Distributional properties of ratios of gamma random variables in the context of quality control

by
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Declaration

I, Philip Albert Mijburgh, declare that this mini dissertation, submitted in partial fulfillment of the degree, MSc Mathematical Statistics, at the University of Pretoria, is my own work and has not been previously submitted at this or any other tertiary institution.

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Abstract

This study emanates from a practical problem in the statistical process control (SPC) environment where the quality of a process is monitored. Specifically, where the variance of a process is being assessed to be the same for all samples. In the traditional SPC environment the parameters of the underlying manufacturing process are usually assumed to be known. If, however, they are not known, they need to be estimated. Estimating these parameters and using them in control charts has many associated problems, especially when the samples that are used to calculate the estimates contain few data points. This study proposes a new control chart that is used to detect a shift in the process's variance, but that does not directly rely on parameter estimates, and as such overcomes many of these problem. The development of this newly proposed control chart gives rise to a new beta type distribution. An overview of the problem statement as identified in the field of SPC is given and the newly developed beta type distribution is proposed. Some statistical properties of this distribution are studied and the effect of different parameter choices on the shape of the distribution are investigated, with the focus specifically on the bivariate case. Through simulation, a comparison study is also performed, comparing the newly proposed model with a generalised version of the Q chart model, which was studied in depth by Adamski (2014).

Keywords Gamma Multivariate beta Shift in process variance Statistical Process Control

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Chapter 1

Introduction

This study emanates from a practical problem in the statistical process control (SPC) environment where the quality of a process is monitored. In SPC the main goal is the detection and elimination of unwanted variation in the production process, specifically when a process moves from an "in control" (IC) state to an "out of control" (OOC) state. A process can move from IC to OOC if the process location and/or the process spread experience a change. This is alternatively referred to as a "shift" in the process mean and/or the process variance. Over the past century many methods have been developed by various authors to aid in the detection of these shifts. One of the most common methods applied in detecting these shifts is the control chart, where the control chart is a graphical representation used to monitor some attribute of a process (such as the mean or variance) over time. The aim of a control chart is to determine whether a process is IC or OOC.

A control chart is a graphical display of successive values of a summary measure (called the charting/plotting statistic), calculated from samples of measurements taken on key quality characteristics, and plotted on the vertical axis of a graph against the sample number/time on the horizontal axis. The control chart traditionally has a centre line (CL) and two additional horizontal lines, one below the CL called the lower control limit (LCL) and one above called the upper control limit (UCL). Traditionally, if all of the charting statistics fall between the LCL and UCL the process is deemed to be IC, and if one of the charting statistics falls below LCL or above the UCL respectively the process is said to be OOC, and the control chart is said to "signal". Note that the description of control charts given above is meant as a general introduction. Many additions and alterations have been proposed to the standard control chart described above, with some bearing only a slight resemblance to what has been described. (An IC process is depicted in Figure 1.1.)

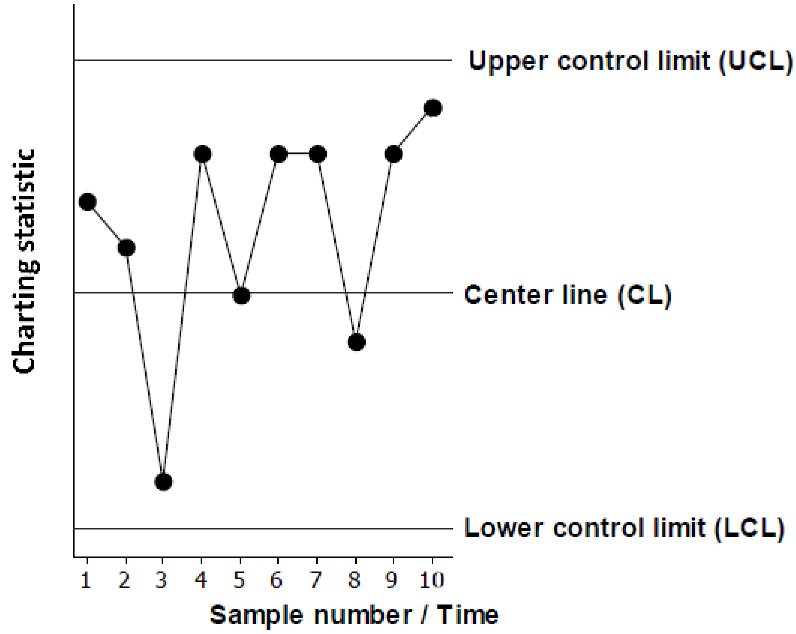

Figure 1.1: IC control chart.

The importance of establishing correct and accurate control limits should be obvious, but cannot be overstated. Having control limits that are overly conservative (narrow) will increase the probability of a control chart signaling that a process has gone OOC when indeed it has not, leading to an inflated type 1 error. However having lenient (wide) control limits will lead to decreased sensitivity in signaling when a process has indeed gone OOC and thus inflates the type 2 error of the procedure. Analytically calculating the true values of the control limits invariably involves the distribution of the charting statistics in some way. A flow diagram of the method used to derive charting statistics and the control limits, in a general parametric case, is given in Figure 1.2.

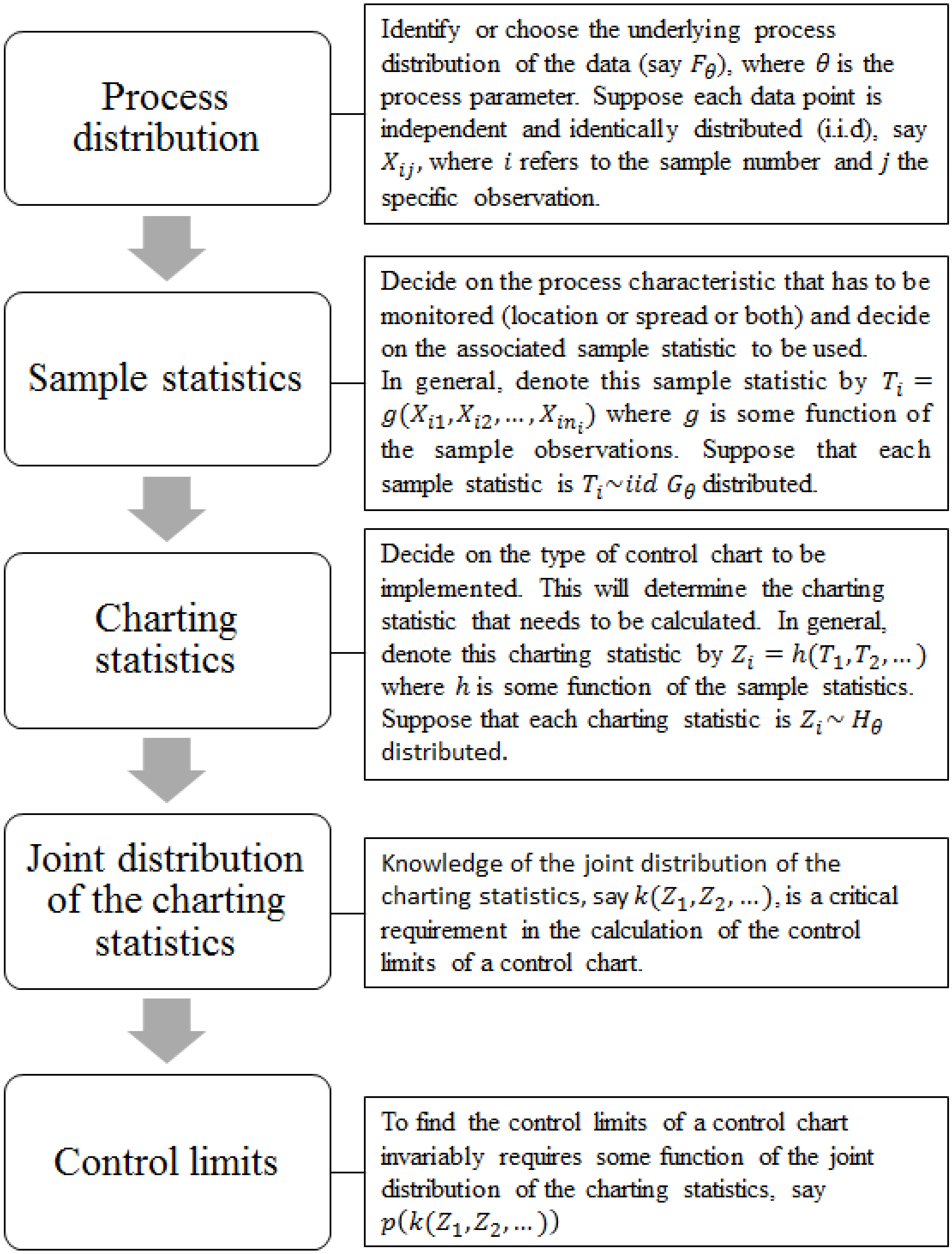

Figure 1.2: SPC derivation process (1).

Note that in general $F_\theta \neq G_\theta \neq H_\theta$, although in some special cases the distributions may be the same. It should also be noted that H_θ depends on G_θ and G_θ depends on F_θ. In other words the distribution of the charting statistics depends on the distribution of the sample statistics and the distribution of the sample statistic depends on the underlying process distribution. The construction of parametric control charts generally follow the method depicted in the flow diagram above (Figure 1.2). Also note that while X_{ij} and T_i are assumed to be i.i.d., this is not necessarily the case for Z_i. In the specific model that this study proposes, Z_i values are dependent non-identically distributed.

Some of the most notable, and commonly used control charts, are the Shewhart $\bar{X}$ and S control charts developed by Shewhart and Deming [51] (which are based on rational subgroups of data), the exponentially weighted moving average (EWMA) chart developed by Roberts [50] and the cumulative sum (CUSUM) chart which was first proposed by Page [46] (the latter two are traditionally based on individual observations). The practical uses of these charts vary, but in essence they are all parametric control charts that are used to detect unwanted variation in the production process. Typically the Shewhart charts are used to detect shifts larger than 1.5σ (where σ is the standard deviation of the process being monitored), while the EWMA and CUSUM charts are used to detect shifts smaller than 1.5σ. While a detailed discussion of these various charts is beyond the scope of this study, it should be noted that these control charting methods traditionally all assume that the true values of process parameters (θ in Figure 1.2) are known exactly. In other words, the assumption is made that process parameters, and thus the expected value and variance of the process, are not random variables. In practice, however, this is hardly ever the case, and these parameters need to be estimated.

If the expected value and variance of a process cannot be assumed to be known, then a two-phase approach is traditionally applied. During the first phase, commonly called "phase I" or the "retrospective phase", samples are drawn, and using these data points, parameters (like the mean and variance) are estimated, and charting statistics and trial control limits are calculated. If a charting statistic plots outside these control limits, the process will be deemed to be OOC. If this occurs, an investigation is conducted into potential causes for the process becoming OOC. If any causes can be found and eliminated, this is done, and any sample data that were affected by this change in the process are discarded. Revised parameter estimates and control limits are then calculated using the smaller data set. This process is repeated until all of the charting statistics fall within the control limits (i.e. until the trial process is deemed to be IC). When this occurs, all the remaining sample data is deemed to come from an IC process. Shewhart [52], p76 wrote, and Jensen et al. [25] reiterated, that "In the majority of practical instances, the most difficult job of all is to choose the sample that is to be used as a basis for establishing the tolerance range (control limits)". During "phase II" or the "prospective phase", a new, final, set of control limits is calculated based on the IC data set from phase I. A charting statistic is then calculated after each sample that is drawn during the production process, and is compared against the final control limits that were established at the start of phase II. If the charting statistic falls outside the control limits, there is strong reason to believe that the process has gone OOC and the reason for this shift in process quality should be investigated.

Montgomery [34] and Quesenberry [48] recommended collecting at least 25 to 30 samples, each containing at least 5 data points, during phase I, before acceptable estimates are obtained for use in phase II. However, Jones et al. [27, 28] and Jensen et al. [25] pointed out that using estimated parameters, even when using a moderate sample size, has an adverse effect on the performance of the control chart during phase II, and reduces the sensitivity of the control chart in detecting changes when the process goes OOC. As such a process is often dependent on very large historical data sets to ensure that the estimates are accurate enough for practical use.

Another problem with parameter estimation, which has been the focus of considerable study (see Cryer and Ryan [12], Sullivan and Woodall [53], Vargas [56], Derman and Ross [14] and Cruthis and Rigdon [11]) is the determination of which parameter estimates are the best suited for various types of control charts. While properties of the estimates, such as the mean square error (MSE), are usually used, another option that has been considered is the ability of certain estimates to detect a shift during phase I.

The practicality of collecting enough data to calculate the parameter estimates is another potential problem. The Shewhart, EWMA and CUSUM charts (that require a phase I analysis to gather data) may require many samples (and thus time, effort and money) before reasonable parameter estimates and control limits can be established to monitor phase II performance. Many processes, however, involve low-volume production, or production where high-volume sampling is too expensive or impractical. For these processes it is desirable to begin charting as early in the production process as possible, with limited if any historical data, and thus the traditional control charts are not ideally suited. To address this problem, many new "self-starting" control charts have been proposed. These charts continuously update their control limits and parameter estimates as each new sample is obtained, eliminating the need for many large phase I samples. The most commonly used of these charts are the self-starting CUSUM proposed by Hawkins [23] and the Q charts proposed by Quesenberry [47, 48, 49]. A lot of study has been done into Q charts in particular (see [47, 48, 49, 10]), and it has been found that one of the shortcomings of the chart is that if a shift occurs early in the production process, and is not detected, the chart becomes "contaminated" with this OOC data. This leads to an insensitivity in detecting shifts later on. This phenomenon is a common problem with self-starting charts and is called "masking of shifts".

To address these problems in the traditional control charts, this mini dissertation proposes and investigates a new control chart procedure to detect a sustained shift in the process variance, and in doing so develops a new multivariate beta distribution which can be seen to consist of ratios of linear combinations gamma random variables.

1.1 Problem statement

Let $\left(X_{i1}, X_{i2}, ..., X_{in_i}\right), i = 0, 1, 2, ..., m$ represent $m+1$ independent samples each of size $n_i \geq 2$ taken on a successive sequence of items. The order of these samples is important and cannot be re-ordered; in other words, they have a set sequence corresponding to the order in which they were taken. Assume that these values are i.i.d., having been collected from a $N\left(\mu, \sigma^2\right)$ distribution where both μ and σ^2 are unknown. (The less general case, when the mean of the process, μ, is known but the variance , σ^2, is unknown, is also considered later). Assume that from some time κ^*, $0 < \kappa^* < m$ onward, the process variance experiences a single sustained shift from σ^2 to $\sigma_1^2 = \lambda\sigma^2$ where $\lambda \neq 1$ and $\lambda > 0$. In essence, σ_1^2 is the process variance after the process experiences the shift at time κ^*. This shift at time κ^* occurs between two successive samples and, for notional simplicity, the sample immediately after the shift in the process variance occurs is called sample κ; therefore, from sample κ onward the process is considered to come from a $N\left(\mu, \lambda\sigma^2\right)$ distribution, which is OOC (see Figure 1.3). It is assumed that the shift in the process variance does not occur at some point during a sample, and thus an entire sample comes from the same distribution. The values of κ^* and λ are also assumed to be unknown, but deterministic in nature, i.e. not random variables.

In practice only an increase in the variance would likely be of concern; in other words, $\lambda > 1$, since if $0 < \lambda < 1$, the implication is that the process has become more stable. A very small variance might indicate, however, that the measurement system is faulty, or that the measuring mechanism in no longer accurate enough to detect the variance, since there will always be some inherent variability in the process. An alternative way in which $0 < \lambda < 1$ may occur is if the process is not in control when monitoring starts, and after some time the process becomes more stable. This will result in the process variance decreasing.

Note

1. At least two samples need to be drawn for a potential shift between them to be possible. As such, $m \geq 1$, so that there are at least two samples.

2. The sample sizes,n_i, need not stay constant between different samples, thus there is the possibility of having varying sample sizes between samples.

3. It was stated that each sample must consist of at least $n_i \geq 2$ observations. This restriction is necessary since the process mean and variance are both assumed to be unknown and have to be estimated. The sample variance with an estimated mean (see Equation (1.2)) by definition requires at least two data points in order to potentially be non-zero. If, however, the process mean is known, this restriction becomes $n_i \geq 1$ (see Equation (1.3)).

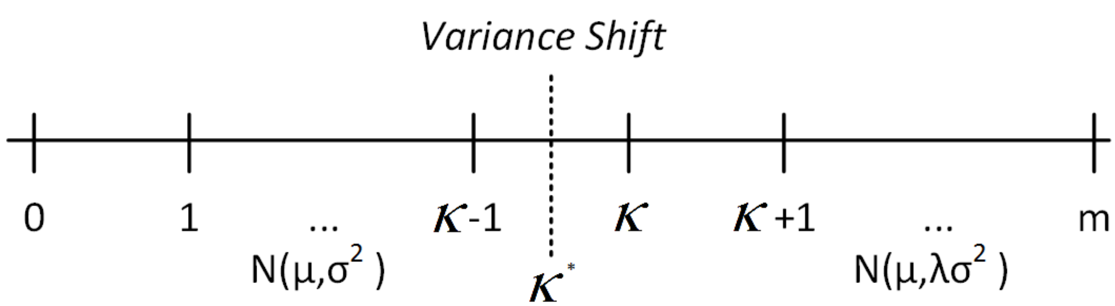

Figure 1.3: Process shift.

The problem of detecting a shift in the process variance, as mentioned above, has been addressed in a myriad of ways by many authors. While many of the methods mentioned vary slightly in their practical applications, in essence, they are all concerned with the detection of shifts in a process's variance. Human et al. [24] proposed a methodology for detecting a shift in the variance based on the Shewhart-type control chart. Lazariv et al. [30] monitored the variance of a process using the generalised likelihood ratio approach, the sequential probability ratio method, a generalised sequential probability ratio procedure, the Shiryaev–Roberts procedure and a generalised modified Shiryaev–Roberts approach. Zafar et al. [59] proposed using two-sided memory control charts, named progressive variance (PV) control charts, which are based on the sample variance, to monitor changes in process's dispersion. Eyvazian et al. [18] used exponentially weighted moving sample variance control charts (the variance analogue of the EWMA chart mentioned above in Chapter 1). Castagliola and Maravelakis [9] used a CUSUM approach, and Adamski

[1] proposed a method by which a closed form expression of the Q chart model (originally developed by Quesenberry [47]) could be calculated.

Besides developing the initial distribution theory that is required to construct the new control chart to detect a shift in the process variance, this study will also compare the performance of the newly proposed model, to that of another method that has been used to detect a sustained shift in a process variance, namely the Q chart model that was investigated in depth by Adamski [1].

1.2 Proposed methodology

What follows is the process that this study proposes in order to construct a new control chart which could be used to detect a shift in a process's variance.

If both the process mean (μ) and variance (σ^2) are unknown, they are estimated by the sample mean and sample variance respectively:

$$\bar{X}_i = \frac{\sum_{j=1}^{n_i} X_{ij}}{n_i}, i = 0, 1, 2, ..., m, \tag{1.1}$$

$$S_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \left(X_{ij} - \bar{X}_i\right)^2, i = 0, 1, 2, ..., m. \tag{1.2}$$

If, however, the mean is a fixed/deterministic value, say μ_0, but the variance is unknown, the variance is estimated as follows:

$$S_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} \left(X_{ij} - \mu_0\right)^2, i = 0, 1, 2, ..., m. \tag{1.3}$$

The problem of determining if a shift in the process variance has occurred can be divided into two segments, namely before the potential shift and after, as indicated below.

Before the shift

Samples:	$i = 0, 1, 2, ..., \kappa - 1$
Distribution:	$X_{ij} \sim N(\mu, \sigma^2)$

If the process mean is not known, then

$$\frac{(n_i - 1) S_i^2}{\sigma^2} \sim \chi^2 (n_i - 1). \tag{1.4}$$

If the process mean is known, then

$$\frac{(n_i) S_i^2}{\sigma^2} \sim \chi^2 (n_i). \tag{1.5}$$

After the shift

Samples:	$i = \kappa, \kappa + 1, ..., m$
Distribution:	$X_{ij} \sim N(\mu, \sigma_1^2 = \lambda \sigma^2)$

If the process mean is not known, then

$$\frac{(n_i - 1) S_i^2}{\sigma_1^2} \sim \chi^2 (n_i - 1). \tag{1.6}$$

If the process mean is known, then

$$\frac{(n_i) S_i^2}{\sigma_1^2} \sim \chi^2 (n_i). \tag{1.7}$$

Where $\chi^2(\alpha)$ is a chi-square random variable with α degrees of freedom, it's density function is defined in the Appendix, Result 1. $\chi^2(\alpha)$ can alternatively be expressed as a gamma random variable, $Gamma\left(\frac{\alpha}{2}, 2\right)$, where the first parameter, $\frac{\alpha}{2}$, is known as the shape parameter, and the second parameter, 2, is known as the scale parameter (see Bain and Engelhardt [5], pp268-269). The gamma random variable's density function is defined in the Appendix, Result 2.

In essence, this newly proposed model compares all the sample variances before a certain point (where the potential shift occurs), with all sample variances after the time of the shift. Thus, the procedure in general can be described as follows:

$$\begin{array}{ccc} S_0^2 & \text{is compared with} & S_1^2, S_2^2, ..., S_m^2 \\ S_0^2, S_1^2 & \text{is compared with} & S_2^2, S_3^2, ..., S_m^2 \\ S_0^2, S_1^2, S_2^2 & \text{is compared with} & S_3^2, S_4^2, ..., S_m^2 \\ & \text{and so forth until} & \\ S_0^2, S_1^2, ..., S_{m-1}^2 & \text{is compared with} & S_m^2. \end{array} \tag{1.8}$$

If there are $m + 1$ samples, and thus $m + 1$ sample variances, there will be m different comparisons made to determine whether, and if so where, the process experiences a change in its variance.

The charting statistics and control limits of a control chart are always derived under the assumption of the null hypothesis; (i.e. under the assumption that no shift in the process has occurred and thus that the process is IC). Assuming that no shift in the process variance has occurred, it is possible to construct a series of two sample statistics that correspond to the general procedure described above in Equation (1.8). Each statistic corresponds to whether at sample $r = \kappa$ the two independent samples (the sample variances before time r and the sample variances after and including time r) are from normal distributions with the same unknown variance σ^2. See Bain and Engelhardt[5] p402. This can alternatively be viewed as whether $\sigma^2 = \sigma_1^2$, implying that $\lambda = 1$. As such, it follows that detecting a shift in the process variance can be reduced to the following sequence of m hypothesis tests:

$$\begin{aligned} H_0 &: \sigma^2 = \sigma_1^2 \\ H_A &: \sigma^2 \neq \sigma_1^2 \end{aligned}$$

or alternatively

$$\begin{aligned} H_0 &: \lambda = 1 \\ H_A &: \lambda \neq 1. \end{aligned}$$

Note that the hypothesis tests described above correspond to detecting whether the process variance experienced an upward or a downward shift. As stated in Section 1.1, practically speaking, only an increase in the process variance will likely be of concern. If this is the case, the sequence of m hypothesis tests become:

$$\begin{aligned} H_0 &: \sigma^2 = \sigma_1^2 \\ H_A &: \sigma^2 < \sigma_1^2 \end{aligned}$$

or alternatively

$$\begin{aligned} H_0 &: \lambda = 1 \\ H_A &: \lambda > 1. \end{aligned}$$

The series of statistics (when both the process mean and variance are unknown) that make up the building blocks of the proposed process are given by

$$U_r^* = \frac{\left(\frac{\sum_{i=r}^{m}(n_i-1)S_i^2}{\lambda\sigma^2\sum_{i=r}^{m}(n_i-1)}\right)}{\left(\frac{\sum_{i=0}^{r-1}(n_i-1)S_i^2}{\sigma^2\sum_{i=0}^{r-1}(n_i-1)}\right)}, r = 1, 2, ..., m-1, m. \quad (1.9)$$

If the process mean is known in advance, these statistics look as follows:

$$U_r^{**} = \frac{\left(\frac{\sum_{i=r}^{m} n_i S_i^2}{\lambda\sigma^2\sum_{i=r}^{m} n_i}\right)}{\left(\frac{\sum_{i=0}^{r-1} n_i S_i^2}{\sigma^2\sum_{i=0}^{r-1} n_i}\right)}, r = 1, 2, ..., m-1, m. \quad (1.10)$$

In essence, the sample variances before the potential shift are pooled together, and the sample variances after the potential shift are pooled together. The numerator of the statistic at time r is the average, weighted by each statistic's degrees of freedom, of all the sample variances between and including times r and m, while the denominator is the corresponding weighted average of all the sample variances between and including times 0 and $r-1$, as graphically presented in Figure 1.4.

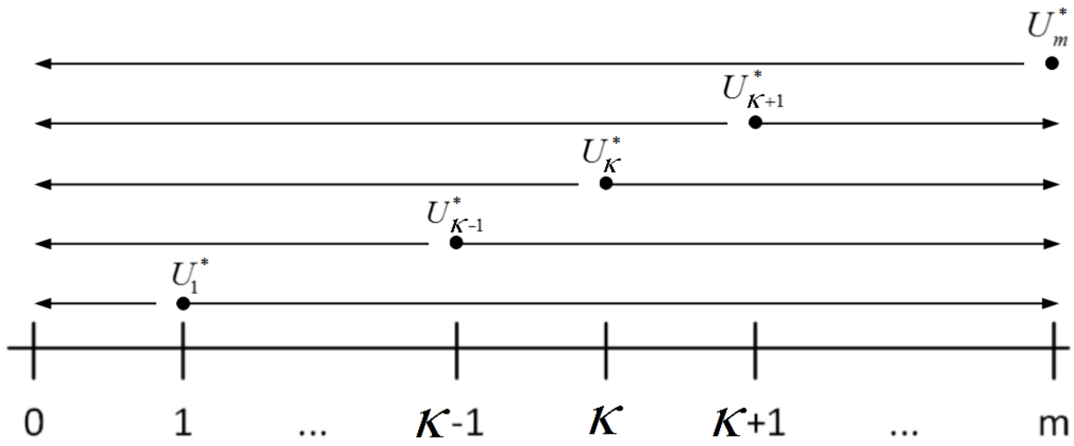

Figure 1.4: Building blocks.

Dividing both the numerator and the denominator by the variance of the process at that time ensures that each statistic follows a known distribution under the null hypothesis:

From equations (1.9), (1.4) and (1.6), it follows that if no shift has occurred in the process variance, $U_r^* = \frac{\left(\frac{\sum_{i=r}^{m}(n_i-1)S_i^2}{\lambda\sigma^2\sum_{i=r}^{m}(n_i-1)}\right)}{\left(\frac{\sum_{i=0}^{r-1}(n_i-1)S_i^2}{\sigma^2\sum_{i=0}^{r-1}(n_i-1)}\right)} = \frac{\left(\frac{\sum_{i=r}^{m}(n_i-1)S_i^2}{\sum_{i=r}^{m}(n_i-1)}\right)}{\left(\frac{\sum_{i=0}^{r-1}(n_i-1)S_i^2}{\sum_{i=0}^{r-1}(n_i-1)}\right)}, r = 1, 2, ..., m-1, m$ and thus each statistic U_r^* is univariate F distributed with $\sum_{i=r}^{m}(n_i-1)$ and $\sum_{i=0}^{r-1}(n_i-1)$ degrees of freedom, respectively. See Bain and Engelhardt [5] p275 and Result 3. Similarly, from equations (1.10), (1.5) and (1.7), it follows that if no shift has occurred in the process variance, and the mean of the process is known, $U_r^{**} = \frac{\left(\frac{\sum_{i=r}^{m} n_i S_i^2}{\lambda\sigma^2\sum_{i=r}^{m} n_i}\right)}{\left(\frac{\sum_{i=0}^{r-1} n_i S_i^2}{\sigma^2\sum_{i=0}^{r-1} n_i}\right)} = \frac{\left(\frac{\sum_{i=r}^{m} n_i S_i^2}{\sum_{i=r}^{m} n_i}\right)}{\left(\frac{\sum_{i=0}^{r-1} n_i S_i^2}{\sum_{i=0}^{r-1} n_i}\right)}, r = 1, 2, ..., m-1, m$ and thus each statistic U_r^{**} is univariate F distributed with $\sum_{i=r}^{m} n_i$ and $\sum_{i=0}^{r-1} n_i$ degrees of freedom, respectively. See Bain and Engelhardt [5] p275 and Result 3. Thus, it follows that under the null hypothesis, each of the statistics (whether the mean is known or not) follows an F distribution.

Note

1. To avoid redundancy in the study going forward, only the case where the mean of the process is unknown will be discussed and considered. It should be obvious, however, that exchanging one case for the other is a very simple procedure with the only difference being the degrees of freedom of the respective chi-square or F distributions.

2. From this point onward the term "charting statistics" will be used interchangeably with "statistics" for the statistics defined in Equation (1.9), since these statistics are analogous to the plotted points in Figure 1.1.

3. The statistics in equations (1.9) and (1.10) are in essence ratios of linear combinations of the sample variances. In the derivation chapter of this study, Chapter 4, the degrees of freedom that the sample variances are weighted by ($\sum_{i=r}^{m}(n_i-1)$ and $\sum_{i=0}^{r-1}(n_i-1)$) are removed. These factors are removed since they do not contain any random variables, and to simplify the derivations as well as to reduce the notational complexity. To indicate this omission of the factors the superscript * in Equation (1.9) is dropped, and therefore the statistics of interest become

$$U_r = \frac{\sum_{i=r}^{m} Y_i}{\sum_{i=0}^{r-1} Y_i}, r = 1, 2, ..., m-1, m, \tag{1.11}$$

where $Y_i \sim \chi^2(n_i - 1), i = 0, 1, ..., m$.

4. During the chapter where the control limits/critical values of the distribution are simulated, Chapter 5, these deterministic multiples are taken into account so as to give accurate, practically relevant values. i.e. The statistics being simulated are those in Equation (1.9), not Equation (1.11).

The first three steps of the SPC derivation process, as seen in Figure 1.2, have been described up to this point. Namely, the process distribution has been defined (Section 1.1), the sample process characteristic has been decided upon and consequently the sample statistics have been selected (Equation (1.2)), and lastly the charting statistics have been defined (Equation (1.9)). To illustrate this the SPC derivation process flow chart is updated with specific details from the proposed model (as it currently stands) in Figure 1.5.

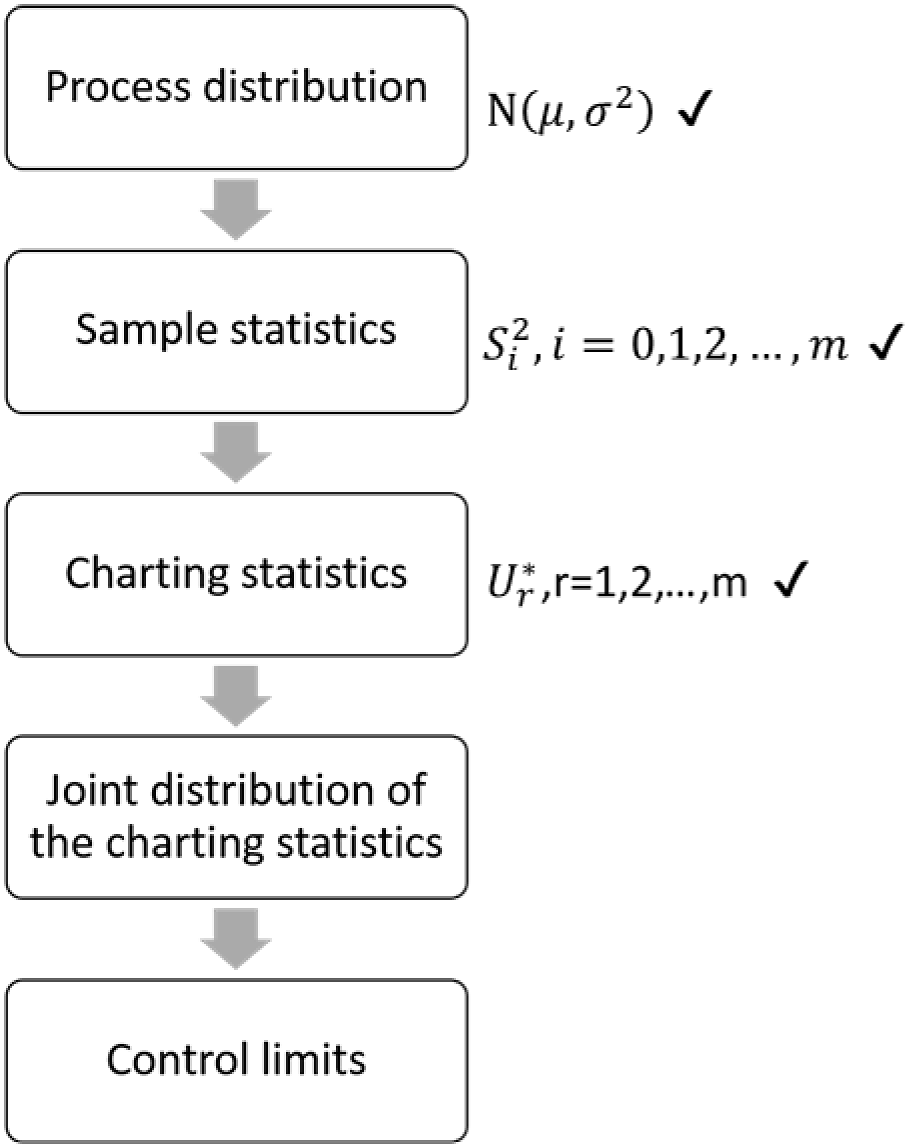

Figure 1.5: Updated SPC derivation process (2).

Now that the charting statistics have been defined, all that is necessary to have a fully functioning control chart is to derive the control limits. (Note that the term "control limits" is often used interchangeably in the relevant literature, as well as in this study, with the term "critical values".) These critical values depend on the joint density function of the charting statistics in Equation (1.9). The joint density function of $u_1, u_2, ..., u_m$, as well as some of its relevant properties are derived and investigated in Chapter 4. This is the main focus of this study. Suppose, for the purpose of this proposed methodology, that $f(u_1^*, u_2^*, ..., u_m^*)$ is the joint density function of the charting statistics in Equation (1.9) and that $F(u_1^*, u_2^*, ..., u_m^*)$ is the cumulative distribution function.

The reasoning behind the critical values that this study proposes is justified by inspecting the sequence of charting statistics in Equation (1.9). Suppose that an increase in the process variance does indeed occur at time $r = \kappa^*$, then:

- The statistic U_r^*'s numerator will contain only sample variances that come from a $N(\mu, \lambda\sigma^2), \lambda > 1$ distribution, whereas the denominator will contain only sample variances that come from a $N(\mu, \sigma^2)$ distribution.

- If k_1 is some integer value such that $1 \leq r-k_1 < r$, then statistic $U_{r-k_1}^* = \frac{\left(\frac{\sum_{i=r-k_1}^{r-1}(n_i-1)S_i^2 + \sum_{i=r}^{m}(n_i-1)S_i^2}{\lambda\sigma^2\left(\sum_{i=r-k_1}^{r-1}(n_i-1)+\sum_{i=r}^{m}(n_i-1)\right)}\right)}{\left(\frac{\sum_{i=0}^{r-k_1-1}(n_i-1)S_i^2}{\sigma^2\sum_{i=0}^{r-k_1-1}(n_i-1)}\right)}$ will contain k_1 sample variances in its numerator that are from a $N(\mu, \sigma^2)$ distribution. This will reduce the weighted average of the sample variances in $U_{r-k_1}^*$'s numerator in comparison to the numerator of U_r^*.

- Similarly, if k_2 is some integer value such that $r < r+k_2 \leq m$, then statistic $U_{r+k_2}^* = \frac{\left(\frac{\sum_{i=r+k_2}^{m}(n_i-1)S_i^2}{\lambda\sigma^2\sum_{i=r+k_2}^{m}(n_i-1)}\right)}{\left(\frac{\sum_{i=0}^{r-1}(n_i-1)S_i^2+\sum_{i=r}^{r+k_2-1}(n_i-1)S_i^2}{\sigma^2\left(\sum_{i=0}^{r-1}(n_i-1)+\sum_{i=r}^{r+k_2-1}(n_i-1)\right)}\right)}$ will contain k_2 sample variances in its denominator that are from a $N(\mu, \lambda\sigma^2)$ distribution. This will increase the weighted average of sample variances in $U_{r+k_2}^*$'s denominator in comparison to the denominator of U_r^*.

- Thus, any statistic other than the one immediately following the shift in the process variance, will contain either smaller (on average) sample variances in its numerator, or larger (on average) sample variances in its denominator. Either of these scenarios result in a high probability that all other statistics are smaller relative to U_r^*.

- This leads to the conclusion that the most probable place where an upwards shift in the process variance will be detected is at the statistic immediately following the shift. The value that this statistic assumes also has a high likelihood of being the maximum value of all the $U_r^*, r = 1, 2, ..., m-1, m$ statistics.

- As such, the most reasonable method of calculating the critical value (to detect an upwards shift in the process variance) of the control chart is to calculate the maximum order statistic of the charting statistics $U_r^*, r = 1, 2, ..., m-1, m$, (under the null hypothesis) and to set the critical value equal to some percentile of the cumulative distribution function of the maximum order statistic.

Using a similar but inverted argument, it can be justified that the critical value of the control chart should be set equal to some percentile of the minimum order statistic of the charting statistics, under the null hypothesis of no shift having occurred if the detection of a downward shift in the process variance is of concern.

Deriving the order statistics of the statistics in Equation (1.9) is a complex task due to the way in which the statistics are defined. Since the series of statistics in Equation (1.9) (or alternatively Equation (1.11)) are neither independent nor identically distributed, the process of finding the order statistics is drastically more complex in comparison to the i.i.d. case, and as such is beyond the scope of this study. Values for the

95^{th} percentile of the maximum order statistics are simulated though, in Chapter 5, for varying numbers of samples and sample sizes, so that the proposed control chart may be practically applicable.

The SPC derivation process flow chart is updated a final time in Figure 1.6 to concisely illustrate the main focus of work that is done in this study.

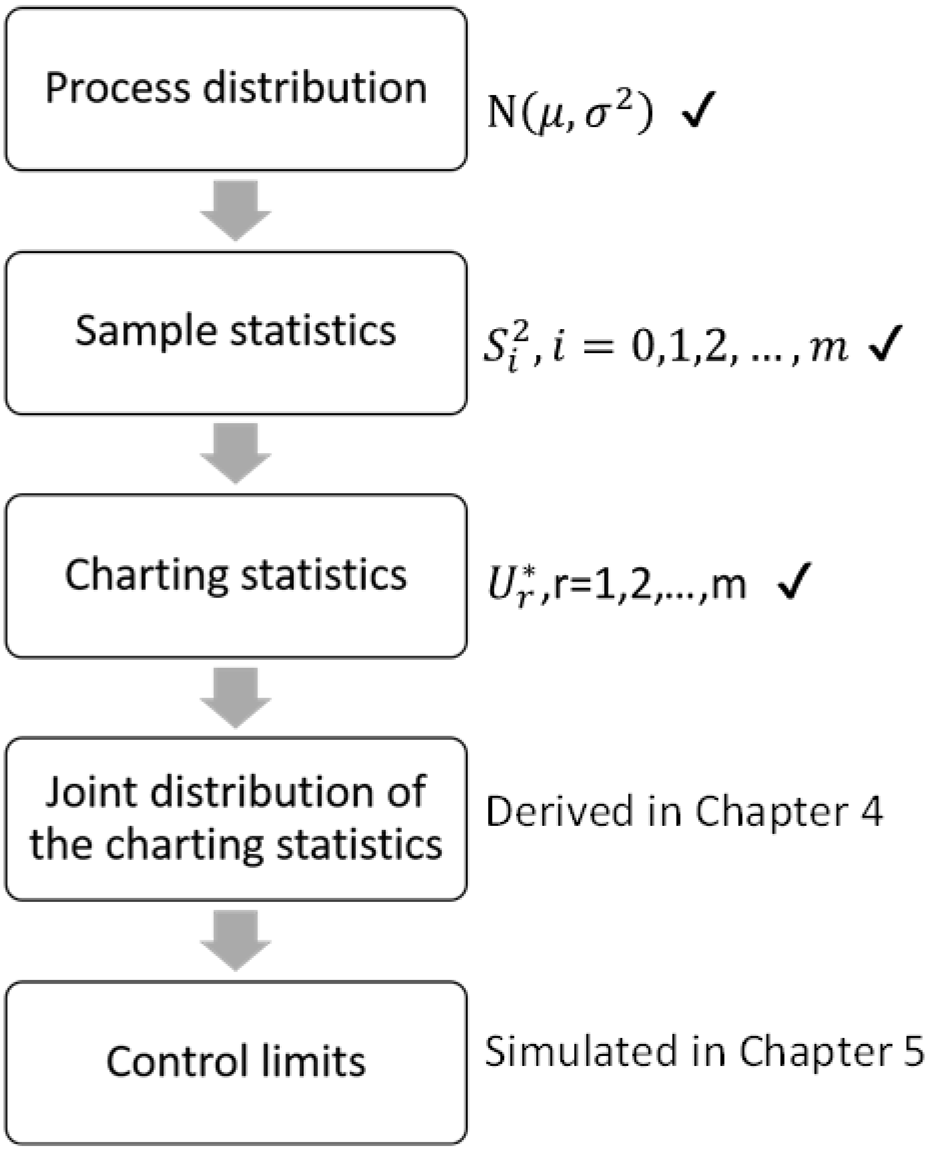

Figure 1.6: Updated SPC derivation process (3).

Although analytically calculating the order statistics (and consequently the critical values) is outside the scope of this mini dissertation, references to potential methods that could be used to derive the order statistics are provided. Methods to calculate dependent non-identically distributed order statistics have been proposed by David and Nagaraja [13], Barakat [7] and Güngör et al. [20]. Yiyu et al [58] also proposed an algorithm by which the order statistics could be computed if traditional Monte Carlo simulation proved to be ineffective or too computation-intensive. In general, the method of deriving the order statistics in the dependent non-identically distributed case (as proposed by David and Nagaraja [13] and expanded upon by Güngör et al. [20]) looks as follows:

1. Suppose that $U_r^*, r = 1, 2, ..., m-1, m$ is a set of dependent variables with joint cumulative distribution function $F\left(u_1^*, u_2^*, ..., u_m^*\right)$. Suppose that $U_{1:m}^* \leq U_{2:m}^* \leq ... \leq U_{m-1:m}^* \leq U_{m:m}^*$ is the corresponding set of order statistics and that $F_{r:m}(x)$ is the notation used for the cumulative distribution function of the r^{th} order statistic out of m possibilities (the cumulative distribution function of $U_{r:m}^*$), at point x. The order statistics can then be calculated using the equations below:

2. Define $F_{j:j}^{(i_{j+1},...,i_m)}(x) = P\left(max\left(U_{i_1}^*, U_{i_2}^*, ..., U_{i_j}^*\right) \leq x\right)$. In other words $F_{j:j}^{(i_{j+1},...,i_m)}(x)$ is the cumulative distribution function of the maximum order statistic, given that $U_{i_{j+1}}^*, U_{i_{j+2}}^*, ..., U_{i_m}^*$ have all been dropped from the sample.

3. Then it is given in David and Nagaraja [13] that

$$
\begin{aligned}
F_{r:m}(x) &= P\left(U_{r:m}^* \leq x\right) \\
&= \sum_{j=r}^{m} (-1)^{j-r} \begin{pmatrix} j-1 \\ r-1 \end{pmatrix} \sum_{1 \leq i_{j+1} < i_{j+2} < ... < i_m \leq m} F_{j:j}^{(i_{j+1},...,i_m)}. \qquad (1.12)
\end{aligned}
$$

Generalisations to Equation (1.12) have been developed by Maurer and Margolin [32] as well as by Barakat [7]. The expression for the order statistics given by Güngör et al. [20] is more simple, but assumes that there must be a discontinuity at point x in Equation (1.12). David and Nagaraja [13] also noted that a reasonably simple expressions for the cumulative distribution function of the order statistics may be possible if the statistics $u_1^*, u_2^*, ..., u_m^*$ are exchangeable; unfortunately, due to the practical interpretation of the statistics in an SPC environment, the statistics in Equation (1.9) are not exchangeable. As such, deriving a closed form expression for Equation (1.12) is a complex process and is beyond the scope of this study. Note however that the function $F_{j:j}^{(i_{j+1},...,i_m)}(x)$ will depend on the joint density function of the statistics $U_r^*, r = 1, 2, ..., m-1, m$ that is derived in Chapter 4 (without the deterministic multipliers).

1.3 Step by step breakdown and example

Breakdown

The process described above in Section 1.2 will now be broken down into practical steps and demonstrated using an example. As has been previously stated, only an increase in the process variance will likely be of practical concern, and thus it is the example that is provided for explanatory purposes.

Step 1: Draw samples from the production process and record the relevant measurements.

Step 2: For each sample, calculate the sample mean and sample variance, using equations (1.1) and (1.2) respectively.

Step 3: Calculate the series of charting statistics, using Equation (1.9). (Note that the first sample variance does not have an accompanying charting statistic.)

Step 4: Compare the largest statistic's value with the value in the simulated reference table, Table 5.2 in Chapter 5, for the respective sample number and sample size.

Step 5: If the sample statistic's value is larger than the critical value in the table, it is likely that a shift in the process variance has indeed occurred.

Example

Suppose that 20 samples are drawn, each of size 5, with the first 10 samples coming from a $N(10,1)$ distribution. Between the tenth and eleventh samples the process variance changes and the process distribution becomes $N(10,2)$. Table 1.1 contains the simulated data set as well as the sample variance and statistics at each time, with the sample variances and statistics rounded to three decimal places.

Sample (i)	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	Statistic (r)	S_r^2	U_r^*
0	9.479599256	10.56691524	9.644227827	9.922192028	8.044010508		0.863	NA
1	10.47194441	11.51790471	11.74472233	10.18951674	8.404691508	1	1.767	2.99
2	10.58891577	11.87454682	10.36923275	8.564766567	9.18642507	2	1.663	2.00
3	10.00081471	9.891379517	9.797217375	10.21570997	10.09582396	3	0.027	1.87
4	9.997518836	8.492887577	9.988882093	9.699631678	10.12378723	4	0.450	2.64
5	9.319976293	9.530450784	8.873042307	9.075308429	9.844033462	5	0.145	3.15
6	11.37215805	9.460822493	9.950656515	9.351992745	10.1385974	6	0.650	3.92
7	9.495036039	11.14637131	8.937737897	11.12620794	10.78840213	7	1.035	4.29
8	9.423366203	10.75234038	10.67888261	9.396195898	10.5978987	8	0.484	4.38
9	10.56624601	10.32832171	10.50961135	9.101225395	9.610517548	9	0.411	4.95
10	10.26408665	11.54398765	10.08200336	9.289054774	9.391387216	10	0.818	5.66
11	7.970629009	11.0515601	10.54150721	6.650280932	7.196981595	11	3.978	6.12
12	6.859576928	11.67550834	14.23916062	8.506222474	7.210240772	12	10.050	4.59
13	9.647151931	7.684586804	6.895711418	10.10194056	10.53416521	13	2.535	2.29
14	8.251119191	6.412758839	12.78404927	5.002383216	11.59927779	14	11.033	2.35
15	12.08430764	9.997772064	11.47053094	10.92906678	10.34158227	15	0.708	1.17
16	12.42280402	9.15204892	8.567232875	8.319021087	12.41161476	16	4.283	1.45
17	8.885708032	8.011719555	7.156831946	9.591389856	6.115358978	17	1.892	1.25
18	8.268276112	11.86606195	9.385458771	8.867808556	13.14192198	18	4.386	1.51
19	9.446564676	8.547714646	6.450908873	10.28868587	10.62451621	19	2.794	1.12

Table 1.1: Simulated data and statistics - Proposed method.

From Table 5.2 in Chapter 5, it follows that the 95^{th} percentile of the maximum order statistic, of the statistics in Equation (1.9), for $m + 1 = 20$ samples, each of size $n = 5$, is 5.863. Since the largest calculated statistic is $U^*_{11} = 6.125$, there is clear evidence to suggest that the process has indeed experienced an increase in its variance. In Figure 1.7 the charting statistics based on the simulated data in Table 1.1 are plotted. The red horizontal line corresponds to the critical value of 5.863, and the blue vertical line is a reference indicating when the process variance experienced the change.

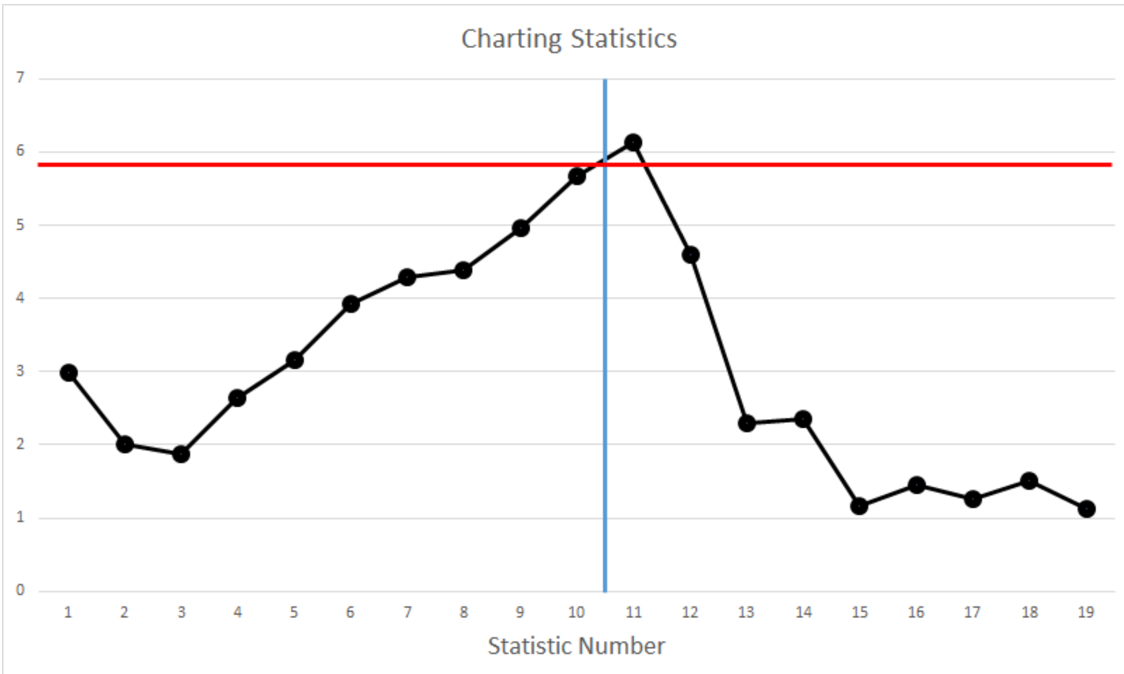

Figure 1.7: Proposed control chart example.

As can be seen, for this specific simulated data set, the control chart does indeed achieve its maximum charting statistic immediately after the change in the process variance, and the control chart does signal that a shift in the process variance occurred.

1.4 Methodology of Q chart

For completeness' sake, the charting statistics that were used and studied in depth by Adamski [1], which lead to a generalised beta distribution, are now shown and briefly described, as they will be applicable later on in this study, in chapters 2, 3 and 5.

The Q chart model studied by Adamski [1] emanates from a practical problem very similar to the one that this study researches. As such, the description of the process going from an IC state to an OOC state, as described in Section 1.1, is still applicable. The difference in methodology lies mainly in the way the Q chart is implemented and in the way sample variances are compared.

In the Q chart, an initial sample is drawn and is used to establish a base line value of the sample variance. A second sample is then drawn and compared to the first. If it is found that the first and second samples come from normal distributions with the same unknown variance σ^2, a third sample is drawn and compared to the weighted average of the first two samples. If the third sample is yet again found to come from a normal distribution with the same unknown variance, σ^2, as the previous pooled variances, this process is repeated, until a sample variance is found to come from a normal distribution with a different unknown variance than its pooled predecessors, and the process is deemed to be OOC. As such, the comparisons made between the sample variances look as follows:

$$\begin{array}{lll} S_1^2 & \text{is compared with} & S_0^2 \\ S_2^2 & \text{is compared with} & S_0^2, S_1^2 \\ S_3^2 & \text{is compared with} & S_0^2, S_1^2, S_2^2 \\ & \text{and so forth until} & \\ S_m^2 & \text{is compared with} & S_0^2, S_1^2, ..., S_{m-1}^2. \end{array} \tag{1.13}$$

Thus, once again, if there are $m+1$ samples and $m+1$ sample variances, there will be m different comparisons made to determine whether, and if so where, process experiences a change in variance.

Assuming that no shift in the process variance has occurred, it is possible to construct a series of two sample statistics that correspond to the general procedure described above in Equation (1.13). Each statistic corresponds to whether at sample $r = \kappa$, the two independent samples (the sample variances before time r and the sample variances after time r) are from normal distributions with the same unknown variance σ^2. (See Bain and Engelhardt[5] p402.) This can alternatively be viewed as whether $\sigma_1^2 = \sigma^2$, implying that $\lambda = 1$. As such, it follows that detecting a shift in the process variance can again be reduced to the following sequence of m hypothesis tests:

$$\begin{array}{ll} H_0: & \sigma^2 = \sigma_1^2 \\ H_A: & \sigma^2 \neq \sigma_1^2 \end{array}$$

or alternatively

$$\begin{array}{ll} H_0: & \lambda = 1 \\ H_A: & \lambda \neq 1. \end{array}$$

Note that the hypothesis tests described above correspond to detecting whether the process variance experienced an upward or a downward shift. As stated in Section 1.1, practically speaking, only an increase in the process variance will likely be of concern. If this is the case, the sequence of m hypothesis tests become:

$$\begin{array}{ll} H_0: & \sigma^2 = \sigma_1^2 \\ H_A: & \sigma^2 < \sigma_1^2 \end{array}$$

or alternatively

$$\begin{array}{ll} H_0: & \lambda = 1 \\ H_A: & \lambda > 1. \end{array}$$

The series of statistics that make up the building blocks of the Adamski [1] distribution are given by

$$T_r^{**} = \frac{\left(\frac{n_r S_r^2}{\sigma^2 n_r}\right)}{\left(\frac{\sum_{i=0}^{r-1} n_i S_i^2}{\lambda\sigma^2 \sum_{i=0}^{r-1} n_i}\right)}, r = 1, 2, ..., m-1, m. \quad (1.14)$$

Adamski [1] assumed that the mean of the underlying process was known. If this is not the case and the sample mean has to be estimated, the degrees of freedom merely change and the statistics become:

$$T_r^{*} = \frac{\left(\frac{(n_r-1) S_r^2}{\sigma^2 (n_r-1)}\right)}{\left(\frac{\sum_{i=0}^{r-1} (n_i-1) S_i^2}{\lambda\sigma^2 \sum_{i=0}^{r-1} (n_i-1)}\right)}, r = 1, 2, ..., m-1, m. \quad (1.15)$$

For a graphical representation of the statistics in Equation (1.15), see Figure 1.8.

Note that these charting statistics are set up under the null hypothesis (i.e. under the assumption that no shift in the process variance has occurred). If this is the case, then the statistics defined by Adamski [1] are known to follow F distributions. Each T_r^{**} statistic is F distributed with n_r and $\sum_{i=0}^{r-1}(n_i)$ degrees of freedom respectively, whereas each T_r^* statistic is F distributed with $n_r - 1$ and $\sum_{i=0}^{r-1}(n_i - 1)$ degrees of freedom respectively. (See Bain and Engelhardt [5] p275.)

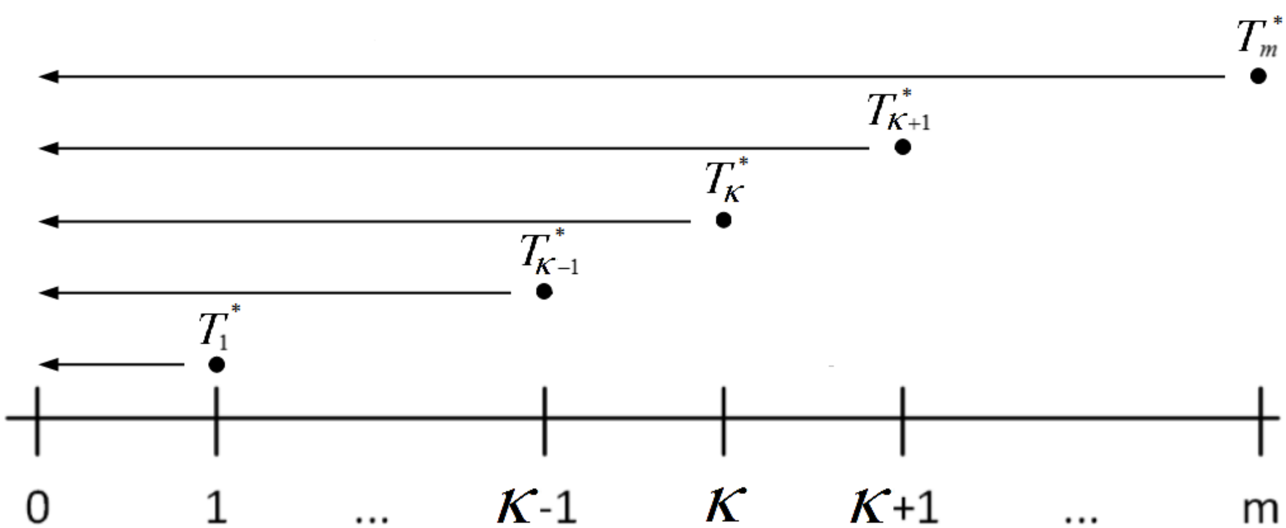

Figure 1.8: Q chart statistics.

The potential for "masking of shifts" to occur with the Adamski [1] model should be apparent. For example, if the statistic at time r indicates that no shift has occurred, when indeed it has, the OOC sample variance will be included in the next statistic's denominator. If, for instance, there was in increase in the process variance, the next statistic's denominator will be inflated resulting in an artificially shrunk test statistic, decreasing the chance of detecting the shift. This result was first observed by Quesenberry [47], and is noted in Adamski [1], where they discerned that the probability of detecting a shift in the process is at its highest immediately after the shift occurs, and before the masking effect can take place.

Note that the statistics in Equation (1.15) are again, in essence, the ratios of linear combinations of the sample variances. As in Adamski [1], from this point onwards the degrees of freedom that the sample variances are weighted by (n_r and $\sum_{i=0}^{r-1}(n_i)$) are removed. These factors are removed to simplify the theoretical derivations and to deduce the notational complexity in Chapter 3. To indicate this omission of the factors we drop the superscript * in Equation (1.15), and therefore the statistics of interest become:

$$T_r = \frac{Y_r}{\sum_{i=0}^{r-1} Y_i}, r = 1, 2, ..., m-1, m, \quad (1.16)$$

where $Y_i \sim \chi^2(n_i - 1), i = 0, 1, ..., m$.

Note

1. During the chapter where the control limits/critical values of the distribution are simulated, Chapter 5, these deterministic multiples are taken into account so as to give accurate, practically relevant values. i.e. the statistics being simulated are those in Equation (1.15), not Equation (1.16).

2. Since the practical application of the of the Adamski [1] distribution differs slightly from the application that this study proposes, the critical values that were used in their study were derived in an alternative way to the method proposed in Section 1.2. To make their model directly comparable to the one that this study proposes, a new set of critical values are simulated in Chapter 5. The proposed critical value of the Q chart in this study follows the same logic that is applied in Section 1.2, namely that the critical value is set to some percentile of the maximum order statistic of the statistics in Equation (1.15). Values for the 95^{th} percentile of these order statistics are simulated in Chapter 5, Table 5.1.

Example

Using the data set from Table 1.1, it can be seen that the Q chart also detects the increase in the process variance. The obvious difference between the two methodologies being that, while it took the Q chart two samples after the shift had occurred to detect the shift in the process variance (see Figure 1.9), the newly proposed methodology detected the shift after a single sample. Note, however, that this observation is made based on a single data set, and as such is insufficient to make general statements about the comparative efficacy of the two competing models. A much more thorough comparison is made in Chapter 5.

Sample (r)	X_{r1}	X_{r2}	X_{r3}	X_{r4}	X_{r5}	S_r^2	U_r^*
0	9.479599256	10.56691524	9.644227827	9.922192028	8.044010508	0.863	NA
1	10.47194441	11.51790471	11.74472233	10.18951674	8.404691508	1.767	2.047
2	10.58891577	11.87454682	10.36923275	8.564766567	9.18642507	1.663	1.264
3	10.00081471	9.891379517	9.797217375	10.21570997	10.09582396	0.027	0.019
4	9.997518836	8.492887577	9.988882093	9.699631678	10.12378723	0.450	0.417
5	9.319976293	9.530450784	8.873042307	9.075308429	9.844033462	0.145	0.152
6	11.37215805	9.460822493	9.950656515	9.351992745	10.1385974	0.650	0.794
7	9.495036039	11.14637131	8.937737897	11.12620794	10.78840213	1.035	1.302
8	9.423366203	10.75234038	10.67888261	9.396195898	10.5978987	0.484	0.587
9	10.56624601	10.32832171	10.50961135	9.101225395	9.610517548	0.411	0.522
10	10.26408665	11.54398765	10.08200336	9.289054774	9.391387216	0.818	1.091
11	7.970629009	11.0515601	10.54150721	6.650280932	7.196981595	3.978	5.264
12	6.859576928	11.67550834	14.23916062	8.506222474	7.210240772	10.050	9.812
13	9.647151931	7.684586804	6.895711418	10.10194056	10.53416521	2.535	1.475
14	8.251119191	6.412758839	12.78404927	5.002383216	11.59927779	11.033	6.209
15	12.08430764	9.997772064	11.47053094	10.92906678	10.34158227	0.708	0.296
16	12.42280402	9.15204892	8.567232875	8.319021087	12.41161476	4.283	1.871
17	8.885708032	8.011719555	7.156831946	9.591389856	6.115358978	1.892	0.786
18	8.268276112	11.86606195	9.385458771	8.867808556	13.14192198	4.386	1.845
19	9.446564676	8.547714646	6.450908873	10.28868587	10.62451621	2.794	1.125

Table 1.2: Simulated data and statistics - Q chart.

Figure 1.9: Q chart of simulated data.

From Table 5.1 in Chapter 5, it follows that the 95^{th} percentile of the maximum order statistic, of the statistics in Equation (1.15), for $m = 20$ samples, each of size $n = 5$, is 7.688. Since the largest calculated statistic is $T_{12}^{*} = 9.812$, there is clear evidence to suggest that the process has indeed experienced an increase in its variance. In Figure 1.9 the charting statistics based on the simulated data in Table 1.2 are plotted. The red horizontal line corresponds to the critical value of 7.688, and the blue vertical line is a reference indicating when the process variance experienced the change.

1.5 Objectives

- Lay the initial distributional foundation needed for a closed form expression of the critical values of the proposed model, as described in Section 1.1.
- Derive the joint density function of the statistics given in Equation (1.11), with a specific focus on the bivariate case.
- Investigate the properties of the above mentioned joint density function, including the relationship between the proposed function and many commonly used bivariate beta densities.
- Compare, through simulation, the efficacy of the control chart this study proposes with that of another self-starting chart, specifically the Q chart form investigated by Adamski [1].

1.6 Key contributions

- A new methodology is proposed to detect a shift in the variance of a process.
- A new bivariate beta distribution is added to the literature.
- The generalised bivariate beta distribution derived by Adamski [1] is further generalised.
- The relationships among some of most commonly used bivariate beta distributions is derived. These relationships have never been published in such detail, and are of importance, especially during theoretical derivations of complex bivariate beta distributions.

1.7 Study outline

- **Chapter 2** serves as a short literature review of some of the most commonly used bivariate beta distributions. It also shows that the model that this study proposes, as well as that of Adamski [1], in the bivariate case, are beta distributions. Relationships between all of the mentioned distributions are derived and presented in a graphical manner.
- In **Chapter 3** the "generalised beta distribution" derived by Adamski [1] is further generalised. The distribution was originally derived in terms of chi-square random variables. This study derives it in terms of gamma random variables. The reason for this is primarily the extra flexibility gained by using gamma random variables, each with two parameters, in contrast to the chi-square random variables, each with only one parameter. This added flexibility will facilitate the ease with which their

model can be compared with the one that this study proposes, as well as to enable the hypothesis being investigated to be easily varied. While the multivariate case is derived and some special cases are given, the focus will mainly be on the bivariate case and its associated properties.

- **Chapter 4** deals with the joint density function which emanates from Equation (1.11). Initially, the bivariate joint density function is derived, and followed by its marginal densities, conditional densities and the product moment. These are accompanied by exploratory shape analyses. Lastly the multivariate joint density function is derived. Special cases are again presented.

  Note that chapters 2 to 4 focus mainly on bivariate distributions. There are many reasons why the focus of this study throughout is mainly on the bivariate cases. The primary one being that focusing on the bivariate cases dramatically reduces the complexity of the theoretical derivations, while allowing insights into the behaviour and properties of the distributions. The methods used during the bivariate derivations are also likely to be similar in nature to the methods required during higher dimensional derivations. Focusing on the bivariate case also allows graphical representations of the distributions to be plotted, which would be impossible for higher dimensions. Another benefit of working with only two dimensions is that there is a vast amount of literature on other types of bivariate beta distributions, which make comparisons between this proposed model and others possible.

- In **Chapter 5** the model that this study proposes is compared to the Q chart studied by Adamski [1], this will be done through a simulation study.

- **Conclusive remarks** and areas for further research are provided.

- **Appendix** contains the series of **Results** used throughout the study. The Appendix contains not only the definitions and theorems used in this study, but also the SAS code used in Chapter 5.

Chapter 2

Positioning

2.1 Introduction

The aims of this chapter are to define some of the most common bivariate beta distributions (Section 2.2), to derive the relationships between them, and to relate these distributions to the bivariate distribution derived by Adamski [1] and the one proposed by this study (Section 2.3).

Most of these relationships, though very useful, have never been published. Their usefulness stems from the fact that many bivariate beta distributions have properties, specifically the product moment and marginal distributions, which are very difficult, if not impossible to derive from first principles. These relationships between the different bivariate beta distributions provide a way of making transformations to overcome this problem. One of the transformations derived in this chapter is essential in deriving the product moment of the model that this study proposes in Section 4.2.

There are many different ways in which bivariate beta distributions have been defined and derived in the literature. Some authors [2, 17] define the distributions as a ratio between chi-square variables, others [4, 21, 31] define them in terms of gamma variables, and a third group [54, 44] state them as specific cases and transformations of the Dirichlet distribution. In essence, however, they are distributions that are comprised of ratios of linear combination of either chi-square or gamma random variables [45, 6, 43].

Suppose that, in Equation (1.16), $m = 2$. It then follows that the two statistics from Adamski [1] are given by

$$\begin{aligned} T_1 &= \frac{Y_1}{Y_0} \\ T_2 &= \frac{Y_2}{Y_0 + Y_1}, \end{aligned} \tag{2.1}$$

where $Y_i \sim \chi^2(n_i - 1), i = 0, 1, 2$.
Similarly suppose that, in Equation (1.11), $m = 2$. It then follows that the two statistics that this study proposes are given by

$$
\begin{aligned}
U_1 &= \frac{Y_1+Y_2}{Y_0}\\
U_2 &= \frac{Y_2}{Y_0+Y_1},
\end{aligned}
\tag{2.2}
$$

where $Y_i \sim \chi^2(n_i-1), i=0,1,2$.
It is obvious that the statistics in equations (2.1) and (2.2) are comprised of ratios of linear combinations chi-square random variables, and thus will follow some form of bivariate beta distribution. From this point onward the joint distribution resulting from Equation (2.1) will be deferred to as the "bivariate beta type VII" distribution, and the joint distribution of Equation (2.2) will be called the "bivariate beta type VIII" distribution. The derivation and investigation of this bivariate beta type VIII distribution is the main focus of this study.

2.2 Bivariate beta distributions: type I to VIII

What follows are some well-known bivariate beta distributions as well as their relationships to each other:

Let $Y_1 \sim \chi^2(\alpha)$, $Y_2 \sim \chi^2(\beta)$ and $Y_3 \sim \chi^2(\gamma)$ be independently distributed chi-square random variables. These are the "building blocks" that make up the bivariate beta distributions in this chapter. (Note that in sections 1.2 and 1.4 the random variables that formed the basis of the statistics in equations (1.11) and (1.16) were $Y_i \sim \chi^2(n_i-1), i=0,1,...,m$, where the indices of the variables were defined to start at 0, to correspond with the sample number in the SPC setting. The degrees of freedom of the random variables were also defined to be related to the sample size of each sample. For the remainder of this chapter, the indices and degrees of freedom of the random variables are presented in a manner more in keeping with the general construction of bivariate beta distributions as in the majority of the applicable literature. (From a practical perspective, however, this change is of no consequence, since equivalence can easily be established by equating the parameters $n_0-1=\alpha, n_1-1=\beta, n_2-1=\gamma$ and subtracting one from the indices.)

Bivariate beta type I

Let

$$Q_1 = \frac{Y_1}{Y_1+Y_2+Y_3} \text{ and } Q_2 = \frac{Y_2}{Y_1+Y_2+Y_3},$$

then the joint distribution of Q_1 and Q_2 is called a bivariate beta type I distribution with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} > 0$. The joint density function of this distribution is given by

$$f_{BI}(q_1,q_2;\tfrac{\alpha}{2},\tfrac{\beta}{2},\tfrac{\gamma}{2}) = \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})} q_1^{\frac{\alpha}{2}-1} q_2^{\frac{\beta}{2}-1} (1-q_1-q_2)^{\frac{\gamma}{2}-1} \quad , \quad 0<q_1,q_2<1 \text{ and } q_1+q_2 \leq 1, \tag{2.3}$$

where $\Gamma(.)$ is the gamma function, as defined in Result 6.

The notation used in this chapter is as follows: $f_{BI}(q_1, q_2; \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ indicates a density function (f), where the type of beta function (type I (BI)) is indicated in the subscript, between two variables (q_1, q_2), with parameters ($\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$). In Figure 2.2 this will be denoted alternatively as $(Q_1, Q_2) \sim B_I(\alpha, \beta, \gamma)$.

The multivariate generalisation of this distribution is called the Dirichlet type I distribution and has been studied in depth (see Gupta and Richards [22] and Balakrishnan and Lai [6]). It has also been used in a wide variety of practical applications. It has been used in consumer behaviour studies (Wrigley and Dunn [57]), to model activity times in programme evaluation and review technique (PERT) networks [33], and in Bayesian statistics [3], to name but a few applications.

Bivariate beta type II

Let

$$V_1 = \frac{Y_1}{Y_3} \text{ and } V_2 = \frac{Y_2}{Y_3},$$

then the joint distribution of V_1 and V_2 is called a bivariate beta type II distribution with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} > 0$. The joint density function of this distribution is given by

$$f_{BII}(v_1, v_2; \tfrac{\alpha}{2}, \tfrac{\beta}{2}, \tfrac{\gamma}{2}) = \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})} v_1^{\frac{\alpha}{2}-1} v_2^{\frac{\beta}{2}-1} (1 + v_1 + v_2)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \quad , \quad v_1, v_2 > 0. \tag{2.4}$$

There are other ways in which this distribution can be obtained. Tiao and Guttman [55], in addition to studying the properties of this distribution in great detail, also noted that the distribution could be obtained as transformation of a bivariate case of the Dirichlet distribution. They subsequently called the above distribution the "inverted Dirichlet distribution".

Bivariate beta type III

If

$$W_1 = \frac{Y_1}{Y_1 + Y_2 + 2Y_3} \text{ and } W_2 = \frac{Y_2}{Y_1 + Y_2 + 2Y_3},$$

then the joint distribution of W_1 and W_2 is called a bivariate beta type III distribution with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} > 0$. The multivariate generalisation was derived and studied by Cardeño et al. [8]. The joint density function of this bivariate distribution is given by

$$
\begin{aligned}
f_{BIII}(w_1, w_2; \tfrac{\alpha}{2}, \tfrac{\beta}{2}, \tfrac{\gamma}{2}) &= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})} 2^{\frac{\alpha}{2}+\frac{\beta}{2}} w_1^{\frac{\alpha}{2}-1} w_2^{\frac{\beta}{2}-1} (1-w_1-w_2)^{\frac{\gamma}{2}-1} \\
&\quad \times \; (1+w_1+w_2)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \qquad , \; 0 < w_1, w_2 < 1 \text{ and } w_1 + w_2 \leq 1.
\end{aligned} \tag{2.5}
$$

Just as in the type I case, this density function is also defined on a bounded interval, and as such, the bivariate beta type III is often used as an alternative to the bivariate beta type I. Ehlers [15] considered the case when $W_1 = \frac{Y_1}{Y_1+Y_2+cY_3}$ and $W_2 = \frac{Y_2}{Y_1+Y_2+cY_3}$, which is a generalisation of the above type III distribution.

Bivariate beta type IV

Let

$$
X_1 = \frac{Y_1}{Y_1+Y_3} \text{ and } X_2 = \frac{Y_2}{Y_2+Y_3},
$$

then the joint distribution of X_1 and X_2 is called a bivariate beta type IV distribution with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} > 0$. The joint density function of this distribution is given by

$$
\begin{aligned}
f_{BIV}(x_1, x_2; \tfrac{\alpha}{2}, \tfrac{\beta}{2}, \tfrac{\gamma}{2}) &= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})} x_1^{\frac{\alpha}{2}-1} x_2^{\frac{\beta}{2}-1} (1-x_1)^{\frac{\beta}{2}+\frac{\gamma}{2}-1} (1-x_2)^{\frac{\alpha}{2}+\frac{\gamma}{2}-1} \\
&\quad \times \; (1-x_1x_2)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \qquad , \; 0 < x_1, x_2 < 1.
\end{aligned} \tag{2.6}
$$

This distribution is also called the Jones model and has its roots in the distribution proposed by Libby and Novick [31]. However, it was more explicitly derived by Jones [29] and Olkin and Liu [44].

Bivariate beta type V

If

$$
C_1 = \frac{aY_1}{aY_1+bY_2+cY_3} \text{ and } C_2 = \frac{bY_2}{aY_1+bY_2+cY_3},
$$

then the joint distribution of C_1 and C_2 is called a bivariate beta type V distribution with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, a, b, c > 0$. The joint density function of this distribution is given by

$$
\begin{aligned}
f_{BV}(c_1, c_2; \alpha, \beta, \gamma, \tfrac{\alpha}{2}, \tfrac{\beta}{2}, \tfrac{\gamma}{2}) &= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})} \left(\frac{c}{a}\right)^{\frac{\alpha}{2}} \left(\frac{c}{b}\right)^{\frac{\beta}{2}} c_1^{\frac{\alpha}{2}-1} c_2^{\frac{\beta}{2}-1} (1-c_1-c_2)^{\frac{\gamma}{2}-1} \\
&\quad \times \; \left(1+\frac{c-a}{a}c_1+\frac{c-b}{b}c_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \qquad , \; 0 < c_1, c_2 < 1 \text{ and } c_1 + c_2 \leq 1.
\end{aligned}
$$

(2.7)

If $a = 1, b = 1$ and $c = 1$, the bivariate beta type V reduces to the bivariate beta type I. If $a = 1, b = 1$ and $c = 2$, the bivariate beta type V reduces to the bivariate beta type III. Thus the bivariate beta type V distribution is a more flexible generalisation of the type I and type III distributions. This distribution, as well as its triply non-central generalisation, was developed in detail by Ehlers et al. [16]. It should be noted that Ehlers et al. [16] experienced difficulties deriving some properties of their distribution, and as such had to utilise the relationships between the beta type V distribution and other beta distributions in order to find closed form expressions of some properties. A similar problem was encountered with this study, and similar transformation methods had to be applied in Section 4.2.5.

Bivariate beta type VI

If

$$Z_1 = \frac{Y_1}{Y_2 + Y_3} \text{ and } Z_2 = \frac{Y_2}{Y_1 + Y_3},$$

then the joint distribution of the above ratios will be called the bivariate beta type VI distribution. This joint density function has not yet been derived in the literature but is a possibly useful addition to the bivariate beta family, that could potentially also be applied to detecting shifts in a process variance. If the statistics are defined as above, practically speaking, each sample variance would be compared to all others in the sequence. Thus (supposing there are $m + 1$ samples), each sample variance would be compared to the other m sample variances. A useful application of a control chart based on these statistics would be to aid in the detection of once off fluctuations in the process variance. This is in contrast to the application of the statistics in Equation (2.2), which is the detection of a sustained shift in the process variance.

Note that since this density function has not been investigated no assumptions will be made about it's domain in this study. However the implicit assumption is made that the parameters are restricted to values for which the probability density function is non negative.

Bivariate beta type VII

The following bivariate beta distribution, which will be referred to as the type VII distribution, was derived by Adamski [1], and was labelled a "generalised beta distribution". If

$$T_1 = \frac{Y_2}{Y_1} \text{ and } T_2 = \frac{Y_3}{Y_1 + Y_2},$$

the joint density function of T_1 and T_2, with parameters $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} > 0$ is given by

$$f_{BVII}(t_1,t_2;\tfrac{\alpha}{2},\tfrac{\beta}{2},\tfrac{\gamma}{2}) = \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})}(t_1)^{\frac{\beta}{2}-1}(t_2)^{\frac{\gamma}{2}-1}(1+t_1)^{\frac{\gamma}{2}} \times (1+t_1+(1+t_1)\,t_2)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \quad , \quad t_1,t_2>0. \tag{2.8}$$

This distribution will be generalised to be expressed in terms of gamma random variables in Section 3.2.

Bivariate beta type VIII

The model that this study proposes in terms of gamma variables is derived in Section 4.2, but when it is reduced to be constructed from chi-square variables, $(\chi^2(\alpha) \equiv Gamma\left(\frac{\alpha}{2},2\right))$, it is also a bivariate beta distribution that is made up of the Y_1, Y_2 and Y_3 building blocks. If

$$U_1 = \frac{Y_2+Y_3}{Y_1} \text{ and } U_2 = \frac{Y_3}{Y_1+Y_2},$$

the joint distribution of U_1 and U_2 will be called a bivariate beta type VIII distribution with parameters $\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}>0$. The joint density function of this distribution is given by

$$f_{BVIII}(u_1,u_2;\tfrac{\alpha}{2},\tfrac{\beta}{2},\tfrac{\gamma}{2}) = \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{\beta}{2}\right)\Gamma(\frac{\gamma}{2})}(u_1-u_2)^{\frac{\beta}{2}-1}u_2^{\frac{\gamma}{2}-1}(1+u_1)^{\frac{\gamma}{2}}(1+u_2)^{\frac{\alpha}{2}} \times ((1+u_2)+(u_1-u_2)+u_2(1+u_1))^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} \quad , \quad u_1>u_2>0. \tag{2.9}$$

The derivation of this density function, and its associated properties in Chapter 4, is the main focus of this study.

Note that the above bivariate beta densities are defined on different domains. These domains influence their practical applications to a very large extent. For example, the bivariate beta type II distribution was initially developed to overcome the limited domain that the bivariate beta type I distribution is defined on, since the $q_1+q_2 \le 1$ restriction of Equation (2.3) severely limits the situations where the bivariate beta type I distribution can be used to model data. The model proposed by this study is defined on a different domain in comparison to the more commonly used bivariate beta distributions, and as such could lead to novel practical applications. In Figure 2.1 the domains of the different bivariate beta distributions are displayed, however the variable names, q_1, q_2, v_1, v_2 etc. on the axes are all replaced with Var_1 and Var_2 to reduce the number of graphs required.

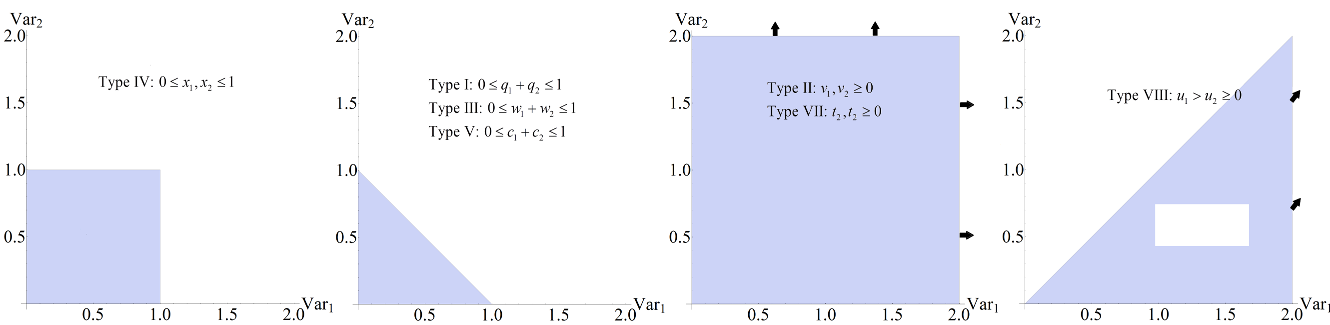

Figure 2.1: Domains of the bivariate beta joint density functions.

There are many more bivariate beta type distributions available in the literature; the ones mentioned in this study are only some of the most commonly used, and as such this study should not be seen as an exhaustive collection of bivariate beta distributions. The bivariate beta types I and II were among the first derived, and are also those which have received the most theoretical interest and practical applications. Types III and V were later developed to be more general cases of the type I. Many authors have discussed and derived a plethora of other bivariate beta distributions, which are mostly generalisations that either offer some theoretical property which the original distributions do not possess, or are applicable to specific practical situations. Nadarajah [35, 36, 37], Nadarajah and Gupta [38] and Nadarajah and Kotz [39, 40, 41, 42] in particular have studied beta distributions in great detail. Balakrishnan and Lai [6] also give a whole host of bivariate beta distributions and their relation to other common distributions like the Laplace and exponential distributions.

2.3 Relationships between bivariate beta type distributions

In Table 2.1 and Figure 2.2 that follow, each of the bivariate beta distributions mentioned Section 2.2 are related to all the other bivariate beta distributions in this study. Since all of them depend on the same chi-square variables it is reasonable to expect that there exists some relation between them. In Table 2.1, the first column gives the two statistics of each bivariate beta distribution in terms of the chi-square random variables (as they are defined in equations (2.3) to (2.9)). Columns two to nine then provide a relationship between the different bivariate beta distributions. For example, the first row, fourth column, provides the relationship between the bivariate beta type I and type III distributions. Suppose that some equation exists that contains both X_1 and X_2; also suppose that, for whatever reason, the equation is unsuitable or impractical for use in a certain situation; it is then possible to rewrite X_1 and X_2 in terms of V_1 and V_2 using the relationships in the second row, fifth column. A simple example where this is applicable is provided after Table 2.1 and Figure 2.2.

	Y_1, Y_2, Y_3	B_I	B_{II}	B_{III}	B_{IV}
B_I	$Q_1 = \frac{Y_1}{Y_1+Y_2+Y_3}$ $Q_2 = \frac{Y_2}{Y_1+Y_2+Y_3}$		$Q_1 = \frac{V_1}{1+V_1+V_2}$ $Q_2 = \frac{V_2}{1+V_1+V_2}$	$Q_1 = \frac{2W_1}{1+W_1+W_2}$ $Q_2 = \frac{2W_2}{1+W_1+W_2}$	$Q_1 = \frac{X_1(1-X_2)}{1-X_1X_2}$ $Q_2 = \frac{X_2(1-X_1)}{1-X_1X_2}$
B_{II}	$V_1 = \frac{Y_1}{Y_3}$ $V_2 = \frac{Y_2}{Y_3}$	$V_1 = \frac{Q_1}{1-Q_1-Q_2}$ $V_2 = \frac{Q_2}{1-Q_1-Q_2}$		$V_1 = \frac{2W_1}{1-W_1-W_2}$ $V_2 = \frac{2W_2}{1-W_1-W_2}$	$V_1 = \frac{X_1}{1-X_1}$ $V_2 = \frac{X_2}{1-X_2}$
B_{III}	$W_1 = \frac{Y_1}{Y_1+Y_2+2Y_3}$ $W_2 = \frac{Y_2}{Y_1+Y_2+2Y_3}$	$W_1 = \frac{Q_1}{2-Q_1-Q_2}$ $W_2 = \frac{Q_2}{2-Q_1-Q_2}$	$W_1 = \frac{V_1}{2+V_1+V_2}$ $W_2 = \frac{V_2}{2+V_1+V_2}$		$W_1 = \frac{X_1(1-X_2)}{2-X_1-X_2}$ $W_2 = \frac{X_2(1-X_1)}{1+X_1+X_2}$
B_{IV}	$X_1 = \frac{Y_1}{Y_1+Y_3}$ $X_2 = \frac{Y_2}{Y_2+Y_3}$	$X_1 = \frac{Q_1}{1-Q_2}$ $X_2 = \frac{Q_2}{1-Q_1}$	$X_1 = \frac{V_1}{1+V_1}$ $X_2 = \frac{V_2}{1+V_2}$	$X_1 = \frac{2W_1}{1+W_1-W_2}$ $X_2 = \frac{2W_2}{1-W_1+W_2}$	
B_V	$C_1 = \frac{aY_1}{aY_1+bY_2+cY_3}$ $C_2 = \frac{bY_2}{aY_1+bY_2+cY_3}$	$C_1 = \frac{aQ_1}{(a-c)Q_1+(b-c)Q_2+c}$ $C_2 = \frac{bQ_2}{(a-c)Q_1+(b-c)Q_2+c}$	$C_1 = \frac{aV_1}{aV_1+bV_2+c}$ $C_2 = \frac{bV_2}{aV_1+bV_2+c}$	$C_1 = \frac{2aW_1}{(2a-c)W_1+(2b-c)W_2+c}$ $C_2 = \frac{2bQ_2}{(2a-c)W_1+(2b-c)W_2+c}$	$C_1 = \frac{aX_1(X_2-1)}{aX_1(X_2-1)+bX_2(X_1-1)-c(X_1-1)(X_2-1)}$ $C_2 = \frac{bX_2(X_1-1)}{aX_1(X_2-1)+bX_2(X_1-1)-c(X_1-1)(X_2-1)}$
B_{VI}	$Z_1 = \frac{Y_1}{Y_2+Y_3}$ $Z_2 = \frac{Y_2}{Y_1+Y_3}$	$Z_1 = \frac{Q_1}{1-Q_1}$ $Z_2 = \frac{Q_2}{1-Q_2}$	$Z_1 = \frac{V_1}{1+V_2}$ $Z_2 = \frac{V_2}{1+V_1}$	$Z_1 = \frac{2W_1}{1-W_1+W_2}$ $Z_2 = \frac{2W_2}{1+W_1-W_2}$	$Z_1 = \frac{X_1(1-X_2)}{1-X_1}$ $Z_2 = \frac{X_2(1-X_1)}{1-X_2}$
B_{VII}	$T_1 = \frac{Y_2}{Y_1}$ $T_2 = \frac{Y_3}{Y_1+Y_2}$	$T_1 = \frac{Q_2}{Q_1}$ $T_2 = \frac{1-Q_1-Q_2}{Q_1+Q_2}$	$T_1 = \frac{V_2}{V_1}$ $T_2 = \frac{1}{V_1+V_2}$	$T_1 = \frac{W_2}{W_1}$ $T_2 = \frac{1-W_1-W_2}{2(W_1+W_2)}$	$T_1 = \frac{(X_1-1)\big(X_2+X_1\big(X_2^2-4X_2+2\big)\big)}{X_1(X_2-1)(X_1X_2-1)}$ $T_2 = \frac{(X_1-1)(X_2-1)}{X_1(1-2X_2)+X_2}$
B_{VIII}	$U_1 = \frac{Y_2+Y_3}{Y_1}$ $U_2 = \frac{Y_3}{Y_1+Y_2}$	$U_1 = \frac{1-Q_1}{Q_1}$ $U_2 = \frac{1-Q_1-Q_2}{Q_1+Q_2}$	$U_1 = \frac{1+V_2}{V_1}$ $U_2 = \frac{1}{V_1+V_2}$	$U_1 = \frac{1-W_1+W_2}{2W_1}$ $U_2 = \frac{1-W_1-W_2}{2(W_1+W_2)}$	$U_1 = \frac{X_1-1}{X_1(1-X_2)}$ $U_2 = \frac{(X_1-1)(X_2-1)}{X_1(1-2X_2)+X_2}$

	B_V	B_{VI}	B_{VII}	B_{VIII}
B_I	$Q_1 = \frac{bcC_1}{b(c-a)C_1+a(c-b)C_2+ab}$ $Q_2 = \frac{abC_2}{b(c-a)C_1+a(c-b)C_2+ab}$	$Q_1 = \frac{Z_1}{1+Z_1}$ $Q_2 = \frac{Z_2}{1+Z_2}$	$Q_1 = \frac{1}{(T_1+1)(T_2+1)}$ $Q_2 = \frac{T_1}{(T_1+1)(T_2+1)}$	$Q_1 = \frac{1}{(U_1+1)}$ $Q_2 = \frac{U_1-U_2}{(U_1+1)(U_2+1)}$
B_{II}	$V_1 = \frac{X_1}{1-X_1}\frac{c}{a}$ $V_2 = \frac{X_2}{1-X_2}\frac{c}{b}$	$V_1 = \frac{Z_1(Z_2+1)}{1-Z_1Z_2}$ $V_2 = \frac{Z_2(Z_1+1)}{1-Z_1Z_2}$	$V_1 = \frac{1}{T_2(T_1+1)}$ $V_2 = \frac{T_1}{T_2(T_1+1)}$	$V_1 = \frac{U_2+1}{U_2(U_1+1)}$ $V_2 = \frac{U_1-U_2}{U_2(U_1+1)}$
B_{III}	$W_1 = \frac{bcC_1}{b(c-2a)C_1+a(c-2b)C_2+2ab}$ $W_2 = \frac{abC_2}{b(c-2a)C_1+a(c-2b)C_2+2ab}$	$W_1 = \frac{Z_1(Z_2+1)}{2+Z_1+Z_2}$ $W_2 = \frac{Z_2(Z_1+1)}{2+Z_1+Z_2}$	$W_1 = \frac{1}{(T_1+1)(2T_2+1)}$ $W_2 = \frac{T_1}{(T_1+1)(2T_2+1)}$	$W_1 = \frac{U_2+1}{(2U_2+1)(U_1+1)}$ $W_2 = \frac{U_1-U_2}{(2U_2+1)(U_1+1)}$
B_{IV}	$X_1 = \frac{cC_1}{(c-a)C_1-a(C_2-1)}$ $X_2 = \frac{cC_2}{(c-b)C_2-b(C_1-1)}$	$X_1 = \frac{Z_1(Z_2+1)}{1+Z_1}$ $X_2 = \frac{Z_2(Z_1+1)}{1+Z_2}$	$X_1 = \frac{1+T_2}{1-T_2(T_1-2)-T_2^2(T_1+1)}$ $X_2 = \frac{T_2(2+T_1)+T_1}{(T_2+1)(T_2(T_1+1)+T_1)}$	$X_1 = \frac{U_2+1}{1+U_2(U_1+2)}$ $X_2 = \frac{U_1-U_2}{U_1(U_2+1)}$
B_V		$C_1 = \frac{aZ_1(Z_2+1)}{aZ_1(Z_2+1)+bZ_2(Z_1+1)+c(1-Z_1Z_2)}$ $C_2 = \frac{bZ_2(Z_1+1)}{aZ_1(Z_2+1)+bZ_2(Z_1+1)+c(1-Z_1Z_2)}$	$C_1 = \frac{a}{a+bT_1+cT_2(T_1+1)}$ $C_2 = \frac{bT_1}{a+bT_1+cT_2(T_1+1)}$	$C_1 = \frac{a(U_2+1)}{a(U_2+1)+b(U_1-U_2)+cU_2(U_1+1)}$ $C_2 = \frac{b(U_1-U_2)}{a(U_2+1)+b(U_1-U_2)+cU_2(U_1+1)}$
B_{VI}	$Z_1 = \frac{-bcC_1}{a(b-c)C_2-ab(C_1-1)}$ $Z_2 = \frac{-acC_2}{b(a-c)C_1-ab(C_2-1)}$		$Z_1 = \frac{1}{T_2(T_1+1)+T_1}$ $Z_2 = \frac{T_1}{T_2(T_1+1)+1}$	$Z_1 = \frac{1}{U_1}$ $Z_2 = \frac{U_1-U_2}{1+U_2(1+U_1)}$
B_{VII}	$T_1 = \frac{aC_2}{bC_1}$ $T_2 = \frac{ab(1-C_1-C_2)}{bcC_1+acC_2}$	$T_1 = \frac{Z_2(Z_1+1)}{Z_1(1+Z_2)}$ $T_2 = \frac{1-Z_1Z_2}{Z_1(1+2Z_2)+Z_2}$		$T_1 = \frac{U_1-U_2}{1+U_2}$ $T_2 = U_2$
B_{VIII}	$U_1 = \frac{a(c-b)C_2-b(C_1-1)}{bcC_1}$ $U_2 = \frac{ab(1-C_1-C_2)}{bcC_1+acC_2}$	$U_1 = \frac{1}{Z_1}$ $U_2 = \frac{1-Z_1Z_2}{Z_1(1+2Z_2)+Z_2}$	$U_1 = T_1 + T_2 + T_1T_2$ $U_2 = T_2$	

Note that Table 2.1 has been split up into two segments to facilitate the physical printing of this study. However, the second segment of the table containing the transformations for B_V to B_{VIII}, should be seen as being spliced onto the right side of the first segment.

Table 2.1: Beta relationships for models (2.3) to (2.9)

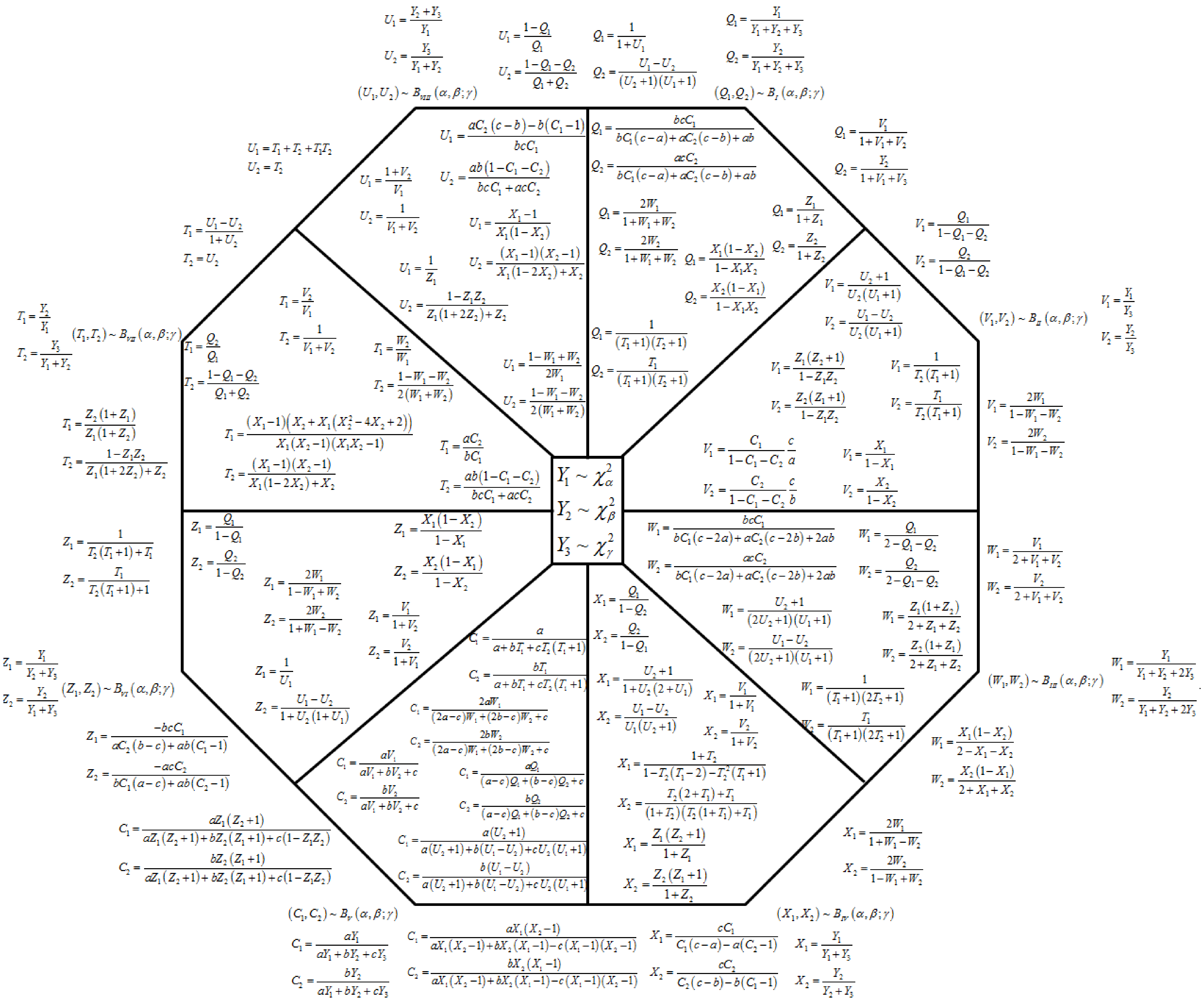

Figure 2.2: Relationships between beta type I to type beta VIII

The example that follows is provided to illustrate a situation were the transformations in Table 2.1 and Figure 2.2 could be used. Note however that in the specific example chosen is rather simplistic, and could in fact be solved using a more traditional first principal approach.

Example

Suppose that the derivation of the product moment of the bivariate beta type IV distribution in Equation (2.6) is required. To derive this from first principles, Equation (2.6) is substituted into the equation for the product moment, Result 5, and thus would be

$$\begin{aligned} E\left(X_1^r X_2^s\right) &= \int_0^1 \int_0^1 x_1^r x_2^s f_{BIV}(x_1, x_2; \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}) dx_2 dx_1 \\ &= \int_0^1 \int_0^1 \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)} x_1^{r+\frac{\alpha}{2}-1} x_2^{s+\frac{\beta}{2}-1} \\ &\times \left(1-x_1\right)^{\frac{\beta}{2}+\frac{\gamma}{2}-1}\left(1-x_2\right)^{\frac{\alpha}{2}+\frac{\gamma}{2}-1}\left(1-x_1x_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} dx_2 dx_1 \\ &= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)} \int_0^1 x_1^{r+\frac{\alpha}{2}-1}\left(1-x_1\right)^{\frac{\beta}{2}+\frac{\gamma}{2}-1} \int_0^1 x_2^{s+\frac{\beta}{2}-1} \\ &\times \left(1-x_2\right)^{\frac{\alpha}{2}+\frac{\gamma}{2}-1}\left(1-x_1x_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)} dx_2 dx_1. \end{aligned} \tag{2.10}$$

While in this specific situation it is possible to solve the integrals in Equation (2.10), for a more complex joint probability density function it my not be.

A solution to overcome this hurdle would be to use the transformations given in Table 2.1 or Figure 2.2. Since the product moment of the bivariate beta type II distribution is easy to derive, using the relationship displayed in row four, column three, of Table 2.1:

$$\begin{aligned} X_1 &= \frac{V_1}{1+V_1} \\ X_2 &= \frac{V_2}{1+V_2}, \end{aligned}$$

Equation (2.10) could be rewritten in terms of Equation (2.4) as

$$
\begin{aligned}
E\left(X_1^r X_2^s\right) &= E\left(\left[\frac{V_1}{1+V_1}\right]^r\left[\frac{V_2}{1+V_2}\right]^s\right) \\
&= \int_0^\infty\int_0^\infty\left(\frac{v_1}{1+v_1}\right)^r\left(\frac{v_2}{1+v_2}\right)^s f_{BII}(v_1,v_2;\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2})dv_2dv_1 \\
&= \int_0^\infty\int_0^\infty\left(\frac{v_1}{1+v_1}\right)^r\left(\frac{v_2}{1+v_2}\right)^s \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)} \\
&\times\ v_1^{\frac{\alpha}{2}-1}v_2^{\frac{\beta}{2}-1}\left(1+v_1+v_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}dv_2dv_1 \\
&= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}\int_0^\infty\int_0^\infty\left(1+v_1\right)^{-r}\left(1+v_2\right)^{-s} \\
&\times\ v_1^{r+\frac{\alpha}{2}-1}v_2^{s+\frac{\beta}{2}-1}\left(1+v_1+v_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}dv_2dv_1 \\
&= \frac{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}\int_0^\infty\left(1+v_1\right)^{-r}v_1^{r+\frac{\alpha}{2}-1} \\
&\times\int_0^\infty\left(1+v_2\right)^{-s}v_2^{s+\frac{\beta}{2}-1}\left(1+v_1+v_2\right)^{-\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)}dv_2dv_1, \qquad (2.11)
\end{aligned}
$$

which may simplify the derivation process. It is a transformation exactly like the one described above that is required later in this study to derive the product moment of the bivariate beta type VIII distribution.

Chapter 3

Generalised beta type VII distribution

3.1 Introduction

As seen in Chapter 2, bivariate beta distributions are most often derived in terms of chi-square random variables, but derivations in terms of gamma variables are not uncommon. In this study all derivations that follow will be derived for the gamma case, since using gamma random variables will allow greater flexibility during the modelling process.

As stated in Section 1.2, in the SPC environment, the joint density function of the charting statistics, as well as the control limits, are constructed under the null hypothesis (i.e. under the assumption that no shift in the process variance has occurred). While deriving the joint density function under this assumption is correct, and often reduces the complexity of the derivations significantly, it does limit the insight that can be gained into the efficacy of the control chart when the process goes OOC. It is for this reason that the derivations in this study will be made in such a way that the joint density function of the charting statistics could come either from an IC process or an OOC process (depending on the parameter choices).

In equations (1.4) and (1.6), it was shown that the sample variance, multiplied by the sample size minus one, divided by the process variance, follows a $\chi^2(n_i-1)$ distribution for each sample, and as a result the series of statistics in Equation (1.15) each follow an F distribution under the null hypothesis. Consequently the statistics in equations (1.11) and (1.16) are constructed to be ratios of linear combinations of chi-square variables (under the null hypothesis). It is a well-established fact that if $Y \sim \chi^2(n-1)$, then Y can also be said to follow a $Gamma(\frac{n-1}{2}, 2)$ distribution (see Bain and Engelhardt [5] pp268-269). Furthermore, it is a well-known property of gamma random variables that if c is some constant, then $cY \sim Gamma(\frac{n-1}{2}, 2c)$. Using these results, suppose that a shift of size λ occurs in the process variance at time κ^*, then the corresponding statistics after the shift would be $\lambda Y_i \sim Gamma(\frac{n_i-1}{2}, 2\lambda), i = \kappa, \kappa{+}1, ..., m$ distributed (see Equation 1.16). Thus using the gamma notation for the random variables, instead of the more traditional chi-square notation, allows for the modelling of the sample variances irrespective of whether the process is IC or OOC. If the process is IC, $\lambda = 1$, and it follows that $\lambda Y_i = Y_i \sim Gamma(\frac{n_i-1}{2}, 2 \times 1) \equiv \chi^2(n_i-1), i = 0, 1, ..., m$. In contrast, if the process is OOC, $\lambda \neq 1$, and it follows that $\lambda Y_i \sim Gamma(\frac{n_i-1}{2}, 2\lambda), i = \kappa, \kappa+1, ..., m$. In essence then, the hypothesis being worked under can be changed depending on the choice of the second parameter of the specific gamma random variables in equations (1.11) and (1.16). If the second parameter is equal to 2 for all gamma random variables, the process is modelled under the IC assumption, while if the second parameter is not always equal to 2, a shift in the process variance can be modelled. Thus, by deriving the bivariate beta type density function in terms of gamma variables, added flexibility is gained. It should be noted, however, that using gamma random variables instead of chi-square leads to more terms in the derived equations and more complex derivations.

Looking at the variables that make up the parameters of the gamma random variables in the previous paragraph, it is obvious that in the SPC context the parameters have a practical interpretation. The parameter of the chi-square variables is related to the sample size of the process samples. Similarly, the first parameter of the gamma random variables is also related to the sample size of the process samples (although the first gamma parameter is half that of the chi-square distribution). As demonstrated above, the second parameter of the gamma variable is related to the variance of the sample. If the sample variance comes from an IC distribution the parameter will be equal to 2, whereas if the process has experienced a shift the parameter will not be equal to 2.

In Adamski [1], the density functions and properties of the statistics in Equation (2.1) were derived in terms of chi-square random variables, with some random variables (those after the potential shift) being multiplied by λ. By using the relationship between the chi-square and gamma distributions noted above, their results can be expanded to allow for the more general gamma case. This will not only add flexibility to their results, but will also allow the comparison between the studied Q chart process and the method that this study proposes to be clearer. In this chapter some the statistics and densities that were derived in Adamski [1] are generalised to allow for gamma variables. The main focus of this chapter will be to expand on the bivariate probability density function and its relevant properties, since, as was stated in the study outline, the methods used during the bivariate derivations are likely to be similar in nature to the methods required during higher dimensional derivations and focusing on the bivariate case also allows graphical representations of the distributions to be plotted, which would be impossible for higher dimensions.

Example

Suppose that a single sample of size 5 is drawn from a process, then the sample mean $\left(\bar{X}_1\right)$ and variance $\left(S_1^2\right)$ are calculated using equations (1.1) and (1.2). From equations (1.4) and (1.6) it is known that $\frac{(5-1)S_1^2}{\sigma^2} \sim \chi^2(5-1)$ and $\frac{(5-1)S_1^2}{\sigma_1^2} \sim \chi^2(5-1)$. It follows, from Bain and Engelhardt [5] pp268-269, that $\chi^2(4)$ can equivalently be written as $Gamma\left(\frac{4}{2}, 2\right)$. Adamski [1] modelled a potential shift in the variance by multiplying the chi-square variable by λ, where $\lambda > 1$ would indicate an increase in the variance, $\lambda < 1$ would indicate a decrease in the process variance, and $\lambda = 1$ would indicate that no shift occurred. Since this study proposes to use gamma random variables the multiplication by λ is no longer necessary. By simply varying the second (scale) parameter of the gamma random variable, a shift in the process variance can be modelled. A $Gamma\left(\frac{4}{2}, 2\right)$ variable would indicate that the process is in control, whereas $Gamma\left(\frac{4}{2}, 2\lambda\right), \lambda > 1$ would imply an increase in the process variance, and similarly $Gamma\left(\frac{4}{2}, 2\lambda\right), \lambda < 1$ would indicate that the process has become more stable.

In Section 3.2.1 the bivariate joint density function of the statistics in Equation (2.1) are generalised to the gamma case. In Chapter 2, T_1 and T_2 were defined in terms of $Y_1 \sim \chi^2(\alpha)$, $Y_2 \sim \chi^2(\beta)$ and $Y_3 \sim \chi^2(\gamma)$. In this chapter they are defined in terms of independent gamma random variables, $W_i \sim Gamma(\alpha_i, \beta_i)$ for $i = 0, 1, 2$. They are, however, still named T_1 and T_2. Special cases of the distribution are investigated, where

- the α's are all equal but β's are not necessarily equal (all sample sizes are the same but shifts in the variance are possible),
- all the β's are equal but the α's may differ (no shift occurs during the three samples, but sample sizes may differ),
- and thirdly where all the α's are equal and all the β's are equal (equal sample sizes, and no shift occurs during the three samples).

In Section 3.2.2 the two marginal density functions of the bivariate density function in Section 3.2.1 are derived, and the three special cases of each of the marginal densities are again investigated. The conditional density functions are derived in Section 3.2.3. In Section 3.2.4 the product moment of the bivariate density function is derived. In Section 3.3.1 the joint multivariate density function from Adamski [1] is also extended to the gamma case, and the three special cases are again discussed.

3.2 Bivariate distribution

3.2.1 Joint density function

Theorem 3.1

Let W_i be independent gamma random variables with parameters $(\alpha_i > 0, \beta_i > 0)$ for $i = 0, 1, 2$. (Note that in Equation (2.8) T_1 and T_2 were constructed out of three independent chi-square random variables; here the theory is extended to allow for gamma random variables.)

Let $T_1 = \frac{W_1}{W_0}$ and $T_2 = \frac{W_2}{W_0 + W_1}$.

Then the joint density function is given by

$$\begin{aligned} g(t_1, t_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} (t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times \left(\beta_1\beta_2 + \beta_0\beta_2 t_1 + \beta_0\beta_1(1+t_1)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \quad , \quad t_1, t_2 > 0. \end{aligned} \tag{3.1}$$

Proof
Since the gamma random variables are independent, the joint density function of W_0, W_1, W_2 is given by

$$g(w_0, w_1, w_2) = \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \left(w_0^{\alpha_0-1}e^{-\frac{w_0}{\beta_0}}\right)\left(w_1^{\alpha_1-1}e^{-\frac{w_1}{\beta_1}}\right)\left(w_2^{\alpha_2-1}e^{-\frac{w_2}{\beta_2}}\right), w_0, w_1, w_2 > 0. \tag{3.2}$$

Let $T = W_0, T_1 = \frac{W_1}{W_0}$ and $T_2 = \frac{W_2}{W_0+W_1}$.

Using the "variables in common" technique (see Olkin and Trikalinos [45]) it is possible to find the joint density function of T, T_1 and T_2 as follows:

The inverse transformation is then given by

$$\begin{aligned} W_0 &= T \\ W_1 &= T_1 T \\ W_2 &= T_2 \left(T + T_1 T\right) = T_2 T \left(1 + T_1\right). \end{aligned}$$

The Jacobian of the transformation is
$J\left(w_0, w_1, w_2 \rightarrow t, t_1, t_2\right)$
$$= \begin{vmatrix} 1 & 0 & 0 \\ t_1 & t & 0 \\ t_2\left(1+t_1\right) & t_2 t & t\left(1+t_1\right) \end{vmatrix}$$
$= t^2\left(1+t_1\right).$

By making the transformation and substituting the equations for W_0, W_1 and W_2 into Equation (3.2), it then follows that the joint density function of T, T_1, T_2 is

$$\begin{aligned} g\left(t, t_1, t_2\right) &= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(t^{\alpha_0-1}e^{-\frac{t}{\beta_0}}\right)\left(\left(t_1 t\right)^{\alpha_1-1}e^{-\frac{t_1 t}{\beta_1}}\right) \\ &\times \left(\left(t_2 t\left(1+t_1\right)\right)^{\alpha_2-1}e^{-\frac{t_2 t\left(1+t_1\right)}{\beta_2}}\right)t^2\left(1+t_1\right) \\ &= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2-1}\left(1+t_1\right)^{\alpha_2} \\ &\times \left(e^{-\frac{t}{\beta_0}}e^{-\frac{t_1 t}{\beta_1}}e^{-\frac{t_2 t\left(1+t_1\right)}{\beta_2}}\right)\left(t^{\alpha_0+\alpha_1+\alpha_2-1}\right). \end{aligned} \tag{3.3}$$

By integrating Equation (3.3) with respect to t, it follows that

$$\begin{aligned} g\left(t_1, t_2\right) &= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2-1}\left(1+t_1\right)^{\alpha_2} \\ &\times \int_0^\infty \left(e^{-t\left(\frac{\beta_1\beta_2+t_1\beta_0\beta_2+t_2\left(1+t_1\right)\beta_0\beta_1}{\beta_0\beta_1\beta_2}\right)}\right)\left(t^{\alpha_0+\alpha_1+\alpha_2-1}\right)dt. \end{aligned} \tag{3.4}$$

By applying Result 11 to Equation (3.4), it then follows that

$$\begin{aligned} g(t_1,t_2) &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times\left(\frac{\beta_1\beta_2+\beta_0\beta_2t_1+\beta_0\beta_1(1+t_1)t_2}{\beta_0\beta_1\beta_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \frac{(\beta_0\beta_1\beta_2)^{\alpha_0+\alpha_1+\alpha_2}\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times(\beta_1\beta_2+\beta_0\beta_2t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times(\beta_1\beta_2+\beta_0\beta_2t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2}. \end{aligned}$$

■

Special cases

1) If $\alpha_i=\alpha$ for $i=0,1,2$, then Equation (3.1) becomes

$$\begin{aligned} g(t_1,t_2) &= \frac{\left(\beta_0^{\alpha+\alpha}\beta_1^{\alpha+\alpha}\beta_2^{\alpha+\alpha}\right)\Gamma(\alpha+\alpha+\alpha)}{\Gamma(\alpha)\Gamma(\alpha)\Gamma(\alpha)}(t_1)^{\alpha-1}(t_2)^{\alpha-1}(1+t_1)^{\alpha} \\ &\times\ (\beta_1\beta_2+\beta_0\beta_2t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha-\alpha-\alpha} \\ &= \frac{(\beta_0^{2\alpha}\beta_1^{2\alpha}\beta_2^{2\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)^3}(t_1)^{\alpha-1}(t_2)^{\alpha-1}(1+t_1)^{\alpha} \\ &\times\ (\beta_1\beta_2+\beta_0\beta_2t_1+\beta_0\beta_1(1+t_1)t_2)^{-3\alpha} \qquad ,\quad t_1,t_2>0. \end{aligned}$$

2) If $\beta_i=\beta$ for $i=0,1,2$, then Equation (3.1) becomes

$$\begin{aligned} g(t_1,t_2) &= \frac{(\beta^{2(\alpha_0+\alpha_1+\alpha_2)})\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times\ (\beta^2(1+t_1+(1+t_1)t_2))^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\ &\times\ (1+t_1+(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2} \qquad ,\quad t_1,t_2>0. \end{aligned}$$

This is the same as the joint density function derived in Adamski [1] with $\lambda=1$, since if all β values are equal no shift has occurred in the process variance.

3) If $\beta_i=\beta$ and $\alpha_i=\alpha$ for $i=0,1,2$, then Equation (3.1) becomes

$$\begin{aligned} g(t_1,t_2) &= \frac{(\beta^{2\alpha}\beta^{2\alpha}\beta^{2\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)^3}(t_1)^{\alpha-1}(t_2)^{\alpha-1}(1+t_1)^{\alpha} \\ &\times\ (\beta^2(1+t_1+(1+t_1)t_2))^{-3\alpha} \\ &= \frac{\Gamma(3\alpha)}{\Gamma(\alpha)^3}(t_1)^{\alpha-1}(t_2)^{\alpha-1}(1+t_1)^{\alpha} \\ &\times\ (1+t_1+(1+t_1)t_2)^{-3\alpha} \qquad ,\quad t_1,t_2>0. \end{aligned}$$

3.2.2 Marginal density functions

Marginal density function of T_1

Theorem 3.2

The marginal density function of T_1 is given by

$$g(t_1) = \frac{(\beta_0^{\alpha_1}\beta_1^{\alpha_0})\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(\beta_1+\beta_0 t_1)^{-\alpha_0-\alpha_1} \quad , \quad t_1 > 0 \ , \tag{3.5}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

Proof

By integrating Equation (3.1) with respect to t_2, and rearranging the terms, it follows that

$$\begin{aligned} g(t_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2} \\ &\times \int_0^\infty (t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2}\,dt_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2} \\ &\times \int_0^\infty (t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_2 t_1+t_2(\beta_0\beta_1+\beta_0\beta_1 t_1))^{-\alpha_0-\alpha_1-\alpha_2}\,dt_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2} \\ &\times \int_0^\infty (t_2)^{\alpha_2-1}\left((\beta_1\beta_2+\beta_0\beta_2 t_1)\left(1+t_2\frac{(\beta_0\beta_1+\beta_0\beta_1 t_1)}{(\beta_1\beta_2+\beta_0\beta_2 t_1)}\right)\right)^{-\alpha_0-\alpha_1-\alpha_2}\,dt_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2 t_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \int_0^\infty (t_2)^{\alpha_2-1}\left(1+t_2\frac{(\beta_0\beta_1+\beta_0\beta_1 t_1)}{(\beta_1\beta_2+\beta_0\beta_2 t_1)}\right)^{-\alpha_0-\alpha_1-\alpha_2}\,dt_2. \end{aligned} \tag{3.6}$$

By applying Result 12 to Equation (3.6), it follows that

$$\begin{aligned} g(t_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2 t_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times B(\alpha_2,\alpha_0+\alpha_1)\,{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_2;\alpha_0+\alpha_1+\alpha_2;1-\frac{(\beta_0\beta_1+\beta_0\beta_1 t_1)}{(\beta_1\beta_2+\beta_0\beta_2 t_1)}\right), \end{aligned} \tag{3.7}$$

where $B(.)$ is the beta function, defined in Result 7, and ${}_2F_1(.)$ is the Gauss hypergeometric function, defined in Result 8.

Since the first and third arguments of the hypergeometric function in Equation (3.7) are the same, the hypergeometric function can be reduced to ${}_1F_0(.)$, which is defined in the Appendix in Result 9. By reducing Equation (3.7) in this manner, it follows that

$$\begin{aligned} g(t_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2 t_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times B(\alpha_2,\alpha_0+\alpha_1)\,{}_1F_0\left(\alpha_2;1-\frac{(\beta_0\beta_1+\beta_0\beta_1 t_1)}{(\beta_1\beta_2+\beta_0\beta_2 t_1)}\right). \end{aligned} \tag{3.8}$$

As is stated in Result 9, the ${}_1F_0$ hypergeometric function can alternatively be expressed as a result of the binomial theorem, and thus, it follows that Equation (3.8) can be rewritten as follows

$$\begin{aligned} g(t_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2 t_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times B(\alpha_2,\alpha_0+\alpha_1)\left(1-\left(1-\frac{(\beta_0\beta_1+\beta_0\beta_1 t_1)}{(\beta_1\beta_2+\beta_0\beta_2 t_1)}\right)\right)^{-\alpha_2} \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_2(\beta_1+\beta_0 t_1))^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times B(\alpha_2,\alpha_0+\alpha_1)\left(\frac{(\beta_0\beta_1(1+t_1))}{(\beta_2(\beta_1+\beta_0 t_1))}\right)^{-\alpha_2} \\ &= \frac{\left(\beta_0^{\alpha_1}\beta_1^{\alpha_0}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(\beta_2(\beta_1+\beta_0 t_1))^{-\alpha_0-\alpha_1}B(\alpha_2,\alpha_0+\alpha_1) \\ &= \frac{\left(\beta_0^{\alpha_1}\beta_1^{\alpha_0}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(\beta_1+\beta_0 t_1)^{-\alpha_0-\alpha_1}\frac{\Gamma(\alpha_2)\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)}. \\ &= \frac{\left(\beta_0^{\alpha_1}\beta_1^{\alpha_0}\right)\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}(t_1)^{\alpha_1-1}(\beta_1+\beta_0 t_1)^{-\alpha_0-\alpha_1}. \end{aligned}$$

■

The marginal density function of T_1 (Equation (3.5)) is very similar in form to the inverted univariate beta distribution, also known as the beta prime distribution or the univariate beta type II distribution's density function (see Johnson et al. [26] for the definition).

Special cases T_1

1) If $\alpha_i = \alpha$ for $i = 0, 1, 2$, then Equation (3.5) reduces to

$$\begin{aligned} g(t_1) &= \tfrac{(\beta_0^\alpha \beta_1^\alpha)\Gamma(2\alpha)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\alpha)\Gamma(\alpha)} (t_1)^{\alpha-1} (\beta_1 + \beta_0 t_1)^{-2\alpha} \\ &= \tfrac{(\beta_0^\alpha \beta_1^\alpha)\Gamma(2\alpha)}{\Gamma(\alpha)^2} (t_1)^{\alpha-1} (\beta_1 + \beta_0 t_1)^{-2\alpha} \qquad , \quad t_1 > 0. \end{aligned}$$

2) If $\beta_i = \beta$ for $i = 0, 1, 2$, then Equation (3.5) reduces to

$$\begin{aligned} g(t_1) &= \tfrac{(\beta^{\alpha_1} \beta^{\alpha_0})\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} (t_1)^{\alpha_1-1} (\beta + \beta t_1)^{-\alpha_0-\alpha_1} \\ &= \tfrac{(\beta^{\alpha_0+\alpha_1})\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} (t_1)^{\alpha_1-1} (\beta(1 + t_1))^{-\alpha_0-\alpha_1} \\ &= \tfrac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} (t_1)^{\alpha_1-1} (1 + t_1)^{-\alpha_0-\alpha_1} \qquad , \quad t_1 > 0. \end{aligned}$$

It can be seen that if $\beta_i = \beta$ for $i = 0, 1, 2$, then $T_1 \sim Beta_{II}(\alpha_1, \alpha_0)$, where $Beta_{II}(.)$ the beta type II density function defined in Result 4.

3) If $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$, then Equation (3.5) reduces to

$$\begin{aligned} g(t_1) &= \tfrac{(\beta^\alpha \beta^\alpha)\Gamma(2\alpha)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\alpha)\Gamma(\alpha)} (t_1)^{\alpha-1} (\beta + \beta t_1)^{-\alpha-\alpha} \\ &= \tfrac{(\beta^{2\alpha})\Gamma(2\alpha)}{\Gamma(\alpha)^2} (t_1)^{\alpha-1} (\beta + \beta t_1)^{-2\alpha} \\ &= \tfrac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} (t_1)^{\alpha-1} (1 + t_1)^{-2\alpha} \qquad , \quad t_1 > 0. \end{aligned}$$

It can be seen that if $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$, then $T_1 \sim Beta_{II}(\alpha, \alpha)$.

Marginal density function of T_2

Theorem 3.3

The marginal density function of T_2 is given by

$$\begin{aligned} g(t_2) &= \tfrac{\left(\beta_0^{\alpha_1+\alpha_2} \beta_1^{\alpha_0+\alpha_2} \beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)} (t_2)^{\alpha_2-1} (\beta_1\beta_2 + \beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, \alpha_1; \alpha_0+\alpha_1; 1 - \tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \qquad , \quad t_2 > 0 \text{ and } \left|1 - \tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right| < 1, \end{aligned} \tag{3.9}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

Proof

By integrating Equation (3.1) with respect to t_1, and rearranging the terms, it follows that

$$\begin{aligned} g(t_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1} \\ &\times \int_0^\infty (t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2}\,dt_1 \end{aligned}$$

$$\begin{aligned} g(t_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1} \\ &\times \int_0^\infty (t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_1 t_2+t_1(\beta_0\beta_2+\beta_0\beta_1 t_2))^{-\alpha_0-\alpha_1-\alpha_2}\,dt_1 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1} \\ &\times \int_0^\infty (t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}\left((\beta_1\beta_2+\beta_0\beta_1 t_2)\left(1+t_1\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \int_0^\infty (t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}\left(1+t_1\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1. \end{aligned} \tag{3.10}$$

By applying Result 12 to Equation (3.10), it follows that

$$\begin{aligned} g(t_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times B(\alpha_1,\alpha_0)\,{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_0+\alpha_1)}\,{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right). \end{aligned}$$

■

Special cases T_2

1) If $\alpha_i = \alpha$ for $i = 0, 1, 2$, then Equation (3.9) reduces to

$$\begin{aligned} g(t_2) &= \frac{\left(\beta_0^{\alpha+\alpha}\beta_1^{\alpha+\alpha}\beta_2^{\alpha+\alpha}\right)\Gamma(\alpha+\alpha+\alpha)}{\Gamma(\alpha+\alpha)\Gamma(\alpha)}(t_2)^{\alpha-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha-\alpha-\alpha} \\ &\times\ {}_2F_1\left(\alpha+\alpha+\alpha,\alpha;\alpha+\alpha;1-\tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \\ &= \frac{(\beta_0^{2\alpha}\beta_1^{2\alpha}\beta_2^{2\alpha})\Gamma(3\alpha)}{\Gamma(2\alpha)\Gamma(\alpha)}(t_2)^{\alpha-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-3\alpha} \\ &\times\ {}_2F_1\left(3\alpha,\alpha;2\alpha;1-\tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \quad , \quad t_2>0 \text{ and } \left|1-\tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right|<1. \end{aligned}$$

2) If $\beta_i = \beta$ for $i = 0, 1, 2$, then Equation (3.9) reduces to

$$\begin{aligned} g(t_2) &= \frac{(\beta^{2(\alpha_0+\alpha_1+\alpha_2)})\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}\beta^{2(-\alpha_0-\alpha_1-\alpha_2)}(1+t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times\ {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\tfrac{(\beta\beta+\beta\beta t_2)}{(\beta\beta+\beta\beta t_2)}\right) \\ &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(1+t_2)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times\ {}_2F_1(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;0) \\ &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(1+t_2)^{-\alpha_0-\alpha_1-\alpha_2} \quad , \quad t_2>0. \end{aligned}$$

Therefore $T_2 \sim Beta_{II}(\alpha_2, \alpha_0+\alpha_1)$ when $\beta_i = \beta$ for $i = 0, 1, 2$.
The marginal density function of T_2 is, also exactly the same as the marginal density function derived by Adamski [1] when $\lambda = 1$. This is to be expected since if no shift occurs in the process variance ($\beta_i = \beta$ for $i = 0, 1, 2$ or equivalently $\lambda = 1$) the gamma variables that this study uses reduces to the chi-square variables used by Adamski [1].

3) If $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$, then Equation (3.9) reduces to

$$\begin{aligned} g(t_2) &= \frac{(\beta^{\alpha+\alpha}\beta^{\alpha+\alpha}\beta^{\alpha+\alpha})\Gamma(\alpha+\alpha+\alpha)}{\Gamma(\alpha+\alpha)\Gamma(\alpha)}(t_2)^{\alpha-1}(\beta\beta+\beta\beta t_2)^{-\alpha-\alpha-\alpha} \\ &\times\ {}_2F_1\left(\alpha+\alpha+\alpha,\alpha;\alpha+\alpha;1-\tfrac{(\beta\beta+\beta\beta t_2)}{(\beta\beta+\beta\beta t_2)}\right) \\ &= \frac{\Gamma(3\alpha)}{\Gamma(2\alpha)\Gamma(\alpha)}(t_2)^{\alpha-1}(1+t_2)^{-3\alpha} \quad , \quad t_2>0. \end{aligned}$$

Therefore $T_2 \sim Beta_{II}(\alpha, 2\alpha)$ when $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$.

3.2.3 Conditional density functions

Conditional density function of T_1 given T_2

Theorem 3.4

The conditional density function of T_1, given T_2, is

$$\begin{aligned} g\left(t_1|t_2\right) &= \tfrac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}\left(t_1\right)^{\alpha_1-1}\left(1+t_1\right)^{\alpha_2}\left(1+\tfrac{\beta_0 t_1(\beta_2+\beta_1 t_2)}{\beta_1\beta_2+\beta_0\beta_1 t_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\div {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right), \quad t_1>0 \text{ and } \left|1-\tfrac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right|<1, \end{aligned} \tag{3.11}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

Proof

From equations (3.1) and (3.9), the conditional density function follows as

$$
\begin{aligned}
g(t_1|t_2) &= \frac{g(t_1,t_2)}{g(t_2)} \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(t_1)^{\alpha_1-1}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1(1+t_1)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div \left[\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(\beta_1\beta_2+\beta_0\beta_1 t_2)^{-\alpha_0-\alpha_1-\alpha_2}\right. \\
&\times \left. {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right)\right] \\
&= \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1(1+t_1)t_2}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \\
&= \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2} \\
&\times \left(\frac{\beta_1\beta_2+\beta_0\beta_1 t_2\left(1+\frac{\beta_0\beta_2 t_1+\beta_0\beta_1 t_1 t_2}{\beta_1\beta_2+\beta_0\beta_1 t_2}\right)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right) \\
&= \frac{\Gamma(\alpha_0+\alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}(t_1)^{\alpha_1-1}(1+t_1)^{\alpha_2}\left(1+\frac{\beta_0 t_1(\beta_2+\beta_1 t_2)}{\beta_1\beta_2+\beta_0\beta_1 t_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_1;\alpha_0+\alpha_1;1-\frac{(\beta_0\beta_2+\beta_0\beta_1 t_2)}{(\beta_1\beta_2+\beta_0\beta_1 t_2)}\right).
\end{aligned}
$$

■

Conditional density function of T_2 given T_1

Theorem 3.5

The conditional density function of T_2, given T_1, is

$$
\begin{aligned}
g(t_2|t_1) &= \frac{\left(\beta_0^{\alpha_2}\beta_1^{\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)}(t_2)^{\alpha_2-1}(1+t_1)^{\alpha_2}(\beta_1+\beta_0 t_1)^{\alpha_0+\alpha_1} \\
&\times (\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1(1+t_1)t_2)^{-\alpha_0-\alpha_1-\alpha_2} \quad , \quad t_2>0,
\end{aligned}
\tag{3.12}
$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

Proof

From equations (3.1) and (3.5), it follows that

$$\begin{aligned} g\left(t_2|t_1\right) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2-1}\left(1+t_1\right)^{\alpha_2} \\ &\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\div \left[\frac{\left(\beta_0^{\alpha_1}\beta_1^{\alpha_0}\right)\Gamma\left(\alpha_0+\alpha_1\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)}\left(t_1\right)^{\alpha_1-1}\left(\beta_1+\beta_0 t_1\right)^{-\alpha_0-\alpha_1}\right] \\ &= \frac{\left(\beta_0^{\alpha_2}\beta_1^{\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0+\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(t_2\right)^{\alpha_2-1}\left(1+t_1\right)^{\alpha_2}\left(\beta_1+\beta_0 t_1\right)^{\alpha_0+\alpha_1} \\ &\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2}. \end{aligned}$$

■

3.2.4 Product moment

Theorem 3.6

The product moment of T_1 and T_2 is given by

$$\begin{aligned} E\left(T_1^r T_2^s\right) &= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{-\alpha_1}\beta_2^{s}\right)\Gamma(\alpha_2+s)\Gamma(\alpha_0+\alpha_1-s)\Gamma(\alpha_1+r)\Gamma(\alpha_0-r)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_0+\alpha_1)} \\ &\times {}_2F_1\left(\alpha_0+\alpha_1-s,\alpha_1+r;\alpha_0+\alpha_1;1-\frac{\beta_0}{\beta_1}\right) \quad , \quad \alpha_0+\alpha_1>s, \\ &\qquad \alpha_0>r \text{ and } \left|1-\frac{\beta_0}{\beta_1}\right|<1, \end{aligned} \tag{3.13}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

Proof

The product moment of a bivariate distribution is defined in Result 5. By substituting Equation (3.1) into Result 5, it follows that

$$
\begin{aligned}
E\left(T_1^r T_2^s\right) &= \int_0^\infty \int_0^\infty g\left(t_1, t_2\right) t_1^r t_2^s dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(t_2\right)^{\alpha_2+s-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \int_0^\infty \left(t_2\right)^{\alpha_2+s-1}\left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \int_0^\infty \left(t_2\right)^{\alpha_2+s-1}\left[\left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)\left(1+\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}t_2\right)\right]^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \int_0^\infty \left(t_2\right)^{\alpha_2+s-1}\left(1+\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1. \quad (3.14)
\end{aligned}
$$

By applying Result 13, Equation (3.14) may be rewritten as

$$
\begin{aligned}
E\left(T_1^r T_2^s\right) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}\right)^{-\alpha_2-s} B\left(\alpha_2+s, \alpha_0+\alpha_1+\alpha_2-\alpha_2-s\right) dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}\right)^{-\alpha_2-s} \frac{\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)} dt_1
\end{aligned}
$$

$$
\begin{aligned}
E\left(T_1^r T_2^s\right) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\frac{\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}\int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{-s} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(\beta_0\beta_1\right)^{-\alpha_2-s}\left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{\alpha_2+s}dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{\alpha_0-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{-s}\left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1+s}dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{\alpha_0-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{-s}\left[\left(\beta_1\beta_2\right)\left(1+\frac{\beta_0\beta_2}{\beta_1\beta_2}t_1\right)\right]^{-\alpha_0-\alpha_1+s}dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{\alpha_0-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(\beta_1\beta_2\right)^{-\alpha_0-\alpha_1+s} \\
&\times \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{-s}\left(1+\frac{\beta_0}{\beta_1}t_1\right)^{-\alpha_0-\alpha_1+s}dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{-\alpha_1}\beta_2^{s}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty \left(t_1\right)^{\alpha_1+r-1}\left(1+t_1\right)^{-s}\left(1+\frac{\beta_0}{\beta_1}t_1\right)^{-\alpha_0-\alpha_1+s}dt_1. \qquad (3.15)
\end{aligned}
$$

Applying Result 12 and Result 7 to Equation (3.15) the following expression follows

$$
\begin{aligned}
E\left(T_1^r T_2^s\right) &= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{-\alpha_1}\beta_2^{s}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times B\left(\alpha_1+r,\alpha_0-r\right){}_2F_1\left(\alpha_0+\alpha_1-s,\alpha_1+r;\alpha_0+\alpha_1;1-\frac{\beta_0}{\beta_1}\right) \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{-\alpha_1}\beta_2^{s}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \frac{\Gamma\left(\alpha_1+r\right)\Gamma\left(\alpha_0-r\right)}{\Gamma\left(\alpha_0+\alpha_1\right)}{}_2F_1\left(\alpha_0+\alpha_1-s,\alpha_1+r;\alpha_0+\alpha_1;1-\frac{\beta_0}{\beta_1}\right) \\
&= \frac{\left(\beta_0^{\alpha_1-s}\beta_1^{-\alpha_1}\beta_2^{s}\right)\Gamma\left(\alpha_2+s\right)\Gamma\left(\alpha_0+\alpha_1-s\right)\Gamma\left(\alpha_1+r\right)\Gamma\left(\alpha_0-r\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_0+\alpha_1\right)} \\
&\times {}_2F_1\left(\alpha_0+\alpha_1-s,\alpha_1+r;\alpha_0+\alpha_1;1-\frac{\beta_0}{\beta_1}\right).
\end{aligned}
$$

■

3.3 Multivariate distribution

3.3.1 Joint density function

Theorem 3.7

Let W_i be independent gamma random variables with parameters $(\alpha_i > 0, \beta_i > 0)$ for $i = 0, 1, 2, ..., m$.

Let $T_1 = \frac{W_1}{W_0}$ and $T_r = \frac{W_r}{\sum_{i=0}^{r-1} W_i}, r = 2, 3, ..., m$.

The joint density function is then given by

$$\begin{aligned} g(t_1, t_2, ..., t_m) &= \frac{\Gamma\left(\sum_{j=0}^{m} \alpha_j\right)}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma(\alpha_j)\right]} \left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right) \\ &\times \left(\frac{1}{\beta_0} + \frac{t_1}{\beta_1} + \sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}[1+t_k]\right]\right)^{-\sum_{j=0}^{m}\alpha_j} \quad , t_1, t_2, ..., t_m > 0. \end{aligned} \tag{3.16}$$

Proof
Since the gamma random variables are independent, the joint density function of $W_i, i = 0, 1, 2, ..., m.$ is given by

$$g(w_0, w_1, ..., w_m) = \prod_{i=0}^{m} \frac{\left(w_i^{\alpha_i - 1} e^{-\frac{w_i}{\beta_i}}\right)}{\beta_i^{\alpha_i}\Gamma(\alpha_i)}, w_0, w_2, ..., w_m > 0. \tag{3.17}$$

Let $T = W_0, T_1 = \frac{W_1}{W_0}$ and $T_r = \frac{W_r}{\sum_{i=0}^{r-1} W_i}, r = 2, 3, ..., m$.

The inverse transformation is then given by

$$\begin{aligned} W_0 &= T \\ W_1 &= T_1 T \\ W_2 &= T_2 T (1 + T_1) \\ W_3 &= T_3 T (1 + T_1)(1 + T_2). \end{aligned}$$

In general, the transformation is

$W_i = T_i T \prod_{k=1}^{i-1} (1 + T_k), i = 2, ..., m.$

The Jacobian of the transformation is then

$J(w_0, ..., w_m \to t, t_1, ..., t_m)$

$$= \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ t_1 & t & 0 & \ldots & 0 \\ t_2(1+t_1) & t_2 t & t(1+t_1) & \ldots & 0 \\ t_3(1+t_2)(1+t_1) & t_3 t(1+t_2) & t_3 t(1+t_1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ t_m \prod_{k=1}^{m-1}(1+t_k) & t_m t \prod_{k=2}^{m-1}(1+t_k) & t_m t(1+t_1)\prod_{k=3}^{m-1}(1+t_k) & \ldots & t\prod_{k=1}^{m-1}(1+t_k) \end{vmatrix}$$
$$= t^m \prod_{k=1}^{m-1}(1+t_k)^{m-k}.$$

By making the transformation and substituting the equations for $W_0, W_1, \ldots W_m$ into Equation (3.17), it follows that the joint density function of $T, T_1, T_2, \ldots, T_m$ is

$$\begin{aligned} g(t,t_1,t_2,\ldots,t_m) &= \frac{\left(t^{\alpha_0-1}e^{-\frac{t}{\beta_0}}\right)}{\beta_0^{\alpha_0}\Gamma(\alpha_0)}\frac{\left((t_1t)^{\alpha_1-1}e^{-\frac{t_1t}{\beta_1}}\right)}{\beta_1^{\alpha_1}\Gamma(\alpha_1)}t^m\prod_{k=1}^{m-1}(1+t_k)^{m-k} \\ &\times \prod_{j=2}^{m}\frac{\left(\left(t_jt\prod_{k=1}^{j-1}(1+t_k)\right)^{\alpha_j-1}e^{-\frac{t_jt\prod_{k=1}^{j-1}(1+t_k)}{\beta_j}}\right)}{\beta_j^{\alpha_j}\Gamma(\alpha_j)} \\ &= = \frac{1}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma(\alpha_j)\right]}\left(t^{\sum_{j=0}^{m}[\alpha_j]-1}\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right) \\ &\times \times\prod_{j=2}^{m}\left[\prod_{k=1}^{j-1}[1+t_k]^{\alpha_j-1}\right]\prod_{k=1}^{m-1}[1+t_k]^{m-k} \\ &\times \times e^{-t\left(\frac{1}{\beta_0}+\frac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}[1+t_k]\right]\right)} \\ &= = \frac{1}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma(\alpha_j)\right]}\left(t^{\sum_{j=0}^{m}[\alpha_j]-1}\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right) \\ &\times \times\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)e^{-t\left(\frac{1}{\beta_0}+\frac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}[1+t_k]\right]\right)}. \end{aligned} \tag{3.18}$$

The last step in Equation (3.18) follows by merely reordering the terms of the equation. For a more detailed discussion of how to perform this reordering see Adamski [1] p19.

By integrating Equation (3.18) with respect to t, it follows that

$$\begin{aligned} g(t_1,t_2,\ldots,t_m) &= \frac{1}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma(\alpha_j)\right]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right) \\ &\times \int_0^\infty t^{\sum_{j=0}^{m}[\alpha_j]-1}e^{-t\left(\frac{1}{\beta_0}+\frac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}[1+t_k]\right]\right)}dt. \end{aligned} \tag{3.19}$$

By applying Result 11 to Equation (3.19) it follows that

$$\begin{aligned}
g\left(t_1,t_2,...,t_m\right) &= \frac{1}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)\right]}\left(\prod_{k=1}^{m-1}\left[\left(1+t_k\right)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right)\\
&\times\ \Gamma\left(\sum_{j=0}^{m}\alpha_j\right)\left(\frac{1}{\beta_0}+\frac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha_j}\\
&= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha_j\right)}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)\right]}\left(\prod_{k=1}^{m-1}\left[\left(1+t_k\right)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right)\\
&\times\left(\frac{1}{\beta_0}+\frac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta_j}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha_j}.
\end{aligned}$$

■

Special cases

1) If $\alpha_i=\alpha$ for $i=0,1,2,...,m$ then Equation (3.16) becomes

$$\begin{aligned}
g\left(t_1,t_2,...,t_m\right) &= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha\right)}{\prod_{j=0}^{m}\left[\beta_j^{\alpha}\Gamma(\alpha)\right]}\left(\prod_{k=1}^{m-1}\left[\left(1+t_k\right)^{\sum_{j=k+1}^{m}\alpha}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha-1}\right]\right)\\
&\times\left(\tfrac{1}{\beta_0}+\tfrac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\tfrac{t_j}{\beta_j}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha}\\
&= \frac{\Gamma((m+1)\alpha)}{(m+1)\Gamma(\alpha)\prod_{j=0}^{m}\left[\beta_j^{\alpha}\right]}\left(\prod_{k=1}^{m-1}\left[\left(1+t_k\right)^{(m-k)\alpha}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha-1}\right]\right)\\
&\times\left(\tfrac{1}{\beta_0}+\tfrac{t_1}{\beta_1}+\sum_{j=2}^{m}\left[\tfrac{t_j}{\beta_j}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-(m+1)\alpha} \quad , \quad t_1,t_2,...,t_m>0.
\end{aligned}$$

2) If $\beta_i=\beta$ for $i=0,1,2,...,m$ then Equation (3.16) becomes

$$
\begin{aligned}
g\left(t_1,t_2,...,t_m\right) &= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha_j\right)}{\prod_{j=0}^{m}\left[\beta^{\alpha_j}\Gamma(\alpha_j)\right]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right)\\
&\times\left(\frac{1}{\beta}+\frac{t_1}{\beta}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha_j}\\
&= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha_j\right)}{\beta^{\sum_{j=0}^{m}[\alpha_j]}\prod_{j=0}^{m}\left[\Gamma(\alpha_j)\right]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right)\\
&\times\left(\frac{1}{\beta}\right)^{-\sum_{j=0}^{m}\alpha_j}\left(1+t_1+\sum_{j=2}^{m}\left[t_j\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha_j}\\
&= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha_j\right)}{\prod_{j=0}^{m}\left[\Gamma(\alpha_j)\right]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha_j}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha_j-1}\right]\right)\\
&\times\left(1+t_1+\sum_{j=2}^{m}\left[t_j\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha_j} \quad , \quad t_1,t_2,...,t_m>0.
\end{aligned}
$$

3) If $\beta_i=\beta$ and $\alpha_i=\alpha$ for $i=0,1,2,...,m$ then Equation (3.16) becomes

$$
\begin{aligned}
g\left(t_1,t_2,...,t_m\right) &= \frac{\Gamma\left(\sum_{j=0}^{m}\alpha\right)}{\prod_{j=0}^{m}\left[\beta^{\alpha}\Gamma(\alpha)\right]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{\sum_{j=k+1}^{m}\alpha}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha-1}\right]\right)\\
&\times\left(\frac{1}{\beta}+\frac{t_1}{\beta}+\sum_{j=2}^{m}\left[\frac{t_j}{\beta}\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-\sum_{j=0}^{m}\alpha}\\
&= \frac{\Gamma((m+1)\alpha)}{(m+1)[\Gamma(\alpha)]\beta^{(m+1)\alpha}}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{(m-k)\alpha}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha-1}\right]\right)\\
&\times\left(\frac{1}{\beta}\right)^{-(m+1)\alpha}\left(1+t_1+\sum_{j=2}^{m}\left[t_j\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-(m+1)\alpha}\\
&= \frac{\Gamma((m+1)\alpha)}{(m+1)[\Gamma(\alpha)]}\left(\prod_{k=1}^{m-1}\left[(1+t_k)^{(m-k)\alpha}\right]\right)\left(\prod_{j=1}^{m}\left[t_j^{\alpha-1}\right]\right)\\
&\times\left(1+t_1+\sum_{j=2}^{m}\left[t_j\prod_{k=1}^{j-1}\left[1+t_k\right]\right]\right)^{-(m+1)\alpha} \quad , \quad t_1,t_2,...,t_m>0.
\end{aligned}
$$

Chapter 4

Generalised beta type VIII distribution

4.1 Introduction

This chapter focuses on the joint density function of the statistics that are defined in Equation (1.11), its respective properties and subsequent derivations. In Section 4.2 the focus is on the bivariate case, where $U_1 = \frac{W_1+W_2}{W_0}$ and $U_2 = \frac{W_2}{W_0+W_1}$ where W_i are independent gamma random variables with parameters $(\alpha_i > 0, \beta_i > 0)$ for $i = 0, 1, 2$. In Section 4.3 the focus is on the multivariate case. These derivations are necessary for creating the theoretical foundation that would be required to derive the closed for expression for the control limits as discussed in Chapter 1.

The main focus of this chapter will be to expand on the bivariate probability density function and its relevant properties, since, as was stated in the study outline, the methods used during the bivariate derivations are likely to be similar in nature to the methods required during higher dimensional derivations and focusing on the bivariate case also allows graphical representations of the distributions to be plotted, which would be impossible for higher dimensions.

4.2 Bivariate distribution

Initially, the joint density function of the statistics in Equation (1.11) is derived in Section 4.2.1. This is followed by a shape analysis and a short investigation into the three special cases described in Chapter 3, namely where the α's are all equal but the β's may be unequal, where all the β's are equal but the α's may differ, and thirdly where all the α's are equal and all the β's are equal. In Section 4.2.3, the marginal density functions for both U_1 and U_2 are derived by integrating out the respective variables from the joint density function. An alternative method for deriving the marginal density functions is mentioned in a note. This alternative method involves constructing the marginal density functions from independent gamma random variables. The derivations are similar in nature to those in sections 3.2.1 and 4.2.1. This is followed an exploratory shape analysis and then by the derivations of the respective conditional density functions. The product moment of U_1 and U_2 is derived in Section 4.2.5. Again two different methods are used. First the traditional method, namely integrating the joint density function with respect to both variables. It is then also derived by using a transformation between the variables defined in sections 4.2.1 and 3.2. It will be shown that deriving the product moment using the traditional method results in a closed-form expression with restrictions that are impossible to meet, and thus the transformation method is necessitated. (This demonstrates the importance and relevance of the table of transformations between the bivariate beta density function derived in Chapter 2, Table 2.1.)

4.2.1 Joint density function

Theorem 4.1

Let W_i be independent gamma random variables with parameters $(\alpha_i > 0, \beta_i > 0)$ for $i = 0, 1, 2$.

Let $U_1 = \frac{W_1+W_2}{W_0}$ and $U_2 = \frac{W_2}{W_0+W_1}$.

The joint density function of U_1 and U_2 is then given by

$$\begin{aligned} f(u_1, u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\ &\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2}, \quad u_1 > u_2 > 0. \end{aligned} \tag{4.1}$$

Proof
Since the gamma random variables are independent, the joint density function of W_0, W_1, W_2 is given by

$$f(w_0, w_1, w_2) = \frac{\left(w_0^{\alpha_0-1}e^{-\frac{w_0}{\beta_0}}\right)\left(w_1^{\alpha_1-1}e^{-\frac{w_1}{\beta_1}}\right)\left(w_2^{\alpha_2-1}e^{-\frac{w_2}{\beta_2}}\right)}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}, w_0, w_1, w_2 > 0. \tag{4.2}$$

Let $U = W_0, U_1 = \frac{W_1+W_2}{W_0}$ and $U_2 = \frac{W_2}{W_0+W_1}$.

Using the "variables in common" technique it is possible to find the joint density function of U, U_1 and U_2 as follows:

The inverse transformation is given by

$W_0 = U$
$W_1 = U_1U - W_2$
$W_2 = U_2(U + W_1)$.

By simultaneous equations, it is possible to solve W_1 and W_2 in terms of U, U_1 and U_2.

$W_1 = U_1U - U_2(U + W_1)$
$W_1 = U_1U - U_2U - U_2W_1$
$W_1 + U_2W_1 = U_1U - U_2U$
$W_1(1 + U_2) = U_1U - U_2U$
$W_1 = \frac{U_1U-U_2U}{(1+U_2)}$
$W_1 = \frac{U(U_1-U_2)}{(1+U_2)}$

$W_2 = U_2\left(U + W_1\right) = U_2\left(U + \frac{U(U_1-U_2)}{(1+U_2)}\right)$
$W_2 = U_2 U + U_2\frac{U(U_1-U_2)}{(1+U_2)}$
$W_2 = \frac{UU_2(1+U_2)+U_2U(U_1-U_2)}{(1+U_2)}$
$W_2 = \frac{UU_2+UU_2U_2+UU_1U_2-UU_2U_2}{(1+U_2)}$
$W_2 = \frac{UU_2+UU_1U_2}{(1+U_2)}$
$W_2 = \frac{UU_2(1+U_1)}{(1+U_2)}.$

The Jacobian of this transformation is then given by

$$
\begin{aligned}
&J\left(w_0, w_1, w_2 \to u, u_1, u_2\right)\\
&= \begin{vmatrix} 1 & 0 & 0 \\ \frac{(u_1-u_2)}{(1+u_2)} & \frac{u}{(1+u_2)} & \frac{-u(1+u_2)-u(u_1-u_2)}{(1+u_2)^2} \\ \frac{u_2(1+u_1)}{(1+u_2)} & \frac{uu_2}{(1+u_2)} & \frac{u(1+u_1)(1+u_2)-uu_2(1+u_1)}{(1+u_2)^2} \end{vmatrix}\\
&= \begin{vmatrix} 1 & 0 & 0 \\ \frac{(u_1-u_2)}{(1+u_2)} & \frac{u}{(1+u_2)} & \frac{-u(1+u_1)}{(1+u_2)^2} \\ \frac{u_1(1+u_1)}{(1+u_1)} & \frac{uu_2}{(1+u_2)} & \frac{u(1+u_1)}{(1+u_2)^2} \end{vmatrix}\\
&= \frac{u}{(1+u_2)}\frac{u(1+u_1)}{(1+u_2)^2} - \frac{-u(1+u_1)}{(1+u_2)^2}\frac{uu_2}{(1+u_2)}\\
&= \frac{uu(1+u_1)}{(1+u_2)^3} + \frac{uuu_2(1+u_1)}{(1+u_2)^3}\\
&= \frac{uu(1+u_1)(1+u_2)}{(1+u_2)^3}\\
&= \frac{u^2(1+u_1)}{(1+u_2)^2}.
\end{aligned}
$$

By making the transformation and substituting the equations for W_0, W_1 and W_2 into Equation (4.2), it follows that the joint density function of U, U_1, U_2 is

$$
\begin{aligned}
f\left(u, u_1, u_2\right) &= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} u^{\alpha_0-1}e^{-\frac{u}{\beta_0}}\left(\frac{u\left(u_1-u_2\right)}{\left(1+u_2\right)}\right)^{\alpha_1-1}\\
&\times \; e^{-\frac{\frac{u(u_1-u_2)}{(1+u_2)}}{\beta_1}}\left(\frac{uu_2\left(1+u_1\right)}{\left(1+u_2\right)}\right)^{\alpha_2-1} e^{-\frac{\frac{uu_2(1+u_1)}{(1+u_2)}}{\beta_2}}\frac{u^2\left(1+u_1\right)}{\left(1+u_2\right)^2}\\
&= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(u_1-u_2\right)^{\alpha_1-1}u_2^{\alpha_2-1}\left(1+u_1\right)^{\alpha_2}\left(1+u_2\right)^{-\alpha_1-\alpha_2}\\
&\times \; u^{(\alpha_0+\alpha_1+\alpha_2)-1}e^{\frac{-u(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1))}{\beta_0\beta_1\beta_2(1+u_2)}}. \qquad (4.3)
\end{aligned}
$$

By integrating Equation (4.3) with respect to u, it follows that

$$\begin{aligned} f(u_1,u_2) &= \frac{1}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{-\alpha_1-\alpha_2} \\ &\quad\times \int_0^{\infty} u^{(\alpha_0+\alpha_1+\alpha_2)-1} e^{\frac{-u(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1))}{\beta_0\beta_1\beta_2(1+u_2)}}du. \end{aligned} \tag{4.4}$$

By applying Result 11 to Equation (4.4), it then follows that

$$\begin{aligned} f(u_1,u_2) &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{-\alpha_1-\alpha_2} \\ &\quad\times \left(\frac{(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1))}{\beta_0\beta_1\beta_2(1+u_2)}\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\ &\quad\times (\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1))^{-\alpha_0-\alpha_1-\alpha_2}. \end{aligned}$$

■

4.2.2 Shape analysis

In this section an exploratory shape analysis is conducted into the joint density function given in Equation (4.1). (To avoid repetition, for the remainder of this chapter whenever reference is made to the "joint density function", it is Equation (4.1) that is being referred to). A standard set of parameters has been chosen as a baseline. The parameters are chosen to be $\alpha_0 = \alpha_1 = \alpha_2 = 5$ and $\beta_0 = \beta_1 = \beta_2 = 2$, in other words, a process where all three samples consist of $5 \times 2 + 1 = 11$ observations (since $\alpha_i = \frac{n_i-1}{2}$), and where no shift has occurred in the process variance. Some of the parameters will then be varied from this baseline in order to investigate the effect that a change in the specific parameters will have on the general shape of the joint density function.

Note that the change in some parameters will be large - so large that they lose practical realism. This is done to emphasise and investigate the general change in the shape, and is not meant to be an indication of the practical applications of the joint density function. For example, suppose that, initially a process is IC and thus $Y_i \sim Gamma(5,2)$. If the process undergoes a 20% increase in variance, this would correspond to a $Y_i^* = 1.2Y_i \sim Gamma(5, 2\times 1.2) \equiv Gamma(5, 2.4)$ random variable. This increase in the scale parameter would probably have a minimal effect on the visual appearance of a bivariate graph of the density function. Instead, to emphasise the potential effect of an increase in the process variance on the shape of the density function, the scale parameter could be varied to 5, for instance. This would correspond to a $Y_i^{**} = 2.5Y_i \sim Gamma(5, 2\times 2.5) \equiv Gamma(5,5)$ random variable, which implies that the process variance increased 250% over the IC baseline values. In practice, a 250% increase in the variance is highly unlikely, but it would serve to illustrate how the joint density function's shape corresponds to certain parameter choices.

The functions will only be plotted on the $u_1 \in [0,3]$, $u_2 \in [0,3]$ domain. This by no means implies that the functions stop at the upper limit of 3. The main purpose of this chapter is the comparison between different parameterisations of the joint density function.

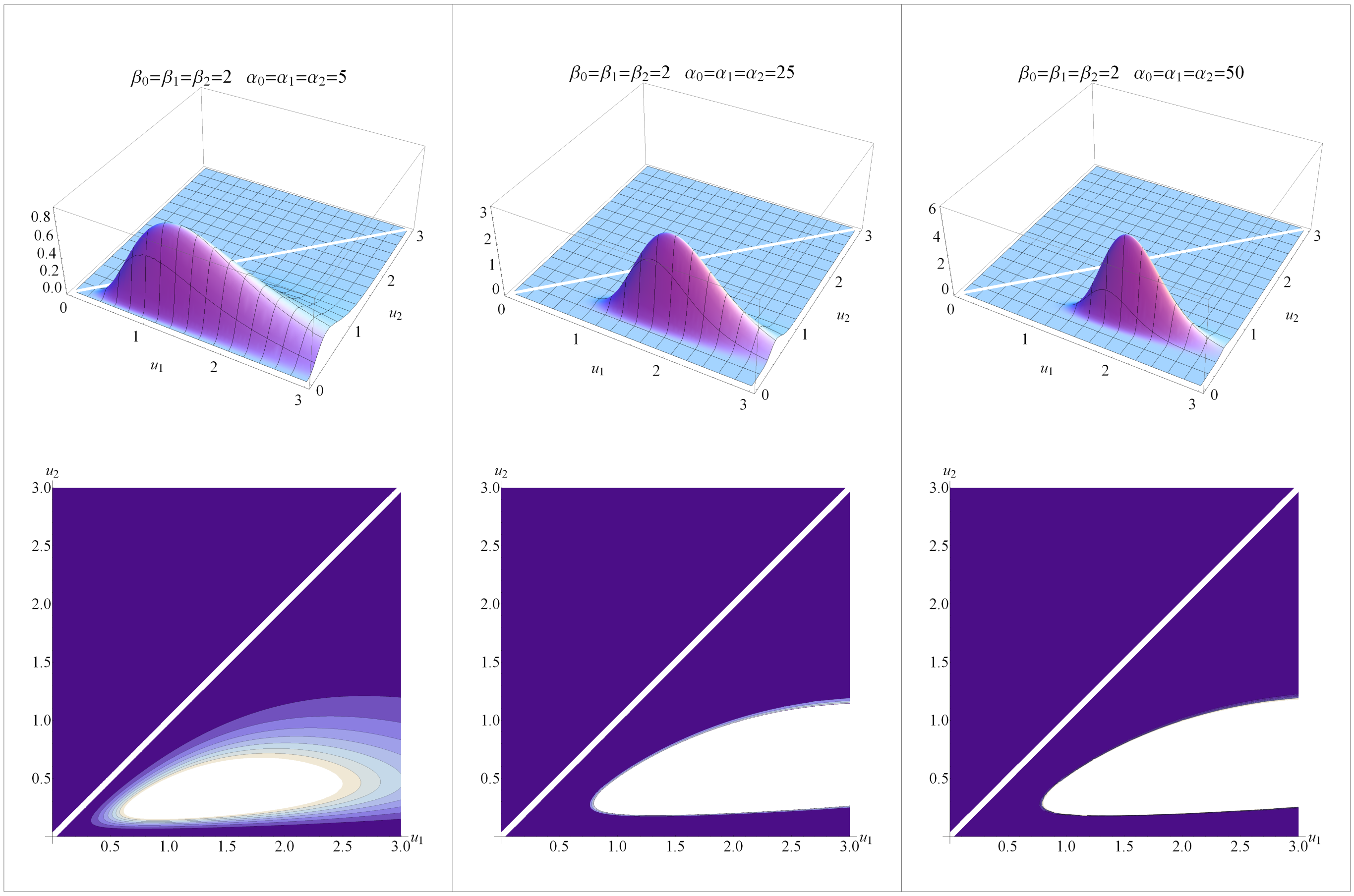

Figure 4.1: Equal sample sizes and in control process.

From Figure 4.1, it is apparent that increasing the sample sizes also increases the height of the peak of the density function. Larger sample sizes also shrink the length and width of the “tails” of the joint density function. In essence, the higher the sample sizes, the smaller the domain on which the function has significant values. Increasing all sample sizes has very little effect in shifting the location of the joint density function.

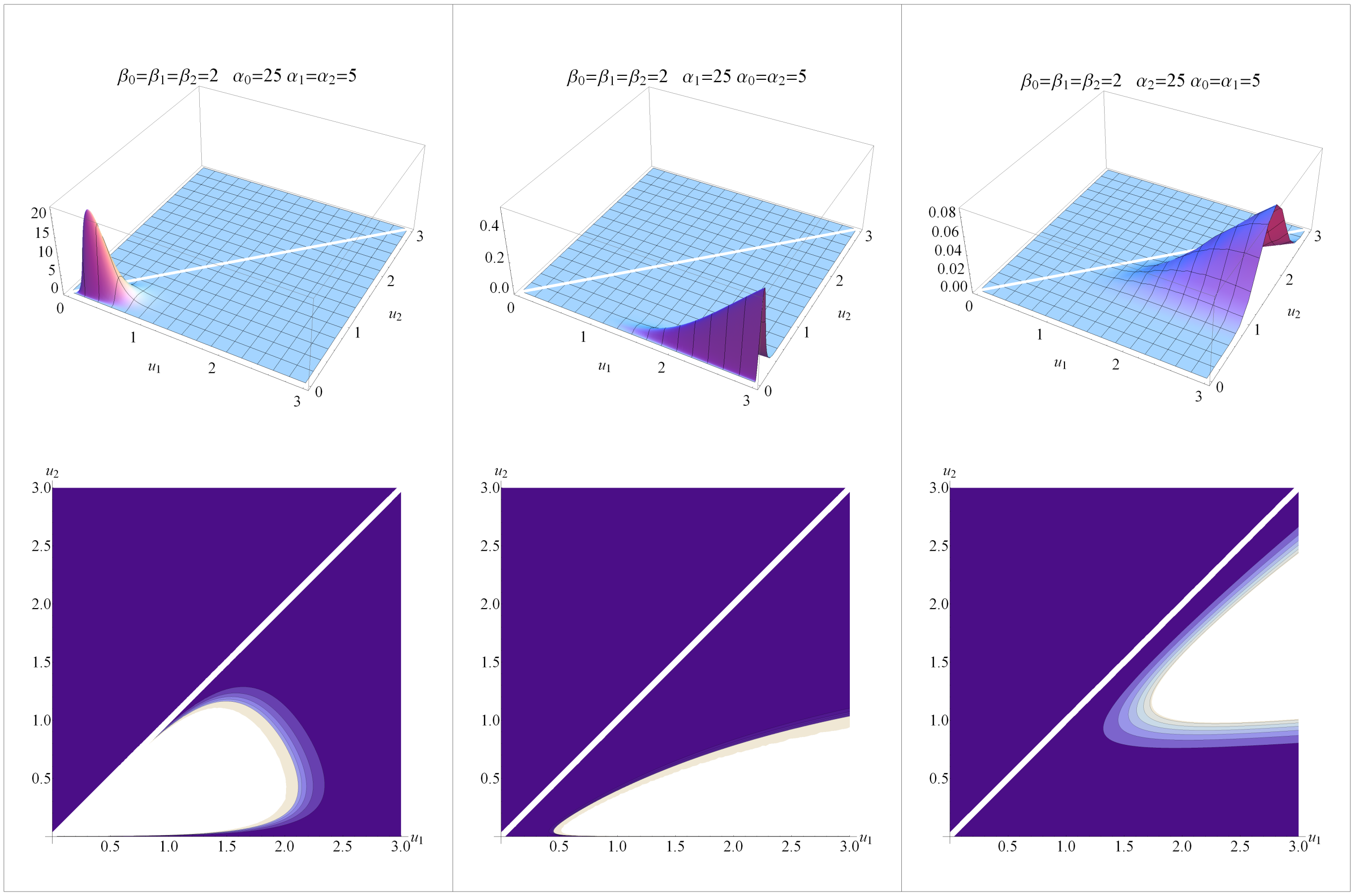

Figure 4.2: Varying sample sizes and in control process.

Figure 4.2 illustrates the effect that having one large (sample relative to the other two small samples) would have on the shape of the joint density function. Having a relatively large first sample would shift the location of the joint density function towards the minimum end of the domain - in other words closer to the [0,0] coordinate. However, this also results in joint density function having a much higher peak. Having a relatively large second sample (relative the first and third samples) elongates the joint density function in the U_1 direction, making it insensitive to large values in U_2. This result coincides with what one would expect given the practical, SPC interpretation of the U_1 statistic, namely that if the sample at time 1 contains a lot of data points, the statistic at this time (U_1) would have a large impact on the joint density function. Similarly, if the third sample is large relative to the first and second samples, the joint density function becomes elongated in the U_2 direction, making it relatively insensitive to large values in U_1. This result again coincides with the general intuition given the practical, SPC interpretation of the U_2 statistic.

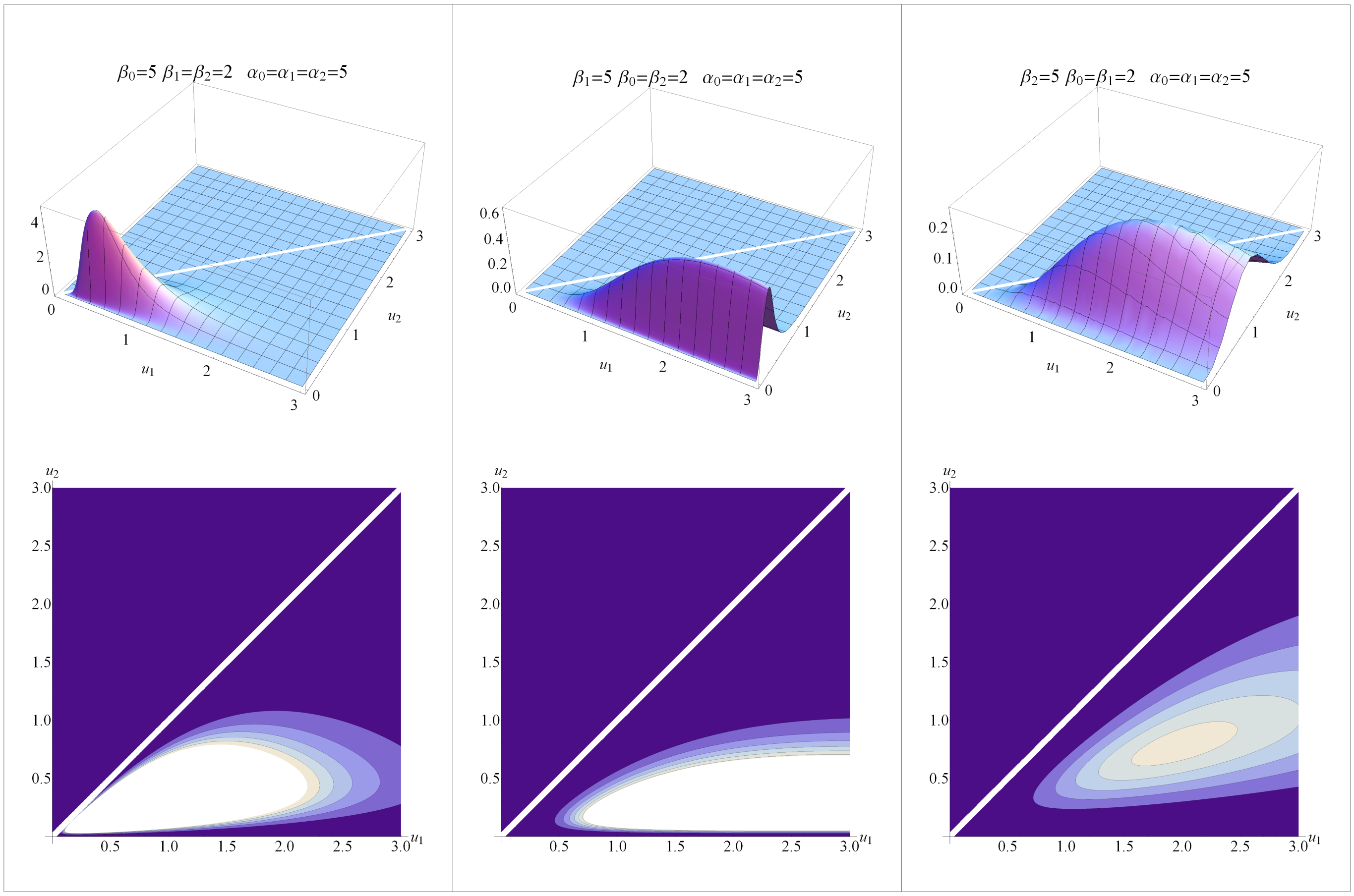

Figure 4.3: Equal sample sizes, unsustained increase in variance.

It should be noted that the scenario in the above image, Figure 4.3, falls outside the practical application of the model that this study proposes (at least for the first two graphs). As previously stated, this study is mainly concerned with detecting a sustained shift in the process variance. These figures are included however since they do yield insight into the workings of the derived joint probability density function. An unsustained increase in the process variance, as depicted in Figure 4.3, would be better modelled by a beta distribution that compares a single sample variance with the other m sample variances. A good example of this would be the bivariate beta type VI distribution mentioned in Chapter 2.

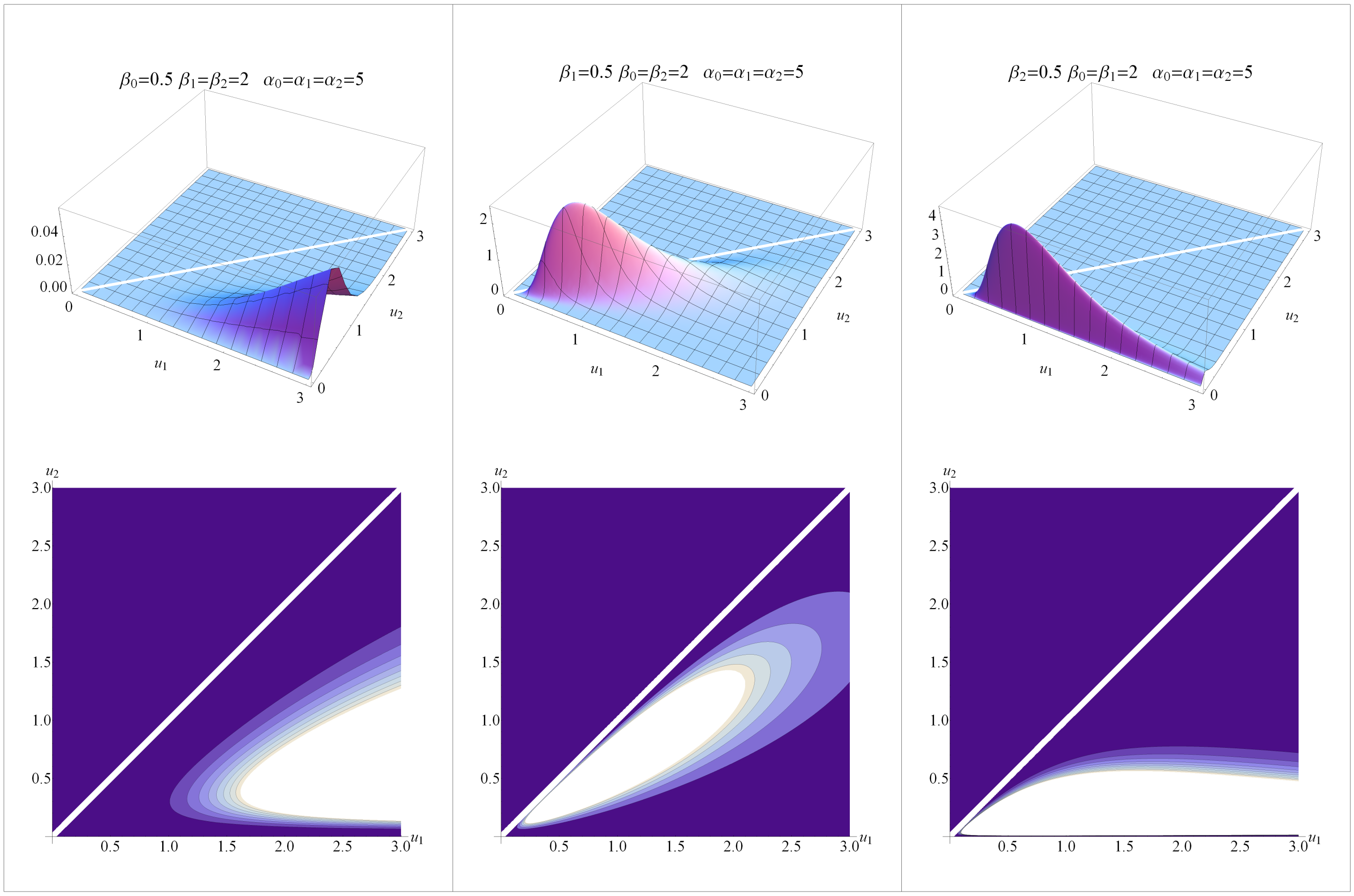

Figure 4.4: Equal sample sizes, unsustained decrease in variance.

Once again, it should be noted that the scenario in Figure 4.4, falls outside the practical application of the model that this study proposes. As previously stated this study is mainly concerned with detecting a sustained shift in the process variance. These figures are included however since they do yield insight into the workings of the derived joint probability density function. An unsustained decrease, as is depicted in Figure 4.4, would be better modeled by the bivariate beta type VI distribution mentioned in Chapter 2.

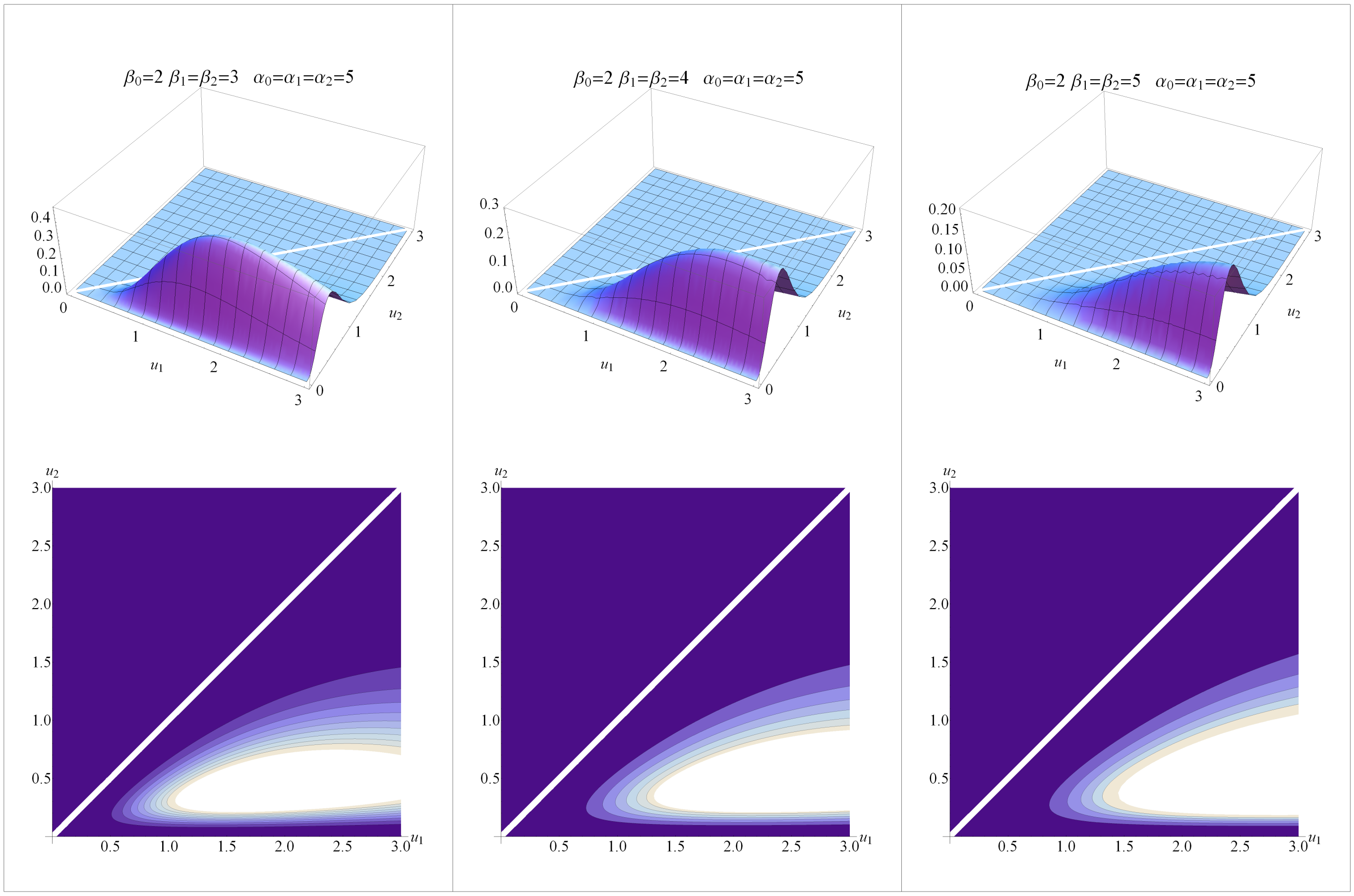

Figure 4.5: Equal sample sizes, sustained increase in variance.

Figure 4.5 illustrates an important advantage of the model that this study proposes over the more traditional Q chart method, namely its resistance to the "masking of shifts" problem that was mentioned in Chapter 1 and again in Section 1.1. The above graphs demonstrate that a sustained increase in the process variance, irrespective of size, minimally affects the general shape of the joint density function, but does affect the location. In the above example, the shift in the process variance occurs at time 1, and as one would hope and expect, the joint density function relies heavily on the value of the statistic at time 1, U_1.

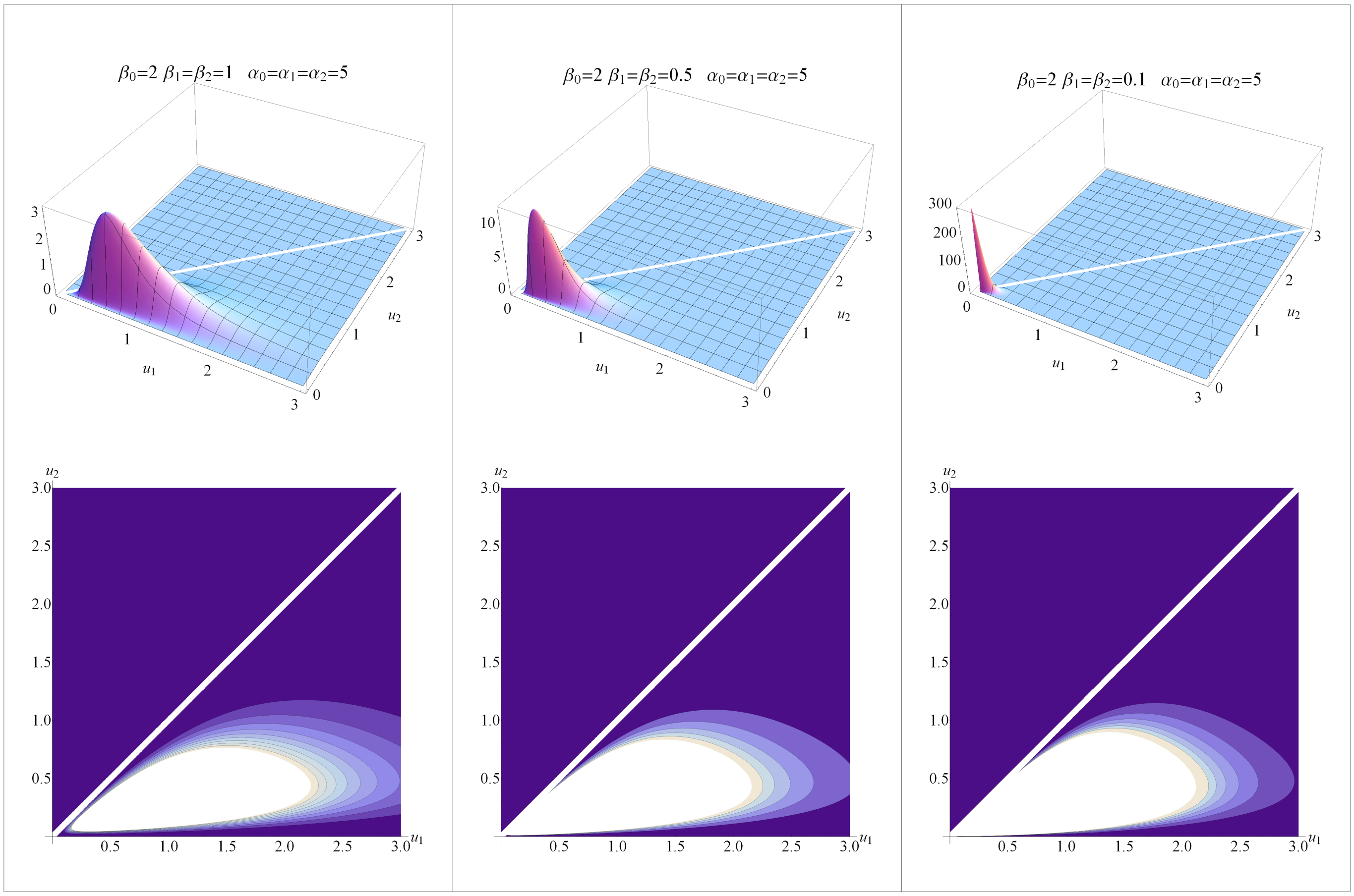

Figure 4.6: Equal sample sizes, sustained decrease in variance.

Figure 4.6 again illustrates the resistance of the proposed model to the "masking of shifts" problem that was mentioned in Chapter 1 and in Section 1.1. The above graphs demonstrate that a sustained decrease in the process variance, irrespective of size, once again does not significantly affect the general shape of the joint density function much, however, the location is affected. In the above example, the shift in the process variance again occurred at time 1, and as one would expect, the joint density function relies heavily on the value of the statistic at time 1, U_1. It should be noted however that a decrease in the process variance does skew the distribution significantly - much more so than an increase in the variance.

Special cases

1) If $\alpha_i = \alpha$ for $i = 0, 1, 2$, Equation (4.1) simplifies to

$$\begin{aligned} f(u_1,u_2) &= \tfrac{(\beta_0^{2\alpha}\beta_1^{2\alpha}\beta_2^{2\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)^3}(u_1-u_2)^{\alpha-1}u_2^{\alpha-1}(1+u_1)^{\alpha}(1+u_2)^{\alpha} \\ &\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-3\alpha} \quad , \quad u_1>u_2>0. \end{aligned}$$

2) If $\beta_i = \beta$ for $i = 0, 1, 2$, Equation (4.1) simplifies to

$$\begin{aligned} f(u_1,u_2) &= \tfrac{(\beta^{2\alpha_0+2\alpha_1+2\alpha_2})\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\ &\times \left(\beta^2((1+u_2)+(u_1-u_2)+u_2(1+u_1))\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \tfrac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\ &\times (1+u_1+u_2(1+u_1))^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \tfrac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\ &\times ((1+u_2)(1+u_1))^{-\alpha_0-\alpha_1-\alpha_2} \\ &= \tfrac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{-\alpha_0-\alpha_1}(1+u_2)^{-\alpha_1-\alpha_2} \quad , \quad u_1>u_2>0. \end{aligned}$$

3) If $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$, Equation (4.1) simplifies to

$$\begin{aligned} f(u_1,u_2) &= \tfrac{(\beta^{6\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(\alpha)\Gamma(\alpha)}(u_1-u_2)^{\alpha-1}u_2^{\alpha-1}(1+u_1)^{\alpha}(1+u_2)^{\alpha} \\ &\times \left(\beta^2((1+u_2)+(u_1-u_2)+u_2(1+u_1))\right)^{-3\alpha} \\ &= \tfrac{\Gamma(3\alpha)}{\Gamma(\alpha)^3}(u_1-u_2)^{\alpha-1}u_2^{\alpha-1}(1+u_1)^{\alpha}(1+u_2)^{\alpha} \\ &\times ((1+u_2)(1+u_1))^{-3\alpha} \\ &= \tfrac{\Gamma(3\alpha)}{\Gamma(\alpha)^3}(u_1-u_2)^{\alpha-1}u_2^{\alpha-1}(1+u_1)^{-2\alpha}(1+u_2)^{-2\alpha} \quad , \quad u_1>u_2>0. \end{aligned}$$

4.2.3 Marginal density functions

Deriving the marginal density functions by integrating out the relevant variables from the joint density function, Equation (4.1), proved to be difficult and time consuming, requiring a lot of trial and error in rearranging the terms of the bivariate beta density function that had to be integrated. The results that this method yields are far less compact than the method mentioned in the note at the end of this section; however, their restrictions are more realistic, and as such this form is of greater practical significance.

Marginal density function of U_1

Theorem 4.2
The marginal density function of U_1 is given by

$$\begin{aligned} f(u_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(\beta_1\beta_2+\beta_0\beta_2u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times (1+u_1)^{\alpha_2}\sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\ &\times \left. {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, k+\alpha_2; k+\alpha_1+\alpha_2; -\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right)\right], \\ & \quad u_1>0, \beta_1(\beta_2+\beta_0(1+u_1))>\beta_0\beta_2 \text{ and } \left|-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right|<1, \end{aligned} \tag{4.5}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

If $\alpha_0 \in \mathbb{N}$, as will be the case in an SPC setting, the sum changes from $\sum_{k=0}^{\infty}$ to $\sum_{k=0}^{\alpha_0}$. (See Result 18.)

Proof

By integrating Equation (4.1) with respect to u_2, rearranging the terms, as well as applying Result 18 and Result 19, it follows that

$$\begin{aligned} f(u_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}\int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\ &\times (\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1))^{-\alpha_0-\alpha_1-\alpha_2}\,du_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}\int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\ &\times (\beta_1\beta_2+\beta_1\beta_2u_2+\beta_0\beta_2u_1-\beta_0\beta_2u_2+\beta_0\beta_1u_2+\beta_0\beta_1u_1u_2)^{-\alpha_0-\alpha_1-\alpha_2}\,du_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}\int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\ &\times (\beta_1\beta_2+\beta_0\beta_2u_1+u_2(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1))^{-\alpha_0-\alpha_1-\alpha_2}\,du_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}+u_2\right)^{-\alpha_0-\alpha_1-\alpha_2}du_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}\sum_{k=0}^{\infty}\left(\binom{\alpha_0}{k}u_2^k\right)\left(\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}+u_2\right)^{-\alpha_0-\alpha_1-\alpha_2}du_2 \end{aligned}$$

$$\begin{aligned} f(u_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}\sum_{k=0}^{\infty}\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}u_2^k\right)\left(\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}+u_2\right)^{-\alpha_0-\alpha_1-\alpha_2}du_2 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \sum_{k=0}^{\infty}\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)\int_0^{u_1}(u_1-u_2)^{\alpha_1-1}u_2^{k+\alpha_2-1}\left(\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}+u_2\right)^{-\alpha_0-\alpha_1-\alpha_2}du_2. \end{aligned} \quad (4.6)$$

By applying Result 14 to Equation (4.6), the density function can be represented as

$$\begin{aligned} f(u_1) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \left(\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}\right)^{-\alpha_0-\alpha_1-\alpha_2}\sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\right. \\ &\times \left. B(\alpha_1,k+\alpha_2)\,{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,k+\alpha_2;k+\alpha_1+\alpha_2;-\frac{u_1}{\frac{\beta_1\beta_2+\beta_0\beta_2u_1}{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1}}\right)\right] \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2u_1)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\ &\times \left. {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,k+\alpha_2;k+\alpha_1+\alpha_2;-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right)\right]. \end{aligned}$$

■

Special cases U_1

1) If $\alpha_i = \alpha$ for $i = 0, 1, 2$, Equation (4.5) simplifies to

$$\begin{aligned} f(u_1) &= \tfrac{(\beta_0^{2\alpha}\beta_1^{2\alpha}\beta_2^{2\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)^3}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-3\alpha} \\ &\times \ (1+u_1)^{\alpha}\sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha!}{k!(\alpha-k)!}\right)(u_1)^{k+2\alpha-1}\tfrac{\Gamma(\alpha)\Gamma(k+\alpha)}{\Gamma(k+2\alpha)}\right. \\ &\times \ \left.{}_2F_1\left(3\alpha, k+\alpha; k+2\alpha; -\tfrac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1 u_1)}{\beta_1\beta_2+\beta_0\beta_2 u_1}\right)\right], \\ & \quad u_1>0 \text{ and } \left|-\tfrac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1 u_1)}{\beta_1\beta_2+\beta_0\beta_2 u_1}\right|<1. \end{aligned}$$

2) If $\beta_i = \beta$ for $i = 0, 1, 2$, Equation (4.5) simplifies to

$$\begin{aligned} f(u_1) &= \tfrac{(\beta^{\alpha_1+\alpha_2}\beta^{\alpha_0+\alpha_2}\beta^{\alpha_0+\alpha_1})\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\beta^2+\beta^2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \ (1+u_1)^{\alpha_2}\sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\tfrac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\ &\times \ \left.{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, k+\alpha_2; k+\alpha_1+\alpha_2; -\tfrac{u_1(\beta^2-\beta^2+\beta^2+\beta^2 u_1)}{\beta^2+\beta^2 u_1}\right)\right] \\ &= \tfrac{(\beta^{2\alpha_0+2\alpha_1+2\alpha_2})\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\beta^{2(-\alpha_0-\alpha_1-\alpha_2)}\left(1+u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\times \ (1+u_1)^{\alpha_2}\sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\tfrac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\ &\times \ \left.{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, k+\alpha_2; k+\alpha_1+\alpha_2; -\tfrac{u_1\beta^2(1+u_1)}{\beta^2(1+u_1)}\right)\right] \\ &= \tfrac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(1+u_1\right)^{-\alpha_0-\alpha_1} \\ &\times \ \sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\tfrac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\ &\times \ \left.{}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, k+\alpha_2; k+\alpha_1+\alpha_2; -u_1\right)\right] \qquad , \quad 0<u_1<1. \end{aligned}$$

3) If $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2$, Equation (4.5) simplifies to

$$\begin{aligned} f(u_1) &= \tfrac{(\beta^{2\alpha}\beta^{2\alpha}\beta^{2\alpha})\Gamma(3\alpha)}{\Gamma(\alpha)^3}\left(\beta^2+\beta^2 u_1\right)^{-3\alpha} \\ &\times \ (1+u_1)^{\alpha}\sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha!}{k!(\alpha-k)!}\right)(u_1)^{k+2\alpha-1}\tfrac{\Gamma(\alpha)\Gamma(k+\alpha)}{\Gamma(k+2\alpha)}\right. \\ &\times \ \left.{}_2F_1\left(3\alpha, k+\alpha; k+2\alpha; -\tfrac{u_1(\beta^2-\beta^2+\beta^2+\beta^2 u_1)}{\beta^2+\beta^2 u_1}\right)\right] \\ &= \tfrac{\Gamma(3\alpha)}{\Gamma(\alpha)^3}\left(1+u_1\right)^{-2\alpha} \\ &\times \ \sum_{k=0}^{\infty}\left[\left(\tfrac{\alpha!}{k!(\alpha-k)!}\right)(u_1)^{k+2\alpha-1}\tfrac{\Gamma(\alpha)\Gamma(k+\alpha)}{\Gamma(k+2\alpha)}\right. \\ &\times \ \left.{}_2F_1\left(3\alpha, k+\alpha; k+2\alpha; -u_1\right)\right] \qquad , \quad 0<u_1<1. \end{aligned}$$

Marginal density function of U_2

Theorem 4.3
The marginal density function of U_2 is given by

$$\begin{aligned} f(u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}(1+u_2)^{\alpha_0}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\right. \\ &\times \sum_{j=0}^{\infty}\left[(-1)^j\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2-j}\right. \\ &\times \left.\left.(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma(\alpha_0+\alpha_2+j-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2+j-k)}\right]\right] \quad , \quad u_2>0, \end{aligned} \tag{4.7}$$

where $\alpha_i, \beta_i > 0$ for $i = 0, 1, 2$.

If $\alpha_2 \in \mathbb{N}$, as will be the case in an SPC setting, the sum changes from $\sum_{k=0}^{\infty}$ to $\sum_{k=0}^{\alpha_2}$. (See Result 18.)

Proof

By integrating Equation (4.1) with respect to u_1, and rearranging the terms, it follows that

$$\begin{aligned} f(u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\ &\times \int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2}\left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2}du_1 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\ &\times \int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2}\left(\beta_1\beta_2+\beta_1\beta_2u_2+\beta_0\beta_2u_1-\beta_0\beta_2u_2+\beta_0\beta_1u_2+\beta_0\beta_1u_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2}du_1 \\ &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2} \\ &\times \left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2+u_1(\beta_0\beta_2+\beta_0\beta_1u_2)\right)^{-\alpha_0-\alpha_1-\alpha_2}du_1. \end{aligned} \tag{4.8}$$

By applying Result 18 and Result 19 to Equation (4.8), it follows that

$$
\begin{aligned}
f(u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2}\left(\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}+u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}\sum_{k=0}^{\infty}\left(\frac{\alpha_2!}{k!(\alpha_2-k)!}u_1^k\right)\left(\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}+u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}u_1^k\left(\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}+u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1\right].
\end{aligned} \tag{4.9}
$$

Applying Result 15 to Equation (4.9) leads to

$$
\begin{aligned}
f(u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \left(\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}+u_2\right)^{-\alpha_0-\alpha_2}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}u_2^kB(-k+\alpha_0+\alpha_2,\alpha_1)\right. \\
&\times \left. {}_2F_1\left(-k,\alpha_1;-k-\alpha_1;-\frac{\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}}{u_2}\right)\right] \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \left(\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2+\beta_0\beta_1u_2}+u_2\right)^{-\alpha_0-\alpha_2}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}u_2^kB(-k+\alpha_0+\alpha_2,\alpha_1)\right. \\
&\times \left. {}_2F_1\left(-k,\alpha_1;-k-\alpha_1;-\frac{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}{\beta_0\beta_2u_2+\beta_0\beta_1u_2^2}\right)\right].
\end{aligned}
$$

Note, however, that in the above expression, the third argument of the hypergeometric function will always be a negative integer, ranging from $-\alpha_1$ to $-\alpha_1-\alpha_2$ (in the case that $\alpha_2 \in \mathbb{N}$). This results in the function being undefined or infinite at these points. An alternative expression can be derived by rearranging Equation (4.8), integrating with respect to u_1, and applying Result 18 and Result 19 as follows

$$
\begin{aligned}
f(u_2) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0} \\
&\times \int_{u_2}^{\infty} (u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2+u_1(\beta_0\beta_2+\beta_0\beta_1u_2)\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \int_{u_2}^{\infty} (u_1-u_2)^{\alpha_1-1}(1+u_1)^{\alpha_2}\left(1+\frac{u_1(\beta_0\beta_2+\beta_0\beta_1u_2)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \int_{u_2}^{\infty} (u_1-u_2)^{\alpha_1-1}\sum_{k=0}^{\infty}\left(\frac{\alpha_2!}{k!(\alpha_2-k)!}u_1^k\right)\left(1+\frac{u_1(\beta_0\beta_2+\beta_0\beta_1u_2)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}u_1^k\left(1+u_1\frac{(\beta_0\beta_2+\beta_0\beta_1u_2)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} du_1\right] \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}u_1^k\sum_{j=0}^{\infty}\left[(-1)^j\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}\right.\right. \\
&\times \left.\left.\left(u_1\frac{(\beta_0\beta_2+\beta_0\beta_1u_2)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2-j}\right] du_1\right] \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}\right.\right. \\
&\times \left.\left.\left(\frac{(\beta_0\beta_2+\beta_0\beta_1u_2)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2-j}\int_{u_2}^{\infty}(u_1-u_2)^{\alpha_1-1}u_1^{k-\alpha_0-\alpha_1-\alpha_2-j}du_1\right]\right]. \qquad (4.10)
\end{aligned}
$$

By applying Result 16 and Result 7 to Equation (4.10), it follows that

$$\begin{aligned}
f\left(u_2\right) &= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} u_2^{\alpha_2-1}\left(1+u_2\right)^{\alpha_0}\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!\left(\alpha_2-k\right)!}\sum_{j=0}^{\infty}\left[(-1)^j\left(\frac{\left(\beta_0\beta_2+\beta_0\beta_1u_2\right)}{\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2}\right)^{-\alpha_0-\alpha_1-\alpha_2-j}\right.\right. \\
&\times \left.\left.\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j} u_2^{k-\alpha_0-\alpha_2-j}\frac{\Gamma\left(\alpha_0+\alpha_2+j-k\right)\Gamma\left(\alpha_1\right)}{\Gamma\left(\alpha_0+\alpha_1+\alpha_2+j-k\right)}\right]\right] \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_2\right)}\left(1+u_2\right)^{\alpha_0}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!\left(\alpha_2-k\right)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right. \\
&\times \binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}\left(\beta_0\beta_2+\beta_0\beta_1u_2\right)^{-\alpha_0-\alpha_1-\alpha_2-j} \\
&\times \left.\left.\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma\left(\alpha_0+\alpha_2+j-k\right)}{\Gamma\left(\alpha_0+\alpha_1+\alpha_2+j-k\right)}\right]\right].
\end{aligned}$$

■

Special cases U_2

1) If $\alpha_i=\alpha$ for $i=0,1,2$, Equation (4.7) simplifies to

$$\begin{aligned}
f\left(u_2\right) &= \frac{\left(\beta_0^{2\alpha}\beta_1^{2\alpha}\beta_2^{2\alpha}\right)\Gamma(3\alpha)}{\Gamma(\alpha)^2}\left(1+u_2\right)^{\alpha}\sum_{k=0}^{\infty}\left[\frac{\alpha!}{k!(\alpha-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right. \\
&\times \binom{3\alpha+j-1}{j}\left(\beta_0\beta_2+\beta_0\beta_1u_2\right)^{-3\alpha-j} \\
&\times \left.\left.\left(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2\right)^j u_2^{k-\alpha-j-1}\frac{\Gamma(2\alpha+j-k)}{\Gamma(3\alpha+j-k)}\right]\right] \quad , \quad u_2>0.
\end{aligned}$$

2) If $\beta_i=\beta$ for $i=0,1,2$, Equation (4.7) simplifies to

$$\begin{aligned}
f\left(u_2\right) &= \frac{\left(\beta^{\alpha_1+\alpha_2}\beta^{\alpha_0+\alpha_2}\beta^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}\left(1+u_2\right)^{\alpha_0}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right. \\
&\times \binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}\left(\beta^2+\beta^2u_2\right)^{-\alpha_0-\alpha_1-\alpha_2-j} \\
&\times \left.\left.\left(\beta^2+\beta^2u_2-\beta^2u_2+\beta^2u_2\right)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma(\alpha_0+\alpha_2+j-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2+j-k)}\right]\right] \\
&= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}\left(1+u_2\right)^{\alpha_0}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right. \\
&\times \binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}\left(1+u_2\right)^{-\alpha_0-\alpha_1-\alpha_2-j} \\
&\times \left.\left.\left(1+u_2\right)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma(\alpha_0+\alpha_2+j-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2+j-k)}\right]\right] \quad , \quad u_2>0.
\end{aligned}$$

3) If $\beta_i=\beta$ and $\alpha_i=\alpha$ for $i=0,1,2$, Equation (4.7) simplifies to

$$\begin{aligned}
f\left(u_2\right) &= \tfrac{\left(\beta^{2\alpha}\beta^{2\alpha}\beta^{2\alpha}\right)\Gamma(3\alpha)}{\Gamma(\alpha)^2}\left(1+u_2\right)^{\alpha}\sum_{k=0}^{\infty}\left[\tfrac{\alpha!}{k!(\alpha-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right.\\
&\times \binom{3\alpha+j-1}{j}\left(\beta^2+\beta^2u_2\right)^{-3\alpha-j}\\
&\times \left.\left.\left(\beta^2+\beta^2u_2-\beta^2u_2+\beta^2u_2\right)^j u_2^{k-\alpha-j-1}\tfrac{\Gamma(2\alpha+j-k)}{\Gamma(3\alpha+j-k)}\right]\right]\\
&= \tfrac{\Gamma(3\alpha)}{\Gamma(\alpha)^2}\left(1+u_2\right)^{\alpha}\sum_{k=0}^{\infty}\left[\tfrac{\alpha!}{k!(\alpha-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\right.\right.\\
&\times \binom{3\alpha+j-1}{j}\left(1+u_2\right)^{-3\alpha-j}\\
&\times \left.\left.\left(1+u_2\right)^j u_2^{k-\alpha-j-1}\tfrac{\Gamma(2\alpha+j-k)}{\Gamma(3\alpha+j-k)}\right]\right] \qquad , \quad u_2>0.
\end{aligned}$$

Note
An alternative method can be used to derive the marginal density functions, namely, constructing them from independent gamma random variables, $W_i \sim Gamma\left(\alpha_i>0,\beta_i>0\right)$ for $i=0,1,2$, similar in nature to the derivations in sections 3.2.1 and 4.2.1. The marginal density functions that are derived in this manner have far more concise closed-form expressions; however, their restrictions are very limiting, to the point where they lose practical applicability. It is for this reason that this method is only mentioned in this cursory note.

Alternative marginal density function of U_1

Let W_i be independent gamma random variables with parameters $(\alpha_i>0,\beta_i>0)$ for $i=0,1,2$.
Let $X_1=W_1+W_2, X_2=W_2$ and $U_1=\frac{W_1+W_2}{W_0}$.
The density function of U_1 can alternatively be given by

$$\begin{aligned}
f\left(u_1\right) &= \tfrac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1+\alpha_2)}\tfrac{(u_1)^{-\alpha_0-1}}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}}\\
&\times \left(\tfrac{1}{\beta_1}+\tfrac{1}{u_1\beta_0}\right)^{-\alpha_0-\alpha_1-\alpha_2}{}_2F_1\left(\alpha_2,\alpha_0+\alpha_1+\alpha_2;\alpha_1+\alpha_2;\tfrac{\beta_0(\beta_2-\beta_1)u_1}{\beta_2(\beta_1+\beta_0u_1)}\right) \quad , \quad \begin{array}{r} u_1>0. \\ \frac{1}{\beta_2}-\frac{1}{\beta_1}\geq 0 \text{ and} \\ \left|\frac{\beta_0(\beta_2-\beta_1)u_1}{\beta_2(\beta_1+\beta_0u_1)}\right|<1. \end{array}
\end{aligned} \tag{4.11}$$

Alternative marginal density function of U_2

Let W_i be independent gamma random variables with parameters $(\alpha_i>0,\beta_i>0)$ for $i=0,1,2$.
Let $X_1=W_0+W_1, X_2=W_1$ and $U_2=\frac{W_2}{W_0+W_1}$.
The density function of U_2 can alternatively be given by

$$\begin{aligned} f(u_2) &= \frac{\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0+\alpha_1)\Gamma(\alpha_2)} \frac{(u_2)^{\alpha_2-1}}{\beta_0^{\alpha_0}\beta_1^{\alpha_1}\beta_2^{\alpha_2}} \\ &\times \left(\frac{1}{\beta_0}+\frac{u_2}{\beta_2}\right)^{-\alpha_0-\alpha_1-\alpha_2} {}_2F_1\left(\alpha_1, \alpha_0+\alpha_1+\alpha_2; \alpha_0+\alpha_1; \frac{\beta_2(\beta_1-\beta_0)}{\beta_1(\beta_2+\beta_0 u_2)}\right), \quad u_2>0, \\ & \qquad \frac{1}{\beta_1}-\frac{1}{\beta_0}\geq 0 \text{ and } \left|\frac{\beta_2(\beta_1-\beta_0)}{\beta_1(\beta_2+\beta_0 u_2)}\right|<1. \end{aligned} \tag{4.12}$$

Note that these equations have the restrictions that $\frac{1}{\beta_2}-\frac{1}{\beta_1}\geq 0$ and $\frac{1}{\beta_1}-\frac{1}{\beta_0}\geq 0$ respectively. These restrictions imply that both of the marginal density functions will only exist in this form when $\beta_0 \geq \beta_1 \geq \beta_2 \geq 0$. This implies that these marginal density functions only exist when a decrease in the process variance is being investigated/modelled.

4.2.4 Shape analysis

In this section, exploratory shape analyses are conducted into the marginal density functions derived in Section 4.2.3. To avoid repetition, for the remainder of this section whenever reference is made to the "marginal density function of the first statistic", it is Equation (4.5) for u_1 that is being referred to. Similarly, when reference is made to the "marginal density function of the second statistic", it is Equation (4.5) for u_2 that is being referred to.

A standard set of parameters has been chosen as a baseline. The parameters are again chosen to be $\alpha_0=\alpha_1=\alpha_2=5$ and $\beta_0=\beta_1=\beta_2=2$; in other words, a process where all three samples consist 11 observations (since since $\alpha_i=\frac{n_i-1}{2}$), and where no shift has occurred in the process variance. Some of the parameters are then varied from this baseline in order to investigate the effect of a change in the specific parameters on the general shape of the marginal density functions. The size of the parameter changes will coincide with the same parameter choices from Section 4.2.2. (Note, however, that for some of the parameter choices chosen below, the marginal density function of the second statistic, as in Equation (4.5), experienced convergence problems. In these cases an alternative expression for the density function of u_2 was plotted.)

Similar to Section 4.2.2, some parameters' changes will be large, so large that they lose practical realism. This is done to emphasise and investigate the general change in the shape, and is not meant to be an indication of the practical applications of the marginal density functions.

The functions will only be plotted on the $u_1 \in [0,5]$ and $u_2 \in [0,5]$ domains respectively. This by no means implies that the functions stop at the upper limit of 5. The main purpose of this chapter is to compare between different parameterisations of the marginal density functions.

Shape analysis of U_1

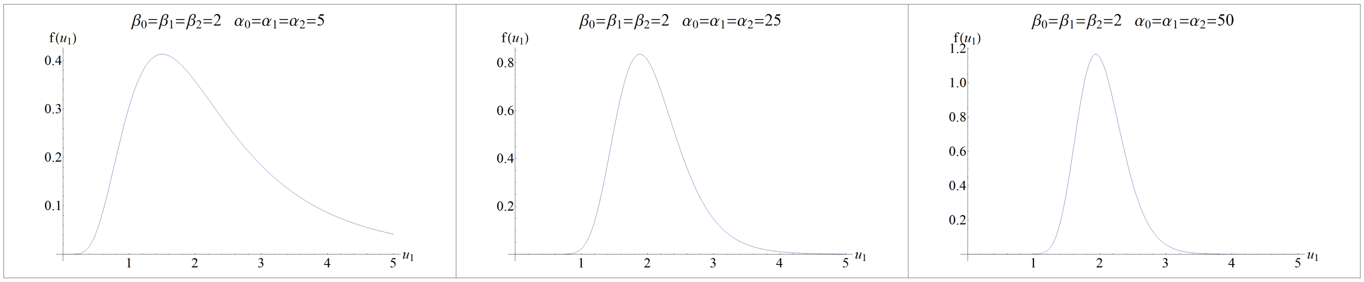

Figure 4.7: Equal sample sizes, in control process.

From Figure 4.7 it is apparent that increasing the sample sizes also increases the height of the peak. Larger sample sizes also shrink the length and width of the tails of the marginal density function. In essence, the higher the sample sizes, the smaller the domain on which the function has significant values. Increasing all sample sizes has very little effect in shifting the location of the peak of the marginal density function of the first statistic. This same effect was observed with the joint density function (see Figure 4.1).

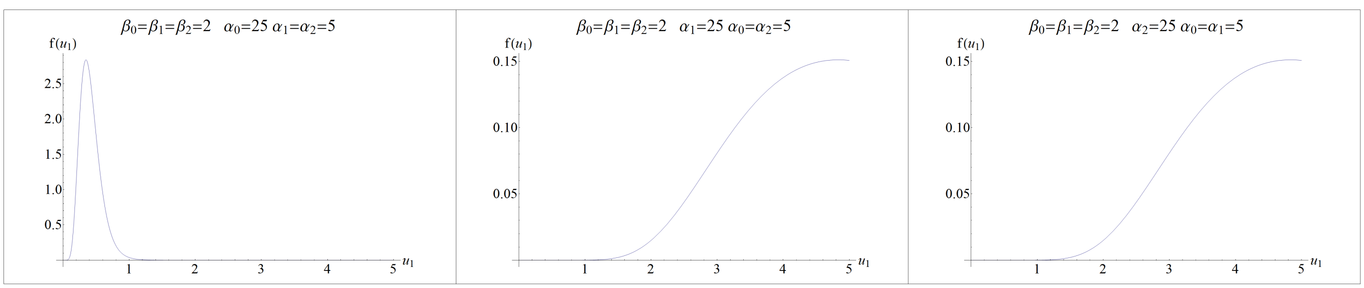

Figure 4.8: Varying sample sizes, in control process.

Figure 4.8 illustrates the effect that having one large sample (relative to the other two small samples) has on the shape of the density function. Having a relatively large first sample would shift the location of the marginal density function towards the minimum end of the domain. However, this also results in the density function having a much higher peak. This same effect was observed with the joint density function (see Figure 4.2).

Having a large second sample relative the first and third samples, or having a large third sample relative to the first and second samples, elongates the density function of U_1. This results in larger, thicker tails. The density function of U_1 seems to be relatively insensitive to whether the larger sample size occurs during the second or third sample (at least when all of the β's are equal).

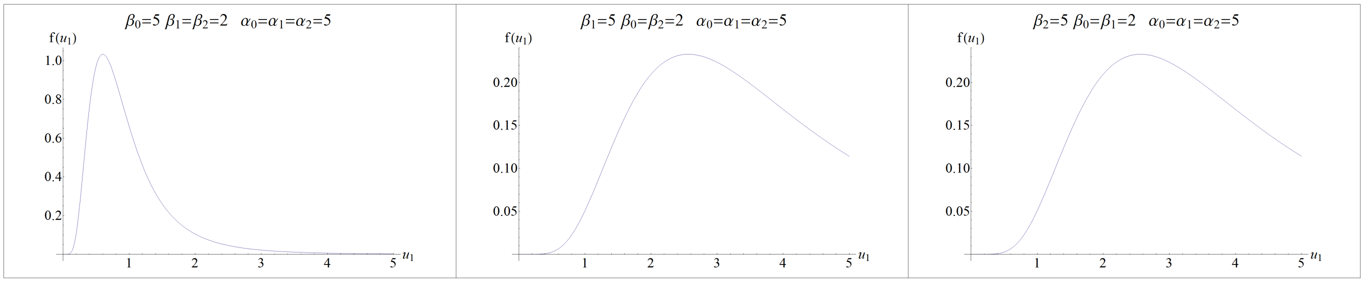

Figure 4.9: Equal sample sizes, unsustained increase in variance.

Figure 4.9 illustrates the effect a single, unsustained increase in the variance would have on the shape of the marginal density function of U_1. If the shift in the variance occurs during the first sample, the location of the density function moves towards the minimum end of the domain, but the peak of the density function is higher. If the shift occurs during the second or third sample, it results in the density function having larger, thicker tails as well as a shift in location towards the right. The density function of U_1 seems to be insensitive to whether the once-off increase in the variance occurs during the second or third sample.

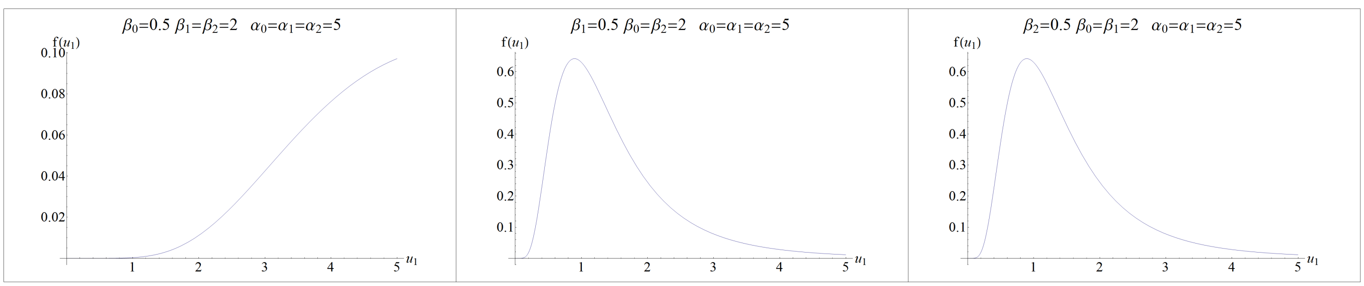

Figure 4.10: Equal sample sizes, unsustained decrease in variance.

Figure 4.10 illustrates the effect that having a single, unsustained decrease in the variance would have on the shape of the marginal density function of U_1. If the shift in the variance occurs during the first sample, the location of the density function shifts towards the right and its peak is lower than the standard parameter choices. The density function also becomes more dispersed. If the shift occurs during the second or third sample, it results in the density function having smaller, thinner tails as well as a shift in location towards the left. The density function of U_1 seems to be insensitive to whether the once-off decrease in the variance occurs during the second or third sample.

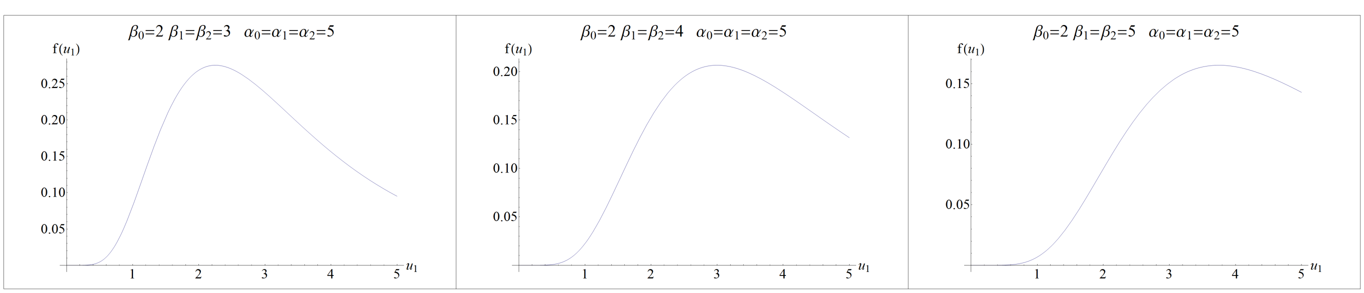

Figure 4.11: Equal sample sizes, sustained increase in variance.

Figure 4.11 illustrates the effect that a sustained increase in the variance would have on the density function of U_1. Very little changes with regard to the general shape on the plotted domain (there is only a slight distortion of the shape), despite a huge increase in the variance. However, on a larger domain it is observed that the tail of the marginal density function of U_1 becomes longer for larger shifts in the process variance. Note that the height of the peak of the density function decreases for larger β values.

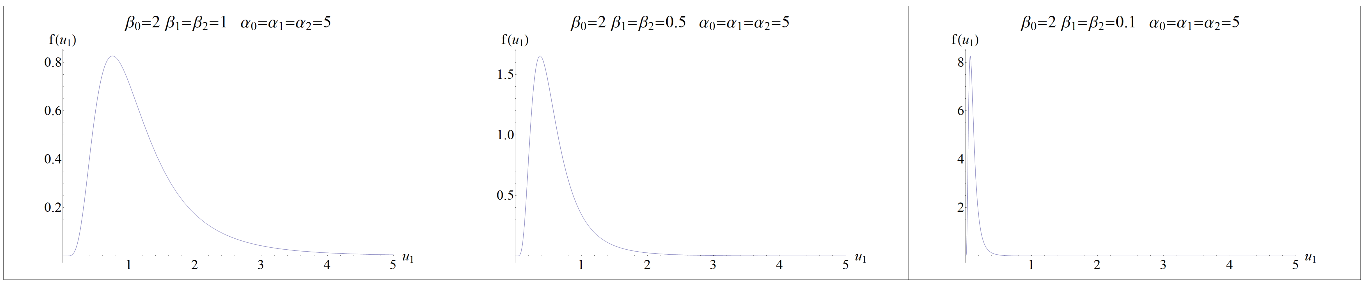

Figure 4.12: Equal sample sizes, sustained decrease in variance.

Figure 4.12 illustrates the effect that a sustained decrease in the variance would have on the density function of U_1. It is apparent that the general shape of the density function of U_1 is not as insensitive to a decrease in the variance as it was to an increase (see Figure 4.11). (A 50% increase in the process variance, for example, increases the length and width of the tails far less dramatically than a 50% decrease in the process variance.) Decreasing the variance of the second and third samples dramatically reduces the length and width of the tails of the density function, while also increasing the height of the peak.

Shape analysis of U_2

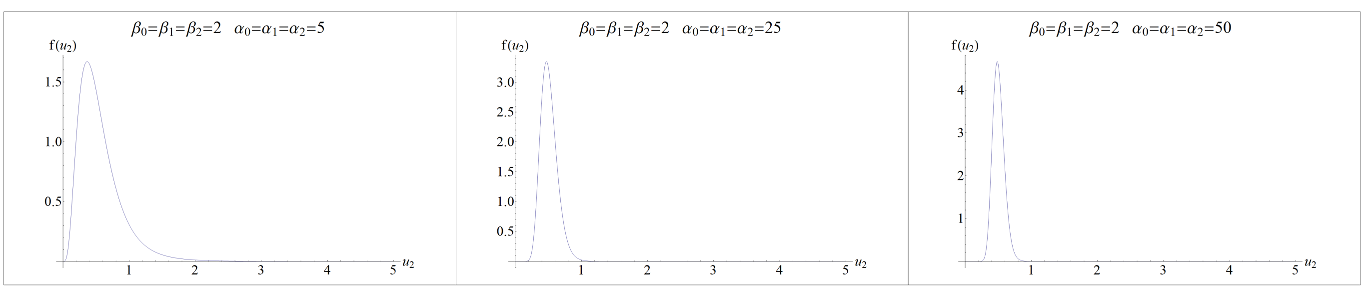

Figure 4.13: Equal sample sizes, in control process.

From Figure 4.13, it is apparent that increasing the sample sizes also increases the height of the peak. Larger sample sizes also shrink the length and width of the tails of the marginal density function of U_2. In essence, the higher the sample sizes, the smaller the domain on which the function has significant values. Increasing all sample sizes has very little effect in shifting the location of the marginal density function.

This same effect was observed with the joint density function (see Figure 4.1) In comparison to the marginal density function of U_1 (Figure 4.7), the peaks of U_2 lie significantly towards the left.

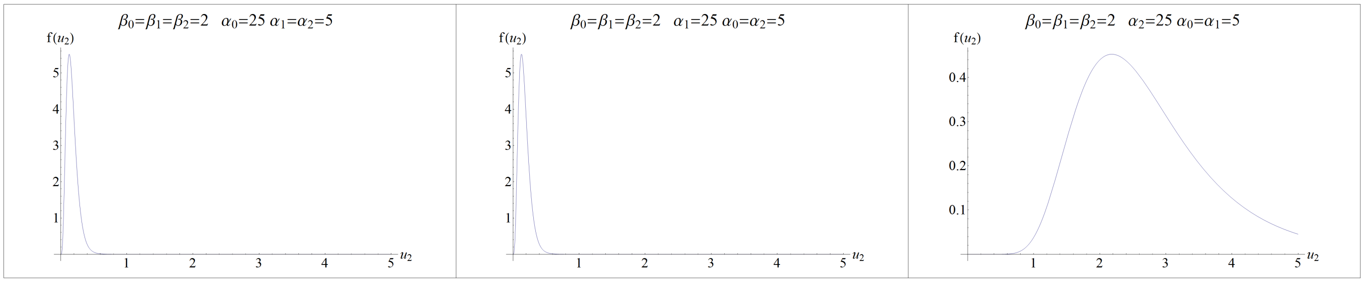

Figure 4.14: Varying sample sizes, in control process.

Figure 4.14 illustrates the effect that having one large sample (relative to the other two small samples) would have on the shape of the marginal density function. Having a relatively large first or second sample shifts the location of the density function towards the minimum end of the domain. However, this also results in a much higher peak of the density function. The density function of U_2 seems to be insensitive to whether the large sample size occurs during the first or second sample (at least when all of the βs are equal).

Having a large third sample relative to the first and second samples elongates the density function of U_2. This results in larger, thicker tails.

It can be seen that the domains on which the marginal densities have significant values (see figures 4.8 and 4.14) correspond to the domain on which the joint density function has significant function values (Figure 4.2).

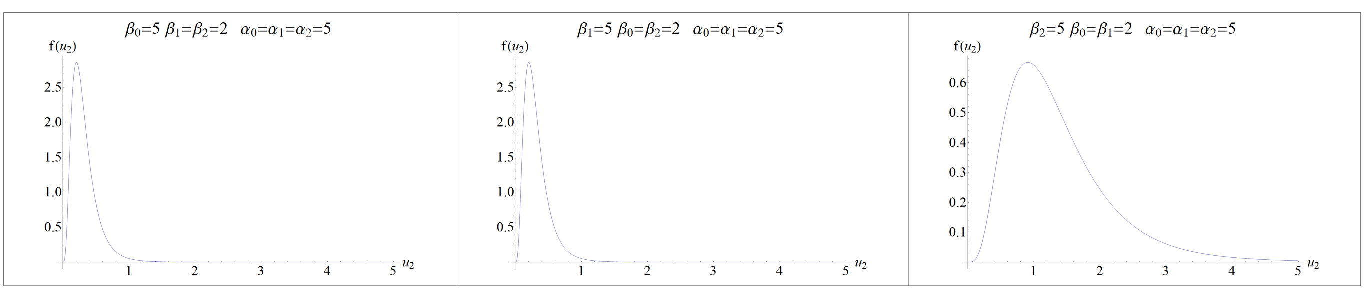

Figure 4.15: Equal sample sizes, unsustained increase in variance.

Figure 4.15 illustrates the effect that a single, unsustained increase in the variance would have on the shape of the marginal density function of U_2. If the shift in the variance occurs during the first or second samples, the location of the density function moves towards the minimum end of the domain, but this also results in the density function having higher peaks. If the shift occurs during the third sample, it results in the density function having larger, thicker tails as well as a shift in location towards the right. The density function of U_2 seems to be insensitive to whether the once-off increase in the variance occurs during the first or second sample.

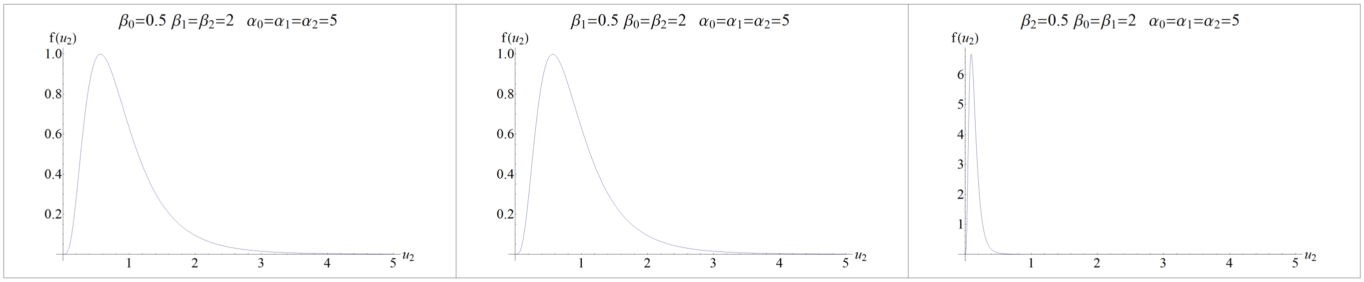

Figure 4.16: Equal sample sizes, unsustained decrease in variance.

Figure 4.16 illustrates the effect that a single, unsustained decrease in the variance would have on the shape of the marginal density function of U_2. If the shift in the variance occurs during the first sample, the location of the density function shifts towards the right relative to the location of the density function when the shift occurs in the third sample. The peak of the density function is lower than the standard parameter choices if the shift occurs during samples one or two, but is much higher if the shift occurs during the third sample. The density function of U_2 seems to be insensitive to whether the once-off decrease in the variance occurs during the first or second sample.

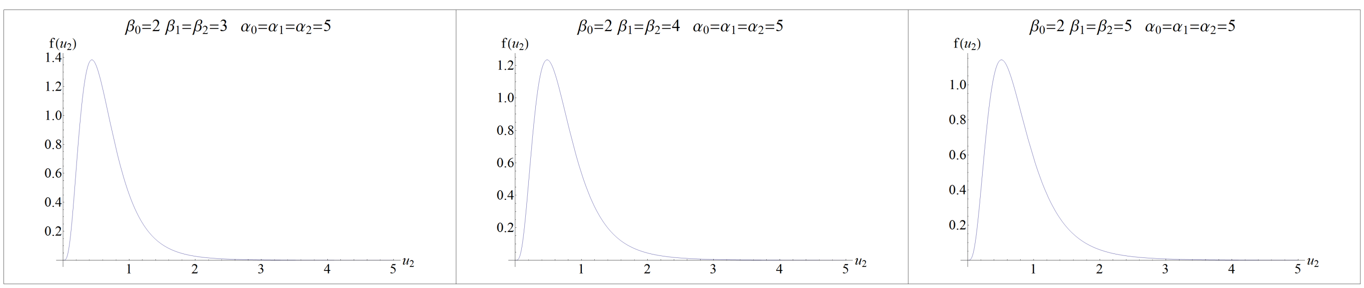

Figure 4.17: Equal sample sizes, sustained increase in variance.

Figure 4.17 illustrates the effect that a sustained increase in the variance would have on the density function of u_2. It is apparent that the general shape of the density function of U_2 is insensitive to a change in the variance. All that is affected by the change in the β_1 and β_2 values is the height of the density function, with the height decreasing for larger β values. In comparison to the density function of U_1 (Figure 4.11), the location of the density function of U_2 is distributed much further to the left, with sharper, higher peaks.

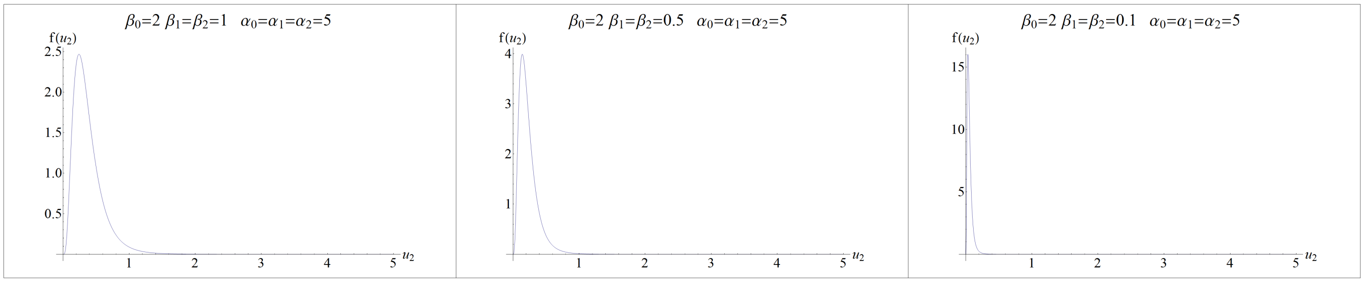

Figure 4.18: Equal sample sizes, sustained decrease in variance.

Figure 4.18 illustrates the effect that a sustained decrease in the variance would have on the density function of U_2. It is apparent that the general shape of the density function of U_2 is not as insensitive to a decrease in the variance as it was to an increase (see Figure 4.17). Decreasing the variance of the second and third samples dramatically reduces the length and width of the tails of the density function, while also increasing the height of the peak.

Conditional density functions

Conditional density function of U_1 given U_2

Theorem 4.4
The conditional density function of U_1 given U_2 is given by

$$\begin{aligned} f\left(u_1|u_2\right) &= \left(u_1-u_2\right)^{\alpha_1-1}\left(1+u_1\right)^{\alpha_2}\left(\beta_0\beta_2+\beta_0\beta_1u_2\right)^{\alpha_0+\alpha_1+\alpha_2} \\ &\times \left(\beta_1\beta_2\left(1+u_2\right)+\beta_0\beta_2\left(u_1-u_2\right)+\beta_0\beta_1u_2\left(1+u_1\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} \\ &\div \left[\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}u_2^{k-\alpha_0-\alpha_2}\frac{\Gamma(\alpha_1)\Gamma(\alpha_0+\alpha_2-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2-k)}\right.\right. \\ &\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,\alpha_0+\alpha_2-k\right. \\ &\left.\left.\left.;\alpha_0+\alpha_1+\alpha_2-k;\frac{\beta_0\beta_2u_2-\beta_1\beta_2-\beta_1\beta_2u_2-\beta_0\beta_1u_2}{u_2(\beta_0\beta_2+\beta_0\beta_1u_2)}\right)\right]\right], \quad u_1>u_2>0 \text{ and } \left|\frac{\beta_0\beta_2u_2-\beta_1\beta_2-\beta_1\beta_2u_2-\beta_0\beta_1u_2}{u_2(\beta_0\beta_2+\beta_0\beta_1u_2)}\right|<1, \end{aligned} \tag{4.13}$$

where $\alpha_i,\beta_i>0$ for $i=0,1,2$.

If $\alpha_2\in\mathbb{N}$, as will be the case in an SPC setting, the sum changes from $\sum_{k=0}^{\infty}$ to $\sum_{k=0}^{\alpha_2}$. (See Result 18.)

Proof
From equations (4.1) and (4.7), it follows that the conditional density function is given as

$$\begin{aligned}
f(u_1|u_2) &= \frac{f(u_1,u_2)}{f(u_2)} \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\
&\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div \left[\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}(1+u_2)^{\alpha_0}\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\right.\right. \\
&\times \sum_{j=0}^{\infty}\left[(-1)^j\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2-j}\right. \\
&\times \left.\left.\left.(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma(\alpha_0+\alpha_2+j-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2+j-k)}\right]\right]\right] \\
&= \frac{(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}}{\Gamma(\alpha_1)}\left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div \left[\sum_{k=0}^{\infty}\left[\frac{\alpha_2!}{k!(\alpha_2-k)!}\sum_{j=0}^{\infty}\left[(-1)^j\binom{\alpha_0+\alpha_1+\alpha_2+j-1}{j}(\beta_0\beta_2+\beta_0\beta_1u_2)^{-\alpha_0-\alpha_1-\alpha_2-j}\right.\right.\right. \\
&\times \left.\left.\left.(\beta_1\beta_2+\beta_1\beta_2u_2-\beta_0\beta_2u_2+\beta_0\beta_1u_2)^j u_2^{k-\alpha_0-j-1}\frac{\Gamma(\alpha_0+\alpha_2+j-k)}{\Gamma(\alpha_0+\alpha_1+\alpha_2+j-k)}\right]\right]\right]. \qquad (4.14)
\end{aligned}$$

■

Conditional density function of U_2 given U_1

Theorem 4.5
The conditional density function of U_2 given U_1 is given by

$$\begin{aligned}
f(u_2|u_1) &= (u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_1\beta_2+\beta_0\beta_2u_1)^{\alpha_0+\alpha_1+\alpha_2} \\
&\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2}. \\
&\div \left[\sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right.\right. \\
&\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,k+\alpha_2\right. \\
&\left.\left.\left.;k+\alpha_1+\alpha_2;-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right)\right]\right], \quad u_1>u_2>0 \text{ and } \left|-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right|<1. \qquad (4.15)
\end{aligned}$$

If $\alpha_0 \in \mathbb{N}$, as will be the case in an SPC setting, the sum changes from $\sum_{k=0}^{\infty}$ to $\sum_{k=0}^{\alpha_0}$. (See Result 18.)

Proof
From equations (4.1) and (4.5), it follows that

$$\begin{aligned}
f(u_2|u_1) &= \frac{f(u_1,u_2)}{f(u_1)} \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\
&\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2}. \\
&\div \left[\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(1+u_1)^{\alpha_2}(\beta_1\beta_2+\beta_0\beta_2u_1)^{-\alpha_0-\alpha_1-\alpha_2}\right. \\
&\times \sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right. \\
&\times \left.\left. {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,k+\alpha_2;k+\alpha_1+\alpha_2;-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right)\right]\right] \\
&= (u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_2)^{\alpha_0}(\beta_1\beta_2+\beta_0\beta_2u_1)^{\alpha_0+\alpha_1+\alpha_2} \\
&\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\div \left[\sum_{k=0}^{\infty}\left[\left(\frac{\alpha_0!}{k!(\alpha_0-k)!}\right)(u_1)^{k+\alpha_1+\alpha_2-1}\frac{\Gamma(\alpha_1)\Gamma(k+\alpha_2)}{\Gamma(k+\alpha_1+\alpha_2)}\right.\right. \\
&\times \left.\left. {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2,k+\alpha_2;k+\alpha_1+\alpha_2;-\frac{u_1(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1+\beta_0\beta_1u_1)}{\beta_1\beta_2+\beta_0\beta_2u_1}\right)\right]\right].
\end{aligned}$$

■

4.2.5 Product moment

4.2.5.1 Construction by integration

Initially, the product moment of U_1 and U_2 is derived by using the definition in Result 5. This method proves to be extremely unwieldy. Another problem, which results in the product moment becoming practically useless when derived in this manner, is that irrespective of the order or method of integration, or any manipulation applied to Equation (4.1), a beta function (see Result 7) always has a negative integer argument, and since beta functions are not defined for negative integer arguments, the entire product moment becomes undefined. This ultimately results in restrictions that are impossible to meet. Out of a multitude of unsuccessful traditional derivation methods attempted, one is included below. This derivation is included to demonstrate that the traditional, first-principle, approach is ineffective at finding a closed-form expression for the product moment of Equation (4.1).

By substituting Equation (4.1), U_1, U_2 and the respective integration limits into Result 5, it follows that

$$\begin{aligned}
E\left(U_1^rU_2^s\right) &= \int_0^{\infty}\int_0^{u_1}u_1^ru_2^s\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}(u_1-u_2)^{\alpha_1-1}u_2^{\alpha_2-1}(1+u_1)^{\alpha_2}(1+u_2)^{\alpha_0} \\
&\times \left(\beta_1\beta_2(1+u_2)+\beta_0\beta_2(u_1-u_2)+\beta_0\beta_1u_2(1+u_1)\right)^{-\alpha_0-\alpha_1-\alpha_2}du_2du_1. \qquad (4.16)
\end{aligned}$$

By applying Result 18 to Equation (4.16) multiple times, and rearranging the terms, it follows that

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \int_0^\infty \int_0^{u_1} u_1^r u_2^s \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \sum_{j=0}^{\alpha_1-1}\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\left(u_1\right)^{\alpha_1-1-j}\left(-u_2\right)^j u_2^{\alpha_2-1} \\
&\times \left(1+u_1\right)^{\alpha_2}\left(1+u_2\right)^{\alpha_0}\left(\beta_1\beta_2\left(1+u_2\right)+\beta_0\beta_2\left(u_1-u_2\right)+\beta_0\beta_1 u_2\left(1+u_1\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2 du_1 \\
&= \sum_{j=0}^{\alpha_1-1}(-1)^j\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\int_0^{u_1} u_2^{s+j+\alpha_2-1}\left(1+u_2\right)^{\alpha_0} \\
&\times \left(\beta_1\beta_2+\beta_1\beta_2 u_2+\beta_0\beta_2 u_1-\beta_0\beta_2 u_2+\beta_0\beta_1 u_2+\beta_0\beta_1 u_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2 du_1 \\
&= \sum_{j=0}^{\alpha_1-1}(-1)^j\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\int_0^{u_1} u_2^{s+j+\alpha_2-1}\left(1+u_2\right)^{\alpha_0} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 u_1+u_2\left(\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2 du_1
\end{aligned}$$

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{j=0}^{\alpha_1-1}(-1)^j\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\int_0^{u_1} u_2^{s+j+\alpha_2-1}\left(1+u_2\right)^{\alpha_0}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \left(1+\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)}{\beta_1\beta_2+\beta_0\beta_2 u_1}u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2 du_1 \\
&= \sum_{j=0}^{\alpha_1-1}(-1)^j\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty \left[u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\right. \\
&\times \left.\int_0^{u_1} u_2^{s+j+\alpha_2-1}\left(1+u_2\right)^{\alpha_0}\left(1+\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)}{\beta_1\beta_2+\beta_0\beta_2 u_1}u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2\right] du_1 \\
&= \sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j\left(\begin{array}{c}\alpha_1-1\\ j\end{array}\right)\left(\begin{array}{c}\alpha_0\\ k\end{array}\right)\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \int_0^{u_1} u_2^{s+j+k+\alpha_2-1}\left(1+\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)}{\beta_1\beta_2+\beta_0\beta_2 u_1}u_2\right)^{-\alpha_0-\alpha_1-\alpha_2} du_2 du_1. \qquad (4.17)
\end{aligned}$$

By applying Result 17, and rewriting the hypergeometric function in terms of Pochhammer symbols, as in Result 4, Equation (4.17) may be rewritten as

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j\begin{pmatrix}\alpha_1-1\\ j\end{pmatrix}\begin{pmatrix}\alpha_0\\ k\end{pmatrix}\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\\
&\times \int_0^\infty u_1^{r+\alpha_1-1-j}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\frac{\left(u_1\right)^{s+j+k+\alpha_2}}{s+j+k+\alpha_2}\\
&\times\ {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, s+j+k+\alpha_2; 1+s+j+k+\alpha_2; -\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)}{\beta_1\beta_2+\beta_0\beta_2 u_1}u_1\right)du_1\\
&= \sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j\begin{pmatrix}\alpha_1-1\\ j\end{pmatrix}\begin{pmatrix}\alpha_0\\ k\end{pmatrix}\frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\\
&\times \int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\\
&\times\ {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2, s+j+k+\alpha_2; 1+s+j+k+\alpha_2; -\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1\left(1+u_1\right)u_1}{\beta_1\beta_2+\beta_0\beta_2 u_1}\right)du_1
\end{aligned}$$

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k} \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \\
&\times \sum_{n=0}^{\infty}\frac{\left(\alpha_0+\alpha_1+\alpha_2\right)_n\left(s+j+k+\alpha_2\right)_n}{\left(1+s+j+k+\alpha_2\right)_n}\frac{\left(-\frac{\beta_1\beta_2-\beta_0\beta_2+\beta_0\beta_1(1+u_1)u_1}{\beta_1\beta_2+\beta_0\beta_2 u_1}\right)^n}{n!}du_1 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\frac{\left(\alpha_0+\alpha_1+\alpha_2\right)_n\left(s+j+k+\alpha_2\right)_n}{\left(1+s+j+k+\alpha_2\right)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{n!\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(-\beta_1\beta_2+\beta_0\beta_2-\beta_0\beta_1\left(1+u_1\right)u_1\right)^n\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-n}du_1 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\frac{\left(\alpha_0+\alpha_1+\alpha_2\right)_n\left(s+j+k+\alpha_2\right)_n}{\left(1+s+j+k+\alpha_2\right)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{n!\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n}\left(-\beta_1\beta_2+\beta_0\beta_2-\beta_0\beta_1\left(1+u_1\right)u_1\right)^n du_1 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\frac{\left(\alpha_0+\alpha_1+\alpha_2\right)_n\left(s+j+k+\alpha_2\right)_n}{\left(1+s+j+k+\alpha_2\right)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{n!\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n\left(1+\frac{\beta_0\beta_1\left(1+u_1\right)u_1}{\beta_1\beta_2+\beta_0\beta_2}\right)^n du_1. \qquad (4.18)
\end{aligned}$$

By applying Result 18 to Equation (4.18) it follows that

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{n=0}^{\infty}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\frac{\left(\alpha_0+\alpha_1+\alpha_2\right)_n\left(s+j+k+\alpha_2\right)_n}{\left(1+s+j+k+\alpha_2\right)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{n!\left(s+j+k+\alpha_2\right)\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n \\
&\times \int_0^\infty u_1^{r+s+\alpha_1+\alpha_2+k-1}\left(1+u_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\beta_0\beta_1\left(1+u_1\right)u_1}{\beta_1\beta_2+\beta_0\beta_2}\right)^l du_1
\end{aligned}$$

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\binom{n}{l}\frac{(\alpha_0+\alpha_1+\alpha_2)_n (s+j+k+\alpha_2)_n}{(1+s+j+k+\alpha_2)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2+l}\beta_1^{\alpha_0+\alpha_2+l}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{n!\,(s+j+k+\alpha_2)\,\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n \\
&\times \int_0^{\infty} u_1^{r+s+\alpha_1+\alpha_2+k+l-1}(1+u_1)^{\alpha_2+l}\left(\beta_1\beta_2+\beta_0\beta_2 u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n}\left(\beta_1\beta_2+\beta_0\beta_2\right)^{-l} du_1 \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\binom{n}{l}\frac{(\alpha_0+\alpha_1+\alpha_2)_n (s+j+k+\alpha_2)_n}{(1+s+j+k+\alpha_2)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2+l}\beta_1^{\alpha_0+\alpha_2+l}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{n!\,(s+j+k+\alpha_2)\,\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n\left(\beta_1\beta_2+\beta_0\beta_2\right)^{-l} \\
&\times \int_0^{\infty} u_1^{r+s+\alpha_1+\alpha_2+k+l-1}(1+u_1)^{\alpha_2+l}\left(\beta_1\beta_2\right)^{-\alpha_0-\alpha_1-\alpha_2-n}\left(1+\frac{\beta_0\beta_2}{\beta_1\beta_2}u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n} du_1 \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\binom{n}{l}\frac{(\alpha_0+\alpha_1+\alpha_2)_n (s+j+k+\alpha_2)_n}{(1+s+j+k+\alpha_2)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2+l}\beta_1^{l-\alpha_1-n}\beta_2^{-\alpha_2-n}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{n!\,(s+j+k+\alpha_2)\,\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n\left(\beta_1\beta_2+\beta_0\beta_2\right)^{-l} \\
&\times \int_0^{\infty} u_1^{r+s+\alpha_1+\alpha_2+k+l-1}(1+u_1)^{\alpha_2+l}\left(1+\frac{\beta_0\beta_2}{\beta_1\beta_2}u_1\right)^{-\alpha_0-\alpha_1-\alpha_2-n} du_1. \qquad (4.19)
\end{aligned}$$

Applying Result 12 to Equation (4.19) leads to the following expression

$$\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\binom{n}{l}\frac{(\alpha_0+\alpha_1+\alpha_2)_n (s+j+k+\alpha_2)_n}{(1+s+j+k+\alpha_2)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2+l}\beta_1^{l-\alpha_1-n}\beta_2^{-\alpha_2-n}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{n!\,(s+j+k+\alpha_2)\,\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n\left(\beta_1\beta_2+\beta_0\beta_2\right)^{-l} \\
&\times B\left(r+s+\alpha_1+\alpha_2+k+l, \alpha_0+\alpha_1+\alpha_2+n-\alpha_2-l-r-s-\alpha_1-\alpha_2-k-l\right) \\
&\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2+n, r+s+\alpha_1+\alpha_2+k+l; \alpha_0+\alpha_1+\alpha_2+n-\alpha_2-l; 1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right) \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{\alpha_1-1}\sum_{k=0}^{\alpha_0}(-1)^j \binom{\alpha_1-1}{j}\binom{\alpha_0}{k}\binom{n}{l}\frac{(\alpha_0+\alpha_1+\alpha_2)_n (s+j+k+\alpha_2)_n}{(1+s+j+k+\alpha_2)_n} \\
&\times \frac{\left(\beta_0^{\alpha_1+\alpha_2+l}\beta_1^{l-\alpha_1-n}\beta_2^{-\alpha_2-n}\right)\Gamma(\alpha_0+\alpha_1+\alpha_2)}{n!\,(s+j+k+\alpha_2)\,\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(-\beta_1\beta_2+\beta_0\beta_2\right)^n\left(\beta_1\beta_2+\beta_0\beta_2\right)^{-l} \\
&\times B\left(r+s+\alpha_1+\alpha_2+k+l, \alpha_0-\alpha_2+n-2l-k-r-s\right) \\
&\times {}_2F_1\left(\alpha_0+\alpha_1+\alpha_2+n, r+s+\alpha_1+\alpha_2+k+l; \alpha_0+\alpha_1+n-l; 1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right). \qquad (4.20)
\end{aligned}$$

The restriction that the second integration technique (Result 12) necessitates is that $\alpha_0+\alpha_1+\alpha_2+n-\alpha_2-l > r+s+\alpha_1+\alpha_2+k+l$. This implies that $-\alpha_2-l > r+s$, and this restriction will never be met. (Note that in the above derivation α_0 and α_1 are assumed to be integers.)

4.2.5.2 Construction by transformation

Using the transformations in Chapter 2, it is possible to derive the product moment in an alternative way. The method that is applied is similar in nature to the example given at the end of Chapter 2. Given that the second statistic of the bivariate beta type VII distribution (T_2 from Equation (2.8)) is the same as the second statistic of the bivariate beta type VIII distribution (U_2 from Equation (2.8)), the transformation between the two sets of variables is the natural choice to use.

Theorem 4.6

The product moment of U_1 and U_2 is given by

$$\begin{aligned} E\left(U_1^r U_2^s\right) &= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{-\alpha_1}\beta_2^{p+s}\right)\Gamma(\alpha_2+p+s)\Gamma(\alpha_0+\alpha_1-p-s)\Gamma(\alpha_1+r-p)\Gamma(\alpha_0-r)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_0+\alpha_1-p)} \\ &\times \; {}_2F_1\left(\alpha_0+\alpha_1-p-s, \alpha_1+r-p; \alpha_0+\alpha_1-p; 1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right) \quad , \quad \alpha_0+\alpha_1 > r+s, \\ & \qquad \alpha_0 > \alpha_1 \text{ and } \left|1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right| < 1. \end{aligned} \tag{4.21}$$

Proof

The relationship between the bivariate beta type VII distribution and the bivariate beta type VIII distribution was derived in Chapter 2. It is stated here again to make the derivation more coherent.

$$\begin{aligned} U_1 &\overset{d}{=} T_1+T_2+T_1T_2 \\ U_2 &\overset{d}{=} T_2 \end{aligned} \tag{4.22}$$

Using the relationship in Equation (4.22), it is possible to rewrite the product moment of U_1 and U_2 in terms of the joint density function of the bivariate beta type VII distribution (Equation (3.1)). It follows that by using Result 5 and Result 18, and then rearranging the terms, the product moment of U_1 and U_2 may be derived as follows

$$
\begin{aligned}
E\left(U_1^r U_2^s\right) &= E\left(\left(T_1+T_2+T_1T_2\right)^r \left(T_2\right)^s\right) \\
&= \int_0^\infty \int_0^\infty \left(t_1+t_2+t_1t_2\right)^r \left(t_2\right)^s g\left(t_1,t_2\right) dt_1 dt_2 \\
&= \int_0^\infty \int_0^\infty \left(t_1+t_2+t_1t_2\right)^r \left(t_2\right)^s \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2-1}\left(1+t_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \int_0^\infty \left(t_1+t_2+t_1t_2\right)^r \\
&\times \left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2+s-1}\left(1+t_1\right)^{\alpha_2}\left(\beta_1\beta_2+\beta_0\beta_2 t_1+\beta_0\beta_1\left(1+t_1\right)t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \int_0^\infty \left(t_1+t_2\left(1+t_1\right)\right)^r \\
&\times \left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2+s-1}\left(1+t_1\right)^{\alpha_2}\left(\beta_0\beta_1\left(1+t_1\right)\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \int_0^\infty \left(\left(1+t_1\right)\left(\frac{t_1}{1+t_1}+t_2\right)\right)^r \\
&\times \left(t_1\right)^{\alpha_1-1}\left(t_2\right)^{\alpha_2+s-1}\left(1+t_1\right)^{\alpha_2}\left(\beta_0\beta_1\left(1+t_1\right)\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_1 dt_2 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{r+\alpha_2}\left(t_1\right)^{\alpha_1-1} \\
&\times \int_0^\infty \left(\frac{t_1}{1+t_1}+t_2\right)^r \left(t_2\right)^{\alpha_2+s-1}\left(\beta_0\beta_1\left(1+t_1\right)\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \frac{\left(\beta_0^{\alpha_1+\alpha_2}\beta_1^{\alpha_0+\alpha_2}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(\beta_0\beta_1\right)^{-\alpha_0-\alpha_1-\alpha_2} \int_0^\infty \left(1+t_1\right)^{r-\alpha_0-\alpha_1}\left(t_1\right)^{\alpha_1-1} \\
&\times \int_0^\infty \left(\frac{t_1}{1+t_1}+t_2\right)^r \left(t_2\right)^{\alpha_2+s-1}\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{r-\alpha_0-\alpha_1}\left(t_1\right)^{\alpha_1-1} \\
&\times \int_0^\infty \left(\frac{t_1}{1+t_1}+t_2\right)^r \left(t_2\right)^{\alpha_2+s-1}\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{r-\alpha_0-\alpha_1}\left(t_1\right)^{\alpha_1-1} \\
&\times \int_0^\infty \sum_{p=0}^{r} \binom{r}{p} \left(t_2\right)^p \left(\frac{t_1}{1+t_1}\right)^{r-p} \left(t_2\right)^{\alpha_2+s-1}\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1
\end{aligned}
$$

$$
\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p}\left(t_1\right)^{\alpha_1+r-p-1} \\
&\times \int_0^\infty \left(t_2\right)^{\alpha_2+p+s-1}\left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}+t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1 \\
&= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p}\left(t_1\right)^{\alpha_1+r-p-1} \\
&\times \left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}\right)^{-\alpha_0-\alpha_1-\alpha_2} \int_0^\infty \left(t_2\right)^{\alpha_2+p+s-1}\left(1+\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}t_2\right)^{-\alpha_0-\alpha_1-\alpha_2} dt_2 dt_1.
\end{aligned}
\tag{4.23}
$$

From Result 13, it follows that Equation (4.23) may be expressed as

$$
\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p}\left(t_1\right)^{\alpha_1+r-p-1} \\
&\times \left(\frac{\beta_1\beta_2+\beta_0\beta_2 t_1}{\beta_0\beta_1\left(1+t_1\right)}\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(\frac{\beta_0\beta_1\left(1+t_1\right)}{\beta_1\beta_2+\beta_0\beta_2 t_1}\right)^{-\alpha_2-p-s} B\left(\alpha_2+p+s,\alpha_0+\alpha_1+\alpha_2-\alpha_2-p-s\right)dt_1 \\
&= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p}\left(t_1\right)^{\alpha_1+r-p-1} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1-\alpha_2}\left(\beta_0\beta_1\left(1+t_1\right)\right)^{\alpha_0+\alpha_1+\alpha_2}\left(\beta_0\beta_1\left(1+t_1\right)\right)^{-\alpha_2-p-s} \\
&\quad \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{\alpha_2+p+s}\frac{\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0+\alpha_1+\alpha_2\right)}dt_1 \\
&= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{-\alpha_0}\beta_1^{-\alpha_1}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p}\left(t_1\right)^{\alpha_1+r-p-1} \\
&\times \left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1+p+s}\left(\beta_0\beta_1\left(1+t_1\right)\right)^{\alpha_0+\alpha_1-p-s}dt_1 \\
&= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{\alpha_0-p-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \int_0^\infty \left(1+t_1\right)^{-\alpha_0-\alpha_1+p+\alpha_0+\alpha_1-p-s}\left(t_1\right)^{\alpha_1+r-p-1}\left(\beta_1\beta_2+\beta_0\beta_2 t_1\right)^{-\alpha_0-\alpha_1+p+s}dt_1 \\
&= \sum_{p=0}^{r} \binom{r}{p} \frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{\alpha_0-p-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \int_0^\infty \left(1+t_1\right)^{-s} \\
&\times \left(t_1\right)^{\alpha_1+r-p-1}\left(\beta_1\beta_2\right)^{-\alpha_0-\alpha_1+p+s}\left(1+\frac{\beta_0\beta_2}{\beta_1\beta_2}t_1\right)^{-\alpha_0-\alpha_1+p+s} dt_1
\end{aligned}
$$

$$
\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{p=0}^{r}\binom{r}{p}\frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{\alpha_0-p-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(\beta_1\beta_2\right)^{-\alpha_0-\alpha_1+p+s} \\
&\times \int_0^\infty \left(1+t_1\right)^{-s}\left(t_1\right)^{\alpha_1+r-p-1}\left(1+\frac{\beta_0\beta_2}{\beta_1\beta_2}t_1\right)^{-\alpha_0-\alpha_1+p+s} dt_1. \qquad (4.24)
\end{aligned}
$$

By applying Result 12 to Equation (4.24), it follows that

$$
\begin{aligned}
E\left(U_1^r U_2^s\right) &= \sum_{p=0}^{r}\binom{r}{p}\frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{\alpha_0-p-s}\beta_2^{\alpha_0+\alpha_1}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}\left(\beta_1\beta_2\right)^{-\alpha_0-\alpha_1+p+s} \\
&\times B\left(\alpha_1+r-p,\alpha_0-r\right){}_2F_1\left(\alpha_0+\alpha_1-p-s,\alpha_1+r-p;\alpha_0+\alpha_1-p-s+s;1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right) \\
&= \sum_{p=0}^{r}\binom{r}{p}\frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{\alpha_0-p-s-\alpha_0-\alpha_1+p+s}\beta_2^{\alpha_0+\alpha_1-\alpha_0-\alpha_1+p+s}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} \\
&\times \frac{\Gamma\left(\alpha_1+r-p\right)\Gamma\left(\alpha_0-r\right)}{\Gamma\left(\alpha_1+r-p+\alpha_0-r\right)}{}_2F_1\left(\alpha_0+\alpha_1-p-s,\alpha_1+r-p;\alpha_0+\alpha_1-p-s+s;1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right) \\
&= \sum_{p=0}^{r}\binom{r}{p}\frac{\left(\beta_0^{\alpha_1-p-s}\beta_1^{-\alpha_1}\beta_2^{p+s}\right)\Gamma\left(\alpha_2+p+s\right)\Gamma\left(\alpha_0+\alpha_1-p-s\right)\Gamma\left(\alpha_1+r-p\right)\Gamma\left(\alpha_0-r\right)}{\Gamma\left(\alpha_0\right)\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_0+\alpha_1-p\right)} \\
&\times {}_2F_1\left(\alpha_0+\alpha_1-p-s,\alpha_1+r-p;\alpha_0+\alpha_1-p;1-\frac{\beta_0\beta_2}{\beta_1\beta_2}\right).
\end{aligned}
$$

■

4.3 Multivariate distribution

Theorem 4.7

Let W_i be independent gamma random variables with parameters $(\alpha_i > 0, \beta_i > 0)$ for $i = 0, 1, 2, ..., m$.

Let $U_r = \frac{\sum_{i=r}^m W_i}{\sum_{i=0}^{r-1} W_i}, r = 1, 2, ..., m-1, m$.

Then the joint density function of $U_1, U_2, ..., U_m$ is given by

$$
\begin{aligned}
f\left(u_1,u_2,...,u_m\right) &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha_j-1}\right](u_m)^{\alpha_m-1}\Gamma\left(\sum_{j=0}^m[\alpha_j]\right)}{\prod_{j=0}^m\left[\beta_j^{\alpha_j}\Gamma(\alpha_j)\right]} \\
&\times \left(1+u_1\right)^{\sum_{j=2}^m[\alpha_j]}\prod_{j=2}^m\left[\left(1+u_j\right)^{-\alpha_{j-1}-\alpha_j}\right] \\
&\times \left(\frac{1}{\beta_0}+\frac{(u_1-u_2)}{\beta_1(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{\beta_j(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{\beta_m(1+u_m)}\right)^{-\sum_{j=0}^m[\alpha_j]}, u_1 > u_2 > ... > u_m > 0. \qquad (4.25)
\end{aligned}
$$

Proof

Since the gamma random variables are independent, the joint density function of $W_i,\ i = 0, 1, 2, ..., m.$ is given by

$$f(w_0, w_1, ..., w_m) = \prod_{i=0}^{m} \frac{\left(w_i^{\alpha_i - 1} e^{-\frac{w_i}{\beta_i}}\right)}{\beta_i^{\alpha_i} \Gamma(\alpha_i)}, w_0, w_1, ..., w_m > 0. \tag{4.26}$$

Let $U = W_0, U_r = \frac{\sum_{i=r}^{m} W_i}{\sum_{i=0}^{r-1} W_i}, r = 1, 2, ..., m-1, m.$

By solving the above set of simultaneous equations it follows that

$$W_0 = U$$

$$W_1 = \frac{U(U_1 - U_2)}{1 + U_2}$$

$$W_r = \frac{U(1 + U_1)(U_r - U_{r+1})}{(1 + U_r)(1 + U_{r+1})}, r = 2, 3, ..., m-1$$

$$W_m = \frac{U(1 + U_1) U_m}{(1 + U_m)}.$$

The Jacobian of the transformation is then

$$J(w_0, ..., w_m \to u, u_1, ..., u_m)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{u_1 - u_2}{1+u_2} & \frac{u}{1+u_2} & -\frac{u(1+u_1)}{(1+u_2)^2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{(1+u_1)(u_2-u_3)}{(1+u_2)(1+u_3)} & \frac{u(u_2-u_3)}{(1+u_2)(1+u_3)} & \frac{u(1+u_1)}{(1+u_2)^2} & -\frac{u(1+u_1)}{(1+u_3)^2} & 0 & 0 & \dots & 0 & 0 \\ \frac{(1+u_1)(u_3-u_4)}{(1+u_3)(1+u_4)} & \frac{u(u_3-u_4)}{(1+u_3)(1+u_4)} & 0 & \frac{u(1+u_1)}{(1+u_3)^2} & -\frac{u(1+u_1)}{(1+u_4)^2} & 0 & \dots & 0 & 0 \\ \frac{(1+u_1)(u_4-u_5)}{(1+u_4)(1+u_5)} & \frac{u(u_4-u_5)}{(1+u_4)(1+u_5)} & 0 & 0 & \frac{u(1+u_1)}{(1+u_4)^2} & -\frac{u(1+u_1)}{(1+u_5)^2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{(1+u_1)(u_{m-1}-u_m)}{(1+u_{m-1})(1+u_m)} & \frac{u(u_{m-1}-u_m)}{(1+u_{m-1})(1+u_m)} & 0 & 0 & 0 & 0 & \dots & \frac{u(1+u_1)}{(1+u_{m-1})^2} & -\frac{u(1+u_1)}{(1+u_m)^2} \\ \frac{(1+u_1)u_m}{1+u_m} & \frac{uu_m}{1+u_m} & 0 & 0 & 0 & 0 & \dots & 0 & \frac{u(1+u_1)}{(1+u_m)^2} \end{vmatrix}$$

$$= \frac{u^m (1+u_1)^{m-1}}{\prod_{j=2}^{m} (1+u_j)^2}.$$

By making the transformation and substituting the equations for $W_0, W_1, ...W_m$ into Equation (4.26), it follows that the joint density function of $U, U_1, U_2, ..., U_m$ is

$$
\begin{aligned}
f\left(u,u_1,u_2,...,u_m\right) &= \frac{\left(u^{\alpha_o-1}e^{-\frac{u}{\beta_0}}\right)}{\beta_0^{\alpha_0}\Gamma\left(\alpha_0\right)}\frac{\left(\left(\frac{u(u_1-u_2)}{1+u_2}\right)^{\alpha_1-1}e^{-\frac{\frac{u(u_1-u_2)}{1+u_2}}{\beta_1}}\right)}{\beta_1^{\alpha_1}\Gamma\left(\alpha_1\right)} \\
&\times \prod_{j=2}^{m-1}\left[\frac{\left(\left(\frac{u(1+u_1)(u_j-u_{j+1})}{(1+u_j)(1+u_{j+1})}\right)^{\alpha_j-1}e^{-\frac{\frac{u(1+u_1)\left(u_j-u_{j+1}\right)}{\left(1+u_j\right)\left(1+u_{j+1}\right)}}{\beta_j}}\right)}{\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)}\right] \\
&\times \frac{\left(\left(\frac{u(1+u_1)(u_m)}{(1+u_m)}\right)^{\alpha_m-1}e^{-\frac{\frac{u(1+u_1)(u_m)}{(1+u_m)}}{\beta_m}}\right)}{\beta_m^{\alpha_m}\Gamma\left(\alpha_m\right)}\frac{u^m\left(1+u_1\right)^{m-1}}{\prod_{j=2}^{m}\left(1+u_j\right)^2}. \\
&= \frac{\prod_{j=1}^{m-1}\left[\left(u_j-u_{j+1}\right)^{\alpha_j-1}\right]\left(u_m\right)^{\alpha_m-1}}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)\right]}u^{\sum_{j=0}^{m}[\alpha_j]-1}\left(1+u_1\right)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[\left(1+u_j\right)^{-\alpha_{j-1}-\alpha_j}\right] \\
&\times e^{-u\left(\frac{1}{\beta_0}+\frac{(u_1-u_2)}{\beta_1(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)\left(u_j-u_{j+1}\right)}{\beta_j\left(1+u_j\right)\left(1+u_{j+1}\right)}\right]+\frac{(1+u_1)u_m}{\beta_m(1+u_m)}\right)}. \qquad (4.27)
\end{aligned}
$$

By integrating Equation (4.27) with respect to u, it follows that

$$
\begin{aligned}
f\left(u_1,u_2,...,u_m\right) &= \frac{\prod_{j=1}^{m-1}\left[\left(u_j-u_{j+1}\right)^{\alpha_j-1}\right]\left(u_m\right)^{\alpha_m-1}}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)\right]}\left(1+u_1\right)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[\left(1+u_j\right)^{-\alpha_{j-1}-\alpha_j}\right] \\
&\times \int_0^{\infty}u^{\sum_{j=0}^{m}[\alpha_j]-1}e^{-u\left(\frac{1}{\beta_0}+\frac{(u_1-u_2)}{\beta_1(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)\left(u_j-u_{j+1}\right)}{\beta_j\left(1+u_j\right)\left(1+u_{j+1}\right)}\right]+\frac{(1+u_1)u_m}{\beta_m(1+u_m)}\right)}du. \qquad (4.28)
\end{aligned}
$$

By applying Result 11 to Equation (4.28), it then follows that

$$
\begin{aligned}
f\left(u_1,u_2,...,u_m\right) &= \frac{\prod_{j=1}^{m-1}\left[\left(u_j-u_{j+1}\right)^{\alpha_j-1}\right]\left(u_m\right)^{\alpha_m-1}\Gamma\left(\sum_{j=0}^{m}\left[\alpha_j\right]\right)}{\prod_{j=0}^{m}\left[\beta_j^{\alpha_j}\Gamma\left(\alpha_j\right)\right]}\left(1+u_1\right)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[\left(1+u_j\right)^{-\alpha_{j-1}-\alpha_j}\right] \\
&\times \left(\frac{1}{\beta_0}+\frac{\left(u_1-u_2\right)}{\beta_1\left(1+u_2\right)}+\sum_{j=2}^{m-1}\left[\frac{\left(1+u_1\right)\left(u_j-u_{j+1}\right)}{\beta_j\left(1+u_j\right)\left(1+u_{j+1}\right)}\right]+\frac{\left(1+u_1\right)u_m}{\beta_m\left(1+u_m\right)}\right)^{-\sum_{j=0}^{m}[\alpha_j]}.
\end{aligned}
$$

■

Special cases

1) If $\alpha_i = \alpha$ for $i = 0, 1, 2, ..., m$ then Equation (4.25) becomes

$$\begin{aligned} f(u_1, u_2, ..., u_m) &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha-1}\right](u_m)^{\alpha-1}\Gamma\left(\sum_{j=0}^{m}[\alpha]\right)}{\prod_{j=0}^{m}[\beta_j{}^{\alpha}\Gamma(\alpha)]}(1+u_1)^{\sum_{j=2}^{m}[\alpha]}\prod_{j=2}^{m}\left[(1+u_j)^{-\alpha-\alpha}\right] \\ &\times \left(\frac{1}{\beta_0}+\frac{(u_1-u_2)}{\beta_1(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{\beta_j(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{\beta_m(1+u_m)}\right)^{-\sum_{j=0}^{m}[\alpha]} \\ &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha-1}\right](u_m)^{\alpha-1}\Gamma((m+1)\alpha)}{\prod_{j=0}^{m}[\beta_j{}^{\alpha}]\Gamma(\alpha)^{(m+1)}}(1+u_1)^{(m-1)\alpha}\prod_{j=2}^{m}\left[(1+u_j)^{-2\alpha}\right] \\ &\times \left(\frac{1}{\beta_0}+\frac{(u_1-u_2)}{\beta_1(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{\beta_j(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{\beta_m(1+u_m)}\right)^{-(m+1)\alpha}, \\ & u_1 > u_2 > ... > u_m > 0. \end{aligned}$$

2) If $\beta_i = \beta$ for $i = 0, 1, 2, ..., m$ then Equation (4.25) becomes

$$\begin{aligned} f(u_1, u_2, ..., u_m) &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha_j-1}\right](u_m)^{\alpha_m-1}\Gamma\left(\sum_{j=0}^{m}[\alpha_j]\right)}{\prod_{j=0}^{m}[\beta^{\alpha_j}\Gamma(\alpha_j)]}(1+u_1)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[(1+u_j)^{-\alpha_{j-1}-\alpha_j}\right] \\ &\times \left(\frac{1}{\beta}+\frac{(u_1-u_2)}{\beta(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{\beta(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{\beta(1+u_m)}\right)^{-\sum_{j=0}^{m}[\alpha_j]} \\ &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha_j-1}\right](u_m)^{\alpha_m-1}\Gamma\left(\sum_{j=0}^{m}[\alpha_j]\right)}{\prod_{j=0}^{m}[\beta^{\alpha_j}\Gamma(\alpha_j)]}(1+u_1)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[(1+u_j)^{-\alpha_{j-1}-\alpha_j}\right] \\ &\times \left(\frac{1}{\beta}\right)^{-\sum_{j=0}^{m}[\alpha_j]}\left(1+\frac{(u_1-u_2)}{(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{(1+u_m)}\right)^{-\sum_{j=0}^{m}[\alpha_j]} \\ &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha_j-1}\right](u_m)^{\alpha_m-1}\Gamma\left(\sum_{j=0}^{m}[\alpha_j]\right)}{\prod_{j=0}^{m}[\Gamma(\alpha_j)]}(1+u_1)^{\sum_{j=2}^{m}[\alpha_j]}\prod_{j=2}^{m}\left[(1+u_j)^{-\alpha_{j-1}-\alpha_j}\right] \\ &\times \left(1+\frac{(u_1-u_2)}{(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{(1+u_m)}\right)^{-\sum_{j=0}^{m}[\alpha_j]}, \\ & u_1 > u_2 > ... > u_m > 0. \end{aligned}$$

3) If $\beta_i = \beta$ and $\alpha_i = \alpha$ for $i = 0, 1, 2, ..., m$ then Equation (4.25) becomes

$$\begin{aligned} f(u_1, u_2, ..., u_m) &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha-1}\right](u_m)^{\alpha-1}\Gamma\left(\sum_{j=0}^{m}[\alpha]\right)}{\prod_{j=0}^{m}[\beta^{\alpha}\Gamma(\alpha)]}(1+u_1)^{\sum_{j=2}^{m}[\alpha]}\prod_{j=2}^{m}\left[(1+u_j)^{-\alpha-\alpha}\right] \\ &\times \left(\frac{1}{\beta}+\frac{(u_1-u_2)}{\beta(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{\beta(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{\beta(1+u_m)}\right)^{-\sum_{j=0}^{m}[\alpha]} \\ &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha-1}\right](u_m)^{\alpha-1}\Gamma((m+1)\alpha)}{\beta^{(m+1)\alpha}\Gamma(\alpha)^{m+1}}(1+u_1)^{(m-1)\alpha}\prod_{j=2}^{m}\left[(1+u_j)^{-2\alpha}\right] \\ &\times \left(\frac{1}{\beta}\right)^{-(m+1)\alpha}\left(1+\frac{(u_1-u_2)}{(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{(1+u_m)}\right)^{-(m+1)\alpha} \\ &= \frac{\prod_{j=1}^{m-1}\left[(u_j-u_{j+1})^{\alpha-1}\right](u_m)^{\alpha-1}\Gamma((m+1)\alpha)}{\Gamma(\alpha)^{m+1}}(1+u_1)^{(m-1)\alpha}\prod_{j=2}^{m}\left[(1+u_j)^{-2\alpha}\right] \\ &\times \left(1+\frac{(u_1-u_2)}{(1+u_2)}+\sum_{j=2}^{m-1}\left[\frac{(1+u_1)(u_j-u_{j+1})}{(1+u_j)(1+u_{j+1})}\right]+\frac{(1+u_1)u_m}{(1+u_m)}\right)^{-(m+1)\alpha}, \\ & u_1 > u_2 > ... > u_m > 0. \end{aligned}$$

Chapter 5

Simulation and comparison

5.1 Introduction

In this chapter, a simulation study is performed to investigate how the proposed model would perform in comparison to another self-starting chart, the Q chart form investigated by Adamski [1]. As mentioned in Section 1.1, from a practical perspective, only an increase in the variance of the process will likely be of concern, and as such it is the only case that is considered in this chapter. In other words, the distributions will be studied under the null hypothesis of no shift occurring ($H_0 : \lambda = 1$), as well as under alternative hypothesis that the process variance has increased ($H_A : \lambda > 1$).

In Section 5.2, some the IC properties of the control charts will be studied, this is imperative since the control limits of a control chart are constructed under the null hypothesis that no shift has occurred, or alternatively that the process is IC. The IC properties that are investigated for each distribution are:

- Where the maximum order statistic is most likely to occur when no shift has occurred in the process variance. Practically, this is a very important question since, if the maximum order statistic consistently occurs at roughly the same place in the sequence of samples, it implies that the control chart should be treated with added suspicion if it indicates that a shift in the process occurs at this location.

- How the 95^{th} percentiles of the maximum order statistics, of the respective distributions, change as the number of samples as well as the sample sizes vary. In essence, these values that are simulated correspond to the UCLs of the control charts during phase I, and as such the terms “UCL”, “critical value” and “95^{th} percentile of the maximum order statistic” are used interchangeably in this chapter. The UCL values are generated for the number of samples (m) equal to 4,9,14,19,24,29,49,99 and 499 as well as each sample size (n) equal to 2,5,10,15,20,25,30,50,100 and 500. Graphs are provided indicating how these simulated 95^{th} percentiles of the maximum order statistic vary as m and n change.

In Section 5.3, the OOC performance of the newly proposed control chart is compared with the Q chart model investigated by Adamski[1]. The probability that the respective control charts will detect a shift in the process variance is investigated for differing sizes of shifts in the process variance. The comparison is made for varying numbers of samples as well as sample sizes.

In this chapter the values that are simulated and the graphs that are drawn are those that correspond to the statistics in equations (1.9) and (1.15), and not those for which the distributions were derived, equations (1.11) and (1.16). This has bee stated before in this study, but is repeated since it is an important distinction to make.

5.2 Distributions when the process is IC

The location of the maximum order statistic when the process is IC is investigated using graphs. For brevity's sake only a few of the 90 possible combinations of the number of samples and sample sizes mentioned above will be included in this study. All of them, however, lead to the same general conclusion. The three graphs that will be included are $m = 29, n = 15$; $m = 19, n = 10$ and $m = 9, n = 5$.

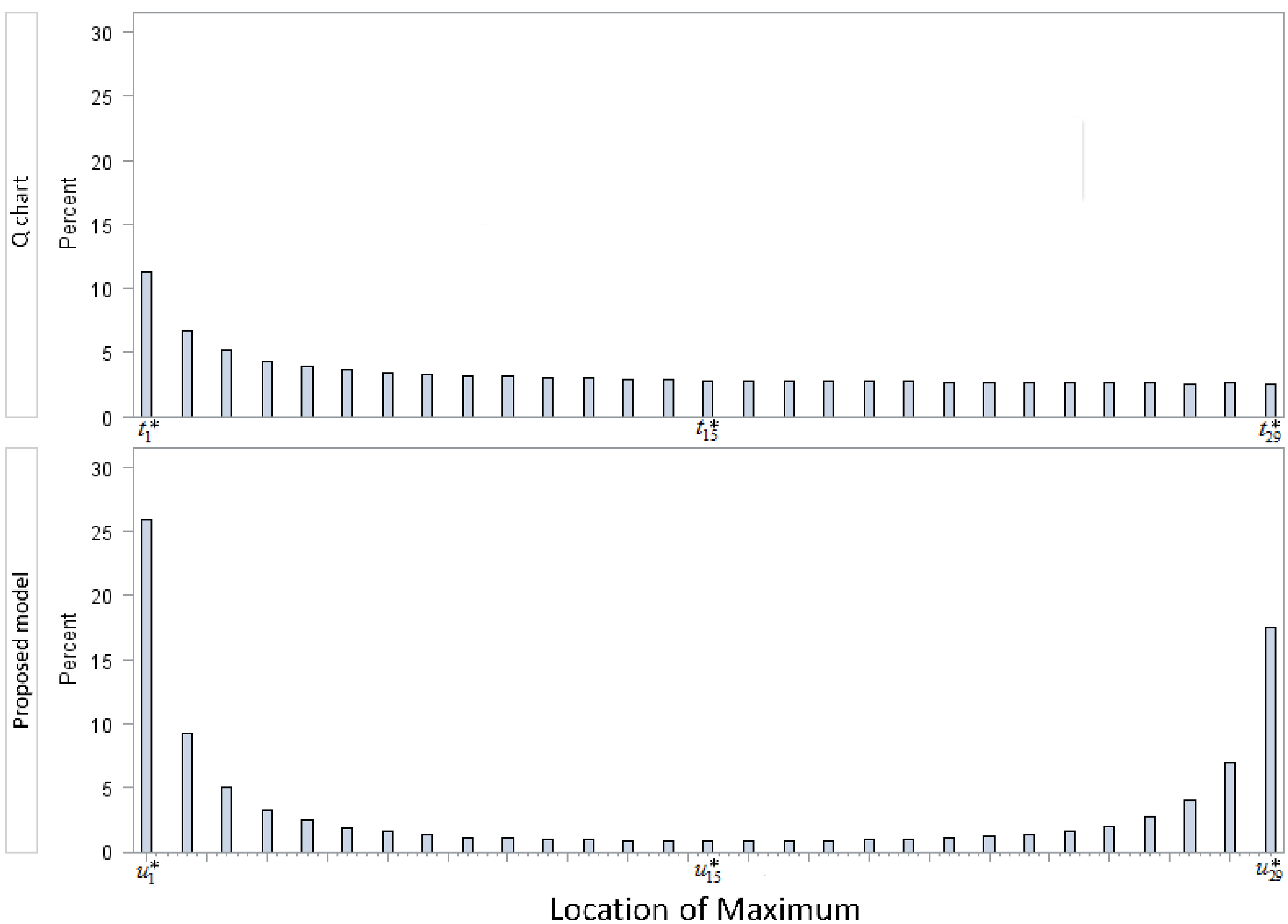

Figure 5.1: Location of maximum order statistic - $m = 29, n = 15$.

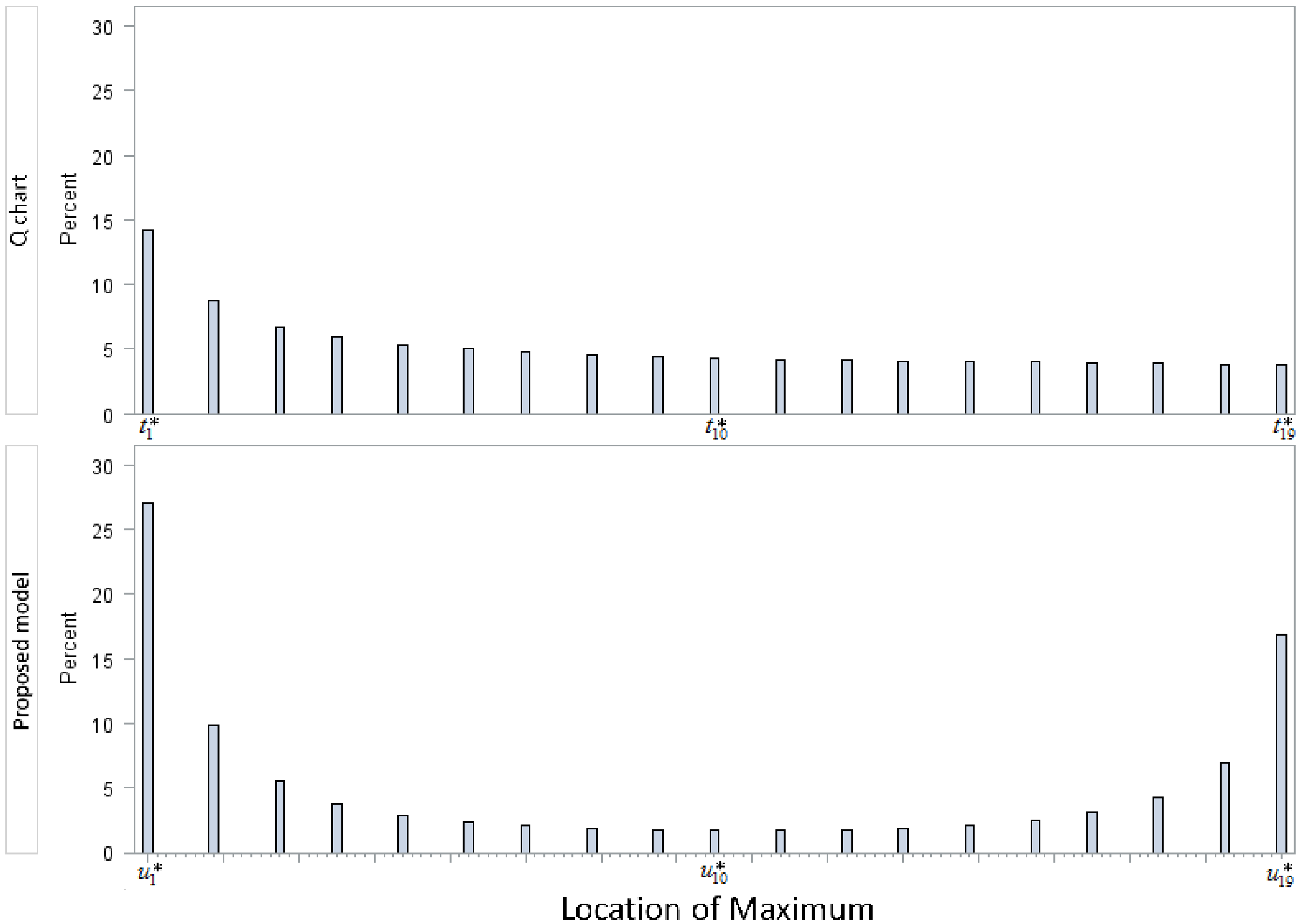

Figure 5.2: Location of maximum order statistic - $m = 19, n = 10$.

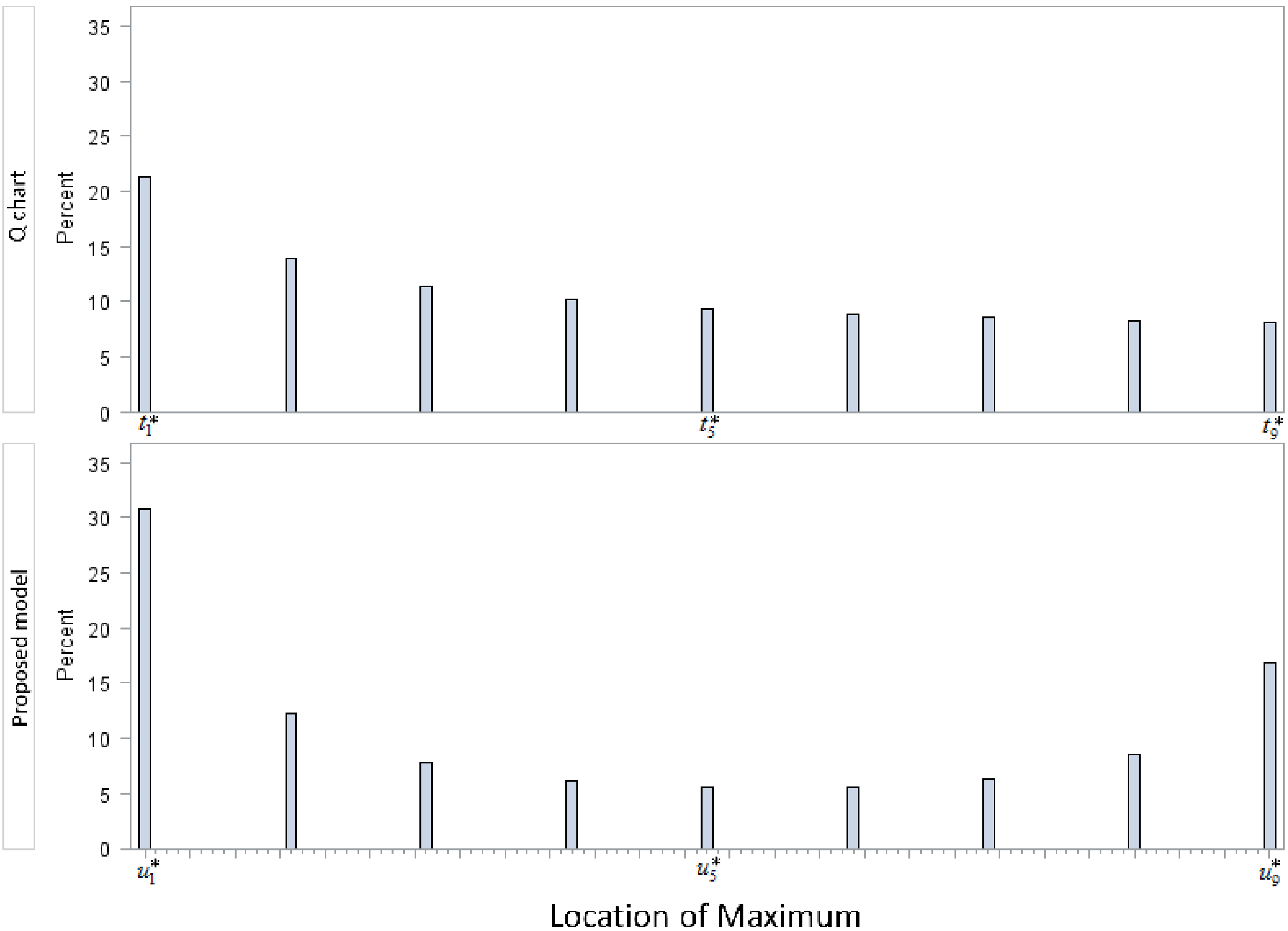

Figure 5.3: Location of maximum order statistic - $m = 9, n = 5$.

As can be seen from figures 5.1, 5.2 and 5.3, the beta type VII distribution's maximum order statistic occurs most often at the first statistic, with the probability of the maximum occurring at subsequent statistics steadily decreasing. This implies that the Q chart becomes more stable as the process progresses, which makes practical sense since each subsequent statistic includes more of the sample data points. The beta type VIII distribution's maximum order statistic occurs most often at the first statistic, and second-most often at the last statistic. This is because of to the way in which the statistics of the beta type VIII distribution are constructed (see Equation (1.9)). The denominator of the first statistic consists of only one sample variance, as does the numerator of the last statistic. Having the numerator or denominator consist of only one sample's data increases the potential for them to fluctuate erratically since they do not average out over a few samples' worth of data. If, for example, the first sample variance is abnormally small in comparison to the other $m - 1$ samples, it will result in a very large first statistic, and vice versa for the last statistic. This implies that while the proposed model may detect shifts at the ends of the samples, signals received at these locations should be treated with a bit of skepticism.

In tables 5.1 and 5.2 which follow, values for the 95^{th} percentiles of the maximum order statistics of the beta type VII and beta type VIII distributions, respectively are simulated (to the third decimal) using Monte Carlo simulation. How these values vary depending on the number of samples, as well as the sample sizes, is shown in figures 5.4, 5.5, 5.6 and 5.7.

	n=2	5	10	15	20	25	30	50	100	500
m=4	202.879	7.515	3.630	2.786	2.410	2.187	2.040	1.732	1.475	1.190
9	204.627	7.619	3.729	2.867	2.478	2.243	2.090	1.770	1.500	1.202
14	203.879	7.685	3.759	2.900	2.503	2.268	2.114	1.789	1.512	1.207
19	202.930	7.688	3.780	2.919	2.523	2.287	2.129	1.800	1.522	1.211
24	203.569	7.685	3.802	2.936	2.536	2.300	2.141	1.811	1.528	1.214
29	201.419	7.666	3.811	2.943	2.546	2.309	2.150	1.818	1.535	1.216
49	207.101	7.706	3.840	2.983	2.580	2.3411	2.179	1.842	1.550	1.223
99	206.304	7.723	3.906	3.038	2.631	2.387	2.221	1.872	1.573	1.233
499	206.026	7.800	4.097	3.213	2.783	2.521	2.343	1.964	1.633	1.257

Table 5.1: 95^{th} percentiles of the maximum order statistic of the multivariate beta type VII distribution.

The values in Table 5.1 were simulated using the SAS code found in Result 20, with 1 000 000 simulations, and are rounded to three decimal places.

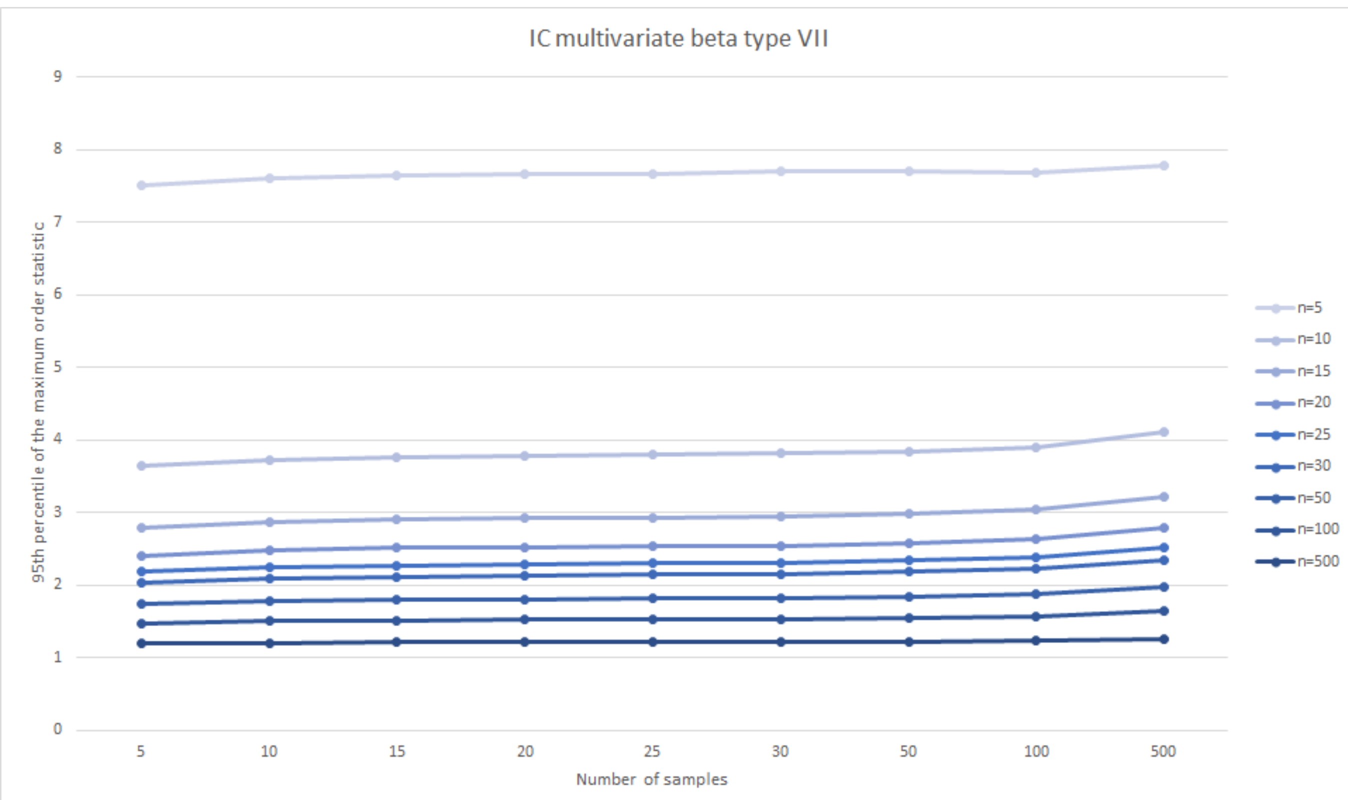

Figure 5.4: Varying 95^{th} percentiles of the maximum order statistic of the multivariate beta type VII distribution as a function of m.

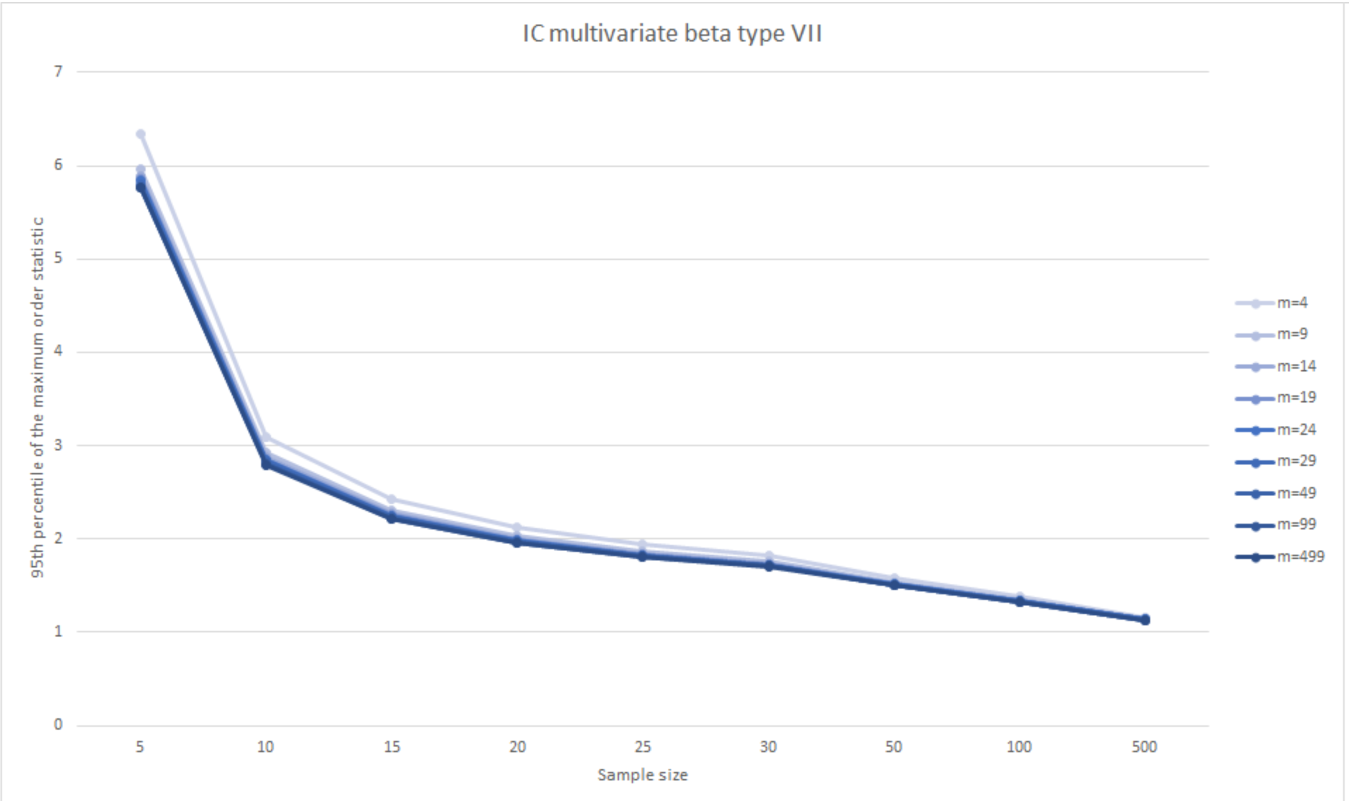

Figure 5.5: Varying 95^{th} percentiles of the maximum order statistic of the multivariate beta type VII distribution as a function of n.

	n=2	5	10	15	20	25	30	50	100	500
m=4	235.290	6.336	3.095	2.421	2.118	1.942	1.826	1.582	1.379	1.153
9	249.969	5.958	2.929	2.311	2.029	1.869	1.761	1.539	1.351	1.143
14	253.365	5.891	2.881	2.272	2.003	1.843	1.739	1.525	1.342	1.139
19	252.002	5.863	2.857	2.254	1.989	1.834	1.731	1.517	1.339	1.138
24	254.506	5.835	2.845	2.247	1.982	1.828	1.724	1.514	1.335	1.137
29	255.251	5.803	2.832	2.239	1.978	1.823	1.723	1.512	1.335	1.136
49	266.453	5.772	2.810	2.228	1.967	1.815	1.715	1.508	1.331	1.135
99	264.693	5.772	2.802	2.218	1.959	1.810	1.709	1.505	1.329	1.134
499	265.135	5.770	2.791	2.216	1.957	1.806	1.706	1.500	1.328	1.134

Table 5.2: 95^{th} percentiles of the maximum order statistic of the multivariate beta type VIII distribution.

The values in Table 5.2 were simulated using the SAS code found in Result 20, with 1 000 000 simulations, and are rounded to three decimal places.

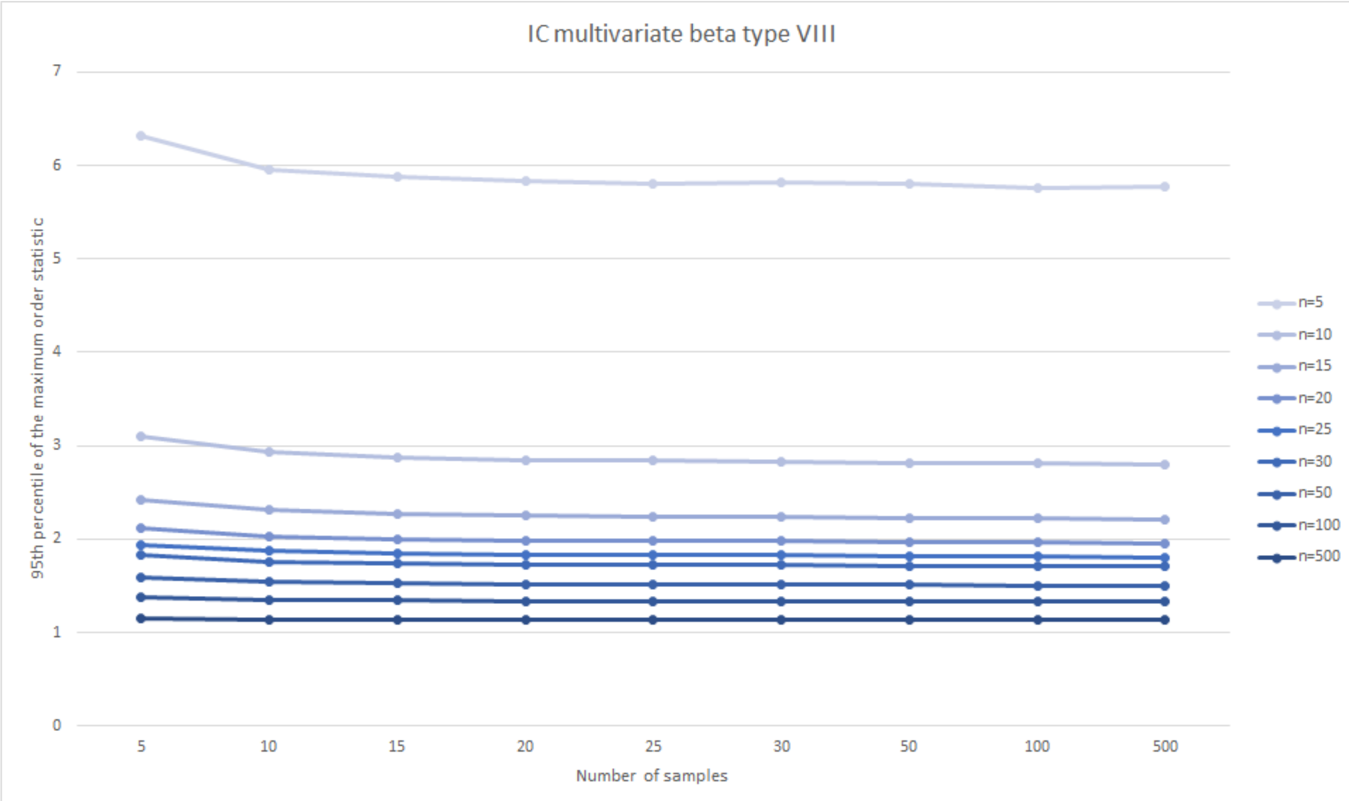

Figure 5.6: Varying 95^{th} percentiles of the maximum order statistic of the multivariate beta type VIII distribution as a function of m.

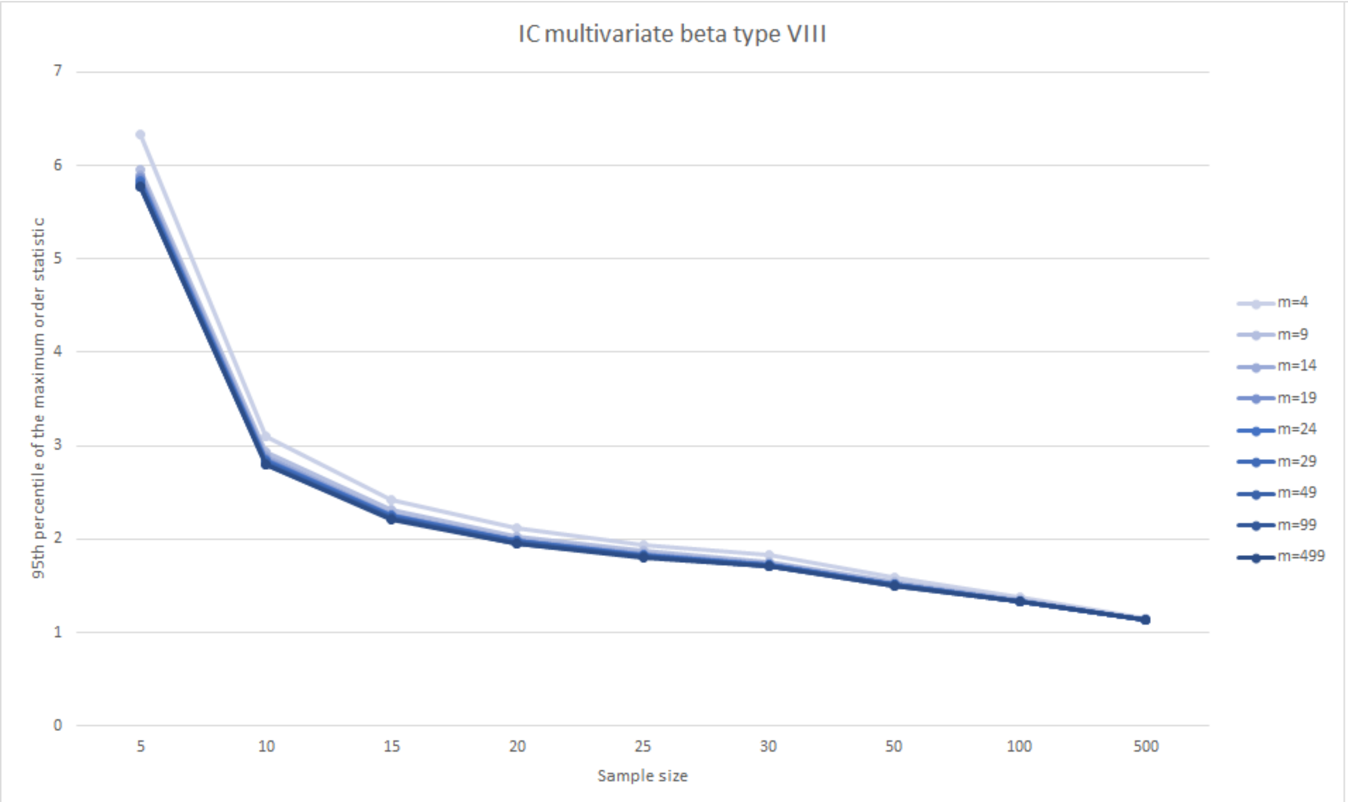

Figure 5.7: Varying 95^{th} percentiles of the maximum order statistic of the multivariate beta type VIII distribution as a function of n.

Some interesting observations and conclusions can be made about the beta type VII and beta type VIII maximum order statistics using figures 5.4, 5.5, 5.6 and 5.7, and tables 5.1 and 5.2:

- Figures 5.4, 5.5, 5.6 and 5.7 do not plot the UCLs for $n = 2$ since they are much larger than the other simulated choices of n. Inducing these plots on the graphs would have inflated the y-axes of the graphs to the point where interpreting the trends in the UCL values would have been difficult from a visual perspective.

- When each sample consists of only two observations ($n = 2$), (the minimum required sample size when the process mean is not known), both the beta type VII and beta type VIII distributions have very large UCL values, irrespective of the number of samples used in the models. This implies that while it is theoretically possible to use extremely small sample sizes, practically, it is inadvisable, especially since increasing the sample sizes by very little, to $n = 5$, for instance - dramatically reduces the UCL values.

- In tables 5.1 and 5.2 for values of n other than $n = 2$ and $n = 5$, the simulated 95^{th} percentiles of the maximum order statistics are monotone, however, when $n = 2$ or $n = 5$ this is not the case. It should also be noted that when $n = 2$ (and to a lesser extent when $n = 5$) the simulated values in tables 5.1 and 5.2 vary greatly and erratically between repeat simulations, with the simulated values varying by as much as 3% between repeated runs. This erratic behaviour continues even if the number of

Monte Carlo simulations are increased from 1 000 000 to 10 000 000. As such this study recommends that using extremely small sample sizes in these control charts should be avoided if at all possible.

- From figures 5.5 and 5.7 it is clear that increasing the sample sizes for both the beta type VII and the beta type VIII distributions lowers the UCL values. This corresponds with the intuitive feeling that increasing the amount of data that the models can rely on results in smaller UCL values, which leads to an increased probability of detecting a shift when it actually does occur. It is also clear that increasing the sample sizes offers diminishing returns in terms of shrinking the UCL values. The largest decreases in UCL values occur between $n = 2$ and $n = 5$, followed by the jump between $n = 5$ and $n = 10$. Practically, this means that increasing the sample size of the samples will only greatly affect the UCLs up to a certain point, and it would probably not be feasible to, for example, increase the sample sizes from 100 to 500, since the added economic cost of doing so might be outweighed by the minimal decrease in the UCL values.

- One would expect (and hope) that, as the number of samples increases, the UCL values would decrease. (The reasoning for this is similar to the reasoning that increasing the sample sizes should shrink the UCL values in the point mentioned above). While it is clear from Figure 5.6 that this is indeed the case for the beta type VIII distribution, Figure 5.4 shows that increasing the number of samples for the beta type VII distribution actually increases the UCL values. This unwanted property of that beta type VII distribution implies that applying the Q chart to a phase I-type setting, as in this study, may not be practical. This unwanted increase in the UCL values as m increases can also be seen in Figure 5.5, where the darker plotted lines (corresponding to higher values of m) plot decreasingly in terms of UCL, whereas in Figure 5.7 the darker lines correspond to lower UCL values.

Note The SAS code that is used to simulate all of the values in this section can be found in Result 20.

5.3 Distributions when the process is OOC

The ability of control charts to quickly and effectively detect shifts in a production process is of paramount importance. As mentioned in Chapter 1, different charts are used to detect different sized shifts. In this section, the proposed model's potential to detect shifts in compared with that of the Q chart form investigated by Adamski [1]. The probability of signaling that a shift in the process variance has occurred depends on a few variables. In this study these variables are: the number of samples, the sample size of the samples, where in the process the shift in the process variance occurs, and the size of the shift. The figures in this section take all of these parameters into account.

In each of the following figures, the probability of signaling a shift in the variance is displayed as a function of the size of the shift, where the shift size λ (see Section 1.1), ranges from $\lambda = 1$ (no shift) to $\lambda = 5$ (a five-fold increase in the process variance). Many different combinations of the number of samples, the sample sizes, as well as the locations of the shift were tested. The chosen parameters that were simulated are: number of samples (m) equal to 10, 20 and 30, the sample sizes (n) equal to 2, 5, 10 and 20, and the location of the shift in the process variance (κ) occurring at (roughly, due to integer sample numbers) 25%, 50% and 75% of the way through the samples. (The SAS code used to generate figures 5.8 to 5.19 can be found in Result 21.)

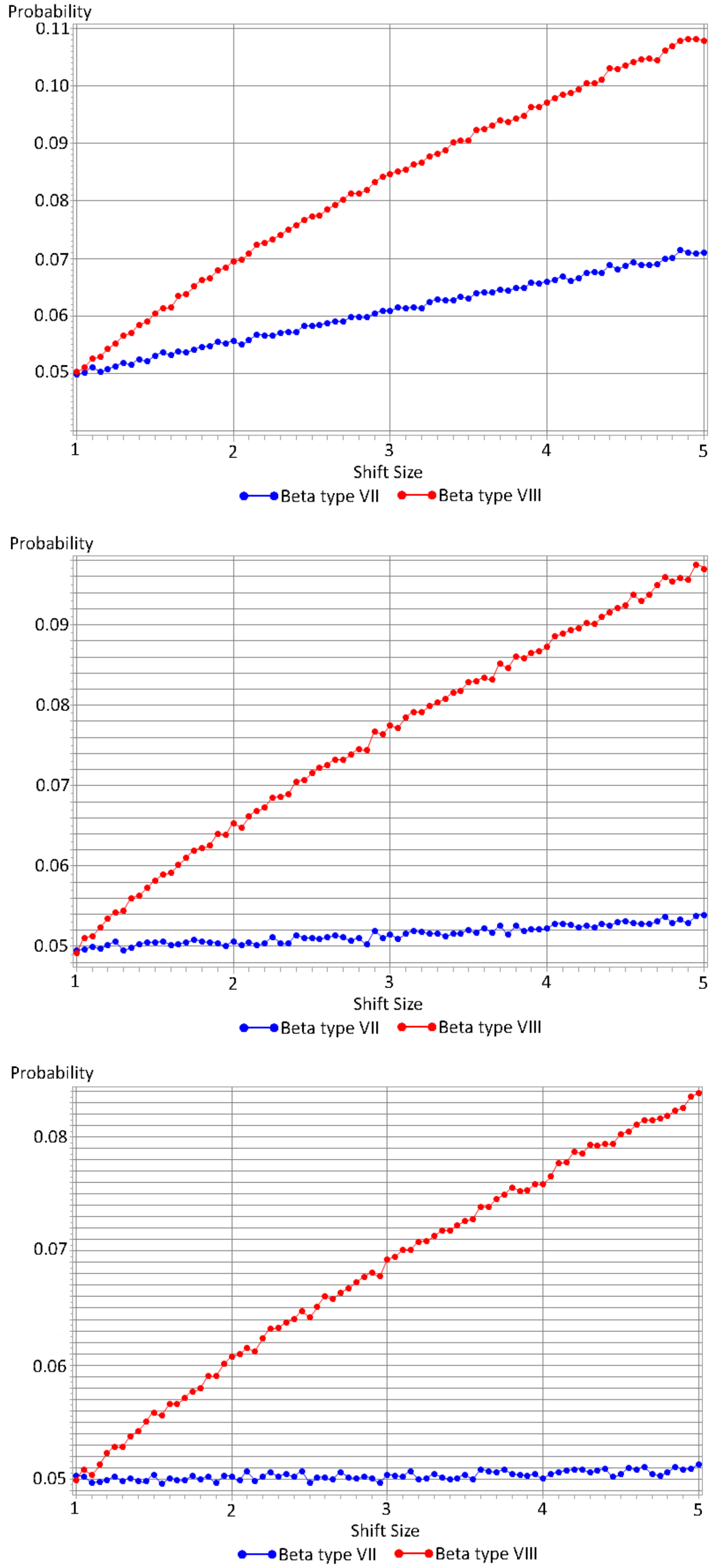

Figure 5.8: Probability of detecting a shift, $m = 9, n = 2, \kappa = 3, 5, 7$.

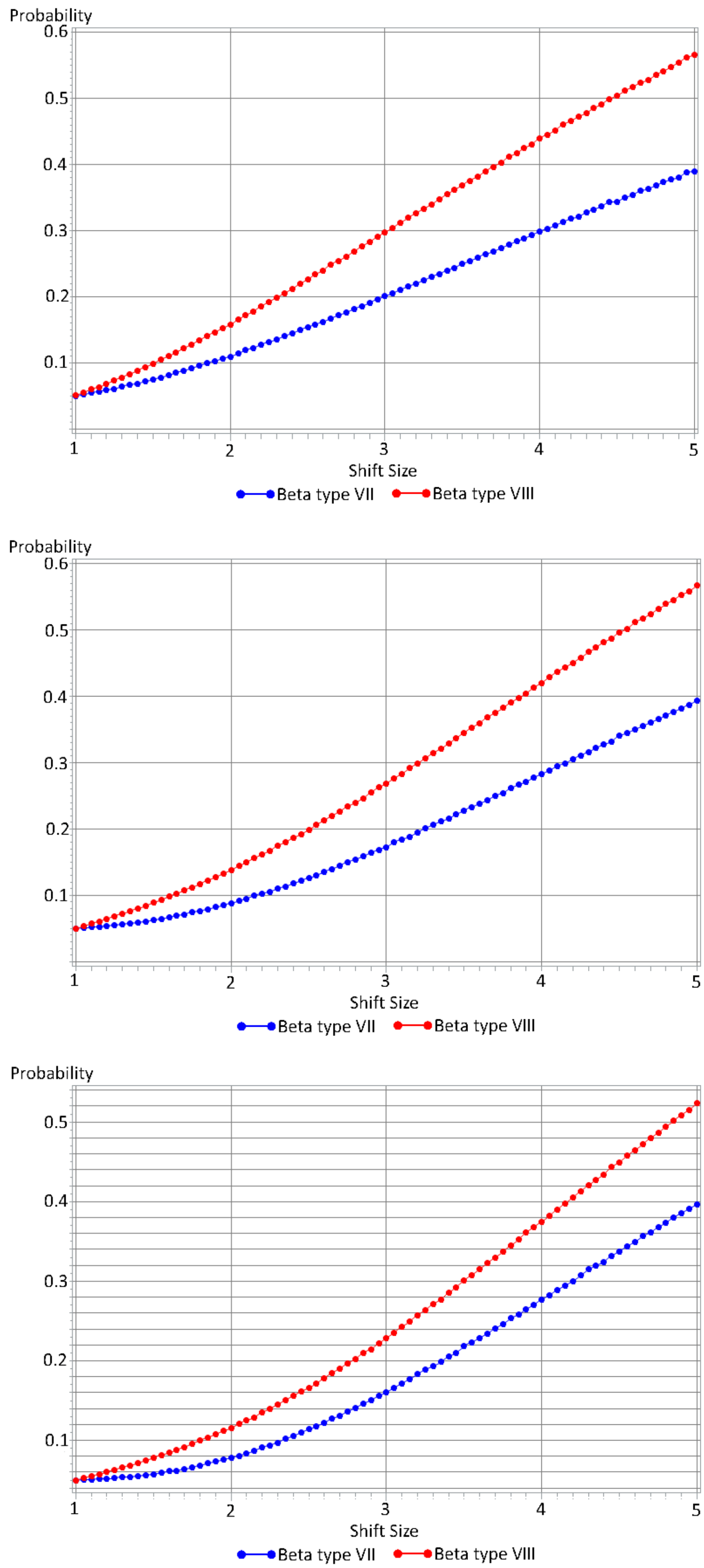

Figure 5.9: Probability of detecting a shift, $m = 9, n = 5, \kappa = 3, 5, 7$.

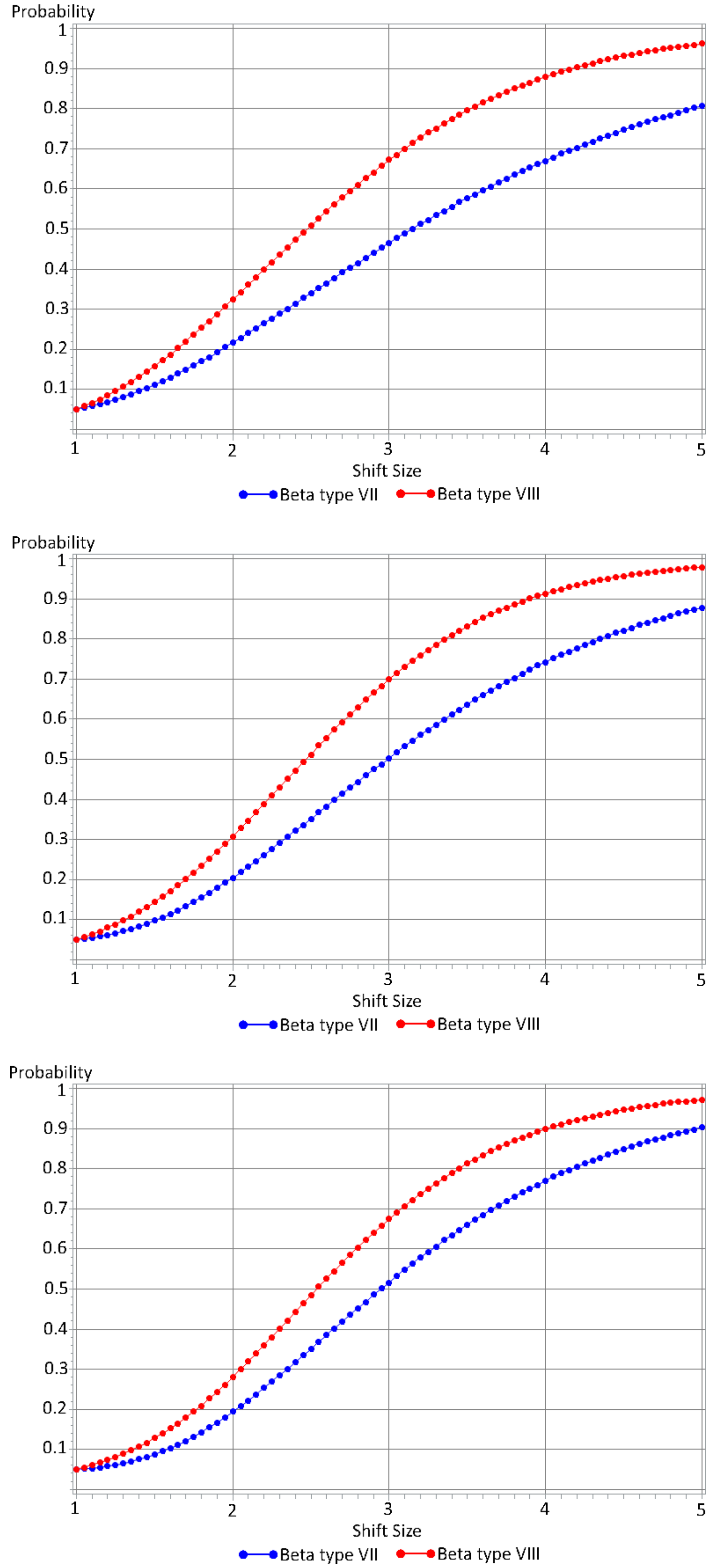

Figure 5.10: Probability of detecting a shift, $m = 9, n = 10, \kappa = 3, 5, 7$.

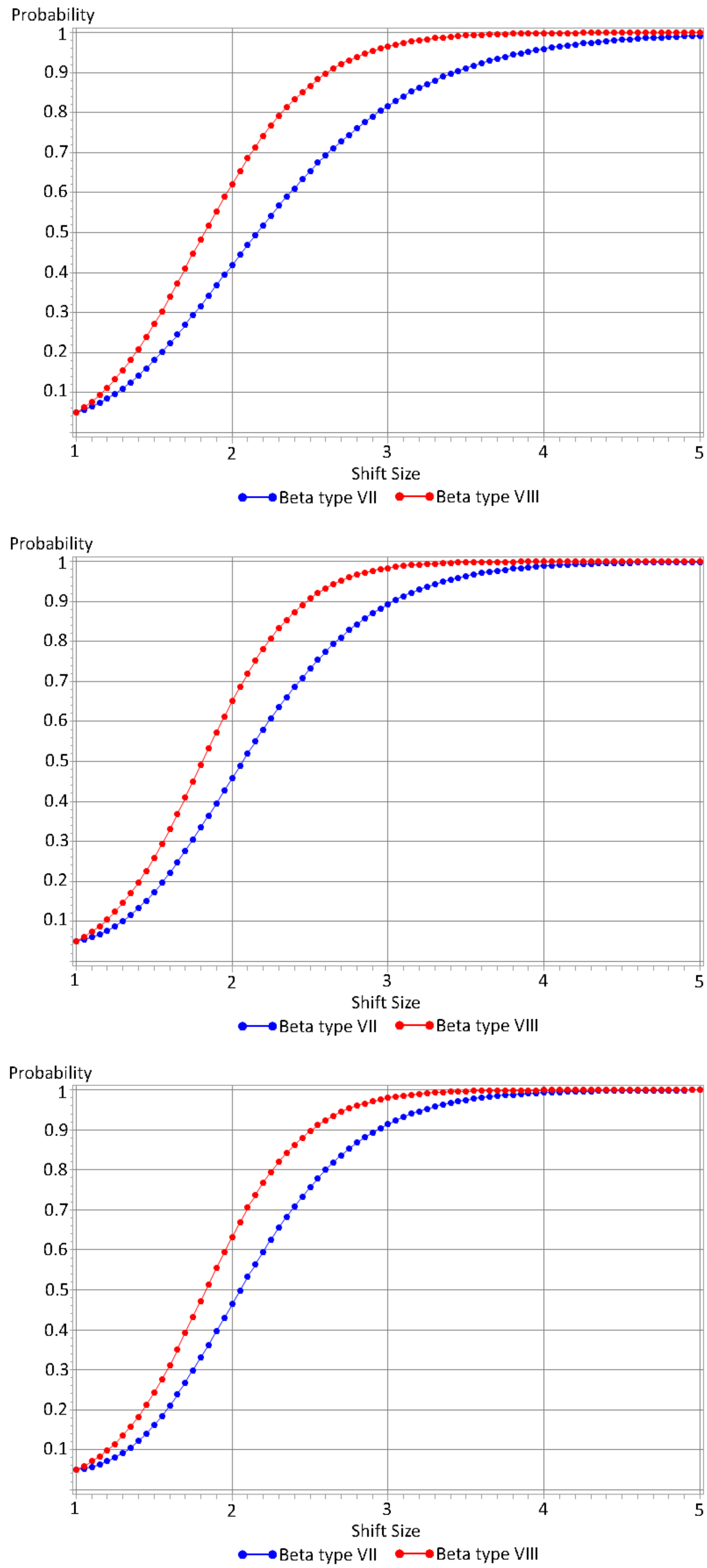

Figure 5.11: Probability of detecting a shift, $m = 9, n = 20, \kappa = 3, 5, 7$.

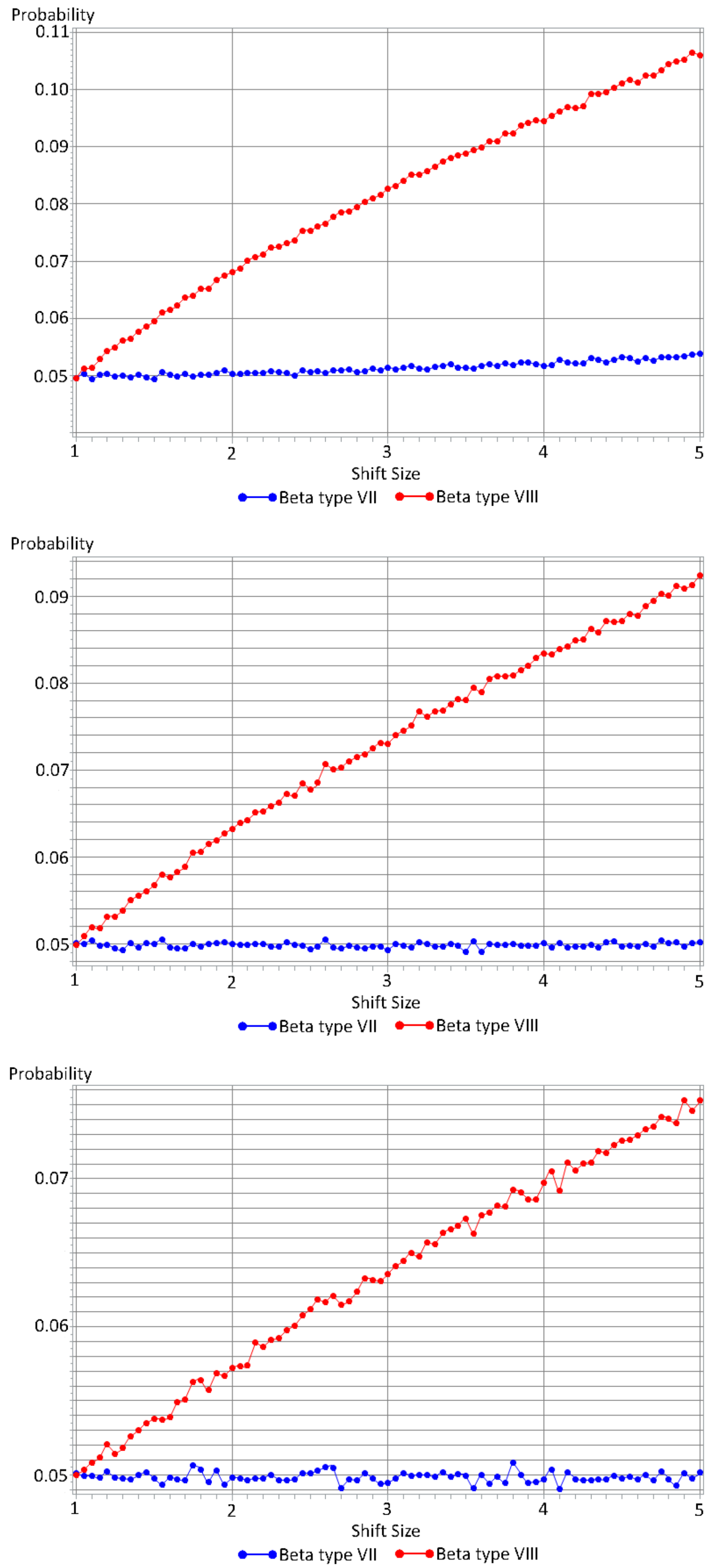

Figure 5.12: Probability of detecting a shift, $m = 19, n = 2, \kappa = 5, 10, 15$.

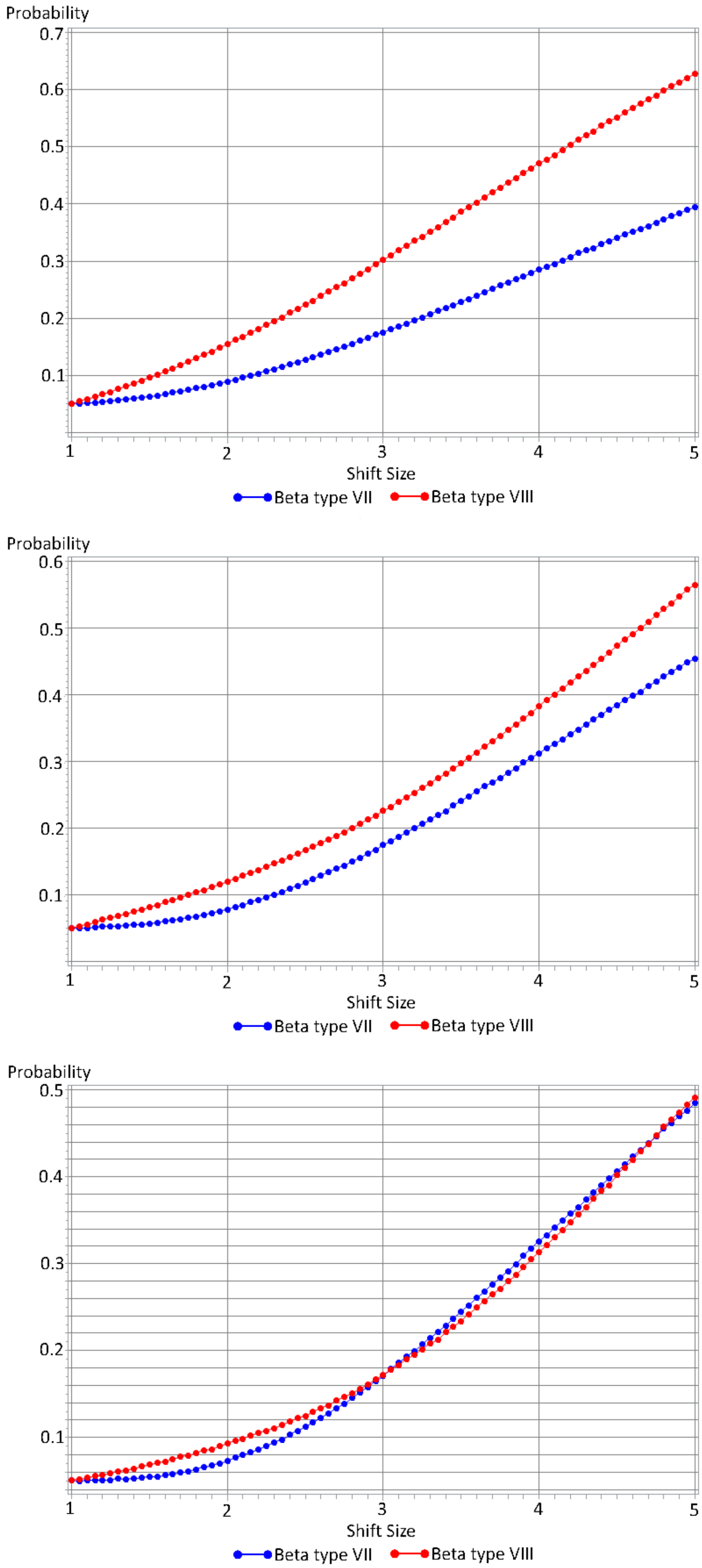

Figure 5.13: Probability of detecting a shift, $m = 19, n = 5, \kappa = 5, 10, 15$.

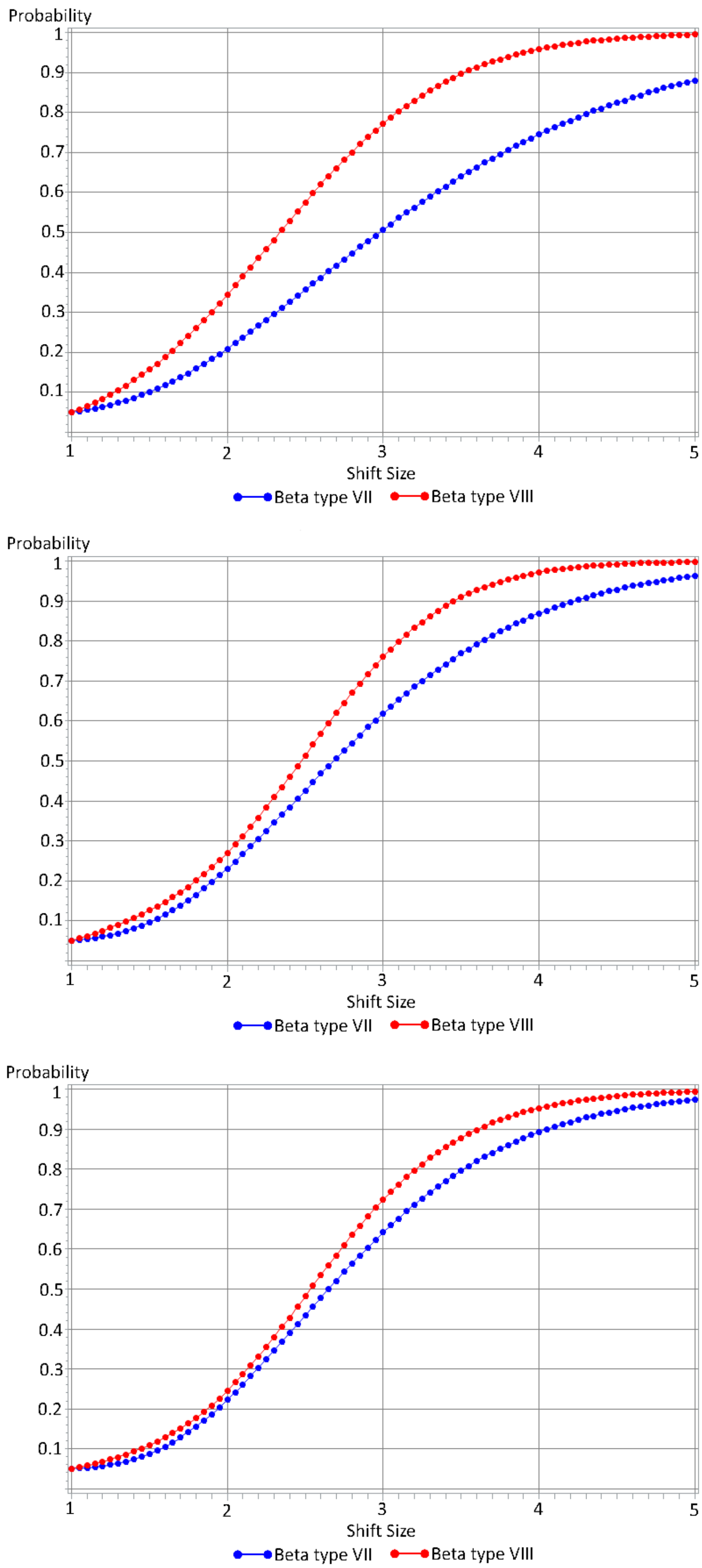

Figure 5.14: Probability of detecting a shift, $m = 19, n = 10, \kappa = 5, 10, 15$.

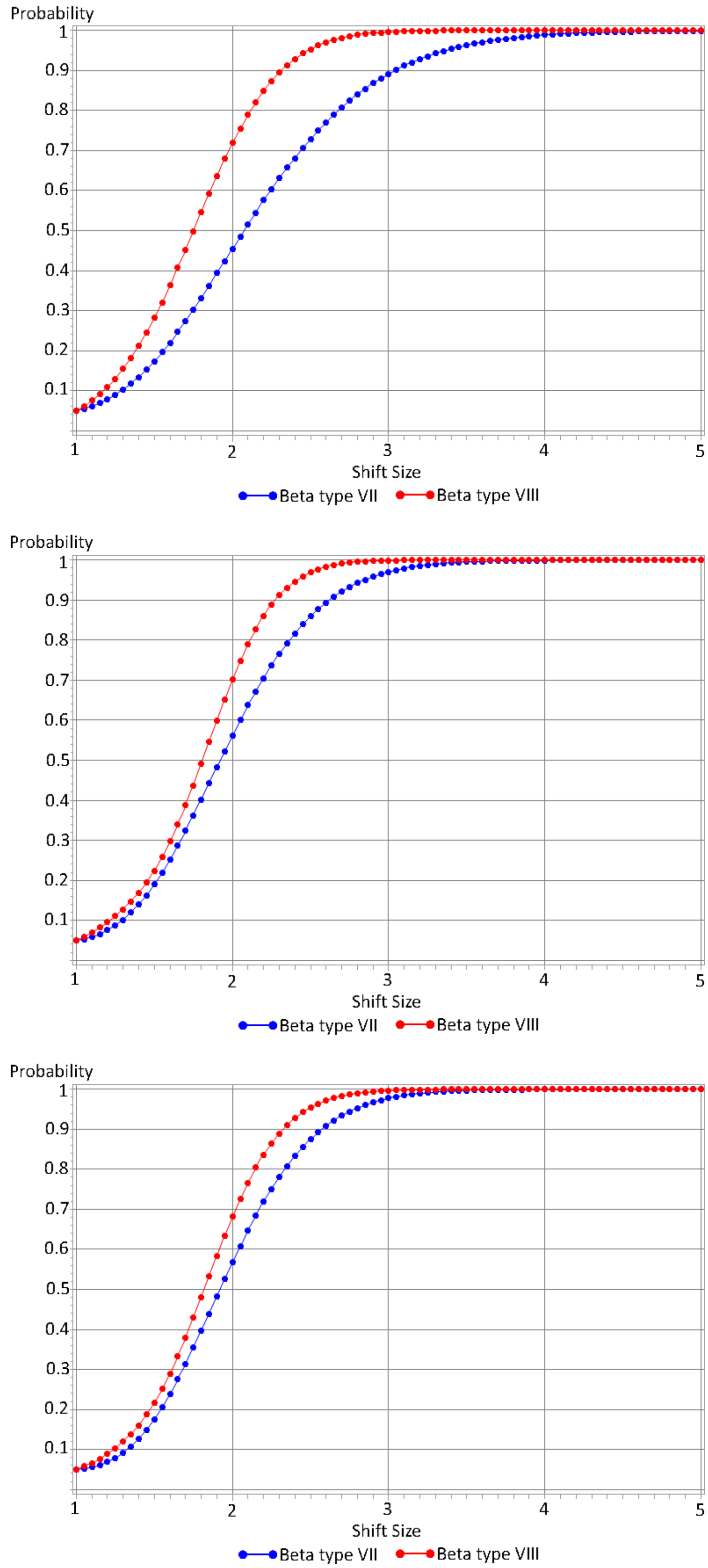

Figure 5.15: Probability of detecting a shift, $m = 19, n = 20, \kappa = 5, 10, 15$.

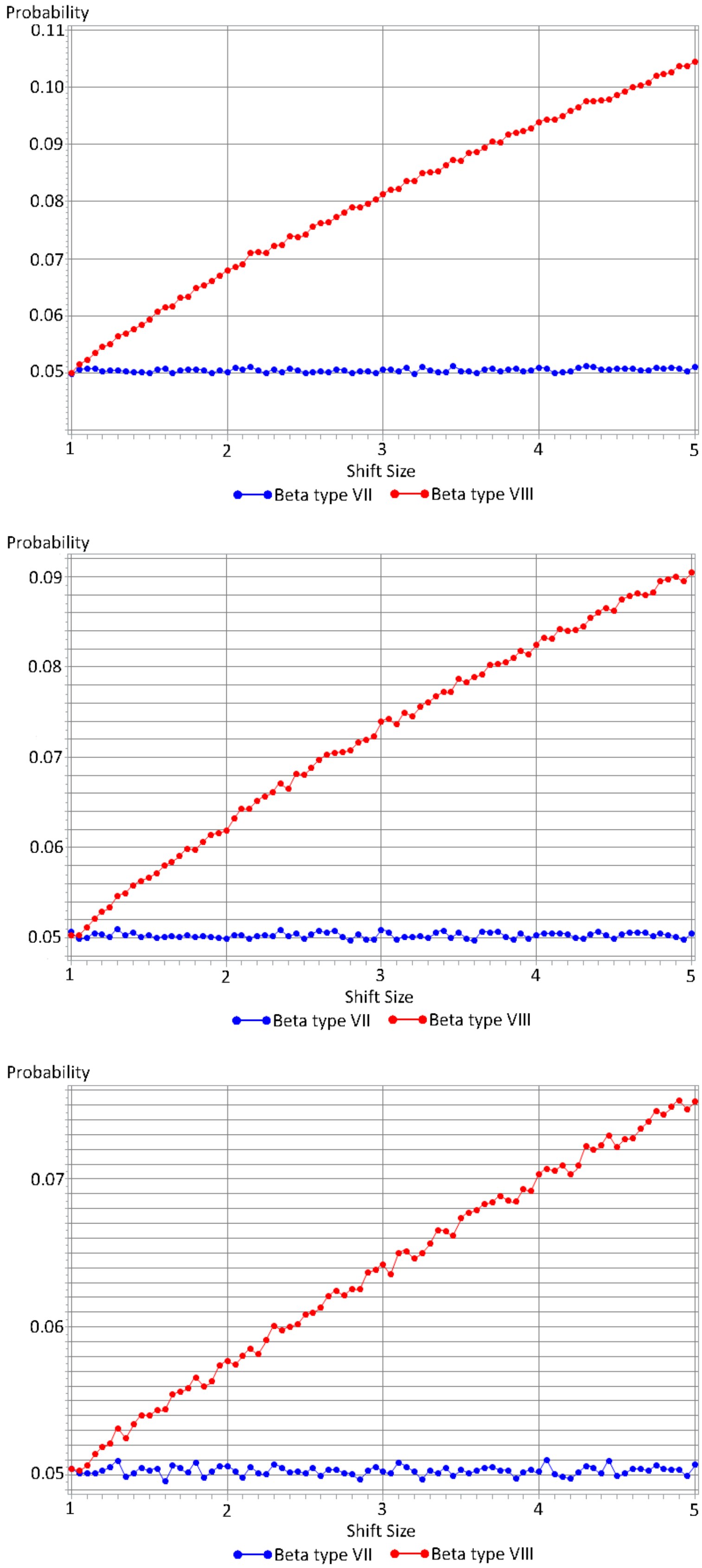

Figure 5.16: Probability of detecting a shift, $m = 29, n = 2, \kappa = 8, 15, 22$.

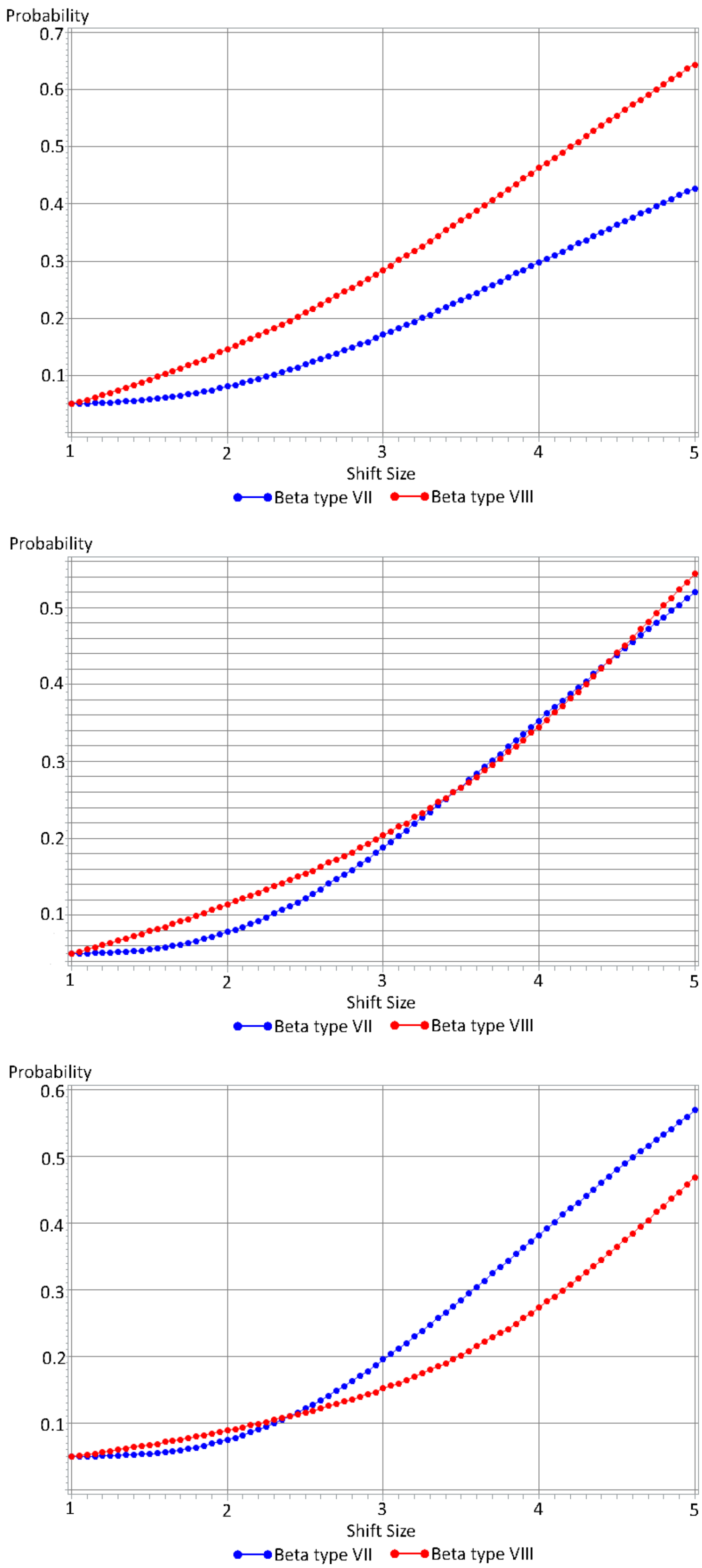

Figure 5.17: Probability of detecting a shift, $m = 29, n = 5, \kappa = 8, 15, 22$.

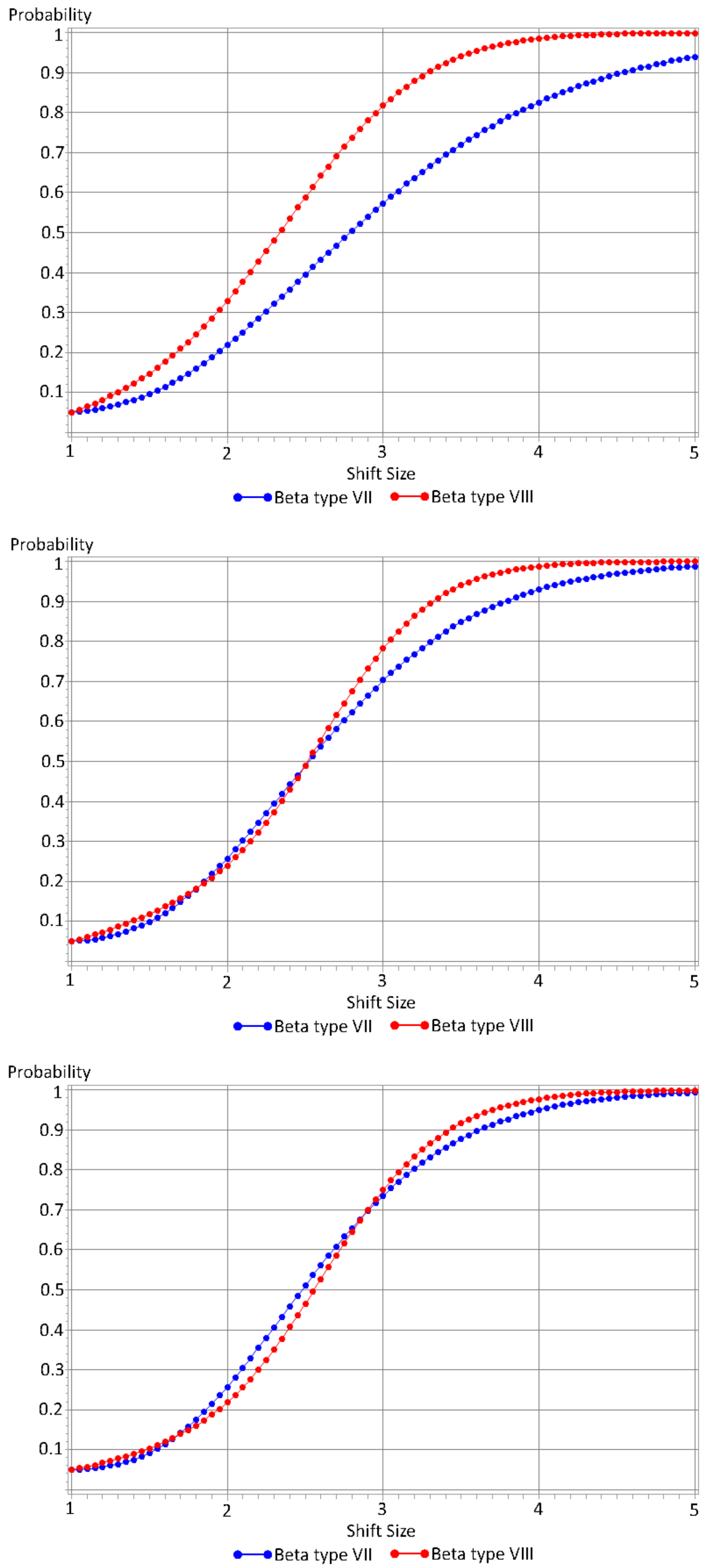

Figure 5.18: Probability of detecting a shift, $m = 29, n = 10, \kappa = 8, 15, 22$.

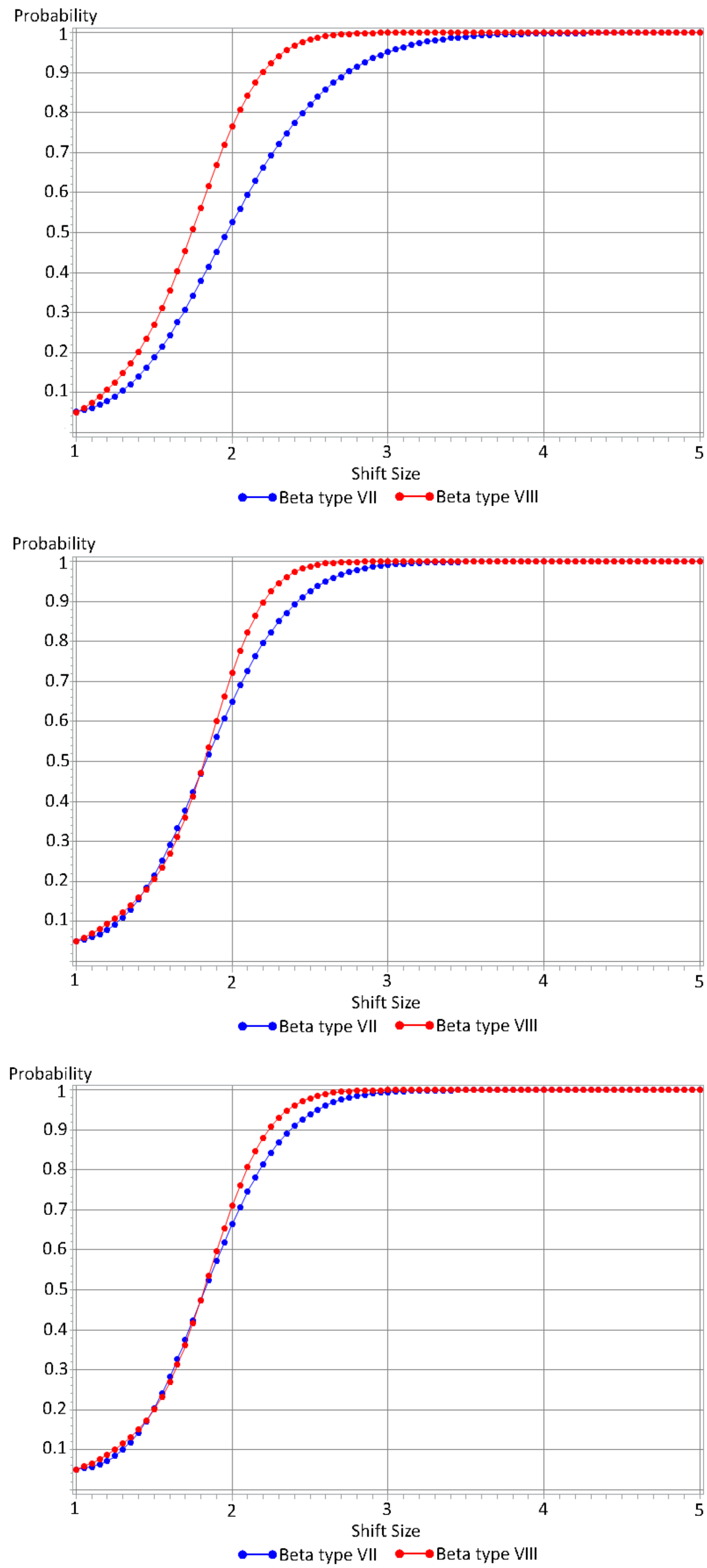

Figure 5.19: Probability of detecting a shift, $m = 29, n = 20, \kappa = 8, 15, 22$.

From figures 5.8 to 5.19, certain conclusions can be reached about the properties and efficacy of the two competing models:

- When $n = 2$, irrespective of the number of samples, the newly proposed model outperforms the Q chart. There are some caveats however that should be noted:

  1. In the above graphs, each plotted point was simulated 1 000 000 times during the Monte Carlo process. For all the $n = 2$ graphs, the points vary erratically between each 0.05 increases in the shift size, and thus even for a large number of simulations the process cannot be described as "stable". This corresponds to and is partially the result of the instability of critical values that were simulated in tables 5.1 and 5.2. The erratic probability of the control charts to detect different sized shifts when $n = 2$ merely reiterates the fact that such small sample sizes are not practically advisable.
  2. The Q chart seems to be completely incapable of detecting an increase in the process variance when $n = 2$, with the probability of detecting a shift staying at roughly 5%, irrespective of the size of the shift.
  3. While the new model's probability of detecting a shift does increase as the size of the shift increases, it remains relatively low, at roughly 7% to 10%, just marginally higher than the 5% chance when the process is actually IC. This implies that while it might be theoretically possible to implement the new model for samples sizes of 2, it would likely not be a practically useful technique.
  4. The new model's probability of detecting a shift does not increase as the number of samples increases, as would be expected (and as is the case for the other choices of n)

- From these points above, it can be concluded that using a sample size of 2 does not lead to an effective control chart.

- For smaller numbers of samples ($m = 9$), the newly proposed model outperforms the Q chart for all simulated sample sizes as well as locations of shifts (for all shift sizes).

- When there are 20 samples ($m = 19$), the newly proposed model outperforms the Q chart in nearly all situations. The Q chart does have a higher probability of detecting a shift in the process variance only when the sample sizes are small ($n = 5$), and the shift occurs relatively late in the process ($\kappa = 15$), for shifts in the process variance between $\lambda = 3$ and $\lambda = 4.75$. Since a 300% to 475% increase in the process variance is unlikely to occur in practice, the newly proposed model would likely be more effective for $m = 19$.

- For $m = 30$, sweeping statements about the performances of the two methods are more difficult to make since the plotted percentage lines cross often. However it can be said that:

  1. For small sample sizes ($n = 5$), the proposed model outperforms the Q chart for small shifts in the process variance, whereas the Q chart performs better for larger shifts.
  2. The Q chart performs at its best when the shift in the process variance occurs late in the series of samples.

3. For larger sample sizes ($n = 20$) the proposed model outperforms the Q chart when the shift in the process variance occurs early, but when the shift occurs roughly half way through the series of samples, or further, the performance of the two methods are very similar.

Note The SAS code that is used to calculate the probabilities that the two control charts will detect a shift in the process variance can be found in Result 21.

Conclusion

In this mini dissertation a new control chart was proposed to aid in the detection of a shift in a process's variance. As a consequence of developing this control chart, new bivariate and multivariate beta distributions were added to the literature, and some properties of the new bivariate beta distribution were derived and investigated. The generalised beta distribution developed by Adamski [1] was also expanded.

The control chart investigated by Adamski [1] was compared to the newly proposed model through a simulation study, and it was found that under the described practical situation the control chart proposed by this study performs favourably in comparison to the Q chart.

Opportunities for further research

- A closed-form expression for the critical values simulated in Chapter 5 still needs to be derived.
- Derivation of the properties of the multivariate beta distribution has yet to be investigated.
- Examination of the proposed model when the process mean experiences a shift has not yet been researched.
- Application of the newly derived beta distributions to practical situations still needs to be investigated.
- An investigation into relaxing some of the underlying assumptions of the proposed model has not yet been researched.

Appendix

Result 1 (Bain and Engelhardt [5] pp268-269)

If $X \sim \chi^2(\alpha)$, then X is said to be chi-square distributed with $\alpha \in \mathbb{N}$ degrees of freedom. It has the following density function

$$f(x) = \frac{1}{2^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)} x^{\frac{\alpha}{2}-1} e^{-\frac{x}{2}} \quad , x > 0,$$

where $\Gamma(.)$ is the gamma function as defined in Result 6.

Result 2 (Bain and Engelhardt [5] p111)

If $X \sim Gamma(\alpha > 0, \beta > 0)$, then X is said to be gamma distributed with degrees of freedom α and β respectively. It has the following density function

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \left(x^{\alpha-1} e^{-\frac{x}{\beta}}\right) \quad , x > 0,$$

where $\Gamma(.)$ is the gamma function as defined in Result 6.

Result 3 (Bain and Engelhardt [5] pp275-276)

If $X \sim F(\alpha > 0, \beta > 0)$, then X is said to be F distributed with degrees of freedom α and β respectively. It has the following density function

$$f(x) = \frac{\Gamma\left(\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)} \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{2}} x^{\frac{\alpha}{2}-1} \left(1 + \frac{\alpha}{\beta}x\right)^{-\frac{\alpha+\beta}{2}} \quad , x > 0,$$

where $\Gamma(.)$ is the beta function as defined in Result 6.

Result 4

If $X \sim Beta_{II}(\alpha > 0, \beta > 0)$, then X is said to be Beta type II distributed with degrees of freedom α and β respectively. It has the following density function

$$f(x) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)} \quad , x > 0,$$

where $B(.)$ is the beta function as defined in Result 7.

Result 5

Suppose that X_1 and X_2 are two random variables, with joint density function $f(x_1, x_2)$, then the product moment of X_1X_2 is defined as

$$E\left(X_1^r X_2^s\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^r x_2^s f\left(x_1, x_2\right) dx_2 dx_1 \ .$$

Result 6 (Gradshteyn and Ryzhik [19] p892, 8.310.1)

The gamma function, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Result 7 (Gradshteyn and Ryzhik [19] p902, 8.380.1)

The beta function, $B(x > 0, y > 0)$, is defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ is the gamma function as defined in Result 6.

Result 8 (Gradshteyn and Ryzhik [19] p1005, 9.101 and 9.111, p1010 9.14)

The Gauss hypergeometric function, ${}_2F_1(a, b; c; z)$, is defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (b)_n}{(c)_n n!} z^n = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt.$$

Note that the Gauss hypergeometric function is undefined/infinite if c is a negative integer. Also, if either a or b are equal to a non-positive number, say $-m$, the series terminates and becomes

$${}_2F_1(-m, b; c; z) = \sum_{n=0}^{m} (-1)^n \begin{pmatrix} m \\ n \end{pmatrix} \frac{(b)_n}{(c)_n} z^n \ ,$$

where $(\alpha)_n$ is the Pochhammer symbol, defined in Result 10 and $B(.)$ is the beta function as defined in Result 7.

Result 9 (Gradshteyn and Ryzhik [19] p1010, 9.14.1 and p25 1.110)

The hypergeometric function, ${}_1F_0(a; z)$, can alternatively be expressed as coming from the binomial theorem:

$${}_1F_0(a; z) = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} = (1-z)^{-a},$$

where $(\alpha)_n$ is the Pochhammer symbol, defined in Result 10.

Result 10 (Gradshteyn and Ryzhik [19] pxliii)

The Pochhammer symbol, $(q)_n$, is defined as:
$(q)_n = \left\{ \begin{array}{cc} 1 & n=0 \\ q(q+1)(q+2)\dots(q+n-1) & n>0 \end{array} \right\} = \frac{\Gamma(q+n)}{\Gamma(q)}$,
where $\Gamma(.)$ is the gamma function as defined in Result 6.

Result 11 (Gradshteyn and Ryzhik [19] p346, 3.381.4)

$\int_0^\infty x^{\alpha-1} e^{-\lambda} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}, \alpha > 0,$
where$\Gamma(.)$ is the gamma function as defined in Result 6.

Result 12 (Gradshteyn and Ryzhik [19] p317, 3.197.5)

$\int_0^\infty x^{\lambda-1}(1+x)^\nu (1+\alpha x)^\mu dx = B(\lambda, -\mu-\nu-\lambda)\, {}_2F_1(-\mu, \lambda; -\mu-\nu; 1-\alpha), -\mu-\nu > \lambda > 0,$
where $B(.)$ is the beta function as defined in Result 7 and ${}_2F_1(.)$ is the Gauss hypergeometric function as defined in Result 8.

Result 13 (Gradshteyn and Ryzhik [19] p315, 3.194.3)

$\int_0^\infty \frac{x^{\mu-1}}{(1+\beta x)^\nu} dx = \beta^{-\mu} B(\mu, \nu-\mu), \nu > \mu > 0,$
where $B(.)$ is the beta function as defined in Result 7.

Result 14 (Gradshteyn and Ryzhik [19] p317, 3.197.8)

$\int_0^u x^{\nu-1}(x+\alpha)^\lambda (u-x)^{\mu-1} dx = \alpha^\lambda u^{\mu+\nu-1} B(\mu,\nu)\, {}_2F_1\left(-\lambda, \nu; \mu+\nu; -\frac{u}{\alpha}\right) \quad , \mu, \nu > 0 \;,$
where $B(.)$ is the beta function as defined in Result 7 and ${}_2F_1(.)$ is the Gauss hypergeometric function as defined in Result 8.

Result 15 (Gradshteyn and Ryzhik [19] p317, 3.197.2)

$$\int_u^\infty x^{-\lambda}(x-u)^{\mu-1}(x+\beta)^\nu dx \;\; = u^{-\lambda}(\beta+u)^{\mu+\nu} B(\lambda-\mu-\nu, \mu)\, {}_2F_1\left(\lambda, \mu; \lambda-\mu; -\frac{\beta}{u}\right), \qquad \begin{array}{c} \left|\frac{\beta}{u}\right| < 1 \\ 0 < \mu < \lambda-\nu, \end{array}$$

where $B(.)$ is the beta function as defined in Result 7 and ${}_2F_1(.)$ is the Gauss hypergeometric function as defined in Result 8.

Result 16 (Gradshteyn and Ryzhik [19] p315, 3.191.2)

$\int_u^\infty x^{-\nu}(x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu-\mu, \mu), \nu > \mu > 0,$
where $B(.)$ is the beta function as defined in Result 7.

Result 17 (Gradshteyn and Ryzhik[19] p315, 3.194.1)

$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^\nu} dx \;\; = \frac{u^\mu}{\mu} {}_2F_1(\nu, \mu; 1+\mu; -\beta u), \quad \mu > 0 \;,$
where ${}_2F_1(.)$ is the Gauss hypergeometric function as defined in Result 8.

Result 18 (Result 18 (Gradshteyn and Ryzhik[19] p25, 1.110 and 1.111)

In general, $(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k$.

However, if $n \in \mathbb{N}$,

- $(a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
- $(a+b)^{-n} \;=\; \sum_{k=0}^{\infty} \binom{-n}{k} a^{-n-k} b^k \;= \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} a^{-n-k} b^k$.

$\binom{n}{k}$ is called the binomial coefficient, and is defined in Result 19.

Result 19 (Bain and Engelhardt [5] p35)

Suppose that n and k are both integers such that $k \leq n$, then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Result 20 (SAS code - IC process simulation)

```
/*IN CONTROL DISTRIBUTION*/
proc iml;
sim = 500000;

** Paramters: dof=sample size, alpha=gamma shape parameter,
beta=gamma scale parameter **;
dof = 5;
alpha = (dof-1)/2;
beta = 2;

** m=Number of samples, kappa=position of shift, lambda=shift size **;
m = 10; kappa  = 1; lambda = 1;

** Initialize the vectors **;
Sample_Vector = j(1,m   ,.);

** Beta type VIII **;
Statistic_VectorA = j(1,m-1,.);
Max_Val_VectorA    = j(1,sim,.);
Max_Loc_VectorA    = j(1,sim,.);
idA                = j(1,sim,1);

** Beta type VII **;
Statistic_VectorK = j(1,m-1,.);
```

```
Max_Val_VectorK     = j(1,sim,.);
Max_Loc_VectorK     = j(1,sim,.);
idK                 = j(1,sim,2);

do Iterations = 1 to sim;
        Seed_Before = j(1,kappa,0);
        Seed_After  = j(1,m-kappa,0);
        X_Before = beta*rangam(Seed_Before,alpha);
        X_After  = lambda*beta*rangam(Seed_After,alpha);

        Sample_Vector = X_Before||X_After;
                do i = 1 to m - 1;
                NumA = Sample_Vector[1,i+1:m];
            DenA = Sample_Vector[1,1:i];
            NumA_DOF = (m-i)*(dof-1);
            DenA_DOF = (i)*(dof-1);
            Statistic_VectorA[i] =
                        NumA[+]/NumA_DOF)/(DenA[+]/DenA_DOF);

                        NumK = Sample_Vector[1,i+1:i+1];
            DenK = Sample_Vector[1,1:i];
                        NumK_DOF = (dof-1);
            DenK_DOF = (i)*(dof-1);
                        Statistic_VectorK[i] = (
                        NumK[+]/NumK_DOF)/(DenK[+]/DenK_DOF);
                end;

        Max_Val_VectorA[Iterations] = Statistic_VectorA[<>];
        Max_Loc_VectorA[Iterations] = Statistic_VectorA[<:>];
        Max_Val_VectorK[Iterations] = Statistic_VectorK[<>];
        Max_Loc_VectorK[Iterations] = Statistic_VectorK[<:>];
end;

dataset = (idA`||Max_Val_VectorA`||Max_Loc_VectorA`)
//(idK`||Max_Val_VectorK`||Max_Loc_VectorK`);

create printable from dataset[colname={'id' 'Max_Values' 'Location_of_Maximum'}];
append from dataset;
quit;

data IC;
set work.printable;
drop id;
if id = 1 then id1 = 'Beta type VIII';
if id = 2 then id1 = 'Beta type VII';
```

```
run;
quit;

proc univariate data = IC noprint;
class id1;
var Max_Values Location_of_Maximum;
output out=percentiles1 pctlpts = 95   pctlpre  = Strength Width
run;
quit;

title 'Maximum Values: Beta type VIII vs. Beta type VII';
ods graphics off;
proc univariate data = IC noprint;
        class id1;
        histogram Max_Values / intertile  = 1.0
        vaxis = 0 10 20 30
        ncols = 1
        nrows = 2;
run;
quit;

title 'Location of Maximums: Beta type VIII vs. Beta type VII';
ods graphics off;
proc univariate data = IC noprint;
        class id1;
        histogram Location_of_Maximum / intertile  = 1.0
        vaxis = 0 5 10 15 20 25 30
        ncols = 1
        midpoints  = 1 2 3 4 5 6 7 8 9 10
        nrows = 2;
run;
quit;
```

Result 21 (SAS code - OOC process simulation)

```
libname albert 'D:';
data albert.power; id1 = 'xxxxxxxxxxxxxxxxxx';
Probability = 0;
Shift = 0;
run;
quit;


%macro power(lambda);
```

```
proc iml;

sim = 500000;

** Paramters **;
dof = 20;
alpha = (dof-1)/2;
beta = 2;

** Number of samples, position of shift and shift size **;
m = 30;
kappa = 22;

** Initialize the vectors **;
Sample_Vector = j(1,m   ,.);

** Beta type VIII **;
Statistic_VectorA = j(1,m-1,.);
Max_Val_VectorA   = j(1,sim,.);
Max_Loc_VectorA   = j(1,sim,.);
idA               = j(1,sim,1);

** Beta type VII **;
Statistic_VectorK = j(1,m-1,.);
Max_Val_VectorK   = j(1,sim,.);
Max_Loc_VectorK   = j(1,sim,.);
idK               = j(1,sim,2);


do Iterations = 1 to sim;
        Seed_Before = j(1,kappa-1,0);
    Seed_After = j(1,m - kappa + 1,0);
    X_Before = beta*rangam(Seed_Before,alpha);
        X_After = &lambda*beta*rangam(Seed_After,alpha);
    Sample_Vector = X_Before||X_After;


    do i = 1 to m - 1;
        NumA = Sample_Vector[1,i+1:m];
      DenA = Sample_Vector[1,   1:i];
      NumA_DOF = (m-i)*(dof-1);
      DenA_DOF = (i)*(dof-1);
      Statistic_VectorA[i] = (NumA[+]/NumA_DOF)/(DenA[+]/DenA_DOF);

          NumK = Sample_Vector[1,i+1:i+1];
```

```
        DenK = Sample_Vector[1,1:i];
            NumK_DOF = (dof-1);
        DenK_DOF = (i)*(dof-1);
            Statistic_VectorK[i] = (NumK[+]/NumK_DOF)/(DenK[+]/DenK_DOF);


    end;


        Max_Val_VectorA[Iterations] = Statistic_VectorA[<>];
    Max_Loc_VectorA[Iterations] = Statistic_VectorA[<:>];
        Max_Val_VectorK[Iterations] = Statistic_VectorK[<>];
    Max_Loc_VectorK[Iterations] = Statistic_VectorK[<:>];
end;

dataset = (idA`||Max_Val_VectorA`||Max_Loc_VectorA`)
//(idK`||Max_Val_VectorK`||Max_Loc_VectorK`);


create printable from dataset[colname={'id' 'Max_Values' 'Max_Loc'}];
append from dataset;


data albert.OOC;
set printable;
drop id;
if id = 1 then id1 = 'Beta Type VIII';
if id = 2 then id1 = 'Beta Type VII';
run;
quit;

data albert.OOCb;
set albert.OOC;

/*Note that the values (1.97... and 2.54...) are the 95th percentiles of the
respective maximum order statistics' distributions.
They are simulated is Result 20, using proc univariate*/

if id1 = 'Beta Type VIII' then do;
if Max_Values >= 1.97437942  then Signal = 1;
else Signal = 0;
end;

if id1 = 'Beta Type VII' then do;
if Max_Values >= 2.542581604  then Signal = 1;
```

```
else Signal = 0;
end;
run;
quit;

proc sort data=albert.OOCb;
by id1;
run;

proc means data = albert.OOCb noprint;
by id1;
var Signal;
output out=work.power mean=Probability;
run;

data work.power;
set work.power;
keep id1 Probability Shift;
shift = &lambda;
run;
quit;


data albert.power;
set albert.power
work.power;
run;
quit;

%mend;

%power(1.00);%power(1.05);%power(1.10);%power(1.15);%power(1.20);
%power(1.25);%power(1.30);%power(1.35);%power(1.40);%power(1.45);
%power(1.50);%power(1.55);%power(1.60);%power(1.65);%power(1.70);
%power(1.75);%power(1.80);%power(1.85);%power(1.90);%power(1.95);
%power(2.00);%power(2.05);%power(2.10);%power(2.15);%power(2.20);
%power(2.25);%power(2.30);%power(2.35);%power(2.40);%power(2.45);
%power(2.50);%power(2.55);%power(2.60);%power(2.65);%power(2.70);
%power(2.75);%power(2.80);%power(2.85);%power(2.90);%power(2.95);
%power(3.00);%power(3.05);%power(3.10);%power(3.15);%power(3.20);
%power(3.25);%power(3.30);%power(3.35);%power(3.40);%power(3.45);
%power(3.50);%power(3.55);%power(3.60);%power(3.65);%power(3.70);
%power(3.75);%power(3.80);%power(3.85);%power(3.90);%power(3.95);
%power(4.00);%power(4.05);%power(4.10);%power(4.15);%power(4.20);
%power(4.25);%power(4.30);%power(4.35);%power(4.40);%power(4.45);
```

```
%power(4.50);%power(4.55);%power(4.60);%power(4.65);%power(4.70);
%power(4.75);%power(4.80);%power(4.85);%power(4.90);%power(4.95);
%power(5.00);

data albert.power1;
set albert.power;
if shift = 0 then delete;
run;
quit;

proc sort data = albert.power1;
by id1 Shift;
run;
quit;

ods output;
symbol1 interpol=spline color=blue value=dot height=1;
symbol2 interpol=spline color=red  value=dot height=1;
proc gplot data = albert.power1;
title 'Probability to Signal vs. Shift Size';
plot Probability*Shift = id1/ overlay grid
vref = 0 cvref=grey lvref=1
href = 1 chref=grey lhref=1;
run;
quit;
```

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