Product of independent generalised gamma random variables

by

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Declaration

I, Vusi Raphael Bilankulu, declare that this mini-dissertation, which I hereby submit for the degree Magister Scientiae in Mathematical Statistics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE:..

DATE:..

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Summary

The generalised gamma distribution has received much attention due to its flexibility and also for having some well-known distributions as special cases. This study originates from a statistic defined as the ratio of products of independent generalised gamma random variables and shows that it can be represented as the product of independent generalised gamma random variables with some re-parametrisation. By decomposing the characteristic function of the negative logarithm of the statistic and then using the distribution of the difference of two independent generalized integer gamma random variables as a basis, accurate and computationally appealing near-exact distributions are derived for the statistic. In the process, a new flexible parameter is introduced in the near-exact distributions which allows to control the degree of precision of these approximations. Furthermore, the performance of the near-exact distributions is assessed using a measure of proximity between cumulative distribution functions; also, by comparison with the exact distribution, empirical distribution and with an approximation developed using a different method and which can only be applied in some particular cases.

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Chapter 0

Abbreviations and notation

cdf	Cumulative distribution function
pdf	Probability density function
$\stackrel{d}{\simeq}$	Approximately equal in distribution
GIG	Generalised integer gamma distribution
$DGIG$	Distrubution of difference of two independent GIG random variables
$SDGIG$	Shifted $DGIG$ distribution
$\approx$	Approximately equal to
$\mathbb{N}$	A set of natural numbers
$\mathbb{R}$	A set of real numbers
$\mathbb{R}^+$	A set of positive real numbers
$\in$	An element of a given set of numbers
$\mathbb{C}$	A set of complex numbers

$\exp(\cdot)$	Exponential function, $e^{(\cdot)}$
$\Gamma(a)$	Gamma function
$\Gamma^n(\cdot)$	$\prod_{j=1}^{n} \Gamma(\cdot)$
$\gamma(\alpha, x)$ and $\Gamma^*(\alpha, x)$	Incomplete gamma functions
$\phi_X(t)$	Characteristic function of the random variable X, $\phi_X(t) = E[\exp(itX)]$
$\binom{x}{n}$	Combination function
$(\alpha)_t$	Pochhammer coefficient
${}_1F_1(\cdot)$	Confluent hypergeometric function
$\mathbf{G}^{m,n}_{r,s}(\cdot)$	Meijer's G-function
$\mathbf{H}^{m,n}_{r,s}(\cdot)$	Fox's H-function
$\mathcal{M}_f(\cdot)$	Mellin transform

Chapter 1

Introduction

1.1 Background and motivation

The distribution of the product or ratio of independent random variables have played an important role in many areas of research. Of particular interest in this study is the distribution of the product of independent generalised gamma distributed random variables. Subsection 1.1.1 provides an overview of substantial contributions to the current theoretical and application understanding of the distribution of product and ratio of generalised gamma random variables. In Subsection 1.1.2, the probability density function (pdf) of the generalised gamma distributed random variable considered in this study is given. Furthermore, it is shown that the inverse of the generalised gamma distributed random variable also follows a generalised gamma distribution. Subsection 1.1.2 ends off by formally defining the statistic of interest for this study.

1.1.1 Literature review

The generalised gamma distribution was introduced by Stacy (see [24]). It is a generalisation of well-known distributions such as gamma, chi-squared, exponential, Rayleigh, Weibull and Nakagami-m. Either in this generalised form or one of its special cases, the generalised gamma distribution has received much interest and wide applications in areas such as hydrological processes, wireless communication, reliability analysis, economics and life testing. This is largely due to its flexibility. In a hydrological application, [1] used the generalised gamma distribution to characterise the duration of a drought, its intensity and successive non-drought duration respectively. In wireless systems, [4] described respectively the fading coefficient of a hop and a channel gain of a hop by using a generalised gamma random variable. A similar application of the generalised gamma distribution can be found in [20] where the performance of multi-hop wireless communication systems in different transmission environments is analysed. [26] unknowingly reintroduced the generalised gamma distribution as a general fading distribution, the so-called $\alpha - \mu$ distribution. In [17], the effects and measurement errors are analysed using the Poisson-gamma hierachical generalised linear model. The authors then use the generalised gamma distribution to model exponents of each these effects.

In many of the applications mentioned above, the product or ratio of independent generalised gamma random variables appears naturally. For instance, [1] considered both the product and ratio of independent generalised gamma random variables to model the magnitude of a drought and relative duration of a drought events respectively. In multi-hop wireless relaying systems, the end-to-end signal-to-noise ratio (SNR) and the rate offset can be modelled as a function of the product of independent generalised gamma, Rayleigh or Nakagami-m random variables (see [4]). Signal-to-interference ratio (SIR) can be modelled as the ratio of either independent generalised gamma, independent Rayleigh or independent Nakagami-m random variables (see [20]). In [17] the authors modelled the intensity of the Poison process in the Poisson-gamma hierarchical generalized linear model as product of independent gamma random variables. In [16], the authors show many applications of the linear combination of independent Gumbel distributed random variables in biology and risk management. Using a rather simple transformation, the distribution of a linear combination of independent Gumbel distributed random variables can be obtained from a product or ratio of independent generalised gamma distributed random variables.

The product and ratio of independent generalised gamma distributed random variables also appear fundamental in the basic statistical theory. Well known distributions such as the beta type I and the Snedecor's F are particular cases of the ratio of independent generalised gamma distributed random variables. The generalised variance is the product of particular independent chi-squared random variables (see [7]). A large number of hypothesis test statistics are distributed as the product or ratio of independent generalised gamma random variables. Examples include equality of two generalised variances, details and some other examples can be found in [8]. In [7], the authors detail a number of applications in statistical theory of the product of independent generalised gamma random variables.

In addition to numerous studies done on independent generalised gamma distributed random variables, some authors (see for example [2] and [3]) have studied the product and sum of correlated generalised gamma distributed random variables.

1.1.2 Statistic of interest

As shown in Subsection 1.1.1, the ratio and product of independent generalised gamma distributed random variables have been widely used to model problems arising in many areas. It is for this reason that a deeper knowledge of the distribution of the ratio and product of independent generalised gamma distributed random variables is necessary.

Let X be a random variable with the pdf given by

$$f_X(x; r, \lambda, \delta) = |\delta| \frac{\lambda^{\delta r} x^{\delta r - 1}}{\Gamma(r)} \exp\left(-(\lambda x)^{\delta}\right) \tag{1.1}$$

for $x \geq 0$, $r > 0$, $\lambda > 0$ and any non-zero quantity δ. r, λ and δ are called shape, rate and power parameters respectively. X is said to follow a generalised gamma distribution denoted by $X \sim G\Gamma(r, \lambda, \delta)$ ([25], p.73).

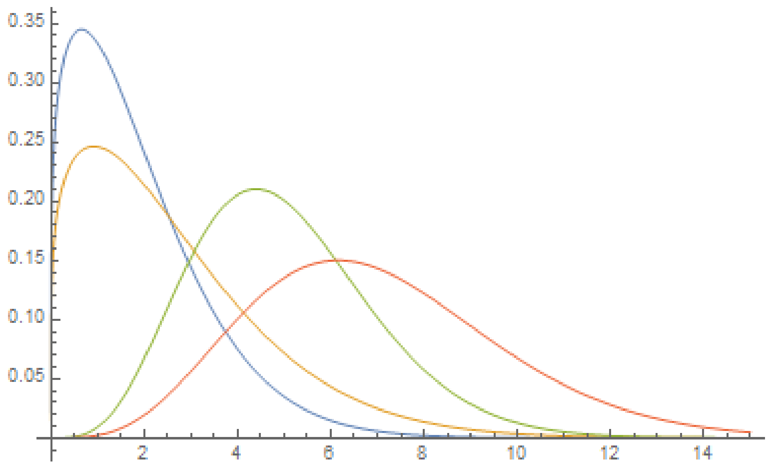

Figure 1.1: Plots of the pdf of generalised gamma distribution (see (1.1)) for various sets of parameter values.

The flexibility of the generalised gamma distribution, as noted in Subsection 1.1.1, can be observed in Figure 1.1. It should be noted that the pdf in (1.1) is an alternative representation of the generalised gamma distribution than that introduced by [24] where the pdf is given as

$$f_X(x; a, d, p) = \frac{p}{a^d \Gamma(d/p)} x^{d-1} \exp\left(-\frac{x}{a}\right)^p$$

for $x > 0$, $a > 0$, $d > 0$ and restricted $p > 0$. Thus the following re-parametrisation is implied

$$\begin{aligned} d &= \delta r \\ a &= \frac{1}{\lambda} \\ p &= \delta . \end{aligned}$$

The generalised gamma distribution has, as special cases, some of the well-known distributions.

Table 1.1: Some of the special cases of the generalised gamma distribution (see (1.1)).

Distribution	r	λ	δ
gamma	r	λ	1
exponential	1	λ	1
Rayleigh	1	$\frac{\lambda}{2}$	2
Nakagami-m	r	$\sqrt{\frac{r}{\Omega}}$	1

Table 1.1 shows some distributions as well as their relationship with the generalised gamma distribution. Further information on these distributions can be found in Appendix B. and [25]. Included in [7] is another detailed list of distributions having particular relationships with the generalised gamma distribution.

The next remark considers the distribution of the inverse of the generalised gamma distributed random variable with pdf given by (1.1).

Remark 1.1 *Let $X \sim G\Gamma(r, \lambda, \delta)$. Define*

$$V = \frac{1}{X}.$$

Then the pdf of V is given by

$$\begin{aligned} f_V(v) &= f_X\left(v^{-1}\right)\left|-v^{-2}\right| \\ &= |\delta| \frac{\lambda^{\delta r}\left(v^{-1}\right)^{\delta r-1}}{\Gamma(r)} \exp\left(-\left(\lambda v^{-1}\right)^{\delta}\right) v^{-2} \\ &= |-\delta| \frac{\left(\lambda^{-1}\right)^{(-\delta)r} v^{(-\delta)r-1}}{\Gamma(r)} \exp\left(-\left(\lambda^{-1} v\right)^{-\delta}\right). \end{aligned} \quad (1.2)$$

It follows from (1.2) that $V \sim G\Gamma\left(r, \lambda^{-1}, -\delta\right)$ i.e. the inverse of a random variable with a generalised gamma distribution is also a generalised gamma distributed random variable.

The following notation will be useful for the remainder of this study. For a power parameter, δ, in a generalised gamma distribution, denote δ by δ^- if $\delta < 0$ and δ^+ if $\delta > 0$. Suppose that

$$\begin{aligned} X_{1j} &\sim G\Gamma(r_{1j}, \lambda_{1j}, \delta_{1j}^+) \qquad j = 1, 2, \ldots, n_1 \\ X_{2t} &\sim G\Gamma(r_{2t}, \lambda_{2t}, \delta_{2t}^+) \qquad t = 1, 2, \ldots, n_2, \end{aligned}$$

where $n_1 + n_2 = n$, $n_1 > 0$ and $n_2 > 0$. Consider the following statistic

$$G = \left(\prod_{j=1}^{n_1} X_{1j}\right)\left(\prod_{t=1}^{n_2} X_{2t}\right)^{-1}. \tag{1.3}$$

In (1.3), G is the ratio of the product of independent generalised gamma distributed random variables. Let

$$G_1 = \prod_{j=1}^{n_1} X_{1j}$$

and

$$\begin{aligned} G_2 &= \left(\prod_{t=1}^{n_2} X_{2t}\right)^{-1} \\ &= \prod_{t=1}^{n_2} X_{2t}^{-1}. \end{aligned}$$

Clearly G_1 is a product of independent generalised gamma distributed random variables. Using Remark 1.1, each of $X_{2t}^{-1} \sim G\Gamma\left(r, \lambda_{2t}^{-1}, \delta_{2t}^-\right)$ where $\delta_{2t}^- = -\delta_{2t}^+$. Therefore G_2 is simply a product of independent generalised gamma random variables itself. Thus another representation of statistic (1.3) is

$$G = G_1 G_2. \tag{1.4}$$

This implies that G is a product of independent generalised gamma distributed random variables such that in at least two of these random variables, one has a positive power parameter and the other has negative power parameter. In effect, one can view the ratio of independent generalised gamma distributed random variables in (1.3) also as the product of independent generalised gamma distributed random variables (1.4).

Next, the statistic of interest for this study is defined. Let $X_1, X_2, \ldots, X_n$ where $X_j \sim G\Gamma\left(r_j, \lambda_j, \delta_j\right)$ such that $\delta_t < 0$ and $\delta_k > 0$ for some $t, k \in \{1, 2, 3, \ldots, n\}$. Define

$$Y = \prod_{j=1}^{n} X_j. \tag{1.5}$$

[15] and [17] considered statistics similar to (1.5). In fact, due to the closeness of the titles and methodologies in these articles and this study, a reader may incorrectly conclude that the content of this study have been considered in the aforementioned articles. [15] and [17] only considered a case where all power parameters are either δ^- or δ^+. Therefore, this study can be view as a generalisation of studies in [15] and [17] since the power parameters are not restricted to be of the same sign (either negative or positive). In fact, this is the first study in literature that the statistic as defined in (1.5) is studied and its distribution evaluated. Noting that a generalised gamma distribution is a special case of the H-function distribution (see (A.35)), [22] studied a more general case of Y where Y is a product of independent H-function distributed random variables. However, the result is of no practical application since Fox's H-function are not computable. In [7], the statistic Y is only noted but never studied. The authors then limit their attention to cases where the power parameters are all either positive or negative.

1.2 Methodology

In the introduction of the generalised gamma distribution, [24] defined and considered the distribution of a ratio of two independent generalised gamma random variables with equal power parameters. However, [24] only went as far as expressing the distribution of Y (see 1.5) for $n = 2$ in terms of a *beta* distribution by making use of the relationship between the *beta* random variable and the ratio of two independent generalised gamma random variables. In particular, [24] noted that if X_1 and X_2 are independent generalised gamma random variables with same power parameters i.e. $X_j \sim G\Gamma(r_j, \lambda_j, \delta)$ for $j = 1, 2$ and

$$Y = \frac{X_1}{X_2},$$

then

$$W = \frac{Y^\delta}{Y^\delta + \left(\frac{\lambda_2}{\lambda_1}\right)^\delta} \sim beta\,(r_1, r_2)\,,$$

where $beta\,(r_1, r_2)$ denote the *beta* distribution with parameters r_1 and r_2. Then [24] expressed the cumulative distribution of Y as

$$P\,(Y \leqslant y) = P\left(W \leqslant \frac{y^\delta}{y^\delta + \left(\frac{\lambda_2}{\lambda_1}\right)^\delta}\right). \tag{1.6}$$

Subsequently, literature on the distribution of Y both exact (see [1], [8], [13], [14],

[18], [21] and [23]) and approximations (see [4], [8], [12] and [17]) has been published. With an exception of few authors, most of the theory on the distribution, or distribution function specifically, of Y is in terms of either special functions or infinite series. [4] also commented on this issue.

To obtain the exact distribution of Y some authors have used the Mellin transformation (see (A.37)) and inverse Mellin transformation (see (A.38)). Subsection 1.2.1 gives a brief comment of results from this methodology. Subsections 1.2.2 and 1.2.3 comment on methodologies used to obtain the approximate distribution of Y.

1.2.1 Exact distribution of the statistic in terms of Meijer's G-function and Fox's H-function

Authors in [18], [22] and [23] used the Mellin and inverse Mellin transformation to derive the exact distribution of Y (see (1.5)) for any value of n. To obtain the computational form of the exact distribution of Y, authors in [23] limited their study exclusively to the product of independent gamma distributed random variables with rate parameter equal to 1 and authors in [18] only considered a product of a random sample from a generalised gamma distribution. Under these special conditions the exact distribution of Y could be expressed in terms of Meijer's G- function (see (A.36)). [11] studied the ratio of generalised gamma distributed random variables. By forcing power parameters of various random various to have some numerical relationship, the authors obtained the pdf of Y in tems of the Meijer's G- function. According to [22], the pdf of Y can be expressed in terms of Fox's H-function (see (A.35)). However Fox's H-functions are not readily computable "even nowadays when good softwares for symbolic and numeric computations are available" [8]. [7] and [8] derived the exact distribution of Y is in terms of infinite series. Though distributions that are in terms of infinite series can be approximated to a high degree of accuracy, this would require a large number of terms in a series to be evaluated in order to get the required accuracy. Therefore, even approximate distributions of Y that are in terms of infinite series are both time and computationally demanding and costly. With reference to the infinite structure of the pdf and the cumulative distribution function (cdf) of Y in [7], the authors commented that the structure is somewhat complicated and a simpler structure is a worthwhile goal.

1.2.2 Approximate distribution of the statistic in terms of elementary functions

Approximate distributions in [4] and [12] are in manageable forms. However [4] and [12] considered only the case where Y (see (1.5)) is a product of independent generalised gamma random variables also with positive power parameters. In [12], Y is specifically a product of n randomly sampled Rayleigh distributed random variables. A new random variable, say Z, which is the n^{th} root of Y divided by the Rayleigh parameter is defined. Using 10^6 simulated values of Z and distributional fitting tool in MATLAB, they determine the distribution of Z and then transform back to Y to obtain the approximate distribution of Y. In [4], the authors noted a relationship between a Rayleigh distributed random variable and an exponentially distributed random variable i.e. if $X \sim Ra\,(1)$ (see (B.4)), then $X^2 \sim EXP\,(1)$ (see (B.1)). Using this relationship and a transformation to allow for exponential parameters other than 1, an approximation of the distribution of product of independent exponentially distributed random variable is obtained. An approximate distribution of the product of independent gamma distributed random variables and product of independent generalised gamma distributed random variables can be obtained respectively in a similar fashion. This approach will be investigated in Section 3.2 and will be referred to as Chen's approximation.

1.2.3 Distribution of the statistic using a characteristic function based method

In [15], [16] and[17], near-exact distributions are derived by using characteristic function based techniques. In this study, a similar approach is also followed to derive near-exact distributions of Y (see (1.5)). Since the characteristic function of the generalised gamma distributed random variable (and hence the characteristic function of Y) is not readily available, the characteristic function of $Z = -\log Y$ will be considered. By decomposing the characteristic function Z, it is shown that Z can be represented as a sum of two independent random variables. To develop near-exact distribution, one random variable in the representation on Z is approximated while the other is left unchanged. In [7] and [8], the characteristic function of Z is decomposed using Result 31 and the derived exact pdf of Z is in terms of infinite sums. Where the exact distribution is represented in terms of infinite series, an approximate distribution is obtained by truncating some of the terms. However, to obtain the required accuracy, a large number of terms need to be evaluated. In reference to this point, authors in [8] commented that "the development of near-exact distribution seems to be much a much desirable goal".

1.3 Aims and outline of this dissertation

1.3.1 Aims

- To develop near-exact distributions for statistic Y (see (1.5)). These near-exact distributions are novel approximations of the exact distribution of Y.
- Assess the quality of near-exact distributions relative to other approximate distribution.
- Recommend the most efficient approximate distribution given the problem being modelled.

1.3.2 Outline

- In Chapter 2, the exact distribution of Y (see (1.5)) in terms of Fox's H-function is presented and the characteristic function of $Z = -\log Y$ is derived. Furthermore, it will be shown that Z can be decomposed into a sum of two independent random variables.
- Chapter 3 uses the representation of Z as a sum of two independent random variables to develop near-exact distributions for Z. By applying a transformation from Z to Y (see(1.5)), near-exact distributions of Y are developed. The chapter end by deriving approximate distribution functions, presented in [4], of Y when the power parameter is positive and fixed.
- Chapter 4 assesses the quality of near-exact distributions relative to approximate distribution in terms of elementary functions, empirical distribution and exact distribution.
- Conclusion and future research opportunities are in Chapter 5.
- For the convenience of the reader, an Appendix with notation and abbreviation (in Appendix 0), useful mathematical results (in Appendix A.) and statistical distributions (in Appendix B.) is included.
- Proposed computational modules for near-exact distributions developed in this study are discussed in Appendix C. Link to Mathematica code for all computational modules implemented in this study is also included

Figure 1.2 summarises graphically the outline of this mini-dissertation

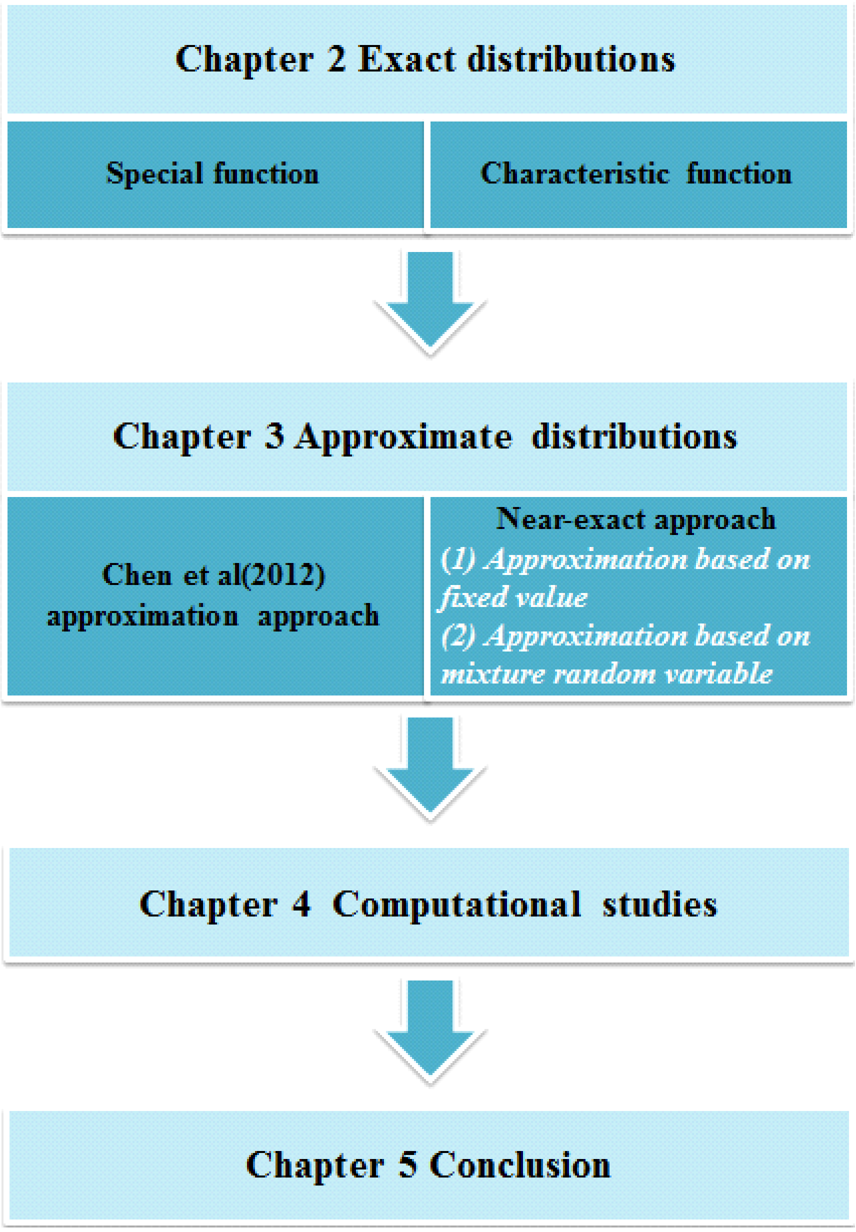

Figure 1.2: Outline of the study.

1.4 Contributions

- Present more detailed and alternative proofs (e.g Results 20 and 37) to some of the popular results in near-exact distribution theory (see Appendix B.).

- Show that the ratio of independent generalised gamma distributed random variables can be represented as the product of independent generalised gamma distributed random variables (see (1.5)).

- Derive and evaluate the near-exact distribution functions for Y as defined in (1.5).

- Contrast exact distribution and classical approximate distributions of Y (see (1.5)) against near-exact distributions.
- Assess the quality and usability of two approximate methods (i.e. near-exact distribution and approximate method presented in [4]) against each other.
- Recommend the most efficient approximate distribution to employ under various conditions given the problem being modelled.
- Develop computational modules for calculating the pdf and cdf of Y (see (1.5)).

Chapter 2

Exact distribution

A number of authors have studied the exact and approximate distributions of the product of independent generalised gamma distributed random variables. In all of such studies, power parameters were all either positive or negative (see [7], [15] and [17]). Fewer authors have consider the ratio of independent generalised gamma distributed random variables. With an exception of [18] and [22] where an arbitrary number of variables is considered, every other study limited their study to a ratio of only two random variables (see for example [1], [13] and [23]). Since a generalised gamma distribution is a special case of the H-function distribution, pdf of the product of independent H-function distributed random variables derived in [22] can be used to obtain the pdf of Y (see (1.5)) in terms of the Fox's H-function. [18] and [23] also derived the pdf of Y in terms of Fox's H-function. However, Fox's H-functions are not readily computable. To obtain the computable form of the pdf of Y, [23] considered only gamma distributed random variables with rate parameter equal to 1 while [18] considered the special case where all the generalised gamma distributed random variables have the same power parameter. In both special cases, Y has an exact pdf that is in terms of the Meijer's-G function which is easily computable using most mathematical softwares. [20] also derived the exact distribution of Y in terms of the Meijer's-G function.

[13] derived the exact distribution of the logarithm of Y (for $n = 2$) and used transformation techniques to obtain the exact distribution of Y. [8] and [17] adopted a newer methodology using characteristic functions to derive the exact distribution and near-exact distributions of $Z = -\log Y$. Via some necessary transformations, near-exact distributions of Y are obtainable from near-exact distribution of Z.

Section 2.1 presents the exact distribution of Y in terms of Fox's H-function. In addition, the pdf of special cases of Y (i.e. as a product of independent exponentially and gamma distributed random variables respectively) in terms of Meijer's-G functions is investigated. Starting from the characteristic function Z, Section 2.2 shows that Z can be decomposed into a sum of two independent random variables. This decomposition will be used in Chapter 3 to obtain near-exact distribution of Z and ultimately near-exact distributions of Y.

2.1 Meijer's-G and Fox's H-functions based method

This section follows the approach in [18] and derives the exact distribution of Y (see (1.5)). Using (A.39) and (B.13), the Mellin transformation of Y is given by

$$\begin{aligned}\mathcal{M}_Y(s) &= \prod_{j=1}^{n} \frac{\Gamma\left(r_j + \frac{(s-1)}{\delta_j}\right)}{\Gamma(r_j)} \lambda_j^{-(s-1)} \\ &= \prod_{j=1}^{n} \Gamma\left(r_j + \frac{(s-1)}{\delta_j}\right) \prod_{j=1}^{n} \lambda_j^{-s} \prod_{j=1}^{n} \frac{\lambda_j}{\Gamma(r_j)}.\end{aligned}$$

Let $B = \prod_{j=1}^{n} \frac{\lambda_j}{\Gamma(r_j)}$ and $C = \prod_{j=1}^{n} \lambda_j$ so that

$$\mathcal{M}_Y(s) = C^{-s} B \prod_{j=1}^{n} \Gamma\left(r_j + \frac{(s-1)}{\delta_j}\right). \tag{2.1}$$

From the relationship between the statistics defined in (1.4) and Y, (2.1) can be decomposed into two parts with one part associated with $\delta_j < 0$ and the other with $\delta_j > 0$. Then (2.1) becomes

$$\mathcal{M}_Y(s) = C^{-s} B \prod_{j=1}^{n_1} \Gamma\left(r_{1j} - \frac{1}{\delta_{1j}^{+}} + \frac{s}{\delta_{1j}^{+}}\right) \prod_{t=1}^{n_2} \Gamma\left(r_{2t} + \frac{1}{\delta_{2t}^{-}} - \frac{s}{\delta_{2t}^{-}}\right), \tag{2.2}$$

where n_1 is the number of random variables with positive power parameters, n_2 is the number of random variables with negative power parameters and $n = n_1 + n_2$. The pdf of Y is obtained by using (2.2) in the inverse Mellin transformation (see (A.38)) as

$$\begin{aligned}f_Y(y) &= \frac{1}{2\pi i} \int_{\mathbb{C}} y^{-s} \mathcal{M}_Y(s)\, ds \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} y^{-s} C^{-s} B \prod_{j=1}^{n_1} \Gamma\left(r_{1j} - \frac{1}{\delta_{1j}^{+}} + \frac{s}{\delta_{1j}^{+}}\right) \prod_{t=1}^{n_2} \Gamma\left(r_{2t} + \frac{1}{\delta_{2t}^{-}} - \frac{s}{\delta_{2t}^{-}}\right) ds \\ &= \frac{B}{2\pi i} \int_{\mathbb{C}} (Cy)^{-s} \prod_{j=1}^{n_1} \Gamma\left(r_{1j} - \frac{1}{\delta_{1j}^{+}} + \frac{s}{\delta_{1j}^{+}}\right) \prod_{t=1}^{n_2} \Gamma\left(r_{2t} + \frac{1}{\delta_{2t}^{-}} - \frac{s}{\delta_{2t}^{-}}\right) ds. \end{aligned} \tag{2.3}$$

where $\mathbb{C}$ indicates the complex contour. Using (A.35), (2.3) can be represented in terms of Fox's H-function as

$$f_Y(y) = B\mathbf{H}_{n_2,n_1}^{n_1,n_2}\left[Cy \mid \begin{array}{l} \left(1-\left(\frac{\delta_{21}^{-}r_{21}-1}{\delta_{21}^{-}}\right),\frac{1}{\delta_{21}^{-}}\right)\left(1-\left(\frac{\delta_{22}^{-}r_{22}-1}{\delta_{22}^{-}}\right),\frac{1}{\delta_{22}^{-}}\right) \\ \times,\ldots,\left(1-\left(\frac{\delta_{2n_2}^{-}r_{2n_2}-1}{\delta_{2n_2}^{-}}\right),\frac{1}{\delta_{2n_2}^{-}}\right) \\ \left(\frac{\delta_{11}^{+}r_{11}-1}{\delta_{11}^{+}},\frac{1}{\delta_{11}^{+}}\right) \\ \times\left(\frac{\delta_{12}^{+}r_{12}-1}{\delta_{12}^{+}},\frac{1}{\delta_{12}^{+}}\right),\ldots,\left(\frac{\delta_{1n_1}^{+}r_{1n_1}-1}{\delta_{1n_1}^{+}},\frac{1}{\delta_{1n_1}^{+}}\right) \end{array}\right] \tag{2.4}$$

for $y > 0$. Therefore according to (B.16), Y is an H-function distributed random variable with parameters

$$\left(\frac{\delta_{11}^{+}r_{11}-1}{\delta_{11}^{+}},\frac{\delta_{12}^{+}r_{12}-1}{\delta_{12}^{+}},\ldots,\frac{\delta_{1n_1}^{+}r_{1n_1}-1}{\delta_{1n_1}^{+}}\right),$$
$$\left(\frac{1}{\delta_{11}^{+}},\frac{1}{\delta_{12}^{+}},\ldots,\frac{1}{\delta_{1n_1}^{+}}\right) \text{ and}$$
$$\left(1-\left(\frac{\delta_{21}^{-}r_{21}-1}{\delta_{21}^{-}}\right),1-\left(\frac{\delta_{22}^{-}r_{22}-1}{\delta_{22}^{-}}\right),\ldots,1-\left(\frac{\delta_{2n_2}^{-}r_{2n_2}-1}{\delta_{2n_2}^{-}}\right)\right),$$
$$\left(\frac{1}{\delta_{21}^{-}},\frac{1}{\delta_{22}^{-}},\ldots,\frac{1}{\delta_{2n_2}^{-}}\right).$$

2.1.1 Special cases

In this subsection, useful forms of (2.4) when Y (see (1.5)) is a product of some of the special cases of the independent generalised gamma random variables are derived. See Table 1.1 for special cases of the generalised gamma distribution.

2.1.1.1 Product of independent exponential random variables

Suppose $X \sim EXP(\lambda)$ (see (B.1)). The Mellin transformation of X is given by (B.3). Let $X_1, X_2, \ldots, X_n$ be a set of independent random variables such that $X_j \sim EXP(\lambda_j)$ for $j = 1, 2, \ldots, n$.

Define

$$Y_{\exp} = \prod_{j=1}^{n} X_j. \tag{2.5}$$

Using (A.39) and (B.3), the Mellin transformation of Y_{exp} is given by

$$\mathcal{M}_{Y_{\text{exp}}}(s) = \prod_{j=1}^{n} \left(\lambda_j^{-s+1}\Gamma(s)\right). \tag{2.6}$$

Using the inverse Mellin transformation of Y_{exp} and (2.6), the exact pdf of Y_{exp} is

$$\begin{aligned} f_{Y_{\text{exp}}}(y) &= \frac{1}{2\pi i}\int_{\mathbb{C}} y^{-s}\mathcal{M}_{Y_{\text{exp}}}(s)\,ds \\ &= \frac{\prod_{j=1}^{n}\lambda_j}{2\pi i}\int_{\mathbb{C}}\left(y\prod_{j=1}^{n}\lambda_j\right)^{-s}\prod_{j=1}^{n}\Gamma(s)\,ds, \end{aligned} \tag{2.7}$$

such that

$$f_{Y_{\text{exp}}}(y) = \prod_{j=1}^{n}\lambda_j \mathbf{G}_{0,n}^{n,0}\left[y\prod_{j=1}^{n}\lambda_j \mid 0\right]. \tag{2.8}$$

where $\mathbf{G}_{r,s}^{m,n}(\cdot)$ is given in (A.36).

2.1.1.2 Product of independent gamma variables

Suppose $X \sim \Gamma(r,\lambda)$ (see (B.6)). The Mellin transformation of X is given by (B.10). Let $X_1, X_2, \ldots, X_n$ be a set of independent gamma random variables with the same shape parameter such that $X_j \sim \Gamma(r,\lambda_j)$ for $j = 1, 2, \ldots, n$.

Define

$$Y_\Gamma = \prod_{j=1}^{n} X_j. \tag{2.9}$$

Using (A.39) and (B.10) the Mellin transformation of Y_Γ is

$$\begin{aligned} \mathcal{M}_{Y_\Gamma}(s) &= \prod_{j=1}^{n}\frac{\Gamma(r+s-1)}{\lambda_j^{s-1}\Gamma(r)} \\ &= \frac{\Gamma^n(r+s-1)}{\Gamma^n(r)}\prod_{j=1}^{n}\lambda_j^{1-s}, \end{aligned} \tag{2.10}$$

where $\Gamma^n(\cdot) = \prod_{j=1}^{n}\Gamma(\cdot)$. By using the inverse Mellin transformation, the exact pdf of Y_Γ is

$$\begin{aligned} f_{Y_\Gamma}(y) &= \frac{1}{2\pi i}\int\limits_{\mathbb{C}} y^{-s}\mathcal{M}_{Y_\Gamma}(s)\,ds \\ &= \frac{1}{2\pi i}\int\limits_{\mathbb{C}} y^{-s}\frac{\Gamma^n(r+s-1)}{\Gamma^n(r)}\prod_{j=1}^{n}\lambda_j^{1-s}ds \\ &= \frac{\prod\limits_{j=1}^{n}\lambda_j}{(2\pi i)\,\Gamma^n(r)}\int\limits_{\mathbb{C}}\left(y\prod_{j=1}^{n}\lambda_j\right)^{-s}\Gamma^n(r-1+s)\,ds. \end{aligned}$$

Let $s' = r-1+s$, then

$$\begin{aligned} f_{Y_\Gamma}(y) &= \frac{\left(y\prod\limits_{j=1}^{n}\lambda_j\right)^{r-1}}{(2\pi i)\,\Gamma^n(r)}\prod_{j=1}^{n}\lambda_j\int\limits_{\mathbb{C}}\left(y\prod_{j=1}^{n}\lambda_j\right)^{-s'}\Gamma^n(s')\,ds' \\ &= \frac{\left(y\prod\limits_{j=1}^{n}\lambda_j\right)^{r-1}}{\Gamma^n(r)}\prod_{j=1}^{n}\lambda_j\mathbf{G}_{0,n}^{n,0}\left[y\prod_{j=1}^{n}\lambda_j \mid 0\right]. \end{aligned} \quad (2.11)$$

By using (2.8), (2.11) can be represented as

$$f_{Y_\Gamma}(y) = \frac{\left(y\prod\limits_{j=1}^{n}\lambda_j\right)^{r-1}}{\Gamma^n(r)}f_{Y_{\exp}}(y), \quad (2.12)$$

where $f_{Y_{\exp}}(\cdot)$ is given in (2.8) and is the pdf of the product of n independent exponential random variables with parameters λ_j for $j = 1,2,3,\ldots,n$.

Remark 2.1 *Representation (2.12) of the exact pdf of the product of n independent gamma distributed random variables will be useful in Section 3.2 when results from [4] are derived.*

2.2 Characteristic functions based method

In this section; the exact distribution of the random variable $Z = -\log Y$, with Y defined in (1.5), is derived by making use of characteristic functions. Furthermore, it will be shown that Z can be decomposed into a sum of two independent random variables. The decomposition of the characteristic function of Z will be useful in Chapter 3 where only one of the two independent random variables representing Z will be approximated in order to obtain the near-exact distributions of Z. Through suitable transformations from Z, near-exact distributions of Y will be obtained.

Let

$$Z = -\sum_{j=1}^{n} \log X_j.$$

By using independence between X_js, the characteristic function of Z can be derived as

$$\begin{aligned} \phi_Z(t) &= E[\exp(itZ)] \\ &= E\left[\exp\left(-it\sum_{j=1}^{n} \log X_j\right)\right] \\ &= E\left[\prod_{j=1}^{n} \exp\left(\log X_j^{-it}\right)\right] \\ &= \prod_{j=1}^{n} E\left[X_j^{-it}\right]. \end{aligned} \tag{2.13}$$

Using (B.12) to evaluate the moment of X_j, (2.13) becomes

$$\phi_Z(t) = \prod_{j=1}^{n} \frac{\Gamma\left(r_j - \frac{it}{\delta_j}\right)}{\Gamma\left(r_j\right)} \lambda_j^{it} \tag{2.14}$$

Another representation of (2.14) can be derived as follows

$$\begin{aligned} \phi_Z(t) &= \prod_{j=1}^{n} \frac{\Gamma\left(r_j + \gamma - \frac{it}{\delta_j}\right)}{\Gamma\left(r_j + \gamma - \frac{it}{\delta_j}\right)} \frac{\Gamma\left(r_j + \gamma\right)}{\Gamma\left(r_j + \gamma\right)} \frac{\Gamma\left(r_j - \frac{it}{\delta_j}\right)}{\Gamma\left(r_j\right)} \lambda_j^{it} \\ &= \prod_{j=1}^{n} \left\{ \frac{\Gamma\left(r_j + \gamma - \frac{it}{\delta_j}\right)}{\Gamma\left(r_j + \gamma\right)} \right\} \frac{\Gamma\left(r_j + \gamma\right)}{\Gamma\left(r_j + \gamma - \frac{it}{\delta_j}\right)} \frac{\Gamma\left(r_j - \frac{it}{\delta_j}\right)}{\Gamma\left(r_j\right)} \lambda_j^{it}, \end{aligned} \tag{2.15}$$

and (2.15) has a new parameter γ that is not present in (2.14). The impact of this parameter will be investigated in Chapter 4. For reasons which will become apparent later, γ will be called a precision parameter. Applying result (A.5) in (2.15), the form of

$\phi_Z(t)$ becomes

$$\begin{aligned}
\phi_Z(t) &= \prod_{j=1}^{n}\left\{\frac{\Gamma\left(r_j+\gamma-\frac{it}{\delta_j}\right)}{\Gamma\left(r_j+\gamma\right)}\right\}\frac{\prod_{k=0}^{\gamma-1}\left(r_j+k\right)\Gamma\left(r_j\right)}{\prod_{k=0}^{\gamma-1}\left(r_j+k-\frac{it}{\delta_j}\right)\Gamma\left(r_j-\frac{it}{\delta_j}\right)}\frac{\Gamma\left(r_j-\frac{it}{\delta_j}\right)}{\Gamma\left(r_j\right)}\lambda_j^{it} \\
&= \left\{\prod_{j=1}^{n}\frac{\Gamma\left(r_j+\gamma-\frac{it}{\delta_j}\right)}{\Gamma\left(r_j+\gamma\right)}\right\}\left\{\prod_{j=1}^{n}\prod_{k=0}^{\gamma-1}\frac{\left(r_j+k\right)}{\left(r_j+k-\frac{it}{\delta_j}\right)}\lambda_j^{it}\right\} \\
&= \phi_{Z_1}(t)\phi_{Z_2}(t). \qquad (2.16)
\end{aligned}$$

Therefore Z can be expressed as a sum of two independent random variables i.e. $Z = Z_1 + Z_2$. Z_1 has characteristic functions $\phi_{Z_1}(t)$ given by

$$\phi_{Z_1}(t) = \prod_{j=1}^{n}\frac{\Gamma\left(r_j+\gamma-\frac{it}{\delta_j}\right)}{\Gamma\left(r_j+\gamma\right)}. \qquad (2.17)$$

From (B.14); it follows that (2.17) is a characteristic function of a random variable Z_1 which is a linear combination of n independent log-gamma distributed random variables with parameters $r_j+\gamma$ and 1, where $\frac{1}{\delta_j}$ (for j =1, 2, 3,. . . ,n) are multipliers.

Z_2 has characteristic function $\phi_{Z_2}(t)$ given by

$$\phi_{Z_2}(t) = \prod_{j=1}^{n}\prod_{k=0}^{\gamma-1}\frac{\left(r_j+k\right)}{\left(r_j+k-\frac{it}{\delta_j}\right)}\lambda_j^{it}. \qquad (2.18)$$

Now (2.18) can be represented as

$$\begin{aligned}
\phi_{Z_2}(t) &= \prod_{j=1}^{n}\prod_{k=0}^{\gamma-1}\frac{\left(r_j+k\right)}{\left(r_j+k-\frac{it}{\delta_j}\right)}\exp\left(\log\left(\lambda_j^{it}\right)\right) \\
&= \prod_{j=1}^{n}\prod_{k=0}^{\gamma-1}\left(r_j+k\right)\left(r_j+k-\frac{it}{\delta_j}\right)^{-1}\exp\left(it\log\left(\lambda_j\right)\right) \\
&= \left[\prod_{j=1}^{n}\prod_{k=0}^{\gamma-1}\left(1-it\frac{1}{\delta_j\left(r_j+k\right)}\right)^{-1}\right]\exp\left(it\sum_{j=1}^{n}\log\left(\lambda_j\right)\right). \qquad (2.19)
\end{aligned}$$

By using (B.2), (2.19) is a product of $n\times\gamma$ characteristic functions of independent exponentially distributed random variables. Therefore Z_2 can be viewed as a shifted sum of $n \times \gamma$ independent exponentially distributed random variables with parameter $\frac{1}{\delta_j (r_j + k)}$ for j =1,2,3,...,n and k =0,1,2,...,$\gamma - 1$ respectively where the shift is given by $\varphi = \sum_{j=1}^{n} \log(\lambda_j)$. By summing the exponentially distributed random variables with the same parameters, the characteristic function of Z_2 can be represented as

$$\phi_{Z_2}(t) = \prod_{j=1}^{\ell} \beta_j^{m_j} (\beta_j - it)^{-m_j} \exp(it\varphi), \qquad (2.20)$$

where ℓ is the number of distinct exponentially distributed random variables, $\beta_j = \delta_j (r_j + k)$ and m_j is the number of exponentially distributed random variables with parameter $\frac{1}{\beta_j}$.

By definition of the statistic Y in (1.5), there is at least one $\delta_j < 0$ and at least one $\delta_j > 0$ for j =1,2,...,n. This implies that there is at least one $\beta_j < 0$ and at least one $\beta_j > 0$ for j =1,2,3,...,n. (2.20) can therefore be decompose as follows

$$\phi_{Z_2}(t) = \left\{ \prod_{j=1}^{\ell^+} (\beta_j^+)^{m_j^+} (\beta_j^+ - it)^{-m_j^+} \right\} \left\{ \prod_{j=1}^{\ell^-} (\beta_j^-)^{m_j^-} (\beta_j^- - it)^{-m_j^-} \right\} \exp(it\varphi), \qquad (2.21)$$

where ℓ^+ and β_j^+ are associated with $\beta_j > 0$ and ℓ^- and β_j^- are associated with $\beta_j < 0$.

Remark 2.2 *It is possible to consider cases where either $\delta_j > 0$ or $\delta_j < 0$, $j = 1, 2, \ldots, n$. These two cases are considered and studied in [15] and [17] where it is shown that Z_2 is a generalised integer gamma distributed random variable.*

Since $\beta_j^- < 0$, the following mathematical manipulation yield a different representation of (2.21)

$$\begin{aligned} \phi_{Z_2}(t) &= \left\{ \prod_{j=1}^{\ell^+} (\beta_j^+)^{m_j^+} (\beta_j^+ - it)^{-m_j^+} \right\} \\ &\quad \times \left\{ \prod_{j=1}^{\ell^-} (-1)^{m_j^-} (-\beta_j^-)^{m_j^-} ((-1)(-\beta_j^- + it))^{-m_j^-} \right\} \exp(it\varphi) \\ &= \left\{ \prod_{j=1}^{\ell^+} (\beta_j^+)^{m_j^+} (\beta_j^+ - it)^{-m_j^+} \right\} \\ &\quad \times \left\{ \prod_{j=1}^{\ell^-} (-\beta_j^-)^{m_j^-} (-1)^{m_j^-} (-1)^{m_j^-} (-\beta_j^- + it)^{-m_j^-} \right\} \exp(it\varphi). \qquad (2.22) \end{aligned}$$

Let $\beta_j^* = -\beta_j^-$ (so that $\beta_j^* > 0$) and $\tau = -t$, then (2.22) can be represented as

$$\begin{aligned}\phi_{Z_2}(t) &= \left\{\prod_{j=1}^{\ell^+}(\beta_j^+)^{m_j^+}\left(\beta_j^+ - it\right)^{-m_j^+}\right\}\left\{\prod_{j=1j}^{\ell^-}\left(\beta_j^*\right)^{m_j^-}\left(\beta_j^* - i\tau\right)^{-m_j^-}\right\}\exp\left(it\varphi\right) \\ &= \phi_{Z_{21}}(t)\phi_{Z_{22}}(\tau)\exp\left(it\varphi\right). \qquad (2.23)\end{aligned}$$

Consider

$$\phi_{Z_{21}}(t) = \prod_{j=1}^{\ell^+}(\beta_j^+)^{m_j^+}\left(\beta_j^+ - it\right)^{-m_j^+}.$$

Using (B.9), $\phi_{Z_{21}}(t)$ is a characteristic function of the sum of ℓ^+ independent gamma random variables with parameters β_j^+ and m_j^+ where m_j^+ are integers. It follows from (B.38) that $Z_{21} \sim GIG\left(\ell^+, \underline{m}^+, \underline{\beta^+}\right)$ with parameters

$$\begin{aligned}\underline{m}^+ &= \left(m_1^+, m_2^+, \ldots, m_{\ell^+}^+\right)' \\ \underline{\beta^+} &= \left(\beta_1^+, \beta_2^+, \ldots, \beta_{\ell^+}^+\right)'.\end{aligned}$$

Similarly, $Z_{22} \sim GIG\left(\ell^-, \underline{m}^-, \underline{\beta^*}\right)$ where

$$\begin{aligned}\underline{m}^- &= \left(m_1^-, m_2^-, \ldots, m_{\ell^-}^-\right)' \\ \underline{\beta^*} &= \left(\beta_1^*, \beta_2^*, \ldots, \beta_{\ell^-}^*\right)'.\end{aligned}$$

Furthermore, from (2.23) Z_{21} and Z_{22} are independent. Note that since $\tau = -t$, then

$$\begin{aligned}&\phi_{Z_{21}}(t)\phi_{Z_{22}}(\tau) \\ =\ &\phi_{Z_{21}}(t)\phi_{Z_{22}}(-t) \\ =\ &E\left[\exp\left(itZ_{21}\right)\right]E\left[\exp\left(-itZ_{22}\right)\right] \\ =\ &E\left[\exp\left(it\left(Z_{21} - Z_{22}\right)\right)\right]. \qquad (2.24)\end{aligned}$$

It follows from Result 38 that (2.24) is the characteristic function of the random variable $Z_{21} - Z_{22}$ such that

$$Z_{21} - Z_{22} \sim DGIG\left(\underline{m}^+, \underline{m}^-, \underline{\beta^+}, \underline{\beta^*}, \ell^+, \ell^-\right).$$

From (2.23)

$$Z_2 = Z_{21} - Z_{22} + \varphi,$$

therefore

$$Z_2 \sim SDGIG\left(\underline{m}^+, \underline{m}^-, \underline{\beta}^+, \underline{\beta}^*, \ell^+, \ell^-, \varphi\right),$$

where $SDGIG$ denoted the shifted version of the $DGIG$ distribution.

2.3 Chapter summary

Contributions in this chapter are summarised as follows:

- In Section 2.1, the exact distribution of Y (see 1.5) is presented in terms of Fox's H-functions by using the inverse Mellin transformation.
- Furthermore, in section 2.1 the exact distribution of special cases of Y (where Y is a product of independent exponentially- and gamma distributed random variables respectively) is given in terms of Meijer's-G functions. Furthermore, the exact pdf of the product of independent gamma distributed random variables is represented in terms of the exact pdf of the product of independent exponentially distributed random variables (see (2.12)).
- In Section 2.2, the characteristic function of $Z = -\log Y$ is derived for the first time. By decomposing its characteristic function, Z is represented as a sum of two independent random variables i.e. $Z = Z_1 + Z_2$. Furthermore, Z_1 is a linear combination of n independent log-gamma distributed random variables.
- The process of decomposing the characteristic function of Z introduces a new parameter, γ, called the precision parameter. The effect of the precision parameter is investigated in Chapter 4 Subsection 4.3.1.

Chapter 3

Approximate distributions

Near-exact distributions are approximate distributions for the exact distribution of the statistic of interest. By decomposing the statistic of interest into a sum of independent random variables and approximating a small part of the decomposition while leaving the rest unchanged, near-exact distributions are obtained. In this chapter, near-exact distributions for Z are developed and from a suitable transformation, near-exact distribution for Y will be obtained.

In Chapter 2, a new representation of the exact characteristic function of $Z = -\log Y$, with Y defined in (1.5), was developed (see (2.23)). In Subsections 3.1.1 and 3.1.2 respectively, the first and the second near-exact distributions are developed by approximating one part of (2.23) and leaving the other part unchanged. Section 3.2 considers the product of independent generalised gamma distributed random variables with equal positive power parameters and confirms results proposed in [4]. These results are useful in Chapter 4 when assessing the relative advantage of near-exact distributions in approximating the exact distribution.

3.1 Near-Exact Distributions

3.1.1 First near-exact distribution

This approach was introduced in [16] on the study of the linear combination of independent Gumbel random distributions.

Consider the following (2.16)

$$Z = Z_1 + Z_2, \tag{3.1}$$

where Z_1 and Z_2 are independent random variables (see (2.16)). (2.17) shows that Z_1 is a linear combination of n independent log-gamma distributed random variables with parameters $r_j + \gamma$ and 1. To obtain the first near-exact distribution of Z, Z_1 will be approximated by its own expected value while Z_2 is left unchanged. Consequently, Z will be approximated by

$$Z^a = E\left[Z_1\right] + Z_2. \tag{3.2}$$

with $E\left[Z_1\right]$ is obtained by

$$E\left[Z_1\right] = \frac{1}{i}\frac{\partial \phi_{Z_1}(t)}{\partial t}\,|_{t=0} \tag{3.3}$$

where ϕ_{Z_1} is given by (2.17). In (3.2) a shift is effectively being added in Z_2 where

$$Z_2 \sim SDGIG\left(\underline{m}^+, \underline{m}^-, \underline{\beta^+}, \underline{\beta^*}, \ell^+, \ell^-, \varphi\right).$$

Therefore

$$Z^a \sim SDGIG\left(\underline{m}^+, \underline{m}^-, \underline{\beta^+}, \underline{\beta^*}, \ell^+, \ell^-, \varphi^*\right),$$

where

$$\begin{aligned}
\underline{m}^+ &= \left(m_1^+, m_2^+, \ldots, m_{\ell+}^+\right)', \\
\underline{\beta^+} &= \left(\beta_1^+, \beta_2^+, \ldots, \beta_{\ell+}^+\right)', \\
\underline{m}^- &= \left(m_1^-, m_2^-, \ldots, m_{\ell-}^-\right)', \\
\underline{\beta^*} &= \left(\beta_1^*, \beta_2^*, \ldots, \beta_{\ell-}^*\right)', \\
\varphi &= \sum_{j=1}^{n} \log\left(\lambda_j\right) \quad \text{and} \\
\varphi^* &= \varphi + E\left[Z_1\right].
\end{aligned} \tag{3.4}$$

The cdf of Y is

$$\begin{aligned}
& P\left(Y \leq y\right) \\
=\ & P\left(Z \geqslant -\log\left(y\right)\right) \\
=\ & 1 - P\left(Z \leq -\log\left(y\right)\right).
\end{aligned} \tag{3.5}$$

The first near-exact cdf of Y is obtained by replacing Z in (3.5) by its approximate random variable Z^a so that

$$P\left(Y \leq y\right) \approx 1 - P\left(Z^a \leq -\log\left(y\right)\right). \tag{3.6}$$

Therefore by considering the shifted version of (B.68), the first near-exact cdf of Y is given by

$$F_Y(y) \approx \begin{cases} 1 - \sum_{j=1}^{\ell^+} \sum_{k=1}^{m^+} \sum_{i=0}^{k-1} p_{jkl} F_{Z_{jkl}} \left(-\log\left(y-\varphi^*\right)\right) \\ -\sum_{j=1}^{\ell^-} \sum_{h=1}^{m^-} \sum_{i=0}^{h-1} p^*_{jkl} & y \leq 1+\varphi^* \\ 1 + \sum_{j=1}^{\ell^-} \sum_{h=1}^{m^-} \sum_{i=0}^{h-1} p^*_{jkl} F_{Z^*_{jkl}} \left(-\log\left(y-\varphi^*\right)\right) \\ -\sum_{j=1}^{\ell^-} \sum_{h=1}^{m^-} \sum_{i=0}^{h-1} p^*_{jkl} & y > 1+\varphi^*, \end{cases} \tag{3.7}$$

p_{jkl} and p^*_{jkl}, are defined as in (B.59) and (B.60) respectively. From (3.5), the pdf of Y is given by

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} P\left(Y \leq y\right) \\ &\approx \frac{d}{dy}\left(1 - P\left(Z^a \leq -\log\left(y\right)\right)\right) \\ &= -\left[\frac{d}{dy} P\left(Z^a \leq -\log\left(y\right)\right)\right] \frac{d}{dy}\left(-\log\left(y\right)\right) \\ &= f_{Z^a}\left(-\log\left(y\right)\right) \frac{1}{y}. \end{aligned} \tag{3.8}$$

Therefore the first near-exact pdf of Y is obtained by applying shifted version of (B.58) in (3.8) as

$$f_Y(y) = \begin{cases} \sum_{j=1}^{\ell^+} \sum_{k=1}^{m_j^+} \sum_{i=0}^{k-1} p_{jkl} \frac{1}{\Gamma(k-i)} \left(-\log\left(y-\varphi^*\right)\right)^{k-i-1} \\ \exp\left(\left(\log\left(y-\varphi^*\right)\right)\beta_j^+\right) \frac{1}{y} & y \leq 1+\varphi^* \\ \sum_{j=1}^{\ell^-} \sum_{h=1}^{m_j^-} \sum_{i=0}^{h-1} p^*_{jkl} \frac{1}{\Gamma(h-i)} \left(\log\left(y-\varphi^*\right)\right)^{h-i-1} \\ \exp\left(-\left(\log\left(y-\varphi^*\right)\right)\beta_j^+\right) \frac{1}{y} & y > 1+\varphi^*. \end{cases} \tag{3.9}$$

As noted in the introductory chapter; many other methods to approximate the distribution of Y are either in terms of infinite series or special functions, therefore difficult to evaluate. The near-exact distribution developed in this section is not only easy to evaluate computationally, but also far more accurate and efficient in terms of computer run-time and resources than most other methods in literature. This will be illustrated in Chapter 4.

3.1.2 Second near-exact distribution

In Subsection 3.1.1, the first near-exact distribution for Y was developed. In this Section, the second near-exact distribution is developed using a similar methodology. However, instead of approximating the random variable Z_1 with its fixed expected value, it will be approximated by a suitable random variable. As noted before, Z_1 is a linear combination of n independent log-gamma distributed random variables. Using Result 31, each of these log-gamma distributed random variables can be represented as a sum of infinite independent exponential random variables. [17] advocates that Z_1 can therefore be represented by a sum of infinite independent gamma random variables. From the infinite gamma distributed random variables representing Z_1, a single random variable (denoted by W) will be selected to approximate Z_1. W will be selected such that it is independent of Z_2 and satisfies the following system of equations

$$\frac{\partial^j \phi_{Z_1}(t)}{\partial t^j}|_{t=0} = \frac{\partial^j \phi_W(t)}{\partial t^j}|_{t=0} \qquad j = 1, 2, 3, \tag{3.10}$$

where ϕ_{Z_1} is given by (2.17) and

$$\phi_W(t) = \left(1 - \frac{it}{\psi}\right)^{-\rho} e^{it\theta},$$

(3.10) will be solved numerically to obtain values of ψ, ρ and θ. The second near-exact approximation of Z is therefore given by

$$Z^b = Z_2 + sign(\psi) \times W,$$

where the function $sign(\cdot)$ is defined as follows

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Z^b is either a sum or difference of a shifted gamma random variable and an independent $SDGIG\left(\underline{m}^+, \underline{m}^-, \underline{\beta}^+, \underline{\beta}^*, \ell^+, \ell^-, \varphi\right)$ distributed random variable. $\underline{m}^+, \underline{m}^-, \underline{\beta}^+, \underline{\beta}^*, \ell^+, \ell^-$ and φ defined on page 24. For $sign(\psi) = 1$, using (3.5) and (B.70) the second near exact cdf of Y is

$$F_Y(y) \approx \begin{cases} 1 - \sum_{j=1}^{\ell+} \sum_{k=1}^{m_j^+} \sum_{i=0}^{k-1} p_{jkl} F_{G_1}(-\log(y-\theta-\varphi)) + \sum_{j=1}^{\ell-} \sum_{h=1}^{m_j^-} \sum_{i=0}^{h-1} p_{jkl}^* \\ \times F_{DG}(-\log(y-\theta-\varphi)) & y \leq 1+\theta+\varphi \\ 1 - \sum_{j=1}^{\ell-} \sum_{h=1}^{m_j^-} \sum_{i=0}^{h-1} p_{jkl}^* F_{DG_1}(-\log(y-\theta-\varphi)) & y > 1+\theta+\varphi, \end{cases} \tag{3.11}$$

p_{jkl} and p_{jkl}^*, are defined as in (B.59) and (B.60). $F_{G_1}(\cdot)$ is given by (B.52) with parameters $\left(k-i, \rho, \beta_j^+, \psi\right)$ and $F_{DG_1}(\cdot)$ is given by (B.18) with gamma distribution's parameters (ρ, ψ) and Erlang distribution's parameters $(h-i, \beta_j^-)$. For $sign(\psi) = -1$, using (3.5) and (B.76) the second near-exact cdf of Y is

$$F_Y(y) \approx \begin{cases} \sum_{j=1}^{\ell+} \sum_{k=1}^{m_j^+} \sum_{i=0}^{k-1} p_{jkl} F_{DG_1}(\log(y-\theta-\varphi)) & y \leq 1+\theta+\varphi \\ \sum_{j=1}^{\ell-} \sum_{h=1}^{m_j^-} \sum_{i=0}^{h-1} p_{jkl}^* F_{G_1}(\log(y-\theta-\varphi)) & \\ -\sum_{j=1}^{\ell+} \sum_{k=1}^{m_j^+} \sum_{i=0}^{k-1} p_{jkl} F_{DG_1}(\log(y-\theta-\varphi)) & y > 1+\theta+\varphi. \end{cases} \tag{3.12}$$

3.2 Chen's approximation

In this section, the methodology used by [4] will be described and applied to obtain an approximation for the distribution of the product of *independent generalised gamma* distributed random variables with equal positive parameters. Firstly, an *exponential distribution* can be represented in terms of a *Rayleigh distribution* (see (B.4)) with parameter equal to 1. Therefore, the exact distribution of $Y_{\exp}$ (see (2.5)) can be represented in terms of the product of n *independent Rayleigh distributed* random variables, denoted by Y_{Ra}. [12] derived the approximation of the exact pdf of $\sqrt[n]{Y_{Ra}}$ in terms of a *Nakagami-m distribution*. Subsequently, by using necessary transformations, the approximate pdf of $Y_{\exp}$ is obtained from the approximate pdf of $\sqrt[n]{Y_{Ra}}$. Using (2.12), the approximate pdf of $Y_{\exp}$ and necessary transformations, the approximate distribution of Y_{Γ} (see (2.9)) can be obtained. Similarly, the approximate distribution of the product of *independent generalised gamma* variables all with equal shape and equal power parameters can be obtained.

Figure 3.1 shows a step-by-step outline of this approximation method.

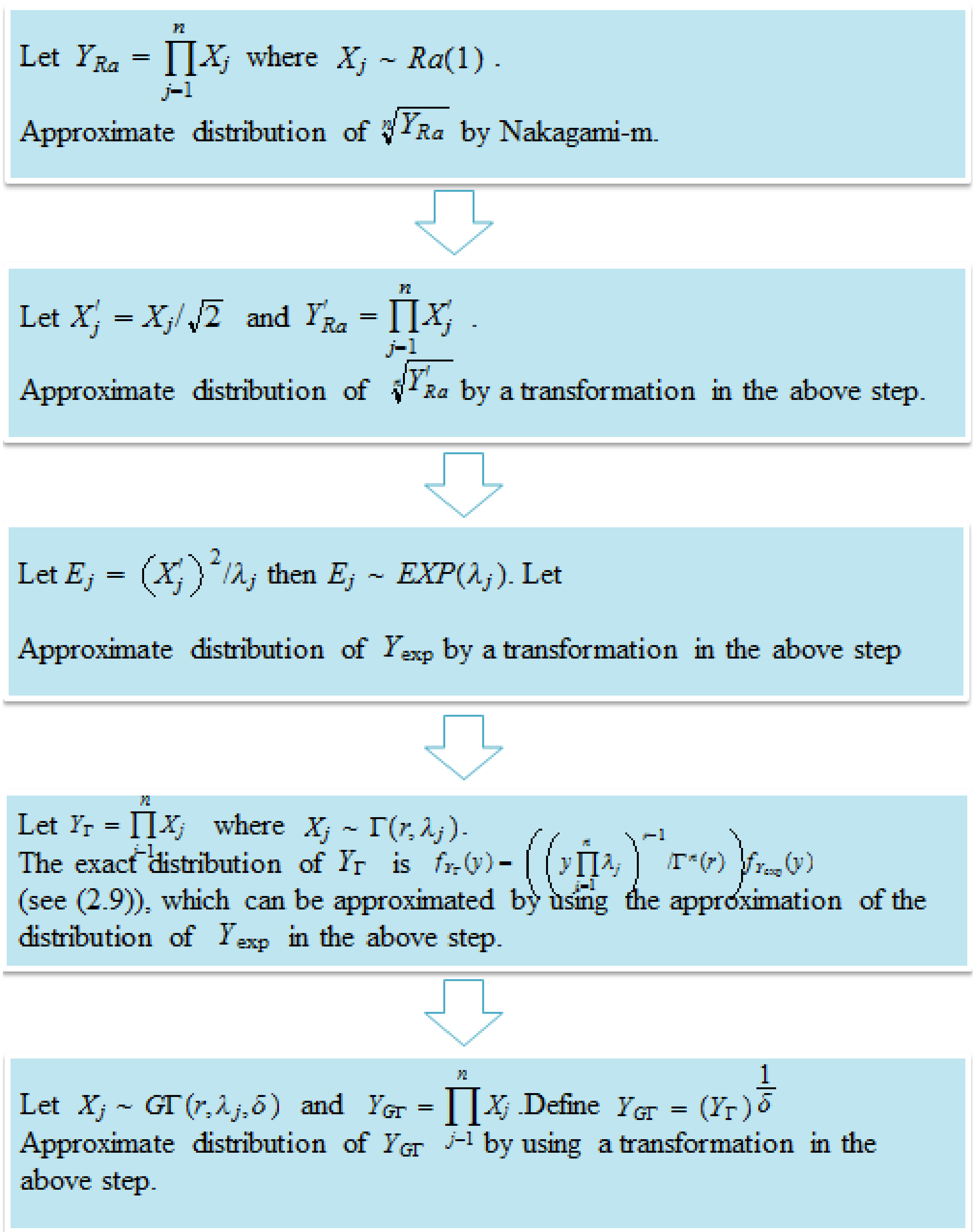

Figure 3.1: Outlines of steps in derivation of Chen's approximation.

3.2.1 Product of independent exponential variables

Let $X_1, X_2, \ldots, X_n$ be a set of independent random variables such that $X_j \sim Ra\,(1)$ (see (B.4))for $j = 1, 2, \ldots, n$. Let

$$Y_{Ra} = \prod_{j=1}^{n} X_j.$$

According to [12] the pdf of $\sqrt[n]{Y_{Ra}}$, denoted by $f_{\sqrt[n]{Y_{Ra}}}(y)$, can be approximated accurately by using the Nakagami-m distribution with pdf

$$f_{\sqrt[n]{Y_{Ra}}}(y) \approx 2\left(\frac{m_0}{\Omega_0}\right)^{m_0} \frac{1}{\Gamma(m_0)} y^{2m_0-1} \exp\left(-\frac{m_0}{\Omega_0}y^2\right); \qquad y > 0. \tag{3.13}$$

where $m_0 = 0.6102n + 0.4263$ and $\Omega_0 = 0.8808n^{-0.9661} + 1.12$. Let X_j' denote a normalised* Rayleigh distributed random variable such that $X_j' = \frac{X_j}{\sqrt{2}}$ for $j = 1, 2, \ldots, n$. The pdf of X_j' is

$$\begin{aligned} f_{X_j'}(x) &= f_{X_j}\left(\sqrt{2}x\right)\sqrt{2} \\ &= \sqrt{2}x \exp\left(-\frac{\left(\sqrt{2}x\right)^2}{2}\right)\sqrt{2} \\ &= 2x \exp\left(-x^2\right); \qquad x > 0. \end{aligned} \tag{3.14}$$

Let a normalised version of Y_{Ra} be

$$Y_{Ra}' = \prod_{j=1}^{n} X_j'. \tag{3.15}$$

From (3.13) pdf of $\sqrt[n]{Y_{Ra}'}$, denoted by $f_{\sqrt[n]{Y_{Ra}'}}(y)$, can be approximated as

$$f_{\sqrt[n]{Y_{Ra}'}}(y) \approx 2\left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{\Gamma(m_0)} y^{2m_0-1} \exp\left(-\frac{2m_0}{\Omega_0}y^2\right); \qquad y > 0. \tag{3.16}$$

Let $E_j = \frac{\left(X_j'\right)^2}{\lambda_j}$ for j =1,2,3,...,n. Through a suitable transformation in (3.14), the pdf of E_j is given by

$$\begin{aligned} f_{E_j}(x) &= f_{X_j'}\left(\sqrt{\lambda_j x}\right)\frac{\sqrt{\lambda_j}}{2\sqrt{\lambda_j x}} \\ &= 2\sqrt{\lambda_j x}\exp\left(-\left(\sqrt{\lambda_j x}\right)^2\right)\frac{\sqrt{\lambda_j}}{2\sqrt{\lambda_j x}} \\ &= \lambda_j \exp\left(-\lambda_j x\right); \qquad x > 0. \end{aligned} \tag{3.17}$$

Therefore $E_j \sim EXP(\lambda_j)$ (see (B.1)). Let

$$Y_{\exp} = \prod_{j=1}^{n} E_j. \tag{3.18}$$

*To make the rate parameter equal to 1

From (3.15), (3.18) can alternatively be represented as

$$\begin{aligned} Y_{\exp} &= \prod_{j=1}^{n} \frac{\left(X_j'\right)^2}{\lambda_j} \\ &= \frac{\left(Y_{Ra}'\right)^2}{\prod\limits_{j=1}^{n} \lambda_j} \\ &= \frac{\left(\sqrt[n]{Y_{Ra}'}\right)^{2n}}{\prod\limits_{j=1}^{n} \lambda_j}. \end{aligned} \tag{3.19}$$

By using the transformation in (3.19) the pdf of $Y_{\exp}$ can be derived from the pdf of $\sqrt[n]{Y_{Ra}'}$

$$f_{Y_{\exp}}(y) = f_{\sqrt[n]{Y_{Ra}'}}\left(\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{2n}}\right)\frac{1}{2n}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{2n}-1}; \qquad y>0. \tag{3.20}$$

By using (3.16), the pdf of $Y_{\exp}$ can be approximated as

$$\begin{aligned} f_{Y_{\exp}}(y) &\approx 2\left(\frac{2m_0}{\Omega_0}\right)^{m_0}\frac{1}{\Gamma(m_0)}\left(\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{2n}}\right)^{2m_0-1} \\ &\quad \times \exp\left(-\frac{2m_0}{\Omega_0}\left(\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{2n}}\right)^2\right)\frac{1}{2n}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{2n}-1} \\ &= \left(\frac{2m_0}{\Omega_0}\right)^{m_0}\frac{1}{n\Gamma(m_0)}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{m_0}{n}-1}\exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{n}}\right) \end{aligned} \tag{3.21}$$

Furthermore; to recognise (3.21) as the pdf of a known distribution, it is adjusted so that the approximation of $f_{Y_{\exp}}(y)$ is

$$f_{Y_{\exp}}(y) \approx \left(\left(\frac{2m_0}{\Omega_0}\right)^n \prod_{j=1}^{n} \lambda_j\right)^{\frac{m_0}{n}} \frac{1}{n\Gamma(m_0)} y^{\frac{m_0}{n}-1}$$
$$\exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{n}}\right); \quad y > 0 \tag{3.22}$$

Therefore; from (B.11)

$$Y_{\exp} \overset{d}{\simeq} G\Gamma\left(m_0, \left(\frac{2m_0}{\Omega_0}\right)^n \prod_{j=1}^{n} \lambda_j, \frac{1}{n}\right).$$

3.2.2 Product of independent gamma variables

Suppose $X_j \sim \Gamma(r, \lambda_j)$ (see (B.6)) for j =1,2,3,...,n. Let

$$Y_\Gamma = \prod_{j=1}^{n} X_j. \tag{3.23}$$

From (2.12), the exact pdf of Y_Γ is given by

$$f_{Y_\Gamma}(y) = \frac{\left(y\prod_{j=1}^{n}\lambda_j\right)^{r-1}}{\Gamma^n(r)} f_{Y_{\exp}}(y), \tag{3.24}$$

where $f_{Y_{\exp}}(y)$ is given by (3.20) and approximated by (3.22). By substituting (3.22) in (3.24), the approximate pdf of Y_Γ is given by

$$f_{Y_\Gamma}(y) \approx \frac{\left(y\prod_{j=1}^{n}\lambda_j\right)^{r-1}}{\Gamma^n(r)} \left(\left(\frac{2m_0}{\Omega_0}\right)^n \prod_{j=1}^{n} \lambda_j\right)^{\frac{m_0}{n}} \frac{1}{n\Gamma(m_0)} y^{\frac{m_0}{n}-1}$$
$$\times \exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{n}}\right); \quad y > 0. \tag{3.25}$$

Furthermore, to recognise (3.25) as the pdf of a known distribution, it is adjusted so that the approximation of $f_{Y_\Gamma}(y)$ is

$$f_{Y_\Gamma}(y) \approx \left(\prod_{j=1}^{n}\lambda_j\left(\frac{2m_0}{\Omega_0}\right)^n\right)^{\frac{m_0}{n}+r}\frac{1}{n\Gamma(m_0+nr)}y^{\frac{m_0}{n}+r-1}$$
$$\times\exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{1}{n}}\right);\quad y>0. \tag{3.26}$$

Therefore; from (B.11)

$$Y_\Gamma \overset{d}{\simeq} G\Gamma\left(m_0+nr,\left(\frac{2m_0}{\Omega_0}\right)^n\prod_{j=1}^{n}\lambda_j,\frac{1}{n}\right).$$

3.2.3 Product of independent generalised gamma variables

Suppose $X_j \sim G\Gamma(r,\lambda_j,\delta)$ (see (B.11)) for $j=1,2,3,\ldots,n$. Let $U_j = X_j^\delta$

$$\begin{aligned} f_{U_j}(u) &= f_{X_j}\left(u^{\frac{1}{\delta}}\right)\left|\frac{1}{\delta}u^{\frac{1}{\delta}-1}\right| \\ &= |\delta|\frac{\lambda_j^{r\delta}u^{\frac{1}{\delta}r\delta-1}}{\Gamma(r)}\exp\left(\left(-\lambda_j u^{\frac{1}{\delta}}\right)^{\delta}\right)\left|\frac{1}{\delta}u^{\frac{1}{\delta}-1}\right| \\ &= \frac{\lambda_j^{r\delta}}{\Gamma(r)}u^{r-1}\exp\left(-\lambda_j^{\delta}u\right), \qquad u>0 \end{aligned} \tag{3.27}$$

It follows from (3.27) that $U_j \sim \Gamma\left(r,\lambda_j^\delta\right)$. Define now

$$Y_{G\Gamma} = \prod_{j=1}^{n}X_j, \tag{3.28}$$

then (3.28) can be re-written as

$$\begin{aligned} Y_{G\Gamma} &= \prod_{j=1}^{n}U_j^{\frac{1}{\delta}} \\ &= (Y_\Gamma)^{\frac{1}{\delta}}, \end{aligned} \tag{3.29}$$

With Y_Γ defined in (3.23) and each component in Y_Γ is a gamma distributed random variables with shape parameter r and rate parameter λ_j^δ. By using a transformation from Y_Γ to $Y_{G\Gamma}$, the exact pdf of $Y_{G\Gamma}$ is given by

$$f_{Y_{G\Gamma}}(y) = f_{Y_\Gamma}\left(y^\delta\right)\left|\delta y^{\delta-1}\right|. \tag{3.30}$$

The approximate pdf of $Y_{G\Gamma}$ can be derived by substituting $f_{Y_\Gamma}\left(y^\delta\right)$ by its approximate form as given by (3.26)

$$\begin{aligned} f_{Y_{G\Gamma}}(y) &\approx \left(\prod_{j=1}^{n}\lambda_j^\delta\left(\frac{2m_0}{\Omega_0}\right)^n\right)^{\frac{m_0}{n}+r}\frac{1}{n\Gamma(m_0+nr)}y^{\delta\left(\frac{m_0}{n}+r-1\right)} \\ &\quad\times\exp\left(-\frac{2m_0}{\Omega_0}\left(y^\delta\prod_{j=1}^{n}\lambda_j^\delta\right)^{\frac{1}{n}}\right)\left|\delta y^{\delta-1}\right| \\ &= \left(\prod_{j=1}^{n}\lambda_j\left(\frac{2m_0}{\Omega_0}\right)^{\frac{n}{\delta}}\right)^{\frac{\delta}{n}m_0+nr}\frac{|\delta|}{n\Gamma(m_0+nr-n)}y^{\frac{\delta}{n}(m_0+nr-n)-1} \\ &\quad\times\exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{\delta}{n}}\right)y^\delta;\quad y>0. \end{aligned} \tag{3.31}$$

Furthermore, to recognise (3.31) as the pdf of a known distribution, it is adjusted so that the approximation of $f_{Y_{G\Gamma}}(y)$ is

$$\begin{aligned} f_{Y_{G\Gamma}}(y) &\approx \left(\left(\frac{2m_0}{\Omega_0}\right)^{\frac{n}{\delta}}\prod_{j=1}^{n}\lambda_j\right)^{\frac{\delta}{n}(m_0+n(r-1))}\frac{|\delta|}{n\Gamma(m_0+n(r-1))} \\ &\quad\times y^{\frac{\delta}{n}(m_0+n(r-1))-1}\exp\left(-\frac{2m_0}{\Omega_0}\left(y\prod_{j=1}^{n}\lambda_j\right)^{\frac{\delta}{n}}\right);\quad y>0. \end{aligned} \tag{3.32}$$

Therefore; from (B.11)

$$Y_{G\Gamma} \overset{d}{\simeq} G\Gamma\left(m_0 + n\left(r-1\right), \left(\frac{2m_0}{\Omega_0}\right)^{\frac{n}{\delta}} \prod_{j=1}^{n} \lambda_j, \frac{\delta}{n}\right).$$

3.3 Chapter summary

The contribution in this chapter are summarised as follows:

- The decomposition (see (2.16)) of Z into $Z_1 + Z_2$ is used to derive near-exact distribution in Section 3.1. Near-exact distributions are obtained by approximating Z_1 while Z_2 is left unchanged.
- For the first near-exact distribution, Z_1 is approximated by its own fixed expected value.
- For the second near-exact distribution, Z_1 is approximated by a gamma distributed random variable.
- Section 3.2 presents results introduced in [4]. This results also approximate the product of independent generalised gamma distributed random variables when the power and shape parameter are fixed.

Chapter 4

Computational studies

This chapter assesses the performance of each of the approximation methods that were proposed in Chapter 3. The performance of the approximate distribution will be measured in terms of

- the accuracy of the distribution in approximating the exact distribution,
- the required computer run-time of the approximate distribution and
- performance of an approximation method relative to other methods.

4.1 Exact distribution

In Section (2.1) of Chapter 2, the exact distribution of Y (see (1.5)) is derived in terms of Fox's H-function. Even with today's powerful computers, Fox's H-functions are still not computable. Instead of the exact distribution of Y, the exact distribution of $Z = -\log Y$ is evaluated using the inversion method in [9] and is given by (4.1)

$$F_Z\left(z\right) = \frac{1}{2} - \frac{1}{2\pi}\int\limits_0^\infty \frac{\phi_Z(-t)\exp\left(itz\right) - \phi_Z(t)\exp\left(-itz\right)}{it}dt, \tag{4.1}$$

where $\phi_Z(\cdot)$ is given by (2.14).

4.2 Empirical distribution

Near-exact distribution as the focus of the study will also be assessed relative to this empirical distribution. This method is outlined below.

To obtain an empirical value z of Z, simulate random variates from each of the given independent generalised gamma distributions. In this study, Mathematica function is used simulate a random variate from a generalised gamma distribution. Re-parametrisation of the Mathematica function is necessary in order ensure similarity with the generalised gamma distribution defined (1.1). Where the power parameter is negative, simulate from an equivalent distribution with a positive parameter by using Remark 1.1. Multiply random variates from distribution with power parameters of the same sign with each other. One product (associated with random variates with positive power parameter) is then divided by the remaining product. The quotient is an empirical variate of the statistic G (see (1.3)) and is denoted by $\mathfrak{g}$. Obtain z as $z = -\log \mathfrak{g}$. 10^6 of z values were simulated to obtain the empirical distribution of Z.

To study the accuracy of the approximate distribution, a measure of proximity given below (see [16] for details of the measure) is calculated.

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi_Z(t) - \phi^*(t)}{t} \right| dt, \tag{4.2}$$

where $\phi_Z(t)$ and $\phi^*(t)$ are respectively the exact and the approximate characteristic functions of the random variable $Z = -\log Y$ with Y defined in (1.5). According to [15]

$$\sup_{z \in \mathbb{R}} |F_Z(z) - F^*(z)| \leq \Delta,$$

where $F_Z(z)$ and $F^*(z)$ are the exact and approximate cdf of the random variable Z respectively. Therefore, Δ provides an upper bound on the proximity between $F_Z(z)$ and $F^*(z)$.

4.3 Near-exact distributions

For computational purposes, three cases of a set of independent gamma distributed random variables will be considered. Similar to the statistic of interest Y (see (1.5)), in each of the three cases there will be at least δ^- and δ^+ in order to ensure that the ratio of generalised gamma distributed random variables is considered. Note that in [4], [17] and [15] only cases where power parameters are either all negative or positive are considered. The three cases studies are summarised in Table 4.1.

Table 4.1: Sets of parameters of independent generalised gamma distribution.

Case	r	λ	δ
I	$\left\{\frac{1}{3}, \frac{22}{7}\right\}$	$\left\{\frac{1}{2}, \frac{1}{4}\right\}$	$\{4, -2\}$
II	$\{2, 3, 5\}$	$\{3, 2, 10\}$	$\left\{\frac{1}{2}, 2, -\frac{1}{4}\right\}$
III	$\left\{2, 3, 5, \frac{1}{2}\right\}$	$\left\{3, 2, 10, \frac{2}{7}\right\}$	$\left\{\frac{1}{2}, 2, -\frac{1}{4}, -\frac{1}{3}\right\}$

4.3.1 Proximity of the near-exact distribution

Table 4.2: Proximity measures for first near-exact distribution of Z.

Precision parameter	Case		
	I	II	III
4	1.6×10^{-2}	5.2×10^{-2}	1.7×10^{-2}
5	1.3×10^{-2}	4.4×10^{-2}	1.4×10^{-2}
10	6.9×10^{-3}	2.5×10^{-2}	7.8×10^{-3}
15	4.8×10^{-3}	1.7×10^{-2}	5.4×10^{-3}
20	3.6×10^{-3}	1.3×10^{-2}	4.1×10^{-3}
50	1.5×10^{-3}	5.5×10^{-3}	1.7×10^{-3}
100	7.5×10^{-4}	2.8×10^{-3}	8.7×10^{-4}
200	3.8×10^{-4}	1.4×10^{-3}	4.4×10^{-4}

Table 4.3: Proximity measures for second near-exact distribution of Z.

Precision parameter	Case		
	I	II	III
4	9.4×10^{-5}	3.4×10^{-4}	5.3×10^{-5}
5	5.6×10^{-5}	2.3×10^{-4}	3.1×10^{-5}
10	1.0×10^{-5}	5.8×10^{-5}	5.4×10^{-6}
15	3.4×10^{-6}	2.3×10^{-5}	1.9×10^{-6}
20	1.6×10^{-6}	1.1×10^{-5}	8.9×10^{-7}
50	1.2×10^{-7}	9.4×10^{-7}	7.1×10^{-8}
100	1.5×10^{-8}	1.3×10^{-7}	9.6×10^{-9}
200	2.0×10^{-9}	1.7×10^{-8}	1.4×10^{-9}

Tables 4.2 and 4.3 presents values of the proximity measure (Δ, see (4.2)) for each of the cases and for various precision parameter values for the first and second near-exact distribution of $Z = -\log Y$ respectively. It is evident in both these tables that in all the cases when the precision parameter increases, values of the proximity measure decreases i.e. the accuracy increases with an increase in the precision parameter. It is for this reason that this parameter is called the precision parameter.

Since all proximity measures in Table 4.2 are far less than their corresponding values in Table 4.3, the second near-exact distribution is considerably more accurate than the first near-exact distribution.

4.3.2 Cumulative probabilities

Tables 4.4, 4.5 and 4.6 further assesses the accuracy of near-exact distributions by considering their cumulative probabilities relative to cumulative probabilities from the exact and empirical distribution. For near-exact distributions, the precision parameter is set at 20.

Table 4.4: Case I cumulative probabilities from each distribution.

Case I				
z	Empirical	First near-exact	Second near-exact	Exact
-2	0.0293930	0.0259231	0.0295135	0.0295136
$-\frac{1}{4}$	0.7967345	0.8005724	0.7981589	0.7981587
0	0.8551040	0.8568766	0.8550986	0.8550984
$\frac{1}{4}$	0.8954174	0.8973874	0.8961004	0.8961003
2	0.9898200	0.9900461	0.9899205	0.9899205

Table 4.5: Case II cumulative probabilities from each distribution.

Case II				
z	Empirical	First near-exact	Second near-exact	Exact
$\frac{5}{2}$	0.007883	0.004827	0.007190	0.007193
5	0.061867	0.049257	0.061869	0.061867
$\frac{15}{2}$	0.297308	0.276796	0.292912	0.292895
10	0.689083	0.704776	0.689083	0.689101
15	0.991684	0.994032	0.991647	0.991644

Table 4.6: Case III cumulative probabilities from each distribution.

Case III				
z	Empirical	First near-exact	Second near-exact	Exact
-15	0.02906600	0.02813162	0.02864619	0.02864621
-7	0.10945100	0.10626696	0.10813867	0.10813874
0	0.33124200	0.32773952	0.33170734	0.33170690
5	0.64419900	0.64527157	0.64485522	0.64485474
10	0.92631100	0.93338995	0.92627116	0.92627268

The first near-exact distribution provides a good approximation to the exact distribution but performs poorly relative to other approximations methods. The second near-exact distribution is more accurate than the empirical distribution. It is accurate to at least the fifth decimal digit whereas the other two methods are accurate to at most three significant digit. This further supports the conclusion in Subsection 4.3.1.

4.3.3 CDF plots

This Subsection contains plots of the empirical cdf, first near-exact cdf, second near-exact cdf and exact cdf of $Z = -\log Y$.

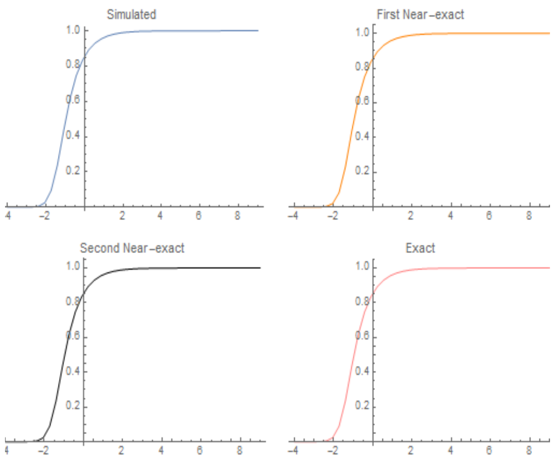

Figure 4.1: Case I cdf plots of various approximate distributions and exact distribution.

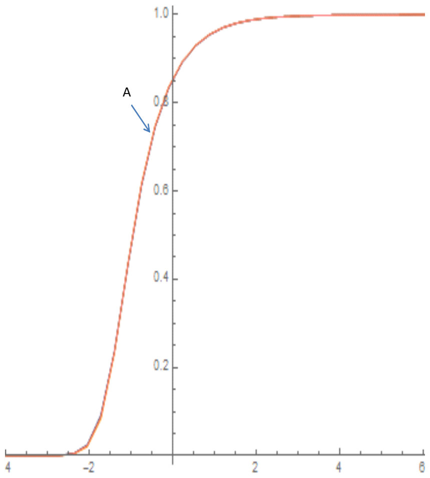

Figure 4.2: Simultaneous plots of cdf plotted on Figure 4.1.

Figure 4.1 provides cdf plots for empirical distribution, first near-exact distribution, second near exact distribution and exact distribution respectively. Figure 4.2 is a plot of the cdf of these distributions on the same set of axes.Thought it is admittedly hard, traces of colours of the first near-exact cdf and the exact cdf in Figure 4.1 can be seen in Figure 4.2. However, the colour of the second near-exact cdf in Figure 4.1 is difficult to see in Figure 4.2. This is because when plotted on the same axes, the second near-exact distribution lies almost completely on top of the exact distribution while it is not the same case with the first near-exact distribution.

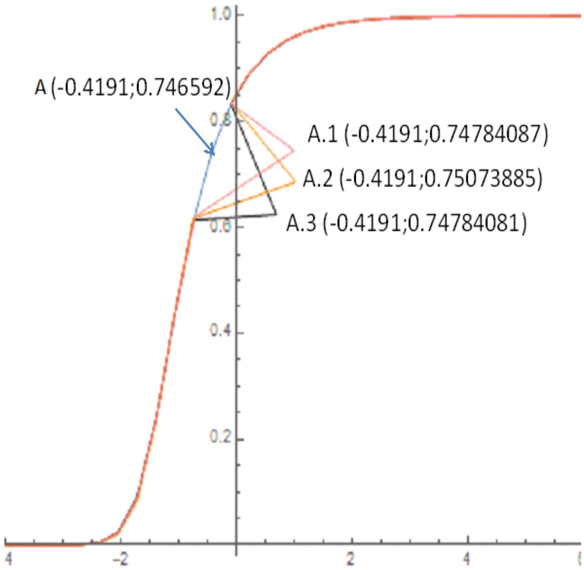

Figure 4.3: Re-plot of Figure 4.2 with point A adjusted for each cdf.

Figure 4.3 is the same plot as Figure 4.2 except point A in Figure 4.2 has been manually adjusted to help show separate cdf plots in Figure 4.3. In Figure 4.3; points A, A.1, A.2 and A.3 lie on the empirical, first near-exact, second near-exact and exact cdf plot respectively. As can be observed in Figure 4.2 (see point A), these points are so close to each other that it is hard to distinguish one from the others. Figures 4.1, 4.2, and 4.3 further emphasises the high level of accuracy of the near-exact distributions in approximating the exact distribution of Z and in turn the exact distribution of Y.

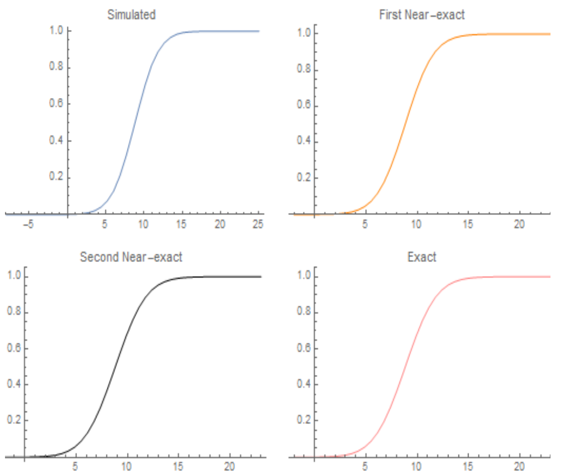

Figure 4.4: Case II cdf plots of various approximate distributions and exact distribution.

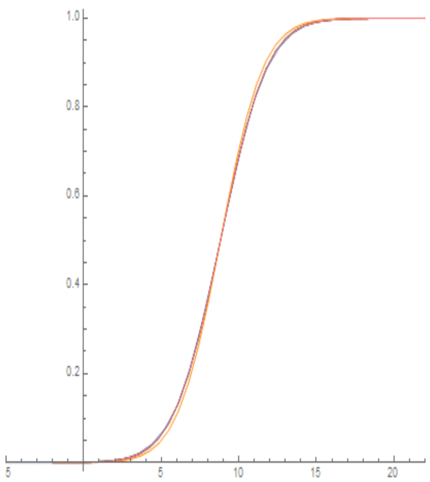

Figure 4.5: Simultaneous plots of cdf plotted on Figure 4.4.

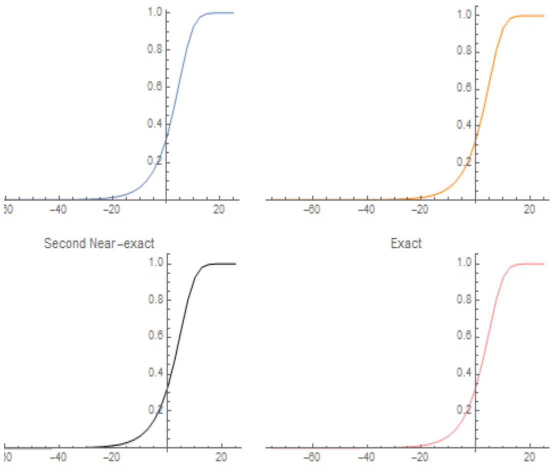

Figure 4.6: Case III cdf plots of various approximate distributions and exact distribution.

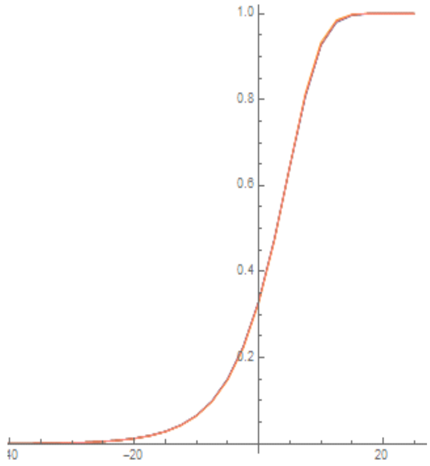

Figure 4.7: Simultaneous plots of cdf plotted on Figure 4.6.

Figures 4.4 and 4.5 and also Figures 4.6 and 4.7 paints a picture similar to that of Figures 4.1 and 4.2. Figure 4.5 provides clear evident on the difference in accuracy of the first near-exact distribution relative to the second nea-exact.distribution. Their cdf plots can clearly be identified from each other whereas the cdf plots of the exact distribution can be seen because it lies almost entirely on top of the cdf plot of the second near-exact distribution.

4.3.4 Computer run-time

(4.1) provides a means to obtain the exact distribution of Z and hence of Y. The natural question would be to ask why it is approximated when it can be obtained exactly. Table 4.1 shows the computer run-time for the exact, first near-exact and second near-exact distributions for Case II (see Table 4.1) and various precision parameter. The computations were run using a Personal Computer Intel Core i7 @ 2.4GHz. The exact distribution is not affected by the choice of the precision parameter. For near-exact distributions, the higher the precision parameter requires more computer run-time. Since the precision parameter controls the accuracy of the near-exact distribution, the accuracy of the near-exact distribution is at the cost of computer run-time. The first near-exact distribution consistently requires relatively less computer run-time than the exact and near-exact distributions. For larger values of the precision parameter, the second near-exact distribution takes longer computer run-time than the exact distribution. The precision parameter of 10 provides a good approximation (see Table 4.3) at half the computer run-time required by the exact distribution. Therefore, the required accuracy can be obtained at an efficient computer run-time by choosing a suitable value of the precision parameter.

Table 4.7: Run-time for exact and near-exact distributions.

Distribution type	Precision parameter			
	4	10	15	20
Exact	2.90625	2.90625	2.90625	2.90625
First near-exact	0.015625	0.0625	0.109375	0.171875
Second near-exact	0.671875	1.54688	2.39063	3.26563

The empirical distribution naturally provides an unstable distribution i.e. different distributional values are obtained with each simulation. The larger the simulated random sample, the more relative stable results can be obtained. However, simulating larger sample means that more computer run-time is required. In situations where stability is of importance, the required sample size to be generated and thus the required computer run-time may be unrealistic. This, combined with lower accuracy level relative to the more stable second near-exact distribution, makes the empirical distribution an unattractive means of approximating the exact distribution.

4.4 Chen's approximation

The accuracy of near-exact distribution relative to the empirical distribution in approximating the exact distribution was dealt with in Section 4.3. This section performs an assessment of the quality of Chen's approximation relative to near-exact distributions as proposed in [15] and [17]. To do the assessment, cumulative probabilities and run-times of each of the approximation method is evaluated.

Chen's approximation requires the power parameter to be the same for all generalised gamma random variables considered. As a results, the statistic of interest in this section will differ to that defined in (1.5). The statistic of interest shall be defined the same as (3.28) so that the fixed power parameter is either positive or negative. This statistic of interest, denoted by $Y_{G\Gamma}$, is just as special case of the statistic of interest which was studied by authors in [17] and [15]. [17] and [15] the power parameters are not necessarily equal but are all either positive or negative whereas in $Y_{G\Gamma}$ (see (3.23)), power parameters are assumed to be equal.

For $Y_{G\Gamma}$, the power parameter is assumed to be positive. Following an approach similar to the one in Section 3.1.1, the first near-exact distribution of $Y_{G\Gamma}$ is $SGIG\left(\ell, \underline{m}, \underline{\beta}, \phi + E\left[Z_1\right]\right)$ where

ℓ	is the number of independent exponential random variables with distinct parameter $\delta(r+k)$ for $k = 0, 2, \ldots, \gamma - 1$ and γ is the precision parameter
$\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_\ell)'$	β_j is the parameter of the j^{th} distinct exponential random variable
$\underline{m} = (m_1, m_2, \ldots, m_\ell)'$	such that m_j is the number of exponential random variables with parameter β_j
ϕ	$\sum\limits_{j=1}^{n} \log(\lambda_j)$; and
$E[Z_1]$	The expected value of the approximated random variable in the decomposition of the random variable $Z = -\log(Y_{G\Gamma})$

Also following an approach similar to the one in Section 3.1.2, the second near-exact distribution of $Y_{G\Gamma}$ is $GNIG\left(\ell, \underline{m}, \underline{\beta}, \rho, \psi, \theta + \phi\right)$ with $\ell, \underline{m}, \underline{\beta}$ and ϕ defined above and ρ, ψ and θ shifted gamma parameter. More detail of these near-exact distribution can be obtained in [15].

Table 4.8: Sets of parameters considered on Chen's approximation.

Case	r	λ	δ
I	$\{1,1,1\}$	$\{1,1,1\}$	$\{1,1,1\}$
II	$\{\frac{3}{2},\frac{3}{2}\}$	$\{\frac{3}{2},3\}$	$\{2,2\}$

For computational purposes, two cases of a set of independent gamma distributed random variables will be considered and are summrised in Table 4.8. Chen's approximation, exact- and near-exact distributions will be evaluated and contrasted against each other. For near-exact distributions, the precision parameter is set at 20.

Table 4.9: Case I cumulative probabilities from various distribution functions.

y	Exact	First near-exact	Second near-exact	Chen's
$\frac{1}{5}$	0.4797166535	0.4780258803	0.4797168322	0.4760854883
1	0.7763872468	0.7801793736	0.7763870227	0.7755972501
$\frac{3}{2}$	0.8383692748	0.8431836528	0.8383689492	0.8384876671
2	0.8761380071	0.8813553388	0.8761376475	0.8767666136
$\frac{5}{2}$	0.9015172592	0.9068328141	0.9015169018	0.9024358694

Table 4.10: Case II cumulative probabilities from various distribution functions.

y	Exact	First near-exact	Second near-exact	Chen's
$\frac{1}{5}$	0.4797166535	0.4780258803	0.4797168322	0.4760854883
1	0.7763872468	0.7801793736	0.7763870227	0.7755972501
$\frac{3}{2}$	0.8383692748	0.8431836528	0.8383689492	0.8384876671
2	0.8761380071	0.8813553388	0.8761376475	0.8767666136
$\frac{5}{2}$	0.9015172592	0.9068328141	0.9015169018	0.9024358694

Tables 4.9 and 4.10 shows cumulative probabilities from exact distribution, first near-exact distribution, second near-exact distribution and Chen's approximation. It is evident from both these tables that Chen's approximation generally provides a better approximation for the cdf of Y_{GT} (see (3.23)) than the first near-exact distribution when the precision parameter is set at 20. However, the accuracy of the first near-exact distribution can be increased by increasing the value of the precision parameter. Chen's approximation performs poorly relative to the second near-exact distribution.

Table 4.11: Run time for each approximation method

Approximation method	Run time (in seconds)	
	Case I	Case II
Chen's	0.00000000000000001	0.00000000000000001
First near-exact	0.10937500000000000	0.09375000000000000
Second near-exact	8.95313000000000000	3.81250000000000000

Table 4.11 show the amount of time each approximation method took to calculate cumulative probabilities at point $y = \frac{1}{5}$. The computations were run using a Personal Computer Intel Core i7 @ 2.4GHz. Chen's approximation is the fastest of the three approximations methods. Since Chen's is computationally faster and more accurate than the first near-exact distribution, it would be sensible to use Chen's approximation. Though the second near-exact distribution takes considerably longer time, it is by far much more accurate method. Hence when high precision is of greater necessity than of run time, the second near-exact distribution should be used.

4.5 Chapter summary

Contribution in this chapter are summarised as follows:

- The precision parameter controls the accuracy of the near-exact distributions, i.e. the accuracy of the near-exact distribution increases when the precision parameter increases (measure by Δ, see (4.2)).

- Given the accuracy and computer run-time, the second near-exact distribution is the better option compared to the first near-exact and empirical distribution.
- Chen's approximation is computationally faster than the near-exact distributions. However, it is less accurate than the second near-exact distribution.
- Chen's approximation restrict the shape and the power parameter to be fixed in all generalised gamma distributed random variable. Near-exact distributions provides flexibility on parameter choices

Chapter 5

Conclusion and future research

Of interest in this study was the statistic Y (see (1.5)) defined as the distribution of the ratio and the product of independent generalised gamma distributed random variables. Chapter 1 gave current literature of the statistic as well as its application. It was also shown in this chapter that the ratio of independent generalised gamma distributed random variables can be represented as the product of independent generalised gamma distributed random variables with some re-parametrisation i.e. at least one power parameter being negative and another be positive.

Chapter 2 presented the exact distribution of the statistic of interest in terms of Fox's H- function, special cases of Y were also considered and their exact distribution given in terms of Meijer's G- functions. Fox's H- functions are not readily computable, hence approximations of the distribution of Y were derived in Chapter 3. Also in Chapter 2, the characteristic function of $Z = -\log Y$ was decomposed so that Z can be represented as the sum of two independent random variables. In the process of decomposing the characteristic function of Z, a new flexible parameter called the precision parameter was introduced. By approximating only one of this two independent random variables and leaving the other one unchanged, the near-exact distribution of Z, and hence of Y, were derived in Chapter 3. An alternative approximation method, Chen's approximation, was also presented. However, this alternative only considered a case where the rate parameters are equal and power parameters are also equal.

Chapter 4 performed numerical studies of the approximate distributions and the exact distribution and compared them against each other. The exact distribution of Y was computed using the inversion theorem. However, this required longer computer run-time and is thus not efficient for regular use. Though providing a better degree of accuracy, empirical distributions were no more efficient in computer run-time. Chen's approximation provided a very good level of accuracy and was most efficient in computer run-time. However, unlike other appproximate methods considered in this study, it can only be applied to special cases of Y. The choice of the precision parameter controls the degree of accuracy for the near-exact distributions. The larger the precision parameter, the more accurate the near-exact distribution but the greater amount of computer run-time is required. The first near-exact distribution is less accurate than all other approximate method. However the efficiency in computer run-time relative to empirical distribution makes it relatively attractive. The second near-exact distribution is the most accurate approximate distribution presented in this study. A suitable choice of the precision parameter in the second near-exact distribution can give an optimal combination of accuracy and computer run-time.

To derive near-exact distributions of Z, the decomposition $Z = Z_1 + Z_2$ was considered. For the first near-exact distribution, Z_2 was left unchanged while Z_1 was approximated by its expected value which is a fixed value (see Section 3.1.1). The second near-exact distribution is an improvement to the first near-exact distribution and is obtained by approximating Z_1 by a single random variable. Future research can improve on the second near-exact distribution by approximating Z_1 by a weighted sum of random variables. A similar approach was considered in [17].

APPENDICES

A. Mathematical functions and expressions

This section contains mathematical results which are useful in this study.

Result 1 *([10], p.xliii)*
x factorial is defined as

$$x! = x \cdot (x-1) \cdot (x-2) \cdots 2 \cdot 1.$$

Result 2 *([10], p.xliii)*
Let $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $n \leq x$, then x combination n is defined as

$$\begin{aligned} \binom{x}{n} &= x(x-1)(x-2)\cdots(x-n+1)\frac{1}{n!} \\ &= \frac{x!}{(x-n)!n!}. \end{aligned} \tag{A.1}$$

Result 3 *([10], p.895 (8.331))*
Let $x \in \mathbb{R}^{+}$,

$$\Gamma(x+1) = x\Gamma(x). \tag{A.2}$$

Result 4 *([10], p.897 (8.339))*
Let $x \in \mathbb{N}$,

$$\Gamma(x) = (x-1)! \tag{A.3}$$

Result 5 *([10], p.xliii)*
Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the Pochhammer coefficient is defined as

$$(x)_n = x(x+1)\ldots(x+n-1) \tag{A.4}$$

$$= \frac{\Gamma(x+n)}{\Gamma(x)} \tag{A.5}$$

(A.5) can be re-arranged as

$$\Gamma(x+n) = (x)_n \Gamma(x).$$

Result 6 *([10], p.1023(9.210))*
Let $x, \alpha, \eta \in \mathbb{R}$, the confluent hypergeometric function is defined as

$$\begin{aligned} {}_1F_1(\alpha;\eta;x) &= 1 + \frac{\alpha}{\eta}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\eta(\eta+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\eta(\eta+1)(\eta+2)}\frac{x^3}{3!} + \cdots \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha)}\left[\frac{\Gamma(\eta)}{\Gamma(\eta)}\right]^{-1}\frac{x^0}{0!} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\left[\frac{\Gamma(\eta+1)}{\Gamma(\eta)}\right]^{-1}\frac{x^1}{1!} \\ &\quad + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\left[\frac{\Gamma(\eta+2)}{\Gamma(\eta)}\right]^{-1}\frac{x^2}{2!} + \cdots \end{aligned} \tag{A.6}$$

Applying (A.5) in the above equation:

$${}_1F_1(\alpha;\eta;x) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\eta)_k}\frac{x^k}{k!} \tag{A.7}$$

where $(\cdot)_k$ is the Pochhammer symbol.

Result 7 *([10], p.899 (8.350))*
Let $z \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$, the lower and upper incomplete gamma functions are respectively defined as

$$\gamma(\alpha, z) = \int_0^z e^{-t}t^{\alpha-1}dt \tag{A.8}$$

and

$$\Gamma^*(\alpha, z) = \int_z^{\infty} e^{-t}t^{\alpha-1}dt. \tag{A.9}$$

Notation $\Gamma^*(\alpha, z)$is used in order to avoid confusion with other notations in this study.

Result 8 *([10], p.900 (8.352))*
Let $z \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$, then

$$\gamma(\alpha, z) = \frac{z^\alpha}{\alpha}\,{}_1F_1(\alpha;\alpha+1;z) \tag{A.10}$$

Result 9 *([10], p.900 (8.354))*
Let $z \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$, a series representation of the incomplete gamma function defined in (A.8) is given by

$$\gamma(\alpha, z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{\alpha+k}}{k!(\alpha+k)}. \tag{A.11}$$

Result 10 *([10], p.25)*
Let $a, x \in \mathbb{R}$ and $n \in \mathbb{N}$, the binomial power series is defined as

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}. \tag{A.12}$$

Result 11 *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=1}^{n} \sum_{j=i}^{n} f(i,j) = \sum_{j=1}^{n} \sum_{i=1}^{j} f(i,j). \tag{A.13}$$

Result 12 *Let $h, r, j_1, j_2, \cdots j_h \in \mathbb{N}$. Define*

$$\Xi(h, r, j_1, j_2, \cdots j_{g-1}) = \sum_{i=1}^{h} \sum_{j_1=1}^{r_i} \sum_{j_2=1}^{j_1} \sum_{j_3=1}^{j_2} \cdots \sum_{j_{h-1}=1}^{j_{h-2}} f(h, r, j_1, j_2, \ldots, j_{h-1}) \tag{A.14}$$

where $f(h, r, j_1, j_2, \cdots j_{h-1})$ is any function depending on $h, r, j_1, j_2, \cdots j_{h-1}$.
(A.14) can be re-arranged as

$$\begin{aligned} \Xi(h, r, j_1, j_2, \cdots j_{h-1}) &= \sum_{i=1}^{h} \sum_{k=1}^{r_i} \sum_{j_1=k}^{r_i} \sum_{j_2=k}^{j_1} \sum_{j_3=k}^{j_2} \\ &\times \sum_{j_4=k}^{j_3} \cdots \sum_{j_{h-2}=k}^{j_{h-3}} f(h, r, j_1, j_2, \cdots j_{h-1}). \end{aligned} \tag{A.15}$$

Proof. Assume that $h = 4$. Then (A.14) becomes

$$\Xi(h, r, j_1, j_2, j_3) = \sum_{i=1}^{4} \sum_{j_1=1}^{r_i} \sum_{j_2=1}^{j_1} \sum_{j_3=1}^{j_2} f(4, r, j_1, j_2, j_3) \tag{A.16}$$

apply (A.13) in the last two summation of (A.16) so that

$$\Xi(4, r, j_1, j_2, j_3) = \sum_{i=1}^{4} \sum_{j_1=1}^{r_i} \sum_{j_3=1}^{j_1} \sum_{j_2=j_3}^{j_1} f(4, r, j_1, j_2, j_3). \tag{A.17}$$

Another application of (A.13) to the middle two summations in (A.17) yields

$$\Xi(4, r, j_1, j_2, j_3) = \sum_{i=1}^{4} \sum_{j_3=1}^{r_i} \sum_{j_1=j_3}^{r_i} \sum_{j_2=j_3}^{j_1} f(4, r, j_1, j_2, j_3)$$

Let $j_3 = k$ the above equation becomes

$$\Xi(4, r, j_1, j_2, j_3) = \sum_{i=1}^{4} \sum_{k=1}^{r_i} \sum_{j_1=k}^{r_i} \sum_{j_2=k}^{j_1} f(4, r, j_1, j_2, j_3).$$

The proof follows similarly for any value of $h \geqslant 2$. ■

Result 13 *([8], p.232)*
Let $h, k \in \mathbb{N}$. then

$$\sum_{i=0}^{h-1} \binom{h-1}{i} (-x)^{h-1-i} \sum_{t=0}^{k+i-1} \frac{(k+i-1)!}{t!} \frac{x^t}{\beta^{k+i-t}} = \sum_{i=0}^{k-1} \binom{k-1}{i} x^{k-1-i} \frac{(h+i-1)!}{\beta^{h+i}}. \tag{A.18}$$

Result 14 *([10], p.340 (3.351))*
Let $\lambda, \alpha \in \mathbb{R}^+$, then

$$\int_0^\infty t^{\alpha-1} \exp(-\lambda t)\, dt = \lambda^{-\alpha} \Gamma(\lambda). \tag{A.19}$$

Result 15 *([10], p.340 (3.381))*
Let $z, \lambda, \alpha \in \mathbb{R}^+$, then

$$\int_z^\infty t^{\alpha-1} \exp(-\lambda t)\, dt = \lambda^{-\alpha} \Gamma^*(\alpha, \lambda z). \tag{A.20}$$

where $\Gamma^*(\cdot)$ is defined in (A.9)

Result 16 *([10], p.346 (3.381))*
Let $\lambda, \alpha \in \mathbb{R}^+$, then

$$\int_0^z t^{\alpha-1} \exp(-\lambda t)\, dt = \lambda^{-\alpha} \gamma(\alpha, \lambda z). \tag{A.21}$$

Result 17 *([10], p.340)*
Let $\lambda \in \mathbb{R}^+$ and $\alpha \in \mathbb{N}$, then

$$\int_0^z t^{\alpha} \exp(-\lambda t)\, dt = \frac{\alpha!}{\lambda^{\alpha+1}} - \exp(z\lambda) \sum_{k=0}^{\alpha} \frac{\alpha!}{k!} \frac{z^k}{\lambda^{\alpha-k+1}}. \tag{A.22}$$

Result 18 *Let* $z, t, \lambda, \alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$,

$$\begin{aligned} & \int_z^\infty (t-z)^n t^{\alpha-1} \exp(-\lambda t)\, dt \\ = & \sum_{k=0}^n \binom{n}{k} (-z)^k \Gamma^*(n+\alpha-k, \lambda z). \end{aligned}$$

Proof. Applying (A.12) on $(t-z)^n$ it follows that

$$\begin{aligned} & \int_z^\infty (t-z)^n t^{\alpha-1} \exp(-\lambda t)\, dt \\ = & \sum_{k=0}^n \binom{n}{k} (-z)^k \int_z^\infty t^{n+\alpha-k-1} \exp(-\lambda t)\, dt. \end{aligned} \tag{A.23}$$

Using (A.20) and (A.23),

$$\begin{aligned} & \int_z^\infty (t-z)^n t^{\alpha-1} \exp(-\lambda t)\, dt \\ = & \sum_{k=0}^n \binom{n}{k} (-z)^k \Gamma^*(n+\alpha-k, \lambda t). \end{aligned} \tag{A.24}$$

■

Result 19 *(A useful representation of the cdf of the gamma distribution (see B.6))*
Let $\lambda, \alpha \in \mathbb{R}^+$, *then*

$$\int_0^z \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t)\, dt = \frac{(\lambda z)^\alpha}{\Gamma(\alpha+1)}\, {}_1F_1(\alpha; \alpha+1; -\lambda z).$$

Proof. From (A.21) it follows that

$$\begin{aligned} & \int_0^z \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t)\, dt \\ = & \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^z t^{\alpha-1} \exp(-\lambda t)\, dt \\ = & \frac{\lambda^\alpha}{\Gamma(\alpha)} \lambda^{-\alpha} \gamma(\alpha, \lambda z) \\ = & \frac{1}{\Gamma(k)} \gamma(\alpha, \lambda z). \end{aligned} \tag{A.25}$$

Using (A.10), (A.25) can be written as

$$\int_0^z \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t)\, dt$$
$$= \frac{(\lambda z)^\alpha}{\Gamma(\alpha)\,\alpha}\, {}_1F_1(\alpha, \alpha+1; -\lambda z) \qquad (A.26)$$

Applying (A.2), (A.26) becomes

$$\int_0^z \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t)\, dt$$
$$= \frac{(\lambda z)^\alpha}{\Gamma(\alpha+1)}\, {}_1F_1(\alpha; \alpha+1; -\lambda z)\,. \qquad (A.27)$$

■

Result 20 *(This is an alternative proof to the proof suggested in [5])*

$$\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t)\, dt$$
$$= (q-1)!\,(p-1)!\,(-1)^p \sum_{j=1}^{q} \frac{a_{q-j,p}}{(j-1)!} z^{j-1} (-\lambda)^{j-q-p}$$
$$+ (q-1)!\,(p-1)!\,(-1)^q \sum_{j=1}^{p} \frac{a_{p-j,q}}{(j-1)!} z^{j-1} \lambda^{j-q-p} \exp(-\lambda z)\,. \qquad (A.28)$$

Proof. Applying (A.12) to $(z-t)^{q-1}$ in the expression $\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t)\, dt$

$$\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt$$
$$= \int_0^z \sum_{j=0}^{q-1} \binom{q-1}{j} (-t)^j\, z^{q-1-j} t^{p-1} \exp(-\lambda t)\, dt$$
$$= \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j\, z^{q-1-j} \int_0^z t^{j+p-1} \exp(-\lambda t)\, dt. \qquad (A.29)$$

By applying (A.22), (A.29) becomes

$$
\begin{aligned}
&\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt \\
&= \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j\, z^{q-1-j} \\
&\times \left[\frac{(j+p-1)!}{\lambda^{j+p}} - \exp(-\lambda z) \sum_{i=0}^{j+p-1} \frac{(j+p-1)!}{i!} \frac{z^i}{\lambda^{j+p-i}} \right]. \qquad \text{(A.30)}
\end{aligned}
$$

Furthermore; (A.30) results from applying $(A.22)$. After some algebraic manipulation and noting that $(-1)^j = (-1)^{-j}$ for $j \in \mathbb{N}$, (A.30) can be written as

$$
\begin{aligned}
&\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt \\
&= \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j\, z^{q-1-j}\, (j+p-1)! k^{-j-p} + (-1)^q\, e^{-kz} \\
&\times \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^{q-j-1}\, z^{q-1-j} \sum_{i=0}^{j+p-1} \frac{(j+p-1)!}{i!} \frac{z^i}{k^{j+p-i}}. \qquad \text{(A.31)}
\end{aligned}
$$

Subsequently, use $(A.18)$ in (A.31)

$$
\begin{aligned}
&\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt \\
&= \sum_{j=0}^{q-1} \binom{q-1}{j} (-1)^j\, z^{q-1-j}\, (j+p-1)! \lambda^{-j-p} \\
&+ (-1)^q \exp(-\lambda z) \sum_{j=0}^{p-1} \binom{p-1}{j} z^{p-1-j}\, (q+j-1)! \lambda^{-j-q} \\
&= \sum_{j=0}^{q-1} \frac{(q-1)!}{j!\,(q-1-j)!} (-1)^j\, z^{q-1-j}\, (j+p-1)! \lambda^{-j-p} \\
&+ (-1)^q \exp(-\lambda z) \sum_{j=0}^{p-1} \frac{(p-1)!}{j!\,(p-1-j)!} z^{p-1-j}\, (q+j-1)! \lambda^{-j-q} \qquad \text{(A.32)}
\end{aligned}
$$

(A.32) is simplified further to obtain

$$
\begin{aligned}
&\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt \\
= &\sum_{j=0}^{q-1} \frac{(q-1)!}{j!\,(q-1-j)!} (-1)^{q-1-j}\, z^j\, (q+p-2-j)! \lambda^{j-q-p+1} \\
&+ (-1)^q \exp(-\lambda z) \sum_{j=0}^{p-1} \frac{(p-1)!}{j!\,(p-1-j)!} z^j\, (q+p-2-j)! \lambda^{j-q-p+1} \\
= &(q-1)!\,(p-1)! \\
&\times \left(\sum_{j=0}^{q-1} \frac{1}{j!} (-1)^{q-1-j}\, z^j \lambda^{j-q-p+1} \frac{(q+p-2-j)!}{(p-1)!\,(q-1-j)!} \right) + (-1)^q \\
&\times (q-1)!\,(p-1)! \left(\sum_{j=0}^{p-1} \frac{1}{j!} z^j \lambda^{j-q-p+1} \frac{(q+p-2-j)!}{(q-1)!\,(p-1-j)!} \right) \exp(-\lambda z) \\
= &(q-1)!\,(p-1)! \sum_{j=0}^{q-1} \frac{1}{j!} (-1)^{q-1-j}\, z^j \lambda^{j-q-p+1} \binom{q+p-2-j}{p-1} \\
&+ (q-1)!\,(p-1)!\,(-1)^q \sum_{j=0}^{p-1} \frac{1}{j!} z^j \lambda^{j-q-p+1} \binom{q+p-2-j}{q-1} \exp(-\lambda z) \\
= &(q-1)!\,(p-1)! \sum_{j=1}^{q} \frac{1}{(j-1)!} (-1)^{q-j}\, z^{j-1} \lambda^{j-q-p} \binom{q+p-1-j}{p-1} + (q-1)!\,(p-1)! \\
&\times (-1)^q \sum_{j=1}^{p} \frac{1}{(j-1)!} z^j \lambda^{j-q-p} \binom{q+p-1-j}{q-1} \exp(-\lambda z)\,. \qquad \text{(A.33)}
\end{aligned}
$$

Define

$$
a_{j,q} = \binom{j+q-1}{q-1} \qquad \text{(A.34)}
$$

so that (A.33) can be simplified using (A.34) as

$$
\begin{aligned}
&\int_0^z (z-t)^{q-1}\, t^{p-1} \exp(-\lambda t) dt \\
= &(q-1)!\,(p-1)!\,(-1)^p \sum_{j=1}^{q} \frac{a_{q-j,p}}{(j-1)!} z^{j-1} (-\lambda)^{j-q-p} \\
&+ (q-1)!\,(p-1)!\,(-1)^q \sum_{j=1}^{p} \frac{a_{p-j,q}}{(j-1)!} z^{j-1} \lambda^{j-q-p} \exp(-\lambda z)\,.
\end{aligned}
$$

■

Definition 1 *([22], p.195)*

The Fox's H- function is defined as

$$\begin{aligned} H(x) &= \mathbf{H}_{p,q}^{m,n}\left[x\middle|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\alpha_j s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\beta_j s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\alpha_j s\right)} x^{-s} ds. \end{aligned} \tag{A.35}$$

where

$$\begin{aligned} i &= \sqrt{-1} \\ 0 &\leqslant m \leqslant q, \\ 0 &\leqslant n \leqslant p, \\ \alpha_j &> 0 \quad \text{for} \quad j=1,2,\cdots,p \\ \beta_j &> 0 \quad \text{for} \quad j=1,2,\cdots,q, \end{aligned}$$

and $a_j\,(j=1,2,\cdots,p)$ *and* $b_j\,(j=1,2,\cdots,q)$ *are complex numbers such that no pole of* $\Gamma\left(b_j-\beta_j s\right)$ *for* $j=1,2,\cdots,m$ *coincides with any pole of* $\Gamma\left(1-a_j+\alpha_j s\right)$ *for* $j=1,2,\cdots,n$. *Furthermore* ω *is some real number such that the points*

$$s=\frac{b_j+k}{\beta_j},$$

for $j=1,2,\cdots,m$ *and* $k=0,1,\cdots$ *and the points*

$$s=\frac{a_j-1-k}{\alpha_j},$$

for $j=1,2,\cdots,n$ *and* $k=0,1,\cdots$, *lie to the right and the left of* $\mathbb{C}$, *respectively.* $\mathbb{C}$ *is the complex contour.*

Result 21 *([22], p.196)*

An important property of Fox's H- function

$$\begin{aligned} &\mathbf{H}_{p,q}^{m,n}\left[x^c\middle|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right] \\ &= \frac{1}{c}\mathbf{H}_{p,q}^{m,n}\left[x\middle|_{\left(b_1,\frac{\beta_1}{c}\right),\cdots,\left(b_q,\frac{\beta_q}{c}\right)}^{\left(a_1,\frac{\alpha_1}{c}\right),\cdots,\left(a_p,\frac{\alpha_p}{c}\right)}\right]; \qquad c>0. \end{aligned}$$

Definition 2 *([19], p.60)*
Meijer's G- function

$$G(x) = \mathbf{G}_{p,q}^{m,n}\left[x\middle|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right] = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)} x^{-s}ds, \tag{A.36}$$

where $z \neq 0$, $i = \sqrt{-1}$, $\mathbb{C}$ *is a suitable complex contour. Parameters* a_1, a_2, $\ldots$, a_p *are complex numbers such that*

$$-b_j - v \neq 1 - a_k + \alpha,$$

for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$; $v, \alpha = 0, 1, \ldots$

Result 22 *([19], p.23) If* $f(x)$ *is a real, singled valued function that is defined almost everywhere for* $x \geqslant 0$ *and is absolutely integrable over the range* $(0, \infty)$, *then the Mellin transformation of* $f(x)$ *is defined by*

$$\mathcal{M}_X(s) = E\left[X^{s-1}\right] = \int_0^\infty x^{s-1} f(x)\, dx. \tag{A.37}$$

Result 23 *([19], p.23) If* $\mathcal{M}_f(s)$ *is a Mellin transformation of* $f(x)$ *and is analytical over* $s \in \mathbb{C}$, *then the Mellin inversion integral is equal to* $f(x)$ *and given by*

$$f(x) = \frac{1}{2\pi i}\int_{\mathbb{C}} x^{-s}\mathcal{M}_f(s)\, ds. \tag{A.38}$$

where $s \in \mathbb{R}$ *and* $i = \sqrt{-1}$.

Result 24 *([18], p.194) The Mellin transformation of the product of independent random variables is the product of the Mellin transformation of the individual random variables*

Proof. Let $X_1, X_2, \cdots, X_n$ be a series of n independent random variables such that X_i has a distribution with pdf $f_{X_i}(x)$. Let

$$Y = \prod_{i=1}^{n} X_i.$$

The Mellin transformation of Y is given by

$$\begin{aligned}\mathcal{M}_Y(s) &= E\left[Y^{s-1}\right] \\ &= E\left[\left(\prod_{i=1}^{n} X_i\right)^{s-1}\right] \\ &= E\left[\prod_{i=1}^{n} X_i^{s-1}\right].\end{aligned}$$

Since $X_1, X_2, \ldots, X_n$ are independent, we have

$$\begin{aligned}\mathcal{M}_Y(s) &= \prod_{i=1}^{n} E\left[X_i^{s-1}\right] \\ &= \prod_{i=1}^{n} \mathcal{M}_{X_i}(s). \qquad \text{(A.39)}\end{aligned}$$

which completes the proof. ■

B. Statistical distributions

This Section contains statistical results that are applied in this study. For the convenience of the reader. Proofs have also been provided for some of non-standard statistical results.

Result 25 *Exponential distribution ([25], p.54).*
Suppose that a random variable X is exponentially distributed, with parameter λ, denoted by

$$X \sim EXP(\lambda).$$

Then the pdf of X is given by

$$f(x;\lambda) = \lambda \exp(-\lambda x); \qquad x > 0 \tag{B.1}$$

where $\lambda \in \mathbb{R}^+$. The characteristic function of X is given by

$$\begin{aligned} \phi_X(t) &= E[\exp(itX)] \\ &= \left(1 - \frac{it}{\lambda}\right)^{-1}. \end{aligned} \tag{B.2}$$

The Mellin transformation (see (A.37)) of X is given by

$$\begin{aligned} \mathcal{M}_X(s) &= \int_0^\infty x^{s-1} \lambda \exp(-x\lambda)\, dx \\ &= \Gamma(s)\lambda^{1-s} \int_0^\infty \frac{\lambda^s}{\Gamma(s)} x^{s-1} \exp(-x\lambda)\, dx \end{aligned}$$

The latter integral equates to a 1 so that

$$\mathcal{M}_X(s) = \frac{\Gamma(s)}{\lambda^{s-1}} \tag{B.3}$$

Result 26 *Rayleigh distribution ([25], p.138)*
Suppose that a random variable X is Rayleigh distributed, with parameter α, denoted by

$$X \sim Ra(\alpha).$$

Then the pdf of X is given by

$$f_X(x;r,\lambda) = \frac{x}{\alpha^2} \exp\left(-\left(\frac{x^2}{2\alpha^2}\right)\right); \qquad x > 0. \tag{B.4}$$

where $\alpha \in \mathbb{R}^+$.

Result 27 *Nakagami–m distribution ([12])*

Suppose that a random variable X is Nakagami–m distributed, with parameters $\Omega > 0$ and $m > 0.5$, denoted by

$$X \sim Nm(\Omega, m)$$

Then the pdf of X is given by

$$f_X(x;\Omega,m) = 2\left(\frac{m}{\Omega}\right)^m \frac{1}{\Gamma(m)} x^{2m-1} \exp\left(-\frac{m}{\Omega}x^2\right); \qquad x > 0. \tag{B.5}$$

Result 28 *Gamma distribution ([25], p.69)*

Suppose that a random variable X has a gamma distribution, with parameters r and λ, denoted by

$$X \sim \Gamma(r, \lambda).$$

Then the pdf of X is given by

$$f_X(x;r,\lambda) = \frac{\lambda(\lambda x)^{r-1}}{\Gamma(r)} \exp(-\lambda x); \qquad x > 0. \tag{B.6}$$

where $r > 0$, $\lambda > 0$. For integer values of r, the distribution of X is known as Erlang distribution with cdf given by

$$F_X(x;r,\lambda) = 1 - \sum_{i=1}^{r-1} \frac{(\lambda x)^i}{i!} \exp(-\lambda x); \qquad x > 0. \tag{B.7}$$

The s^{th} moment of X is given by

$$E[X^s] = \frac{\Gamma(s+r)}{\lambda^s \Gamma(r)} \tag{B.8}$$

and its characteristic function is given by

$$\begin{aligned} \phi_X(t) &= E[\exp(itX)] \\ &= \left(1 - \frac{it}{\lambda}\right)^{-r}. \end{aligned} \tag{B.9}$$

Using (B.8), the Mellin transformation of X is

$$\mathcal{M}_X(s) = \frac{\Gamma(r+s-1)}{\lambda^{s-1}\Gamma(r)}. \tag{B.10}$$

Result 29 *Generalised gamma distribution ([25], p.73)*

Suppose that a random variable X has a generalised gamma distribution, with parameters r, λ and δ, denoted by

$$X \sim G\Gamma(r, \lambda, \delta)$$

Then the pdf of X is given by

$$f_X(x; r, \lambda, \delta) = |\delta| \frac{\lambda^{\delta r} x^{\delta r - 1}}{\Gamma(r)} \exp\left(-(\lambda x)^{\delta}\right); \qquad x > 0 \tag{B.11}$$

where $r > 0$, $\lambda > 0$ and δ any real number. The s^{th} moment of X is given by

$$\begin{aligned} E[X^s] &= \int_{-\infty}^{\infty} x^s f_X(x, r, \lambda, \delta)\, dx \\ &= \int_{-\infty}^{\infty} x^s |\delta| \frac{\lambda^{\delta r} x^{\delta r - 1}}{\Gamma(r)} \exp\left(-(\lambda x)^{\delta}\right) dx \\ &= \lambda^{-s} \frac{\Gamma\left(r + \frac{s}{\delta}\right)}{\Gamma(r)} \int_{-\infty}^{\infty} |\delta| \lambda^{\delta\left(r + \frac{s}{\delta}\right)} \frac{x^{\left(r + \frac{s}{\delta}\right)\delta - 1}}{\Gamma\left(r + \frac{s}{\delta}\right)} \exp\left(-(\lambda x)^{\delta}\right) dx \end{aligned}$$

The latter integral equates to 1 so that

$$E[X^s] = \frac{\Gamma\left(r + \frac{s}{\delta}\right)}{\Gamma(r)\, \lambda^{-s}} \tag{B.12}$$

From (B.12) the Mellin transformation of X is given by

$$\begin{aligned} \mathcal{M}_X(s) &= E\left[X^{s-1}\right] \\ &= \frac{\Gamma\left(r + \frac{s-1}{\delta}\right)}{\Gamma(r)} \lambda^{-(s-1)} \end{aligned} \tag{B.13}$$

Result 30 *If $X \sim \Gamma(r, \lambda)$ (see (B.6)) then $V = -\log X$ has log-gamma distribution denoted by $V \sim Log\Gamma(r, \lambda)$. The characteristic function of V is given by*

$$\phi_V(t) = \frac{\Gamma(r - it)}{\Gamma(r)\, \lambda^{-it}}$$

(See [16])

Proof.

$$\begin{aligned} \phi_V(t) &= E[\exp(-it \log X)] \\ &= E\left[X^{-it}\right] \\ &= \frac{\Gamma(r - it)}{\Gamma(r)} \lambda^{it} \end{aligned} \tag{B.14}$$

which completes the proof. ■

Result 31 *If $X \sim \Gamma(r,\lambda)$ (see (B.6)) then $V = -\log X$ has log-gamma distribution denoted by $V \sim Log\Gamma(r,\lambda)$. V can be represented as an infinite sum of independent exponentially distributed random variables (see [16]). (See [16])*

Proof. Using the result from [16]

$$\Gamma(z) = \frac{1}{z}\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{z}\left(1+\frac{z}{k}\right)^{-1}; \qquad \text{for } z \in \mathbb{C}$$

we get another representation of the characteristic of V, (B.14) as

$$\begin{aligned}
\phi_V(t) &= \frac{1}{r-it}\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{r-it}\left(1+\frac{r-it}{k}\right)^{-1} \\
&\quad\times\left[\frac{1}{r}\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{r}\left(1+\frac{r}{k}\right)^{-1}\right]^{-1}\lambda^{it} \\
&= \frac{r}{r-it}\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{r-it}\left(\frac{k}{k+r-it}\right)\left(1+\frac{1}{k}\right)^{-r}\frac{k+r}{k}\exp\left(\log\left(\lambda^{it}\right)\right) \\
&= \left[\frac{r}{r-it}\exp(it\log\lambda)\right]\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{-it}\left(\frac{k+r}{k+r-it}\right) \\
&= \left\{\frac{1}{1-it\left(\frac{1}{r}\right)}\exp(it\log\lambda)\right\} \\
&\quad\times\left\{\prod_{k=1}^{\infty}\frac{1}{\left(1-it\left(\frac{1}{k+r}\right)\right)}\exp\left(-it\log\left(1+\frac{1}{k}\right)\right)\right\}.
\end{aligned}$$

Let

$$\begin{aligned}
\phi_{V_1}(t) &= \frac{1}{1-it\left(\frac{1}{r}\right)}\exp(it\log\lambda) \\
\phi_{V_2}(t) &= \prod_{k=1}^{\infty}\frac{1}{\left(1-it\left(\frac{1}{k+r}\right)\right)}\exp\left(-it\log\left(1+\frac{1}{k}\right)\right), \qquad \text{(B.15)}
\end{aligned}$$

$\phi_{V_1}(t)$ is a characteristic function of an exponential random variable with parameter r and shift $\exp(it\log\lambda)$. $\phi_{V_2}(t)$ is a characteristic function of the sum of infinite independent exponential random variables with parameter k and shift $\exp\left(-it\log\left(1+\frac{1}{k}\right)\right)$. Therefore $V = V_1 + V_2$ is the sum of infinite independent exponential random variables. ∎

Result 32 *Consider a random variable X with pdf given by*

$$f_X(x) = \begin{cases} k\mathbf{H}cx|^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}; & \text{for } x > 0 \\ 0; & \text{otherwise,} \end{cases} \tag{B.16}$$

where $H(\cdot)$ is Fox's $H-$ function defined in (A.35). $k, c, a_j\ (j = 1,2,\ldots,p)$, $\alpha_j\ (j = 1,2,\ldots,p)$, $b_j\ (j = 1,2,\ldots,p)$ and $\beta_j\ (j = 1,2,\ldots,p)$ are parameters of the distribution then X is said to follow the H-function distribution. (See [22])

Result 33 *Let $W \sim \Gamma(\rho,\delta)$ and $X \sim \Gamma(r,\lambda)$ be independent random variables where $\rho \in \mathbb{R}^+ \backslash \mathbb{N}$ and $r \in \mathbb{N}$. Define*

$$Z = W - X.$$

The pdf of Z is given by

$$f_Z(z) = \begin{cases} \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma_i(r-1-k+\rho,(\lambda+\delta)z)}{(\lambda+\delta)^{\rho+r-k-1}} & \\ \times \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z); & z \geqslant 0 \\ \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma(\rho+r-k-1)}{(\lambda+\delta)^{\rho+r-k-1}} & \\ \times \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z); & z < 0, \end{cases} \tag{B.17}$$

and the cdf of Z is given by

$$F_Z(z) = \begin{cases} \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k - \frac{\Gamma(\rho,\delta z)}{\Gamma(\rho)} + 1 & \\ \times \frac{\Gamma(i-k+\rho,(\lambda+\delta)z)}{(\lambda+\delta)^{i-k+\rho}}; & z \geqslant 0 \\ \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k & \\ \times \frac{\Gamma(i-k+\rho)}{(\lambda+\delta)^{i-k+\rho}}; & z < 0. \end{cases} \tag{B.18}$$

(see [16])

Proof. The pdf of Z will be presented first, followed by the presentation of the cdf of Z.

From the independence of W and X, the pdf of Z can be written as

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w-z) f_W(w)\, dw. \tag{B.19}$$

To evaluate (B.19), two cases will be considered i.e. $z > 0$ and $z < 0$.

Consider the first case; $z > 0$. (B.19) becomes

$$\begin{aligned} f_Z(z) &= \int_0^z \underbrace{f_X(w-z)}_{=0} f_W(w)\,dw + \int_z^\infty f_X(w-z)\,f_W(w)\,dw \\ &= \int_z^\infty f_X(w-z)\,f_W(w)\,dw. \end{aligned} \tag{B.20}$$

Using the pdf of W and X (see (B.6)), (B.20) becomes

$$f_Z(z) = \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z) \int_z^\infty (w-z)^{r-1} w^{\rho-1} \exp(-(\lambda+\delta) w)\,dw \tag{B.21}$$

(A.24) simplifies the integral in (B.21) so that (B.21) can be represented as

$$f_Z(z) = \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z) \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma_i(r-1-k+\rho, (\lambda+\delta) z)}{(\lambda+\delta)^{r-k+\rho-1}}. \tag{B.22}$$

Consider the second case; $z < 0$. (B.19) becomes.

$$f_Z(z) = \int_0^\infty f_X(w-z)\,f_W(w)\,dw, \tag{B.23}$$

using the pdf of W and X (see (B.6)), (B.23) becomes

$$f_Z(z) = \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z) \int_0^\infty (w-z)^{r-1} w^{\rho-1} \exp(-(\lambda+\delta) w)\,dw. \tag{B.24}$$

By applying (A.12), (B.24) is

$$\begin{aligned} f_Z(z) &= \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z) \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \\ &\quad \times \int_0^\infty w^{\rho+r-k-2} \exp(-(\lambda+\delta) w)\,dw. \end{aligned} \tag{B.25}$$

(A.19) simplifies the integral in (B.25) so that (B.25) can be represented as

$$f_Z(z) = \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z) \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma(\rho+r-k-1)}{(\lambda+\delta)^{\rho+r-k-1}}. \tag{B.26}$$

From (B.22) and (B.26), the pdf of Z is given by

$$f_Z(z) = \begin{cases} \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma_i(r-1-k+\rho, (\lambda+\delta) z)}{(\lambda+\delta)^{\rho+r-k-1}} \\ \times \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z); & z \geqslant 0 \\ \sum_{k=0}^{r-1} \binom{r-1}{k} (-z)^k \frac{\Gamma(\rho+r-k-1)}{(\lambda+\delta)^{\rho+r-k-1}} \\ \times \frac{\lambda^r \delta^\rho}{\Gamma(r)\Gamma(\rho)} \exp(\lambda z); & z < 0, \end{cases}$$

Similarly for the cdf of Z, two cases of z will be considered i.e $z > 0$ and $z < 0$. From the independence of W and X, the cdf of Z can be written as

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} F_X(w-z) f_W(w)\, dw \\ &= \int_{0}^{\infty} F_X(w-z) f_W(w)\, dw. \end{aligned} \quad \text{(B.27)}$$

Consider $z > 0$. Using (B.7), (B.27) becomes

$$\begin{aligned} F_Z(z) &= 1 - \int_0^z \underbrace{F_X(w-z)}_{=0} f_W(w)\, dw + \int_z^\infty F_X(w-z) f_W(w)\, dw \\ &= 1 - \int_z^\infty F_X(w-z) f_W(w)\, dw. \end{aligned} \quad \text{(B.28)}$$

Using (B.7) and (B.6), (B.28) becomes

$$\begin{aligned} F_Z(z) &= 1 - \int_z^\infty \left[1 - \sum_{i=1}^{r-1} \frac{(\lambda(w-z))^i}{i!} \exp(-\lambda(w-z))\right] \\ &\quad \times \frac{\delta^\rho}{\Gamma(\rho)} w^{\rho-1} \exp(-\delta w)\, dw \\ &= 1 - \frac{\delta^\rho}{\Gamma(\rho)} \int_z^\infty w^{\rho-1} \exp(-\delta w)\, dw \\ &\quad + \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \int_z^\infty (w-z)^i w^{\rho-1} \exp(-(\lambda+\delta) w)\, dw. \end{aligned} \quad \text{(B.29)}$$

Integral in (B.29) is simplified using (A.24) so that (B.29) becomes

$$\begin{aligned} F_Z(z) &= 1 - \frac{\Gamma_i(\rho, \delta z)}{\Gamma(\rho)} + \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \\ &\quad \times \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k \frac{\Gamma_i(i-k+\rho, (\lambda+\delta) z)}{(\lambda+\delta)^{i-k+\rho}}. \end{aligned} \tag{B.30}$$

Next, consider $z < 0$, hence $w - z > 0$. Using (B.7) and (B.6), B.27 can be written as

$$\begin{aligned} F_Z(z) &= 1 - \int_0^\infty \left[1 - \sum_{i=1}^{r-1} \frac{(\lambda(w-z))^i}{i!} \exp(-\lambda(w-z))\right] \\ &\quad \times f_W(w) \\ &= 1 - \int_0^\infty f_W(w)\, dw + \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \\ &\quad \times \int_0^\infty (w-z)^i w^{\rho-1} \exp(-(\lambda+\delta) w)\, dw. \end{aligned} \tag{B.31}$$

The former integral equates to 1. Applying (A.12) in $(w-z)^i$ in (B.31), we can rewrite (B.31) as

$$\begin{aligned} F_Z(z) &= \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} \\ &\quad (-z)^k \int_0^\infty w^{i-k+\rho-1} \exp(-(\lambda+\delta) w)\, dw. \end{aligned} \tag{B.32}$$

Adopting the integrand in (B.32) to be of the form of (B.6), if follows that

$$\begin{aligned} F_Z(z) &= \frac{\delta^\rho}{\Gamma(\rho)} e^{\lambda z} \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k \frac{\Gamma(i-k+\rho)}{(\lambda+\delta)^{i-k+\rho}} \\ &\quad \times \int_0^\infty \frac{(\lambda+\delta)^{i-k+\rho}}{\Gamma(i-k+\rho)} w^{i-k+\rho-1} \exp(-(\lambda+\delta) w)\, dw \\ &= \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k \frac{\Gamma(i-k+\rho)}{(\lambda+\delta)^{i-k+\rho}}. \end{aligned} \tag{B.33}$$

From (B.30) and (B.33), the complete cdf for z is

$$F_Z(z) = \begin{cases} \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k - \frac{\Gamma(\rho, \delta z)}{\Gamma(\rho)} + 1 \\ \times \frac{\Gamma(i-k+\rho, (\lambda+\delta) z)}{(\lambda+\delta)^{i-k+\rho}}; & z \geqslant 0 \\ \frac{\delta^\rho}{\Gamma(\rho)} \exp(\lambda z) \sum_{i=1}^{r-1} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-z)^k \\ \times \frac{\Gamma(i-k+\rho)}{(\lambda+\delta)^{i-k+\rho}}; & z < 0. \end{cases}$$

■

Result 34 *([5], p. 88) Let Y_1 and Y_2 be independent random variables such that $Y_i \sim \Gamma(r_i, \lambda_i)$ for $\lambda_i > 0$, $i = 1, 2$ and integer valued r_i. Define the random variable Z as*

$$Z = Y_1 + Y_2.$$

The pdf of Z is given by

$$f_Z(z) = \begin{cases} \lambda_1^{r_1} \lambda_2^{r_2} \sum_{i=1}^{2} (-1)^{S-r_i} \sum_{i=1}^{2} (-1)^{S-r_i} \exp(-z\lambda_i) \\ \times \sum_{j=1}^{r_i} \frac{a_{r_i-j,S-r_i}}{(j-1)!} z^{j-1} (2\lambda_i - L)^{j-S}; & \text{for } z > 0 \\ 0; & \text{for } z < 0, \end{cases} \tag{B.34}$$

where $S = r_1 + r_2$, $L = \lambda_1 + \lambda_2$ and $a_{r_i-j,S-r_i}$ defined as in (A.34). (See [5])

Proof. From the independence between Y_1 and Y_2 and (A.28), the pdf of Z is

$$\begin{aligned} f_Z(z) &= \int_0^z f_{Y_1}(y_1) f_{Y_2}(z-y_1) \, dy_1 \\ &= \int_0^z \frac{\lambda_1 (\lambda_1 y_1)^{r_1-1}}{\Gamma(r_1)} \exp(-\lambda_1 y_1) \frac{\lambda_2 (\lambda_2 (z-y_1))^{r_2-1}}{\Gamma(r_2)} \\ &\quad \exp(-\lambda_1 (z-y_1)) \, dy_1. \end{aligned} \tag{B.35}$$

By using (A.3), (B.35) can be written as

$$\begin{aligned} f_Z(z) &= \frac{\lambda_1^{r_1} \lambda_2^{r_2} \exp(-\lambda_2 z)}{(r_1-1)!(r_2-1)!} \\ &\quad \times \int_0^z (z-y_1)^{r_2-1} y_1^{r_1-1} \exp(-(\lambda_1-\lambda_2) y_1) \, dy_1. \end{aligned} \tag{B.36}$$

The integral in (B.36) is simplified using (A.28) so that

$$\begin{aligned} f_Z(z) &= \frac{\lambda_1^{r_1}\lambda_2^{r_2}\exp(-\lambda_2 z)}{(r_1-1)!\,(r_2-1)!}(r_1-1)!\,(r_2-1)! \\ &\quad\times(-1)^{r_1}\sum_{j=1}^{r_2}\frac{a_{r_2-j,r_1}}{(j-1)!}z^{j-1}\left(-(\lambda_1-\lambda_2)\right)^{j-r_1-r_2} \\ &\quad+\frac{\lambda_1^{r_1}\lambda_2^{r_2}\exp(-\lambda_2 z)}{(r_1-1)!\,(r_2-1)!}(r_1-1)!\,(r_2-1)! \\ &\quad\times(-1)^{r_2}\sum_{j=1}^{r_1}\frac{a_{r_1-j,r_2}}{(j-1)!}z^{j-1}(\lambda_1-\lambda_2)^{j-r_1-r_2}\exp\left(-(\lambda_1-\lambda_2)z\right) \end{aligned} \tag{B.37}$$

(B.37) is simplified further to obtain

$$\begin{aligned} f_Z(z) &= \lambda_1^{r_1}\lambda_2^{r_2}(-1)^{r_1}\sum_{j=1}^{r_2}\frac{a_{r_2-j,r_1}}{(j-1)!}z^{j-1}\left(-(\lambda_1-\lambda_2)\right)^{j-r_1-r_2}\exp(-\lambda_2 z) \\ &\quad+\lambda_1^{r_1}\lambda_2^{r_2}(-1)^{r_2}\sum_{j=1}^{r_1}\frac{a_{r_1-j,r_2}}{(j-1)!}z^{j-1}(\lambda_1-\lambda_2)^{j-r_1n-r_2}\exp(-\lambda_1 z) \\ &= \lambda_1^{r_1}\lambda_2^{r_2}(-1)^{(r_1+r_2)-r_2}\sum_{j=1}^{r_2}\frac{a_{r_2-j,(r_1+r_2)-r_2}}{(j-1)!}z^{j-1} \\ &\quad\times\left(2\lambda_2-(\lambda_1-\lambda_2)\right)^{j-r_1-r_2}\exp(-\lambda_2 z) \\ &\quad+\lambda_1^{r_1}\lambda_2^{r_2}(-1)^{(r_1+r_2)-r_1}\sum_{j=1}^{r_1}\frac{a_{r_1-j,(r_1+r_2)-r_1}}{(j-1)!}z^{j-1} \\ &\quad\times\left(2\lambda_1-(\lambda_1-\lambda_2)\right)^{j-r_1-r_2}\exp(-\lambda_1 z)\,. \end{aligned}$$

Let $S = r_1 + r_2$ and $L = \lambda_1 + \lambda_2$

$$f_Z(z) = \begin{cases} \lambda_1^{r_1}\lambda_2^{r_2}\sum\limits_{i=1}^{2}(-1)^{S-r_i}\sum\limits_{i=1}^{2}(-1)^{S-r_i}\exp(-z\lambda_i) & \\ \times\sum\limits_{j=1}^{r_i}\frac{a_{r_i-j,S-r_i}}{(j-1)!}z^{j-1}(2\lambda_i-L)^{j-S}\,; & \text{for } z>0 \\ 0; & \text{for } z<0. \end{cases}$$

■

Result 35 *Let*

$$Y_i \sim \Gamma(r_i, \lambda_i) \qquad i = 1, 2, 3..., g \geqslant 2$$

be independent gamma random variables. Then, if $r_i\,(i = 1, 2, 3..., g)$ *are integers, the pdf of* $Z = Y_1 + Y_2 + \ldots + Y_g$ *is given by*

$$\begin{aligned} f_Z(z) &= K \sum_{i=1}^{g} (-1)^{S-r_i} \sum_{j_1=1}^{r_i} a_{r_i - j_1, r_1^{*i}} \left(\lambda_i - \lambda_1^{*i}\right)^{j_1 - r_i - r_1^*} \\ &\quad \times \sum_{j_2=1}^{j_1} a_{j_1 - j_2, r_2^{*i}} \left(\lambda_i - \lambda_2^{*i}\right)^{j_2 - j_1 - r_2^*} \cdots \sum_{j_{g-1}=1}^{j_{g-2}} \frac{a_{j_{g-2} - j_{g-1}, r_{g-1}^{*i}}}{(j_{g-1} - 1)!} \\ &\quad \left(\lambda_i - \lambda_{g-1}^{*i}\right)^{j_{g-1} - j_{g-2} - r_{g-1}^*} z^{j_{g-1} - 1} \exp(-\lambda_i z), \end{aligned} \tag{B.38}$$

for $z \geqslant 0$*, with*

$$K = \prod_{i=1}^{g} \lambda_i^{r_i}, \tag{B.39}$$

$$S = \sum_{i=1}^{g} r_i,$$

and a_{r^*} *are defined as in (A.34).* λ_j^{*i} *is the* j^{th} *element of the set* $\{\lambda_1, \lambda_2, \cdots \lambda_g\} \setminus \{\lambda_i\}$ *and* r_j^{*i} *is the* j^{th} *element of set* $\{r_1, r_2, \cdots, r_g\} \setminus \{r_i\}$ *where "\" denote set difference.* Z *is said to follow the generalised integer gamma with parameters* $g, \underline{r} = (r_1, r_2, \cdots, r_g)'$ *and* $\underline{\lambda} = (\lambda_1, \lambda_2, \cdots \lambda_g)'$*, denoted by*

$$Z \sim GIG(g, \underline{r}, \underline{\lambda}).$$

(See [5])

Proof. Result 34 obtains the distribution of $Z_1 = Y_1 + Y_2$. For $g = 3$, define $Z_2 = Y_1 + Y_2 + Y_3 = Z_1 + Y_3$. The pdf of Z_1 is given by (B.34)

$$\begin{aligned} f_{Z_1}(z_1) &= K \sum_{i=1}^{2} (-1)^{S^* - r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i - j_1, S^* - r_i}}{(j_1 - 1)!} \\ &\quad \times z^{j_1 - 1} (2\lambda_i - L)^{j_1 - r_1 - r_2} \exp(-\lambda_i z_1), \end{aligned} \tag{B.40}$$

where $S^* = \sum\limits_{i=1}^{2} r_i$ and $K = \lambda_1^{r_1} \lambda_2^{r_2}$ so that pdf of Z_2 is given by

$$f_{Z_2}(z_2) = \int_0^{z_2} f_{Z_1}(z_1)\, f_{Y_3}(z_2 - z_1)\, d_{z_1}. \tag{B.41}$$

First consider $f_{Z_1}(z_1)\, f_{Y_3}(z_2 - z_1)$ above. Using (B.40), (B.6) and the independence between Z_1 and Y_3, it follows that

$$
\begin{aligned}
& f_{Z_1}(z_1) f_{Y_3}(z_2 - z_1) \\
= \; & K \sum_{i=1}^{2} (-1)^{S^*-r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i-j_1,S^*-r_i}}{(j_1-1)!} z^{j_1-1} (2\lambda_i - L)^{j_1-r_1-r_2} \\
& \times \exp(-\lambda_i z) \frac{\lambda_3 (\lambda_3 (z_2 - z_1))^{r_3-1}}{\Gamma(r_3)} \exp(-\lambda_3 (z_2 - z_1)) \\
= \; & K \frac{\lambda_3^{r_3}}{\Gamma(r_3)} \sum_{i=1}^{2} (-1)^{S^*-r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i-j_1,S^*-r_i}}{(j_1-1)!} (2\lambda_i - L)^{j_1-r_1-r_2} \\
& \times \exp(-\lambda_3 z_2) z_1^{j_1-1} (z_2 - z_1)^{r_3-1} \exp(-(\lambda_i - \lambda_3) z_1), \qquad \text{(B.42)}
\end{aligned}
$$

Substituting (B.42) into (B.41), the latter can be written as

$$
\begin{aligned}
f_{Z_2}(z_2) &= \int_0^z f_{Z_1}(z_1) f_{Y_3}(z_2 - z_1) d_{z_1} \\
&= K \frac{\lambda_3^{r_3}}{\Gamma(r_3)} \sum_{i=1}^{2} (-1)^{S^*-r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i-j_1,S-r_i}}{(j_1-1)!} (2\lambda_i - L)^{j_1-r_1-r_2} \\
& \quad \times \exp(-\lambda_3 z_2) \int_0^z z_1^{j_1-1} (z_2 - z_1)^{r_3-1} e^{-(\lambda_i-\lambda_3)z_1} d_{z_1}. \qquad \text{(B.43)}
\end{aligned}
$$

The integral in (B.43) is simplified using (A.28) to obtain the following representation of $f_{Z_2}(z_2)$

$$
\begin{aligned}
f_{Z_2}(z_2) &= K \frac{\lambda_3^{r_3}}{\Gamma(r_3)} \sum_{i=1}^{2} (-1)^{S^*-r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i-j_1,S^*-r_i}}{(j_1-1)!} (2\lambda_i - L)^{j_1-r_1-r_2} \exp(-\lambda_3 z_2) \\
& \quad \times (j_1-1)! (r_3-1)! (-1)^{j_1} \sum_{j_2=1}^{r_3} \frac{a_{r_3-j_2,j_1}}{(j_2-1)!} z_2^{j_2-1} (\lambda_3 - \lambda_i)^{j_2-r_3-j_1} \\
& \quad + K \frac{\lambda_3^{r_3}}{\Gamma(r_3)} \sum_{i=1}^{2} (-1)^{S^*-r_i} \sum_{j_1=1}^{r_i} \frac{a_{r_i-j_1,S^*-r_i}}{(j_1-1)!} (2\lambda_i - L)^{j_1-r_1-r_2} \exp(-\lambda_3 z_2) \\
& \quad \times (j_1-1)! (r_3-1)! (-1)^{r_3} \sum_{j_2=1}^{j_1} \frac{a_{j_1-j_2,r_3}}{(j_2-1)!} z^{j_2-1} \\
& \quad \times (\lambda_i - \lambda_3)^{j_2-r_3-j_1} \exp(-(\lambda_i - \lambda_3) z_2). \qquad \text{(B.44)}
\end{aligned}
$$

Using (A.3) and the fact that $(2\lambda_i - L)^{j_1-r_1-r_2} (-1)^{j_1} = (L - 2\lambda_i)^{j_1-r_1-r_2} (-1)^{r_1+r_2}$

$$\begin{aligned} f_{Z_2}(z_2) &= K\lambda_3^{r_3}(-1)^{r_1+r_2}\sum_{i=1}^{2}(-1)^{S^*-r_i}\sum_{j_1=1}^{r_i}a_{r_i-j_1,S-r_i}(L-2\lambda_i)^{j_1-r_1-r_2}\exp(-\lambda_3 z_2) \\ &\quad\times\sum_{j_2=1}^{r_3}\frac{a_{r_3-j_2,j_1}}{(j_2-1)!}z_2^{j_2-1}(\lambda_3-\lambda_i)^{j_2-r_3-j_1}+K\lambda_3^{r_3}\sum_{i=1}^{2}S_i^* \\ &\quad\times\sum_{j_1=1}^{r_i}a_{r_i-j_1,S^*-r_i}(2\lambda_i-L)^{j_1-r_1-r_2}(-1)^{r_3}\sum_{j_2=1}^{j_1}\frac{a_{j_1-j_2,r_3}}{(j_2-1)!} \\ &\quad\times z^{j_2-1}(\lambda_i-\lambda_3)^{j_2-r_3-j_1}\exp(-\lambda_i z_2) \\ &= K\lambda_3^{r_3}(-1)^{r_1+r_2}\sum_{j_1=1}^{r_3}a_{r_3-j_1,r_1}(\lambda_3-\lambda_1)^{j_1-r_1-r_3}\sum_{j_2=1}^{j_1}\frac{a_{j_1-j_2,r_2}}{(j_2-1)!} \\ &\quad\times z_2^{j_2-1}(\lambda_3-\lambda_2)^{j_2-r_2-j_1}\exp(-\lambda_3 z_2) \\ &\quad+K\lambda_3^{r_3}\sum_{i=1}^{2}(-1)^{r_i+r_3}\sum_{j_1=1}^{r_i}a_{r_i-j_1,S^*-r_i}(2\lambda_i-L)^{j_1-r_1-r_2} \\ &\quad\times\sum_{j_2=1}^{j_1}\frac{a_{j_1-j_2,r_3}}{(j_2-1)!}z^{j_2-1}(\lambda_i-\lambda_3)^{j_2-r_3-j_1}\exp(-\lambda_i z_2). \end{aligned}$$

Assume that elements in a set are ordered in chronological order of their subscripts i.e. element associated with smallest subscript is taken as the first elements and so on. In the above derivations, $\lambda_i \in \{\lambda_1,\lambda_2\}$. Therefore, set $\{\lambda_1,\lambda_2\}$ is an intersection of mutually exclusive sets $\{\lambda_i\}$ and $\{\lambda_1,\lambda_2\}\setminus\{\lambda_i\}$ with the latter set having only one element as well. We can therefore express $2\lambda_i - L$ as the difference between λ_i and the (first) element in set $\{\lambda_1,\lambda_2\}\setminus\{\lambda_i\}$. $\lambda_3-\lambda_1$ is the difference between λ_3 and the first element of $\{\lambda_1,\lambda_2,\lambda_3\}\setminus\{\lambda_3\}$. Similarly statements can be made with reference to set $\{r_1,r_2,r_3\}$. Let λ_j^{*i} and r_j^{*i} denote the j^{th} element of set $\{\lambda_1,\lambda_2,\lambda_3\}\setminus\{\lambda_j\}$ and $\{r_1,r_2,r_3\}\setminus\{r_j\}$ respectively. Furthermore, let $S=\sum_{i=1}^{3}r_i$, $S_i=(-1)^{S-r_i}$ and $K=\lambda_1^{r_1}\lambda_2^{r_2}\lambda_3^{r_3}$, then the pdf of Z_2 can be written as

$$\begin{aligned} f_{Z_2}(z_2) &= K\sum_{i=1}^{3}(-1)^{S-r_i}\sum_{j_1=1}^{r_i}a_{r_i-j_1,r_1^{*i}}\left(\lambda_i-\lambda_1^{*i}\right)^{j_1-r_i-r_1^{*i}} \\ &\quad\times\sum_{j_2=1}^{j_1}\frac{a_{j_1-j_2,r_2^{*i}}}{(j_2-1)!}\left(\lambda_i-\lambda_2^{*i}\right)^{j_2-j_1-r_2^{*i}}z_2^{j_2-1}\exp(-\lambda_i z_2). \end{aligned}$$

Similarly, pdf of $Z=Y_1+Y_2+\ldots+Y_g$ for $g\geqslant 2$ can be derived as

$$
\begin{aligned}
f_Z(z) &= K\sum_{i=1}^{g}(-1)^{S-r_i}\sum_{j_1=1}^{r_i}a_{r_i-j_1,r_1^{*i}}\left(\lambda_i-\lambda_1^{*i}\right)^{j_1-r_i-r_1^{*}}\sum_{j_2=1}^{j_1}a_{j_1-j_2,r_2^{*i}}\\
&\times\left(\lambda_i-\lambda_2^{*i}\right)^{j_2-j_1-r_2^{*i}}\cdots\sum_{j_{g-1}=1}^{j_{g-2}}\frac{a_{j_{g-2}-j_{g-1},r_{g-1}^{*i}}}{(j_{g-1}-1)!}\left(\lambda_i-\lambda_{g-1}^{*i}\right)^{j_{g-1}-j_{g-2}-r_{g-1}^{*}}\\
&\times z^{j_{g-1}-1}\exp\left(-\lambda_i z\right)\\
&= K\sum_{i=1}^{g}(-1)^{S-r_i}\sum_{j_1=1}^{r_i}\sum_{j_2=1}^{j_1}\cdots\sum_{j_{g-1}=1}^{j_{g-2}}a_{r_i-j_1,r_1^{*i}}\left(\lambda_i-\lambda_1^{*i}\right)^{j_1-r_i-r_1^{*i}}a_{j_1-j_2,r_2^{*i}}\\
&\times\left(\lambda_i-\lambda_2^{*i}\right)^{j_2-j_1-r_2^{*i}}\cdots\frac{a_{j_{g-2}-j_{g-1},r_{g-1}^{*i}}}{(j_{g-1}-1)!}\left(\lambda_i-\lambda_{g-1}^{*i}\right)^{j_{g-1}-j_{g-2}-r_{g-1}^{*i}}\\
&z^{j_{g-1}-1}\exp\left(-\lambda_i z\right). \qquad \text{(B.45)}
\end{aligned}
$$

where $S=\sum_{i=1}^{g}r_i$, $S_i=(-1)^{S-r_i}$, $K=\prod_{i=1}^{g}\lambda_i^{r_i}$ and $a_{r^{*i}}$ are defined as in $(A.34)$. ■

If $\theta\in\mathbb{R}$ is a shift parameter then $Z+\theta\sim SGIG(g,\underline{r},\underline{\lambda},\theta)$. Next, the computational form of (B.45) is obtained
Using $(A.15)$ with $j_{g-1}=k$, $(B.38)$ can be written as

$$
\begin{aligned}
f_Z(z) &= K\sum_{i=1}^{g}(-1)^{S-r_i}\sum_{k=1}^{r_i}\sum_{j_1=k}^{r_i}\sum_{j_2=k}^{j_1}\sum_{j_3=k}^{j_2}\cdots\sum_{j_{g-2}=k}^{j_{g-3}}a_{r_i-j_1,r_1^{*i}}\left(\lambda_i-\lambda_1^{*i}\right)^{j_1-r_i-r_1^{*i}}a_{j_1-j_2,r_2^{*i}}\\
&\times\left(\lambda_i-\lambda_2^{*i}\right)^{j_2-j_1-r_2^{*i}}\cdots\frac{a_{j_{g-2}-k,r_{g-1}^{*i}}}{(k-1)!}\left(\lambda_i-\lambda_{g-1}^{*i}\right)^{k-j_{g-2}-r_{g-1}^{*i}}z^{k-1}\exp\left(-\lambda_i z\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
\alpha_{j_1} &= a_{r_i-j_1,r_1^{*i}}\left(\lambda_i-\lambda_1^{*i}\right)^{j_1-r_i-r_1^{*i}}\\
\beta_{j_t} &= a_{j_{t-1}-j_t,r_t^{*i}}\left(\lambda_i-\lambda_t^{*i}\right)^{j_t-j_{t-1}-r_t^{*i}} \qquad t=2,3,4,\ldots,g-2\\
\delta_k &= \frac{a_{j_{g-2}-k,r_{g-1}^{*i}}}{(k-1)!}\left(\lambda_i-\lambda_{g-1}^{*i}\right)^{k-j_{g-2}-r_{g-1}^{*i}}.
\end{aligned}
$$

Therefore $f_Z(z)$ can be written in the form

$$
\begin{aligned}
f_Z(z) &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i}\left((-1)^{S-r_i}\sum_{j_1=k}^{r_i}\sum_{j_2=k}^{j_1}\sum_{j_3=k}^{j_2}\cdots\sum_{j_{g-2}=k}^{j_{g-3}}\alpha_{j_1}\delta_k\prod_{t=2}^{g-2}\beta_{j_t}\right)\\
&\times z^{k-1}\exp\left(-\lambda_i z\right).
\end{aligned}
$$

Furthermore, let

$$\begin{aligned}c_{i,k}\left(g,\underline{r},\underline{\lambda}\right) &= \left((-1)^{S-r_i}\sum_{j_1=k}^{r_i}\sum_{j_2=k}^{j_1}\sum_{j_3=k}^{j_2}\cdots\sum_{j_{g-2}=k}^{j_{g-3}}\alpha_{j_1}\delta_k\prod_{t=2}^{g-2}\beta_{j_t}\right) \quad \text{and} \\ P_i\left(z\right) &= \sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right) z^{k-1},\end{aligned} \tag{B.46}$$

Computational form of (B.46) is given by (B.50) and (B.51) below. Hence

$$f_Z\left(z\right) = K\sum_{i=1}^{g} P_i\left(z\right)\exp\left(-\lambda_i z\right). \tag{B.47}$$

The cdf of Z is derived as

$$\begin{aligned}F_Z\left(z\right) &= \int_0^z f_Z\left(t\right)dt \\ &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right)\int_0^z t^{k-1}\exp\left(-\lambda_i t\right)dt \\ &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right)\frac{\Gamma\left(k\right)}{\lambda_i^k}\int_0^z \frac{\lambda_i^k}{\Gamma\left(k\right)}t^{k-1}\exp\left(-\lambda_i t\right)dt \\ &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right)\frac{\Gamma\left(k\right)}{\lambda_i^k}F_{Z_{ik}}\left(z\right),\end{aligned} \tag{B.48}$$

where $F_{Z_{ik}}\left(z\right)$ is the cdf of a random variable $Z_{ik} \sim \Gamma\left(k,\lambda_i\right)$. Further note that

$$\begin{aligned}1 &= \int_0^\infty f_Z\left(z\right)dz \\ &= \int_0^\infty K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right) z^{k-1}\exp\left(-\lambda_i z\right)dz \\ &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right)\int_0^\infty z^{k-1}\exp\left(-\lambda_i z\right)dz \\ &= K\sum_{i=1}^{g}\sum_{k=1}^{r_i} c_{i,k}\left(g,\underline{r},\underline{\lambda}\right)\frac{\Gamma\left(k\right)}{\lambda_i^k}\int_0^\infty \frac{\lambda_i^k}{\Gamma\left(k\right)}z^{k-1}\exp\left(-\lambda_i z\right)dz.\end{aligned}$$

The latter integral equates to 1 so that

$$1 = K \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \frac{\Gamma\left(k\right)}{\lambda_i^k},$$

hence

$$\left(K\right)^{-1} = \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \frac{\Gamma\left(k\right)}{\lambda_i^k}. \tag{B.49}$$

Note that (B.46) in its current form is not easy to compute. The computation form of (B.46) can be found in [5] and is given by:

$$c_{i,r_i}\left(g, \underline{r}, \underline{\lambda}\right) = \frac{1}{\left(r_i - 1\right)!} \prod_{j=1, j\neq i}^{g} \left(\lambda_j - \lambda_i\right)^{-r_j}, \tag{B.50}$$

and for $k = 0,\ 1,\ 2\ , \ldots, r_i - 1$

$$\begin{aligned} c_{i,r_i-k}\left(g, \underline{r}, \underline{\lambda}\right) &= \frac{1}{k} \sum_{j=1}^{k} \frac{\left(r_i - k + j - 1\right)!}{\left(r_i - k - 1\right)!} R\left(j-1, i, g, \underline{r}, \underline{\lambda}\right) \\ &\quad \times c_{i,r_i-(k-j)}\left(g, \underline{r}, \underline{\lambda}\right), \end{aligned} \tag{B.51}$$

where

$$R\left(n, i, g, \underline{r}, \underline{\lambda}\right) = \sum_{i=1, i\neq j}^{g} r_i \left(\lambda_j - \lambda_i\right)^{-n-1}, \quad n = 0, 1, 2, \ldots, r_i - 1.$$

Result 36 *(Generalised near-integer gamma with only two variables). Let*

$$\begin{aligned} W &\sim \Gamma\left(\rho, \delta\right) \quad \text{and} \\ X_1 &\sim \Gamma\left(r_1, \lambda_1\right), \end{aligned}$$

where $\delta, \lambda_1 \in \mathbb{R}^+$, $r_1 \in \mathbb{N}$ and ρ is positive non-integer number. Define

$$Z = X_1 + W.$$

The cdf is given by

$$\begin{aligned} F_Z\left(z\right) &= \frac{\left(\delta z\right)^{\rho}}{\Gamma\left(\rho + 1\right)}\ {}_1F_1\left(\rho; \rho + 1; -\delta z\right) - \exp\left(-\lambda_1 z\right) \\ &\quad \times \sum_{i=0}^{r_1 - 1} \frac{\delta^{\rho} \lambda_1^i z^{i+\rho}}{\Gamma\left(i + \rho + 1\right)}\ {}_1F_1\left(\rho; i + \rho + 1; -\left(\delta - \lambda_1\right) z\right); \quad z > 0, \end{aligned} \tag{B.52}$$

where ${}_1F_1\left(\cdot\right)$ is defined as (A.7)

Result 36 is a special case of Result 37. It is included in this study for the first time and will be used to give and alternative proof of Result 37 presented in [6].

Proof. Using independence between X_1 and W and their respective pdf, the cdf of Z is given by

$$\begin{aligned} F_Z(z) &= \int_0^z F_{X_1}(z-w)\, f_W(w)\, dw \\ &= \int_0^z \left(1 - \sum_{i=0}^{r_1-1} \frac{(\lambda_1 (z-w))^i}{i!} \exp(-\lambda_1 (z-w))\right) \\ &\quad \times \frac{\delta^\rho w^{\rho-1}}{\Gamma(\rho)} \exp(-\delta w)\, dw \\ &= \int_0^z \frac{\delta^\rho w^{\rho-1}}{\Gamma(\rho)} \exp(-\delta w)\, dw - \sum_{i=0}^{r_1-1} \frac{\lambda_1^i \delta^\rho}{i!\Gamma(\rho)} \exp(-\lambda_1 z) \\ &\quad \times \int_0^z (z-w)^i\, w^{\rho-1} \exp(-(\delta-\lambda_1) w)\, dw. \end{aligned}$$

Using ([10] p.347 (3.383)), it follows that

$$\begin{aligned} &\int_0^z (z-w)^i\, w^{\rho-1} \exp(-(\rho-\lambda_1) w)\, dw \\ &= \frac{i!\Gamma(\rho)}{\Gamma(i+\rho+1)} z^{i+\rho}\ {}_1F_1(\rho; i+\rho+1; -(\delta-\lambda_1) z), \end{aligned}$$

then

$$\begin{aligned} &F_Z(z) \\ &= F_W(w) - \exp(-\lambda_1 z) \sum_{i=0}^{r_1-1} \frac{\lambda_1^i \delta^\rho z^{i+\rho}}{\Gamma(i+\rho+1)} \\ &\quad \times ({}_1F_1(\rho; i+\rho+1; -(\delta-\lambda_1) z)). \end{aligned}$$

Using (A.27), $F_W(w)$ can be written as

$$F_W(w) = \frac{(\delta z)^\rho}{\Gamma(\rho+1)}\ {}_1F_1(\rho, \rho+1, -\delta z),$$

therefore

$$\begin{aligned} F_Z(z) &= \frac{(\delta z)^\rho}{\Gamma(\rho+1)}\ {}_1F_1(\rho; \rho+1; -\delta z) - \exp(-\lambda_1 z) \\ &\quad \times \sum_{i=0}^{r_1-1} \frac{\lambda_1^i \delta^\rho z^{i+\rho}}{\Gamma(i+\rho+1)}\ {}_1F_1(\rho; i+\rho+1; -(\delta-\lambda_1) z). \end{aligned}$$

■

Result 37 *(Generalised near-integer gamma with at least two variables) Let V and W be independent random variables such that $V \sim GIG\left(g, \underline{r}, \underline{\lambda}\right)$ where $\underline{r} = (r_1, r_2 \cdots r_g)$ and $\underline{\lambda} = (\lambda_1, \lambda_2 \cdots \lambda_g)$ and $W \sim \Gamma\left(\rho, \delta\right)$ where ρ is non-integer. Define the random variable*

$$Z = V + W.$$

then the random variable Z follows a generalised near integer gamma distribution denoted by

$$Z \sim GNIG\left(g, \underline{r}, \underline{\lambda}, \rho, \delta\right).$$

The pdf of Z is given by

$$\begin{aligned} f_Z\left(z\right) &= K\delta^\rho \sum_{i=1}^{g} \exp\left(-z\lambda_i\right) \sum_{k=1}^{r_i} c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \frac{\Gamma\left(k\right)}{\Gamma\left(k+\rho\right)} \\ &\quad \times z^{k+\rho-1}\ {}_1F_1\left(\rho; k+\rho; -\left(\delta-\lambda_i\right) z\right); \qquad z > 0, \end{aligned} \tag{B.53}$$

and the cdf of Z is given by

$$\begin{aligned} F_Z\left(z\right) &= \frac{\left(\delta z\right)^\rho}{\Gamma\left(\rho+1\right)}\ {}_1F_1\left(\rho, \rho+1, -\delta z\right) - K\delta^\rho \sum_{i=1}^{g} \sum_{k=1}^{r_i} c^*_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \\ &\quad \times \exp\left(-\lambda_i z\right) \sum_{j=0}^{k-1} \frac{\lambda_i^j z^{j+\rho}}{\Gamma\left(j+\rho+1\right)}\ {}_1F_1\left(\rho; j+\rho+1; -\left(\delta-\lambda_i\right) z\right), \end{aligned} \tag{B.54}$$

for $z > 0$, where K is defined as in (B.39), $c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right)$ is defined as in (B.46) and ${}_1F_1\left(\cdot\right)$ is defined as (A.7). Furthermore $c^*_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) = c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \frac{\Gamma(k)}{\lambda_i^k}$. (See [6] for further details.)

Proof. Given the independent between V and W, we can use the pdf of V in (B.47) and pdf of in W in (B.6) to get the pdf of Z as

$$\begin{aligned} f_Z\left(z\right) &= \int_0^z f_Y\left(z-w\right) f_W\left(w\right) dw \\ &= \int_0^z \left[K \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \left(z-w\right)^{k-1} \exp\left(-\left(z-w\right)\lambda_i\right)\right] \\ &\quad \times \frac{\delta^\rho w^{\rho-1}}{\Gamma\left(\rho\right)} \exp\left(-\delta w\right) dw \\ &= K \frac{\delta^\rho}{\Gamma\left(\rho\right)} \sum_{i=1}^{g} \exp\left(-z\lambda_i\right) \sum_{k=1}^{r_i} c_{i,k}\left(g, \underline{r}, \underline{\lambda}\right) \\ &\quad \times \int_0^z \left(z-w\right)^{k-1} w^{\rho-1} \exp\left(-\left(\delta-\lambda_i\right) w\right) dw, \end{aligned} \tag{B.55}$$

where K is defined as in (B.39) in terms of $(\underline{r}, \underline{\lambda})$ and $c_{i,k}(g, \underline{r}, \underline{\lambda})$ is defined as in (B.46). Note that from ([10] p.347)

$$\begin{aligned}&\int_0^z (z-w)^{k-1} w^{\rho-1} \exp\left(-(\delta-\lambda_i)w\right) dw \\ =\;& \frac{\Gamma(k)\Gamma(\rho)}{\Gamma(k+\rho)} z^{k+\rho-1} \; {}_1F_1\left(\rho; k+\rho; -(\delta-\lambda_i)z\right).\end{aligned}$$

Therefore (B.55) becomes

$$\begin{aligned}f_Z(z) &= K\delta^\rho \sum_{i=1}^{g} \exp(-\lambda_i z) \sum_{k=1}^{r_i} c_{i,k}(g, \underline{r}, \underline{\lambda}) \\ &\quad \times \frac{\Gamma(k)}{\Gamma(k+\rho)} z^{k+\rho-1} \; {}_1F_1\left(\rho; k+\rho; -(\delta-\lambda_i)z\right).\end{aligned}$$

The cdf of Z is given by

$$\begin{aligned}F_Z(z) &= \int_0^z F_Y(z-w) f_W(w) dw \\ &= \int_0^z K \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}(g, \underline{r}, \underline{\lambda}) \frac{\Gamma(k)}{\lambda_i^k} F_{Z_{ik}}(z-w) f_W(w) dw \\ &= K \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}(g, \underline{r}, \underline{\lambda}) \frac{\Gamma(k)}{\lambda_i^k} \int_0^z F_{Z_{ik}}(z-w) f_W(w) dw. \qquad \text{(B.56)}\end{aligned}$$

$\int_0^z F_{Z_{ik}}(z-w) f_W(w) dw$ is the cdf of a sum of two independent gamma random variables i.e. $Z_{ik} + W$ where $Z_{ik} \sim \Gamma(k, \lambda_i)$ and can therefore be represented by (B.52). Let $c_{i,k}^*(g, \underline{r}, \underline{\lambda}) = c_{i,k}(g, \underline{r}, \underline{\lambda}) \frac{\Gamma(k)}{\lambda_i^k}$ in (B.56) such that

$$\begin{aligned}F_Z(z) &= K \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}^*(g, \underline{r}, \underline{\lambda}) \frac{(\delta z)^\rho}{\Gamma(\rho+1)} \left[{}_1F_1(\rho, \rho+1, -\delta z)\right] \\ &\quad - K\delta^\rho \sum_{i=1}^{g} \sum_{k=1}^{r_i} c_{i,k}^*(g, \underline{r}, \underline{\lambda}) \exp(-\lambda_i z) \sum_{j=0}^{k-1} \frac{\lambda_i^j z^{j+\rho}}{\Gamma(j+\rho+1)} \\ &\quad \times {}_1F_1\left(\rho; j+\rho+1; -(\delta-\lambda_i)z\right). \qquad \text{(B.57)}\end{aligned}$$

Next, use (B.49) in (B.57)

$$\begin{aligned}
F_Z(z) &= K(K)^{-1}\frac{(\delta z)^{\rho}}{\Gamma(\rho+1)}\,{}_1F_1(\rho;\rho+1;-\delta z) \\
&\quad -K\delta^{\rho}\sum_{i=1}^{g}\sum_{k=1}^{r_i}c^*_{i,k}(g,\underline{r},\underline{\lambda})\exp(-\lambda_i z)\sum_{j=0}^{k-1}\frac{\lambda_i^j z^{j+\rho}}{\Gamma(j+\rho+1)} \\
&\quad \times {}_1F_1(\rho;j+\rho+1;-(\lambda_i+\delta)z) \\
&= \frac{(\delta z)^{\rho}}{\Gamma(\rho+1)}\,{}_1F_1(\rho;\rho+1;-\delta z)-K\delta^{\rho}\sum_{i=1}^{g}\sum_{k=1}^{r_i}c^*_{i,k}(g,\underline{r},\underline{\lambda}) \\
&\quad \times\exp(-\lambda_i z)\sum_{j=0}^{k-1}\frac{\lambda_i^j z^{j+\rho}}{\Gamma(j+\rho+1)}\,{}_1F_1(\rho;j+\rho+1;-(\delta-\lambda_i)z).
\end{aligned}$$

■

Result 38 *[8] Let* $X_1 \sim GIG\left(p_1, \underline{r_{1j}}, \underline{\lambda}_j\right)$ *where* $\underline{r_{1j}} = (r_{11}, r_{12}\cdots r_{1p_1})$ *and* $\underline{\lambda}_j = (\lambda_1, \lambda_2\cdots\lambda_{p_1})$ *and* $X_2 \sim GIG(p_2, \underline{r_{2l}}, \underline{\nu_l})$ *where* $\underline{r_{2_l}} = (r_{11}, r_{12}\cdots r_{1p_2})$ *and* $\underline{\nu}_j = (\nu_1, \nu_2\cdots\nu_{p_2})$ *be independent random variables and define*

$$Z = X_1 - X_2.$$

Then the pdf of Z *is given by*

$$f_Z(z) = \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=01}^{k-1} p_{jkl} f_{Z_{jki}}(z); & z \geqslant 0 \\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} f_{Z^*_{lhi}}(-z); & z < 0, \end{cases} \tag{B.58}$$

where $Z_{jki} \sim \Gamma(k-i, \lambda_j)$ *and* $Z^*_{lhi} \sim \Gamma(h-i, \nu_l)$*. Furthermore* K_1 *and* c_{jk} *are defined in a similar manner as (B.39) and (B.46) and* K_2 *and* d_{lh} *are defined in a corresponding manner, replacing* p_1 *by* p_2 *and* r_{1j} *by* r_{2j}*.* p_{jkl} *and* p^*_{jkl} *are respectively given by*

$$p_{jkl} = K_1K_2c_{jk}\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}} d_{lh}\frac{(k-1)!}{i!}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}}\frac{1}{\lambda_j^{k-i}} \tag{B.59}$$

and

$$p^*_{jkl} = K_1K_2d_{lh}\sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}} c_{jk}\binom{h-1}{i}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}\frac{1}{\nu_l^{h-i}}. \tag{B.60}$$

Z *is said to follow the difference generalised integer gamma distribution and is denoted by*

$$Z \sim DGIG\left(\underline{r_{1j}}, \underline{r_{2l}}, \underline{\lambda}_j, \underline{\nu_l}, p_1, p_2\right).$$

(See [8])

Proof. Let $X_2 = X_1 - Z$. Using independence between X_1 and X_2, the pdf of Z is given by

$$\begin{aligned}
f_Z(z) &= \int_{\max(z,0)}^{\infty} f_{X_1}(x_1) f_{X_2}(x_1 - z)\, dx_1 \\
&= \int_{\max(z,0)}^{\infty} K_1 \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} c_{jk} x_1^{k-1} \exp(-\lambda_j x_1) K_2 \\
&\quad \times \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} d_{lh} (x_1 - z)^{h-1} \exp(-\nu_l (x_1 - z))\, dx_1 \\
&= K_1 K_2 \int_{\max(z,0)}^{\infty} \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \left(\sum_{k=1}^{r_{1j}} c_{jk} x_1^{k-1} \right) \\
&\quad \left(\sum_{h=1}^{r_{2l}} d_{lh} (x_1 - z)^{h-1} \right) \exp(-(\nu_l + \lambda_j) x_1) \exp(\nu_l z)\, dx_1 \\
&= K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \int_{\max(z,0)}^{\infty} \exp(-(\nu_l + \lambda_j) x_1) \exp(\nu_l z) \\
&\quad \left(\sum_{k=1}^{r_{1j}} \sum_{h=1}^{r_{2l}} c_{jk} d_{lh} x_1^{k-1} (x_1 - z)^{h-1} \right) dx_1. \qquad \text{(B.61)}
\end{aligned}$$

Using (A.12), it follows that

$$(x_1 - z)^{h-1} = \sum_{i=0}^{h-1} \binom{h-1}{i} x_1^i (-z)^{h-i-1},$$

hence (B.61) can be written as

$$\begin{aligned}
f_Z(z) &= K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \int_{\max(z,0)}^{\infty} \sum_{k=1}^{r_{1j}} \sum_{h=1}^{r_{2l}} c_{jk} d_{lh} x_1^{k-1} \\
&\quad \times \sum_{i=0}^{h-1} \binom{h-1}{i} x_1^i (-z)^{h-i-1} \exp(-(\nu_l + \lambda_j) x_1) \exp(\nu_l z)\, dx_1 \\
&= K_1 K_2 \sum_{j=1}^{p_1} \sum_{l=1}^{p_2} \exp(\nu_l z) \sum_{k=1}^{r_{1j}} \sum_{h=1}^{r_{2l}} c_{jk} d_{lh} \sum_{i=0}^{h-1} \binom{h-1}{i} (-z)^{h-i-1} \\
&\quad \times \int_{\max(z,0)}^{\infty} x_1^{k+i-1} \exp(-(\nu_l + \lambda_j) x_1)\, dx_1. \qquad \text{(B.62)}
\end{aligned}$$

Using (A.19) to simplify the integral in (B.62), (B.62) becomes

$$f_Z(z) = \begin{cases} K_1K_2 \sum_{j=1}^{p_1}\sum_{l=1}^{p_2} \exp(\nu_l z) \sum_{k=1}^{r_{1j}}\sum_{h=1}^{r_{2l}} c_{jk}d_{lh} \sum_{i=0}^{h-1}\binom{h-1}{i} \exp\left(-\left(\nu_l+\lambda_j\right)z\right) \\ \times(-z)^{h-i-1}\sum_{t=0}^{k+i-1}\frac{(k+i-1)!}{t!}\frac{z^t}{(\nu_l+\lambda_j)^{k+i+t}}; & z\geqslant 0 \\ K_1K_2\sum_{j=1}^{p_1}\sum_{l=1}^{p_2} e^{z\nu_l}\sum_{k=1}^{r_{1j}}\sum_{h=1}^{r_{2l}} c_{jk}d_{lh}\sum_{i=0}^{h-1}\binom{h-1}{i}(-z)^{h-i-1} \\ \times\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}; & z<0. \end{cases} \tag{B.63}$$

From (A.18) follows that

$$\begin{aligned} &\sum_{i=0}^{h-1}\binom{h-1}{i}(-z)^{h-i-1}\sum_{t=0}^{k+i-1}\frac{k+i-1!}{t!}\frac{z^t}{(\nu_l+\lambda_j)^{k-i+t}} \\ =\;&\sum_{i=0}^{k-1}\binom{k-1}{i}z^{k-i-1}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}} \end{aligned}$$

therefore (B.63) can be rewritten as

$$\begin{aligned} f_Z(z) &= \begin{cases} K_1K_2\sum_{j=1}^{p_1}\sum_{l=1}^{p_2}\exp(-z\lambda_j)\sum_{k=1}^{r_{1j}}\sum_{h=1}^{r_{2l}}c_{jk}d_{lh}\sum_{i=0}^{k-1}\binom{k-1}{i} \\ \times z^{k-i-1}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}}; & z\geqslant 0 \\ K_1K_2\sum_{j=1}^{p_1}\sum_{l=1}^{p_2}\exp(z\nu_l)\sum_{k=1}^{r_{1j}}\sum_{h=1}^{r_{2l}}c_{jk}d_{lh}\sum_{i=0}^{h-1}\binom{h-1}{i} \\ \times(-z)^{h-i-1}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}; & z<0 \end{cases} \\ &= \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1}\left(K_1K_2c_{jk}\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}d_{lh}\binom{k-1}{i}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}}\right) \\ \times z^{k-i-1}\exp(-z\lambda_j); & z\geqslant 0 \\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1}\left(K_1K_2d_{lh}\sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}c_{jk}\binom{h-1}{i}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}\right) \\ \times(-z)^{h-i-1}\exp(z\nu_l); & z<0. \end{cases} \end{aligned} \tag{B.64}$$

Next, use (A.1) and (A.3) in $\binom{h-1}{i}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}$:

$$\begin{aligned}
&\binom{h-1}{i}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}} \\
=&\ \frac{(h-1)!}{i!\,(h-i-1)!}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}} \\
=&\ \frac{(h-1)!}{i!}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}\frac{1}{\Gamma(h-i)}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\binom{k-1}{i}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}} \\
=&\ \frac{(k-1)!}{i!}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}}\frac{1}{\Gamma(k-i)}.
\end{aligned}$$

Let

$$\begin{aligned}
p_{jkl}^{1} &= K_1K_2c_{jk}\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}} d_{lh}\frac{(k-1)!}{i!}\frac{(h+i-1)!}{(\nu_l+\lambda_j)^{h+i}} \\
p_{jkl}^{1*} &= K_1K_2d_{lh}\sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}} c_{jk}\binom{h-1}{i}\frac{(k+i-1)!}{(\nu_l+\lambda_j)^{k+i}}.
\end{aligned}$$

Then $f_Z(z)$ in (B.64) can be written as

$$f_Z(z)=\begin{cases}\sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl}^{1}\frac{1}{\Gamma(k-i)}z^{k-i-1}\exp(-\lambda_j z); & z\geqslant 0\\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p_{jkl}^{1*}\frac{1}{\Gamma(h-i)}(-z)^{h-i-1}\exp(\nu_l z); & z<0.\end{cases} \tag{B.65}$$

Thus Z with pdf given by (B.65) can be seen as a mixture of particular independent random variables. To make this variables come from a familiar distribution, we can make the following definitions

$$\begin{aligned}
p_{jkl} &= p_{jkl}^{1}\frac{1}{\lambda_j^{k-i}} \\
p_{jkl}^{*} &= p_{jkl}^{1*}\frac{1}{\nu_l^{h-i}}.
\end{aligned}$$

Therefore, Z is a mixture of independent gamma random variables and has pdf given by

$$
\begin{aligned}
f_Z(z) &= \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl}\frac{\lambda_j^{k-i}}{\Gamma(k-i)} z^{k-i-1}\exp(-\lambda_j z)\,; & z \geqslant 0 \\ \sum_{j=1}^{p_1}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl}\frac{\nu_l^{h-i}}{\Gamma(h-i)} (-z)^{h-i-1}\exp(\nu_l z)\,; & z < 0 \end{cases} \\
&= \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=01}^{k-1} p_{jkl} f_{Z_{jki}}(z)\,; & z \geqslant 0 \\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} f_{Z^*_{lhi}}(-z)\,; & z < 0, \end{cases}
\end{aligned}
$$

where $f_{Z_{jki}}(z)$ is the pdf of $Z_{jki} \sim \Gamma(k-i, \lambda_j)$ and $f_{Z^*_{lhi}}(-z)$ is the pdf of $Z^*_{lhi} \sim \Gamma(h-i, \nu_l)$. ■

If $\theta \in \mathbb{R}$ is a shift parameter then

$$
Z + \theta \sim SDGIG\left(\underline{r_{1j}}, \underline{r_{2l}}, \underline{\lambda_j}, \underline{\nu_l}, p_1, p_2, \theta\right).
$$

Next, the cdf of Z is determined using the pdf of Z in (B.58) as

$$
F_Z(z) = \int_{-\infty}^{z} f_Z(t)\,dt.
$$

Two cases, i.e. $z > 0$ and $z < 0$ will be considered.

When $z > 0$;

$$
\begin{aligned}
&F_Z(z) \\
&= \int_{-\infty}^{z} f_Z(t)\,dt \\
&= \int_{-\infty}^{0} f_Z(t)\,dt + \int_{0}^{z} f_Z(t)\,dt \\
&= \int_{-\infty}^{0} \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} f_{Z^*_{lhi}}(-t)\,dt + \int_{0}^{z} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} f_{Z_{jkl}}(t)\,dt \\
&= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_{-\infty}^{0} f_{Z^*_{lhi}}(-t)\,dt + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_{0}^{z} f_{Z_{jkl}}(t)\,dt \\
&= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{Z_{jkl}}(z),
\end{aligned}
$$

where $F_{Z_{jkl}}(z)$ is the cdf of Z_{jkl}. Therefore when $z > 0$, then

$$F_Z(z) = \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{Z_{jkl}}(z). \tag{B.66}$$

If $z < 0$;

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} f_Z(t)\, dt \\ &= \int_{-\infty}^{z} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} f_{Z^*_{lhi}}(-t)\, dt \\ &= \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} \int_{-\infty}^{z} f_{Z^*_{lhi}}(-t)\, dt \\ &= \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} \left(\int_{-\infty}^{0} f_{Z^*_{lhi}}(-t)\, dt - \int_{z}^{0} f_{Z^*_{lhi}}(-t)\, dt \right) \\ &= \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} \left(1 - F_{Z^*_{lhi}}(-z)\right). \end{aligned}$$

Therefore if $z < 0$:

$$F_Z(z) = \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} \left(1 - F_{Z^*_{jkl}}(-z)\right). \tag{B.67}$$

Combining (B.66) and (B.67), the cdf of Z is given by

$$F_Z(z) = \begin{cases} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{Z_{jkl}}(z) & z \geqslant 0 \\ \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} \left(1 - F_{Z^*_{jkl}}(-z)\right) & z < 0. \end{cases} \tag{B.68}$$

Lastly, the weights in (B.58) and (B.68) are shown to add to 1, and hence valid weights. Integrating $f_Z(z)$ over the entire state space yields

$$\begin{aligned}\int_{-\infty}^{\infty} f_Z(z)\,dz &= \int_{-\infty}^{0} f_Z(z)\,dz + \int_{0}^{\infty} f_Z(z)\,dz \\ &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_{-\infty}^{0} f_{Z^*_{lhi}}(-z)\,dz + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_{0}^{\infty} f_{Z_{jki}}(z)\,dz \\ &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl}. \qquad \text{(B.69)}\end{aligned}$$

Since $\int_{-\infty}^{\infty} f_Z(z)\,dz = 1$, the weights sum to 1.

Result 39 *[16] Let Y and W be independent random variables such that*

$$Y \sim DGIG\left(\underline{r_{1j}}, \underline{r_{2l}}, \underline{\lambda_j}, \underline{\nu_l}, p_1, p_2\right)$$

and $W \sim \Gamma(\rho, \delta)$. Define the random variable Z_1 such that

$$Z_1 = Y + W.$$

The cdf of Z_1 is given by

$$F_{Z_1}(z) = \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{G_1}(z) \\ +\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{DG_1}(z); & z \geqslant 0 \\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{DG_1}(z); & z < 0, \end{cases} \qquad \text{(B.70)}$$

where p_{jkl} and p^*_{jkl} are defined in (B.59) and (B.60) respectively. $F_{G_1}(z)$ and $F_{DG_1}(z)$ is of form (B.52) and (B.18) respectively. The pdf of Z is given by

$$f_{Z_1}(z) = \begin{cases} \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} f_{G_1}(z) \\ +\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} f_{DG_1}(z); & z \geqslant 0 \\ \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} f_{DG_1}(z); & z < 0, \end{cases}$$

where $f_{G_1}(z)$ and $f_{DG_1}(z)$ is of form (B.53) and (B.17) respectively. (See [16] for further details.)

Proof. Using independence of Y and W, cdf of Z_1 is given by

$$F_{Z_1}(z) = \int_{-\infty}^{\infty} F_Y(z-w) f_W(w)\, dw \qquad z \in (-\infty, \infty).$$

When $z > 0$, (B.68) can be use and if follows that

$$\begin{aligned}
F_{Z_1}(z) &= \int_{-\infty}^{z} F_Y\underbrace{(z-w)}_{\geqslant 0} f_W(w)\, dw + \int_{z}^{\infty} F_Y\underbrace{(z-w)}_{\leq 0} f_W(w)\, dw \\
&= \int_{-\infty}^{z} F_Y\underbrace{(z-w)}_{\geqslant 0} f_W(w)\, dw + \int_{-\infty}^{\infty} F_Y\underbrace{(z-w)}_{\leq 0} f_W(w)\, dw \\
&\quad - \int_{-\infty}^{z} F_Y\underbrace{(z-w)}_{\leq 0} f_W(w)\, dw \\
&= \int_{-\infty}^{z} \left[\sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{Z_{jki}}(z-w) \right] f_W(w)\, dw \\
&\quad + \int_{-\infty}^{\infty} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \left(1 - F_{Z_{lhi}^{*}}(w-z)\right) f_W(w)\, dw \\
&\quad - \int_{-\infty}^{z} \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \left(1 - F_{Z_{lhi}^{*}}(w-z)\right) f_W(w)\, dw
\end{aligned} \tag{B.71}$$

(B.71) can be simplified further to obtain:

$$\begin{aligned}
F_{Z_1}(z) &= \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \int_{0}^{z} f_W(w)\, dw + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} \\
&\quad \times \int_{-\infty}^{z} F_{Z_{jkl}}(z-w) f_W(w)\, dw + \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \int_{0} f_W(w)\, dw \\
&\quad - \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \int_{-\infty}^{\infty} F_{Z_{lhi}^{*}}(w-z) f_W(w)\, dw \\
&\quad - \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \int_{0}^{z} f_W(w)\, dw \\
&\quad + \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} \int_{-\infty}^{z} F_{Z_{lhi}^{*}}(w-z) f_W(w)\, dw,
\end{aligned} \tag{B.72}$$

where $Z_{jki} \sim \Gamma(k-i, \lambda_j)$ and $Z^*_{lhi} \sim \Gamma(h-i, \nu_l)$. Note that

$$\begin{aligned} & \int_0^z F_{Z_{jkl}}(z-w) f_W(w)\,dw \\ = \;& P(Z_{jkl} \leq z - W) \\ = \;& P(Z_{jkl} + W \leq z). \end{aligned} \tag{B.73}$$

(B.73) is the cdf of a sum of two independent gamma random variables, therefore $Z_{jkl} + W \sim GNIG(2, \underline{r}^*, \underline{\lambda}^*)$ where $\underline{r}^* = (k-i, \rho)$ and $\underline{\lambda}^* = (\lambda_j, \delta)$. Denote this cdf by $F_{G_1}(z)$ and is given by (B.52). (B.72) can be simplied using (B.73) and the fact that $\int_{-\infty}^{\infty} f_W(w)\,dw = \int_{-\infty}^{\infty} F_{Z^*_{lhi}}(w-z) f_W(w)\,dw = 1$. Therefore (B.72) becomes

$$\begin{aligned} F_{Z_1}(z) &= \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_0^z F_{Z_{jkl}}(z-w) f_W(w)\,dw \\ &\quad + \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^z F_{Z^*_{jkl}}(w-z) f_W(w)\,dw \\ &= \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{G_1}(z) + \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{DG_1}(z). \end{aligned}$$

When $z < 0$, use (B.68) and derive

$$\begin{aligned} F_{Z_1}(z) &= \int_z^{\infty} F_Y \underbrace{(z-w)}_{\leq 0} f_W(w)\,dw \\ &= \int_z^{\infty} \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \left(1 - F_{Z^*_{lhi}}(w-z)\right) f_W(w)\,dw \\ &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_z^{\infty} f_W(w)\,dw \\ &\quad - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_z^{\infty} F_{Z^*_{lhi}}(w-z) f_W(w)\,dw. \end{aligned} \tag{B.74}$$

Further note that

$$\begin{aligned}
&\int_z^\infty F_{Z^*_{jkl}}\left(w-z\right) f_W\left(w\right) dw \\
=\; & P\left(Z^*_{jkl} < W - z\right) \\
=\; & 1 - P\left(W - Z^*_{jkl} < z\right). \qquad \text{(B.75)}
\end{aligned}$$

(B.75) is the cdf of $W - Z^*_{jkl}$ and will be denoted by $F_{DG_1}\left(z\right)$, the form of $F_{DG_1}\left(z\right)$ is given in (B.18).
Integrals in (B.74) equates to 1. Using (B.75); the cdf in (B.74) can be represented as

$$\begin{aligned}
F_{Z_1}\left(z\right) &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl}\left(1 - F_{DG_1}\left(z\right)\right) \\
&= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{DG_1}\left(z\right),
\end{aligned}$$

and the cdf of Z_1 is therefore given by

$$F_{Z_1}\left(z\right) = \begin{cases} \sum\limits_{j=1}^{p_1}\sum\limits_{k=1}^{r_{1j}}\sum\limits_{i=0}^{k-1} p_{jkl} F_{G_1}\left(z\right) \\ + \sum\limits_{l=1}^{p_2}\sum\limits_{h=1}^{r_{2l}}\sum\limits_{i=0}^{h-1} p^*_{jkl} F_{DG_1}\left(z\right); & z \geqslant 0 \\ \sum\limits_{l=1}^{p_2}\sum\limits_{h=1}^{r_{2l}}\sum\limits_{i=0}^{h-1} p^*_{jkl} F_{DG_1}\left(z\right); & z < 0, \end{cases}$$

The pdf of Z_1 is given by

$$\begin{aligned}
f_{Z_1}\left(z\right) &= \frac{d}{dz} F_{Z_1}\left(z\right) \\
&= \begin{cases} \sum\limits_{j=1}^{p_1}\sum\limits_{k=1}^{r_{1j}}\sum\limits_{i=0}^{k-1} p_{jkl} f_{G_1}\left(z\right) \\ + \sum\limits_{l=1}^{p_2}\sum\limits_{h=1}^{r_{2l}}\sum\limits_{i=0}^{h-1} p^*_{jkl} f_{DG_1}\left(z\right); & z \geqslant 0 \\ \sum\limits_{l=1}^{p_2}\sum\limits_{h=1}^{r_{2l}}\sum\limits_{i=0}^{h-1} p^*_{jkl} f_{DG_1}\left(z\right); & z < 0, \end{cases}
\end{aligned}$$

where $f_{G_1}\left(z\right)$ and $f_{DG_1}\left(z\right)$ is of form (B.53) and (B.17) respectively. ■

Result 40 *[16] Let* Y *and* W *be independent random variables such that*

$$Y \sim DGIG\left(\underline{r_{1j}}, \underline{r_{2l}}, \underline{\lambda_j}, \underline{\nu_l}, p_1, p_2\right)$$

and $W \sim \Gamma(\rho, \delta)$. *Define the random variable* Z_1 *such that*

$$Z_2 = Y - W.$$

The cdf of Z_1 *is given by*

$$F_{Z_2}(z) = \begin{cases} 1 - \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z); & z \geq 0 \\ 1 - \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} F_{G_2}(-z) & \\ \quad + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z); & z > 0, \end{cases} \tag{B.76}$$

where p_{jkl} *and* p^*_{jkl} *are defined in (B.59) and (B.60) respectively.* $F_{G_2}(z)$ *and* $F_{DG_2}(z)$ *are defined in (B.52) and (B.18) respectively. The pdf of* Z_2 *is given by*

$$f_{Z_2}(z) = \begin{cases} \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} f_{DG_2}(-z); & z \geq 0 \\ \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p^*_{jkl} f_{G_2}(-z) & \\ \quad - \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} f_{DG_2}(-z); & z > 0, \end{cases} \tag{B.77}$$

where $f_{G_2}(z)$ *and* $f_{DG_2}(z)$ *is defined by (B.53) and (B.17) respectively. (See [16] for further details.)*

Proof. Two cases, i.e. $z > 0$ and $z < 0$ will be considered.

Consider first when $z < 0$.Using independence between Y and W, the cdf of Z_2 is given by

$$\begin{aligned}
F_{Z_2}(z) &= \int_0^{-z} F_Y \underbrace{(w+z)}_{\leq 0} f_W(w)\,dw + \int_{-z}^{\infty} F_Y \underbrace{(w+z)}_{\geq 0} f_W(w)\,dw \\
&= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^{-z} f_W(w)\,dw - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^{-z} F_{Z^*_{jkl}}(-z-w) f_W(w)\,dw \\
&\quad + \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_{-z}^{\infty} f_W(w)\,dw + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_{-z}^{\infty} F_{Z_{jkl}}(z+w) f_W(w)\,dw \\
&= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^{\infty} f_W(w)\,dw - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^{-z} F_{Z^*_{jkl}}(-z-w) f_W(w)\,dw \\
&\quad + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_{-z}^{\infty} F_{Z_{jkl}}(z+w) f_W(w)\,dw,
\end{aligned} \tag{B.78}$$

where $Z_{jki} \sim \Gamma(k-i, \lambda_j)$ and $Z^*_{lhi} \sim \Gamma(h-i, \nu_l)$. Note that

$$\begin{aligned}
& \int_0^{-z} F_{Z^*_{jkl}}(-z-w) f_W(w) \\
&= P\left(Z^*_{jkl} \leq -z-W\right) \\
&= P\left(Z^*_{jkl} + W \leq -z\right) \\
&= F_{G_2}(-z),
\end{aligned} \tag{B.79}$$

$Z_{jkl} + W \sim GNIG(2, \underline{r}^*, \underline{\lambda}^*)$ where $\underline{r}^* = (h-i, \rho)$ and $\underline{\lambda}^* = (v_l, \delta)$. Therefore the cdf of $Z_{jkl} + W$ is given by (B.79) and is of form (B.52) and shall be denoted by $F_{G_2}(-z)$. Next, consider

$$\begin{aligned}
& \int_{-z}^{\infty} F_{Z_{jkl}}(z+w) f_W(w)\,dw \\
&= 1 - P(Z_{jkl} \leq W - z) \\
&= 1 - P(Z_{jkl} - W \leq -z) \\
&= 1 - F_{DG_2}(-z),
\end{aligned} \tag{B.80}$$

where $F_{DG_2}(-z)$ is the cdf of $F_{DG_2}(-z)$ with its form given by (B.18). Substituting (B.79) and (B.80) in (B.78) so that it becomes

$$F_{Z_2}(z) = \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{G_2}(-z) + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl}\left(1 - F_{DG_2}(-z)\right). \tag{B.81}$$

From (B.69), it follows that

$$\sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} = 1,$$

therefore (B.81) becomes

$$F_{Z_2}(z) = 1 - \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} F_{G_2}(-z) + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z). \tag{B.82}$$

If $z \geq 0$. Using independence between Y and W, the cdf of Z_2 is given by

$$\begin{aligned} F_{Z_2}(z) &= \int_0^\infty F_Y\underbrace{(w+z)}_{\geq 0} f_w(w)\,dw \\ &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} \int_0^\infty f_w(w)\,dw \\ &\quad + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} \int_0^\infty F_{Z_{jkl}}(z+w)\, f_W(w)\,dw. \end{aligned} \tag{B.83}$$

Using (B.80) to simplify the second integral in (B.83), (B.83) becomes

$$\begin{aligned} F_{Z_2}(z) &= \sum_{l=1}^{p_2}\sum_{h=1}^{r_{2l}}\sum_{i=0}^{h-1} p^*_{jkl} + \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} - \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z) \\ &= 1 - \sum_{j=1}^{p_1}\sum_{k=1}^{r_{1j}}\sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z). \end{aligned} \tag{B.84}$$

From (B.84) and (B.82) the cdf of Z_2 is given by

$$F_{Z_2}(z) = \begin{cases} 1 - \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z); & z \geq 0 \\ 1 - \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} F_{G_2}(-z) & \\ + \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} F_{DG_2}(-z); & z > 0, \end{cases}$$

The pdf of Z_2 is given by

$$\begin{aligned} f_{Z_2}(z) &= \frac{d}{dz} F_{Z_2}(z) \\ &= \begin{cases} \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} f_{DG_2}(-z); & z \geq 0 \\ \sum_{l=1}^{p_2} \sum_{h=1}^{r_{2l}} \sum_{i=0}^{h-1} p_{jkl}^{*} f_{G_2}(-z) & \\ - \sum_{j=1}^{p_1} \sum_{k=1}^{r_{1j}} \sum_{i=0}^{k-1} p_{jkl} f_{DG_2}(-z); & z > 0, \end{cases} \end{aligned}$$

where $f_{G_2}(z)$ and $f_{DG_2}(z)$ is defined by (B.53) and (B.17) respectively. ■

C. Computational modules

This section contains details of the proposed computational modules for the near-exact distributions of Y (see (1.5)) and their implementations. The proposed computational modules are implemented using Mathematica, Version 10.

D1: All modules for the implementation of the first and second near-exact distributions are available at the following link:

http://www.up.ac.za/media/shared/115/product-of-independent-gen-gamma.zp101826.pdf

D2: The link also contains modules for the implementation some of important statistical distributions used in near-exact distributions developed in this study. Among these statistical distributions is the generalised near-integer gamma. Modules for implementation of its pdf and cdf are denoted, respectively, by `GNIGpdf[z_,elλ_,Γλ_,elr_,Γr_]` and `GNIGcdf[z_,elλ_,Γλ_,elr_,Γr_]` where `z` is the running value, the vector `elλ` and the parameter $\Gamma\lambda$ are the rate parameters, the vector `elr` is a list of the integer shape parameters and `Γr` is a non-integer shape parameter. Alternative modules can be found at the following link:

https://sites.google.com/site/nearexactdistributions/

D3: The module to implement cumulative probabilities for the first near-exact distribution is denoted by `SDGIGcdf[z_,r1_,r2_,λ1_,λ2_]`. `z` is the running value, the vectors `r1` and `r2` are list of parameters as indicated in (3.4) (specifically β_j^+ and β_j^*). The vectors λ1 and λ2 contains parameters as indicated in (3.4) (specifically m_j^+ and m_j^-).

D4: The proposed module to implement cumulative probabilities for the second near-exact distribution is denoted by `SDGIGcdf2[z_,r1_,r2_,λ1_,λ2_,Γλ_,Γr_]`
where `z, r1, r2,` λ1 and λ2 are defined above. $\Gamma\lambda$ and `Γr` denote, respectively, the rate and the shape parameters for the independent gamma distributed random variables.

D5: Parameters used in modules `GNIGpdf[z_,elλ_,Γλ_,elr_,Γr_]`, `GNIGcdf[z_-,elλ_,Γλ_,elr_,Γr_]` and `SDGIGcdf2[z_,r1_,r2_,λ1_,λ2_,Γλ_,Γr_]` are obtained using a module denoted by `ParameterPrep[rn_,rd_,λn_,λd_,δn_,δd_,precision_]`, where `rn,λn` and `δn` denote vectors for shape, rate and power parameters, respectively, of generalised gamma distributed random variables that are in of the numerator of G (see (1.3)) or having positive power parameters in Y (see (1.1.2)). `rd,λd` and `δd` denote vectors for shape, rate and power parameters, respectively, of generalised gamma random variables in the denominator of G (see (1.3)) or with negative power parameter in Y (see (1.1.2)). The precision parameter is denoted by `precision.`

D6: To evaluate modules for both the first- and second near-exact distribution, some preliminary modules in the link need to be evaluated first. Table 5.1 contains a list of such modules with an indication by `X` of whether they should be evaluated before a module for a given near-exact distribution is evaluated.

Table 5.1: List of preliminary modules to be evaluated before a near-exact distribution module is evaluated

Module	First near-exact	Second near-exact
FactorK[λ_,r_]	X	X
cik[t_,r_,λ_]	X	X
GNIGcdf[z_,elλ_,Γλ_,elr_,Γr_]		X
DGamErlangcdf[z_,elλ_,Γλ_,elr_,Γr_]		X
SDGIGcdfpos[z_,r1_,r2_,λ1_,λ2_,Γλ_,Γr_]		X
SDGIGcdfpos[z_,r1_,r2_,λ1_,λ2_,Γλ_,Γr_]		X

D7: In addition to these preliminary modules, the module for the first near-exact distribution uses a Mathematica function to evaluate the cdf of the Erlang distribution (see B.7).

As an example of an application of these modules, consider Case III in Table 4.1

$$\mathtt{r}=\left\{2,3,5,\frac{1}{2}\right\}, \quad \lambda=\left\{3,2,10,\frac{2}{7}\right\}, \quad \delta=\{\frac{1}{2},2,-\frac{1}{4},-\frac{1}{3}\}$$

By using $\mathtt{ParameterPrep}[\{2,3\},\left\{5,\frac{1}{2}\right\},\{3,2\},\left\{10,\frac{2}{7}\right\},\{\frac{1}{2},2\},\left\{\frac{1}{4},\frac{1}{3}\right\},20]$, the parameters for near-exact distributions are obtained. Various computational studies (similar to those in Section 4.3) can be perfomed using these parameters and a relevant module for a particular near-exact distribution.

Modules for the implementation of empirical distribution can be obtained in the link below

http://www.up.ac.za/media/shared/115/product-of-independent-gen-gamma.zp101826.pdf

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