance firm is given by the following stochastic delay differential equation (SDDE) with regime-switching

$$\begin{aligned} dX(t) &= [p(t)+r(t)X(t)+\pi(t)(\alpha^\Lambda(t)-r(t))-\varphi(t,X(t),\bar{Y}(t),U(t))]dt \\ &\quad +\pi(t)\beta(t)dW(t)+\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)N(dt,dz)-\int_0^\infty zN^0(dt,dz) \\ &= \Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\alpha^\Lambda(t)-r(t))+\vartheta(t)\bar{Y}(t) \\ &\quad -\xi U(t)+\sum_{j=1}^D\langle\Lambda(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\varepsilon_j(t)\nu_j(dz) \\ &\quad -\int_0^\infty\lambda_j(t)zf_j(dz)\Big)\Big]dt+\pi(t)\beta(t)dW(t) \\ &\quad +\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\tilde{N}_\Lambda(dt,dz)-\int_0^\infty z\tilde{N}^0_\Lambda(dt,dz)\,,\quad t\in[0,T]\,, \\ X(t) &= x_0>0,\quad t\in[-\varrho,0]\,. \end{aligned} \tag{2.4}$$

We assume that the function $x_0 := x_0(t)$ is continuous and deterministic.

Definition 2.1. The portfolio process $\{\pi(t),\, t \in [0,T]\}$ is said to be admissible if it satisfies the following:

(1) $\pi(t)$ is $\{\mathcal{F}_t^Z\}_{t\in[0,T]}$-progressively measurable and $\int_0^T |\pi(t)|^2 dt < \infty$, $\mathbb{P}$-a.s.
(2) The SDDE (2.4) admits a unique strong solution;
(3)

$$\sum_{j=1}^{D}\Big\{\int_0^T |p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\alpha_j(t)-r(t)) \\ +\bar{\vartheta}(t)Y(t)-\xi U(t)|dt+\int_0^T\Big[\int_{\mathbb{R}}(\pi(t))^2\Big(e^z-1\Big)^2\varepsilon_j(t)\nu_j(dz) \\ +\int_0^\infty z^2\lambda_j(t)f_j(dz)+\pi^2(t)\beta^2(t)\Big]dt\Big\} < \infty\,, \quad \mathbb{P}-a.s.$$

We denote the space of admissible investment strategies by $\mathcal{A}$.

We end this section, clarifying the information structure of our main problem. We define $\mathbb{F} := \{\mathcal{F}_t \mid t \in [0,T]\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the risky asset $S(t)$ and the insurance risk process $R(t)$. This denotes the observable filtration in the market model. Let $\mathcal{G}_t := \mathcal{F}_t^\Lambda \vee \mathcal{F}_t \vee \mathcal{F}_t^Z$. $\mathbb{G} := \{\mathcal{G}_t \mid t \in [0,T]\}$ represents the full information structure of the model, where $\mathcal{F}^\Lambda$ is the filtration generated by the Markov chain Λ.

3. Reduction by the filtering theory

As we are working with an unobservable Markov regime-switching model, one needs to reduce the model into one with complete observations. We adopt the filtering theory for this reduction. This is a classical approach and it has been widely applied in stochastic control problems. See, for example, Bäuerle and Rieder [3], Elliott *et. al.* [15], Elliott and Siu [17], Siu [34], and references therein. We proceed as in Siu [36].

Consider the following $\mathcal{F}_t$-adapted process $\widehat{W} := \{\widehat{W}(t),\, t \in [0,T]\}$ defined by

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\alpha^\Lambda(s)-\hat{\alpha}^\Lambda(s)}{\beta(s)}ds\,, \quad t\in[0,T]\,,$$

where $\hat{\alpha}$ is the optional projection of α under $\mathbb{P}$, with respect to the filtration $\mathbb{F}$, i.e., $\hat{\alpha}^\Lambda(t) = \mathbb{E}[\alpha^\Lambda(t) \mid \mathcal{F}_t]$, $\mathbb{P}$-a.s.. Then it was shown that $\widehat{W}$ is a Brownian motion. See e.g., Elliott and Siu [17] or Kallianpur [22], Lemma 11.3.1.

Let $\hat{\Lambda}$ be the optional projection of the Markov chain Λ with respect to the observable filtration $\mathbb{F}$. For the jump part of the risky share N and the insurance risk N^0, we consider the following:

$$\hat{\nu}(dt,dz) := \sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\varepsilon_j(t)\nu_j(dz)dt \quad \text{and}$$

$$\hat{\nu}^0(dt,dz) := \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j \rangle \lambda_j(t) \nu_j^0(dz) dt \,.$$

Define the compensated random measures $\widehat{N}(dt,dz)$ and $\widehat{N}^0(dt,dz)$ by

$$\begin{aligned} \widehat{N}(dt,dz) &:= N(dt,dz) - \hat{\nu}(dt,dz) \\ \widehat{N}^0(dt,dz) &:= N^0(dt,dz) - \hat{\nu}^0(dt,dz) \,. \end{aligned}$$

Then, it can be shown that the processes $\widehat{M}$ and $\widehat{M}^0$ are martingales with respect to the filtration $\mathbb{F}$ under the probability $\mathbb{P}$. (See Elliott [13]):

$$\begin{aligned} \widehat{M}(t) &:= \int_0^t \int_{\mathbb{R}} \Big(e^z - 1\Big) \widehat{N}(dt,dz) \\ \widehat{M}^0(t) &:= \int_0^t \int_0^\infty z \widehat{N}^0(dt,dz) \,. \end{aligned}$$

Therefore, the surplus process $X(t)$ for any $t \in [0,T]$, can be written, under $\mathbb{P}$, as:

$$\begin{aligned} (3.1) dX(t) &= \Big[p(t) + (r(t) - \vartheta(t) - \xi) X(t) + \pi(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) + \bar{\vartheta}(t) Y(t) - \xi U(t) \\ &\quad + \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j \rangle \Big(\pi(t) \int_{\mathbb{R}} \Big(e^z - 1\Big) \varepsilon_j(t) \nu_j(dz) - \int_0^\infty \lambda_j(t) z f_j(dz) \Big) \Big] dt \\ &\quad + \pi(t)\beta(t) d\widehat{W}(t) + \pi(t) \int_{\mathbb{R}} \Big(e^z - 1\Big) \widehat{N}_{\Lambda}(dt,dz) - \int_0^\infty z \widehat{N}^0_{\Lambda}(dt,dz) \,, \\ X(t) &= x_0 > 0, \quad t \in [-\varrho, 0] \,. \end{aligned}$$

Note that the dynamics (2.3) remains, since it is driven by an independent Brownian motion.

We then use the reference probability approach to derive a filtered estimate $\hat{\Lambda}$ of the Markov chain Λ following the discussions in Siu [36].

Let $\varphi(t) \in \mathbb{R}^D$, such that $\varphi_j(t) = \alpha_j(t) - \frac{1}{2}\beta^2(t)$, $j = 1, 2, \ldots, D$. Define, for any $t \in [0,T]$, the following functions

$$\begin{aligned} \Psi_1(t) &:= \int_0^t \langle \varphi(s), \Lambda(s) \rangle ds + \int_0^t \beta(s) dW(s) \,; \\ \Psi_2(t) &:= \int_0^t \int_{\mathbb{R}} \Big(e^z - 1\Big) N(ds,dz) \,; \\ \Psi_3(t) &:= \int_0^t \int_0^\infty z N^0(ds,dz) \,. \end{aligned}$$

Write $\mathbb{P}^*$, for a probability measure on $(\Omega, \mathcal{F})$, under which the observation process does not depend on the Markov chain Λ. Define, for each $j = 1, 2, \ldots, D$,

$$F_j(t,z) := \frac{\lambda_j(t) f_j(dz)}{f(dz)} \quad \text{and} \quad \mathcal{E}_{\mathrm{j}}(\mathrm{t},\mathrm{z}) := \frac{\varepsilon_{\mathrm{j}}(\mathrm{t}) \nu_{\mathrm{j}}(\mathrm{dz})}{\nu(\mathrm{dz})} \,.$$

Consider the following $\mathbb{G}$-adapted processes Γ_1, Γ_2 and Γ_3 defined by putting

$$
\begin{aligned}
\Gamma_1(t) &:= \exp\Big(\int_0^t \beta^{-2}(s)\langle\varphi(s),\Lambda(s)\rangle d\Psi_1(s) - \frac{1}{2}\int_0^t \beta^{-4}(s)\langle\varphi(s),\Lambda(s)\rangle^2 ds\Big)\,;\\
\Gamma_2(t) &:= \exp\Big[-\int_0^t \sum_{j=1}^{D}\langle\Lambda(s-),e_j\rangle\Big(\int_{\mathbb{R}}(\mathcal{E}_j(s,z)-1)\nu(dz)\Big)ds\\
&\quad + \int_0^t\int_{\mathbb{R}}\Big(\sum_{j=1}^{D}\langle\Lambda(s-),e_j\rangle\ln(\mathcal{E}_j(s,z))\Big)N(ds,dz)\Big]\,;\\
\Gamma_3(t) &:= \exp\Big[-\int_0^t \sum_{j=1}^{D}\langle\Lambda(s-),e_j\rangle\Big(\int_0^\infty(F_j(s,z)-1)f(dz)\Big)ds\\
&\quad + \int_0^t\int_0^\infty\Big(\sum_{j=1}^{D}\langle\Lambda(s-),e_j\rangle\ln(F_j(s,z))\Big)N^0(ds,dz)\Big]\,.
\end{aligned}
$$

Consider the $\mathbb{G}$-adapted process $\Gamma := \{\Gamma(t),\, t\in[0,T]\}$ defined by

$$\Gamma(t) := \Gamma_1(t)\cdot\Gamma_2(t)\cdot\Gamma_3(t).$$

Note that the process Γ is a local martingale. After imposing some technical conditions on some model coefficients, Γ is a true martingale. Consequently, $\mathbb{E}[\Gamma(T)] = 1$. See, for instance, Proposition 2.5.1 in Delong [11].

The main goal of the filtering process is to evaluate the $\mathbb{F}$-optional projection of the Markov chain Λ under $\mathbb{P}$. To that end, let, for each $t\in[0,T]$,

$$\mathbf{q}(t) := \mathbb{E}^*[\Gamma(t)\Lambda(t)\mid\mathcal{F}_t]\,,$$

where $\mathbb{E}^*$ is an expectation under the reference probability measure $\mathbb{P}^*$. The process $\mathbf{q}(t)$ is called an unnormalized filter of $\Lambda(t)$.

Define, for each $j = 1,2,\ldots,D$ the scalar valued process $\gamma_j := \{\gamma_j(t),\, t\in[0,T]\}$ by

$$
\begin{aligned}
\gamma_j(t) &:= \exp\Big(\int_0^t \varphi_j(s)\beta^{-2}(s)d\Psi_1(s) - \frac{1}{2}\int_0^t\varphi_j^2(s)\beta^{-4}(s)ds + \int_0^t(1-\varepsilon_j(s))ds\\
&\quad + \int_0^t(1-f_j(s))ds + \int_0^t\ln(\mathcal{E}_j(s))dN(s) + \int_0^t\ln(F_j(s))dN^0(s)\Big)\,.
\end{aligned}
$$

Consider a diagonal matrix $\mathbf{L}(t) := \mathbf{diag}(\gamma_1(\mathrm{t}),\gamma_2(\mathrm{t}),\ldots,\gamma_{\mathrm{D}}(\mathrm{t}))$, for each $t\in[0,T]$. Define the transformed unnormalized filter $\{\bar{\mathbf{q}}(t),\, t\in[0,T]\}$ by

$$\bar{\mathbf{q}}(t) := \mathbf{L}^{-1}(t)\mathbf{q}(t)\,.$$

Note that the existence of the inverse $\mathbf{L}^{-1}(t)$ is guaranteed by the definition of $\mathbf{L}(t)$ and the positivity of $\gamma_j(t)$, $j = 1,2,\ldots,D$, for all $t\in[0,T]$.

Then, it has been shown (see Elliott and Siu [17]), that the transformed unnormalized filter $\bar{\mathbf{q}}$ satisfies the following linear order differential equation

$$\frac{d\bar{\mathbf{q}}(t)}{dt} := \mathbf{L}^{-1}(t)A(t)\mathbf{L}(t)\bar{\mathbf{q}}(t)\,, \qquad \bar{\mathbf{q}}(0) = \mathbf{q}(0) = \mathbb{E}[\Lambda(0)]\,.$$

Hence, by a version of the Bayes rule, the optimal estimate $\hat{\Lambda}(t)$ of the Markov chain $\Lambda(t)$ is given by

$$\hat{\Lambda} := \mathbb{E}[\Lambda(t) \mid \mathcal{F}_t] = \frac{\mathbb{E}^*[\Gamma(t)\Lambda(t) \mid \mathcal{F}_t]}{\mathbb{E}^*[\Gamma(t) \mid \mathcal{F}_t]} = \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1}\rangle}\,,$$

where $\mathbf{1} = (1, 1, \cdots, 1)^T$.

4. Risk-based optimal investment problem

In this section, we introduce the optimal investment problem of an insurance firm with regime-switching and delay. We consider a problem where the objective is to minimize the risk described by the convex risk measure, with the insurance firm not only concerned with the terminal wealth, but also with the integrated noisy memory surplus over the period $[T - \varrho, T]$. This **problem** is then described as follows: Find the investment strategy $\pi(t) \in \mathcal{A}$ which minimizes the risks of the terminal surplus and the integrated surplus, i.e., $X(T) + \kappa Y(T)$, where $\kappa \geq 0$ denotes the weight between $X(T)$ and $Y(T)$. This allows us to incorporate the terminal wealth as well as the delayed wealth at the terminal time T in the performance functional.

Since we are dealing with a measure of risk, we will use the notion of convex risk measures introduced in Föllmer and Schied [19] and Frittelli and Rosazza [20]. Which is the generalization of the notion of coherent risk measures proposed by Artzner *et. al.* [1].

Definition 4.1. Let $\mathcal{S}$ be a space of all lower bounded $\mathcal{F}_t$-measurable random variables. A convex risk measure on $\mathcal{S}$ is a map $\rho : \mathcal{S} \to \mathbb{R}$ such that:

(1) (*translation*) If $\epsilon \in \mathbb{R}$ and $X \in \mathcal{S}$, then $\rho(X + \epsilon) = \rho(X) - \epsilon$;
(2) (*monotonicity*) For any $X_1, X_2 \in \mathcal{S}$, if $X_1(\omega) \leq X_2(\omega)$; $\omega \in \Omega$, then $\rho(X_1) \geq \rho(X_2)$;
(3) (*convexity*) For any $X_1, X_2 \in \mathcal{S}$ and $\varsigma \in (0, 1)$,

$$\rho(\varsigma X_1 + (1 - \varsigma)X_2) \leq \varsigma\rho(X_1) + (1 - \varsigma)\rho(X_2)\,.$$

Following the general representation of the convex risk measures (see e.g., Theorem 3, Frittelli and Rosazza [20]), also applied by Mataramvura and Øksendal [25], Elliott and Siu [16], Meng and Siu [26], among others, we assume that the risk measure ρ under consideration in this paper, is as follows:

$$\rho(X) = \sup_{Q \in \mathcal{M}_a} \{\mathbb{E}^Q[-X] - \eta(Q)\},$$

where $\mathbb{E}^Q$ is the expectation under Q, for the family $\mathcal{M}_a$ of probability measures and for some penalty function $\eta : \mathcal{M}_a \to \mathbb{R}$. We assume that the penalty function η is bounded from bellow and is not identically equal to $+\infty$. The following result gives the representation of the convex risk measure. For its proof, we refer to Föllmer and Schied [19], Theorem 3.2.

Theorem 4.1. *Let $\mathcal{M}_a$ be a family of probability measures with probability measure Q. Any convex risk measure $\rho \in \mathbb{R}$ is of the form*

$$\rho(X) = \sup_{Q \in \mathcal{M}_a} \{\mathbb{E}^Q[-X] - \eta(Q)\},$$

where $\eta : \mathcal{M}_a \to \mathbb{R}$ is a function such that $\eta(Q) < \infty$.

In order to specify the penalty function, we first describe a family $\mathcal{M}_a$ of all measures Q of Girsanov type. We consider a robust modeling setup, given by a probability measure $Q := Q^{\theta_0,\theta_1,\theta_2}$, with the Radon-Nikodym derivative given by

$$\frac{dQ}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = G^{\theta_0,\theta_1,\theta_2}(T).$$

The density process $G^{\theta_0,\theta_1,\theta_2}(t)\,,\ t \in [-\varrho, T]$, is given by

$$\begin{aligned} dG^{\theta_0,\theta_1,\theta_2}(t) &= G^{\theta_0,\theta_1,\theta_2}(t-)\Big[\theta_0(t)d\widehat{W}(t) + \theta_1(t)dW_1(t) + \int_0^\infty \theta_0(t)\widehat{N}^0_\Lambda(dt,dz) \\ &\quad + \int_{\mathbb{R}} \theta_2(t,z)\widehat{N}_\Lambda(dt,dz)\Big]\,, \\ G^{\theta_0,\theta_1,\theta_2}(0) &= 1, \\ G^{\theta_0,\theta_1,\theta_2}(t) &= 0\,, \quad t \in [-\varrho, 0)\,. \end{aligned} \tag{4.1}$$

The set $\Theta := \{\theta_0, \theta_1, \theta_2\}$ is considered as a set of scenario control. We say that Θ is admissible if $\theta_2(t,z) > -1$ and

$$\mathbb{E}\left[\int_0^T \Big\{\theta_0^2(t) + \theta_1^2(t) + \int_{\mathbb{R}} \theta_2^2(t,z)\nu_\Lambda(dz)\Big\}dt\right] < \infty\,.$$

Then, the family $\mathcal{M}_a$ of probability measures is given by

$$\mathcal{M}_a := \mathcal{M}(\Theta) = \{Q^{\theta_0,\theta_1,\theta_2} \;:\; (\theta_0,\theta_1,\theta_2) \in \Theta\}\,.$$

Let us now specify the penalty function η. Suppose that for each $(\pi, \theta_0, \theta_1, \theta_2) \in \mathcal{A} \times \Theta$ and $t \in [0,T]$, $\pi(t) \in \mathbf{U}_1$ and $\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \in \mathbf{U_2}$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are compact metric spaces in $\mathbb{R}$ and $\mathbb{R}^3$.

Let $\ell : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two bounded measurable convex functions in $\theta(t) \in \mathbf{U}_2$ and $(X(T), Y(T)) \in \mathbb{R} \times \mathbb{R}$, respectively. Then, for each $(\pi, \theta) \in \mathcal{A} \times \Theta$,

$$\mathbb{E}\left[\int_0^T |\ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))|dt + |h(X(T), Y(T))|\right] < \infty\,.$$

As in Mataramvura and Øksendal [25], we consider, for each $(\pi, \theta) \in \mathcal{A} \times \Theta$, a penalty function η of the form

$$\eta(\pi, \theta_0, \theta_1, \theta_2) \;:=\; \mathbb{E}\Big[\int_0^T \ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))dt$$

$$+h(X(T),Y(T)) \mid \mathcal{F}_t\Big].$$

Then, we define a convex risk measure for the terminal wealth and the integrated wealth of an insurance firm, i.e., $X(T)+\kappa Y(T)$, for $\kappa \geq 0$, given the information $\mathcal{F}_t$ associated with the family of probability measures $\mathcal{M}_a$ and the penalty function η, as follows:

$$\rho(X(T),Y(T)) := \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \left\{\mathbb{E}^Q[-(X^\pi(T)+\kappa Y^\pi(T)) \mid \mathcal{F}_t] - \eta(\pi,\theta_0,\theta_1,\theta_2)\right\}.$$

As in Elliott and Siu [16], the main objective of the insurance firm is to select the optimal investment process $\pi(t) \in \mathcal{A}$ so as to minimizes the risks described by $\rho(X(T),Y(T))$. That is, the optimal problem of an insurance firm is:

$$\mathcal{J}(x) := \inf_{\pi\in\mathcal{A}} \left\{ \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \left\{\mathbb{E}^Q[-(X^\pi(T)+\kappa Y^\pi(T)) \mid \mathcal{F}_t] - \eta(\pi,\theta_0,\theta_1,\theta_2)\right\} \right\}. \tag{4.2}$$

Note that $\mathbb{E}^Q[-(X^\pi(T)+\kappa Y^\pi(T))] = \mathbb{E}[-(X^\pi(T)+\kappa Y^\pi(T))G^{\theta_0,\theta_1,\theta_2}(T)]$ (See Cuoco [7] or Karatzas and Shreve [23] for more details). Then from the form of the penalty function,

$$\begin{aligned}\bar{\mathcal{J}}(x) &= \inf_{\pi\in\mathcal{A}} \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \mathbb{E}\Big[-(X^\pi(T)+\kappa Y^\pi(T))G^{\theta_0,\theta_1,\theta_2}(T) \\ &\quad - \int_0^T \ell(t,X(t),Y(t),Z(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))dt - h(X(T),Y(T))\Big] \\ &= \mathcal{J}(x).\end{aligned}$$

For each $(\pi,\theta) \in \mathcal{A}\times\Theta$, suppose that

$$\begin{aligned}\mathcal{V}^{\pi,\theta}(x) &:= \mathbb{E}\Big[-(X^\pi(T)+\kappa Y^\pi(T))G^{\theta_0,\theta_1,\theta_2}(T) \\ &\quad - \int_0^T \ell(t,X(t),Y(t),Z(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))dt - h(X(T),Y(T))\Big].\end{aligned}$$

Then,

$$\mathcal{J}(x) = \inf_{\pi\in\mathcal{A}} \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \mathcal{V}^{\pi,\theta}(x) = \mathcal{V}^{\pi^*,\theta^*}(x), \tag{4.3}$$

that is, the insurance firm selects an optimal investment strategy π so as to minimize the maximal risks, whilst the market reacts by selecting a probability measure indexed by $((\theta_0,\theta_1,\theta_2)) \in \Theta$ corresponding to the worst-case scenario, where the risk is maximized. Actually, "the market" or even better "Nature" plays first and chooses $((\theta_0,\theta_1,\theta_2)) \in \Theta$ in order to create the most unfavorable scenario for the controller. Then, the controller (insurance firm) tries to solve a minimization problem over this worst case scenario. This is a classical robust control setting, see e.g., Hansen and Sargent [21]. To solve this game problem, one must select the optimal strategy $(\pi^*,\theta_0^*,\theta_1^*,\theta_2^*)$ from the insurance firm and the market, respectively, as well as the optimal value function $\mathcal{J}(x)$.

5. The BSDE approach to a game problem

In this section, we solve the risk-based optimal investment problem of an insurance firm using delayed BSDE with jumps. Delayed BSDEs may arise in insurance and finance, when one wants to find an investment strategy which should replicate a liability or meet a purpose depending on the past values of the portfolio. For instance, under participating contracts in life insurance endowment contracts, we have a so called *performance-linked payoff*, that is, the payoff from the policy is related to the performance of the portfolio held by the insurance firm. Thus, the current portfolio and the past values of the portfolio have an impact on the final value of the liability. For more discussions on this and more applications of delayed BSDEs see Delong [10].

We first consider the following notation in order to establish the existence and uniqueness result of a delayed BSDE with jumps.

- $\mathbb{L}^2_{-\varrho}(\mathbb{R})$- the space of measurable functions $k : [-\varrho, 0] \mapsto \mathbb{R}$, such that $\int_{-\varrho}^0 |k(t)|^2 dt < \infty$.
- $\mathbb{S}^2_{-\varrho}(\mathbb{R})$- the space of bounded measurable functions $y : [-\varrho, 0] \mapsto \mathbb{R}$ such that
$$\sup |y(t)|^2 < \infty\,;$$
- $\mathbb{H}^2_{-\varrho,\nu}$- the space of product measurable functions $\upsilon : [-\varrho, 0] \times \mathbb{R} \mapsto \mathbb{R}$, such that
$$\int_{-\varrho}^0 \int_{\mathbb{R}} |\upsilon(t, z)|^2 \nu(dz)dt < \infty\,;$$
- $\mathbb{L}^2(\mathbb{R})$- the space of random variables $\xi : \Omega \mapsto \mathbb{R}$, such that $\mathbb{E}[\,|\xi|^2] < \infty$;
- $\mathbb{H}^2(\mathbb{R})$- the space of measurable functions $K : \mathbb{R} \mapsto \mathbb{R}$ such that
$$\mathbb{E}\left[\int_{\mathbb{R}} |K(t)|^2 dt\right] < \infty\,;$$
- $\mathbb{S}^2(\mathbb{R})$- the space of adapted càdlàg processes $Y : \Omega \times [0, T] \mapsto \mathbb{R}$ such that
$$\mathbb{E}[\sup |Y(t)|^2] < \infty$$
and
- $\mathbb{H}^2_{\nu}$- the space of predictable processes $\Upsilon : \Omega \times [0, T] \times \mathbb{R} \mapsto \mathbb{R}$, such that
$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\Upsilon(t, z)|^2 \nu(dz)dt\right] < \infty.$$

Define the following delayed BSDE with jumps:

$$\begin{aligned} d\mathcal{Y}(t) &= -\mathcal{W}(t, \pi(t), \theta(t))dt + K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) \\ &\quad + \int_{\mathbb{R}} \Upsilon_1(t, z)\widehat{N}_\Lambda(dt, dz) + \int_0^\infty \Upsilon_2(t, z)\widehat{N}^0_\Lambda(dt, dz); \\ \mathcal{Y}(T) &= h(X(T), Y(T))\,, \end{aligned} \tag{5.1}$$

where

$$\mathcal{W}(t, \pi(t), \theta(t)) \;:=\; \mathcal{W}(t, \mathcal{Y}(t), \mathcal{Y}(t-\varrho), K_1(t), K_1(t-\varrho), K_2(t), K_2(t-\varrho), \Upsilon_1(t, \cdot),$$

$$\Upsilon_1(t-\varrho,\cdot),\Upsilon_2(t,\cdot),\Upsilon_2(t-\varrho,\cdot),\pi(t),\theta(t))\,.$$

We assume that the generator $\mathcal{W}:\Omega\times[0,T]\times\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2_{-\varrho}(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_{-\varrho}(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})\times\mathbb{H}^2_{-\varrho,\nu}(\mathbb{R})\mapsto\mathbb{R}$ satisfy the following Lipschtz continuous condition, i.e., there exists a constant $C>0$ and a probability measure η on $([-\varrho,0]\times\mathcal{B}([-\varrho,0]))$ such that

$$\begin{aligned}
&\Big|\mathcal{W}(t,\pi(t),\theta(t))-\tilde{\mathcal{W}}(t,\pi(t),\theta(t))\Big|^2\\
\le\ & C\Big(\int_{-\varrho}^0|y(t+\zeta)-\tilde{y}(t+\zeta)|^2\eta(d\zeta)+\int_{-\varrho}^0|k_1(t+\zeta)-\tilde{k}_1(t+\zeta)|^2\eta(d\zeta)\\
&+\int_{-\varrho}^0|k_2(t+\zeta)-\tilde{k}_2(t+\zeta)|^2\eta(d\zeta)+\int_{-\varrho}^0\int_{\mathbb{R}}|\upsilon_1(t+\zeta,z)-\tilde{\upsilon}_1(t+\zeta,z)|^2\nu(dz)\eta(d\zeta)\\
&+\int_{-\varrho}^0\int_{\mathbb{R}}|\upsilon_2(t+\zeta,z)-\tilde{\upsilon}_2(t+\zeta,z)|^2\nu(dz)\eta(d\zeta)+\int_0^T|y(t)-\tilde{y}(t)|^2dt\\
&+\int_0^T|k_2(t)-\tilde{k}_2(t)|^2dt+\int_0^T\int_{\mathbb{R}}|\upsilon_1(t,z)-\tilde{\upsilon}_1(t,z)|^2\nu(dz)dt\\
&+\int_0^T\int_{\mathbb{R}}|\upsilon_2(t,z)-\tilde{\upsilon}_2(t,z)|^2\nu(dz)dt\Big)\,.
\end{aligned}\tag{5.2}$$

Then, if $h\in\mathbb{L}^2$ and the above Lipschitz condition is satisfied, one can prove the existence and uniqueness solution $(\mathcal{Y},K_1,K_2,\Upsilon_1,\Upsilon_2)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})$ of a delayed BSDE with jumps (5.1). See Delong and Imkeller [12] and Delong [11] for more details. In practice, $\mathcal{Y}$ denotes a replicating portfolio, $K_1,K_2,\Upsilon_1,\Upsilon_2$ represent the replicating strategies, $h(X(T),Y(T))$ is a terminal liability and $\mathcal{W}$ models the stream liability that has to be covered during the contract life-time, such as payments of the claims during the investment period.

The key result for solving our delayed stochastic differential game problem is based on the following theorem.

Theorem 5.1. *Assume that the Isaacs conditions condition holds, i.e., suppose that there exists a strategy $(\hat{\pi}(t),\hat{\theta}(t))\in\mathbf{U}_1\times\mathbf{U}_2$ such that*

$$\begin{aligned}
\mathcal{W}(t,\hat{\pi}(t),\hat{\theta}(t)) &= \inf_{\pi\in\mathcal{A}}\sup_{(\theta_0,\theta_1,\theta_2)\in\Theta}\mathcal{W}(t,\pi,\theta)\\
&= \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta}\inf_{\pi\in\mathcal{A}}\mathcal{W}(t,\pi,\theta)\,.
\end{aligned}\tag{5.3}$$

Then, there exists a unique solution $(\mathcal{Y}^{\pi,\theta}(t),K_1^{\pi,\theta}(t),K_2^{\pi,\theta}(t),\Upsilon_1^{\pi,\theta}(t,\cdot),\Upsilon_2^{\pi,\theta}(t,\cdot))\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})$ of the BSDE (5.1), for all $(\pi,\theta)\in\mathcal{A}\times\Theta$.

Furthermore, the value function $\mathcal{J}(x)$ is given by $\mathcal{Y}^{\hat{\pi},\hat{\theta}}(t)$, and the pair of strategies of the problem (4.2) given by

$$\begin{cases}\pi^*(t)=\hat{\pi}(t,Y(t),Y(t-\varrho),K_1(t),K_1(t-\varrho),K_2(t),K_2(t-\varrho),\Upsilon_1(t,\cdot),\Upsilon_1(t-\varrho,\cdot),\Upsilon_2(t,\cdot),\Upsilon_2(t-\varrho,\cdot))\,,\\ \theta^*(t)=\hat{\theta}(t,Y(t),Y(t-\varrho),K_1(t),K_1(t-\varrho),K_2(t),K_2(t-\varrho),\Upsilon_1(t,\cdot),\Upsilon_1(t-\varrho,\cdot),\Upsilon_2(t,\cdot),\Upsilon_2(t-\varrho,\cdot))\,.\end{cases}\tag{5.4}$$

is a saddle point of the zero-sum stochastic differential game (4.3).

Proof. The existence and uniqueness of the solution follows from the Lipschitz condition above. The remainder of the proof is based on the comparison principle for BSDEs with jumps as follows, (see Theorem 3.2.1 in Delong [11]). Define three generators ϕ_1, ϕ_2 and ϕ_3 by

$$\begin{aligned}\phi_1(t,\cdot) &= \mathcal{W}(t,\hat{\pi}(t),\theta(t))\\ \phi_2(t,\cdot) &= \mathcal{W}(t,\hat{\pi}(t),\hat{\theta}(t))\\ \phi_3(t,\cdot) &= \mathcal{W}(t,\pi(t),\hat{\theta}(t))\end{aligned}$$

and the corresponding BSDEs

$$\begin{aligned}d\mathcal{Y}_1(t) &= -\phi_1(t,\cdot)dt + K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) + \int_{\mathbb{R}} \Upsilon_1(t,z)\widehat{N}_\Lambda(dt,dz)\\ &\quad + \int_0^\infty \Upsilon_2(t,z)\widehat{N}^0_\Lambda(dt,dz)\,,\\ \mathcal{Y}_1(T) &= h(X(T),Y(T))\,.\end{aligned}$$

$$\begin{aligned}d\mathcal{Y}_2(t) &= -\phi_2(t,\cdot)dt + Z_1(t)d\widehat{W}(t) + Z_2(t)dW_1(t) + \int_{\mathbb{R}} \Phi_1(t,z)\widehat{N}_\Lambda(dt,dz)\\ &\quad + \int_0^\infty \Phi_2(t,z)\widehat{N}^0_\Lambda(dt,dz)\\ \mathcal{Y}_2(T) &= h(X(T),Y(T))\,.\end{aligned}$$

and

$$\begin{aligned}d\mathcal{Y}_3(t) &= -\phi_3(t,\cdot)dt + L_1(t)d\widehat{W}(t) + L_2(t)dW_1(t) + \int_{\mathbb{R}} \Psi_1(t,z)\widehat{N}_\Lambda(dt,dz)\\ &\quad + \int_0^\infty \Psi_2(t,z)\widehat{N}^0_\Lambda(dt,dz)\,,\\ \mathcal{Y}_3(T) &= h(X(T),Y(T))\,.\end{aligned}$$

It is well known that $\mathcal{W}$ satisfy the Isaac's condition if and only if there exist two measurable functions π^* and θ^* such that

$$\phi_1(t,\cdot) \leq \phi_2(t,\cdot) \leq \phi_3(t,\cdot).$$

Then, by comparison principle, $\mathcal{Y}_1(t) \leq \mathcal{Y}_2(t) = \mathcal{J}(x) \leq \mathcal{Y}_3(t)$, for all $t \in [0,T]$. By uniqueness, we get $\mathcal{Y}_2(t) = \mathcal{V}^{\pi^*,\theta^*}$. Hence, the saddle point for the game problem is given by (5.4). □

In order to solve our main problem, note that from the dynamics of the processes $X(t)$, $Y(t)$ and $G^{\theta_0,\theta_1,\theta_2}$ in (2.4), (2.3) and (4.1), respectively and applying the Itô's differentiation rule for delayed SDEs with jumps (See Baños *et. al.* [2], Theorem 3.8), we have:

$$\begin{aligned}
& d[(X(t)+\kappa Y(t))G^{\theta_0,\theta_1,\theta_2}(t)] \\
= \ & G^{\theta_0,\theta_1,\theta_2}(t-)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t)) \\
& +(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& +\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& -\int_0^{\infty}\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big]dt+G^{\theta_0,\theta_1,\theta_2}(t-)[(\pi(t)\beta(t)+X(t)\theta_0(t))d\widehat{W}(t) \\
& +(\theta_1(t)+X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}))dW_1(t)] \\
& +G^{\theta_0,\theta_1,\theta_2}(t-)\int_{\mathbb{R}}[(1+\theta_2(t,z))\pi(t)\Big(e^z-1\Big)+X(t)\theta_2(t,z)]\widehat{N}_{\Lambda}(dt,dz) \\
& -G^{\theta_0,\theta_1,\theta_2}(t-)\int_0^{\infty}[(1+\theta_0(t))z-X(t)\theta_0(t)]\widehat{N}^0_{\Lambda}(dt,dz)\ ,.
\end{aligned}$$

Thus, for each $(\pi,\theta_0,\theta_1,\theta_2)$,

$$\begin{aligned}
& \mathcal{J}(x) \\
= \ & \mathbb{E}\Big\{-\int_0^T\Big[G^{\theta_0,\theta_1,\theta_2}(t-)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t)) \\
& +(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& +\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& -\int_0^{\infty}\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big] \\
& +\ell(t,X(t),Y(t),U(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))\Big]dt-h(X(T),Y(T))\Big\}.
\end{aligned}$$

We now define, for each $(t,X,Y,U,\pi,\theta_0,\theta_1,\theta_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbf{U}_1\times\mathbf{U}_2$, a function

$$\begin{aligned}
& \tilde{\ell}(t,X(t),Y(t),U(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot)) \\
= \ & G^{\theta_0,\theta_1,\theta_2}(t-)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t)) \\
& +(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& +\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& -\int_0^{\infty}\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big] \\
& +\ell(t,X(t),Y(t),U(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))\,.
\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{J}(x) &= -x_0 + \mathbb{E}\Big[-\int_0^T \tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))dt \\ &\qquad -h(X(T), Y(T))\Big] .\end{aligned}$$

Define, for each $(\pi, \theta) \in \mathcal{A} \times \Theta$, a functional

$$\tilde{\mathcal{J}}(x) = \mathbb{E}\Big[-\int_0^T \tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))dt - h(X(T), Y(T))\Big] .$$

Then, the stochastic differential delay game problem discussed in the previous section is equivalent to the following problem:

$$\tilde{\mathcal{V}}(t,x) = \inf_{\pi\in\mathcal{A}} \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \tilde{\mathcal{J}}(x) .$$

We now define the Hamiltonian of the aforementioned game problem $\mathcal{H} : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ as follows:

$$\begin{aligned}&\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\ :=\ & -\tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) .\end{aligned}$$

In order for the Hamiltonian $\mathcal{H}$ to satisfy the Issac's condition, we require that $\mathcal{H}$ is convex in π and concave in $(\theta_0, \theta_1, \theta_2)$. Moreover, for the existence and uniqueness solution of the corresponding delayed BSDE with jumps, the Hamiltonian should satisfy the Lipschitz condition. From the boundedness of the associate parameters, we prove that $\mathcal{H}$ is indeed Lipschitz.

Lemma 5.2. *The Hamiltonian $\mathcal{H}$ is Lipschitz in $(K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot))$ in the sense of (5.2) and uniformly in $(t, X(t), Y(t), U(t))$.*

Proof. For each $(\pi, (\theta_0, \theta_1, \theta_2)) \in \mathbf{U}_1 \times \mathbf{U}_2$, let

$$\begin{aligned}&\ell_1(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\ =\ & G^{\theta_0,\theta_1,\theta_2}(t-)\Big[p(t) + (r(t) - \vartheta(t) - \xi)X(t) + \pi(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) \\ & +(\bar{\vartheta}(t) - \kappa\zeta)Y(t) - \xi U(t) + \pi(t)\beta(t)\theta_0(t) + \theta_1(t)\kappa X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\ & + \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j\rangle \Big(\pi(t)\int_{\mathbb{R}} \Big(e^z - 1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\ & - \int_0^\infty \lambda_j(t)z(1 + \theta_0(t))f_j(dz)\Big)\Big] .\end{aligned}$$

Then

$$\begin{aligned}&\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\ =\ & -(\ell_1(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))\end{aligned}$$

$$+\ell(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)))\,.$$

By definition of the admissible strategy $(\pi, (\theta_0, \theta_1, \theta_2)) \in \mathbf{U}_1 \times \mathbf{U}_2$, we conclude that ℓ_1 is bounded. Furthermore, by definition, ℓ is also bounded. This implies that $\mathcal{H}$ is uniformly bounded with respect to $(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2$.

Now, suppose that $\mathcal{H}$ were not Lipschitz in $(K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot))$, uniformly in $(t, X(t), Y(t), U(t))$. Then, there exist two points $(K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot))$ and $(\tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot))$ such that

$$\begin{aligned}
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \\
&-\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))|
\end{aligned}$$

is unbounded. However, since $\mathcal{H}$ is uniformly bounded with respect to $(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2$, we have

$$\begin{aligned}
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \\
&-\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \quad \leq \\
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \\
&+|\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \quad \leq \quad 2D\,,
\end{aligned}$$

for some positive constant D.

Therefore, we must have that $\mathcal{H}$ is Lipschitz in $(K_1(t),\ K_2(t),\ \Upsilon_1(t, \cdot),\ \Upsilon_2(t, \cdot))$ and uniformly in $(t, X(t), Y(t), U(t))$. □

Then, following Theorem 5.1 above, we establish the relationship between the value function of the game problem and the solution of a delayed BSDE with jumps. Thus, the value function $\tilde{\mathcal{J}}(t)$ is given by the following noisy memory BSDE:

$$\begin{aligned}
&d\tilde{\mathcal{J}}(t) \\
= \;&-\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi^*(t), \theta_0^*(t), \theta_1^*(t), \theta_2^*(t, \cdot))dt \\
&+K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) + \int_{\mathbb{R}} \Upsilon_1(t, z)\widehat{N}_\Lambda(dt, dz) + \int_{\mathbb{R}} \Upsilon_2(t, z)\widehat{N}^0_\Lambda(dt, dz)\,,
\end{aligned}$$

with the terminal condition $\tilde{\mathcal{J}}(T) = h(X(T), Y(T))$.

In fact, the existence and uniqueness of the solution to the above delayed BSDE with jumps is guaranteed from the Lipschitz condition proved in Lemma 5.2. Then, the solution of the delayed BSDE is given by

$$\begin{aligned}
\tilde{\mathcal{J}}(t) \;&=\; \mathbb{E}\Big[h(X(T), Y(T)) - \int_t^T \tilde{\ell}(s, X(s), Y(s), U(s), \pi^*(s), \theta_0^*(s), \theta_1^*(s), \theta_2^*(s, \cdot))ds \mid \mathcal{F}_t\Big] \\
&=\; \mathcal{V}(\pi^*, \theta_1^*, \theta_2^*, \theta_3^*)\,,
\end{aligned}$$

which is the optimal value function from Theorem 5.1.

6. A quadratic penalty function case

In this section, we consider a convex risk measure with quadratic penalty. We derive explicit solutions when ℓ is quadratic in θ_0, θ_1, θ_2 and h is identical zero. The penalty function under consideration here, may be related to the entropic penalty function considered, for instance, by Delbaen *et. al.* [9]. It has also been adopted by Elliott and Siu [16], Siu [34] and Meng and Siu [26]. We obtain the explicit optimal investment strategy and the optimal risks for this case of a risk-based optimization problem with jumps, regime switching and noisy delay. Finally, we consider some particular cases and we see using some numerical parameters, how an insurance firm can allocate his portfolio.

Suppose that the penalty function is given by

$$\begin{aligned}&\ell(t,X(t),Y(t),Z(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))\\ :=\ &\frac{1}{2(1-\delta)}\Big(\theta_0^2(t)+\theta_1^2(t)+\int_{\mathbb{R}}\theta_2^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big)G^{\theta_0,\theta_1,\theta_2}(t)\,,\end{aligned}$$

where $1-\delta$ is a measure of an insurance firm's relative risk aversion and $\delta<1$. Then, the Hamiltonian $\mathcal{H}$ becomes:

$$\begin{aligned}&\mathcal{H}(t,X(t),Y(t),U(t),K_1(t),K_2(t),\Upsilon_1(t,\cdot),\Upsilon_2(t,\cdot),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))\\ =\ &-G^{\theta_0,\theta_1,\theta_2}(t-)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t))\\ &+(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]})\\ &+\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz)\\ &-\int_0^\infty\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big]\\ &-\frac{1}{2(1-\delta)}\Big(\theta_0^2(t)+\theta_1^2(t)+\int_{\mathbb{R}}\theta_2^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big)G^{\theta_0,\theta_1,\theta_2}(t)\,.\end{aligned}$$

Applying the first order condition for maximizing the Hamiltonian with respect to θ_0, θ_1 and θ_2, and minimizing with respect to π, we obtain the following

$$\begin{aligned}\pi^*(t)\ &=\ \frac{\hat{\alpha}^{\Lambda}(t)-r(t)+(1-\delta)\beta(t)\Big(\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\int_0^\infty z\lambda_j(t)f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t)+\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\int_{\mathbb{R}}\Big(e^z-1\Big)^2\varepsilon_j(t)\nu_j(dz)\Big)}\,,\\ \theta_0^*(t)\ &=\ (1-\delta)\Big[\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\int_0^\infty z\lambda_j(t)f_j(dz)\\ &-\frac{\hat{\alpha}^{\Lambda}(t)-r(t)+(1-\delta)\beta(t)\Big(\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\int_0^\infty z\lambda_j(t)f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t)+\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\int_{\mathbb{R}}\Big(e^z-1\Big)^2\varepsilon_j(t)\nu_j(dz)\Big)}\beta(t)\Big]\,,\end{aligned}$$

$$\theta_1^*(t) \;=\; (\delta-1)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]})$$

and

$$\theta_2^*(t,z) = (\delta-1)z\frac{\hat{\alpha}^{\Lambda}(t)-r(t)+(1-\delta)\beta(t)\Big(\sum_{j=1}^D\langle\hat{\Lambda}(t-),e_j\rangle\int_0^\infty z\lambda_j(t)f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t)+\sum_{j=1}^D\langle\hat{\Lambda}(t-),e_j\rangle\int_{\mathbb{R}}\Big(e^z-1\Big)^2\varepsilon_j(t)\nu_j(dz)\Big)}\,.$$

Then, the value function of the game problem is given by the following BSDE:

$$\begin{aligned}
d\mathcal{J}(t) \;=\;& \Big\{G^{\theta_0^*,\theta_1^*,\theta_2^*}(t-)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi^*(t)(\hat{\alpha}^{\Lambda}(t)-r(t))\\
&+(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi^*(t)\beta(t)\theta_0^*(t)+\theta_1^*(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]})\\
&+\sum_{j=1}^D\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi^*(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2^*(t,z)\varepsilon_j(t)\nu_j(dz)\\
&-\int_0^\infty\lambda_j(t)z(1+\theta_0^*(t))f_j(dz)\Big)\Big]\\
&+\frac{1}{2(1-\delta)}\Big((\theta_0^*)^2(t)+(\theta_1^*)^2(t)+\int_{\mathbb{R}}(\theta_2^*)^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big)\Big\}dt+K_1(t)d\widehat{W}(t)\\
&+K_2(t)dW_1(t)+\int_{\mathbb{R}}\Upsilon_1(t,z)\widehat{N}_{\Lambda}(dt,dz)+\int_{\mathbb{R}}\Upsilon_2(t,z)\widehat{N}^0_{\Lambda}(dt,dz)\,,\\
\mathcal{J}(T) \;=\;& h(X(T),Y(T))\,.
\end{aligned}$$

Note that the generator of above BSDE is independent of the control variables $(K_1(t),\, K_2(t),\, \Upsilon_1(t,\cdot),\, \Upsilon_2(t,\cdot))$. Then following similar arguments as in Dahl *et. al.* [8], Example 7.2, $K_1(t)=K_2(t)=\Upsilon_1(t,\cdot)=\Upsilon_2(t,\cdot)=0$. Furthermore, the value function $\mathcal{J}(t)$ is given by:

$$\begin{aligned}
\mathcal{J}(t) \;=\;& \mathbb{E}\Big\{\int_t^T\Big\{G^{\theta_0^*,\theta_1^*,\theta_2^*}(s-)\Big[p(s)+(r(s)-\vartheta(s)-\xi)X(s)+\pi^*(s)(\hat{\alpha}^{\Lambda}(s)-r(s))\\
&+(\bar{\vartheta}(s)-\kappa\zeta)Y(s)-\xi U(s)+\pi^*(s)\beta(s)\theta_0^*(s)+\theta_1^*(s)\kappa X(s)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]})\\
&+\sum_{j=1}^D\langle\hat{\Lambda}(s-),e_j\rangle\Big(\pi^*(s)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2^*(s,z)\varepsilon_j(s)\nu_j(dz)\\
&-\int_0^\infty\lambda_j(s)z(1+\theta_0^*(s))f_j(dz)\Big)\Big]+\frac{1}{2(1-\delta)}\Big((\theta_0^*)^2(s)+(\theta_1^*)^2(s)\\
&+\int_{\mathbb{R}}(\theta_2^*)^2(s,z)\nu_{\hat{\Lambda}}(dz)\Big)\Big\}dt+h(X(T),Y(T))\mid\mathcal{F}_t\Big\}\,.
\end{aligned}$$

Example 6.1. Suppose that the the driving processes $\tilde{N}$ and $\tilde{N}^0$ are Poisson processes N and N^0, with the jump intensities λ and λ^0. Under noisy delay modeling, we consider the following cases:

Case 1. We suppose that there is no regime switching in the model, then the optimal investment strategy is given by

$$\pi^*(t) = \frac{\alpha(t) - r(t)}{(1-\delta)(\beta^2(t) + \lambda)}\,. \tag{6.1}$$

To be concrete, we assume that the interest rate $r = 4.5\%$, the appreciation rate $\alpha = 11\%$, the volatility $\beta = 20\%$, the insurance firm's relative risk aversion $\delta = 0.5$ and the jump intensity given by $\lambda = 0.5$. Then the optimal portfolio invested in the risky asset is given by $\pi^* = 0.24074$, i.e., 24.074% of the wealth should be invested in the risky share.

Note that, if we assume the absence of jump, i.e., $\lambda = 0$, the optimal investment strategy is equal to that in [34], Example 1.

Case 2. We suppose existence of two state Markov chain $\mathcal{S} = \{e_1, e_2\}$, where the states e_1 and e_2 represent the expansion and recession of the economy respectively. By definition, $\langle\hat{\Lambda}(t), e_1\rangle = \mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)$ and $\langle\hat{\Lambda}(t), e_2\rangle = 1 - \mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)$. Let $\alpha_i, r_i, \lambda_i, \lambda_i^0$ be the associate parameters when the economy is in state e_i, $i = 1, 2$. Then the optimal portfolio is given by

$$\begin{aligned}\pi^*(t) &= \frac{[\alpha_1(t) - r_1(t) - (\alpha_2(t) - r_2(t)) + (1-\delta)\beta(t)(\lambda_1^0(t) - \lambda_2^0(t))]\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)}{(1-\delta)[\beta^2(t) + \lambda_2(t) + (\lambda_1(t) - \lambda_2(t))\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)]} \\ &\quad + \frac{\alpha_2(t) - r_2(t) + (1-\delta)\beta(t)\lambda_2^0}{(1-\delta)[\beta^2(t) + \lambda_2(t) + (\lambda_1(t) - \lambda_2(t))\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)]}\,.\end{aligned}$$

In this case, we consider the following parameters: $\alpha_1 = 13\%, \alpha_2 = 9\%, r_1 = t\%, r_2 = 9\%, \beta = 20\%, \lambda_1^0 = \lambda_1 = 0.5, \lambda_2 = \lambda_2^0 = 0.7, \delta = 0.5$ and $\mathbb{P}(X = e_1) = 70\%$. Then $\pi^* = 0.28$, i.e., 28% of the wealth should be invested in the risky share.

From the optimal strategy (6.1), we can see that the volatility β and the jump intensity λ are in the denominator, which implies that when these parameters increase, the proportion of the wealth invested in the risky asset will reduce. This is due to the uncertainty that the risk factors β and α may cause. However, this may not be the case when the regime switching case is applied, since it brings more information on the situation of the economy.

Finally, one can see that the delay has an impact only on the total wealth of the insurance firm available at time t, not in the investment strategy.

7. Conclusion

In this paper, we considered a risk-based optimization problem of an insurance firm in a regime-switching model with noisy memory. Using a robust optimization modelling, we formulated the problem as a zero-sum stochastic differential delay game problem between the insurer and the market with a convex risk measure of the terminal surplus and delay. This type of risk measure allows that a diversification of investments does not increase the risks. To turn the model from partial observation to complete observation setup, we used the filtering theory techniques, then, by the BSDE approach, we solved the game problem. The model in this paper combined and generalized several components:

- an asset market where prices follow a regime-switching jump-diffusions with unobservable states;

- capital inflows/outflows which are subject to delays of different forms;
- premium and claim processes which are close to standard actuarial settings.

Then, we considered an example to illustrate the applicability of the model for the quadratic penalty case.

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