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Measures on Boolean algebras

by

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I declare that this dissertation, which I hereby submit for the degree *MSc Mathematics* at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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Abstract

This thesis deals with a number of related results on Boolean algebras. First, we prove the Stone Representation Theorem, which shows that every Boolean algebra is isomorphic to an algebra of sets, namely the clopen algebra of its Stone space. Then we prove the Loomis-Sikorski Theorem, which shows exactly how the Stone Representation Theorem may be extended to represent countable suprema and infima in terms of unions and intersections of sets. Finally, we discuss strictly positive measures. We provide a characterisation, in terms of intersection numbers and covering numbers, of those Boolean algebras which admit strictly positive measures, and we conclude by showing that a σ-complete Boolean algebra admits a strictly positive σ-additive measure if and only if it admits a strictly positive measure and it is weakly σ-distributive.

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Introduction

Boolean algebras are algebraic and order-theoretic generalisations of collections of sets which are closed under unions, intersections, and complementation. They have a natural place in measure theory, as every measure is defined on some kind of Boolean algebra—usually a σ-algebra. In fact, many aspects of measure theory can be formulated in the language of Boolean algebras. Given a probability space (X, Σ, μ), denote by N_μ the collection of μ-null sets in Σ. Then N_μ is a σ-ideal in the σ-algebra Σ, and so Σ/N_μ is a Boolean algebra. Furthermore, μ induces a strictly positive, countably additive real valued function on Σ/N_μ, that is, a measure on Σ/N_μ. One approach to measure theory is to work with an abstract Boolean algebra $\mathbb{L}$ and a (finite or countably) additive measure on it. Fremlin [Fre02] makes a forceful case for this approach.

This leads to two questions. Firstly, which Boolean algebras can be represented as the quotient of a σ-algebra Σ with a σ-ideal N in Σ? Secondly, which Boolean algebras admit strictly positive finitely additive, respectively countably additive, measures? The first question was answered by Loomis and Sikorski ([Loo47] and [Sik69], respectively): every σ-complete Boolean algebra can be represented in this way. The second question dates back to the 1930s. It was posed by von Neumann as Problem 163 in the Scottish Book [Ula][1]. It has attracted the attention of many mathematicians, including Horn and Tarski [HT48], Maharam [Mah47], Gaifman [Gai64], and Kelley [Kel59]. Jech [Jec17] has also made a recent contribution. In this thesis, we present a proof of the Loomis-Sikorksi Theorem, and Kelley's combinatorial characterization of Boolean algebras which admit strictly positive (countably) additive measures.

An indispensable tool in our work, as in much of the theory of Boolean algebras, is the remarkable Stone Representation Theorem [Sto36]. It states that every Boolean algebra is (isomorphic to) a Boolean algebra whose elements are sets and whose operations are union, intersection and set-theoretic complementation. It allows the use of set-theoretic arguments in general Boolean algebras, and proves indispensable in our further discussions. Chapter 1 is dedicated to this result and its proof. Our approach is based on that found in [HG09].

The second chapter is devoted to the Loomis-Sikorski Theorem, which answers the question of how to represent countable suprema and infima using countable unions and intersections. It shows that countable operations in Boolean algebras cannot generally be represented exactly as unions and intersections, but they can be represented by taking an appropriate quotient of an associated σ-algebra. Our approach is based on [Fre02] (Theorem 314M).

[1]The Scottish Book was a notebook used by mathematicians in Lwów, Poland in the 1930s for writing down problems that they had discussed in *the Scottish Café*, the book's namesake. The authors of the book had a habit of promising prizes for the solutions of the problems they recorded in it, usually something like "three small beers" or "a bottle of wine". For Problem 163, von Neumann promised "a bottle of whiskey of measure > 0." For the solution of Problem 153, posed by Stanisław Mazur on approximations of continuous functions, the prize was "a live goose". The problem was solved by Per Enflo in 1972, and Mazur kept his promise: a photograph of Mazur presenting Enflo with his prize can be found at https://upload.wikimedia.org/wikipedia/commons/e/ec/MazurGes.jpg.

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The third chapter is devoted to strictly positive measures. Here, we consider the results of Kelley [Kel59]. We begin by defining measures and the so-called countable chain condition, and demonstrate the relationship between the countable chain condition and the existence of a strictly positive measure on a Boolean algebra. We then exhibit a bijection between measures on a Boolean algebra and positive linear functionals on an associated Banach space. Next, we examine intersection numbers and covering numbers, and conclude with a characterisation, in terms of intersection numbers and covering numbers, of exactly which Boolean algebras admit strictly positive measures. Finally, we define weak σ-distributivity and show that a σ-complete Boolean algebra which admits a strictly positive measure admits a strictly positive σ-additive measure if and only if it is weakly σ-distributive.

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1 The Stone Representation Theorem

In this chapter, we prove the Stone Representation Theorem, which shows that every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of its Stone space. It allows arguments about general Boolean algebras to be made purely in terms of set algebras (at least when only finitary unions and intersections are concerned).

1.1 Preliminaries

We begin with some definitions and preliminary facts needed for the Stone Representation Theorem.

1.1.1 Lattices

Definition 1.1.1: Semilattices

A **join-semilattice** is a partially ordered set in which every pair of elements has a supremum (i.e. a least upper bound). A **meet-semilattice** is a partially ordered set in which every pair of elements has an infimum (i.e. a greatest lower bound).

Definition 1.1.2: Lattices

A **lattice** is a partially ordered set in which every pair of elements has a supremum and an infimum.

Let $\mathbb{L}$ be a set and let $\leq$ be a partial order on $\mathbb{L}$. We say that $\mathbb{L}$ is a **lattice** if, for all $A, B \in \mathbb{L}$, there exist elements $A \vee B$ and $A \wedge B$ of $\mathbb{L}$ which are the supremum and infimum respectively of $\{A, B\}$ in $\mathbb{L}$. The expressions $A \vee B$ and $A \wedge B$ are read as "A join B" and "A meet B" respectively.

A useful mnemonic for distinguishing the symbols $\vee$ and $\wedge$ is to remember the symbols used for union and intersection of sets. The join symbol ($\vee$) resembles the union symbol ($\cup$), and the set $A \cup B$ can be thought of as the result of *joining* the elements of the sets A and B. The meet symbol ($\wedge$) resembles the intersection symbol ($\cap$), and the set $A \cap B$ can be thought of as "the set of points where A and B meet". Indeed, the following example reinforces this analogy.

Example 1.1.3: The power-set lattice

Let X be a set. The set $\mathcal{P}(X)$ of all subsets of X is partially ordered by inclusion. Under this partial order, $\mathcal{P}(X)$ is a lattice, and for all $A, B \subseteq X$,

$$A \vee B = A \cup B$$
$$A \wedge B = A \cap B.$$

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Indeed, $A, B \subseteq A \cup B$, and any subset of X which includes both A and B as subsets includes $A \cup B$ as well. Furthermore, $A \cap B \subseteq A, B$ and any subset of X which is included in both A and B is included in their intersection.

In the context of lattices, we use the term "dual" to refer to an expression obtained from some other expression by swapping $\vee$ and $\wedge$. For example, the dual of the expression "$A \vee (B \wedge C) = D$" is "$A \wedge (B \vee C) = D$".

There is an alternative algebraic definition of lattices, which we give below. The definition used is ultimately a matter of choice—the two definitions are equivalent.

Definition 1.1.4: Lattices (algebraic version)

A **semilattice** is a commutative idempotent[2] semigroup. If $\mathbb{L}$ is a semilattice and its operation is denoted $\vee$, then we say that $\mathbb{L}$ is a **join-semilattice**. If the operation is denoted $\wedge$, we say that $\mathbb{L}$ is a **meet-semilattice**.

A **lattice** is a set which is simultaneously a join-semilattice and a meet-semilattice, which additionally satisfies the absorption law: for all $A, B \in \mathbb{L}$,

$$A \vee (A \wedge B) = A.$$

The absorption law is equivalent to its dual. Thus, to show that $\mathbb{L}$ is a lattice, one can alternatively show that

$$A \wedge (A \vee B) = A$$

for all $A, B \in \mathbb{L}$.

The order-theoretic definition of a lattice can be recovered from the algebraic one by defining the partial order $\leq$ with the following assertion:

$$A \leq B \quad \text{if and only if} \quad A \vee B = B.$$

(Or, equivalently, $A \leq B$ if and only if $A \wedge B = A$.) This definition is natural in the context of power-sets: given sets A and B, $A \subseteq B$ if and only if $A \cup B = B$ (if and only if $A \cap B = A$).

It is easy to check that these definitions are equivalent. The idempotence of the operations $\vee$ and $\wedge$ ensures that the resulting relation is reflexive, and the absorption law ensures that the resulting relation is transitive and antisymmetric.

An elementary yet significant property of lattices is the fact that the operations of join and meet are order-preserving. That is, if $A \leq B$ and $C \in \mathbb{L}$, then $(C \vee A) \leq (C \vee B)$ and $(C \wedge A) \leq (C \wedge B)$.

[2] We say that the semigroup $(S, \cdot)$ is **idempotent** if, for all $x \in S$, $x \cdot x = x$.

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1.1.2 Boolean Algebras

Definition 1.1.5: Boolean algebras

A **Boolean algebra** $\mathbb{L}$ is a lattice such that:

BA1) $\mathbb{L}$ has a least element. The least element is denoted 0, and is called the "zero" or "bottom" of $\mathbb{L}$.

BA2) $\mathbb{L}$ has a greatest element. The greatest element is denoted 1, and is called the "one" or "top" or "unit" of $\mathbb{L}$.

BA3) There exists a unary operation $\neg$ on $\mathbb{L}$ (called the "complementation" operation) such that, for all $A \in \mathbb{L}$,

$$A \vee \neg A = 1,$$
$$A \wedge \neg A = 0.$$

BA4) Join and meet distribute over each other. That is, for all $A, B, C \in \mathbb{L}$,

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C),$$
$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$$

Axioms BA1 and BA2 say that $\mathbb{L}$ is a **bounded** lattice. BA1 and BA2 are satisfied if and only if 0 is an identity for the join operation and 1 is an identity for the meet operation. We can abbreviate axiom BA3 by saying that $\mathbb{L}$ is **complemented**, and BA4 by saying that $\mathbb{L}$ is **distributive**. Only one of the two identities in BA4 is necessary—if one holds for all $A, B, C \in \mathbb{L}$, then the other also holds for all $A, B, C \in \mathbb{L}$. Thus we can abbreviate the definition by saying that a Boolean algebra is a complemented distributive lattice. (Note that boundedness is implied by complementedness.)

Strictly speaking, a Boolean algebra is a tuple of the form $(\mathbb{L}, \vee, \wedge, 0, 1, \neg)$ satisfying the above axioms, but we often just say that $\mathbb{L}$ is a Boolean algebra without specifying what the operations and identities are.

The prototypical example of a lattice—the power set of some set—is also an example of a Boolean algebra.

Example 1.1.6: Power set Boolean algebra

Let X be a set. Then the lattice $\mathcal{P}(X)$ has least element $\emptyset$ and greatest element X. With the operation of set exclusion $(\mathcal{P}(X) \ni A \mapsto X \setminus A)$ as its complementation operation, $\mathcal{P}(X)$ is a Boolean algebra.

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Owing to this example, we say that two elements A and B of a Boolean algebra are **disjoint** if $A \wedge B = 0$.

Remark 1.1.7

The definition of a Boolean algebra can be visualised in another way, as the "algebra of logic". We can imagine that the elements of $\mathbb{L}$ are predicates over some language, and we can imagine the join operation is disjunction (OR) and the meet operation is conjunction (AND). In this perspective, the greatest and least elements are the Boolean values *true* and *false* respectively, and the complement operation is negation. The partial order $\leq$ of $\mathbb{L}$ is logical implication ($IMPLIES$).

In fact, this perspective is not far off from the power-set perspective. We can identify the set $A \subseteq X$ in the power-set perspective with the formula $x \in A$ in the logic perspective. Then, the statement $A \subseteq B$ corresponds to the sentence $\forall x : (x \in A) \Rightarrow (x \in B)$, the set $X \backslash A$ corresponds to the formula $x \notin A$, the set $A \cup B$ corresponds to the formula $(x \in A) \vee (x \in B)$, and so on.

In the power set example of a Boolean algebra, the complement of a set $A \in \mathcal{P}(X)$ is the largest subset of X which is disjoint from A. It is also the smallest set whose union with A is X. This fact generalises to all Boolean algebras, as the following lemma shows.

Lemma 1.1.8

Let $\mathbb{L}$ be a Boolean algebra and let $A \in \mathbb{L}$. Then

(i) $\neg A$ is the least element of the set $\{B \in \mathbb{L} : B \vee A = 1\}$, and

(ii) $\neg A$ is the greatest element of the set $\{B \in \mathbb{L} : B \wedge A = 0\}$.

Proof:
(i): Suppose $B \vee A = 1$. Taking the meet with $\neg A$ on both sides and then distributing gives

$$\begin{aligned} \neg A \wedge (B \vee A) &= \neg A \wedge 1 \\ (\neg A \wedge B) \vee (\neg A \wedge A) &= \neg A \\ (\neg A \wedge B) \vee 0 &= \neg A \\ \neg A \wedge B &= \neg A. \end{aligned}$$

Thus $\neg A \leq B$. This shows that $\neg A$ is the least element of the set $\{B \in \mathbb{L} : B \vee A = 1\}$.

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(ii): Suppose $B \wedge A = 0$. Then

$$\begin{aligned} \neg A \vee (B \wedge A) &= \neg A \\ (\neg A \vee B) \wedge (\neg A \vee A) &= \neg A \\ \neg A \vee B &= \neg A. \end{aligned}$$

Thus $B \leq \neg A$, so $\neg A$ is the greatest element of the set $\{B \in \mathbb{L} : B \wedge A = 0\}$.

Corollary 1.1.9

Let $\mathbb{L}$ be a Boolean algebra and $A \in \mathbb{L}$. Then, for all $B \in \mathbb{L}$, B is disjoint from A if and only if $B \leq \neg A$.

Proof:
If $B \leq \neg A$, then taking the meet with A on both sides gives $B \wedge A \leq \neg A \wedge A = 0$, so $B \wedge A = 0$.

If $B \wedge A = 0$, then by point (ii) of the preceding lemma, $B \leq \neg A$.

A number of familiar properties of power sets are possessed by Boolean algebras in general. Importantly, Boolean algebras satisfy De Morgan's Laws:

Theorem 1.1.10: De Morgan's Laws for Boolean algebras

Let $\mathbb{L}$ be a Boolean algebra. Then, for all $A, B, C \in \mathbb{L}$,

$$\begin{aligned} \neg(A \wedge B) &= \neg A \vee \neg B \\ \neg(A \vee B) &= \neg A \wedge \neg B. \end{aligned}$$

Proof:
First, we show that $\neg A \vee \neg B \leq \neg(A \wedge B)$.

By the definition of complementation,

$$\neg(A \wedge B) \vee (A \wedge B) = 1.$$

Distributing gives

$$(\neg(A \wedge B) \vee A) \wedge (\neg(A \wedge B) \vee B) = 1.$$

This shows that 1 is the greatest lower bound of $\neg(A \wedge B) \vee A$ and $\neg(A \wedge B) \vee B$. Since 1 is the greatest element of $\mathbb{L}$,

$$\neg(A \wedge B) \vee A = \neg(A \wedge B) \vee B = 1.$$

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By Lemma 1.1.8(i), this implies that

$$\neg A \leq \neg(A \wedge B)$$
$$\neg B \leq \neg(A \wedge B).$$

By the definition of the supremum, we have $\neg A \vee \neg B \leq \neg(A \wedge B)$.

Next, we show that $\neg(A \wedge B) \leq \neg A \vee \neg B$.

By distributivity, we have

$$\begin{aligned}(\neg A \vee \neg B) \vee (A \wedge B) &= (A \vee (\neg A \vee \neg B)) \wedge (B \vee (\neg A \vee \neg B)) \\ &= (A \vee \neg A \vee \neg B) \wedge (B \vee \neg A \vee \neg B) \\ &= 1 \wedge 1 \\ &= 1.\end{aligned}$$

By Lemma 1.1.8(i), this shows that $\neg(A \wedge B) \leq \neg A \vee \neg B$.

Hence $\neg A \vee \neg B = \neg(A \wedge B)$.

The proof of $\neg A \wedge \neg B = \neg(A \vee B)$ is similar. For $\neg A \wedge \neg B \leq \neg(A \vee B)$, we have

$$\begin{aligned}(\neg A \wedge \neg B) \wedge (A \vee B) &= (A \wedge \neg A \wedge \neg B) \vee (B \wedge \neg A \wedge \neg B) \\ &= 0 \vee 0 \\ &= 0,\end{aligned}$$

so, by Lemma 1.1.8(ii), $\neg A \wedge \neg B \leq \neg(A \vee B)$. For $\neg(A \vee B) \leq \neg A \wedge \neg B$, we have

$$\begin{aligned}(\neg A \wedge \neg B) \vee (A \vee B) &= (\neg A \vee (A \vee B)) \wedge (\neg B \vee (A \vee B)) \\ &= 1 \wedge 1 \\ &= 1,\end{aligned}$$

so, by Lemma 1.1.8(i), $\neg(A \vee B) \leq \neg A \wedge \neg B$. Therefore $\neg A \wedge \neg B = \neg(A \vee B)$.

This leads to an important property of complementation, namely that it is an order anti-isomorphism (that is, it is an order isomorphism from a Boolean algebra $(\mathbb{L}, \leq)$ to its dual $(\mathbb{L}, \geq)$).

Theorem 1.1.11: Complementation is an order anti-isomorphism

Let $\mathbb{L}$ be a Boolean algebra. Then for all $A, B \in \mathbb{L}$,

(i) $\neg\neg A = A$, and

(ii) $A \leq B \iff \neg B \leq \neg A$.

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Proof:

(i): By definition, $A \wedge \neg A = 0$. By Lemma 1.1.8 (ii), it follows that $A \leq \neg\neg A$ (since $\neg\neg A$ is the greatest element which is disjoint from $\neg A$). Again by definition, $A \vee \neg A = 1$; thus by Lemma 1.1.8 (i), $\neg\neg A \leq A$ (since $\neg\neg A$ is the least element whose join with $\neg A$ is 1). Thus $\neg\neg A = A$.

(ii): Suppose $A \leq B$. Then, by De Morgan's Laws,

$$\begin{aligned} \neg A \vee \neg B &= \neg(A \wedge B) \\ &= \neg A. \end{aligned}$$

Thus $\neg B \leq \neg A$.

Conversely, suppose $\neg B \leq \neg A$. Then, again by De Morgan's Laws,

$$\begin{aligned} A \vee B &= \neg(\neg A \wedge \neg B) \\ &= \neg(\neg B) \\ &= B. \end{aligned}$$

Thus $A \leq B$.

The fact that complementation is an order anti-isomorphism allows us to generalise De Morgan's Laws to suprema and infima of arbitrary sets.

Theorem 1.1.12: Generalised De Morgan's Laws for Boolean algebras

Let $\mathbb{L}$ be a Boolean algebra, and let $\mathcal{S} \subseteq \mathbb{L}$. Define $\neg\mathcal{S} := \{\neg A : A \in \mathcal{S}\}$. Then:

(i) $\mathcal{S}$ has a supremum $\bigvee \mathcal{S}$ in $\mathbb{L}$ if and only if $\neg\mathcal{S}$ has an infimum $\bigwedge \neg\mathcal{S}$, in which case $\neg \bigvee \mathcal{S} = \bigwedge \neg\mathcal{S}$.

(ii) $\mathcal{S}$ has an infimum in $\mathbb{L}$ if and only if $\neg\mathcal{S}$ has a supremum, in which case $\neg \bigwedge \mathcal{S} = \bigvee \neg\mathcal{S}$.

Proof:

(i): Suppose $\mathcal{S}$ has a supremum $\bigvee \mathcal{S}$ in $\mathbb{L}$. Then $A \leq \bigvee \mathcal{S}$ for all $A \in \mathcal{S}$. Since complementation is an order anti-isomorphism (Theorem 1.1.11), we have $\neg \bigvee \mathcal{S} \leq \neg A$ for all $A \in \mathcal{S}$. Thus $\neg \bigvee \mathcal{S}$ is a lower bound of $\neg\mathcal{S}$.

Now, suppose that B is a lower bound of $\neg\mathcal{S}$, so $B \leq \neg A$ for all $A \in \mathcal{S}$. Taking complements again, we have $A \leq \neg B$ for all $A \in \mathcal{S}$. Hence $\neg B$ is an upper bound of $\mathcal{S}$, so $\bigvee \mathcal{S} \leq \neg B$. Taking complements again, we have $B \leq \neg \bigvee \mathcal{S}$. This shows that $\neg \bigvee \mathcal{S}$ is the greatest lower bound of

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$\neg\mathcal{S}$, i.e.

$$\bigwedge \neg\mathcal{S} = \neg \bigvee \mathcal{S}.$$

Conversely, suppose $\neg\mathcal{S}$ has an infimum $\bigwedge \neg\mathcal{S}$. By Theorem 1.1.11, $\neg \bigwedge \neg\mathcal{S}$ is an upper bound of $\mathcal{S}$. Now, suppose B is an upper bound of $\mathcal{S}$. Applying Theorem 1.1.11 again, $\neg B \leq \neg A$ for all $A \in \mathcal{S}$. Thus $\neg B \leq \bigwedge \neg\mathcal{S}$. Taking complements again gives $\neg \bigwedge \neg\mathcal{S} \leq B$. This shows that $\neg \bigwedge \neg\mathcal{S}$ is the least upper bound of $\mathcal{S}$, as required.

(ii): Suppose $\mathcal{S}$ has an infimum in $\mathbb{L}$. Since complementation is an involution (Theorem 1.1.11 (i)), $\neg\neg\mathcal{S}$ has an infimum $\bigwedge \neg\neg\mathcal{S}$ in $\mathbb{L}$. By (i), $\neg\mathcal{S}$ has a supremum, and $\neg \bigvee \neg\mathcal{S} = \bigwedge \neg\neg\mathcal{S} = \bigwedge \mathcal{S}$. Taking complements gives

$$\neg \bigwedge \mathcal{S} = \bigvee \neg\mathcal{S},$$

as required.

We can generalise distributivity in a similar way.

Theorem 1.1.13: Generalised distributivity

Let $\mathbb{L}$ be a Boolean algebra, and let $\mathcal{S} \subseteq \mathbb{L}$ and $A \in \mathbb{L}$. Then:

$(i$: binary joins over arbitrary joins)

$$\bigvee_{B\in\mathcal{S}} (A \vee B) = A \vee \left(\bigvee_{B\in\mathcal{S}} B \right).$$

That is, if $\mathcal{S}$ has a supremum in $\mathbb{L}$, then $\{A \vee B : B \in \mathcal{S}\}$ has a supremum in $\mathbb{L}$ which is equal to $A \vee (\bigvee \mathcal{S})$.

$(ii$: binary meets over arbitrary meets)

$$\bigwedge_{B\in\mathcal{S}} (A \wedge B) = A \wedge \left(\bigwedge_{B\in\mathcal{S}} B \right).$$

That is, if $\mathcal{S}$ has an infimum in $\mathbb{L}$, then $\{A \wedge B : B \in \mathcal{S}\}$ has an infimum in $\mathbb{L}$ which is equal to $A \wedge (\bigwedge \mathcal{S})$.

$(iii$: binary meets over arbitrary joins)

$$\bigvee_{B\in\mathcal{S}} (A \wedge B) = A \wedge \left(\bigvee_{B\in\mathcal{S}} B \right).$$

That is, if $\mathcal{S}$ has a supremum in $\mathbb{L}$, then $\{A \wedge B : B \in \mathcal{S}\}$ has a supremum in $\mathbb{L}$ which is equal to $A \wedge (\bigvee \mathcal{S})$.

$(iv$: binary joins over arbitrary meets)

$$\bigwedge_{B\in\mathcal{S}} (A \vee B) = A \vee \left(\bigwedge_{B\in\mathcal{S}} B \right).$$

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That is, if $\mathcal{S}$ has an infimum in $\mathbb{L}$, then $\{A \vee B : B \in \mathcal{S}\}$ has an infimum in $\mathbb{L}$ which is equal to $A \vee (\bigwedge \mathcal{S})$.

Proof:

(i): Suppose $\mathcal{S}$ has a supremum $\bigvee \mathcal{S}$ in $\mathbb{L}$. Then, for all $B \in \mathcal{S}$,

$$\begin{aligned} B &\leq \bigvee \mathcal{S} \\ (A \vee B) &\leq A \vee \left(\bigvee \mathcal{S}\right). \end{aligned}$$

Thus $A \vee (\bigvee \mathcal{S})$ is an upper bound of the set $\{A \vee B : B \in \mathcal{S}\}$. Now suppose that C is an upper bound of $\{A \vee B : B \in \mathcal{S}\}$. Then $(A \vee B) \leq C$ for all $B \in \mathcal{S}$. Since $A \vee B$ is the least upper bound of A and B, $B \leq C$ for all $B \in \mathcal{S}$. Thus $\bigvee \mathcal{S} \leq C$ and $A \leq C$, so

$$A \vee \left(\bigvee \mathcal{S}\right) \leq C.$$

This shows that $A \vee (\bigvee \mathcal{S})$ is the least upper bound of the set $\{A \vee B : B \in \mathcal{S}\}$.

(ii): Suppose $\mathcal{S}$ has an infimum in $\mathbb{L}$. By De Morgan's laws and (i),

$$\begin{aligned} A \wedge \left(\bigwedge_{B \in \mathcal{S}} B\right) &= \neg\left(\neg A \vee \neg\left(\bigwedge_{B \in \mathcal{S}} B\right)\right) \\ &= \neg\left(\neg A \vee \left(\bigvee_{B \in \mathcal{S}} \neg B\right)\right) \\ &= \neg \bigvee_{B \in \mathcal{S}} (\neg A \vee \neg B) \\ &= \bigwedge_{B \in \mathcal{S}} \neg(\neg A \vee \neg B) \\ &= \bigwedge_{B \in \mathcal{S}} (A \wedge B), \end{aligned}$$

as required.

(iii): Suppose $\bigvee \mathcal{S}$ exists in $\mathbb{L}$. Then, for all $B \in \mathcal{S}$,

$$A \wedge B \leq A \wedge \bigvee \mathcal{S},$$

so $A \wedge \bigvee \mathcal{S}$ is an upper bound of the set $\{A \wedge B : B \in \mathcal{S}\}$. Now let C be an upper bound of that set. Then, for all $B \in \mathcal{S}$,

$$A \wedge B \leq C.$$

Taking the join with $\neg A$ gives

$$\begin{aligned} (A \wedge B) \vee \neg A &\leq C \vee \neg A \\ (A \vee \neg A) \wedge (B \vee \neg A) &\leq C \vee \neg A \\ B \vee \neg A &\leq C \vee \neg A. \end{aligned}$$

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Since $B \leq B \vee \neg A$, it follows that $B \leq C \vee \neg A$. Hence

$$\bigvee \mathcal{S} \quad \leq \quad C \vee \neg A.$$

Taking the meet with A gives

$$\begin{aligned} A \wedge \bigvee \mathcal{S} \quad &\leq \quad (C \vee \neg A) \wedge A \\ A \wedge \bigvee \mathcal{S} \quad &\leq \quad C \wedge A \quad \leq \quad C. \end{aligned}$$

This shows that $A \wedge \bigvee \mathcal{S}$ is the least upper bound of $\{A \wedge B : B \in \mathcal{S}\}$. That is,

$$\bigvee_{B \in \mathcal{S}} (A \wedge B) = A \wedge \bigvee_{B \in \mathcal{S}} B$$

(iv): Suppose $\bigwedge \mathcal{S}$ exists in $\mathbb{L}$. By De Morgan's laws and (iii),

$$\begin{aligned} A \vee \bigwedge_{B \in \mathcal{S}} B &= \neg \left(\neg A \wedge \neg \bigwedge_{B \in \mathcal{S}} B \right) \\ &= \neg \left(\neg A \wedge \bigvee_{B \in \mathcal{S}} \neg B \right) \\ &= \neg \bigvee_{B \in \mathcal{S}} (\neg A \wedge \neg B) \\ &= \bigwedge_{B \in \mathcal{S}} \neg(\neg A \wedge \neg B) \\ &= \bigwedge_{B \in \mathcal{S}} (A \vee B), \end{aligned}$$

as required.

Next, we define homomorphisms and isomorphisms of Boolean algebras.

Definition 1.1.14: Boolean algebra homomorphisms

Let $(\mathbb{L}, \vee_{\mathbb{L}}, \wedge_{\mathbb{L}}, 0_{\mathbb{L}}, 1_{\mathbb{L}}, \neg_{\mathbb{L}})$ and $(\mathbb{M}, \vee_{\mathbb{M}}, \wedge_{\mathbb{M}}, 0_{\mathbb{M}}, 1_{\mathbb{M}}, \neg_{\mathbb{M}})$ be Boolean algebras, and let $f : \mathbb{L} \to \mathbb{M}$ be a function. We say that f is a **Boolean algebra homomorphism** if the following equalities hold for all $A, B \in \mathbb{L}$:

$$f(A \vee_{\mathbb{L}} B) = f(A) \vee_{\mathbb{M}} f(B) \qquad\qquad f(A \wedge_{\mathbb{L}} B) = f(A) \wedge_{\mathbb{M}} f(B)$$
$$f(0_{\mathbb{L}}) = 0_{\mathbb{M}} \qquad\qquad f(1_{\mathbb{L}}) = 1_{\mathbb{M}}.$$

In this case, we also have $f(\neg_{\mathbb{L}} A) = \neg_{\mathbb{M}} f(A)$ for all $A \in \mathbb{L}$.

A Boolean algebra homomorphism also happens to be monotone: if $A \leq B$ in $\mathbb{L}$, then $f(A) \leq f(B)$ in $\mathbb{M}$.

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Definition 1.1.15: Boolean algebra isomorphisms

Let $\mathbb{L}$ and $\mathbb{M}$ be Boolean algebras, and let $f : \mathbb{L} \to \mathbb{M}$ be a Boolean algebra homomorphism. We say that f is a **Boolean algebra isomorphism** if there exists a Boolean algebra homomorphism $g : \mathbb{M} \to \mathbb{L}$ (the **inverse** of f) such that $g \circ f : \mathbb{L} \to \mathbb{L}$ and $f \circ g : \mathbb{M} \to \mathbb{M}$ are the identity maps on $\mathbb{L}$ and $\mathbb{M}$ respectively. In this case, we may write f^{-1} in place of g.

The idea of this definition is that an isomorphism is a bijection between the underlying sets which preserves the structure of both Boolean algebras.

Theorem 1.1.16

Let $f : \mathbb{L} \to \mathbb{M}$ be a function between Boolean algebras. Then f is an isomorphism if and only if it is a bijective homomorphism.

Proof (sketch):
This is essentially a basic universal algebra result. Every bijection has an inverse function, and if a bijection preserves some binary operation between two algebraic structures, then its inverse certainly does as well. Hence the inverse of a bijective homomorphism is itself a (bijective) homomorphism.

Theorem 1.1.17: Boolean algebra isomorphisms preserve all suprema and infima where they exist

Let $f : \mathbb{L} \to \mathbb{M}$ be a Boolean algebra isomorphism, and let $\mathcal{S} \subseteq \mathbb{L}$. Then:

(i) If $\mathcal{S}$ has a supremum $\bigvee_{\mathbb{L}} \mathcal{S}$ in $\mathbb{L}$, then $f[\mathcal{S}]$ has a supremum in $\mathbb{M}$, and $\bigvee_{\mathbb{M}} f[\mathcal{S}] = f\left(\bigvee_{\mathbb{L}} \mathcal{S}\right)$.

(ii) If $\mathcal{S}$ has an infimum $\bigwedge_{\mathbb{L}} \mathcal{S}$ in $\mathbb{L}$, then $f[\mathcal{S}]$ has an infimum in $\mathbb{M}$, and $\bigwedge_{\mathbb{M}} f[\mathcal{S}] = f\left(\bigwedge_{\mathbb{L}} \mathcal{S}\right)$.

Proof:
(i): Suppose $\mathcal{S}$ has a supremum $\bigvee_{\mathbb{L}} \mathcal{S}$ in $\mathbb{L}$. Since f is monotone, $f(A) \leq f\left(\bigvee_{\mathbb{L}} \mathcal{S}\right)$ for all $A \in \mathcal{S}$. Hence $f\left(\bigvee_{\mathbb{L}} \mathcal{S}\right)$ is an upper bound of $f[\mathcal{S}]$ in $\mathbb{M}$. Now let $f(B)$ be an upper bound of $f[\mathcal{S}]$ in $\mathbb{M}$. Since f has an inverse which is a homomorphism (and thus monotone), it follows that B is an upper bound of $\mathcal{S}$ in $\mathbb{L}$, and thus $\bigvee_{\mathbb{L}} \mathcal{S} \leq B$. Thus $f\left(\bigvee_{\mathbb{L}} \mathcal{S}\right) \leq f(B)$. This shows that $f\left(\bigvee_{\mathbb{L}} \mathcal{S}\right)$ is the least upper bound of $f[\mathcal{S}]$ in $\mathbb{M}$.

(ii): The argument is identical to (i), with the words "supremum", "upper bound", and "least" replaced with "infimum", "lower bound", and "greatest" respectively, and with $\leq$ and $\bigvee_{\mathbb{L}}$ replaced with $\geq$ and $\bigwedge_{\mathbb{L}}$ respectively.

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1.1.3 Boolean Rings

We will exclusively be concerned with *unital* rings (i.e. rings with a multiplicative identity), so every ring in this thesis is assumed to be unital.

Definition 1.1.18: Boolean rings

A **Boolean ring** is a ring in which every element is idempotent. That is, if R is a ring, then we say that R is a Boolean ring if $x * x = x$ for all $x \in R$.

A Boolean ring is really a tuple of the form $(R, +, *, 0, 1)$ satisfying the axioms, but we will usually just say "R is a Boolean ring" without specifying the operations and identity. We will often denote multiplication with juxtaposition, writing xy for $x * y$.

Theorem 1.1.19

Let R be a Boolean ring. Then:

(i) Addition in R is involutive. That is, for every $x \in R$, $x + x = 0$.

(ii) Multiplication in R is commutative.

Proof:
(i): Let $x \in X$. Then

$$\begin{aligned} x + x &= (x + x)^2 \\ &= x^2 + x^2 + x^2 + x^2 \\ &= x + x + x + x. \end{aligned}$$

Subtracting $x + x$ from both sides gives

$$0 = x + x.$$

(ii): Let $x, y \in X$. Then $x + y = (x + y)^2$ and $x + y = x^2 + y^2$. Hence

$$\begin{aligned} (x + y)^2 &= x^2 + y^2 \\ x^2 + xy + yx + y^2 &= x^2 + y^2 \\ xy + yx &= 0 \\ xy &= -yx = yx. \end{aligned}$$

The prototypical example of a Boolean ring is, as for Boolean algebras, the power set of a set.

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Example 1.1.20: Power set Boolean ring

Let X be a set and let $\triangle$ be the symmetric difference operator (i.e. $A \triangle B = (A \cup B) \setminus (A \cap A)$ for all sets A and B). Then $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a Boolean ring. Symmetric difference is associative, intersection is idempotent and distributes over symmetric difference, and the empty set is an identity for the symmetric difference operator.

We will not focus greatly on the ring-theoretic properties of Boolean rings in this thesis, but we include some here for interest's sake.

Remark 1.1.21: Some properties of Boolean rings

Let R be a Boolean ring with at least two elements.

(i) Since $1 + 1 = 0$, R has characteristic 2. Hence R is a commutative algebra over the two-element field $\mathbb{Z}/2\mathbb{Z}$.

(ii) For every $x \in R$, we have $x(1 + x) = x + x^2 = x + x = 0$. This shows that every element of R—other than 0 and 1—is a zero-divisor. Hence the only Boolean ring which is an integral domain (also the only Boolean ring which is a *field*) is the two-element field. No element of R is irreducible, since $x = x^2$ for all $x \in R$.

(iii) Every quotient ring of R is Boolean. Every subring and localisation of R is also a Boolean ring.

(iv) An ideal of a ring is radical if and only if the corresponding quotient ring is reduced (i.e. has no non-trivial nilpotents). Hence every ideal of R is radical.

(v) If I is a prime ideal of R, then R/I is an integral domain and a Boolean ring. Thus R/I is the two-element field. It follows that I is maximal. This shows that every prime ideal in R is maximal. Since (in every ring) every maximal ideal is prime, it follows that prime ideals and maximal ideals coincide in Boolean rings.

(vi) A non-trivial ring is a local ring if and only if the sum of any two non-units in that ring is a non-unit. If there exists an $x \in R \setminus \{0, 1\}$, then x and $1 - x$ are non-units while $x + (1 - x) = 1$ is a unit. Hence the only Boolean ring which is a local ring is the two-element field.

(vii) A Boolean ring is Noetherian (and Artinian) if and only if it is finite. This follows from the Stone Representation Theorem, which will be proved later.

If R is a Boolean ring of sets (i.e. its elements are sets and its operations are $\triangle$ and $\cap$), then (unless specified otherwise) the expressions $A + B$ and $A \cdot B$ (or just AB) denote $A \triangle B$ and $A \cap B$ respectively.

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If A and B are elements of a power set Boolean ring, the expression $A - B$ has the same value as $A + B$ and should not be confused with *set exclusion*, which we instead denote with a backslash: $A \backslash B = \{x \in A : x \notin B\}$.

A Boolean ring homomorphism is just a ring homomorphism, and likewise for isomorphisms. For completness's sake, we include the definitions here.

Definition 1.1.22: Boolean ring homomorphisms

Let $(R, +_R, *_R, 0_R, 1_R)$ and $(S, +_S, *_S, 0_S, 1_S)$ be Boolean rings, and let $f : R \to S$ be a function. We say that f is a **Boolean ring homomorphism** if, for all $x, y \in R$,

$$\begin{aligned} f(x +_R y) &= f(x) +_S f(y) \\ f(x *_R y) &= f(x) *_S f(y) \\ f(1_R) &= 1_S. \end{aligned}$$

This is equivalent to f simply being a homomorphism of (unital) rings.

Definition 1.1.23: Boolean ring isomorphisms

A **Boolean ring isomorphism** is defined analogously to how Boolean algebra isomorphisms are defined. That is, a Boolean ring homomorphism $f : R \to S$ is an isomorphism if there exists a Boolean ring homomorphism $g : S \to R$ such that $g \circ f$ and $f \circ g$ are the identity maps on R and S respectively.

As in the case of Boolean algebras, a Boolean ring homomorphism is an isomorphism if and only if it is bijective.

1.1.4 Correspondence Between Boolean Rings and Boolean Algebras

Every Boolean algebra is a Boolean ring and vice versa. More specifically, given a Boolean algebra $\mathbb{L}$, we can define Boolean ring operations on $\mathbb{L}$ to yield a Boolean ring with the same underlying set. Conversely, given a Boolean ring R, we can define a partial order on R that makes it into a Boolean algebra. Furthermore, these constructions are inverses of each other.

Let $(\mathbb{L}, \vee, \wedge, 0, 1)$ be a Boolean algebra. Define the operations $\oplus$ and $*$ on $\mathbb{L}$ as follows: for all $A, B \in \mathbb{L}$,

$$\begin{aligned} A \oplus B &:= (A \wedge \neg B) \vee (\neg A \wedge B) \\ A * B &:= A \wedge B. \end{aligned}$$

Using the power set example of a Boolean algebra (i.e. where $\vee$ and $\wedge$ are union and intersection respectively), this amounts to defining $A \oplus B$ as the symmetric difference of A and B and defining $A * B$ to be their intersection.

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It is elementary to show that $(\mathbb{L}, \oplus, *)$ is a Boolean ring, with additive identity equal to 0 and multiplicative identity equal to 1.

Now let $(R, +, \cdot, 0, 1)$ be a Boolean ring. Define the operations $\vee'$ and $\wedge'$ on R as follows: for all $A, B \in R$,

$$A \vee' B := A + B + A \cdot B$$
$$A \wedge' B := A \cdot B.$$

Using the power set example of a Boolean ring (where $+$ and $\cdot$ are symmetric difference and intersection respectively), it is easy to see that this makes $\vee'$ the union operation. Indeed, the symmetric difference $A \triangle B$ of the sets A and B is $(A \cup B) \setminus (A \cap B)$, and since $A \triangle B$ is disjoint from $A \cap B$, we have $A \triangle B \triangle (A \cap B) = (A \triangle B) \cup (A \cap B) = A \cup B$. This also illustrates why these two constructions (defining a Boolean ring from a Boolean algebra and vice versa) are inverses of each other. Furthermore, it follows that the complementation operation is $\neg : A \mapsto 1 + A$, which is consistent with the observation that, for each set X, the complement of $A \subseteq X$ is $X \setminus A = X \triangle A$.

This also illustrates why Boolean algebra homomorphisms and isomorphisms are the same as Boolean ring homomorphisms and isomorphisms respectively. Indeed, the operations of each structure can be written in terms of the operations of the other structure, so any equality containing operations in one structure can be translated directly into an equality with operations in the other.

1.1.5 Filters

The Stone Representation Theorem allows every Boolean algebra to be represented as an algebra of sets. Given a Boolean algebra $\mathbb{L}$, every element of $\mathbb{L}$ is represented as a particular set. Specifically, the elements are represented as sets of ultrafilters, which we now define. The elements can be equivalently represented as sets of homomorphisms—we will handle that shortly.

Definition 1.1.24: Filters

Let $(P, \leq)$ be a partially ordered set, and let $\mathbf{F} \subseteq P$. We say that $\mathbf{F}$ is a **filter** if

(*i*) $\mathbf{F}$ is non-empty,

(*ii*) $\mathbf{F}$ is downward directed: for every $x, y \in \mathbf{F}$ there exists a $z \in \mathbf{F}$ such that $z \leq x$ and $z \leq y$, and

(*iii*) $\mathbf{F}$ is upward closed: if $x \in \mathbf{F}$, $y \in P$, and $x \leq y$, then $y \in \mathbf{F}$.

A **principal** filter is a set of the form $\{y \in P : x \leq y\}$ for some $x \in P$. It is easy to verify that every set of this form is indeed a filter. The principal filter of elements greater than or equal to x is sometimes denoted $\uparrow x$.

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Often we are interested in filters in the power set of a given set, which is partially ordered by inclusion. Unless specified otherwise, a "filter of subsets" of a set X refers to a filter in the poset $(\mathcal{P}(X), \subseteq)$.

We say that a filter $\mathbf{F} \subseteq P$ is **proper** if it is not equal to the whole poset P. An **ultrafilter** is a maximal filter, i.e. a proper filter which is not a subset of any filter other than itself and the whole poset P.

If $\mathbb{L}$ is a lattice, then $\mathbf{F} \subseteq \mathbb{L}$ is a filter if and only if it is non-empty, upward closed, and closed under finite meets. Furthermore, given a subset $\mathcal{S}$ of a lattice, there is a smallest filter containing that subset, called the **filter generated by** $\mathcal{S}$. The existence of such a filter is easy to prove in a bounded lattice. Indeed, there always exists at least one filter containing $\mathcal{S}$ (namely the filter $\mathbb{L}$), and it is easy to see that the intersection of any collection of filters is itself a filter. Thus the intersection of all filters containing $\mathcal{S}$ is the smallest filter containing $\mathcal{S}$. A direct construction of the filter generated by a subset is given below.

Theorem 1.1.25: Filter generated by a subset of a lattice

Let $\mathbb{L}$ be a lattice and let $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Define

$$\mathbf{F}_{\mathcal{S}} := \left\{ x \in \mathbb{L} : x \geq \bigwedge \mathcal{G} \text{ for some finite } \mathcal{G} \subseteq \mathcal{S} \right\}.$$

Then $\mathbf{F}_{\mathcal{S}}$ is a filter containing $\mathcal{S}$, and if $\mathcal{F} \supseteq \mathcal{S}$ is a filter, then $\mathcal{F} \supseteq \mathbf{F}_{\mathcal{S}}$.

Proof:
For all $x \in \mathcal{S}$, we have $x \geq \bigwedge \{x\}$, so $x \in \mathbf{F}_{\mathcal{S}}$. Hence $\mathcal{S} \subseteq \mathbf{F}_{\mathcal{S}}$.

The fact that $\mathbf{F}_{\mathcal{S}}$ is upward closed follows immediately from the fact that $\geq$ is transitive.

Let $x, y \in \mathbf{F}_{\mathcal{S}}$. Then $x \geq \bigwedge \mathcal{G}_1$ and $y \geq \bigwedge \mathcal{G}_2$ for some finite sets $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{S}$. Since $\wedge$ is order-preserving, we have

$$x \wedge y \quad \geq \quad x \wedge \left(\bigwedge \mathcal{G}_2 \right) \quad \geq \quad \left(\bigwedge \mathcal{G}_1 \right) \wedge \left(\bigwedge \mathcal{G}_2 \right) \quad = \quad \bigwedge (\mathcal{G}_1 \cup \mathcal{G}_2).$$

Hence $x \wedge y \in \mathbf{F}_{\mathcal{S}}$. This shows that $\mathbf{F}_{\mathcal{S}}$ is closed under finite meets (and thus downward directed). Therefore $\mathbf{F}_{\mathcal{S}}$ is a filter containing $\mathcal{S}$.

Suppose $\mathcal{F} \supseteq \mathcal{S}$ is a filter. Since $\mathcal{F}$ is closed under finite meets, $\bigwedge \mathcal{G} \in \mathcal{F}$ for every finite set $\mathcal{G} \subseteq \mathcal{S}$. Since $\mathcal{F}$ is upward closed, $x \in \mathcal{F}$ whenever $x \geq \bigwedge \mathcal{G}$ and $\mathcal{G} \subseteq \mathcal{S}$ is finite. Hence $\mathbf{F}_{\mathcal{S}} \subseteq \mathcal{F}$.

In a Boolean algebra, there is a useful property of proper filters which is equivalent to maximality.

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Theorem 1.1.26

Let $\mathbb{L}$ be a Boolean algebra and let $\mathcal{F} \subsetneq \mathbb{L}$ be a proper filter in $\mathbb{L}$. Then $\mathcal{F}$ is an ultrafilter if and only if, for every $A \in \mathbb{L}$, exactly one of A and $\neg A$ is in $\mathcal{F}$.

Proof:

Suppose $\mathcal{F}$ is an ultrafilter and let $A \in \mathcal{F}$. If $\neg A \in \mathcal{F}$, then, since $\mathcal{F}$ is downward directed, $A \wedge \neg A = 0 \in \mathcal{F}$, and thus $\mathcal{F} = \mathbb{L}$, contradicting the properness of $\mathcal{F}$. Thus, for all $A \in \mathbb{L}$, $A \in \mathcal{F}$ implies $\neg A \notin \mathcal{F}$.

Now suppose that $A \notin \mathcal{F}$. Then the filter $\mathbf{F}_{\mathcal{F} \cup \{A\}}$ generated by $\mathcal{F} \cup \{A\}$ is a filter and a proper superset of $\mathcal{F}$. By the maximality of $\mathcal{F}$, we must have $\mathbf{F}_{\mathcal{F} \cup \{A\}} = \mathbb{L}$. Hence $0 \in \mathbf{F}_{\mathcal{F} \cup \{A\}}$, so there exists a finite set $\mathcal{G} \subseteq \mathcal{F} \cup \{A\}$ such that $0 \geq \bigwedge \mathcal{G}$, i.e. $\bigwedge \mathcal{G} = 0$. We cannot have $\mathcal{G} \subseteq \mathcal{F}$, because that would imply that $0 \in \mathcal{F}$ (which, by the upward closure of $\mathcal{F}$, would imply that $\mathcal{F} = \mathbb{L}$, contradicting the properness of $\mathcal{F}$). Hence $A \in \mathcal{G}$.

Suppose $\mathcal{G} = \{A, B_1, \dots, B_n\}$, where $n \in \mathbb{N}$ and $B_i \in \mathcal{F}$ for all $i \in \{1, \dots, n\}$. Set $B = B_1 \wedge \cdots \wedge B_n$. Then $B \in \mathcal{F}$ and $A \wedge B = 0$. That is, A and B are disjoint. Now recall Lemma 1.1.8 (ii): $\neg A$ is the largest element of $\mathbb{L}$ which is disjoint from A. Hence $\neg A \geq B$. Since $\mathcal{F}$ is upward closed, $\neg A \in \mathcal{F}$.

Thus we have shown that, for every $A \in \mathbb{L}$, $A \in \mathcal{F}$ if and only if $\neg A \notin \mathcal{F}$.

Conversely, suppose that, for all $A \in \mathbb{L}$, we have $A \in \mathcal{F}$ if and only if $\neg A \notin \mathcal{F}$. Let $\mathcal{E} \supsetneq \mathcal{F}$ be a filter strictly containing $\mathcal{F}$. Then there exists an $A \in \mathcal{E} \setminus \mathcal{F}$. By assumption, $\neg A \in \mathcal{F}$. But since $\mathcal{F} \subseteq \mathcal{E}$, $\neg A \in \mathcal{E}$. Hence $A \wedge \neg A = 0 \in \mathcal{E}$, so $\mathcal{E} = \mathbb{L}$. Hence $\mathcal{F}$ is maximal, i.e. $\mathcal{F}$ is an ultrafilter.

A prototypical example of a filter is a neighbourhood filter. Given a topological space X and a point $x \in X$, the **neighbourhood filter of** x, (denoted $\mathbf{N}(x)$) is the collection of all neighbourhoods of x. It is a filter in the lattice $\mathcal{P}(X)$ of subsets of X.

Remark 1.1.27

A filter can be interpreted as a "locating scheme". If we are, say, trying to locate a particular point or subset of a topological space, then if two sets A and B contain the point we are looking for, then their intersection must also contain it. Moreover, if A contains what we are looking for, then any superset of A certainly also contains what we are looking for. This is one motivation for the definition. In this interpretation, an ultrafilter is a "strong locating scheme", in the sense that it respects the idea that if a set does not contain what we are looking for, then its complement must contain what we are looking for.

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Another interpretation of the definition of a filter is as a collection of sets which we can label as "non-negligible". For example, if X is a measure space with measure ∞, then the subsets of X whose complements have finite measure form a filter of subsets of X.

Definition 1.1.28: Convergence of filters in topology

Let X be a topological space, let $\mathcal{F}$ be a filter of subsets of X, and let $x \in X$. We say that $\mathcal{F}$ **converges to** x (written $\mathcal{F} \to x$) if every neighbourhood of x is in $\mathcal{F}$ (in other words, the neighbourhood filter $\mathbf{N}(x)$ is a subset of $\mathcal{F}$).

The convergence of filters generalises the notion of the convergence of sequences. A filter may converge to more than one point. A useful property of ultrafilters is due to their relation to compactness, as the following theorem shows. A proof of this theorem can be found in [Mor85] (Corollary A6.1.24).

Theorem 1.1.29: Convergence of ultrafilters in compact spaces

A space X is compact if and only if every ultrafilter of subsets of X converges to at least one point.

1.1.6 Stone Spaces

We now define an important class of topological spaces which will be indispensable for us: Stone spaces. They are important because, as the Stone Representation Theorem will show, every Boolean algebra is isomorphic to an algebra of subsets of a Stone space.

Definition 1.1.30: Zero-dimensional spaces

Let X be a topological space. We say that X is **zero-dimensional** if it has a base consisting of clopen sets. That is, if, for every open set $U \subseteq X$, there exists a collection $\mathcal{V}$ of clopen subsets of X such that $U = \bigcup \mathcal{V}$.

Definition 1.1.31: Stone spaces

A **Stone space** is a compact Hausdorff space which is zero-dimensional.

This should not be confused with the term "Stonean space", which refers to a compact Hausdorff space in which the closure of every open set is open.

A compact Hausdorff space is also known as a **compactum**, so a Stone space can alternatively be called a *zero-dimensional compactum*.

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Topological spaces provide another example of Boolean algebras.

Definition 1.1.32: Clopen algebras

Let X be a topological space. The set Cl_X of all clopen (i.e. open and closed) subsets of X is called the **clopen algebra** of X.

The clopen algebra of any topological space is a Boolean algebra under the usual set-theoretical operations. Furthermore, every continuous function between topological spaces corresponds to a homomorphism of Boolean algebras in the reverse direction. That is, if $f : X \to Y$ is a continuous function, then the inverse image map

$$f^{-1}[-] \ : \ Cl_Y \ \ni \ A \ \mapsto \ f^{-1}[A] \ \in \ Cl_X$$

is a Boolean algebra homomorphism. In category-theoretical language, the “clopen algebra” and “inverse image map” constructions form a contravariant functor from the category of topological spaces to the category of Boolean algebras. When restricted to the subcategory of Stone spaces, it turns out that this functor is actually an equivalence, as the Stone Representation Theorem will show.

The claims above are substantiated by the following theorem. Given a function f, we denote its inverse image map by Cl_f. We will not make much use of the fact that the map Cl is a functor—however, the category-theoretical language allows us to neatly express its properties.

Theorem 1.1.33: Cl is a functor

Cl is a contravariant functor from the category **Top** of topological spaces and continuous functions to the category **BAlg** of Boolean algebras and homomorphisms. That is,

(i) If X is a topological space, then Cl_X is a Boolean algebra under the operations of union, intersection, and complementation, with join identity the empty set and meet identity X.

(ii) If $f : X \to Y$ is a continuous function of topological spaces, then Cl_f is a Boolean algebra homomorphism from Cl_Y to Cl_X.

(iii) If X is a topological space and $i : X \to X$ is the identity map, then $Cl_i : Cl_X \to Cl_X$ is the identity homomorphism.

(iv) If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $Cl_{g\circ f} = Cl_f \circ Cl_g$.

Proof:
(i): The power set of X is a Boolean algebra under the operations of union, intersection, and complementation, with join identity $\emptyset$ and meet identity X. The union and intersection of two clopen sets is a clopen set, and the complement of a clopen set is clopen. The empty set and X are clopen in X, so they are both in Cl_X. Hence Cl_X is closed under the Boolean algebra operations of the power set Boolean algebra of X, and contains the identities. Thus Cl_X is a

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Boolean algebra.

(ii): Let $U, V \in Cl_Y$. Then,

$$\begin{aligned} Cl_f(U \vee V) &= Cl_f(U \cup V) \\ &= f^{-1}[U \cup V] \\ &= f^{-1}[U] \cup f^{-1}[V] \\ &= Cl_f(U) \vee Cl_f(V). \end{aligned} \qquad \begin{aligned} Cl_f(U \wedge V) &= Cl_f(U \cap V) \\ &= f^{-1}[U \cap V] \\ &= f^{-1}[U] \cap f^{-1}[V] \\ &= Cl_f(U) \wedge Cl_f(V). \end{aligned}$$

$$\begin{aligned} Cl_f(0_{Cl_Y}) &= f^{-1}[\emptyset] \\ &= \emptyset \\ &= 0_{Cl_X}. \end{aligned} \qquad \begin{aligned} Cl_f(1_{Cl_Y}) &= f^{-1}[Y] \\ &= X \\ &= 1_{Cl_X}. \end{aligned}$$

$$\begin{aligned} Cl_f(\neg U) &= Cl_f(Y \backslash U) \\ &= f^{-1}[Y \backslash U] \\ &= X \backslash f^{-1}[U] \\ &= \neg Cl_f(U). \end{aligned}$$

Thus $Cl_f : Cl_Y \to Cl_X$ is a Boolean algebra homomorphism.

(iii): The inverse image of any set under the identity map is itself. Hence Cl_i is the identity homomorphism on Cl_X.

(iv): Let $W \in Cl_Z$. Then,

$$\begin{aligned} Cl_{g \circ f}(W) &= (g \circ f)^{-1}[W] \\ &= f^{-1}\left[g^{-1}[W]\right] \\ &= (Cl_f \circ Cl_g)(W). \end{aligned}$$

Thus $Cl_{g \circ f} = Cl_f \circ Cl_g$.

We now define the core construction of the theorem: the Stone space *of a Boolean algebra*. It can be shown that this construction can be extended to a contravariant functor from the category of Boolean algebras to the category of Stone spaces, and that this yields the inverse functor to Cl.

The Stone space of a Boolean algebra can be defined either as the set of ultrafilters in that Boolean algebra, or the set of homomorphisms from that Boolean algebra into the unique two-element Boolean algebra. The unique two-element Boolean algebra is just the Boolean algebra corresponding to the unique two-element field mentioned in Remark 1.1.21 (ii). In any case, we include its definition here.

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Definition 1.1.34: $\mathbb{2}$

Define $\mathbb{2}$ as the (up to isomorphism) unique Boolean algebra with two elements. Denote its elements by 0 and 1, with join and meet given by the following tables:

$\vee$	**0**	**1**
0	0	1
1	1	1

$\wedge$	**0**	**1**
0	0	0
1	0	1

This definition makes 0 the least element and 1 the greatest element (hence also defining complementation). It can also be defined as the set $\{true, false\}$ with the logical operations of disjunction (OR) and conjunction (AND). Incidentally (though not relevant in this thesis), any identity which holds in $\mathbb{2}$ holds in every Boolean algebra ([HG09], Chapter 15, Theorem 9)

The notation $\mathbb{2}$ is consistent with the fact that the von Neumann definition of the number 2 is exactly the set $\{0, 1\}$.

Definition 1.1.35: Stone spaces of Boolean algebras

Let $\mathbb{L}$ be a Boolean algebra. We define the **Stone space** K of $\mathbb{L}$ to be the set of Boolean algebra homomorphisms from $\mathbb{L}$ to $\mathbb{2}$:

$$K := \{u : \mathbb{L} \to \mathbb{2} \mid u \text{ is a homomorphism}\}.$$

We define ψ (called the **Stone isomorphism**, a name which will be justified later) as follows: for each $A \in \mathbb{L}$, let

$$\psi(A) := \{u \in K : u(A) = 1\}.$$

We endow K with a topology (called the **Stone topology**) having the following base:

$$\mathcal{B} = \{\psi(A) : A \in \mathbb{L}\}.$$

Remark 1.1.36: Ultrafilter definition of the Stone space of a Boolean algebra

It is worth noting that we can alternatively define the Stone space K of a Boolean algebra $\mathbb{L}$ as follows. Let

$$K_U = \{\mathbf{F} \subseteq \mathbb{L} : \mathbf{F} \text{ is an ultrafilter on } \mathbb{L}\}$$
$$\psi_U : \quad \mathbb{L} \ni A \ \mapsto \ \{\mathbf{F} \in K_U : A \in \mathbf{F}\} \subseteq K_U.$$

The two definitions are equivalent because every ultrafilter corresponds uniquely to a homomorphism, and vice versa. Indeed, let $\mathbf{F} \subseteq \mathbb{L}$ be an ultrafilter. Let $u_{\mathbf{F}}$ be the indicator

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function of $\mathbf{F}$. That is, $u_{\mathbf{F}}(A) = \begin{cases} 1 & A \in \mathbf{F} \\ 0 & A \notin \mathbf{F} \end{cases}$ for all $A \in \mathbb{L}$. We now show that $u_{\mathbf{F}}$ is a Boolean algebra homomorphism from $\mathbb{L}$ to $\mathbb{2}$.

Let $A \in \mathbb{L}$. By the maximality of $\mathbf{F}$, exactly one of A and $\neg A$ is in $\mathbf{F}$ (Theorem 1.1.26). If $A \in \mathbf{F}$, then $u_{\mathbf{F}}(\neg A) = 0$ and thus $\neg u_{\mathbf{F}}(A) = \neg 1 = 0 = u_{\mathbf{F}}(\neg A)$. If $A \notin \mathbf{F}$, then $u_{\mathbf{F}}(\neg A) = 1$ and $\neg u_{\mathbf{F}}(A) = \neg 0 = 1 = u_{\mathbf{F}}(\neg A)$. Thus $u_{\mathbf{F}}$ preserves complementation.

Let $B \in \mathbb{L}$. If $u_{\mathbf{F}}(A \wedge B) = 1$, then $A \wedge B \in \mathbf{F}$. By the upward closure of $\mathbf{F}$, A and B are both in $\mathbf{F}$, whence $u_{\mathbf{F}}(A) \wedge u_{\mathbf{F}}(B) = 1 \wedge 1 = 1 = u_{\mathbf{F}}(A \wedge B)$. If $u_{\mathbf{F}}(A \wedge B) = 0$, then $A \wedge B \notin \mathbf{F}$. Since $\mathbf{F}$ is closed under meets, at least one of A and B is not in $\mathbf{F}$. Hence at least one of $u_{\mathbf{F}}(A)$ and $u_{\mathbf{F}}(B)$ is 0, so $u_{\mathbf{F}}(A) \wedge u_{\mathbf{F}}(B) = 0 = u_{\mathbf{F}}(A \wedge B)$. Thus $u_{\mathbf{F}}$ preserves meets.

By De Morgan's Laws, $u_{\mathbf{F}}$ preserves joins as well (because $u_{\mathbf{F}}(A \vee B) = u_{\mathbf{F}}(\neg(\neg A \wedge \neg B)) = \neg(\neg u_{\mathbf{F}}(A) \wedge \neg u_{\mathbf{F}}(B)) = u_{\mathbf{F}}(A) \vee u_{\mathbf{F}}(B)$). Furthermore, $u_{\mathbf{F}}(0) = 0$ (because $\mathbf{F}$ is a *proper* filter) and $u_{\mathbf{F}}(1) = 1$ (because $\mathbf{F}$ is upward closed and non-empty).

Thus $u_{\mathbf{F}}$ is a homomorphism.

It follows easily that $\mathbf{F}$ is the inverse image of $\{1\} \subseteq \mathbb{2}$ under $u_{\mathbf{F}}$. Conversely, if $u : \mathbb{L} \to \mathbb{2}$ is a homomorphism, then the inverse image of $\{1\}$ under u is an ultrafilter, and u is the indicator function of this ultrafilter.

The next remark is included for the sake of interest, and may be skipped.

Remark 1.1.37: Aside on an additional definition of the Stone space of a Boolean algebra

The equivalence between the two definitions of the Stone space of a Boolean algebra can be taken further. Since every ultrafilter on $\mathbb{L}$ is the inverse image of 1 under a homomorphism from $\mathbb{L}$ to $\mathbb{2}$, we can alternatively associate each ultrafilter with the *kernel* (i.e. inverse image of 0) of the corresponding homomorphism. It turns out that the kernels of homomorphisms from $\mathbb{L}$ to $\mathbb{2}$ are precisely the prime ideals of $\mathbb{L}$ (considered as a Boolean ring). This gives a bijection between ultrafilters and prime ideals in $\mathbb{L}$. Hence we can alternatively define the Stone space K as the set of prime ideals of $\mathbb{L}$, which in algebraic geometry is known as $\mathrm{Spec}(\mathbb{L})$ (the **spectrum of** $\mathbb{L}$). Endowing $\mathrm{Spec}(\mathbb{L})$ with a topology whose base consists of the sets of the form $\{\mathcal{I} \in \mathrm{Spec}(\mathbb{L}) : A \notin \mathcal{I}\}$ for $A \in \mathbb{L}$ yields a topological space homeomorphic to the ultrafilter version of the Stone space. This topology on $\mathrm{Spec}(\mathbb{L})$ happens to be the so-called **Zariski topology**, which is the standard topology used in algebraic geometry to study the spectra of rings.

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Notice that we have defined ψ as a function on $\mathbb{L}$. It will turn out that ψ is in fact an isomorphism from $\mathbb{L}$ to the clopen algebra of K, but we first need to show that K is indeed a topological space, and that the sets in the image of ψ are clopen in K.

We first show that the Stone isomorphism is a homomorphism from $\mathbb{L}$ to the power set Boolean algebra of K. We do this first because it makes it easier to show that K is a Stone space and to prove other results.

Theorem 1.1.38: ψ is a Boolean algebra homomorphism

Let $\mathbb{L}$ be a Boolean algebra. Define K and ψ as in Definition 1.1.35. Then ψ is a Boolean algebra homomorphism from $\mathbb{L}$ to $\mathcal{P}(K)$.

Proof:
Let $A, B \in \mathbb{L}$. Then

$$\begin{aligned}
\psi(A \vee B) &= \{u \in K : u(A \vee B) = 1\} \\
&= \{u \in K : u(A) \vee u(B) = 1\} \\
&= \{u \in K : u(A) = 1 \text{ or } u(B) = 1\} \\
&= \{u \in K : u(A) = 1\} \\
&\qquad \cup \{u \in K : u(B) = 1\} \\
&= \psi(A) \cup \psi(B).
\end{aligned}
\qquad
\begin{aligned}
\psi(A \wedge B) &= \{u \in K : u(A \wedge B) = 1\} \\
&= \{u \in K : u(A) \wedge u(B) = 1\} \\
&= \{u \in K : u(A) = 1 \text{ and } u(B) = 1\} \\
&= \{u \in K : u(A) = 1\} \\
&\qquad \cap \{u \in K : u(B) = 1\} \\
&= \psi(A) \cap \psi(B).
\end{aligned}$$

$$\begin{aligned}
\psi(\neg A) &= \{u \in K : u(\neg A) = 1\} \\
&= \{u \in K : u(A) = 0\} \\
&= \{u \in K : u(A) \neq 1\} \\
&= K \backslash \psi(A).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\psi(0) &= \{u \in K : u(0) = 1\} \\
&= \emptyset.
\end{aligned}
\qquad
\begin{aligned}
\psi(1) &= \{u \in K : u(1) = 1\} \\
&= K.
\end{aligned}$$

Since this holds for all $A, B \in \mathbb{L}$, this shows that ψ is a Boolean algebra homomorphism.

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Theorem 1.1.39: K is a topological space

Let $\mathbb{L}$ be a Boolean algebra, and define K as in Definition 1.1.35. Then K is a topological space.

Proof:

The collection

$$\mathcal{B} = \{\psi(A) : A \in \mathbb{L}\}$$

covers K, because $\psi(1) = \{u \in K : u(1) = 1\} = K$, hence $\bigcup_{A \in \mathbb{L}} \psi(A) = K$. It is also closed under intersection, because for all $A, B \in \mathbb{L}$, $\psi(A) \cap \psi(B) = \psi(A \wedge B) \in \mathcal{B}$. Hence $\mathcal{B}$ is a base for a topology on K. This makes K into a topological space.

It follows that a subset U of K is open if and only if, for all $u \in U$, there exists an $A \in \mathbb{L}$ such that $u \in \psi(A)$ (i.e., $u(A) = 1$) and $\psi(A) \subseteq U$.

We now show that for a Boolean algebra $\mathbb{L}$, K is indeed a Stone space, thus justifying the use of the name "Stone space" for that construction.

Theorem 1.1.40: K is a Stone space

Let $\mathbb{L}$ be a Boolean algebra and K its Stone space as in Definition 1.1.35. Then K is a Stone space.

Proof:

K is zero-dimensional:

Let $A \in \mathbb{L}$. Then $\psi(A)$ is open in K, and

$$\begin{aligned} K \backslash \psi(A) &= \{u \in K : u(A) = 0\} \\ &= \{u \in K : u(\neg A) = 1\} \\ &= \psi(\neg A). \end{aligned}$$

This shows that $K \backslash \psi(A)$ is also open. Hence $\psi(A)$ is clopen, and thus $\mathcal{B}$ is a base for the topology on K consisting of clopen sets.

K is compact:

Let $\mathbf{F}$ be an ultrafilter of subsets of K. Suppose $\mathbf{F}$ does not converge to any point in K. That is, for all $u \in K$, there exists a neighbourhood U of u such that $U \notin \mathbf{F}$.

Let u be the indicator function of $\mathbf{F}$. Then u is a homomorphism (as shown in Remark 1.1.36). Since u is a homomorphism and $\mathbf{F}$ does not converge to u, there exists a neighbourhood U of u such that $U \notin \mathbf{F}$. There exists an $A \in \mathbb{L}$ such

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that $u \in \psi(A) \subseteq U$, and thus $\psi(A) \notin \mathbf{F}$. By the definition of u, $u(A) = 0$. But this implies that $u \notin \psi(A)$, contradicting the fact that $u \in \psi(A)$. Hence $\mathbf{F}$ must converge to at least one point of K.

Hence every ultrafilter of subsets of K converges to at least one point in K. Therefore K is compact.

K is Hausdorff:

Let u and v be any distinct elements of K. Then there exists an $A \in \mathbb{L}$ such that $u(A) \neq v(A)$. Suppose $u(A) = 1$ and $v(A) = 0$. Then $u \in \psi(A)$ and $v \notin \psi(A)$. It follows that $v \in \psi(\neg A)$, and $\psi(A)$ and $\psi(\neg A)$ are disjoint. Thus $\psi(A)$ and $\psi(\neg A)$ are disjoint neighbourhoods of u and v respectively. Thus K is Hausdorff.

Thus K is a zero-dimensional compact Hausdorff space, i.e. a Stone space.

To recap: we have shown that the clopen algebra of a topological space is a Boolean algebra, and that this construction yields a contravariant functor from topological spaces to Boolean algebras. We have also shown that, given a Boolean algebra $\mathbb{L}$, the Stone space K of $\mathbb{L}$ is a Stone space, and the Stone isomorphism $\psi : A \mapsto \{u \in K : u(A) = 1\}$ is a homomorphism from $\mathbb{L}$ to $\mathcal{P}(K)$. The Stone Representation Theorem is essentially the statement that ψ is an isomorphism from $\mathbb{L}$ to Cl_K.

We proceed with another theorem which will be needed to prove the Stone Representation Theorem.

Theorem 1.1.41

Let X and Y be Stone spaces, and let $\phi : Cl_Y \to Cl_X$ be a Boolean algebra homomorphism. Then there exists a unique continuous function $f : X \to Y$ such that $\phi = Cl_f$.

Proof:

We wish to define $f : X \to Y$ such that $\phi(U) = f^{-1}[U] = \{x \in X : f(x) \in U\}$ for all $U \in Cl_Y$. This property is satisfied by f if and only if

$$f(x) \in U \quad \text{if and only if} \quad x \in \phi(U)$$

for all $x \in X$.

Let $x \in X$. We claim that the collection $W = \{U \in Cl_Y : x \in \phi(U)\}$ has the finite intersection property. The purpose of this claim is to show, by the compactness of Y, that the collection itself has non-empty intersection, i.e. that it has at least one element. We will then proceed to show that it has at *most* one element, and we will define $f(x)$ to be this unique element. We now verify the claim.

Let $\{U_1, \ldots, U_n\}$ be a finite subset of W. Then $x \in \phi(U_1) \cap \cdots \cap \phi(U_n)$. Since ϕ is a Boolean algebra homomorphism, this implies $x \in \phi(U_1 \cap \cdots \cap U_n)$. Hence

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$\phi(U_1 \cap \cdots \cap U_n)$ is non-empty, and thus $U_1 \cap \cdots \cap U_n$ is non-empty. Hence W has the finite intersection property.

Since Y is compact, W has non-empty intersection. That is, $\bigcap W$ has at least one element. It remains to show that there is at most one (and thus exactly one) element in this intersection.

Suppose $a, b \in \bigcap W$ with $a \neq b$. Since Y is Hausdorff, there exist disjoint open sets A and B such that $a \in A$ and $b \in B$. Since Y is zero-dimensional, we may assume that A and B are closed as well. Since Y is a compact Hausdorff space and A and B are closed, it follows that A and B are compact. Since A and B are disjoint, we have $x \notin \phi(A \cap B) = \phi(A) \cap \phi(B)$. Assume $x \notin \phi(A)$. Then $x \in \phi(Y \backslash A)$, so $Y \backslash A \in W$. Hence $\bigcap W \subseteq Y \backslash A$. Therefore A is disjoint from $\bigcap W$. This is a contradiction, since $a \in A$ and $a \in \bigcap W$; thus there must be exactly one element of $\bigcap W$.

We define $f(x)$ to be this unique element.

Now that we have a function $f : X \to Y$, it remains to show that f is continuous, and that $\phi = Cl_f$.

Let $U \in Cl_Y$. Then

$$Cl_f(U) = f^{-1}[U] = \{x \in X : f(x) \in U\}$$

Observe that $f(x) \in U$ if and only if $x \in \phi(U)$. Hence this becomes

$$\begin{aligned} Cl_f(U) &= \{x \in X : x \in \phi(U)\} \\ &= \phi(U). \end{aligned}$$

This shows that $\phi = Cl_f$.

Moreover, since $\phi : Cl_Y \to Cl_X$ is a Boolean algebra homomorphism, this shows that Cl_f maps clopen subsets of Y to clopen subsets of X. Hence the inverse image of every clopen subset of Y under f is a clopen subset of X.

Recall that Y is zero-dimensional. Thus the topology on Y has a base, say $\mathcal{B}$, consisting of clopen sets. Hence the inverse image of any element of $\mathcal{B}$ under f is a clopen subset of X. In particular, the inverse image of every basic open set in Y under f is open in X. This shows that f is continuous.

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Before moving on to the Stone Representation Theorem, we prove a central and extremely useful result about Stone spaces.

Theorem 1.1.42

Let X be a Stone space and let $\mathcal{S}$ be a subset of Cl_X. Then $\mathcal{S}$ has a supremum in Cl_X if and only if $\overline{\bigcup \mathcal{S}}$ is open in X. Furthermore, if $\mathcal{S}$ has a supremum in Cl_X, then its supremum is $\overline{\bigcup \mathcal{S}}$.

Proof:

Suppose $\overline{\bigcup \mathcal{S}}$ is open in X. Then it is a clopen set, so $\overline{\bigcup \mathcal{S}} \in Cl_X$. Since $A \subseteq \bigcup \mathcal{S} \subseteq \overline{\bigcup \mathcal{S}}$ for each $A \in \mathcal{S}$, $\overline{\bigcup \mathcal{S}}$ is an upper bound of $\mathcal{S}$.

Now let $B \in Cl_X$ be an upper bound of $\mathcal{S}$ in Cl_X. Then $A \subseteq B$ for all $A \in \mathcal{S}$, so $\bigcup_{A \in \mathcal{S}} A = \bigcup \mathcal{S} \subseteq B$. Since $B \in Cl_X$, taking closures shows

$$\overline{\bigcup \mathcal{S}} \subseteq B.$$

This shows that $\overline{\bigcup \mathcal{S}}$ is the least upper bound of $\mathcal{S}$ in Cl_X.

Suppose $\mathcal{S}$ has a supremum $\bigvee \mathcal{S}$ in Cl_X. Then

$$A \subseteq \bigvee \mathcal{S}$$

for all $A \in \mathcal{S}$. Thus $\bigcup \mathcal{S} \subseteq \bigvee \mathcal{S}$. Since $\bigvee \mathcal{S}$ is closed, taking closures gives

$$\overline{\bigcup \mathcal{S}} \subseteq \bigvee \mathcal{S}.$$

Now suppose that $\overline{\bigcup \mathcal{S}} \neq \bigvee \mathcal{S}$. Then $\bigvee \mathcal{S} \backslash \overline{\bigcup \mathcal{S}}$ is non-empty. Let $V = \bigvee \mathcal{S} \backslash \overline{\bigcup \mathcal{S}}$. Being the intersection of two open sets, V is a non-empty open set. Let $u \in V$. Since X is zero-dimensional, there exists a clopen set U containing u such that $U \subseteq V$.

Since V is disjoint from $\overline{\bigcup \mathcal{S}}$, it follows that U is also disjoint from $\overline{\bigcup \mathcal{S}}$. Together with the fact that $\overline{\bigcup \mathcal{S}} \subseteq \bigvee \mathcal{S}$, this shows that

$$\overline{\bigcup \mathcal{S}} \subseteq \left(\bigvee \mathcal{S}\right) \backslash U.$$

Moreover, $A \subseteq \overline{\bigcup \mathcal{S}}$ for all $A \in \mathcal{S}$, so $A \subseteq (\bigvee \mathcal{S}) \backslash U$ for all $A \in \mathcal{S}$. Since $\bigvee \mathcal{S}$ and U are both in Cl_X it follows that $(\bigvee \mathcal{S}) \backslash U \in Cl_X$. Hence $(\bigvee \mathcal{S}) \backslash U$ is also an upper bound of $\mathcal{S}$. But U is a non-empty subset of $\bigvee \mathcal{S}$, so $(\bigvee \mathcal{S}) \backslash U$ is a proper subset of $\bigvee \mathcal{S}$. This contradicts the fact that $\bigvee \mathcal{S}$ is the supremum of $\mathcal{S}$ in Cl_X. Thus V must be empty. That is,

$$\overline{\bigcup \mathcal{S}} = \bigvee \mathcal{S}.$$

This shows that $\overline{\bigcup \mathcal{S}}$ is the supremum of $\mathcal{S}$ in Cl_X. This also shows that $\overline{\bigcup \mathcal{S}}$ is open in X.

Using De Morgan's Laws and the fact that the closure and interior are dual operations (in the sense

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that $X \setminus (A^o) = \overline{X \setminus A}$ for each topological space X and each subset A of X), it follows that, if X is a Stone space and $\mathcal{S} \subseteq Cl_X$, then $\mathcal{S}$ has an infimum in Cl_X if and only if $\left(\bigcap \mathcal{S}\right)^o$ is closed in X. Furthermore, if $\mathcal{S}$ has an infimum in Cl_X, then its infimum is $\left(\bigcap \mathcal{S}\right)^o$.

1.2 The Stone Representation Theorem

We now have the necessary background for the proof of the Stone Representation Theorem.

Theorem 1.2.1: The Stone Representation Theorem

Let $\mathbb{L}$ be a Boolean algebra. Then $\mathbb{L}$ is isomorphic to the clopen algebra of its Stone space via the Stone isomorphism $\psi : A \mapsto \{u \in K : u(A) = 1\}$.

Furthermore, if X is a Stone space and $\theta : \mathbb{L} \to Cl_X$ is a Boolean algebra homomorphism, then there exists a unique continuous function $f : X \to K$ such that $Cl_f \circ \psi = \theta$.

Proof:

We first show that ψ is a Boolean algebra isomorphism.

ψ is injective:

Let $A \neq 0$ be an element of $\mathbb{L}$. We show that $\psi(A) \neq \emptyset$ by finding a homomorphism $u : \mathbb{L} \to 2$ such that $u(A) = 1$.

Define the following filter on $\mathbb{L}$:

$$\mathbf{F}_A = \{B \in \mathbb{L} : B \geq A\} \subseteq \mathbb{L}.$$

By the Ultrafilter Lemma, there exists an ultrafilter $\mathbf{F}$ on $\mathbb{L}$ such that $\mathbf{F}_A \subseteq \mathbf{F}$. Define $u_A : \mathbb{L} \to 2$ to be the indicator function of $\mathbf{F}$. By Remark 1.1.36, u_A is a homomorphism. Clearly $A \geq A$, so $A \in \mathbf{F}$, and thus $u_A(A) = 1$. Hence $u_A \in \psi(A)$, so $\psi(A) \neq \emptyset$.

This shows that $\psi(A)$ is empty if and only if $A = 0$.

Let $A, B \in \mathbb{L}$, and suppose $\psi(A) = \psi(B)$. Then $\psi(A \setminus B) = \psi(A) \setminus \psi(B) = 0 = \emptyset$. By the above, $A \setminus B = 0$, so $A \leq B$. By an analogous argument, $B \leq A$. Thus $A = B$. This shows that ψ is injective.

[3] Note that this is an ultrafilter on $\mathbb{L}$, not on the power set of $\mathbb{L}$.

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ψ is surjective:

Let U be a clopen subset of K. If $U = \emptyset$, then $U = \psi(0)$. Suppose $U \neq \emptyset$. Then, for each $u \in U$, there exists an $A_u \in U$ such that $\psi(A_u) \subseteq U$. Then the set $\{\psi(A_u) : u \in U\}$ is an open cover of U. In fact, since $\psi(A_u) \subseteq U$ for all $u \in U$, it follows that

$$U = \bigcup_{u \in U} \psi(A_u).$$

Since U is a closed subset of a compact Hausdorff space, U is compact, and thus there exist finitely many elements, say $A_{u_1}, \ldots, A_{u_n} \in \mathbb{L}$, such that

$$U = \bigcup_{i=1}^{n} \psi(A_{u_i}).$$

Since ψ is a Boolean algebra homomorphism, it follows that

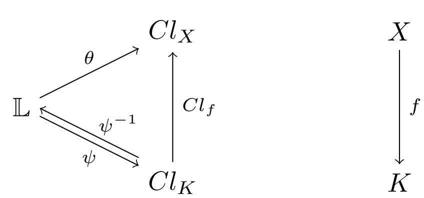
$$U = \psi\left(A_{u_1} \vee \cdots \vee A_{u_n}\right).$$

Thus, there exists an $A \in \mathbb{L}$ (namely, $A_{u_1} \vee \cdots \vee A_{u_n}$) such that $U = \psi(A)$. This shows that ψ is surjective. (Moreover, this also shows that $Cl_K = \mathcal{B}$, the base for the Stone topology on K.)

Thus ψ is a bijective homomorphism, and thus an isomorphism of $\mathbb{L}$ to Cl_K.

The second conclusion of the present theorem is now easy to prove. Let X be a Stone space and $\phi : \mathbb{L} \to Cl_X$ be a homomorphism. Then $\psi^{-1} \circ \phi$ is a homomorphism from Cl_K to Cl_X. By Theorem 1.1.41, there exists a unique continuous function $f : X \to K$ such that $\psi^{-1} \circ \phi = Cl_f$.

The situation in the Stone Representation Theorem is represented by the following commutative diagram:



We conclude with some corollaries of the Stone Representation Theorem. The first one, which we will not prove, is a more concrete statement of the correspondence between Boolean algebras and Stone spaces.

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Corollary 1.2.2: Stone Duality

The category **Stone** of Stone spaces and continuous maps is dual (i.e. contravariantly equivalent) to the category **BAlg** of Boolean algebras and homomorphisms.

Corollary 1.2.3: An isomorphism from $\mathbb{L}$ to Cl_X yields a homeomorphism from K to X

Let $\mathbb{L}$ be a Boolean algebra and K its Stone space. If X is a Stone space and $\theta : \mathbb{L} \to Cl_X$ is an isomorphism, then there exists a homeomorphism $f : X \to K$ such that $Cl_f \circ \psi = \theta$.

Proof:

By the Stone Representation Theorem, there exists a unique continuous function $f : X \to K$ such that $Cl_f \circ \psi = \theta$. It thus remains to show that f is a homeomorphism.

Since θ is an isomorphism, the inverse θ^{-1} exists and is an isomorphism from Cl_X to $\mathbb{L}$. Hence $\psi \circ \theta^{-1}$ is an isomorphism from Cl_X to Cl_K. By Theorem 1.1.41, there exists a continuous function $g : K \to X$ such that $Cl_g = \psi \circ \theta^{-1}$. Pre-composing (i.e. composing on the right) with θ shows that $Cl_g \circ \theta = \psi$. Hence

$$\begin{aligned} Cl_g \circ (Cl_f \circ \psi) &= \psi \\ Cl_g \circ Cl_f &= \mathrm{id}_{Cl_K} \\ Cl_{f \circ g} &= \mathrm{id}_{Cl_K}, \end{aligned}$$

where id_{Cl_K} is the identity function on Cl_K.

Thus the inverse image of any clopen subset of K under $f \circ g$ is itself. Since the topology on K is generated by Cl_K, this shows that the inverse image of any open subset and any closed subset of K under $f \circ g$ is itself. Since K is a Hausdorff space, each singleton is closed in K. Hence, for all $u \in K$, the inverse image of $\{u\}$ under $f \circ g$ is $\{u\}$. This shows that $f \circ g = \mathrm{id}_K$. In particular, this shows that f is surjective.

Substituting $\psi = Cl_g \circ \theta$ into $Cl_f \circ \psi = \theta$ yields

$$\begin{aligned} Cl_f \circ (Cl_g \circ \theta) &= \theta \\ Cl_f \circ Cl_g &= \mathrm{id}_{Cl_X} \\ Cl_{g \circ f} &= \mathrm{id}_{Cl_X}. \end{aligned}$$

By a similar argument to that of the previous paragraph, this shows that $g \circ f = \mathrm{id}_X$. In particular, f is injective and thus bijective. Since the domain of f is compact and the codomain is Hausdorff, it follows that f is a homeomorphism.

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The next corollary is a topological charaterisation of the Stone spaces of complete Boolean algebras. The complete Boolean algebras correspond to the extremally disconnected Stone spaces.

Definition 1.2.4: Extremally disconnected

Let X be a topological space. We say that X is **extremally disconnected** if the closure of every open subset of X is open.

A Stone space which is extremally disconnected is called a **Stonean space**.

Definition 1.2.5: Dedekind complete

Let P be a partially ordered set. We say that P is **Dedekind complete** if every subset of P which is bounded above has a least upper bound.

Dedekind completeness is self-dual for complemented lattices: a complemented lattice is Dedekind complete if and only if every subset which is bounded below has a greatest lower bound. In a bounded lattice—and, in particular, in a Boolean algebra—every subset is bounded above and below, so Dedekind completeness of a Boolean algebra is equivalent to ordinary completeness.

Corollary 1.2.6: Equivalence between Dedekind completeness of Cl_X and extremal disconnectedness of X

Let X be a Stone space. Then X is extremally disconnected if and only if Cl_X is Dedekind complete.

Proof:

Suppose X is extremally disconnected, and let $\mathcal{S}$ be a subset of Cl_X. Then $\bigcup \mathcal{S}$ is an open subset of X. Since X is extremally disconnected, the closure $\overline{\bigcup \mathcal{S}}$ is open. Hence $\overline{\bigcup \mathcal{S}} \in Cl_X$. By Theorem 1.1.42, $\mathcal{S}$ has a supremum in Cl_X, and its supremum is equal to $\overline{\bigcup \mathcal{S}}$.

This shows that every subset of Cl_X has a supremum in Cl_X. Hence Cl_X is Dedekind complete.

Suppose Cl_X is Dedekind complete, and let U be an open subset of X.

Since X is zero-dimensional, we can write U as a union of clopen sets:

$$U = \bigcup \mathcal{U},$$

where $\mathcal{U} \subseteq Cl_X$ is some set of clopen sets. Since Cl_X is Dedekind complete, $\mathcal{U}$ has a supremum $\bigvee \mathcal{U}$ in Cl_X. By Theorem 1.1.42,

$$\bigvee \mathcal{U} = \overline{\bigcup \mathcal{U}} \in Cl_X.$$

Hence $\overline{\bigcup \mathcal{U}} = \overline{U}$ is open. Since U was an arbitrary open set, this shows that X is Dedekind complete.

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2 The Loomis-Sikorski Theorem

The Stone Representation Theorem establishes that every Boolean algebra is isomorphic to the clopen algebra of its Stone space. This result provides a representation of Boolean algebras as algebras of sets. It is natural (especially in the context of measure theory) to ask whether there exists an analogous representation for the σ-complete Boolean algebras as σ-algebras. (Recall that a Boolean algebra is said to be **σ-complete** if every countable subset has a supremum and an infimum.)

Having seen the Stone Representation Theorem, the reader might wonder why we need a representation of σ-complete Boolean algebras as algebras of sets. After all, we already know that every Boolean algebra is isomorphic to an algebra of sets, and the isomorphism in question preserves *all* suprema and infima where they exist. Thus, one might expect that if $\mathbb{L}$ is a σ-complete Boolean algebra and K is its Stone space, then Cl_K should be a σ-algebra, and countably infinite suprema and infima in $\mathbb{L}$ correspond to countably infinite unions and intersections in Cl_K, respectively. However, this is not the case. To illustrate why, let $\mathbb{L}$ be a σ-complete Boolean algebra with Stone space K, and let $\mathcal{S}$ be a countably infinite subset of Cl_K. Then the union $\bigcup \mathcal{S}$ of $\mathcal{S}$ is not necessarily closed, and thus not necessarily clopen. Hence we are not guaranteed that $\bigcup \mathcal{S} \in Cl_K$. In fact, it is easy to construct an example where $\bigcup \mathcal{S} \notin Cl_K$. Suppose $\mathbb{L}$ is an infinite σ-complete Boolean algebra, and let $\mathcal{S} = \{A_n : n \in \mathbb{N}\}$ be a collection of countably infinitely many disjoint non-zero elements of $\mathbb{L}$. Let $\psi : \mathbb{L} \to Cl_K$ be the Stone isomorphism of $\mathbb{L}$. Then $\{\psi(A_n) : n \in \mathbb{N}\}$ is a countably infinite disjoint collection of clopen subsets of K, but the union of these subsets is *not* clopen in K. This is because it is (by definition) covered by infinitely many disjoint open sets, and therefore it is not compact. However, in a compact space (in particular, in K), every closed set is compact. Thus $\bigcup_{n\in\mathbb{N}} \psi(A_n)$ is not closed and thus not clopen. This shows that the supremum of $\mathcal{S}$ in $\mathbb{L}$ does not correspond to the union of the corresponding subsets of K. That is,

$$\psi\left(\bigvee\nolimits^{\mathbb{L}}_{n\in\mathbb{N}} A_n\right) \quad = \quad \bigvee\nolimits^{Cl_K}_{n\in\mathbb{N}} \psi(A_n) \quad \neq \quad \bigcup_{n\in\mathbb{N}} \psi(A_n).$$

The correct answer is given by Theorem 1.1.42: in the clopen algebra of a Stone space, the supremum (if it exists) of a collection of clopen sets is the *closure* of its union. In other words,

$$\psi\left(\bigvee\nolimits^{\mathbb{L}}_{n\in\mathbb{N}} A_n\right) \quad = \quad \bigvee\nolimits^{Cl_K}_{n\in\mathbb{N}} \psi(A_n) \quad = \quad \overline{\bigcup_{n\in\mathbb{N}} \psi(A_n)}.$$

Consequently, the clopen algebra of the Stone space of $\mathbb{L}$ is not a σ-regular subalgebra of the *power set* algebra of K. That is, the inclusion function $i : Cl_K \to \mathcal{P}(K)$ of Cl_K into the power set of K does not preserve countable suprema.

This discrepancy is resolved by a closer examination of the "small" difference between open sets and their closures. More specifically, if U is an open set in a topological space X, then $\overline{U} \setminus U$ is a meagre

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set in X. The Loomis-Sikorski Theorem essentially says that, while we cannot in general represent a σ-complete Boolean algebra as a σ-algebra, we *can* represent it as the quotient of a σ-algebra by a collection of meagre sets. In particular, if $\mathbb{L}$ is σ-complete, then the meagre subsets of K form a σ-ideal in a certain σ-algebra, and $\mathbb{L}$ is isomorphic to the quotient of Cl_K by this ideal. Consequently, the supremum of a sequence of elements does correspond—modulo a meagre set—to the union of the corresponding clopen sets. This is what we will investigate in this chapter.

2.1 Preliminaries

2.1.1 Sigma-algebras

The definition of a σ-algebra is standard, but we include it here for the sake of clarity.

Definition 2.1.1: σ-algebras

Let X be a set, and let Σ be a collection of subsets of X. We say that Σ is a **σ-algebra on** X (or just a **σ-algebra** if the set X is implicit) if

- the empty set is in Σ,
- Σ is closed under complementation (i.e. if $S \in \Sigma$ then $X \backslash S \in \Sigma$), and
- Σ is closed under countable unions (i.e. if $\mathcal{S}$ is a countable subset of Σ, then $\bigcup \mathcal{S} \in \Sigma$).

By applying De Morgan's Laws, it follows immediately that every σ-algebra is closed under countable intersections.

In measure theory, σ-algebras provide the usual setting for measures.

Given a collection C of subsets of a set X, the intersection of all σ-algebras containing every set in C is a σ-algebra known as **the σ-algebra generated by** C. It is the smallest σ-algebra containing every set in C.

Definition 2.1.2: σ-complete Boolean algebras

Let $\mathbb{L}$ be a Boolean algebra. We say that $\mathbb{L}$ is **σ-complete** if every countable subset of $\mathbb{L}$ has a supremum and an infimum.

A few examples of σ-algebras are worth mentioning.

Example 2.1.3: σ-algebras

(i) For any set X, $\{\emptyset, X\}$ is a σ-algebra.

(ii) For any set X, the power set $\mathcal{P}(X)$ of X is a σ-algebra.

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For any set X, (i) and (ii) are the smallest and largest σ-algebras on X respectively.

(iii) Given a subset A of a set X, the set $\{\emptyset, A, X\backslash A, X\}$ is a σ-algebra.

(iv) For any topological space X, the **Borel σ-algebra on X** is the σ-algebra generated by the open subsets of X. (This terminology is found in, for example, [Fre00].) In general, this is not the power set of X. (For example, take the group $\mathbb{R}$ of real numbers under addition. The rational numbers $\mathbb{Q}$ form a normal subgroup of this group, and the elements of the quotient group $\mathbb{R}/\mathbb{Q}$ are equivalence classes of the form $\{r + q : q \in \mathbb{Q}\}$ for some $r \in \mathbb{R}$. These equivalence classes are called *Vitali sets*, and they are not Borel sets under the standard topology on $\mathbb{R}$.)

2.1.2 Ideals

In order to use the phrase "modulo clopen sets", we proceed by examining ideals and quotients by ideals in Boolean algebras and σ-complete Boolean algebras.

Definition 2.1.4: Ideals in order theory

Let $(P, \leq)$ be a partially ordered set, and let $I \subseteq P$. We say that I is an **ideal** (or an **order ideal**) of P if

- I is non-empty,
- for all $x \in I$ and all $y \in P$, $y \leq x$ implies $y \in I$ (I is **downward closed**), and
- for all $x, y \in I$, there exists a $z \in I$ such that $x \leq z$ and $y \leq z$ (I is **upward directed**).

If P is a lattice, then $I \subseteq P$ is an ideal in P if and only if it is non-empty, downward closed, and closed under finite joins.

If P is a Boolean algebra, then I is an ideal if and only if it is a *ring* ideal when P is considered as a Boolean ring. This is the origin of the terminology.

Theorem 2.1.5

Let $\mathbb{L}$ be a Boolean algebra, and let $I \subseteq \mathbb{L}$. Then I is an order ideal in $\mathbb{L}$ if and only if it is a ring ideal when $\mathbb{L}$ is considered as a Boolean ring.

Proof:
Suppose I is an order ideal in $\mathbb{L}$. Let $A, B \in I$. Recall that

$$A + B := (A \wedge \neg B) \vee (\neg A \wedge B).$$

Notice that $(A \wedge \neg B) \leq A$ and $(\neg A \wedge B) \leq B$. Since I is downward closed, $(A \wedge \neg B) \in I$ and

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$(\neg A \wedge B) \in I$. Since I is upward directed, there exists an $C \in I$ which is an upper bound of $(A \wedge \neg B)$ and $(\neg A \wedge B)$. Again by the downward closure of I, the *least* upper bound $(A \wedge \neg B) \vee (\neg A \wedge B)$ of $(A \wedge \neg B)$ and $(\neg A \wedge B)$ is also in I. Hence $A + B \in I$. This shows that I is closed under addition.

Let $C \in \mathbb{L}$. Recall that $AC := A \wedge C$. Moreover, $(A \wedge C) \leq A$, while $A \in I$ and I is downward closed. Hence $AC \in I$. This shows that I absorbs multiplication in $\mathbb{L}$. Thus I is a ring ideal in $\mathbb{L}$ (under the Boolean ring operations).

Suppose I is a ring ideal in $\mathbb{L}$ (under the Boolean ring operations). Then $0 \in I$, so I is non-empty. Let $A \in I$ and $B \in \mathbb{L}$ with $B \leq A$. Then $AB = A \wedge B = B$. Since $A \in I$ and I absorbs multiplication in $\mathbb{L}$, it follows that $AB = B \in I$. This shows that I is downward closed.

Let $A, B \in I$. Recall that

$$A \vee B := A + B + AB.$$

Since I is closed under addition and absorbs multiplication, $A \vee B \in I$. Since $A \leq A \vee B$ and $B \leq A \vee B$, this shows that I is upward directed (equivalently, closed under finite joins). Thus I is an order ideal in $\mathbb{L}$ (under the Boolean algebra operations).

This allows us to speak of a subset of a Boolean algebra as simply being an ideal, without specifying whether it is an order ideal or a ring ideal.

As is the case for rings, one can speak of the **ideal generated by a subset** of a Boolean algebra. This is the smallest ideal containing the subset, and it is equal to the intersection of all ideals which include the subset. The next theorem gives a useful order-theoretic description of such an ideal. (Compare this theorem with Theorem 1.1.25.)

Theorem 2.1.6: Ideals generated by subsets

Let $\mathbb{L}$ be a Boolean algebra and $\mathcal{S} \subseteq \mathbb{L}$. Let $\mathbf{I}_{\mathcal{S}}$ be the ideal generated by $\mathcal{S}$. Then

$$\mathbf{I}_{\mathcal{S}} = \left\{ A \in \mathbb{L} : A \leq \bigvee \mathcal{G} \text{ for some finite } \mathcal{G} \subseteq \mathcal{S} \right\}.$$

Proof:

It is straightforward to see that the set $\{A \in \mathbb{L} : A \leq \bigvee \mathcal{G}$ for some finite $\mathcal{G} \subseteq \mathcal{S}\}$ is an ideal which includes $\mathcal{S}$. Now, let $\mathcal{I} \supseteq \mathcal{S}$ be an ideal. If $\mathcal{G}$ is any finite subset of $\mathcal{S}$, then $\bigvee \mathcal{G} \in \mathcal{I}$, so $A \in \mathcal{I}$ whenever $A \leq \bigvee \mathcal{G}$ (since $\mathcal{I}$ is downward closed). Since this applies to all finite $\mathcal{G} \subseteq \mathcal{S}$, this shows that every $A \in \mathbb{L}$ which is less than or equal to the supremum of finitely many

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elements of $\mathcal{S}$ is also in $\mathcal{I}$. Therefore

$$\left\{A \in \mathbb{L} : A \leq \bigvee \mathcal{G} \text{ for some finite } \mathcal{G} \subseteq \mathcal{S}\right\} \subseteq \mathcal{I}.$$

Since this holds for all ideals $\mathcal{I}$ such that $\mathcal{I} \supseteq \mathcal{S}$, we therefore conclude that $\{A \in \mathbb{L} : A \leq \bigvee \mathcal{G} \text{ for some finite } \mathcal{G} \subseteq \mathcal{S}\} = \mathbf{I}_{\mathcal{S}}$.

Since every Boolean algebra is also a Boolean ring, it is natural to define the quotient of a Boolean algebra by an ideal in the same way that a quotient ring is defined.

Definition 2.1.7: Quotient of a Boolean algebra by an ideal

Let $\mathbb{L}$ be a Boolean algebra, and let $\mathcal{I}$ be an ideal in $\mathbb{L}$. The **quotient Boolean algebra** of $\mathbb{L}$ by $\mathcal{I}$ (denoted $\mathbb{L}/\mathcal{I}$) is the Boolean algebra of equivalence classes of elements of $\mathbb{L}$ under the equivalence relation $\sim$ given by

$$A \sim B \iff A - B \in \mathcal{I},$$

where the subtraction operation is Boolean ring subtraction in $\mathbb{L}$. Since every element of $\mathbb{L}$ is its own additive inverse, this is equivalent to

$$A \sim B \iff A + B \in \mathcal{I}.$$

We use the notations $[A]$ and $A + \mathcal{I}$ to denote the equivalence class of $A \in \mathbb{L}$ under this equivalence relation.

We must of course show that the quotient of a Boolean algebra by an ideal is indeed a Boolean algebra.

Theorem 2.1.8: Quotient of a Boolean algebra

Let $\mathbb{L}$ be a Boolean algebra and let $\mathcal{I}$ be an ideal in $\mathbb{L}$. Then the quotient $\mathbb{L}/\mathcal{I}$ is a Boolean algebra, and the quotient map $q : \mathbb{L} \ni A \mapsto [A] \in \mathbb{L}/\mathcal{I}$ is a homomorphism.

Proof:

If $\mathbb{L}$ is considered as a Boolean ring, then certainly $\mathbb{L}/\mathcal{I}$ is also a ring. It thus remains to show that it is a Boolean ring, and that the quotient map is a Boolean algebra homomorphism.

Let $A \in \mathbb{L}$. Then, since the quotient map is a ring homomorphism, we have $[A]^2 = [A^2] = [A]$. This shows that multiplication in $\mathbb{L}/\mathcal{I}$ is idempotent. Furthermore, $[A][1] = [1A] = [A]$ for all $A \in \mathbb{L}$, so $\mathbb{L}/\mathcal{I}$ is unital with unit $[1]$. Thus a $\mathbb{L}/\mathcal{I}$ is a Boolean ring.

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Let $A, B \in \mathbb{L}$. Then

$$[A] \vee [B] := [A] + [B] + [A][B] = [A + B + AB] = [A \vee B]. \qquad [A] \wedge [B] := [A][B] = [AB] = [A \wedge B].$$

Thus the quotient map preserves the operations $\vee$ and $\wedge$. Furthermore, for all $A \in \mathbb{L}$,

$$[0] \vee [A] = [0 \vee A] = [A],$$

so $[0]$ is an identity of the operation $\vee$.

Thus the quotient map preserves the operations of $\vee$ and $\wedge$ as well as the zero and unit. By the definition of Boolean algebra homomorphisms, this shows that the quotient map is a homomorphism.

The fact that the quotient map is a homomorphism shows that the Boolean algebra operations on the quotient Boolean algebra are as one would expect them to be: $[A] \vee [B] = [A \vee B]$, $[A] \wedge [B] = [A \wedge B]$, and $\neg[A] = [\neg A]$.

The next theorem gives a useful characterisation of the statement $[A] \leq [B]$.

Theorem 2.1.9

Let $\mathbb{L}$ be a Boolean algebra and $\mathcal{I}$ an ideal in $\mathbb{L}$. Then, for all A and B in $\mathbb{L}$, $A + \mathcal{I} \leq B + \mathcal{I}$ if and only if there exists an $P \in \mathcal{I}$ such that $A + P \leq B$.

Proof:
Suppose there exists a $P \in \mathcal{I}$ such that $A + P \leq B$. Since the quotient map is monotone, it follows that $A + \mathcal{I} = (A + P) + \mathcal{I} \leq B + \mathcal{I}$.

Now suppose that $A + \mathcal{I} \leq B + \mathcal{I}$. Then there exists an $A' \in A + \mathcal{I}$ and a $B' \in B + \mathcal{I}$ such that $A' \leq B'$. Since $A' \in A + \mathcal{I}$ and $B' \in B + \mathcal{I}$, there exist M and N in $\mathcal{I}$ such that $A' = A + M$ and $B' = B + N$. Hence $A + M \leq B + N$. Thus $(A + M)(B + N) = A + M$. (Recall that juxtaposition is used to denote the meet operation, i.e. multiplication.) Adding $A + M$ on both sides gives $(A + M)(1 + B + N) = 0$.
Let $P = A(M \vee N)$ (recall that $M \vee N = M + N + MN$). Then $P \in \mathcal{I}$ and

$$A + P = A + A(M + N + MN) = A(1 + M + N + MN) = A(1 + M)(1 + N).$$

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Now, due to the idempotence of multiplication, we have

$$\begin{aligned} A(1+M) &= A + AM \\ &= A^2 + AM \\ &= A(A+M). \end{aligned}$$

Combining these two results, we get

$$\begin{aligned} (A+P)(1+B) &= A(1+M)(1+N)(1+B) \\ &= A(A+M)(1+N)(1+B) \\ &= A(1+N)(A+M)(1+B) \\ &= A(1+N)(A+M)(1+B+N+N) \\ &= A(1+N)(A+M)(1+B+N) + (1+N)A(A+M)N \\ &= A(1+N)(0) + N(1+N)A(A+M) \\ &= (N+N)A(A+M) \\ &= 0. \end{aligned}$$

This shows that $(A+P)(1+B) = 0$, i.e. $(A+P)B = A+P$. Hence $A+P \leq B$.

Ideals are dual to filters. That is, $\mathcal{I}$ is an ideal in a poset $(P, \leq)$ if and only if it is a filter in the dual poset $(P, \geq)$. This suggests an interpretation of ideals: analogous to how a filter $\mathcal{F}$ represents "non-negligible" subsets of a set X, ideals can be understood as collections of "negligible" subsets of X. Indeed, if a set is negligible, all its subsets should also be negligible, and the union of any two negligible sets should also be negligible. In fact, some authors define negligible sets as elements of particular ideals.

2.1.3 Nowhere-dense sets and meagre sets

Definition 2.1.10: Nowhere-dense sets

Let X be a topological space and let $A \subseteq X$. We say that A is **nowhere-dense** if its closure has empty interior. That is, if

$$\left(\overline{A}\right)^{\circ} = \emptyset.$$

There are a number of useful and instructive statements which are equivalent to nowhere-density.

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Theorem 2.1.11

Let X be a topological space and let $A \subseteq X$. Then the following are equivalent:

(i) A is nowhere dense.

(ii) $\overline{A}$ has dense complement.

(iii) Every non-empty open set $U \subseteq X$ contains a non-empty open subset which is disjoint from A.

(iv) For every non-empty open set $U \subseteq X$, the intersection $A \cap U$ is not a dense subset of U (i.e. A is not dense in any open set U. This is the origin of the terminology.)

(v) A is a subset of the boundary of some open set.

Proof:

$(i) \implies (ii)$:

This follows from the fact that the closure of $X \setminus \overline{A}$ is just $X \setminus \left(\overline{A}\right)^{\circ}$.

$(ii) \implies (iii)$:

Suppose $\overline{A}$ has dense complement. Since $\overline{A}$ is closed, $X \backslash \overline{A}$ is open. Let U be a non-empty open set. Then $U \cap (X \backslash \overline{A})$ is a non-empty open set which is disjoint from A.

$(iii) \implies (iv)$:

Let U be a non-empty open set. Then there exists a non-empty open subset $U' \subseteq U$ which is disjoint from A. In particular, U' is disjoint from $A \cap U$. Thus $A \cap U$ is not a dense subset of U.

$(iv) \implies (i)$:

Suppose that (i) is false. Then $\left(\overline{A}\right)^{\circ}$ is non-empty. Now, observe that $\left(\overline{A}\right)^{\circ} \subseteq \overline{A}$, so every open set which meets $\left(\overline{A}\right)^{\circ}$ contains a point of $\overline{A}$. If an open set contains a point of $\overline{A}$, then it must contain a point of A, since every point of $\overline{A}$ is either interior to A or in the boundary of A. Hence every open set which meets $\left(\overline{A}\right)^{\circ}$ also meets A. In particular, every non-empty open subset of $\left(\overline{A}\right)^{\circ}$ meets $A \cap \left(\overline{A}\right)^{\circ}$. Thus $A \cap \left(\overline{A}\right)^{\circ}$ is dense in $\left(\overline{A}\right)^{\circ}$. Thus (iv) is false, and we conclude that (iv) implies (i).

This shows that (i) to (iv) are all equivalent. It remains to show that (i) and (v) are equivalent.

Suppose A is nowhere dense. By (ii), $X \backslash \overline{A}$ is dense in X. Let $U = X \backslash \overline{A}$. Then $\overline{U} = X$, so every element of X is either an interior point of U or a boundary point of U. Since A is disjoint from U, no element of A can be an interior point of U, so every element of A must be

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a boundary point of U. Hence A is a subset of the boundary of an open set (namely $X\backslash\overline{A}$). This shows that $(i) \implies (v)$.

Suppose there exists an open set U such that $A \subseteq \partial U$. Taking closures shows that $\overline{A} \subseteq \overline{\partial U}$. Since ∂U is closed (being the intersection of the closed sets $\overline{U}$ and $\overline{X\backslash U}$), we have $\overline{A} \subseteq \partial U$. Next we show that ∂U has empty interior.

> Suppose x is an interior point of $\partial U = \overline{U}\backslash U$. Then there exists an open neighbourhood V of x which is a subset of $\overline{U}\backslash U$. Hence $V \subseteq \overline{U}$. But by the definition of closure, this implies that $V \cap U$ is non-empty. This contradicts the fact that V is a subset of $\overline{U}\backslash U$. Hence ∂U has no interior points.

It follows that $\overline{A}$ also has no interior points, so A is nowhere dense. Hence $(v) \implies (i)$, and we conclude that (i) and (v) are equivalent.

Given a topological space X, the collection of nowhere-dense sets forms an ideal of subsets of X: any subset of a nowhere-dense set is nowhere dense, and the union of two nowhere-dense sets is nowhere dense. However, the union of countably many nowhere-dense sets is not necessarily nowhere dense—such a union is known as a **meagre set**.

Definition 2.1.12: Meagre sets

Let X be a topological space, and let $A \subseteq X$. We say that A is **meagre** (in X) if it is the union of countably many nowhere-dense sets.

Some examples of meagre sets:

- The ternary Cantor set is meagre in the standard topology on $[0, 1]$.
- $\mathbb{Q}$ is a meagre (and dense) subset of $\mathbb{R}$, because $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}$ and $\mathbb{Q}$ is countable.
- More generally, any countable subset of any Euclidean space is meagre in the standard topology.
- In $C[0, 1]$ (the space of continuous real-valued functions with domain $[0, 1]$ under the supremum norm), the set of functions which have a derivative at at least one point in $[0, 1]$ is meagre.

The set of meagre subsets of a topological space forms a σ**-ideal** (defined below) in the power set of the space.

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Definition 2.1.13: σ-ideals

Let P be a partially ordered set, and let I be a subset of P. We say that I is a **σ-ideal** (of P) if it is a countably upward-directed ideal. That is, if

- I is an ideal of P, and
- for every countable subset $\{x_1, x_2, \ldots\}$ of I, there exists a $z \in I$ such that $x_i \leq z$ for all $i \in \mathbb{N}$.

If P is a lattice, then an ideal I is a σ-ideal if and only if it is an ideal which is closed under countable joins. Note that this does not require every countable subset of P to have a join—it only requires that every countable subset of I has a join, and that each such join is also in I.

We list the following two facts here in order to refer to them later.

Lemma 2.1.14

(i) Every nowhere-dense set is meagre.

(ii) Let $\mathbb{L}$ be a Boolean algebra, and let $\mathcal{M}$ be the set of meagre subsets of its Stone space. Then $\mathcal{M}$ is a σ-ideal in the clopen algebra of the Stone space of $\mathbb{L}$.

Often we are interested in σ-ideals of subsets of a given set X. Unless specified otherwise, a "σ-ideal of subsets of X" refers to a σ-ideal in the lattice $(\mathcal{P}(X), \subseteq, \cup, \cap)$.

Next we show that the quotient of a σ-complete Boolean algebra by a σ-ideal is σ-complete.

Theorem 2.1.15: Quotient of a σ-complete Boolean algebra

Let $\mathbb{L}$ be a σ-complete Boolean algebra and let $\mathcal{I}$ be a σ-ideal in $\mathbb{L}$. Then $\mathbb{L}/\mathcal{I}$ is a σ-complete Boolean algebra, and the quotient map is a σ-homomorphism (i.e., a Boolean algebra homomorphism that preserves countable joins and meets).

Proof:
Clearly $\mathbb{L}/\mathcal{I}$ is a Boolean algebra, so it remains only to show that $\mathbb{L}/\mathcal{I}$ is σ-complete and that the quotient map preserves countable meets and countable joins.

Let $\mathcal{S}$ be a countable subset of $\mathbb{L}$. Since $\mathbb{L}$ is σ-complete, the supremum $\bigvee \mathcal{S}$ of $\mathcal{S}$ in $\mathbb{L}$ exists. Because the quotient map is monotone, $q(A) \leq q(\bigvee \mathcal{S})$ for all $A \in \mathcal{S}$. Thus $q(\bigvee \mathcal{S})$ is an upper bound of $q[\mathcal{S}]$ in $\mathbb{L}/\mathcal{I}$.

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Now suppose that $B \in \mathbb{L}$ and that $q(B)$ is an upper bound of $q[\mathcal{S}]$. Then, for all $A \in \mathcal{S}$,

$$\begin{gathered} q(A) \leq q(B) \\ \neg q(B) \leq \neg q(A) \\ q(A) \wedge \neg q(B) \leq \neg q(A) \wedge q(A) = 0, \end{gathered}$$

so $q(A \wedge \neg B) = q(0)$, i.e., $A \wedge \neg B \in \mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal, it is closed under countable joins: $\bigvee_{A \in S} (A \wedge \neg B) \in \mathcal{I}$. Thus (using generalized distributivity, Theorem 1.1.13),

$$\begin{aligned} q\left(\bigvee_{A \in \mathcal{S}} A\right) \wedge \neg q(B) &= q\left(\neg B \wedge \bigvee_{A \in \mathcal{S}} A\right) \\ &= q\left(\bigvee_{A \in \mathcal{S}} (\neg B \wedge A)\right) \\ &= 0. \end{aligned}$$

It follows that

$$q\left(\bigvee_{A \in \mathcal{S}} A\right) \leq q(B).$$

Hence $q\left(\bigvee_{A \in \mathcal{S}} A\right)$ is the least upper bound of $q[S]$ in $\mathbb{L}/\mathcal{I}$. That is,

$$\bigvee q[\mathcal{S}] = q\left(\bigvee \mathcal{S}\right).$$

This shows that the quotient map preserves countable joins.

By the monotonicity of q, $q(\bigwedge \mathcal{S})$ is a lower bound of $q[\mathcal{S}]$. Now suppose that $q(B)$ is a lower bound of $q[\mathcal{S}]$. Then, for all $A \in \mathcal{S}$, $q(B) \leq q(A)$. Taking complements gives $\neg q(A) = q(\neg A) \leq \neg q(B)$. Since q preserves countable joins, we have

$$\bigvee_{A \in \mathcal{S}} q(\neg A) = q\left(\bigvee_{A \in \mathcal{S}} \neg A\right) \leq \neg q(B).$$

Taking complements again (and using De Morgan's law) gives

$$\begin{aligned} q(B) &\leq \neg q\left(\bigvee_{A \in \mathcal{S}} \neg A\right) \\ &= \neg q\left(\neg \bigwedge_{A \in \mathcal{S}} A\right) \\ &= q\left(\bigwedge \mathcal{S}\right). \end{aligned}$$

This shows that $q\left(\bigwedge \mathcal{S}\right)$ is the greatest lower bound of $q[\mathcal{S}]$ in $\mathbb{L}/\mathcal{I}$. That is,

$$\bigwedge q[\mathcal{S}] = q\left(\bigwedge \mathcal{S}\right).$$

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Thus, every countable subset of $\mathbb{L}/\mathcal{I}$ has a supremum and an infimum in $\mathbb{L}/\mathcal{I}$, so $\mathbb{L}/\mathcal{I}$ is a σ-complete Boolean algebra. This also shows that the quotient map preserves countable suprema and infima.

2.2 The Loomis-Sikorski Theorem

We now have the necessary background for the Loomis-Sikorski Theorem. For its proof, we need a final lemma.

Lemma 2.2.1

Let K be a Stone space whose clopen algebra is σ-complete. Let $\mathcal{M}$ be the set of meagre subsets of K, and let Σ be the collection of all subsets of K which differ from a clopen set by a meagre set. That is,

$$\Sigma = \{S \subseteq K : S \triangle N \in Cl_K \text{ for some } N \in \mathcal{M}\}.$$

Then:

(i) $\Sigma = \{S \subseteq K : S \triangle C \in \mathcal{M} \text{ for some } C \in Cl_K\} = \{C \triangle N : C \in Cl_K, N \in \mathcal{M}\}$,

(ii) Σ is a σ-algebra, and

(iii) $\mathcal{M}$ is a σ-ideal in Σ.

Proof:

(i): Let $S \in \Sigma$. Then $C := S \triangle N \in Cl_K$ for some $N \in \mathcal{M}$, and $S \triangle C = S \triangle S \triangle N = N \in \mathcal{M}$. This shows that

$$\Sigma \subseteq \{S \subseteq K : S \triangle C \in \mathcal{M} \text{ for some } C \in Cl_K\}.$$

Next, let $S \subseteq K$ such that $N := S \triangle C \in \mathcal{M}$ for some $C \in Cl_K$. Taking the symmetric difference with $S \triangle N$ on both sides gives

$$(S \triangle N) \triangle N = (S \triangle N) \triangle (S \triangle C)$$
$$S = N \triangle C.$$

This shows that

$$\{S \subseteq K : S \triangle C \in \mathcal{M} \text{ for some } C \in Cl_K\} \subseteq \{C \triangle N : C \in Cl_K, N \in \mathcal{M}\}.$$

Finally, let $S \in \{C \triangle N : C \in Cl_K, N \in \mathcal{M}\}$. Then there exist $C \in Cl_K$ and $N \in \mathcal{M}$ such that $S = C \triangle N$, which implies $S \triangle N = C \triangle N \triangle N = C \in Cl_K$. This shows that

$$\{C \triangle N : C \in Cl_K, N \in \mathcal{M}\} \subseteq \Sigma,$$

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hence showing that all three sets are equal.

(ii): The empty set is in Σ, because it is the symmetric difference of a clopen set (namely, itself) and a meagre set (itself).

Let $S \in \Sigma$. There exists a clopen set C such that $S \triangle C \in \mathcal{M}$. Hence $(K \backslash S) \triangle (K \backslash C) = S \triangle C \in \mathcal{M}$, and $K \backslash C$ is clopen, so $K \backslash S$ differs from a clopen set by a meagre set. Hence $K \backslash S \in \Sigma$. This shows that Σ is closed under complementation.

Let $\{S_n : n \in \mathbb{N}\}$ be a countable subset of Σ. There exists a countable collection of clopen sets $\{C_n : n \in \mathbb{N}\}$ such that $S_n \triangle C_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. Since the union of countably many meagre sets is meagre, $\bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$ is also in $\mathcal{M}$. Let $S = \bigcup_{n \in \mathbb{N}} S_n$ and $C = \bigcup_{n \in \mathbb{N}} C_n$. We now show that $S \triangle C \subseteq \bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$.

> Let $x \in S \triangle C$. Then x is in exactly one of S and C.
>
> Suppose that $x \in S$. Then there exists a $k \in \mathbb{N}$ such that $x \in S_k$. Furthermore, $x \notin \bigcup_{n \in \mathbb{N}} C_n$, so $x \notin C_n$ for all $n \in \mathbb{N}$. Hence $x \in S_k \triangle C_k$, so $x \in \bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$.
>
> Now suppose that $x \in C$. Apply the same argument as in the previous paragraph with the roles of C_n and S_n exchanged to show that $x \in \bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$.
>
> This shows that $S \triangle C \subseteq \bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$.

Since $\bigcup_{n \in \mathbb{N}} (S_n \triangle C_n)$ is meagre and any subset of a meagre set is meagre, $S \triangle C \in \mathcal{M}$. By the σ-completeness of Cl_K, the closure of C is open (by Theorem 1.1.42) and is thus clopen.

Observe that, if two sets A and B are disjoint, then $A \triangle B = A \cup B$. In particular, since C is an open set, it is disjoint from its boundary, so $\overline{C} = C \triangle \partial C$. By Theorem 2.1.11, $\partial C \in \mathcal{M}$. Thus $S \triangle \overline{C} = S \triangle C \triangle \partial C = (S \triangle C) \triangle \partial C$ is the symmetric difference of two meagre sets. Hence $S \triangle \overline{C} \in \mathcal{M}$, and we conclude that $S \in \Sigma$. This shows that Σ is closed under countable unions.

Σ contains the empty set, is closed under complementation, and is closed under countable unions. Therefore Σ is a σ-algebra.

(iii): Each meagre set differs from a clopen set (the empty set) by a meagre set (itself), so $\mathcal{M} \subseteq \Sigma$. The empty set is meagre, the symmetric difference of two meagre sets is meagre, and the intersection of a meagre set with any set is meagre (so $\mathcal{M}$ is an ideal in Σ). The union of countably many meagre sets is also meagre, therefore $\mathcal{M}$ is a σ-ideal in Σ.

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Suppose that K is a Stone space whose clopen algebra is σ-complete. Let Σ and $\mathcal{M}$ be as in Lemma 2.2.1. Then, as mentioned at the beginning of this section, Cl_K is not necessarily a sub-σ-algebra of Σ. This is because, given a countable collection $\mathcal{S} \subseteq Cl_K$ of clopen sets, the supremum of $\mathcal{S}$ in Σ is $\bigcup \mathcal{S}$, while the supremum in Cl_K is $\overline{\bigcup \mathcal{S}}$. Similarly, the infimum in Σ is $\bigcap \mathcal{S}$ while the infimum in Cl_K is $(\bigcap \mathcal{S})^\circ$.

We now proceed with the proof of the Loomis-Sikorski Theorem. It is essentially the statement that Cl_K is isomorphic to $\Sigma/\mathcal{M}$, and this immediately implies that $\mathbb{L}$ is isomorphic to $\Sigma/\mathcal{M}$ due to the Stone Representation Theorem.

Theorem 2.2.2

Let K be a Stone space whose clopen algebra is σ-complete, and let Σ and $\mathcal{M}$ be as in Lemma 2.2.1. Then Cl_K is isomorphic to the quotient $\Sigma/\mathcal{M}$.

Proof:

Since each clopen set differs from a clopen set (itself) by a meagre set (the empty set), every clopen subset of K is in Σ. Hence Cl_K is a sub-algebra of Σ.

Let $q : \Sigma \to \Sigma/\mathcal{M}$ be the quotient map. By Theorem 2.1.15, q is a Boolean algebra homomorphism, so the restriction of q to Cl_K is also a Boolean algebra homomorphism into $\Sigma/\mathcal{M}$. We show that q is a bijection from Cl_K to $\Sigma/\mathcal{M}$. (Strictly speaking, we show that *the restriction* $q|_{Cl_K}$ *of* q *to* Cl_K is a bijection, but this abuse of notation should not lead to any confusion.)

> We use the symbol $+$ to denote the Boolean ring operation of symmetric difference in Σ. Recall that every element of a Boolean ring is its own additive inverse; therefore $A + B = A - B$ for all $A, B \in \Sigma$.
>
> Let $C, D \in Cl_K$ and suppose $q(C) = q(D)$, i.e. $C + \mathcal{M} = D + \mathcal{M}$. Then $C + D = C - D \in \mathcal{M}$, but also $C + D \in Cl_K$, so $C + D$ is a meagre open set. By the Baire Category Theorem for locally compact Hausdorff spaces, every non-empty open subset of K is non-meagre. Therefore $C + D$ must be empty, so $C = D$. This shows that q is injective.
>
> Let $S + \mathcal{M}$ be an equivalence class in $\Sigma/\mathcal{M}$. By Lemma 2.2.1 (i), there exist $C \in Cl_K$ and $N \in \mathcal{M}$ such that $S = C + N$. Hence $q(C) = q(S) = S + \mathcal{M}$. This shows that q is surjective.

Since q is a bijective homomorphism, it is thus a Boolean algebra isomorphism. Thus Cl_K is isomorphic to $\Sigma/\mathcal{M}$.

Thanks to the Stone Representation Theorem, we finally conclude that every σ-complete Boolean algebra is isomorphic to the quotient of a σ-algebra by the σ-ideal of meagre subsets of its Stone space.

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Corollary 2.2.3: The Loomis-Sikorski Theorem

Suppose that

- $\mathbb{L}$ is a σ-complete Boolean algebra,
- K is its Stone space,
- Σ is the σ-algebra of subsets of K which differ from a clopen set by a meagre set, and
- $\mathcal{M}$ is the σ-ideal in Σ of meagre subsets of K.

Then $\mathbb{L}$ is isomorphic to $\Sigma/\mathcal{M}$.

Proof:
By the Stone Representation Theorem, $\mathbb{L}$ is isomorphic to Cl_K. Hence K is a Stone space whose clopen algebra is σ-complete. It follows from Theorem 2.2.2 that $\mathbb{L}$ is isomorphic to $\Sigma/\mathcal{M}$.

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3 Measures

We now turn to measures on Boolean algebras. We first discuss the countable chain condition, which is a useful condition implied by the existence of a strictly positive measure. It also implies completeness for σ-complete Boolean algebras. We then exhibit a bijection between measures on a Boolean algebra and positive linear functionals on the space of continuous real-valued functions on its Stone space. Next, we define intersection numbers and covering numbers, and we show that intersection numbers and covering numbers provide a characterisation of the Boolean algebras which admit strictly positive measures. We conclude with a section on weak σ-distributivity, which provides a characterisation of the σ-complete Boolean algebras which admit strictly positive σ-additive measures.

3.1 Preliminaries

A measure can be thought of as an assignment of "size" to a set. It generalises all similar notions of size: length, area, volume, mass, charge, and probability. A measure is usually defined as a non-negative function from a σ-algebra of sets to the non-negative extended reals which is countably additive and maps the empty set to zero (for instance, in [Fre00] or in [GR16]). However, we will require them to be merely finitely additive, in addition to attaining a maximum value of 1.

Recall that two elements A and B of a Boolean algebra $\mathbb{L}$ are **disjoint** if $A \wedge B = 0_{\mathbb{L}}$. We say that a set $\mathcal{S} \subseteq \mathbb{L}$ is **disjoint** if all elements of $\mathcal{S}$ are mutually disjoint.

Definition 3.1.1: Measures on Boolean algebras

Let $\mathbb{L}$ be a Boolean algebra, and let $\mu : \mathbb{L} \to [0, \infty)$ be a function with codomain the non-negative real numbers. We say that μ is a **measure (on $\mathbb{L}$)** if:

(*i*) $\mu(0_{\mathbb{L}}) = 0$ (where $0_{\mathbb{L}}$ is the least element of $\mathbb{L}$),

(*ii*) $\mu(1_{\mathbb{L}}) = 1$ (where $1_{\mathbb{L}}$ is the greatest element of $\mathbb{L}$), and

(*iii*) if A and B are disjoint elements of $\mathbb{L}$, then

$$\mu(A \vee B) = \mu(A) + \mu(B).$$

Axiom (*iii*) in Definition 3.1.1 implies that $\mu(A_1 \vee \cdots \vee A_n) = \mu(A_1) + \cdots + \mu(A_n)$ whenever $A_1, \ldots, A_n$ are disjoint elements of $\mathbb{L}$. This condition is therefore often stated by saying that μ is **finitely additive**. Some authors define measures to be **countably additive** (defined below), but we do not adopt this convention.

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It is easy to see that every measure is order-preserving; if $A \leq B$ then

$$\begin{aligned}\mu(B) &= \mu(A \vee (B \wedge \neg A)) \\ &= \mu(A) + \mu(B \wedge \neg A) \\ &\geq \mu(A).\end{aligned}$$

The familiar concept of "measure" from measure theory is a special case of measures as defined here. Recall that a σ-algebra of subsets of a set is a Boolean algebra, with meet and join given by intersection and union respectively. This means that every measure in the usual measure-theoretical sense is a measure as defined here.

Definition 3.1.2: Countably additive measures

Let μ be a measure on a σ-complete Boolean algebra $\mathbb{L}$. If, for every countable set $\{A_i : i \in \mathbb{N}\}$ of disjoint elements of $\mathbb{L}$,

$$\mu\left(\bigvee_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i),$$

then we say that μ is a **countably additive measure** or a **σ-additive measure**.

Note that the sum in the above definition is always guaranteed to converge. This is because the sequence of partial sums $\left(\sum_{i=1}^{n} \mu(A_i)\right)_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-negative real numbers, and each partial sum is at most $\mu(1_{\mathbb{L}}) = 1$. Hence the sequence of partial sums has a limit, so the sum converges.

3.2 Strictly Positive Measures and the Countable Chain Condition

A measure is said to be strictly positive if it assigns positive measure to every non-zero element of the Boolean algebra. In this brief section, we will show that the existence of a strictly positive measure implies a certain countability condition called the countable chain condition. We will also show that the countable chain condition implies completeness for σ-complete Boolean algebras.

Definition 3.2.1: Strictly positive measures

Let μ be a measure on a Boolean algebra $\mathbb{L}$. We say that μ is **strictly positive** if the only element of $\mathbb{L}$ mapped to zero by μ is $0_{\mathbb{L}}$.

Definition 3.2.2: The countable chain condition

A Boolean algebra $\mathbb{L}$ is said to satisfy **the countable chain condition** if every disjoint subset of $\mathbb{L}$ is countable (i.e. finite or countably infinite).

We will often abbreviate the countable chain condition as "CCC". We start with a few results involving the CCC.

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Theorem 3.2.3

Let $\mathbb{L}$ be a Boolean algebra. Then the following are equivalent:

(i) $\mathbb{L}$ satisfies the CCC.

(ii) For every subset $\mathcal{S}$ of $\mathbb{L}$, there exists a countable set $\mathcal{T} \subseteq \mathcal{S}$ so that $\mathcal{S}$ and $\mathcal{T}$ have the same upper bounds.

Proof:
$(ii) \Longrightarrow (i)$:

Suppose for every subset $\mathcal{S}$ of $\mathbb{L}$, there exists a countable set $\mathcal{T} \subseteq \mathcal{S}$ so that $\mathcal{S}$ and $\mathcal{T}$ have the same upper bounds.

Let $\mathcal{S}$ be a disjoint subset of $\mathbb{L}$. By assumption, there exists a countable subset $\mathcal{T} \subseteq \mathcal{S}$ with the same upper bounds as $\mathcal{S}$. Now, suppose there exists an $A \in \mathcal{S} \setminus \mathcal{T}$. Since $A \in \mathcal{S}$, A is disjoint from every element of $\mathcal{T}$.

Recall Corollary 1.1.9: the elements of $\mathbb{L}$ disjoint from A are precisely the lower bounds of $\neg A$. Hence $T \leq \neg A$ for all $T \in \mathcal{T}$, so $\neg A$ is an upper bound of $\mathcal{T}$. But $\mathcal{T}$ has the same upper bounds as $\mathcal{S}$, so $\neg A$ is an upper bound of $\mathcal{S}$. In particular, $A \leq \neg A$. This implies that $A = 0_{\mathbb{L}}$ (because the meet operation is order-preserving, so $A \wedge A \leq \neg A \wedge A = 0_{\mathbb{L}}$). Hence $\mathcal{S} \setminus \mathcal{T} \subseteq \{0_{\mathbb{L}}\}$. Since $\mathcal{T} \subseteq \mathcal{S}$, we have $\mathcal{T} \subseteq \mathcal{S} \subseteq \mathcal{T} \cup \{0_{\mathbb{L}}\}$. Since $\mathcal{T}$ is countable, it follows that $\mathcal{S}$ is countable.

$(i) \Longrightarrow (ii)$:

Suppose $\mathbb{L}$ satisfies the CCC and $\mathcal{S} \subseteq \mathbb{L}$. Let $\mathbf{I}_{\mathcal{S}}$ be the ideal in $\mathbb{L}$ generated by $\mathcal{S}$. Then, by Theorem 2.1.6,

$$\mathbf{I}_{\mathcal{S}} = \left\{ A \in \mathbb{L} : A \leq \bigvee \mathcal{G} \text{ for some finite } \mathcal{G} \subseteq \mathcal{S} \right\}.$$

Since $\mathcal{S} \subseteq \mathbf{I}_{\mathcal{S}}$, every upper bound of $\mathbf{I}_{\mathcal{S}}$ is an upper bound of $\mathcal{S}$. Conversely, every upper bound of $\mathcal{S}$ is an upper bound of $\mathbf{I}_{\mathcal{S}}$. This is easily verified:

> Let U be an upper bound of $\mathcal{S}$ and let $\mathcal{G}$ be a finite subset of $\mathcal{S}$. Then $F \leq U$ for all $F \in \mathcal{G}$ because $\mathcal{G} \subseteq \mathcal{S}$. Hence $\bigvee \mathcal{G} \leq U$. Since this holds for all finite $\mathcal{G} \subseteq \mathcal{S}$, it follows that U is an upper bound of $\mathbf{I}_{\mathcal{S}}$.

Thus $\mathcal{S}$ and $\mathbf{I}_{\mathcal{S}}$ have the same upper bounds.

Now, let $\mathcal{M}$ be a maximal disjoint subset of $\mathbf{I}_{\mathcal{S}}$. The existence of $\mathcal{M}$ follows from Zorn's Lemma:

> Let $\mathbf{M}$ be the collection of all disjoint subsets of $\mathbf{I}_{\mathcal{S}}$, partially ordered by inclusion. If $\mathbf{N}$ is a totally ordered subset of $\mathbf{M}$, then $\bigcup \mathbf{N}$ is an upper bound of $\mathbf{N}$, and its

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elements are certainly all disjoint: if $A, B \in \bigcup \mathbf{N}$, then there exist $\mathbf{N}_A, \mathbf{N}_B \in \mathbf{N}$ such that $A \in \mathbf{N}_A$ and $B \in \mathbf{N}_B$. Since $\mathbf{N}$ is totally ordered, we can assume without loss of generality that $\mathbf{N}_A \subseteq \mathbf{N}_B$, so $A, B \in \mathbf{N}_B$. Since $\mathbf{N}_B \in \mathbf{M}$, A and B are disjoint. Hence $\bigcup \mathbf{N} \in \mathbf{M}$, so $\mathbf{N}$ has an upper bound in $\mathbf{M}$. By Zorn's Lemma, there exists a maximal element $\mathcal{M}$ of $\mathbf{M}$.

We now show that $\mathbf{I}_\mathcal{S}$ and $\mathcal{M}$ have the same upper bounds. Let U be an upper bound of $\mathbf{I}_\mathcal{S}$. Since $\mathcal{M} \subseteq \mathbf{I}_\mathcal{S}$, U is an upper bound of $\mathcal{M}$ as well. Conversely, let U be an upper bound of $\mathcal{M}$, and let $A \in \mathbf{I}_\mathcal{S}$. Let $B := A \wedge (\neg U) \leq A$. Since $\mathbf{I}_\mathcal{S}$ is an ideal in $\mathbb{L}$, $B \in \mathbf{I}_\mathcal{S}$. Since $B \leq \neg U$, B is disjoint from U. Thus B is disjoint from every $C \leq U$; in particular, B is disjoint from each element of $\mathcal{M}$. But since $\mathcal{M}$ is a *maximal* disjoint subset of $\mathbf{I}_\mathcal{S}$ and $B \in \mathbf{I}_\mathcal{S}$, it follows that $B \in \mathcal{M}$, so $B \leq U$. But $B \leq \neg U$ as well, so $B \leq (U \wedge \neg U) = 0$. Thus $B = A \wedge (\neg U) = 0$, so A is disjoint from $\neg U$. Thus $A \leq U$. Since this applies to all $A \in \mathbf{I}_\mathcal{S}$, it follows that U is an upper bound of $\mathbf{I}_\mathcal{S}$. Thus $\mathbf{I}_\mathcal{S}$ and $\mathcal{M}$ have the same upper bounds.

Since we assumed that $\mathbb{L}$ satisfies the CCC, $\mathcal{M}$ is countable. Thus $\mathcal{M}$ is a countable set with the same upper bounds as $\mathcal{S}$ (because $\mathcal{S}$ and $\mathbf{I}_\mathcal{S}$ have the same upper bounds). We now finally show that there exists a countable subset of $\mathcal{S}$ with the same upper bounds as $\mathcal{S}$.

For each $A \in \mathcal{M}$, there exists a finite set $\mathcal{G}_A \subseteq \mathcal{S}$ such that $A \leq \bigvee \mathcal{G}_A$. Let $\mathcal{S}' = \bigcup_{A \in \mathcal{M}} \mathcal{G}_A$. Then $\mathcal{S}'$ is a countable subset of $\mathcal{S}$, and it has the same upper bounds as $\mathcal{S}$, as we now show.

Every upper bound of $\mathcal{S}$ is an upper bound of $\mathcal{S}'$ because $\mathcal{S}' \subseteq \mathcal{S}$. Conversely, let U be an upper bound of $\mathcal{S}'$. Then $B \leq U$ for all $B \in \mathcal{S}'$, and thus $\bigvee \mathcal{G} \leq U$ for all finite sets $\mathcal{G} \subseteq \mathcal{S}'$. But since every element of $\mathcal{M}$ is less than $\bigvee \mathcal{G}$ for some finite set $\mathcal{G} \subseteq \mathcal{S}'$, it follows that U is an upper bound of $\mathcal{M}$. Since $\mathcal{M}$ and $\mathcal{S}$ have the same upper bounds, it follows that U is an upper bound of $\mathcal{S}$. Hence $\mathcal{S}$ and $\mathcal{S}'$ have the same upper bounds.

Thus $\mathcal{S}'$ is a countable subset of $\mathcal{S}$ with the same upper bounds as $\mathcal{S}$.

Theorem 3.2.4

Let $\mathbb{L}$ be a Boolean algebra. If $\mathbb{L}$ is σ-complete and satisfies the CCC, then $\mathbb{L}$ is complete.

Proof:
Suppose $\mathbb{L}$ is σ-complete and satisfies the CCC, and let $\mathcal{S}$ be a subset of $\mathbb{L}$. By Theorem 3.2.3, there exists a countable set $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{S}$ and $\mathcal{T}$ have the same upper bounds. Since $\mathbb{L}$ is σ-complete, the set of upper bounds of $\mathcal{T}$ has a least element $\bigvee \mathcal{T}$. Since $\mathcal{S}$ and $\mathcal{T}$ have the same upper bounds, it follows that $\mathcal{S}$ has a least upper bound as well. Since $\mathcal{S}$ was an arbitrary subset of $\mathbb{L}$, this shows that every subset of $\mathbb{L}$ has a least upper bound.

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With $\mathcal{S}$ still an arbitrary subset of $\mathbb{L}$, $\neg\mathcal{S} := \{\neg A : A \in \mathcal{S}\}$ has a supremum in $\mathbb{L}$ (by the previous paragraph). By De Morgan's Laws (Theorem 1.1.12 (i)), $\neg\neg\mathcal{S} = \mathcal{S}$ has an infimum in $\mathbb{L}$.

Thus $\mathbb{L}$ is complete.

Theorem 3.2.5

Let $\mathbb{L}$ be a Boolean algebra. If there exists a strictly positive measure on $\mathbb{L}$, then $\mathbb{L}$ satisfies the CCC.

Proof:
Suppose there exists a measure μ on $\mathbb{L}$ which is strictly positive. Let $\mathcal{S}$ be a disjoint subset of $\mathbb{L}$. To show that $\mathbb{L}$ satisfies the CCC, we wish to show that $\mathcal{S}$ is countable.

For each $n \in \mathbb{N}$, let $\mathcal{S}_n := \left\{A \in \mathcal{S} : \mu(A) > \frac{1}{n}\right\}$. We show that each $\mathcal{S}_n$ is be finite.

Let $n \in \mathbb{N}$ and suppose $\mathcal{S}_n$ is infinite. Then $\mathcal{S}_n$ has a finite subset with at least n elements. Let $\mathcal{F} = \{B_1, \ldots, B_m\}$ be any such subset of $\mathcal{S}_n$ (i.e. $m \geq n$). Then

$$\begin{aligned} \mu\left(\bigvee \mathcal{F}\right) &= \sum_{k=1}^{m} \mu(B_k) \\ &> \sum_{k=1}^{m} \frac{1}{n} \\ &= m \cdot \frac{1}{n} \\ &\geq 1. \end{aligned}$$

Hence $\mu\left(\bigvee \mathcal{F}\right) > 1 = \mu\left(1_{\mathbb{L}}\right)$. But $\bigvee \mathcal{F} \leq 1_{\mathbb{L}}$, so this contradicts the monotonicity of μ. Thus $\mathcal{S}_n$ must be finite.

Now, since μ is strictly positive, every non-zero element of $\mathcal{S}$ has positive measure. Thus

$$\mathcal{S} \subseteq \{0_{\mathbb{L}}\} \cup \bigcup_{n \in \mathbb{N}} \mathcal{S}_n,$$

which is a union of countably many finite sets. Thus $\mathcal{S}$ is countable, and we conclude that $\mathbb{L}$ satisfies the CCC.

Corollary 3.2.6

If $\mathbb{L}$ is a σ-complete Boolean algebra which admits a strictly positive measure, then $\mathbb{L}$ is complete.

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3.3 Measures on $\mathbb{L}$ and Positive Linear Functionals on $C(K)$

Recall that the Stone Representation Theorem allows us to view an arbitrary Boolean algebra as the clopen algebra of its Stone space. Unless otherwise stated, we view every Boolean algebra as the clopen algebra of its Stone space, and we view *finitary* suprema and infima as unions and intersections, respectively. (In view of the Loomis-Sikorski Theorem, this cannot be done in general for infinite suprema and infima.)

Let $\mathbb{L}$ be a Boolean algebra with Stone space K and let Cl_K be the clopen algebra of K. Consider $C(K)$, the set of continuous real-valued functions on K. It is a vector space over $\mathbb{R}$ under pointwise addition, and it is partially ordered under the pointwise ordering: $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in K$. It is in fact a *vector lattice*, i.e. a partially ordered vector space in which every pair of elements has a supremum. It is also a Banach space under the supremum norm $|| \cdot || : f \mapsto \sup_{x\in K} |f(x)|$.

The function with domain K which is zero everywhere is denoted by $\mathbf{0}$. The function with domain K which is one everywhere is denoted by $\mathbf{1}$. For each $A \subseteq K$, the indicator function of A (which is 1 on A and 0 elsewhere) is denoted by $\mathbb{1}_A$. A functional $f : C(K) \to \mathbb{R}$ is said to be **positive** if $f(g) \geq 0$ for all $g \geq \mathbf{0}$. It is a standard result that a linear functional on a partially ordered vector space is positive if and only if it is monotone.

The purpose of this section is to establish a bijection between measures on $\mathbb{L}$ and positive linear functionals on $C(K)$ with unit norm. This result will then be used in the following section to establish a characterisation of the Boolean algebras which admit a strictly positive measure.

In the rest of this section, $I(K)$ denotes the set of indicator functions of clopen subsets of K.

Remark 3.3.1

The supremum of a given f and g in $C(K)$ is the pointwise maximum:

$$f \vee g : x \mapsto \max\{f(x), g(x)\}.$$

The fact that this is indeed an element of $C(K)$ is straightforward but still worth proving.

> Let $a < b \in \mathbb{R}$ and $x \in K$ such that $(f \vee g)(x) \in (a, b)$. We show that there exists a neighbourhood U of x such that $(f \vee g)[U] \subseteq (a, b)$.
>
> At least one of $f(x)$ and $g(x)$ is equal to $(f \vee g)(x)$; suppose without loss of generality that $f(x) \geq g(x)$, so $(f \vee g)(x) = f(x)$. By the continuity of f, there exists an open neighbourhood U_f of x such that $f[U_f] \subseteq (a, b)$.
>
> Since $g(x) \leq f(x) < b$ and g is continuous, there is an open neighbourhood U_g of x such that $g[U_g] \subseteq (-\infty, b)$. Let $U = U_f \cap U_g$. Then U is an open neighbourhood of x such that $(f \vee g)[U] \subseteq (a, b)$.
>
> Thus, for every interval $(a, b) \subseteq \mathbb{R}$ and every $x \in (f \vee g)^{-1}[(a, b)]$, there exists an

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open neighbourhood U of x such that $(f \vee g)[U] \subseteq (a, b)$. It follows that $f \vee g$ is continuous.

We begin with an embedding of $\mathbb{L}$ into $C(K)$. First, observe that $I(K) \subseteq C(K)$, because for each $A \in Cl_K$, the inverse image of any subset of $\mathbb{R}$ under $\mathbb{1}_A$ is either $\emptyset$, A, $K \setminus A$, or K, all of which are open in K. This shows that each element of $I(K)$ is continuous, and thus $I(K) \subseteq C(K)$.

Theorem 3.3.2

Let $\mathbb{L}$ be a Boolean algebra with Stone space K. Then $I(K)$ is a Boolean algebra and is isomorphic to Cl_K.

Proof:
To every $A \in Cl_K$, assign the indicator function $\mathbb{1}_A$. It is straightforward to see that this gives a bijection between Cl_K and $I(K)$.

The Boolean algebra structure of $I(K)$ is also straightforward. Given indicator functions $\mathbb{1}_A, \mathbb{1}_B \in I(K)$, their join is their pointwise maximum, which is equal to $\mathbb{1}_{A\cup B}$; and their meet is their pointwise minimum, which is equal to $\mathbb{1}_{A\cap B}$. The greatest element is $\mathbb{1}_K$, which is equal to the constant function $\mathbf{1} : K \to \mathbb{R}$, and the least element is $\mathbb{1}_\emptyset$, which is equal to the constant function $\mathbf{0} : K \to \mathbb{R}$. For each $A \in Cl_K$, the complement of $\mathbb{1}_A$ is $\mathbf{1} - \mathbb{1}_A = \mathbb{1}_{K\setminus A}$.

This shows that the assignment $A \mapsto \mathbb{1}_A$ is a Boolean algebra isomorphism from Cl_K to $I(K)$.

It also follows that $\mathbb{L} \cong I(K)$. Now, consider the linear span of $I(K)$ in $C(K)$. This is the set

$$S(K) := \{\alpha_1 \mathbb{1}_{A_1} + \cdots + \alpha_n \mathbb{1}_{A_n} \ : \ n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ A_1, \ldots, A_n \in Cl_K\}$$

of all linear combinations of elements of $I(K)$. It follows that $S(K)$ is a linear subspace (and sublattice) of $C(K)$.

$C(K)_+$ and $S(K)_+$ denote, respectively, the subsets of $C(K)$ and $S(K)$ of functions which are everywhere non-negative.

Next, we will show that a measure μ on $I(K)$ extends uniquely to a positive linear functional on $S(K)$. In order to do so, we make use of the following definition and lemma.

Definition 3.3.3: Finite clopen disjoint decomposition (FCDD)

Let X be a topological space. A **finite clopen disjoint decomposition (or FCDD) of** X is a finite collection of disjoint clopen sets with union X. In other words, an **FCDD** of X is a cover of X consisting of finitely many disjoint clopen sets.

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Lemma 3.3.4

Let μ be a measure on $\mathbb{L} \cong Cl_K \cong I(K)$. If A is clopen and $\{B_1, \ldots, B_l\}$ is an FCDD of K, then

$$\mu(A) = \sum_{j=1}^{l} \mu(A \cap B_j).$$

Proof:

Since $\{B_1, \ldots, B_l\}$ is an FCDD (in particular, a finite cover) of K, we know that

$$A = \bigcup_{j=1}^{l} (A \cap B_j).$$

Since the B_j's are disjoint, this is a disjoint union. By the finite additivity of μ, we get

$$\mu(A) = \sum_{j=1}^{l} \mu(A \cap B_j).$$

Theorem 3.3.5

Let μ be a measure on $\mathbb{L} \cong Cl_K \cong I(K)$. Then there exists a unique positive linear functional $\phi_\mu : S(K) \to \mathbb{R}$ on $S(K)$ which agrees with μ on $I(K)$.

In order to prove this theorem, the perhaps obvious approach is the following: for each $s \in S(K)$, there exist scalars $\alpha_1, \ldots, \alpha_n$ and clopen sets $A_1, \ldots, A_n$ such that $s = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$. We then set $\phi_\mu(s) := \sum_{i=1}^{n} \alpha_i \mu(A_i)$. However, the scalars $\alpha_1, \ldots, \alpha_n$ and clopen sets $A_1, \ldots, A_n$ do not necessarily form a *unique* representation of s. We also need to show that $\phi_\mu(s)$ is well-defined, i.e. that the value of $\sum_{i=1}^{n} \alpha_i \mu(A_i)$ does not depend on the choice of scalars $\alpha_1, \ldots, \alpha_n$ and sets $A_1, \ldots, A_n$. It turns out that this approach makes it difficult to show that ϕ_μ is linear. However, using representations derived from FCDDs makes this easier.

The proof will work as follows: in part (i) of the proof, we will show that, given an $s \in S(K)$, there exists a representation of s using indicator functions of sets from an FCDD of K. Then, in part (ii), we will show that every representation of s using indicator functions of sets from an FCDD gives the same value for $\phi_\mu(s)$. We will then be able to unambiguously define the value of $\phi_\mu(s)$ in part (iii), and then we will then show that ϕ_μ is linear in part (iv).

Note that, once linearity is shown, we will no longer need to rely on a representation of s via FCDDs to evaluate $\phi_\mu(s)$. Any representation of s as a linear combination of indicator functions will suffice.

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Theorem 3.3.5

Proof:

(i)

Let $s \in S(K)$. Then there exist non-zero scalars $\alpha_1, \ldots, \alpha_n$ and clopen sets $A_1, \ldots, A_n$ such that

$$s = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}.$$

Observe that, for all $x \in K$, $s(x)$ is the sum of some sub-collection of $\alpha_1, \ldots, \alpha_n$. In particular, the range of s is finite. If the range of s is $\{\beta_1, \ldots, \beta_l\}$, then letting $B_j = s^{-1}[\{\beta_j\}]$ for each $j \in \{1, \ldots, l\}$ gives

$$s = \sum_{j=1}^{l} \beta_j \mathbb{1}_{B_j}$$

with $\beta_1, \ldots, \beta_l$ distinct. We now show that $B_1, \ldots, B_l$ form an FCDD of K.

> For all $x \in K$, $x \in s^{-1}[\{s(x)\}]$, so x is in at least one of $B_1, \ldots, B_l$.
>
> If $x \in B_j \cap B_k$ for some $j, k \in \{1, \ldots, l\}$, then $x \in B_j$, so $s(x) = \beta_j$. But also $x \in B_k$, so $\beta_k = s(x) = \beta_j$. Hence $B_j = B_k$, so x is in at most one of $B_1, \ldots, B_l$.
>
> Thus each $x \in K$ is in exactly one of $B_1, \ldots, B_l$, so $\{B_1, \ldots, B_l\}$ is a partition of K.
>
> Each B_j is clopen. Indeed, if $j \in \{1, \ldots, l\}$, then B_j is the inverse image of a singleton (i.e. a closed set in $\mathbb{R}$) and is thus closed. B_j is also the inverse image of an open set under the continuous function s (because, there exists an open interval I such that $I \cap \{\beta_1, \ldots, \beta_l\} = \{\beta_j\}$, so $B_j = s^{-1}[I]$) and is thus open. It follows that $\{B_1, \ldots, B_l\}$ is an FCDD of K.

This shows that s has a representation via sets from an FCDD of K.

(ii)

We now show that defining $\phi_\mu(s)$ via FCDDs is unambiguous.

Suppose $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_l$ are scalars and $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_l\}$ are FCDDs of K, such that

$$s = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i} = \sum_{j=1}^{l} \beta_j \mathbb{1}_{B_j}.$$

Suppose $1 \leq i \leq n$, $1 \leq j \leq l$, and $x \in A_i \cap B_j$. Then $s(x) = \alpha_i$ because $x \in A_i$, and $s(x) = \beta_j$ because $x \in B_j$. Hence $\alpha_i = \beta_j$. Thus, for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, l\}$, either

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$A_i \cap B_j = \emptyset$ or $\alpha_i = \beta_j$. In both cases,

$$\alpha_i \mu(A_i \cap B_j) = \beta_j \mu(A_i \cap B_j).$$

Since $\{B_1, \dots, B_l\}$ is an FCDD of K, by Lemma 3.3.4 we have $\mu(A_i) = \sum_{j=1}^{l} \mu(A_i \cap B_j)$ for all i. Since $\{A_1, \dots, A_n\}$ is an FCDD of K, we have $\mu(B_j) = \sum_{i=1}^{n} \mu(A_i \cap B_j)$ for all j. Thus

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i \mu(A_i) &= \sum_{i=1}^{n} \sum_{j=1}^{l} \alpha_i \mu(A_i \cap B_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{l} \beta_j \mu(A_i \cap B_j) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{n} \beta_j \mu(A_i \cap B_j) \\
&= \sum_{j=1}^{l} \beta_j \mu(B_j).
\end{aligned}$$

This shows that we may unambiguously define $\phi_\mu(s)$ via FCDDs.

(iii)

Let $\alpha_1, \dots, \alpha_n$ be scalars and let $\{A_1, \dots, A_n\}$ be an FCDD of K such that

$$s = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}.$$

Such a representation of s exists by part (i) of this proof. Set

$$\phi_\mu(s) := \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

By part (ii), this defines $\phi_\mu(s)$ unambiguously, i.e. it is independent of the choice of FCDD $\{A_1, \dots, A_n\}$ and scalars $\alpha_1, \dots, \alpha_n$. It remains to show that ϕ_μ is linear and positive.

(iv)

Let $s, t \in S(K)$ with

$$s = \sum_{i=1}^{n} \gamma_i \mathbb{1}_{C_i} \qquad\qquad t = \sum_{j=1}^{l} \delta_j \mathbb{1}_{D_j},$$

with $\{C_1, \dots, C_n\}$ and $\{D_1, \dots, D_l\}$ two FCDDs of K, and $\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_l$ scalars. By the

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definition of ϕ_μ in part (iii) and using Lemma 3.3.4,

$$\begin{aligned}\phi_\mu(s) + \phi_\mu(t) &= \sum_{i=1}^{n} \gamma_i \mu(C_i) + \sum_{j=1}^{l} \delta_j \mu(D_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{l} \gamma_i \mu(C_i \cap D_j) + \sum_{j=1}^{l}\sum_{i=1}^{n} \delta_j \mu(C_i \cap D_j) \\ &= \sum_{\substack{1\le i\le n\\ 1\le j\le l}} (\gamma_i + \delta_j)\mu(C_i \cap D_j).\end{aligned}$$

Now, observe that

$$s + t = \sum_{i=1}^{n} \gamma_i \mathbb{1}_{C_i} + \sum_{j=1}^{l} \delta_j \mathbb{1}_{D_j} = \sum_{\substack{1\le i\le n\\ 1\le j\le l}} (\gamma_i + \delta_j)\mathbb{1}_{C_i \cap D_j}.$$

The sets of the form $C_i \cap D_j$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, l\}$ are disjoint, so $\{C_i \cap D_j : 1 \le i \le n \text{ and } 1 \le j \le l\}$ is an FCDD of K. By the definition of ϕ_μ,

$$\begin{aligned}\phi_\mu(s + t) &= \sum_{\substack{1\le i\le n\\ 1\le j\le l}} (\gamma_i + \delta_j)\mu(C_i \cap D_j) \\ &= \phi_\mu(s) + \phi_\mu(t).\end{aligned}$$

This shows that ϕ_μ is additive. Homogeneity of ϕ_μ is similar: if $\alpha \in \mathbb{R}$, then

$$\begin{aligned}\phi_\mu(\alpha s) &= \phi_\mu\left(\sum_{i=1}^{n} \alpha\gamma_i \mathbb{1}_{C_i}\right) \\ &= \sum_{i=1}^{n} \alpha\gamma_i \mu(C_i) \\ &= \alpha \sum_{i=1}^{n} \gamma_i \mu(C_i) \\ &= \alpha\phi_\mu(s).\end{aligned}$$

Thus ϕ_μ is linear.

Since ϕ_μ extends μ, it is easy to show that ϕ_μ is a positive functional (i.e. that $\phi_\mu(s) \ge 0$ for all $s \ge \mathbf{0}$). If $s(x) \ge 0$ for all $x \in K$, then there exist non-negative scalars $\alpha_1, \dots, \alpha_n$ and disjoint clopen sets $A_1, \dots, A_n$ such that $s = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$. Since $\mu(A_i)$ is non-negative for each i, it follows that $\phi_\mu(s) \ge 0$.

Finally, ϕ_μ is the *only* linear extension of μ to $S(K)$. This is because it agrees with μ on $I(K)$, which is a spanning set of $S(K)$, and every linear operator is uniquely determined by its action on a spanning set.

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As previously stated, now that we have shown that ϕ_μ is linear, it is no longer necessary to represent a given step function $s \in S(K)$ using indicator functions of sets from an FCDD in order to evaluate $\phi_\mu(s)$. Any representation of s as a linear combination of indicator functions will suffice.

Lemma 3.3.6

The linear functional ϕ_μ given in Theorem 3.3.5 is bounded and has norm 1.

Proof:

Recall that a linear functional on a partially ordered vector space is positive if and only if it is monotone. Thus ϕ_μ is monotone. Therefore, for every $f \in C(K)$ with $||f|| = 1$, we have $f \leq \mathbf{1}$, so $\phi_\mu(f) \leq \phi_\mu(\mathbf{1}) = 1$. Hence $||\phi_\mu|| \leq 1$. But of course $\phi_\mu(\mathbf{1}) = 1$; thus $||\phi_\mu|| = 1$.

Theorem 3.3.7

Let $f \in C(K)$ and $\varepsilon > 0$. Then there exists an $s \in S(K)$ such that $||s - f|| < \varepsilon$ and $s\,[K] \subseteq f\,[K]$. In particular, $S(K)$ is dense in $C(K)$, and $S(K)_+$ is dense in $C(K)_+$.

Proof:

Let $x \in K$. Since f is continuous, there exists a neighbourhood U_x of x such that $f[U_x] \subseteq \Big(f(x) - \varepsilon, f(x) + \varepsilon\Big)$. Now, recall that K is zero-dimensional, i.e. it has a base consisting of clopen sets. Thus we can assume that U_x is clopen (if U_x is not clopen, then there exists a clopen subset of U_x containing x; replace U_x with this clopen subset). Since this holds for all $x \in K$, this gives an open cover of K:

$$K = \bigcup_{x \in K} U_x.$$

Since K is compact, there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$ such that

$$K = \bigcup_{i=1}^{n} U_{x_i}.$$

Furthermore, for each $i \in \{1, \ldots, n\}$, $f[U_{x_i}]$ is a subset of an open interval of radius ε.

Next, let $V_{x_1} := U_{x_1}$, and recursively define

$$V_{x_i} := U_{x_i} \setminus \big(V_{x_1} \cup \cdots \cup V_{x_{i-1}}\big)$$

for each $i \in \{2, \ldots, n\}$. It follows that each V_{x_i} is clopen, and $\{V_{x_1}, \ldots, V_{x_n}\}$ is a refinement of $\{U_{x_1}, \ldots, U_{x_n}\}$ which is disjoint. Finally, let

$$s_f := \sum_{i=1}^{n} f(x_i) \mathbb{1}_{V_{x_i}}.$$

Let $x \in K$. Then there exists exactly one i such that $x \in V_{x_i}$, so $s_f(x) = f(x_i)$, and $f(x) \in f[V_{x_i}] \subseteq f[U_{x_i}]$. Thus $f(x) \in \Big(f(x_i) - \varepsilon, f(x_i) + \varepsilon\Big) = \Big(s_f(x) - \varepsilon, s_f(x) + \varepsilon\Big)$, so

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$|s_f(x) - f(x)| < \varepsilon$. Since this holds for all $x \in K$, it follows that $||s_f - f|| < \varepsilon$. Moreover, by the definition of s_f, $s_f[K] \subseteq f[K]$.

Thus every open ball in $C(K)$ contains a step function, so $S(K)$ is dense in $C(K)$.

It also follows easily that $S(K)_+$ is dense in $C(K)_+$; if $f \in C(K)_+$, then $s_f \in S(K)_+$, because $s[K] \subseteq f[K] \subseteq [0, \infty)$.

Corollary 3.3.8

Let μ be a measure on $\mathbb{L} \cong I(K)$. Then there exists a unique positive linear functional $\widehat{\mu}$ on $C(K)$ with unit norm which extends μ.

Proof:
By Theorem 3.3.5 and Lemma 3.3.6, there exists a unique positive linear extension ϕ_μ of μ to $S(K)$ with unit norm. By a standard result in functional analysis, a bounded linear functional on a dense subspace of a normed space has a unique linear extension to the whole space with the same norm. Hence, by the density of $S(K)$ in $C(K)$ (Theorem 3.3.7), there exists a unique linear extension $\widehat{\mu}$ of ϕ_μ to $C(K)$ with norm 1. It also follows from the density of $S(K)_+$ in $C(K)_+$ that $\widehat{\mu}$ is positive.

Finally, we show a bijection between measures on $\mathbb{L}$ and positive linear functionals on $C(K)$ with unit norm. The forward direction of this bijection is given by Corollary 3.3.8, while the reverse direction is given by restriction to $I(K)$.

Theorem 3.3.9

Let $\mathbb{L}$ be a Boolean algebra and K its Stone space. Then there is a bijection between measures on $\mathbb{L}$ and positive linear functionals on $C(K)$ with unit norm.

Proof:
Let μ be a measure on $\mathbb{L} \cong I(K)$. By Corollary 3.3.8, there exists a unique extension $\widehat{\mu}$ of μ to $C(K)$ which is a positive linear functional with unit norm. This gives a mapping from measures on $\mathbb{L}$ to positive linear functionals on $C(K)$ with unit norm.

Let ϕ be a positive linear functional on $C(K)$ such that $||\phi|| = 1$. Then $\phi(1_{\mathbb{L}})$ (or rather, $\phi(\mathbb{1}_K)$) is equal to 1. It is easily seen that the restriction $\phi|_{\mathbb{L}}$ of ϕ to $\mathbb{L}$ satisfies the requirements of a measure on $\mathbb{L}$. This gives the mapping in the reverse direction.

To see that these two mappings are inverses, let μ be a measure on $\mathbb{L}$. Then $\widehat{\mu}$ agrees with μ on $\mathbb{L}$, so clearly $\widehat{\mu}|_{\mathbb{L}} = \mu$. Conversely, let ϕ be a positive linear functional on $C(K)$ such

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that $||\phi|| = 1$. Then $\phi|_{\mathbb{L}}$ is the restriction of ϕ to $I(K) \cong \mathbb{L}$, so ϕ is of course a positive linear extension of $\phi|_{\mathbb{L}}$ to $C(K)$. However, $\widehat{\phi|_{\mathbb{L}}}$ is also a positive linear extension of $\phi|_{\mathbb{L}}$ to $C(K)$ with unit norm. Since such an extension of $\phi|_{\mathbb{L}}$ is unique, it follows that $\widehat{\phi|_{\mathbb{L}}} = \phi$.

Remark 3.3.10: Measures on $\mathbb{L}$ and regular Borel probability measures on K

Theorem 3.3.9 will be used in the next section. However, it is worth mentioning that that theorem, together with the Riesz-Markov-Kakutani (RMK) Theorem for compact Hausdorff spaces, yields a correspondence between measures on $\mathbb{L}$ and regular Borel probability measures on K.

The RMK Theorem for compact Hausdorff spaces is as follows ([GR16], Theorem 10.35): If X is a compact Hausdorff space and $\phi : C(X) \to \mathbb{R}$ is a positive linear functional with norm 1, then there exists a unique regular Borel probability measure P_ϕ on X such that

$$\phi(f) = \int_X f dP_\phi$$

for all $f \in C(X)$.

Let μ be a measure on $\mathbb{L}$. By Theorem 3.3.9, there exists a unique positive linear extension $\widehat{\mu}$ of μ to $C(K)$ which has norm 1. By the RMK Theorem, there is a unique regular Borel probability measure $P_{\widehat{\mu}}$ on K such that

$$\widehat{\mu}(f) = \int_K f dP_{\widehat{\mu}}$$

for all $f \in C(K)$. This gives a mapping in the forward direction: $\mu \mapsto P_{\widehat{\mu}}$.

Conversely, let P be a regular Borel probability measure on K. Then the mapping

$$\phi_P : f \mapsto \int_K f dP$$

on $C(K)$ is a positive linear functional with unit norm. By Theorem 3.3.9, we can restrict to $\mathbb{L}$ to get a measure $\phi_P|_{\mathbb{L}}$ on $\mathbb{L}$. This gives a mapping in the reverse direction: $P \mapsto \phi_P|_{\mathbb{L}}$.

To see that these mappings are inverses, let μ be a measure on $\mathbb{L}$. Then clearly the mapping $\phi_{P_{\widehat{\mu}}} : f \mapsto \int_K f dP_{\widehat{\mu}}$ is just $\widehat{\mu}$. Restricting to $\mathbb{L}$ gives $\widehat{\mu}|_{\mathbb{L}} = \mu$, as required. Conversely, if P is a regular Borel probability measure on K, then $\widehat{\phi_P|_{\mathbb{L}}}$ is of course just ϕ_P, and $P_{\phi_P} = P_{\widehat{\phi_P|_{\mathbb{L}}}}$ is the unique regular Borel probability measure such that

$$\phi_P(f) = \int_K f dP_{\phi_P}$$

for all $f \in C(K)$. But we also have

$$\phi_P(f) = \int_K f dP$$

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for all $f \in C(K)$. By the uniqueness guaranteed by the RMK Theorem, $P_{\phi_P} = P$. This proves that the two mappings are inverses.

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3.4 Intersection Numbers and Covering Numbers

In this section, we arrive at the promised characterisation of Boolean algebras which admit strictly positive measures. We will show that a Boolean algebra admits a strictly positive measure if and only if the Boolean algebra can be expressed as a union of countably many sets with positive intersection number (which we now define).

Definition 3.4.1: The intersection number of a subset of a Boolean algebra

Let $\mathbb{L}$ be a Boolean algebra and let $\mathcal{S} \subseteq \mathbb{L}$ be non-empty. For each tuple $(A_1, \ldots, A_n)$ of (not necessarily distinct) elements of $\mathcal{S}$, let $i(A_1, \ldots, A_n)$ be the maximum number of terms in the tuple $(A_1, \ldots, A_n)$ with non-empty intersection. Define $I(\mathcal{S})$, the **intersection number of $\mathcal{S}$**, as follows.

$$I(\mathcal{S}) := \inf \left\{ \frac{i(A_1, \ldots, A_n)}{n} : n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S} \right\}$$

The intersection number of a subset $\mathcal{S}$ of a Boolean algebra $\mathbb{L}$ is, loosely speaking, a particular measure of the "congestion" of the elements of $\mathcal{S}$ relative to the union of the elements of $\mathcal{S}$. Given $A_1, \ldots, A_n \in \mathcal{S}$, $i(A_1, \ldots, A_n)$ is the maximum number of times a point in K is contained in the sets $A_1, \ldots, A_n$. This immediately implies the following.

Lemma 3.4.2

Let $\mathbb{L}$ be a Boolean algebra and $\mathcal{S} \subseteq \mathbb{L}$. Let $C(K)$ be the Banach space of continuous real-valued functions on K. Then, for all $n \in \mathbb{N}$ and all $A_1, \ldots, A_n \in \mathcal{S}$, we have

$$i(A_1, \ldots, A_n) \quad = \quad \max_{x \in K} \sum_{k=1}^{n} \mathbb{1}_{A_k}(x) \quad = \quad \left\| \sum_{k=1}^{n} \mathbb{1}_{A_k} \right\|.$$

An equivalent definition of the intersection number of $\mathcal{S}$ is thus

$$I(\mathcal{S}) := \inf \left\{ \max_{x \in K} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{A_k}(x) \ : \ n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S} \right\}.$$

Remark 3.4.3: Visualisation of the intersection number

In order to obtain a small value of $\frac{i(A_1, \ldots, A_n)}{n}$, we need a large value of n and a small value of $i(A_1, \ldots, A_n)$. Thus we can imagine obtaining the intersection number of $\mathcal{S}$ by choosing as many elements of $\mathcal{S}$ as possible while minimising the amount of overlap between them.

For instance, suppose that $\mathbb{L}$ is the power set of $[0, 1] \subseteq \mathbb{R}$, and suppose that $\mathcal{S}$ is the set of all open intervals in $[0, 1]$ with length greater than or equal to $\frac{1}{m}$ (where m is some large natural number). Given any collection $A_1, \ldots, A_n$ of elements of $\mathcal{S}$ such that $[0, 1] \setminus \bigcup_{k=1}^{n} A_k$ contains

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an element of $\mathcal{S}$ (say A_{n+1}), then $\frac{i(A_1,\ldots,A_{n+1})}{n+1} < \frac{i(A_1,\ldots,A_n)}{n}$. We have thus reduced the ratio by adding an element of $\mathcal{S}$ which is disjoint from $A_1, \ldots, A_n$. We can keep reducing the ratio by adding more elements of $\mathcal{S}$ while avoiding overlaps.

The notion of intersection numbers gives a condition which is equivalent to the existence of a strictly positive measure on $\mathbb{L}$. We will show that there exists a strictly positive measure on $\mathbb{L}$ if and only if the non-zero elements of $\mathbb{L}$ are the union of a countable collection of subsets, each of which has positive intersection number.

Theorem 3.4.4

If $\mathbb{L}$ is a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$, then for every measure μ on $\mathbb{L}$,

$$\inf_{A\in\mathcal{S}} \mu(A) \leq I(\mathcal{S}).$$

Proof:

Let μ be a measure on $\mathbb{L}$. By Theorem 3.3.9, there exists a positive linear functional ϕ_μ on $C(K)$ such that $||\phi_\mu|| = 1$, $\phi_\mu(\mathbf{1}) = 1$. Recall that the functional ϕ_μ is the unique positive linear functional on $C(K)$ which agrees with μ on $\mathbb{L} \cong I(K)$ (where $I(K)$ is the set of indicator functions of clopen subsets of K).

Let $A_1, \ldots, A_n \in \mathcal{S}$. Then

$$\begin{aligned}
\inf_{A\in\mathcal{S}} \mu(A) &= \frac{1}{n}\sum_{k=1}^{n} \inf_{A\in\mathcal{S}} \mu(A) \\
&\leq \frac{1}{n}\sum_{k=1}^{n} \mu(A_k) \\
&= \frac{1}{n}\sum_{k=1}^{n} \phi_\mu(\mathbb{1}_{A_k}) \\
&= \phi_\mu\left(\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{A_k}\right) \\
&\leq ||\phi_\mu|| \cdot \left|\left|\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{A_k}\right|\right| \\
&= \left|\left|\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{A_k}\right|\right| \\
&= \frac{i(A_1,\ldots,A_n)}{n}.
\end{aligned}$$

Since this holds for all $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$, it follows that $\inf_{A\in\mathcal{S}} \mu(A)$ is a lower bound of

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the set $\left\{\frac{i(A_1,\ldots,A_n)}{n} : n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S}\right\}$. Hence

$$\inf_{A\in\mathcal{S}} \mu(A) \leq \inf\left\{\frac{i(A_1,\ldots,A_n)}{n} : n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S}\right\} = I(\mathcal{S}).$$

Since the above theorem holds for *all* measures on $\mathbb{L}$, we can take the supremum of $\inf\{\mu(A) : A \in \mathcal{S}\}$ over all measures μ on $\mathbb{L}$, giving the following corollary.

Corollary 3.4.5

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Then

$$\sup\Big\{\inf\{\mu(A) : A \in \mathcal{S}\} : \mu \text{ is a measure on } \mathbb{L}\Big\} \leq I(\mathcal{S}).$$

Next, we will show that this inequality is actually an equality, and that the supremum is actually a maximum. In order to do this, the following lemma is required.

Lemma 3.4.6

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Let $F \subseteq C(K)$ be the set of indicator functions of the elements of $\mathcal{S}$. Let G be the convex hull of F in $C(K)$. Finally, let

$$D := \left\{\sum_{i=1}^{n} \frac{1}{n}\mathbb{1}_{A_i} : n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S}\right\}.$$

Then $D \subseteq G$ is dense in G.

Proof:
We first verify another equality. Let

$$C := \left\{\sum_{i=1}^{n} r_i\mathbb{1}_{A_i} : n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S}, r_1, \ldots, r_n \in \mathbb{Q} \cap [0,1], \sum_{i=1}^{n} r_i = 1\right\}.$$

We show that $C = D$.

It is clear that $D \subseteq C$, because $\frac{1}{n} \in \mathbb{Q} \cap [0,1]$ for all $n \in \mathbb{N}$. Now let

$$f = \sum_{i=1}^{n} r_i\mathbb{1}_{A_i}$$

be an element of C. For each $i \in \{1, \ldots, n\}$, write r_i as a ratio of integers $\frac{p_i}{q_i}$ with $p_i \geq 0$ and $q_i > 0$. Let $q = \prod_{i=1}^{n} q_i$. Then

$$f = \sum_{i=1}^{n} \frac{r_i q}{q}\mathbb{1}_{a_i}.$$

Since each q_i divides q, $r_i q = \frac{p_i}{q_i} \cdot q$ is an integer for all $i \leq n$. Moreover, since

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$\sum_{i=1}^n r_i = 1$, it follows that $\sum_{i=1}^n r_i q = q$. Hence we have

$$f = \sum_{i=1}^n \frac{r_i q}{q} \mathbb{1}_{a_i} = \sum_{i=1}^n \sum_{j=1}^{r_i q} \frac{1}{q} \mathbb{1}_{a_i}$$

so $f \in D$ as required.

This shows that $C \subseteq D$. We conclude that $C = D$.

Now let $f \in G$ and $\varepsilon > 0$. Then

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$$

for non-negative scalars $\lambda_1, \dots, \lambda_n$ summing to 1, and $A_1, \dots, A_n \in \mathcal{S}$. We will obtain an element of D within ε of f. That is, we want coefficients $r_1, \dots, r_n \in \mathbb{Q} \cap [0,1]$ such that $\sum_{i=1}^n r_i = 1$ and $||f - \sum_{i=1}^n r_i \mathbb{1}_{A_i}|| < \varepsilon$.

For each $i \leq n$, let $r_i \in \left(\lambda_i - \frac{1}{n^2}\varepsilon, \lambda_i + \frac{1}{n^2}\varepsilon\right) \cap [0,1] \cap \mathbb{Q}$ (which exists by the density of $\mathbb{Q}$ in $\mathbb{R}$). In particular, $r_i \in \left(\lambda_i - \frac{1}{n}\varepsilon, \lambda_i + \frac{1}{n}\varepsilon\right)$ for each $i \leq n$. Thus $|\lambda_i - r_i| < \frac{1}{n}\varepsilon$ for each $i \leq n$, and hence

$$\begin{aligned}
\left|\left| f - \sum_{i=1}^n r_i \mathbb{1}_{A_i} \right|\right| &= \max_{x \in K} \left| f(x) - \sum_{i=1}^n r_i \mathbb{1}_{A_i}(x) \right| \\
&= \max_{x \in K} \left| \sum_{i=1}^n (\lambda_i - r_i) \mathbb{1}_{A_i}(x) \right| \\
&\leq \sum_{i=1}^n |\lambda_i - r_i| \\
&< \sum_{i=1}^n \frac{1}{n}\varepsilon \ = \ \varepsilon.
\end{aligned}$$

We are not guaranteed that $\sum_{i=1}^n r_i = 1$, but this is easily remedied as follows: let $r_i' = r_i$ for each $i \in \{1, \dots, n-1\}$, and set $r_n' := 1 - \sum_{i=1}^{n-1} r_i' \in \mathbb{Q}$.

Since $|r_i' - \lambda_i| < \frac{1}{n^2}\varepsilon$ for each $i < n$, we have

$$\begin{aligned}
\left| \sum_{i=1}^{n-1} r_i' - \sum_{i=1}^{n-1} \lambda_i \right| &= \left| \sum_{i=1}^{n-1} r_i' - \lambda_i \right| \\
&\leq \sum_{i=1}^{n-1} |r_i' - \lambda_i| \\
&< \frac{n-1}{n^2}\varepsilon.
\end{aligned}$$

In other words, $\sum_{i=1}^{n-1} r_i'$ is within $\frac{n-1}{n^2}\varepsilon$ of $\sum_{i=1}^{n-1} \lambda_i$. It follows that $1 - \sum_{i=1}^{n-1} r_i'$ is within $\frac{n-1}{n^2}\varepsilon$

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of $1 - \sum_{i=1}^{n-1} \lambda_i$. That is,

$$\begin{aligned} \frac{n-1}{n^2}\varepsilon &> \left| \left(1 - \sum_{i=1}^{n-1} r_i'\right) - \left(1 - \sum_{i=1}^{n-1} \lambda_i\right) \right| \\ &= |r_n' - \lambda_n| . \end{aligned}$$

This shows that $|r_i' - \lambda_i| < \frac{1}{n}\varepsilon$ for all $i \in \{1, \ldots, n\}$. Thus $\sum_{i=1}^n r_i' \mathbb{1}_{A_i} \in D$, and $||f - \sum_{i=1}^n r_i' \mathbb{1}_{A_i}|| < \varepsilon$. In particular, D intersects $B_\varepsilon(f)$. Since this holds for all $f \in G$, this shows that D is dense in G.

Theorem 3.4.7

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Then there exists a measure μ on $\mathbb{L}$ such that

$$\inf \{\mu(A) : A \in \mathcal{S}\} = I(\mathcal{S}).$$

Proof:
As in Lemma 3.4.6 let $F \subseteq C(K)$ be the set of indicator functions of the elements of $\mathcal{S}$, let G be the convex hull of F, and let

$$D := \left\{ \sum_{i=1}^n \frac{1}{n} \mathbb{1}_{A_i} \ : \ n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S} \right\}.$$

We first show that $I(\mathcal{S}) = \inf_{f \in G} ||f||$.

First, observe that $I(\mathcal{S}) = \inf_{f \in D} ||f||$. Indeed, as shown previously,

$$I(\mathcal{S}) = \inf \left\{ \frac{1}{n} \cdot \max_{x \in K} \sum_{k=1}^n \mathbb{1}_{A_k}(x) \ : \ n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S} \right\}.$$

Now this can be rewritten as

$$\begin{aligned} I(\mathcal{S}) &= \inf \left\{ \left|\left| \sum_{k=1}^n \frac{1}{n} \mathbb{1}_{A_k} \right|\right| \ : \ n \in \mathbb{N}, A_1, \ldots, A_n \in \mathcal{S} \right\} \\ &= \inf_{f \in D} ||f||, \end{aligned}$$

as required.

By the density of D in G (Lemma 3.4.6) and the continuity of the norm, we have $\inf\limits_{f \in G} ||f|| = \inf\limits_{f \in D} ||f|| = I(A)$.

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We now show the existence of the required measure.

Let $B_r(f) = \{g \in C(K) : ||g - f|| < r\}$ be the open ball in $C(K)$ centred at f with radius r, and let $B_r[f] = \{g \in C(K) : ||g - f|| \leq r\}$ be the closed ball. Consider the following two sets.

$$P = \left\{t(g + f) : t \in [0, \infty), g \in G, f \in B_{I(\mathcal{S})}[\mathbf{0}]\right\}$$
$$Q = B_1(-\mathbf{1}).$$

We will use the Hahn-Banach Separation Theorem ([NB10], Theorem 1.6) to separate these sets. The resulting linear functional will correspond to the required measure.

First we show that P and Q are disjoint.

Each element of Q is negative everywhere. Indeed, if $q \in Q$ and $x \in K$, then $|q(x) + 1| < 1$, so $q(x) + 1 \in (-1, 1)$, and thus $q(x) \in (-2, 0)$.

Let $g \in G$. Then $||g|| \geq I(\mathcal{S})$ (and g is non-negative everywhere by the definition of G), so there exists an $x \in K$ such that $g(x) \geq I(\mathcal{S})$. Let $f \in B_{I(\mathcal{S})}[\mathbf{0}]$. Then for all $x \in K$, $f(x) \geq -I(\mathcal{S})$. Hence, for all $g \in G$ and all $f \in B_{I(\mathcal{S})}[\mathbf{0}]$, there exists an $x \in K$ such that $g(x) + f(x) \geq I(\mathcal{S}) - I(\mathcal{S}) = 0$. Hence every element of P is somewhere non-negative. In particular, P is disjoint from Q.

It is clear that Q is non-empty, convex, and open. Since G and $B_{I(\mathcal{S})}[\mathbf{0}]$ are both convex sets in $C(K)$, their element-wise sum $\left\{g + f : g \in G, f \in B_{I(\mathcal{S})}[0]\right\}$ is also convex. It therefore follows easily that P is convex. By the Hahn-Banach Separation Theorem, there exists a continuous linear functional $\phi : C(K) \to \mathbb{R}$ such that $\phi(q) < \phi(h)$ for all $h \in P$ and all $q \in Q$.

Since $\mathbf{0} \in P$ and $-\mathbf{1} \in Q$, we have $\phi(-\mathbf{1}) < \phi(\mathbf{0}) = 0$. Assume without loss of generality that $\phi(-\mathbf{1}) = -1$ $\left(\text{if not, replace } \phi \text{ with } \phi' := \frac{1}{\phi(\mathbf{1})}\phi\right)$, so $\phi(\mathbf{1}) = 1$. Furthermore, for all $h \in P$, $\phi(h) \geq 0$. Indeed, if $\phi(h) < 0$ for some $h \in P$, then $-\frac{1}{\phi(h)}h \in P$ and $-1 = \phi(-\mathbf{1}) < \phi\left(-\frac{1}{\phi(h)}h\right) = -1$, a contradiction.

Now, let $g \in G$. We have $|| - I(\mathcal{S}) \cdot \mathbf{1}|| = I(\mathcal{S})$, so $g - I(\mathcal{S}) \cdot \mathbf{1} \in P$. Hence $\phi(g - I(\mathcal{S}) \cdot \mathbf{1}) \geq 0$, so

$$\phi(g) \geq \phi(I(\mathcal{S}) \cdot \mathbf{1}) = I(\mathcal{S}).$$

Thus $\phi(g) \geq I(\mathcal{S})$ for all $g \in G$.

For each $A \in \mathbb{L}$, let $\mu(A) := \phi(\mathbb{1}_A)$. Then μ is non-negative, finitely additive, and $\mu(1_{\mathbb{L}}) = \phi(\mathbf{1}) = 1$. Moreover, $\mu(A) \geq I(\mathcal{S})$ for all $A \in \mathcal{S}$, because in this case $\mathbb{1}_A \in G$. Hence $\inf\{\mu(A) : A \in \mathcal{S}\} \geq I(\mathcal{S})$. By Theorem 3.4.4, $\inf\{\mu(A) : A \in \mathcal{S}\} = I(\mathcal{S})$.

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Corollary 3.4.8

For each non-empty subset $\mathcal{S}$ of a Boolean algebra $\mathbb{L}$,

$$I(\mathcal{S}) = \max\left\{\inf_{A\in\mathcal{S}} \mu(A) : \mu \text{ is a measure on } \mathbb{L}\right\}.$$

We now have what we need to prove the promised characterisation of Boolean algebras which admit a strictly positive measure.

Theorem 3.4.9: Strictly positive measures and intersection numbers

Let $\mathbb{L}$ be a Boolean algebra. There exists a strictly positive measure on $\mathbb{L}$ if and only if there exists a countable collection $\{\mathcal{S}_n : n \in \mathbb{N}\}$ of non-empty subsets of $\mathbb{L}$ such that $I(\mathcal{S}_n) > 0$ for all $n \in \mathbb{N}$, and

$$\mathbb{L} \setminus \{0_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

Proof:
Suppose μ is a strictly positive measure on $\mathbb{L}$, and let $\mathcal{S}_n := \left\{A \in \mathbb{L} : \mu(A) \geq \frac{1}{n}\right\}$ for each $n \in \mathbb{N}^+$. Then

$$\mathbb{L} \setminus \{0_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

By Theorem 3.4.4,

$$\begin{aligned} I(\mathcal{S}_n) &\geq \inf\{\mu(A) : A \in \mathcal{S}_n\} \\ &\geq \frac{1}{n} \\ &> 0 \end{aligned}$$

for each n. Thus $\mathbb{L}\setminus\{0_{\mathbb{L}}\}$ is the union of countably many sets with positive intersection number.

Suppose $\mathbb{L} \setminus \{0_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$, with $I(\mathcal{S}_n) > 0$ for all n. For each n, let μ_n be a measure on $\mathbb{L}$ with $\inf\{\mu_n(A) : A \in \mathcal{S}_n\} = I(\mathcal{S}_n) > 0$ (such a measure exists by Theorem 3.4.7). Define

$$\mu : \mathbb{L} \ni A \ \mapsto \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n}.$$

We now shaw that μ is a strictly positive measure on $\mathbb{L}$.

Clearly $\mu(A) \in [0, 1]$ for all $A \in \mathbb{L}$, and we have $\mu(0_{\mathbb{L}}) = 0$ and $\mu(1_{\mathbb{L}}) = 1$. Let

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$A, B \in \mathbb{L}$ with $A \wedge B = 0_{\mathbb{L}}$. Then

$$\begin{aligned}\mu(A \vee B) &= \sum_{n=1}^{\infty} \frac{\mu_n(A \vee B)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{\mu_n(A) + \mu_n(B)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n} + \sum_{n=1}^{\infty} \frac{\mu_n(B)}{2^n} \\ &= \mu(A) + \mu(B).\end{aligned}$$

Thus μ is finitely additive.

Now let $0_{\mathbb{L}} \neq C \in \mathbb{L}$. Then there exists an $n \in \mathbb{N}^+$ such that $C \in \mathcal{S}_n$. Hence $\mu_n(C) \geq \inf_{A \in \mathcal{S}_n} \mu_n(A) = I(\mathcal{S}_n) > 0$, so $\mu(C) \geq \frac{\mu_n(C)}{2^n} > 0$. Hence μ is strictly positive.

Covering Numbers

We now define covering numbers, which are in some sense dual to intersection numbers. Recall that, for $A_1, \ldots, A_n \in \mathcal{S}$, $i(A_1, \ldots, A_n)$ is the maximum number of times a point in K is contained in the sets $A_1, \ldots, A_n$. The covering number uses $m(A_1, \ldots, A_n)$, which is defined as the *minimum* number of times a point in K is contained in the sets $A_1, \ldots, A_n$. We will show that the covering number gives an equivalent condition for the existence of a strictly positive measure on a Boolean algebra.

Definition 3.4.10: The covering number of a subset of a Boolean algebra

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. For each $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$, let

$$m(A_1, \ldots, A_n) := \min_{x \in K} \sum_{k=1}^{n} \mathbb{1}_{A_k}(x).$$

The **covering number** $C(\mathcal{S})$ of $\mathcal{S}$ is

$$C(\mathcal{S}) := \sup \left\{ \frac{m(A_1, \ldots, A_n)}{n} : n \in \mathbb{N}, A_1, \ldots, A_n \right\}.$$

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The relationship between the covering number and the intersection number is illustrated by the following theorem.

Theorem 3.4.11

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Let $\neg\mathcal{S} := \{\neg A : A \in \mathcal{S}\}$ be the set of complements of elements of $\mathcal{S}$. Then

$$I(\mathcal{S}) + C(\neg\mathcal{S}) = 1.$$

Proof:

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$. Recall that the negative of the supremum of a bounded set Y of real numbers is the same as the infimum of the negatives: $-\left(\bigvee Y\right) = \bigwedge \{-y : y \in Y\}$. Also notice that $\mu(\neg A) = 1 - \mu(A)$ for all $A \in \mathbb{L}$, because A and $\neg A$ are disjoint and their join is $1_{\mathbb{L}}$, whose measure is 1. Hence

$$\begin{aligned} m(\neg A_1, \ldots, \neg A_n) &= \min_{x \in K} \sum_{k=1}^{n} \mathbb{1}_{\neg A_k}(x) \\ &= \min_{x \in K} \sum_{k=1}^{n} 1 - \mathbb{1}_{A_k}(x) \\ &= n + \min_{x \in K} \sum_{k=1}^{n} -\mathbb{1}_{A_k}(x) \\ &= n - \max_{x \in K} \sum_{k=1}^{n} \mathbb{1}_{A_k}(x) \\ &= n - i(A_1, \ldots, A_n). \end{aligned}$$

Thus $\frac{1}{n} \cdot m(\neg A_1, \ldots, \neg A_n) = 1 - \frac{1}{n} \cdot i(A_1, \ldots, A_n)$. Since this holds for all $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$, it follows that $C(\neg\mathcal{S}) = 1 - I(\mathcal{S})$.

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Corollary 3.4.12

Let $\mathbb{L}$ be a Boolean algebra and $\emptyset \neq \mathcal{S} \subseteq \mathbb{L}$. Then

$$C(\mathcal{S}) = \min\left\{\sup_{A\in\mathcal{S}} \mu(A) : \mu \text{ is a measure on } \mathbb{L}\right\}.$$

Proof:

By Corollary 3.4.8 and Theorem 3.4.11, we have

$$\begin{aligned} C(\mathcal{S}) &= 1 - I(\neg\mathcal{S}) \\ &= 1 - \max\left\{\inf_{A\in\mathcal{S}} \mu(\neg A) : \mu \text{ is a measure on } \mathbb{L}\right\} \\ &= 1 - \max\left\{1 - \sup_{A\in\mathcal{S}} \mu(A) : \mu \text{ is a measure on } \mathbb{L}\right\} \\ &= 1 - \left(1 - \min\left\{\sup_{A\in\mathcal{S}} \mu(A) : \mu \text{ is a measure on } \mathbb{L}\right\}\right) \\ &= \min\left\{\sup_{A\in\mathcal{S}} \mu(A) : \mu \text{ is a measure on } \mathbb{L}\right\}. \end{aligned}$$

An immediate corollary of Theorem 3.4.11 is the following equivalent condition for the existence of a strictly positive measure on a Boolean algebra.

Corollary 3.4.13

Let $\mathbb{L}$ be a Boolean algebra. Then there exists a strictly positive measure on $\mathbb{L}$ if and only if there exists a countable collection $\{\mathcal{S}_n : n \in \mathbb{N}\}$ of non-empty subsets of $\mathbb{L}$ such that $C(\mathcal{S}_n) < 1$ for all $n \in \mathbb{N}$, and

$$\mathbb{L} \setminus \{\mathbb{1}_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

Proof:

Suppose

$$\mathbb{L} \setminus \{\mathbb{1}_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

with $C(\mathcal{S}_n) < 1$ for all n. Then $I(\neg\mathcal{S}_n) > 0$ for all n (by Theorem 3.4.11). Furthermore,

$$\begin{aligned} \neg\left(\mathbb{L} \setminus \{\mathbb{1}_{\mathbb{L}}\}\right) &= \{\neg A : A \in \mathbb{L}\} \setminus \{0_{\mathbb{L}}\} \\ &= \mathbb{L} \setminus \{0_{\mathbb{L}}\}. \end{aligned}$$

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because $0_{\mathbb{L}}$ is the only element of $\mathbb{L}$ whose complement is $1_{\mathbb{L}}$. Hence

$$\begin{aligned}\mathbb{L} \setminus \{0_{\mathbb{L}}\} &= \neg(\mathbb{L} \setminus \{1_{\mathbb{L}}\}) \\ &= \neg\left(\bigcup_{n=1}^{\infty} \mathcal{S}_n\right) \\ &= \bigcup_{n=1}^{\infty} \neg\mathcal{S}_n\end{aligned}$$

is the union of a countable collection of sets, each of which has positive intersection number. By Theorem 3.4.9, there exists a strictly positive measure on $\mathbb{L}$.

The converse is similar: if there exists a strictly positive measure on $\mathbb{L}$, then Theorem 3.4.9 gives

$$\mathbb{L} \setminus \{0_{\mathbb{L}}\} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

with $I(\mathcal{S}_n) > 0$ for all n. Hence

$$\begin{aligned}\mathbb{L} \setminus \{1_{\mathbb{L}}\} &= \neg(\mathbb{L} \setminus \{0_{\mathbb{L}}\}) \\ &= \neg\left(\bigcup_{n=1}^{\infty} \mathcal{S}_n\right) \\ &= \bigcup_{n=1}^{\infty} \neg\mathcal{S}_n\end{aligned}$$

is the union of a countable collection of sets, each of which has covering number less than 1.

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3.5 Weak σ-distributivity and the Existence of a σ-additive Strictly Positive Measure

We now turn to the existence of σ-additive strictly positive measures on σ-complete Boolean algebras.[4] We will first define weak σ-distributivity, and then show a characterisation of the weakly σ-distributive Boolean algebras in terms of a striking topological property of their Stone spaces. We conclude by showing that, if $\mathbb{L}$ is a σ-complete Boolean algebra which admits a strictly positive measure, then $\mathbb{L}$ admits a σ-additive strictly positive measure if and only if it is weakly σ-distributive.

In the following, $\mathbb{N}^{\mathbb{N}}$ is the set of all sequences of positive integers.

Definition 3.5.1: Weak countable distributivity

Let $\mathbb{L}$ be a Boolean algebra. A **double sequence in** $\mathbb{L}$ is a function from $\mathbb{N}^2$ to $\mathbb{L}$, usually expressed in the form $(A_{i,j})_{i,j\in\mathbb{N}}$. A double sequence $(A_{i,j})_{i,j\in\mathbb{N}}$ is **decreasing in the second index** if, for every $i \in \mathbb{N}$, the sequence $(A_{i,j})_{j\in\mathbb{N}}$ is decreasing, i.e. $A_{i,j} \geq A_{i,j+1}$ for all $j \in \mathbb{N}$.

We say that $\mathbb{L}$ is **weakly countably distributive** or **weakly σ-distributive** if, for every double sequence $(A_{i,j})_{i,j\in\mathbb{N}}$ in $\mathbb{L}$ which is decreasing in the second index,

$$\bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} A_{i,j} = \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)},$$

whenever the appropriate suprema and infima exist.

Remark 3.5.2

In any Boolean algebra, the inequality

$$\bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} A_{i,j} \leq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)}$$

holds for any double sequence $(A_{i,j})_{i,j\in\mathbb{N}}$ (provided the appropriate suprema and infima exist). Indeed, $\bigwedge_{j\in\mathbb{N}} A_{i,j} \leq \bigvee_{k\in\mathbb{N}} A_{k,s(k)}$ for any $i \in \mathbb{N}$ and any $s \in \mathbb{N}^{\mathbb{N}}$. Taking the infimum over all $s \in \mathbb{N}^{\mathbb{N}}$ gives $\bigwedge_{j\in\mathbb{N}} A_{i,j} \leq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{k\in\mathbb{N}} A_{k,s(k)}$ for all $i \in \mathbb{N}$. Taking the supremum over all $i \in \mathbb{N}$ gives $\bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} A_{i,j} \leq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)}$.

Weak σ-distributivity means that the reverse inequality

$$\bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} A_{i,j} \geq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)}$$

holds as well.

[4] Recall that a σ-complete Boolean algebra which admits a strictly positive measure is complete. Hence we are really dealing with complete Boolean algebras here.

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Weak σ-distributivity is equivalent to its dual statement: for any double sequence $(A_{i,j})_{i,j\in\mathbb{N}}$ which is *increasing* in the second index,

$$\bigwedge_{i\in\mathbb{N}}\bigvee_{j\in\mathbb{N}} A_{i,j} \leq \bigvee_{s\in\mathbb{N}^{\mathbb{N}}}\bigwedge_{i\in\mathbb{N}} A_{i,s(i)}$$

whenever the appropriate suprema and infima exist. (Of course, this implies that the two sides are actually equal.)

A Boolean algebra is **σ-distributive** if the equality in the above definition holds for all double sequences in $\mathbb{L}$ (not just ones which are decreasing in the second index).

The next lemma is fairly well known, but we include it anyway.

Lemma 3.5.3: σ-additive measures are continuous

Let $\mathbb{L}$ be a Boolean algebra and let μ be a σ-additive measure on $\mathbb{L}$. Then, for every increasing sequence $(A_i)_{i\in\mathbb{N}}$ in $\mathbb{L}$ with a supremum in $\mathbb{L}$ and every decreasing sequence $(B_i)_{i\in\mathbb{N}}$ in $\mathbb{L}$ with an infimum in $\mathbb{L}$, we have

$$\mu\left(\bigvee_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty}\mu(A_i) \qquad \mu\left(\bigwedge_{i\in\mathbb{N}} B_i\right) = \lim_{i\to\infty}\mu(B_i).$$

Proof:

Set $A_0 := \emptyset$. Observe that, since $(A_i)_{i\in\mathbb{N}}$ is increasing, $A_n = (A_n \setminus A_{n-1}) \vee A_{n-1}$ for each $n \in \mathbb{N}$. Expanding A_{n-1} using the same equation gives $A_n = (A_n \setminus A_{n-1}) \vee (A_{n-1} \setminus A_{n-2}) \vee A_{n-2}$. Repeating this expansion until we reach A_0, we get

$$A_n = \bigvee_{i=1}^{n} A_i \setminus A_{i-1}$$

for each n. Moreover, the set $\{A_i \setminus A_{i-1} : i \in \mathbb{N}\}$ is a disjoint set. Hence

$$\begin{aligned}
\mu\left(\bigvee_{i\in\mathbb{N}} A_i\right) &= \mu\left(\bigvee_{i\in\mathbb{N}} A_i \setminus A_{i-1}\right) \\
&= \sum_{i=1}^{\infty}\mu(A_i \setminus A_{i-1}) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\mu(A_i \setminus A_{i-1}) \\
&= \lim_{n\to\infty}\mu\left(\bigvee_{i=1}^{n} A_i \setminus A_{i-1}\right) \\
&= \lim_{n\to\infty}\mu(A_n).
\end{aligned}$$

For the decreasing sequence $(B_i)_{i\in\mathbb{N}}$, observe that $(\neg B_i)_{i\in\mathbb{N}}$ is an increasing sequence. By the

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above, we have

$$\begin{aligned}\mu\left(\bigwedge_{i\in\mathbb{N}} B_i\right) &= 1-\mu\left(\neg\bigwedge_{i\in\mathbb{N}} B_i\right)\\ &= 1-\mu\left(\bigvee_{i\in\mathbb{N}} \neg B_i\right)\\ &= 1-\lim_{i\to\infty}\mu(\neg B_i)\\ &= 1-\lim_{i\to\infty} 1-\mu(B_i)\\ &= \lim_{i\to\infty}\mu(B_i).\end{aligned}$$

The proofs of the remaining results in this section are adapted from [HT48].

Theorem 3.5.4

Let $\mathbb{L}$ be a Boolean algebra. If there exists a σ-additive strictly positive measure on $\mathbb{L}$, then $\mathbb{L}$ is weakly σ-distributive.

Proof:
Suppose μ is a σ-additive strictly positive measure on $\mathbb{L}$. Let $(A_{i,j})_{i,j\in\mathbb{N}}$ be a double sequence in $\mathbb{L}$ which is decreasing in the second index, and suppose that the suprema and infima in Definition 3.5.1 exist. For each $i\in\mathbb{N}$, set $A_{i,\infty} := \bigwedge_{j\in\mathbb{N}} A_{i,j}$.

Let $\varepsilon > 0$. By the continuity of μ (Lemma 3.5.3), we have

$$\mu(A_{i,\infty}) = \lim_{j\to\infty}\mu(A_{i,j})$$

for each $i\in\mathbb{N}$. Thus, for every $i\in\mathbb{N}$, there exists a $k_i\in\mathbb{N}$ such that $\mu(A_{i,k_i}) < \mu(A_{i,\infty}) + \varepsilon/2^i$, that is, $\mu(A_{i,k_i}\setminus A_{i,\infty}) < \varepsilon/2^i$. It follows that

$$\begin{aligned}\mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i}\setminus A_{i,\infty}\right) &\le \sum_{i\in\mathbb{N}}\mu(A_{i,k_i}\setminus A_{i,\infty})\\ &= \sum_{i\in\mathbb{N}}\varepsilon/2^i\\ &= \varepsilon.\end{aligned}$$

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Let $A := \bigvee_{i\in\mathbb{N}} A_{i,\infty}$, and let B be a lower bound of the set $\left\{\bigvee_{i\in\mathbb{N}} A_{i,s(i)} : s \in \mathbb{N}^{\mathbb{N}}\right\}$. Then

$$\begin{aligned}
\mu(B) &\leq \mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i}\right) \\
&= \mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i} \setminus A\right) + \mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i} \wedge A\right) \\
&\leq \mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i} \setminus A\right) + \mu\left(\bigvee_{n\in\mathbb{N}} A\right) \\
&\leq \mu\left(\bigvee_{i\in\mathbb{N}} A_{i,k_i} \setminus A_{i,\infty}\right) + \mu(A) \\
&= \mu(A) + \varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, it follows that $\mu(B) \leq \mu(A)$. Because B was an arbitrary lower bound, this holds in particular for the greatest lower bound $S := \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)}$. Thus $\mu(S) \leq \mu(A)$. But, by Remark 3.5.2, $A \leq S$. Thus $\mu(A) \leq \mu(S)$, and we conclude that

$$\mu(A) = \mu(S).$$

Since $A \leq S$, it follows that $\mu(S \setminus A) = \mu(S) - \mu(A) = 0$. Since μ is strictly positive, $S \setminus A = \emptyset$, so $S = A$. That is,

$$\bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} A_{i,j} = A = \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} A_{i,s(i)}.$$

Thus $\mathbb{L}$ is weakly σ-distributive.

We will now show the topological characterisation of weakly σ-distributive Boolean algebras in terms of their Stone spaces. Recall that a subset of a topological space is said to be **nowhere dense** if its closure has empty interior, and a subset is said to be **meagre** if it is the union of countably many nowhere-dense sets. Clearly every nowhere-dense set is meagre. We will show that a Boolean algebra is weakly σ-distributive if and only if there are no "properly meagre" sets in its Stone space, i.e. if every meagre subset of its Stone space is nowhere dense.

Lemma 3.5.5

Let Y be a zero-dimensional topological space and Cl_Y the set of clopen subsets of Y. Then

(i) For all subsets $B \subseteq Y$,

$$\overline{B} = \bigcap \{A \in Cl_Y : A \supseteq B\}.$$

(ii) If $\mathbb{L}$ is a complete Boolean algebra satisfying the CCC and K is its Stone space, then for all $N \subseteq K$, N is nowhere dense if and only if there exists a decreasing sequence in $\mathbb{L}$ of supersets of N with infimum $\emptyset$.

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Proof:

(i): Since every clopen set is closed, and the closure of a set is the intersection of its closed supersets, $\overline{B} \subseteq \bigcap\{A \in Cl_Y : A \supseteq B\}$. Now let $x \in \bigcap\{A \in Cl_Y : A \supseteq B\}$, and let V be a closed superset of B. Since Y is zero-dimensional, V is the intersection of some collection of clopen sets, each of which is a superset of B. By assumption, x is in every clopen superset of B, so x is in V. This shows that x is in every closed superset of B, i.e. $x \in \overline{B}$. Hence $\bigcap\{A \in Cl_Y : A \supseteq B\} \subseteq \overline{B}$. Combining these results, we conclude that $\overline{B} = \bigcap\{A \in Cl_Y : A \supseteq B\}$.

(ii): Let $N \subseteq K$. By (i), $\overline{N} = \bigcap\{A \in \mathbb{L} : A \supseteq N\}$. By Theorem 1.1.42 and the completeness of $\mathbb{L}$,

$$\text{int}\left(\overline{N}\right) = \bigwedge\{A \in \mathbb{L} : A \supseteq N\}.$$

If there exists a sequence $(A_i)_{i\in\mathbb{N}}$ in $\mathbb{L}$ of supersets of N such that $\bigwedge_{i\in\mathbb{N}} A_i = \emptyset$, then $\text{int}\left(\overline{N}\right) = \bigwedge\{A \in \mathbb{L} : A \supseteq N\} \leq \bigwedge\{A_i : i \in \mathbb{N}\} = \emptyset$, so N is nowhere dense. Conversely, if N is nowhere dense, then $\text{int}\left(\overline{N}\right) = \bigwedge\{A \in \mathbb{L} : A \supseteq N\} = \emptyset$. By the CCC and Theorem 3.2.3, there exists a countable subset of $\{A \in \mathbb{L} : A \supseteq N\}$, say $\{A_i : i \in \mathbb{N}\}$, such that $\bigwedge_{i\in\mathbb{N}} A_i = \bigwedge\{A \in \mathbb{L} : A \supseteq N\} = \emptyset$. For each $i \in \mathbb{N}$, set $B_i = \bigwedge_{j=1}^{i} A_j$. The sequence $(B_i)_{i\in\mathbb{N}}$ is a decreasing sequence of supersets of N in $\mathbb{L}$, and $\bigwedge_{i\in\mathbb{N}} B_i = \emptyset$ in $\mathbb{L}$, as required.

Lemma 3.5.6

Let $\mathbb{L}$ be a complete Boolean algebra which satisfies the CCC, and let K be its Stone space. Then the following are equivalent:

(i) $\mathbb{L}$ is weakly σ-distributive.

(ii) Every meagre subset of K is nowhere dense.

Proof:

Suppose $\mathbb{L}$ is weakly σ-distributive, and let M be a meagre subset of K. Then $M = \bigcup_{i\in\mathbb{N}} N_i$ where each N_i is some nowhere-dense set in K. By Lemma 3.5.5 (ii), for each $i \in \mathbb{N}$ there exists a decreasing sequence $(P_{i,j})_{j\in\mathbb{N}}$ of supersets of N_i such that $\bigwedge_{j\in\mathbb{N}} P_{i,j} = \emptyset$. Hence $N_i \subseteq \bigcap_{j\in\mathbb{N}} P_{i,j}$ for each i, which implies that $M = \bigcup_{i\in\mathbb{N}} N_i \subseteq \bigvee_{i\in\mathbb{N}} P_{i,s(i)}$ for each $s \in \mathbb{N}^{\mathbb{N}}$. Since $\mathbb{L}$ is weakly σ-distributive,

$$\emptyset = \bigvee_{i\in\mathbb{N}} \bigwedge_{j\in\mathbb{N}} P_{i,j} = \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} P_{i,s(i)} = \left(\bigcap_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} P_{i,s(i)}\right)^{\circ}.$$

The last equality is due to Theorem 1.1.42. Thus $\bigcap_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} P_{i,s(i)}$ is a closed set with empty interior. Since $M \subseteq \bigvee_{i\in\mathbb{N}} P_{i,s(i)}$ for each $s \in \mathbb{N}^{\mathbb{N}}$, we have $M \subseteq \bigcap_{s\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i\in\mathbb{N}} P_{i,s(i)}$. Thus M is nowhere dense.

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Conversely, suppose every meagre subset of K is nowhere dense. Let $(A_{i,j})_{i,j\in\mathbb{N}}$ be a double sequence in $\mathbb{L}$ decreasing in the second index. As shown in Remark 3.5.2, $\bigvee_{i\in\mathbb{N}}\bigwedge_{j\in\mathbb{N}} A_{i,j} \leq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}}\bigvee_{i\in\mathbb{N}} A_{i,s(i)}$. Let

$$B := \left(\bigwedge_{s\in\mathbb{N}^{\mathbb{N}}}\bigvee_{i\in\mathbb{N}} A_{i,s(i)}\right) \setminus \left(\bigvee_{i\in\mathbb{N}}\bigwedge_{j\in\mathbb{N}} A_{i,j}\right).$$

We wish to show that $B = \emptyset$.

Since B is disjoint from $\bigwedge_{j\in\mathbb{N}} A_{i,j}$ for each i, B is disjoint from the interior of $\bigcap_{j\in\mathbb{N}} A_{i,j}$ for each i (Theorem 1.1.42 again). Hence $B\cap\bigcap_{j\in\mathbb{N}} A_{i,j}$ is included in the boundary of $\bigcap_{j\in\mathbb{N}} A_{i,j}$ and is therefore nowhere dense (Theorem 2.1.11 (v)). Thus $\bigcup_{i\in\mathbb{N}}\left(B\cap\bigcap_{j\in\mathbb{N}} A_{i,j}\right) = B\cap\left(\bigcup_{i\in\mathbb{N}}\bigcap_{j\in\mathbb{N}} A_{i,j}\right)$ is meagre, and therefore nowhere dense by assumption. Let $C = \bigcup_{i\in\mathbb{N}}\bigcap_{j\in\mathbb{N}} A_{i,j}$. Supposing, for the sake of a contradiction, that $B \neq \emptyset$, the fact that $B\cap C$ is nowhere dense implies that $B \not\subseteq \overline{B\cap C}$ (because B is a non-empty open set, while $\overline{B\cap C}$ has empty interior). Hence $B\setminus\overline{B\cap C}$ is a non-empty open set. Since K is zero-dimensional, there exists a non-empty clopen set U such that $U \subseteq B\setminus\overline{B\cap C}$. It follows U is a non-empty clopen subset of B which is disjoint from $\overline{B\cap C}$. Thus U is disjoint from $B\cap C$ and therefore from $C = \bigcup_{i\in\mathbb{N}}\bigcap_{j\in\mathbb{N}} A_{i,j}$. Hence U is disjoint from $\bigcap_{j\in\mathbb{N}} A_{i,j}$ for each i. Because $(A_{i,j})_{i,j\in\mathbb{N}}$ is decreasing in the second index, this means that, for each i, there exists a natural number, say $s(i)$, such that U is disjoint from $A_{i,s(i)}$. Thus $U\wedge\bigvee_{i\in\mathbb{N}} A_{i,s(i)} = \bigvee_{i\in\mathbb{N}} U\wedge A_{i,s(i)} = \bigvee_{i\in\mathbb{N}}\emptyset = \emptyset$. It follows that U is disjoint from $\bigwedge_{s\in\mathbb{N}^{\mathbb{N}}}\bigvee_{i\in\mathbb{N}} A_{i,s(i)}$, contradicting the fact that $\emptyset \neq U \subseteq B \subseteq \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}}\bigvee_{i\in\mathbb{N}} A_{i,s(i)}$. Thus B must be empty, showing that $\bigvee_{i\in\mathbb{N}}\bigwedge_{j\in\mathbb{N}} A_{i,j} = \bigwedge_{s\in\mathbb{N}^{\mathbb{N}}}\bigvee_{i\in\mathbb{N}} A_{i,s(i)}$ as required.

We finally conclude with the promised characterisation of the (σ-complete) Boolean algebras which admit a σ-additive strictly positive measure.

Theorem 3.5.7

Let $\mathbb{L}$ be a σ-complete Boolean algebra which admits a strictly positive measure μ, and let K be the Stone space of $\mathbb{L}$. If $\mathbb{L}$ is weakly σ-distributive, then there exists a strictly positive σ-additive measure on $\mathbb{L}$.

Proof:
For each $P \subseteq K$, let

$$\lambda(P) := \inf\left\{\sum_{i\in\mathbb{N}}\mu(A_i) : \{A_i : i\in\mathbb{N}\}\subseteq\mathbb{L} \text{ and } \bigcup_{i\in\mathbb{N}} A_i \supseteq P\right\}.$$

We claim that λ is an outer measure on K. Note that we are using the convention that an outer measure is a monotone, countably subadditive function on a *power set* algebra which maps the

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empty set to zero. We now verify this claim.

It is easy to see that $\lambda(\emptyset) = 0$. To see that λ is monotone, let $P \subseteq Q \subseteq K$. Every countable subset of $\mathbb{L}$ which covers Q is also a countable subset of $\mathbb{L}$ which covers P. Thus

$$\left\{\{A_i : i \in \mathbb{N}\} \subseteq \mathbb{L} : \bigcup_{i \in \mathbb{N}} A_i \supseteq Q\right\} \subseteq \left\{\{A_i : i \in \mathbb{N}\} \subseteq \mathbb{L} : \bigcup_{i \in \mathbb{N}} A_i \supseteq P\right\},$$

whence $\lambda(P) \leq \lambda(Q)$.

Now let $P_i \subseteq K$ for each $i \in \mathbb{N}$.

Let $\varepsilon > 0$. By the characterising property of the infimum, for each $i \in \mathbb{N}$ there exists a collection $\{A_{i,j} : j \in \mathbb{N}\} \subseteq \mathbb{L}$ with $\bigcup_{j \in \mathbb{N}} A_{i,j} \supseteq P_i$ such that $\sum_{j \in \mathbb{N}} \mu(A_{i,j}) \in \big[\lambda(P_i), \lambda(P_i) + \varepsilon/2^i\big)$. It follows that $\{A_{i,j} : i,j \in \mathbb{N}\}$ is a countable subset of $\mathbb{L}$ with

$$\bigcup_{i,j \in \mathbb{N}} A_{i,j} \supseteq \bigcup_{i \in \mathbb{N}} P_i.$$

By the definition of λ, we have

$$\begin{aligned}
\lambda\left(\bigcup_{i \in \mathbb{N}} P_i\right) &\leq \sum_{i,j \in \mathbb{N}} \mu(A_{i,j}) \\
&= \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \mu(A_{i,j}) \\
&\leq \sum_{i \in \mathbb{N}} \Big(\lambda(P_i) + \varepsilon/2^i\Big) \\
&= \left(\sum_{i \in \mathbb{N}} \lambda(P_i)\right) + \varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, it follows that

$$\lambda\left(\bigcup_{i \in \mathbb{N}} P_i\right) \leq \sum_{i \in \mathbb{N}} \lambda(P_i).$$

Thus λ is countably subadditive. Therefore λ is an outer measure on K.

Next, we show that each element of $\mathbb{L}$ is measurable with respect to λ. That is, for all $A \in \mathbb{L}$, we have $\lambda(P) = \lambda(P \cap A) + \lambda(P \setminus A)$ whenever $P \subseteq K$.

Let $A \in \mathbb{L}$ and $P \subseteq K$. By the countable subadditivity of λ,

$$\begin{aligned}
\lambda(P) &= \lambda\Big((P \cap A) \cup (P \setminus A) \cup \emptyset \cup \emptyset \cup \cdots\Big) \\
&\leq \lambda(P \cap A) + \lambda(P \setminus A).
\end{aligned}$$

To show the reverse inequality, let $\varepsilon > 0$. There exists a cover $\{B_i : i \in \mathbb{N}\} \subseteq \mathbb{L}$ of P such that $\sum_{i \in \mathbb{N}} \mu(B_i) \leq \lambda(P) + \varepsilon$. Then $\{B_i \cap A : i \in \mathbb{N}\}$ and $\{B_i \setminus A : i \in \mathbb{N}\}$

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are subsets of $\mathbb{L}$ and covers of $P \cap A$ and $P \setminus A$ respectively, and of course $B_i \cap A$ and $B_i \setminus A$ are disjoint for all i. Hence

$$\begin{aligned}\lambda(P \cap A) + \lambda(P \setminus A) &\leq \sum_{i \in \mathbb{N}} \mu(B_i \cap A) + \sum_{i \in \mathbb{N}} \mu(B_i \setminus A) \\ &= \sum_{i \in \mathbb{N}} \mu(B_i \cap A) + \mu(B_i \setminus A) \\ &= \sum_{i \in \mathbb{N}} \mu(B_i) \\ &\leq \lambda(P) + \varepsilon.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, it follows that $\lambda(P \cap A) + \lambda(P \setminus A) \leq \lambda(P)$. Therefore we conclude that $\lambda(P) = \lambda(P \cap A) + \lambda(P \setminus A)$. Since this holds for all $P \subseteq X$, this shows that A is measureable with respect to λ.

The measurable sets of an outer measure form a σ-algebra, and an outer measure becomes σ-additive when restricted to its measurable sets. This is a standard measure theory result; for a proof, see [Fre00, **113C**]. This implies that the λ-measurable sets form a σ-algebra containing every element of $\mathbb{L}$. It follows that $\sigma(\mathbb{L})$ (the smallest σ-algebra on K containing each element of $\mathbb{L}$) is a sub-σ-algebra of the σ-algebra of λ-measurable sets. In other words, the restriction of λ to $\sigma(\mathbb{L})$ is a σ-additive measure.

Next, let $b = \sup\{\lambda(B) : B \in \sigma(\mathbb{L}) \text{ and } B \text{ is nowhere dense}\}$. There exists a nowhere-dense set $C \in \sigma(\mathbb{L})$ such that $\lambda(C) = b$. This is easily verified:

For every $i \in \mathbb{N}$, there exists a nowhere-dense $B_i \in \sigma(\mathbb{L})$ such that $\lambda(B_i) > b - 1/i$. It follows that $\sup\{\lambda(B_i) : i \in \mathbb{N}\} = b$. Let $C = \bigcup_{i \in \mathbb{N}} B_i$. Then $\lambda(C) = b$, and by Lemma 3.5.6, C is nowhere dense.

Define $\lambda' : A \mapsto \lambda(A \setminus C)$ on $\sigma(\mathbb{L})$. Then λ' is a measure on $\mathbb{L}$, $\lambda' \leq \lambda$, and λ' vanishes on every nowhere-dense set. Indeed, let $B \in \sigma(\mathbb{L})$ be nowhere dense. By the measurability of C with respect to λ, we have $\lambda(B \cup C) = \lambda(C) + \lambda(B \setminus C) = \lambda(C) + \lambda'(B) = b + \lambda'(B)$. But since $B \cup C$ is a nowhere-dense superset of C, we have $\lambda(B \cup C) = b$, and thus $\lambda'(B) = 0$.

We now show that λ' is strictly positive on $\mathbb{L}$.

Let A be a non-empty element of $\mathbb{L}$. Then A is a non-empty open set, so $A \not\subseteq \overline{C}$, and thus there exists an x in the open set $A \setminus \overline{C}$. Since X is zero-dimensional, there exists a clopen set A' such that $x \in A'$ and $A' \subseteq A \setminus \overline{C} \subseteq A \setminus C$. Thus A' is a non-empty clopen subset of $A \setminus C$. It follows that $\lambda'(A) \geq \lambda'(A') = \lambda(A') > 0$. Since this holds for all non-empty $A \in L$, we conclude that λ' is strictly positive on $\mathbb{L}$.

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Finally, we show that λ' is σ-additive on $\mathbb{L}$.

Let $(A_i)_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathbb{L}$. Then $\left(\bigvee_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcup_{i\in\mathbb{N}} A_i\right)$ is nowhere dense, so

$$\begin{aligned}
\lambda'\left(\bigvee_{i\in\mathbb{N}} A_i\right) &= \lambda'\left(\left(\bigvee_{i\in\mathbb{N}} A_i\right) \setminus \left(\bigcup_{i\in\mathbb{N}} A_i\right)\right) + \lambda'\left(\bigcup_{i\in\mathbb{N}} A_i\right) \\
&= \lambda'\left(\bigcup_{i\in\mathbb{N}} A_i\right) \\
&= \lambda\left(\bigcup_{i\in\mathbb{N}} (A_i \setminus C)\right) \\
&= \sum_{i\in\mathbb{N}} \lambda(A_i \setminus C) \\
&= \sum_{i\in\mathbb{N}} \lambda'(A_i).
\end{aligned}$$

Thus, we have a strictly positive σ-additive measure λ' on $\mathbb{L}$.

Combining Theorems 3.5.4 and 3.5.7, we get the following:

Corollary 3.5.8

Let $\mathbb{L}$ be a σ-complete Boolean algebra with a strictly positive measure. Then there exists a strictly positive σ-additive measure on $\mathbb{L}$ if and only if $\mathbb{L}$ is weakly σ-distributive.

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