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Application of the finite element method to second order hyperbolic type partial differential equations

by

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Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own independent work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature: ________________________________

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Summary

In this dissertation various models with variational forms similar to that of the wave equation are considered, i.e. second order hyperbolic type partial differential equations. These models include several linear vibration problems and heat conduction models taking phase-lag into account.

Clearly numerical methods need to be used to solve these problems and the Finite Element Method (FEM) is used in this study. Before applying such a method, existence of a solution needs to be established. Therefore, a review of the work by Van Rensburg and Van der Merwe (2002) on general second order hyperbolic type problems was done. The results were not only presented, but additional remarks and a discussion which assists in applying the theory were also included. To obtain convergence results and error estimates when FEM is applied to the various models, general convergence results were presented. For this the article by Basson and Van Rensburg (2013) was used.

The first model considered consists of two serially connected Timoshenko beams. One of the beams was modelled as embedded in an elastic material, while the other beam is either free or subjected to a prescribed external load. This model can be adapted for a single beam with different loads on separate parts. To apply the convergence theory though it was necessary to use the double beam model, while a single beam model can be used when FEM is applied. This was demonstrated when these models were used to model a plant with a tap root system. In this biological application various things were investigated, including different forms of FEM, a comparison of the results for the static double beam and static single beam, and the dynamics of the beam. These experiments indicated that the two models compare well and gave insight into how the parameter modelling the resistance of

the soil influences key aspects of how the plant reacts due to external forces.

Models for rigid bodies attached to beams were also investigated. The equations used to describe the dynamics of a beam with a tip body were derived, with special attention given to the interface conditions. Consequently, a model problem for an intermediate rigid body between two Timoshenko beams was investigated.

Hyperbolic heat conduction models were also considered and the application to bio-heat transfer in skin was discussed. Specifically, a model from the work by Dekka and Dutta (2019) was investigated. Their approach to existence of solutions was scrutinized and it was found that their application of existence results from the 2002 article by Van Rensburg and Van der Merwe is incomplete. Due to this the exposition of the theory is improved in the dissertation.

For all the mentioned models, the existence and uniqueness of a solution were obtained by defining the relevant function spaces and proving the required properties. Convergence was also established from the general convergence results and the systems of ordinary differential equations were obtained which can be used to obtain numerical approximations.

Contents

Chapter 1

Hyperbolic mathematical models

1.1 Introduction

The vibration of structures consisting of elastic bodies is of great importance in applied mathematics and engineering. To model vibrating structures, partial differential equations or systems of partial differential equations are used. It is well known that problems of this kind can seldom be solved exactly and it is necessary to consider the numerical approximation of solutions. It is generally agreed that the finite element method (FEM) is most suitable.

This dissertation (a literature study) forms part of an ongoing research project, *Vibration analysis*. The project encompasses theoretical analysis, modelling and finite element computation. The idea is to obtain theoretical and practical results.

All linear vibration problems have a variational form that resembles the variational form of the wave equation. The abstract formulation of linear vibration problems is referred to as the general linear vibration problem in [VV02]. This problem may also be referred to as a general linear second order hyperbolic problem (see eg. [VS19]).

Regarding FEM, generalised convergence results are discussed in [BV13]. This article is a generalisation of previous articles. It emerged in [BV13] that the results in [VV02] are appropriate to show the existence of a weak solution. Furthermore, the estimates necessary for existence theory are essential also for convergence theory. Results on existence theory are found in for example [Sho77] and [AKS96]. For this dissertation the theory in [VV02] is ideal, since existence results are formulated in terms of the variational theory.

Applications considered in this dissertation deal mostly with vibrating systems of elastic bodies. We also consider the biological application of using the Timoshenko beam theory to model plants with a tap root system. Remarkably, heat conduction can also be modelled by hyperbolic type partial differential equations. Examples of such models are presented and the application of FEM considered.

1.2 Wave equation

In this section, the wave equation is considered to introduce the concept of a variational form, which is essential for the existence theory discussed in this dissertation. Problems with variational forms similar to that of the wave equation are referred to as second order hyperbolic problems. (See the end of this section for more details).

Consider a string of length ℓ that is fixed at the endpoints. The point x represents a cross-section of the string, and $w(x,t)$ denotes the transverse displacement of x at time t. The wave equation below is used to model the vibrating string

$$\rho\partial_t^2 w = \vartheta\partial_x^2 w + q \text{ in } (0,\ell), \text{ for each } t \in (0,t_*] \text{ where } t_* < \infty. \tag{1.2.1}$$

Here ρ is the mass per unit length of the string, ϑ is the tensile force of the string, and q is the external force density experienced by the string.

For a well-posed problem, boundary conditions and initial values for the displacement w and velocity $\partial_t w$ are required. A problem can be formulated as:

Given a function q and positive constants ρ and ϑ, find w such that

$$\rho\partial_t^2 w = \vartheta\partial_x^2 w + q \ \text{ in } (0,\ell) \text{ for each } t \in (0,t_*]$$

with boundary conditions

$$w(0,t) = w(\ell,t) = 0 \text{ for each } t \in (0,t_*], \tag{1.2.2}$$

and initial conditions

$$w(x,0) = A(x) \text{ and } \partial_t w(x,0) = B(x) \text{ for each } x \in (0,\ell). \tag{1.2.3}$$

The presence of viscous damping changes Equation (1.2.1) into

$$\rho\partial_t^2 w = \vartheta\partial_x^2 w - \gamma\partial_t w + q, \tag{1.2.4}$$

where γ is the damping parameter (a positive constant).

Remark. *Interestingly, Equation* (1.2.4) *is also a mathematical model for hyperbolic heat conduction, see Section 1.5.*

To apply the existence theory in Chapter 2, the partial differential equation (1.2.4) has to be written in variational form. To this end, multiply Equation (1.2.4) by a function $v \in C^1[0,\ell]$ and integrate. Using integration by parts yields

$$\begin{aligned}\int_0^\ell \rho\partial_t^2 w(\cdot,t)v =& \vartheta\partial_x w(\ell,t)v(\ell) - \vartheta\partial_x w(0,t)v(0) - \int_0^\ell \vartheta\partial_x w(\cdot,t)v' \\ & - \int_0^\ell \gamma\partial_t w(\cdot,t)v + \int_0^\ell q(\cdot,t)v. \end{aligned} \tag{1.2.5}$$

Substituting the boundary conditions in Equation (1.2.2) necessitates the introduction of a test function space

$$\mathcal{T}[0,\ell] = \{v \in C^1[0,\ell] \mid v(0) = v(\ell) = 0\}.$$

The variational form of the problem is: Given a function q and positive constants γ, ρ and ϑ, find w such that $w(\cdot,t) \in \mathcal{T}[0,\ell]$ for each $t > 0$ and

$$\int_0^\ell \rho\partial_t^2 w(\cdot,t)v = -\int_0^\ell \vartheta\partial_x w(\cdot,t)v' - \int_0^\ell \gamma\partial_t w(\cdot,t)v + \int_0^\ell q(\cdot,t)v, \tag{1.2.6}$$

for all $v \in \mathcal{T}[0,\ell]$, with initial conditions

$$w(\cdot,0) = w_0 \text{ and } \partial_t w(\cdot,0) = w_d.$$

Note that other boundary and initial conditions are also possible and that the boundary conditions influence the space of test functions.

The $\mathcal{L}^2(0,\ell)$ inner product is denoted by $(\cdot,\cdot)$ and the following bilinear forms are defined

$$\begin{aligned}& c(u,v) = (\rho u, v), b(u,v) = (\vartheta u', v'), \text{ and} \\ & a(u,v) = (\gamma u, v).\end{aligned}$$

Variational equation (1.2.6) can now be written as

$$c(\partial_t^2 w(\cdot,t),v) + a(\partial_t w(\cdot,t),v) + b(w(\cdot,t),v) = (q(t),v), \qquad (1.2.7)$$

for each $v \in \mathcal{T}[0,\ell]$.

The variational form of the wave equation is typical of many applications of linear vibration problems. That is, second order hyperbolic type problems have a variational equation with the same form as Equation (1.2.7). This form is identified by the three bilinear forms c, a and b. See Subsections 1.3.4, 1.4.2 and 1.5.5. In Chapter 2 the general formulation for a second order hyperbolic problem is defined with the consequence that these mentioned problems become special cases.

1.3 Timoshenko beam theory

A beam is a three-dimensional body that is often used to support a load. The Timoshenko beam model is a one-dimensional model which is a simplification. In this section, we consider the Timoshenko beam model and introduce concepts relevant for the well-posedness of such beam models.

In their study, [LVV09] consider distinct linear theories for a cantilever beam, and conclude that the Timoshenko theory may be used as a guide to estimate or determine the limitations of the Euler-Bernoulli theory. A justification of the applicability of the linear Timoshenko model was investigated by [SP06]. The authors conclude that the eigenvalues of a one-dimensional and three-dimensional linear Timoshenko model compare well, provided specific modes are identified.

1.3.1 Equations of motion and constitutive equations

The Timoshenko beam model consists of two partial differential equations. The first equation is for the deflection w, and the other for the angle ϕ, which results from the rotation of a cross-section. The equations of motion are

$$\rho A \partial_t^2 w = \partial_x F + q, \qquad (1.3.1)$$
$$\rho I \partial_t^2 \phi = F + \partial_x M, \qquad (1.3.2)$$

where ρ, A and I correspondingly denote the density, area of a cross-section and area moment of inertia.

The constitutive equations are

$$F = AG\kappa^2(\partial_x w - \phi), \tag{1.3.3}$$
$$M = EI\partial_x \phi, \tag{1.3.4}$$

here E and G are elastic constants, κ^2 is the shear correction factor, M is the moment and F the shear force. We refer the reader to [Tim37, p 337-8] and [Inm94, p 337-8] for more detail.

The equations of motion and the constitutive equations yield a system of partial differential equations

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x \big(AG\kappa^2(\partial_x w - \phi) \big), \\ \rho I \partial_t^2 \phi &= AG\kappa^2(\partial_x w - \phi) + \partial_x (EI \partial_x \phi). \end{aligned}$$

It is worth noting that the system of partial differential equations above is not used.

1.3.2 Dimensionless form

For the purpose of numerical experiments, it is convenient to write the model in its dimensionless form. More importantly, this form permits the use of fewer parameters to characterise the beam.

Let

$$\tau = \frac{t}{\zeta}, \quad \xi = \frac{x}{\ell},$$
$$w^*(\xi, \tau) = \frac{w(x,t)}{\ell} \quad \text{and } \phi^*(\xi,\tau) = \phi(x,t),$$

where

$$\zeta = \sqrt{\frac{\rho \ell^2}{G\kappa^2}}.$$

The dimensionless form of the shear force, moment and load respectively is

$$F^*(\xi,\tau) = \frac{F(x,t)}{AG\kappa^2}, \quad M^*(\xi,\tau) = \frac{M(x,t)}{AG\kappa^2 \ell} \quad \text{and} \quad q^*(\xi,\tau) = \frac{q(x,t)\ell}{AG\kappa^2}.$$

We introduce the following dimensionless constants

$$\alpha = \frac{A\ell^2}{I}, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \text{ and } \gamma_0 = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}.$$

Remark. *According to [VV06], the values for κ^2 range between $\frac{1}{2}$ and 1, while for isotropic materials it is assumed that $\dfrac{G}{E} = \dfrac{1}{2(1+\nu)}$. Realistic values for γ_0 range between $\frac{1}{6}$ and $\frac{1}{2}$, whereas α may vary significantly.*

Using the original notation, the dimensionless form of the equations of motion

$$\partial_t^2 w = \partial_x F + q, \tag{1.3.5}$$

$$\frac{1}{\alpha}\,\partial_t^2 \phi = F + \partial_x M. \tag{1.3.6}$$

Constitutive equations

$$F = \partial_x w - \phi, \tag{1.3.7}$$

$$M = \frac{1}{\beta}\,\partial_x \phi. \tag{1.3.8}$$

1.3.3 Boundary conditions

Boundary conditions are applied specific to the problem of interest. The possible boundary conditions for a dimensionless beam model are:

Cantilever

$$w(0,t) = \phi(0,t) = F(1,t) = M(1,t) = 0. \tag{1.3.9}$$

Pinned-Pinned

$$w(0,t) = M(0,t) = w(1,t) = M(1,t) = 0. \tag{1.3.10}$$

Clamped-Clamped

$$w(0,t) = \phi(0,t) = w(1,t) = \phi(1,t) = 0.$$

A cantilever Timoshenko beam problem is formulated as: Given a function q and positive constants α and β, find w and ϕ such that Equations (1.3.5) to (1.3.8) are satisfied with boundary conditions (1.3.9), and initial conditions

$$w(\cdot,0) = w_0, \partial_t w(\cdot,0) = w_d, \phi(\cdot,0) = \phi_0, \partial_t \phi(\cdot,0) = \phi_d \text{ for } x \in (0,1),$$

for given functions w_0, w_d, ϕ_0, ϕ_d.

1.3.4 Variational form

To obtain the variational form of the cantilever problem, multiply Equations (1.3.5) and (1.3.6) by functions v and $\nu \in C^1[0,1]$ respectively. Integrate using integration by parts to get the result

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v =& F(1,t)v(1) - F(0,t)v(0) - \int_0^1 F(\cdot,t)v' \\ &+ \int_0^1 q(\cdot,t)v, \qquad (1.3.11)\\ \frac{1}{\alpha}\int_0^1 \partial_t^2\phi(\cdot,t)\nu =& M(1,t)\nu(1) - M(0,t)\nu(0) + \int_0^1 F(\cdot,t)\nu \\ &- \int_0^1 M(\cdot,t)\nu'. \qquad (1.3.12)\end{aligned}$$

Substituting boundary conditions (1.3.9) into Equations (1.3.11) and (1.3.12) yields

$$\int_0^1 \partial_t^2 w(\cdot,t)v = -F(0,t)v(0) - \int_0^1 F(\cdot,t)v' + \int_0^1 q(\cdot,t)v, \qquad (1.3.13)$$

$$\int_0^1 \frac{1}{\alpha}\partial_t^2\phi(\cdot,t)\nu = -M(0,t)\nu(0) + \int_0^1 F(\cdot,t)\nu - \int_0^1 M(\cdot,t)\nu'. \qquad (1.3.14)$$

The space of test functions for the problem is defined as

$$\mathcal{T}[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}.$$

The substitution of Equations (1.3.7) and (1.3.8) into Equations (1.3.13) and (1.3.14) results in the standard variational form of the problem: Given a function q and positive constants α and β, find w, ϕ where $w(\cdot,t)$ and $\phi(\cdot,t) \in \mathcal{T}[0,1]$ for each $t > 0$ such that

$$\begin{aligned}\int_0^1 \partial_t^2 w(\cdot,t)v &= -\int_0^1 (\partial_x w(\cdot,t) - \phi(\cdot,t))v' + \int_0^1 q(\cdot,t)v,\\ \int_0^1 \frac{1}{\alpha}\partial_t^2\phi(\cdot,t)\nu &= \int_0^1 (\partial_x w(\cdot,t) - \phi)(\cdot,t)\nu - \int_0^1 \frac{1}{\beta}\partial_x\phi(\cdot,t)\nu',\end{aligned}$$

for each v and $\nu \in \mathcal{T}[0,1]$.

To write the Timoshenko beam model in terms of bilinear forms we add the equations above to obtain

$$\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\partial_t^2\phi(\cdot,t)\nu = -\int_0^1 (\partial_x w(\cdot,t) - \phi(\cdot,t))(v' - \nu) - \int_0^1 \frac{1}{\beta}\partial_x\phi(\cdot,t)\nu' + \int_0^1 q(\cdot,t)v.$$

Define the product space $\mathcal{T} = \mathcal{T}[0,1] \times \mathcal{T}[0,1]$ and the notation $y = \langle y_1, y_2\rangle$ for $y \in \mathcal{T}$. The following bilinear forms are needed

$$c(u,v) = (u_1, v_1) + \left(\frac{1}{\alpha}u_2, v_2\right) \quad \text{and}$$
$$b(u,v) = (u_1' - u_2, v_1' - v_2) + \left(\frac{1}{\beta}u_2', v_2'\right).$$

The variational problem is:
Find $u(\cdot,t) = \langle w(\cdot,t), \phi(\cdot,t)\rangle \in \mathcal{T}$ such that

$$c(\partial_t^2 u(\cdot,t), v) + b(u(\cdot,t), v) = (q(\cdot,t), v_1) \quad \text{for all} \quad v \in \mathcal{T}.$$

1.4 Rayleigh and Euler-Bernoulli beam models

The Rayleigh and Euler-Bernoulli models are special cases of the Timoshenko model. A difference between the beam models is that the Timoshenko model accounts for shear deformation while the Euler-Bernoulli and Rayleigh models do not.

For convenience, we recall the equations of motion and constitutive for the Timoshenko model.

$$\partial_t^2 w = \partial_x F + q, \tag{1.4.1}$$
$$\frac{1}{\alpha}\partial_t^2\phi = F + \partial_x M, \tag{1.4.2}$$
$$F = \partial_x w - \phi, \tag{1.4.3}$$
$$M = \frac{1}{\beta}\partial_x\phi. \tag{1.4.4}$$

To derive the Rayleigh model from the Timoshenko model, it is necessary to eliminate F in Equation (1.4.1) and Equation (1.4.2). Assume that every cross-section of a beam remains perpendicular to the neutral surface, i.e. $\partial_x w = \phi$. A consequence of this assumption is that

Equation (1.4.3) cannot be used. Differentiating Equation (1.4.2) then subtracting the result from Equation (1.4.1) yields

$$\partial_t^2 w - \frac{1}{\alpha}\partial_t^2 \partial_x \phi = -\partial_x^2 M + q. \tag{1.4.5}$$

Substituting the assumption $\partial_x w = \phi$ on the cross-sections in Equation (1.4.5) gives the dimensionless equation of motion for the Rayleigh model

$$\partial_t^2 w - \frac{1}{\alpha}\partial_t^2 \partial_x^2 w = -\partial_x^2 M + q. \tag{1.4.6}$$

From the assumption $\partial_x w = \phi$ and the only constitutive equation equation that can be used, we have that

$$\begin{aligned} M &= \frac{1}{\beta}\partial_x \phi \\ &= \frac{1}{\beta}\partial_x(\partial_x w) \\ &= \frac{1}{\beta}\partial_x^2 w. \end{aligned} \tag{1.4.7}$$

The term $-\frac{1}{\alpha}\partial_t^2\partial_x^2 w$ in Equation (1.4.6) is known as rotary inertia.

The Euler-Bernoulli model is obtained by substituting Equation (1.4.7) into Equation (1.4.6), provided the rotary inertia term is neglected. Particularly, $\partial_t^2 w = -\frac{1}{\beta}\partial_x^4 w + q$.

Remark. *In some applications, Equation* (1.4.2) *is used as a "boundary condition". It is useful to consider the following example, assume that a beam is free at one of its endpoints, say x^*. The boundary condition becomes $F(x^*, t) = 0$, applying this result and allowing $\partial_x w = \phi$ in Equation* (1.4.2) *implies*

$$\partial_x M(x^*, t) = \frac{1}{\alpha}\partial_t^2 \partial_x w(x^*, t). \tag{1.4.8}$$

1.4.1 Damped Rayleigh beam model

To determine the influence of damping on the well-posedness of a Rayleigh model, we discuss three types of damping namely: Kelvin-Voigt, viscous and boundary damping. We use an example to introduce relevant and useful concepts.

Given non-negative dimensionless constants μ^*, γ^*, k_0^* and k_1^* and positive constants α and β, find w such that

$$\partial_t^2 w - \frac{1}{\alpha}\partial_t^2\partial_x^2 w = -\partial_x^2 M - \mu^*\partial_t\partial_x^4 w - \gamma^*\partial_t w, \tag{1.4.9}$$

$$w(0,t) = \partial_x w(0,t) = 0, \tag{1.4.10}$$

$$\frac{1}{\alpha}\partial_t^2\partial_x w(1,t) - \partial_x M(1,t) = \mu^*\partial_t\partial_x^3 w(1,t) - k_0^*\partial_t w(1,t), \tag{1.4.11}$$

$$M(1,t) = -\mu^*\partial_t\partial_x^2 w(1,t) - k_1^*\partial_t\partial_x w(1,t), \tag{1.4.12}$$

$$w(\cdot,0) = w_0 \text{ and } \partial_t w(\cdot,0) = w_d. \tag{1.4.13}$$

According to [VV02], the term $\mu^*\partial_t\partial_x^4 w$ is used to model Kelvin-Voigt damping, $\gamma^*\partial_t w$ viscous damping and the terms $k_0^*\partial_t w(1,t)$ and $k_1^*\partial_t\partial_x w(1,t)$ boundary damping. The constants μ^*, γ^*, k_0^* and k_1^* are referred to as damping coefficients.

1.4.2 Variational form

The variational form of the damped Rayleigh model problem is obtained by multiplying Equation (1.4.9) by a function $v \in C^1[0,1]$. Integrating and using integration by parts we have that

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v - \left(\frac{1}{\alpha}\partial_t^2\partial_x w(1,t)v(1) - \frac{1}{\alpha}\partial_t^2\partial_x w(0,t)v(0) - \int_0^1 \frac{1}{\alpha}\partial_t^2\partial_x w(\cdot,t)v'\right)\\
&= \int_0^1 \partial_x M(\cdot,t)v' + \int_0^1 \mu^*\partial_t\partial_x^3 w(\cdot,t)v' - \int_0^1 \gamma^*\partial_t w(\cdot,t)v\\
&\quad - \Big(\mu^*\partial_t\partial_x^3 w(1,t) + \partial_x M(1,t)\Big)v(1) + \Big(\mu^*\partial_t\partial_x^3 w(0,t) + \partial_x M(0,t)\Big)v(0)\\
&= -\int_0^1 M(\cdot,t)v'' - \int_0^1 \mu^*\partial_t\partial_x^2 w(\cdot,t)v'' - \int_0^1 \gamma^*\partial_t w(\cdot,t)v\\
&\quad + \Big(\mu^*\partial_t\partial_x^2 w(1,t) + M(1,t)\Big)v'(1) - \Big(\mu^*\partial_t\partial_x^2 w(0,t) + M(0,t)\Big)v'(0)\\
&\quad - \Big(\mu^*\partial_t\partial_x^3 w(1,t) + \partial_x M(1,t)\Big)v(1) + \Big(\mu^*\partial_t\partial_x^3 w(0,t) + \partial_x M(0,t)\Big)v(0)
\end{aligned} \tag{1.4.14}$$

Regrouping the terms of Equation (1.4.14) results in

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\partial_t^2\partial_x w(\cdot,t)v' \\
&= -\int_0^1 M(\cdot,t)v'' - \int_0^1 \mu^*\partial_t\partial_x^2 w(\cdot,t)v'' \\
&\quad - \int_0^1 \gamma^*\partial_t w(\cdot,t)v - \Big(\partial_x M(1,t) - \frac{1}{\alpha}\partial_t^2\partial_x w(1,t) + \mu^*\partial_t\partial_x^3 w(1,t)\Big)v(1) \\
&\quad + \Big(\mu^*\partial_t\partial_x^3 w(0,t) - \frac{1}{\alpha}\partial_t^2\partial_x w(0,t) + \partial_x M(0,t)\Big)v(0) \\
&\quad + \Big(\mu^*\partial_t\partial_x^2 w(1,t) + M(1,t)\Big)v'(1) + \Big(-\mu^*\partial_t\partial_x^2 w(0,t) - M(0,t)\Big)v'(0).
\end{aligned}$$

The space of test functions is defined by

$$\mathcal{T}[0,1] = \{v \in C^2[0,1] \mid v(0) = v'(0) = 0\}.$$

Substituting Equations (1.4.7), (1.4.11) and (1.4.12) into the modified Equation (1.4.14) results in the variational form: Given non-negative damping coefficients μ^*, γ^*, k_0^* and k_1^* and positive constants α and β, find w where $w(\cdot,t) \in \mathcal{T}[0,1]$ for each $t > 0$ such that

$$\begin{aligned}
&\int_0^1 \partial_t^2 w(\cdot,t)v + \int_0^1 \frac{1}{\alpha}\partial_t^2\partial_x w(\cdot,t)v' \\
&= -\int_0^1 \frac{1}{\beta}\partial_x^2 w(\cdot,t)v'' - \int_0^1 \mu^*\partial_t\partial_x^2 w(\cdot,t)v'' - \int_0^1 \gamma^*\partial_t w(\cdot,t)v \\
&\quad - k_0^*\partial_t w(1,t)v(1) - k_1^*\partial_t\partial_x w(1,t)v'(1)
\end{aligned} \tag{1.4.15}$$

for each $v \in \mathcal{T}[0,1]$.

Define the following bilinear forms

$$\begin{aligned}
c(u,v) &= (u,v) + \left(\frac{1}{\alpha}u', v'\right), \\
b(u,v) &= \left(\frac{1}{\beta}u'', v''\right) \quad \text{and} \\
a(u,v) &= (\mu^* u'', v'') + (\gamma^* u, v) + k_0^* u(1,t)v(1) + k_1^* u'(1,t)v'(1).
\end{aligned}$$

Variational equation (1.4.15) can now be written as

$$c(\partial_t^2 w(\cdot,t), v) + a(\partial_t w(\cdot,t), v) + b(w(\cdot,t), v) = 0 \quad \forall\, v \in \mathcal{T}. \tag{1.4.16}$$

Remark. *Variational equations* (1.2.7) *and* (1.4.16) *look identical, but the definition of the bilinear forms differs.*

1.5 Heat conduction

In this section, we discuss various heat conduction models. A summary of the nomenclature that is used throughout the section is included in the table below.

Symbol	Description	Units
$[\rho]$	Volume density of a material	$kg \cdot m^{-3}$
$[c_\rho]$	Specific heat capacity of a material	$J \cdot kg^{-1} \cdot {}^\circ C^{-1}$
$[k]$	Thermal conductivity	$W \cdot m^{-1} \cdot {}^\circ C^{-1}$
$[\alpha]$	Thermal diffusivity	$m^2 \cdot s^{-1}$
$[q]$	Heat flux	$W \cdot m^{-2}$
$[T]$	Temperature	${}^\circ C$
$[\tau_q]$	Phase-lag associated to q	s
$[\tau_T]$	Phase-lag associated to T	s

Table 1.1: Nomenclature for heat transfer.

1.5.1 Conservation law for heat conduction

Let Ω be an arbitrary region in space with a boundary defined by $\partial\Omega$. Suppose Σ is a part of $\partial\Omega$. The quantity of heat energy required to raise the temperature in region Ω from 0 to T is

$$\iiint_\Omega \rho c_p T dV,$$

where $T(\bar{r})$ is the temperature at $\bar{r}$ in region Ω. For convenience, zero can be taken as the ambient temperature.

The heat flux into region Ω is described by

$$-\iint_{\partial\Omega} q \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit outward normal vector along $\partial\Omega$.

The conservation law of heat energy is given by

$$\frac{d}{dt}\iiint_\Omega \rho c_\rho T dV = -\iint_{\partial\Omega} q \cdot \mathbf{n} dS.$$

From the conservation law it follows

$$\rho c_\rho \partial_t T = -\text{div}(q). \tag{1.5.1}$$

Fourier's law of heat conduction is the supposed constitutive equation

$$q = -k\nabla T, \tag{1.5.2}$$

where k is a constant term for thermal conductivity. Substituting Equation (1.5.2) into Equation (1.5.1) yields a partial differential equation known as the classical heat equation:

$$\partial_t T = \alpha \nabla^2 T, \quad \text{with } \alpha = \frac{k}{\rho c_p}.$$

1.5.2 Hyperbolic heat conduction model

Fourier's law is generally considered to yield a reliable model for heat conduction. However some scientists, for instance Cattaneo and Vernotte, criticized the fact that energy can be transplanted at infinite speed according to the classical heat equation model.

The authors of [Cat48] and [Ver58] independently suggested an alternative constitutive equation to Fourier's law. Their proposed constitutive equation is:

$$q + \tau_q \partial_t q = -k\nabla T, \tag{1.5.3}$$

where τ_q is the "time delay" in the heat flux, see Subsection 1.5.3.

According to [Tzo95], Equation (1.5.3) is referred to as the Cattaneo-Vernotte equation, while the authors of [DWJMB08] refer to Equation (1.5.3) as the Maxwell-Cattaneo equation. In this dissertation, Equation (1.5.3) will be referred to as the Maxwell-Cattaneo equation.

The hyperbolic heat conduction model is derived as follows: Differentiating Equation (1.5.1) yields

$$\rho c_\rho \partial_t^2 T = -\text{div}(\partial_t q). \tag{1.5.4}$$

Taking the divergence of Equation (1.5.3) it follows

$$\text{div}(q) + \text{div}(\tau_q \partial_t q) = -k\nabla^2 T. \tag{1.5.5}$$

Finally, combining Equations (1.5.4) and (1.5.5), then applying Equation (1.5.1), the law of conservation of heat energy results in

$$\tau_q \rho c_\rho \partial_t^2 T + \rho c_\rho \partial_t T - k\nabla^2 T = Q, \tag{1.5.6}$$

where Q denotes an externally generated heat source.

One dimensional model

The one-dimensional hyperbolic heat conduction equation is given by

$$\tau_q \rho c_p \partial_t^2 T + \rho c_p \partial_t T - k \partial_x^2 T = Q. \tag{1.5.7}$$

Note that Equation (1.5.7) is similar to the one-dimensional wave equation with viscous damping as seen in Section 1.2, only with a change in parameters.

Single-phase-lag model

In 1995, Tzou [Tzo95] proposed the following model:

$$q(\bar{r}, t + \tau_q) = -k \nabla T(\bar{r}, t), \tag{1.5.8}$$

where τ_q is the phase-lag in the heat flux.

Equation (1.5.8) is referred to as the single-phase-lag (SPL) model. The model accounts for the brief time delay in the conduction of heat by a material. Equation (1.5.3) can be derived from Equation (1.5.8) by taking the linear approximation

$$q(\bar{r}, t + \tau_q) = q(\bar{r}, t) + \tau_q \partial_t q(\bar{r}, t).$$

1.5.3 Dual-phase-lag model

In addition to the SPL model, the author of [Tzo95] suggested a Dual-Phase-Lag (DPL) model. For a DPL model, the phase-lag in the heat flux (τ_q) and gradient of the temperature (τ_T) are taken into account. The proposed model is of the form:

$$q(\bar{r}, t + \tau_q) = -k \nabla T(\bar{r}, t + \tau_T). \tag{1.5.9}$$

The linearised constitutive equation for the model is:

$$q + \tau_q \partial_t q = -k \nabla T - k \tau_T \partial_t \big(\nabla T\big). \tag{1.5.10}$$

Combining the conservation law of heat energy (Equation (1.5.4)) with the linearised constitutive equation (Equation (1.5.10)), results in the Dual Phase Lag model:

$$\tau_q \rho c_p \partial_t^2 T + \rho c_p \partial_t T = \text{div}(kT) + \partial_t \text{div}(k \tau_T \nabla T). \tag{1.5.11}$$

Remark. *In literature, SPL models are classified as hyperbolic partial differential equations, whereas DPL models are not, see [LWC12].*

As mentioned, in this dissertation all partial differential equations with second order time derivatives and a variational form similar to that of the wave equation will be referred to as second order hyperbolic problems. This approach follows the work by various authors using the finite element method.

1.5.4 Dimensionless forms of heat conduction models

To write heat conduction models in their respective dimensionless forms, we introduce the following dimensionless variables in Equation (1.5.1) and Equation (1.5.3), using the following parameter transformation:

$$\begin{aligned} T^* &= \frac{T}{T_0}, \\ q^* &= \frac{qt_0}{\rho c_p T_0 L}, \\ x^* &= \frac{x}{L}, \end{aligned}$$

$$\begin{aligned} t^* &= \frac{t}{t_0}, \\ \tau_q^* &= \frac{\tau_q}{t_0} \qquad \text{and} \\ \alpha^* &= \frac{\alpha t_0}{L^2}. \end{aligned} \tag{1.5.12}$$

Recall that $\alpha = \dfrac{k}{\rho c_p}$. In Equation (1.5.12), T_0 is an appropriate reference temperature (e.g, the steady-state or initial temperature) and t_0 is a reference time (e.g, the time required to reach a specific temperature). L is reference distance (e.g, the length or diameter of a specimen).

The dimensionless form of the law of conservation of heat energy (Equation (1.5.1)) becomes

$$\partial_{t^*} T^* = -\text{div}(q^*). \tag{1.5.13}$$

For Fourier's law of heat conduction (Equation (1.5.2)) we obtain

$$q^* = -\alpha^* \nabla^* T^*. \tag{1.5.14}$$

The constitutive equation for the linearised DPL model takes the form

$$q^* + \tau_q^* \partial_{t^*} q^* = -\alpha^* \nabla^* T^* - \alpha^* \tau_T^* \partial_{t^*} (\nabla^* T^*). \tag{1.5.15}$$

Returning to the original notation, Equations (1.5.13) to (1.5.15) respectively become

$$q = -\alpha \nabla T, \tag{1.5.16}$$
$$\partial_t T = -\text{div}(q) \quad \text{and} \tag{1.5.17}$$
$$\partial_t T + \tau_q \partial_t T = -\alpha \nabla T - \alpha \tau_T \partial_t (\nabla T). \tag{1.5.18}$$

Setting $\tau_T = 0$ in Equation (1.5.18) yields the dimensionless and linearised equation for the SPL model

$$q + \tau_q \partial_t q = -\alpha \nabla T. \tag{1.5.19}$$

The combination of the law of heat conservation with the respective constitutive equation yields the following dimensionless heat conduction models:

$$\partial_t T = \text{div}(\alpha \nabla T), \tag{1.5.20}$$
$$\partial_t T + \tau_q \partial_t^2 T = \text{div}(\alpha T) + \alpha \tau_T \partial_t \text{div}(\nabla T) \quad \text{and} \tag{1.5.21}$$
$$\partial_t T + \tau_q \partial_t^2 T = \text{div}(\alpha \nabla T). \tag{1.5.22}$$

Similarly the one-dimensional single-phase-lag model in dimensionless form is:

$$\partial_t T + \tau_q \partial_t^2 T = \alpha \partial_x^2 T.$$

1.5.5 Boundary condition and variational form

In this subsection we discuss specific boundary conditions and the variational form of a multi-dimensional single-phase-lag model. Recall that the boundary of the domain is denoted by $\partial\Omega$ and that Σ is a part of the boundary.

The boundary condition for heat conduction models using Equation (1.5.16) is given by

$$-\alpha \nabla T \cdot \mathbf{n} = 0 \quad \text{along } \Sigma. \tag{1.5.23}$$

A problem can be formulated as:

Given positive constants α and τ_q, find T for each $t > 0$ such that

$$\tau_q \partial_t^2 T = \text{div}(\alpha T) - \partial_t T, \text{ in } \Omega \text{ for } t \in (0, t_*] \text{ with } t_* < \infty. \quad (1.5.24)$$

with boundary conditions (1.5.23) and

$$\nabla T \cdot \mathbf{n} = 0 \text{ along } \partial\Omega - \Sigma \text{ for } t \in (0, t_*], \quad (1.5.25)$$

and initial conditions

$$T(x, 0) = T_0 \text{ and } \partial_t T(x, 0) = T_d \text{ for } x \in \Omega. \quad (1.5.26)$$

The variational form of Equation (1.5.24) with boundary conditions (1.5.23) and (1.5.25) and initial conditions (1.5.26),is derived by multiplying Equation (1.5.24) by a function $v \in C^1(\bar{\Omega})$. Integrate, if $T \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$, use Green's formula to obtain

$$\begin{aligned}\iiint_\Omega \tau_q \partial_t^2 T(\cdot, t) v dV = &- \iiint_\Omega \alpha \nabla T(\cdot, t) \cdot \nabla v dV - \iiint_\Omega \partial_t T(\cdot, t) v dV \\ &+ \iint_\Sigma v(\alpha \nabla T(\cdot, t) \cdot \mathbf{n}) dS \\ &+ \iint_{\partial\Omega - \Sigma} v(\alpha \nabla T(\cdot, t) \cdot \mathbf{n}) dS, \end{aligned} \quad (1.5.27)$$

where $\mathbf{n}$ as the unit outward normal vector to $\partial\Omega$.

The space of test functions is defined to be

$$\mathcal{T}(\Omega) = \{v \in C^1(\bar{\Omega}) \mid v = 0 \text{ along } \partial\Omega - \Sigma\}.$$

Substituting the boundary condition (1.5.23) into Equation (1.5.27) we have that the variational form of the problem is: Given a function α and a positive constant τ_q, find T for each $t > 0$ where $T(\cdot, t) \in \mathcal{T}(\Omega)$ such that

$$\iiint_\Omega \tau_q \partial_t^2 T(\cdot, t) v dV = - \iiint_\Omega \alpha \nabla T(\cdot, t) \cdot \nabla v dV - \iiint_\Omega \partial_t T(\cdot, t) v dV, \quad (1.5.28)$$

for each $v \in \mathcal{T}(\Omega)$ with $T(\cdot, 0) = T_0$ and $\partial_t T(\cdot, 0) = T_d$.

Let the $\mathcal{L}^2(\Omega)$ inner product be denoted by $(\cdot, \cdot)$ and define bilinear forms

$$\begin{aligned} c(u, v) &= (\tau_q u, v), \\ b(u, v) &= \iiint_\Omega \alpha \nabla u \cdot \nabla v dV, \quad \text{and} \\ a(u, v) &= (u, v). \end{aligned}$$

The variational equation (1.5.28) can now be written as:

$$c(\partial_t^2 T(\cdot,t),v) + a(\partial_t T(\cdot,t),v) + b(T(\cdot,t),v) = 0, \qquad (1.5.29)$$

for all $v \in \mathcal{T}(\Omega)$.

Remark. *The variational form of the one-dimensional case of the problem in dimensionless form is discussed in Section 2.1.*

1.6 Biological application of hyperbolic heat conduction

In this section, we consider biological applications of the Single-Phase Lag (SPL) and Dual-Phase Lag (DPL) models, specifically in bio-heat transfer in skin. Heat transfer in living tissue is known as bio-heat transfer see [DWJMB08] and [LWC12].

To introduce bio-heat transfer, we consider the mathematical models in [DWJMB08] and [LWC12] and will refer to them as model in Dai et al. and model in Liu et al. respectively. In these articles, the skin is modelled as a tri-layered structure consisting of: epidermis, dermis and subcutaneous fat correspondingly.

According to the authors of [DWJMB08] and [LWC12], complexities arise as a result of the differences in the physiological and thermal properties of each layer of skin. Both authors insist that blood perfusion affects the thermal response in living tissues.

To mathematically describe bio-heat transfer with expected difficulties, characteristics of the complexities are accommodated by introducing constraints on a model. These constraints differ depending on the objective of the model.

To illustrate the influence of additional constraints on a model, consider a layered structure described by a linearised DPL model of the form:

$$\tau_{q,i}\partial_t^2 T_i = \alpha_i \nabla^2 T_i + \alpha_i \tau_{T,i}(\nabla^2 T_i) - \partial_t T_i, \qquad (1.6.1)$$

where i denotes each quantity for a corresponding layer.

A special case of Equation (1.6.1) is an SPL model of the form:

$$\tau_{q,i}\partial_t^2 T_i = \alpha_i \nabla^2 T_i - \partial_t T_i, \qquad (1.6.2)$$

Additional conditions are required at the interface between the layers for Equations (1.6.1) and (1.6.2).

For the rest of this section, we consider specified applications of Equations (1.6.1) and (1.6.2) and the various constraints that are encountered for some bio-heat transfer models.

1.6.1 Model by Dai et al. [DWJMB08]

In [DWJMB08], a three-dimensional SPL model is applied to investigate the effect of high thermal radiation on skin.

Equation (1.6.2) may be applied for this investigation, where the skin is described on a three-dimensional domain characterising the three composites of skin. The composites are considered on a x, y, z system, where radiation is assumed to occur along the z-axis.

To account for blood perfusion, the authors consider metabolic heat generation and blood flow in the arteries and veins in their model. The proposed non-dimensionless model by Dai et al. is:

$$k_i \nabla^2 T_i = \rho_i c_i \partial_t T_i + \tau_q \rho_i c_i \partial_t^2 T_i + \tau_q w_i c_i^b \partial_t T_i + w_i c_i^b (T_i - T_b) - R_i, \tag{1.6.3}$$

where w_i is the blood perfusion rate in the i^{th} skin layer, T_b is the temperature of the blood at exit or at entrance of the third level vessel for the artery or vein. The specific heat of blood at each layer is denoted by c_i^b, and R_i is the volumetric heat at each layer.

An additional constraint on the model by Dai et al. is a result of the assumption that heat transfer on the skin surface is described by:

$$-k_1 \partial_z T_1 = h(T_a - T_1) + \varepsilon\sigma(T_a^4 - T_1^4), \quad z = 0,$$

here h is the convective heat transfer co-efficient, T_a is the ambient temperature, σ is the Stefan-Boltzmann constant, and ε is the emissivity. For thermal radiation $T_a > T_1$.

An additional assumption is that heat flux approaches zero as the depth of tissue being heated increases.

The boundary condition for the problem is

$$\nabla T_1 \cdot n = 0,$$

where n is the unit outward normal to the first skin layer.

The interface conditions are given by

$$\begin{aligned} T_1 &= T_2, \quad k_1\partial_z T_1 = k_2\partial_z T_2, \quad z = L_1,\\ T_2 &= T_3, \quad k_2\partial_z T_2 = k_3\partial_z T_3, \quad z = L_1 + L_2. \end{aligned}$$

The authors of [DWJMB08], imposed a constraint on the first and second skin layers, the respective lengths of veins and arteries along the z-axis.

The prescribed initial conditions are

$$T_i = T_0, \quad t = 0, \quad i = 1, 2, 3.$$

1.6.2 Model by Liu et al. [LWC12]

As previously mentioned, the authors of [LWC12] also consider a tri-layered structure describing bio-heat transfer in skin. Unlike [DWJMB08], the model is regarded on a one-dimensional domain. For their model, Liu et al. also considered blood perfusion and metabolic heat generation.

Note that Equation (1.6.1) may be applied for this investigation, where the skin is described on a one-dimensional domain characterising the three composites. In this case, heat transfer is considered along the x-axis.

The proposed non-dimensionless model, on a generalised finite dimensional domain by Liu et al. is of the form

$$\begin{aligned} \tau_{q,i}\rho_i c_i \partial_t^2 T_i =& \operatorname{div}(k_i T_i) + \tau_{T,i}\partial_t \nabla^2 T_i - \rho_i c_i \partial_t T_i\\ &+ \left(\rho_b w_b c_b + \tau_q \rho_b w_b c_b \frac{\partial}{\partial_t}\right)(T_i - T_b)\\ &- \left(1 + \tau_q \frac{\partial}{\partial_t}\right)(r_m + r_r), \end{aligned} \tag{1.6.4}$$

where ρ_b, c_b and w_b are respectively, the density, specific heat and perfusion rate of blood. r_m is the metabolic heat generation, r_r is the heat source for spatial heating and T_b denotes the arterial temperature.

An additional constraint for their model is that r_m is a constant thermal parameter while $r_r = 0$. The linearised one dimensional DPL model for bio-heat transfer is

$$\begin{aligned}\tau_{q,i}\rho_i c_i \partial_t^2 T_i =& k_i \partial_x^2 T_i + \tau_{T,i}\partial_t \partial_x^2 T_i - \rho_i c_i \partial_t T_i - r_m \\ &+ \left(\rho_b w_b c_b + \tau_q \rho_b w_b c_b \frac{\partial}{\partial_t}\right)(T_i - T_b).\end{aligned}$$

for $i = 1, 2, 3$.

Additional conditions were also imposed at the boundaries of the extreme layers

$$\begin{aligned} T(0,t) &= 100[1 - u(t-15)] \quad 0 < t \leq 45, \\ T(L_3, t) &= T_b, \ t > 0. \end{aligned}$$

where L_i denotes the dimension of each i^{th} layer.

In 2012, Liu et al. considered heat conduction in the x direction and use L_i to denote the respective length of the skin layer.

The prescribed interface conditions are

$$\begin{aligned} T_1(L_1, t) = T_2(0, t), \quad T_2(L_1 + L_2, t) = T_3(0, t), \\ r_1(L_1) = r_2(0, t) \text{ and } r_2(L_1 + L_2, t) = r_2(0, t). \end{aligned}$$

Initial conditions are

$$T(x, 0) = T_b \ \text{ and } \ \partial_t T(x, 0) = 0.$$

Remark. *A detailed study of these complex biological models is beyond the scope of this dissertation. However, we consider a simplified model by [DD19] in Chapter 7.*

Chapter 2

Second order hyperbolic type problems

This dissertation is mainly about the application of the finite element method to hyperbolic type problems. However, we consider it prudent to establish the existence and uniqueness of solutions to such problems (whenever possible). In this chapter, the variational approach to the existence for second order hyperbolic type problems is considered.

2.1 Weak variational form

In this section, we introduce the idea of a weak variational form which serves as motivation for the theory in the next section.

In Chapter 1, various model problems were written in variational form. Since our interest is in the existence of solutions, we also investigate when will a solution of a problem in variational form be a solution of the corresponding boundary value problem. The one-dimensional hyperbolic heat conduction problem is used as an example.

Problem HHE (Hyperbolic Heat Equation)

Given a function Q and parameters τ_q and α, find T such that

$$\tau_q \partial_t^2 T = \alpha \partial_x^2 T - \partial_t T + Q \text{ in } (0,1) \text{ for each } t > 0, \tag{2.1.1}$$

$$T(0,t) = \partial_x T(1,t) = 0 \text{ for each } t > 0, \tag{2.1.2}$$

$$T(\cdot,0) = T_0 \text{ and } \partial_t T(\cdot,0) = T_d \text{ for all } x \in (0,1). \tag{2.1.3}$$

As previously mentioned, from a mathematical perspective Problem HHE is the same as the wave equation with viscous damping. In his book [Wei95], the author rigorously showed how a solution may be obtained using Fourier series. The result depended on the properties of the functions T_0 and T_d. The analysis by Weinberger is not at all trivial and in this chapter, we will show how a general theory can be applied to obtain the uniqueness and existence of a solution.

To obtain the variational form of Problem HHE, multiply Equation (2.1.1) by a function $v \in C^1[0,1]$ and integrate. From integration by parts we have

$$\int_0^1 \alpha \partial_x^2 T(\cdot,t) v = \alpha \partial_x T(1,t) v(1) - \alpha \partial_x T(0,t) v(0) - \int_0^1 \alpha \partial_x T(\cdot,t) v', \tag{2.1.4}$$

provided $\partial_x^2 T(\cdot,t)$ is integrable. Since $T(\cdot,t)$ is a solution of Equation (2.1.1)

$$\begin{aligned}\int_0^1 \tau_q \partial_t^2 T(\cdot,t) v = & \alpha \partial_x T(1,t) v(1) - \alpha \partial_x T(0,t) v(0) \\ & - \int_0^1 (\alpha \partial_x T(\cdot,t) v' + \partial_t T(\cdot,t) v) + \int_0^1 Q(\cdot,t) v.\end{aligned} \tag{2.1.5}$$

The space of test functions is defined by

$$\mathcal{T}[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}. \tag{2.1.6}$$

Note that for any $v \in \mathcal{T}[0,1]$, Equation (2.1.5) takes the form

$$\int_0^1 \tau_q \partial_t^2 T(\cdot,t) v = - \int_0^1 (\alpha \partial_x T(\cdot,t) v' + \partial_t T(\cdot,t) v) + \int_0^1 Q(\cdot,t) v,$$

(since $\partial_x T(1,t) = 0$).

Notation
The $\mathcal{L}^2(0,1)$ inner product is denoted by $(\cdot,\cdot)$ and the bilinear form b is defined as

$$b(u,v) = (\alpha u', v').$$

We can now write the variational form of Problem HHE.

Problem HHE-V

Find T where $T(\cdot,t) \in \mathcal{T}[0,1]$ for each $t > 0$, such that for each $v \in \mathcal{T}[0,1]$,

$$(\tau_q \partial_t^2 T(\cdot,t), v) + (\partial_t T(\cdot,t), v) + b(T(\cdot,t), v) = (Q(\cdot,t), v), \quad \text{while } T(\cdot,0) = T_0 \text{ and } \partial_t T(\cdot,0) = T_d. \tag{2.1.7}$$

We have actually proved the following result.

Proposition 2.1.1. *A solution of Problem HHE is a solution of Problem HHE-V.*

Proposition 2.1.2. *A solution T of Problem HHE-V is a solution of Problem HHE provided $T(\cdot,t) \in C^2[0,1]$ for each $t > 0$.*

Proof. Since T is a solution of Problem HHE-V, $T(\cdot,t) \in \mathcal{T}[0,1]$ and hence $T(0,t) = 0$.

Next, we prove that T satisfies the partial differential equation (2.1.1). In this proof, the set $\mathcal{S} = \{v \in C^1[0,1] \mid v(0) = v(1) = 0\}$ is useful. Since T is a solution of Problem HHE-V and $\mathcal{S}$ is a subset of $\mathcal{T}[0,1]$ we obtain

$$(\tau_q \partial_t^2 T(\cdot,t), v) + (\partial_t T(\cdot,t), v) + b(T(\cdot,t), v) = (Q(\cdot,t), v) \quad \text{for each } v \in \mathcal{S}. \tag{2.1.8}$$

Observe that from Equation (2.1.4)

$$b(T(\cdot,t), v) = -(\alpha \partial_x^2 T(\cdot,t), v) \text{ for all } v \in \mathcal{S}. \tag{2.1.9}$$

Substituting Equation (2.1.9) into Equation (2.1.8) yields

$$(\tau_q \partial_t^2 T(\cdot,t) + \partial_t T(\cdot,t) - \alpha \partial_x^2 T(\cdot,t), v) = (Q(\cdot,t), v) \text{ for each } v \in \mathcal{S}. \tag{2.1.10}$$

By Theorem A.1.1 (in Appendix A.1), the set $\mathcal{S}$ is dense in $\mathcal{L}^2(0,1)$ and hence Equation (2.1.10) holds for each $v \in \mathcal{L}^2(0,1)$. It follows that

$$\tau_q \partial_t^2 T(\cdot,t) + \partial_t T(\cdot,t) - \alpha \partial_x^2 T(\cdot,t) = Q(\cdot,t).$$

Lastly, we consider the boundary condition at the right endpoint. Since $T(\cdot,t)$ satisfies Equation (2.1.1), it also satisfies Equation (2.1.5), for each $v \in \mathcal{T}[0,1]$ and hence

$$(\tau_q \partial_t^2 T(\cdot,t), v) + (\partial_t T(\cdot,t), v) + b(T(\cdot,t), v) = \alpha \partial_x T(1,t) v(1) + (Q(\cdot,t), v). \tag{2.1.11}$$

Subtracting Equation (2.1.11) from Equation (2.1.7) yields

$$\alpha \partial_x T(1,t) v(1) = 0. \tag{2.1.12}$$

But, by the definition of $\mathcal{T}[0,1], v(1)$ is arbitrary. Thus, $\partial_x T(1,t) = 0$.

It follows that T is a solution of Problem HHE. □

It is natural to ask about the relevance of a solution of Problem HHE-V, which is not a solution of Problem HHE. At this point, the proof of the validity of the series solution for Problem HHE in [Wei95] provides some insight. To obtain a classical solution for Problem HHE, restrictive conditions are imposed on the initial conditions T_0 and T_d. To be precise, it is required that the function $T_0 \in C^2[0,1]$ satisfies the boundary conditions (2.1.2) and $T_0''(0) = 0$. From a practical point of view, this appears to be too restrictive.

Problem HHE-V is used to implement the Finite Element Method.

Next, we introduce the weak variational form of the problem.

The function T is defined on $[0,1] \times [0,t_*]$, we now consider a function $u(t)(x) = T(x,t)$ such that

$$u : [0,t_*] \to \mathcal{T}[0,1] \text{ for each } t_* < \infty.$$

Similarly for $Q(x,t)$, we consider a function $\tilde{q}(t)(x) = Q(x,t)$.

To prepare for the abstract problem in the next section we define bilinear forms

$$a(u,v) = (u,v), \text{ and } c(u,v) = (\tau_q u, v).$$

Note that the bilinear form b is defined in the sense of weak derivatives.

Let $\mathcal{J}$ be an interval containing zero and Z be any Hilbert space. Consider a function u on $\mathcal{J}$ with values in Z. We write $u'(t) \in Z$ if the derivative exists with respect to the norm of Z. Note that all norms are not equivalent in an infinite dimensional space. We use the following notation:

1. $u \in C^k(\mathcal{J}, Z)$ if $u^{(k)}(t) \in C(\mathcal{J}, Z)$,
2. $u^{(k)} \in \mathcal{L}^2(\mathcal{J}, Z)$ if $u^{(k)}(t) \in Z$ for each $t \in \mathcal{J}$ and $\int_{\mathcal{J}} \|u^{(k)}\|_Z^2 < \infty$.

Let $V(0,1)$ be the closure of $\mathcal{T}[0,1]$ in the Sobolev space $H^1(0,1)$ (see Appendix A.1 for more details). For the weak variational form of Problem HHE, let Z be the Hilbert space $\mathcal{L}^2(0,1)$.

Problem HHE-W

Find $u \in C^2(\mathcal{J}, \mathcal{L}^2(0,1))$ such that for each $t \in \mathcal{J}, u(t) \in V(0,1)$ and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (\tilde{q}(t),v) \quad \text{for all } v \in V(0,1), \tag{2.1.13}$$

$$\text{while } u(0) = T_0 \text{ and } u'(0) = T_d.$$

Remark. *Note the differences between Problem HHE-V and Problem HHE-W, especially the requirements on the function to be a solution. In literature, a solution of Problem HHE-W is referred to as a weak solution of Problem HHE.*

2.2 The general second order hyperbolic problem

In Section 2.1 we derived the weak variational form of a hyperbolic heat problem. In this section, we consider the solvability of the general case for second order hyperbolic problems as proposed by the authors of [VV02].

Suppose there exists real Hilbert spaces V, W and X such that $V \subset W \subset X$. Consider bilinear forms a, b and c, with a and b defined on V and c defined on W. The basic assumptions are that b is an inner product for V and c is an inner product for W.

Problem GH below is an abstract weak problem which characterises hyperbolic vibration type problems.

Problem GH (General Hyperbolic)

Given $f : \mathcal{J} \to X$, find $u \in C(\mathcal{J}, V)$ such that $u'(t)$ is continuous at zero with respect to $\|\cdot\|_W$ and for each $t \in \mathcal{J}$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (f(t),v)_X \text{ for each } v \in V,$$
$$\text{while } u(0) = u_0, \text{ and } u'(0) = u_d.$$

In the formulation above, $(\cdot,\cdot)_X$ is the inner product for X. It may be helpful to compare Problem GH with the variational form of the wave equation in Section 1.2, where a corresponds to a viscous damping term.

Note that Problem HHE-W in Section 2.1 is a special case of the problem above.

Assumptions

In [VV02] the following assumptions are used to obtain existence results for Problem GH.

A1 V is dense in W and W is dense in X.

A2 There exists a positive constant C_W such that for each $u \in W$, $\|u\|_X \leq C_W \|u\|_W$.

A3 There exists a positive constant C_V such that for each $u \in V$, $\|u\|_W \leq C_V \|u\|_V$.

A4 The bilinear form a is bounded on V i.e., there exists a positive constant C_A such that for each u and $v \in V$,

$$|a(u,v)| \leq C_A \|u\|_V \|v\|_V.$$

Additionally, the bilinear form a is non-negative and symmetric on V.

We state theorems from [VV02] without proofs. However, some discussion of the theory is in the next section.

Theorem 2.2.1. *Suppose Assumptions A1, A2, A3 and A4 hold. If for $u_0 \in V$ and $u_d \in V$, there exists some $y \in W$ such that*

$$b(u_0, v) + a(u_d, v) = c(y, v) \quad \text{for all } v \in V, \tag{2.2.1}$$

then for each $f \in C^1([0,\infty), X)$, there exists a unique solution

$$u \in C^1([0,\infty), V) \cap C^2([0,\infty), W)$$

for Problem GH.

Note that the condition in Equation (2.2.1) is also a necessary condition, as proven in [VV02].

The type of damping has an effect on the theory. Viscous damping as previously mentioned is an example of so called weak damping.

Assumption A4W (Weak damping) The bilinear form a is bounded on W, i.e. there exists a positive constant K_W such that for all u and $v \in W$,

$$|a(u,v)| \leq K_W \|u\|_W \|v\|_W.$$

Additionally, the bilinear form a is non-negative and symmetric on W.

Remark. *A special case of weak damping is when* $a(u,v) = 0$ *for each* u *and* v *in* W. *This is discussed further in Subsection 2.3.2.*

To formulate an improved result, we define the subspace E_b of V :
$E_b := \{x \in V : \text{for some } y \in W, \text{ such that } b(x,v) = c(y,v) \text{ for all } v \in V.\}$

Theorem 2.2.2. *(Weak damping) Suppose Assumptions A1, A2, A3, and A4W hold, then there exists a unique solution*

$$u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W)$$

for Problem GH for all $u_0 \in E_b$, $u_d \in V$ *and* $f \in C^1(\mathcal{J}, X)$.

Note that, as in Theorem 2.2.1, the conditions on u_0 and u_d are necessary.

In some applications, the bilinear form a is positive definite with respect to the V norm. This occurrence is referred to as strong damping.

Assumption A4S (Strong damping) There exists a positive constant K_S such that for each $u \in V$

$$a(u,u) \geq K_S \|u\|_V^2.$$

Theorem 2.2.3. *(Strong damping) Suppose Assumptions A1, A2, A3, and A4S hold, then there exists a unique solution*

$$u \in C^1([0,\infty), V) \cap C^2([0,\infty), W)$$

for Problem GH for all $u_0 \in V$, $u_d \in W$ *and* f *which is Hölder continuous on* V. *If* $f = 0$, *then*

$$u \in C^1([0,\infty), V) \cap C^2([0,\infty), W) \cap C^\infty((0,\infty), V).$$

Remarks.

a. *In Theorem 2.2.3, observe that if* f *is Lipschitz, i.e. for some non-negative constant* κ, $\|f(t_2) - f(t_1)\|_W \leq \kappa|t_2 - t_1|$, *then* f *will be Hölder continuous.*

b. *It is important to note that the spaces V and W must be defined for each application.*

c. *If b is non-symmetric, there exists two bilinear forms b_0 and b_n such that $b = b_0 + b_n$, where b_0 is symmetric. The inner product of V is defined by the bilinear form b_0. Due to the assumption made in [VS19], the results in [VV02] become a special case where $b_n = 0$. In this dissertation and application b is symmetric.*

2.3 First order system

In this section, we consider a first-order system that is equivalent to Problem GH. This makes it possible to show how semigroup theory is used.

2.3.1 General case

It is necessary to construct an operator A such that Problem GH is equivalent to a system of the form

$$U'(t) = AU(t) + F(t),$$
$$U(0) = U_0.$$

Let $H = V \times W$ with inner product

$$(x, y)_H = b(x_1, y_1) + c(x_2, y_2) \quad \text{for any } x \text{ and } y \ \in H.$$

For $x \in H$ denote x and its components by $x = \langle x_1, x_2 \rangle$.

In [VV02], the existence of an operator A with the following properties is proved:

1. A is a densely defined closed linear operator on H.
2. For any $x \in \mathcal{D}(A)$, $Ax = y$ if and only if

$$x_2 = y_1, \quad \text{and}$$
$$b(x_1, v) + a(x_2, v) = -c(y_2, v) \text{ for each } v \in V.$$

   This characterises the operator A and its domain.
3. For any $\lambda \geq 0$, $R(\lambda I - A) = H$.

4. $x \in \mathcal{D}(A)$ if and only if $x_2 \in V$ and there exists $z \in W$ such that
$$b(x_1, v) + a(x_2, v) = c(z, v) \text{ for each } v \in V.$$

5. For each x and $y \in \mathcal{D}(A)$,
$$(Ax, y)_H = b(x_2, y_1) - b(x_1, y_2) - a(x_2, y_2).$$

6. $(Ax, x)_H = -a(x_2, x_2)$ for each $x \in \mathcal{D}(A)$. This is a consequence of the symmetry of b.

7. A is dissipative whenever a is non-negative. This implies that the semigroup generated by A satisfies the inequality $\|T(t)x\| \leq \|x\|$, for all $t > 0$.

Remarks.

a. *To prove the properties above, the authors of [VV02] used Assumptions A1-A4.*

b. *Observe that Equation* (2.2.1) *is a result of property (2).*

Recall that the interval $\mathcal{J}$ contains zero. Using the properties above, we present the initial value problem of the first order system for Problem GH.

Problem IVP (Initial Value Problem)

Given a function $F : \mathcal{J} \to H$, find $U \in C(\mathcal{J}, H)$ such that for each $t \in \mathcal{J}$, $U(t) \in \mathcal{D}(A), U'(t) \in H$ and
$$\begin{aligned} U'(t) &= AU(t) + F(t), \\ U(0) &= U_0. \end{aligned}$$

To relate Problem GH to Problem IVP, we use the following result.

Lemma 2.3.1. *Suppose $F(t) = \langle 0, f(t) \rangle$ for each $t \in \mathcal{J}$.*

a. *If u is a solution of Problem GH, then $U = \langle u, u' \rangle$ is a solution of Problem IVP, with $U_0 = \langle u_0, u_d \rangle$.*

b. *If U is a solution of Problem IVP with $U_0 = \langle u_0, u_d \rangle$, then the first component of $u = U_1$ is a solution of Problem GH.*

The authors of [VV02] and [VS19] used results from semigroup theory (see e.g [Paz83] or [Sho77]), to prove that A is an infinitesimal generator of a C_0 semigroup of contractions in H. Therefore, Problem IVP has a solution if $U_0 \in \mathcal{D}(A)$ and $f \in C^1(\mathcal{J}, H)$.

Remarks

a. *The condition* $U_0 \in \mathcal{D}(A)$ *is necessary, while the condition* $f \in C^1(\mathcal{J}, H)$ *is not. The two conditions together are sufficient.*

b. *The solution* U *of Problem IVP satisfies* $U(t) \in \mathcal{D}(A)$ *for each* t.

2.3.2 Weak damping

In the case of weak damping, better results are proved in [VV02]. Recall that a is non-negative, symmetric and bounded on W.

Property (2) from the general case changes to: $\mathcal{D}(A) = E_b \times V$.

Once more, the authors of [VV02] used results from semigroup theory to prove that A is an infinitesimal generator of a C_0 group in H. (Note that a group is a special case of a semigroup.)

Therefore, we have the following result for weak damping.

Theorem 2.3.1. *Let* A *be the infinitesimal generator of a* C_0 *group and*
$F \in C^1(\mathcal{J}, H)$. *Then Problem IVP has a unique solution* $U \in C^1(\mathcal{J}, H)$ *for all* $U_0 \in \mathcal{D}(A)$. *If* $F = 0$, *then* $U \in C^1((-\infty, \infty), H)$.

Using Lemma 2.3.1, Theorem 2.2.2 follows from the above result.

It may be asked, what happens in the absence of damping ($a = 0$)? By the definition of weak damping, it appears obvious that Theorem 2.2.2 remains true for the case of no damping. To be on the safe side, we investigated the construction of the operator A in [VV02], and the process is not influenced by setting $a = 0$.

2.4 Application of existence theory to the hyperbolic heat conduction problem

In this section, the general theory in Section 2.2 is applied. To this end, we consider Problem HHE-W as derived in Section 2.1. Recall bilinear forms a, b and c and the space $V(0, 1)$ defined in Section 2.1.

2.4.1 Function spaces

As mentioned in Section 2.2, the spaces X, W and V must be defined. Now, for Problem HHE-V, X is simply the space $\mathcal{L}^2(0,1)$. In the next result we prove that the bilinear from c is an inner product for $\mathcal{L}^2(0,1)$. Let W be equal to the linear space $\mathcal{L}^2(0,1)$ but use $c(\cdot,\cdot)$ as an inner product. The space V is the space $V(0,1)$ defined in Section 2.1.

Proposition 2.4.1. *The bilinear form c is an inner product for the space X.*

Proof. Since the bilinear form c is symmetric, it is an inner product if $c(u,u) = 0$ implies $u = 0$. Let $u \in X$, then

$$c(u,u) = (\tau_q u, u) \geq K_C \|u\|_X^2, \tag{2.4.1}$$

where $K_C = \min\{\tau_q\}$. □

Definition 2.4.1 (Inertia space W)**.** *The vector space X equipped with the inner product c is referred to as the Inertia space W. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.*

Proposition 2.4.2. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent on W.*

Proof. First for $u \in W$,

$$\|u\|_W^2 = (\tau_q u, u) \leq K_W \|u\|_X^2,$$

where $K_W = \max\{\tau_q\}$. Using Inequality (2.4.1), the result follows. □

Corollary 2.4.1. *The space W is complete.*

Proof. Applying Proposition A.1.2 in Appendix A and the fact that $\mathcal{L}^2(0,1)$ is complete, it follows that X is complete. Since in this application, $W = X$ and the norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent and the desired result is obtained. □

Proposition 2.4.3. *There exists a positive constant K_B, such that*

$$b(u,u) \geq K_B \|u\|_X^2 \text{ for each } u \in V.$$

Proof. The space V is the closure of the set of test functions. Since $v(0) = 0$, we can use Lemma A.2.4 (in Appendix A.2) to obtain

$$\|u\|_X \leq \|u'\|_X, \tag{2.4.2}$$

with $\ell = 1$. From the definition of the bilinear form b,

$$b(u,u) = (\alpha u', u') \geq K_B \|u'\|_X^2,$$

where $K_B = \min\{\alpha\}$. Using Inequality (2.4.2) it follows

$$b(u,u) \geq K_B \|u'\|_X^2 \geq K_B \|u\|_X^2.$$

□

Corollary 2.4.2. *The bilinear form b is an inner product for the space V.*

Definition 2.4.2 (Energy space V)**.** *The vector space V equipped with the inner product b is referred to as the Energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.*

Proposition 2.4.4. *There exists a positive constant K_I, such that*

$$b(u,u) \geq K_I \|u\|_W^2 \text{ for each } u \in V.$$

Proof. Using Propositions 2.4.3 and 2.4.2, for any $u \in V$,

$$b(u,u) \geq K_B (K_W)^{-1} \|u\|_W^2.$$

□

Proposition 2.4.5. *The norms $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent on V.*

Proof. Let $u \in V$, by the definition of the norm on H^1

$$\|u\|_1^2 = \|u\|_X^2 + \|u'\|_X^2. \tag{2.4.3}$$

From the proof of Proposition 2.4.3 we obtain

$$\|u'\|_X^2 \leq (K_B)^{-1} b(u,u). \tag{2.4.4}$$

Substituting Inequality (2.4.4) and the result in Proposition 2.4.3 into (2.4.3) yields

$$\|u\|_1^2 \leq K_E b(u,u) = K_E \|u\|_V^2,$$

where $K_E = 2(K_B)^{-1}$.

Consider $u \in V$, from Equation (2.4.3) we have

$$\|u\|_V^2 = (\alpha u', u') \leq K_V (u', u') \leq K_V \|u\|_1^2,$$

where $K_V = \max\{\alpha\}$. □

Corollary 2.4.3. *The space V is complete.*

Proof. The space V is a closed subspace of the Hilbert space $H^1(0,1)$. Using Proposition A.1.1 we have that V is complete with respect to the $\|\cdot\|_1$ norm. Since the norms $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent (Proposition 2.4.5) we have the desired result. □

Proposition 2.4.6. *The bilinear form a is non-negative, symmetric and bounded on W.*

Proof. The definition of a is the inner product of $\mathcal{L}^2(0,1)$. The bilinear form a is therefore symmetric and non-negative on W. Let u and $v \in W$, using the Cauchy-Schwartz inequality on a yields

$$|a(u,v)| = |(u,v)| \leq \|u\|_X \|v\|_X. \tag{2.4.5}$$

By the equivalence of the norms on X and W, a is bounded on W. □

2.4.2 Existence

Proposition 2.4.7. *V is dense in X.*

Proof. The set $C_0^\infty(0,1)$ is contained in $\mathcal{T}[0,1]$ and $\mathcal{T}[0,1]$ is contained in V. Since $C_0^\infty(0,1)$ is dense in $X = \mathcal{L}^2(0,1)$ (Theorem A.1.1) we have V is dense in X. □

Remark. *In this application, the three sets W, X and $\mathcal{L}^2(0,1)$ are the same.*

For convenience, we recall a second order hyperbolic heat problem.

Problem HHE-W

Find $u \in C^2(\mathcal{J}, \mathcal{L}^2(0,1))$ such that for each $t \in \mathcal{J}, u(t) \in V(0,1)$ and

$$c(u''(t),v) + a(u'(t),v) + b(u(t),v) = (\tilde{q}(t),v) \quad \text{for all } v \in V(0,1), \tag{2.4.6}$$

$$\text{while } u(0) = u_0 \text{ and } u'(0) = u_d.$$

Assumptions A1-A3 are respectively satisfied due to Propositions 2.4.7, 2.4.2 and 2.4.4. As a result of Proposition 2.4.6, Assumption A4W holds for Problem HHE-W. Using Theorem 2.2.2, we have a unique solution

$$u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W),$$

provided $u_0 \in E_b, u_d \in V$ and $f \in C^1(\mathcal{J}, X)$.

It needs to be emphasised that $u_0 \in E_b$ and $u_d \in V$ are necessary conditions as proven in [VV02].

2.4.3 Nonexistence

The benchmark problem below is well known for the fact that numerical approximations show spurious oscillations (see [SV12]). Some authors claim to have derived an improved numerical technique that reduces the oscillations. In [SV12], the authors dismiss the claim and proved that the benchmark problem is not well-posed. There is either no solution or many solutions.

Benchmark problem.

$$\begin{aligned} \partial_t^2 T + 2\partial_t T - \partial_x^2 T = 0 \text{ for all } x \in (0,1) \text{ for each } t > 0, \\ T(0,t) = 1,\ \partial_x T(1,t) = 0 \text{ for each } t > 0, \\ T(x,0) = \partial_t T(x,0) = 0 \text{ for all } x \in (0,1). \end{aligned}$$

The following transformation is made in [SV12]: $\theta(x,t) = T(x,t) - 1$ and the Benchmark problem is written in terms of θ. This is done to obtain a problem with homogeneous boundary conditions that is equivalent to the Benchmark problem.

Special case of Problem HHE

$$\begin{aligned} \partial_t^2 \theta + 2\partial_t \theta - \partial_x^2 \theta = 0 \text{ for all } x \in (0,1) \text{ for each } t > 0, \\ \theta(0,t) = \partial_x \theta(1,t) = 0 \text{ for each } t > 0, \\ \theta(x,0) = \theta_{in}(x) = -1,\ \partial_t \theta(x,0) = 0 \text{ for all } x \in (0,1). \end{aligned}$$

For convenience, we refer to the problem above as Problem SC. Note that Problem SC is a special case of Problem HHE when $\tau_q = 1, \alpha = 1$ and the weak damping co-efficient is 2. Since $\theta_{in}(x) = -1$, it follows that $T = \theta + 1$ is a solution of the Benchmark problem if and only if θ is a solution of Problem SC.

As previously mentioned, it was proved by [Wei95] that a classical solution for Problem SC can be obtained if $\theta_{in} \in C^2[0,1]$ and $\theta_{in}(0) = \theta'_{in}(1) = 0$ and $\theta''_{in}(0) = 0$. Note that as a result of transforming the benchmark problem $\theta_{in}(0) \neq 0$. In [Wei95], the conditions on θ_{in} are sufficient and were not proved to be necessary. To this end, we do not expect to obtain a classical solution.

Consider the possibility that Problem SC has a weak solution. To apply the theory in Subsections 2.4.1 and 2.4.2, it is important to note that Problem SC is a special case of Problem HHE-W where $f = 0$. The initial conditions still need to be considered.

Recall the space of test functions for Problem SC is defined as $\mathcal{T}[0,1] = \{v \in C^1[0,1] \mid v(0) = 0\}$. The space $V(0,1)$ is the closure of the space of test functions in the Sobolev space $H^1(0,1)$. Lastly, $E_b(0,1) = \{y \in V(0,1) : \text{for some } z \in W(0,1), \text{ such that } b(y,v) = c(z,v)$ for all $v \in V(0,1)\}$.

Note the following relation holds for the three sets $\mathcal{T}[0,1] \subset V(0,1)$ and $E_b(0,1) \subset V(0,1)$. We are required to show that $\theta_{in}(x) \in E_b(0,1)$ and $\partial_t\theta(x,0) \in V(0,1)$. Let $\theta_d(x) = \partial_t\theta(x,0) = 0$ for all $x \in [0,1]$.

Proposition 2.4.8. *$\theta_d \in V(0,1)$ and $\theta_{in} \notin E_b(0,1)$.*

Proof. To prove that $\theta_d \in V(0,1)$, we show that $\theta_d \in \mathcal{T}[0,1]$. By the definition of the set $\mathcal{T}(0,1)$, θ_d is in $\mathcal{T}[0,1]$ a subset of $V(0,1)$. Therefore, $\theta_d \in V(0,1)$.

To prove that $\theta_{in} \notin E_b(0,1)$, we show that $\theta_{in} \notin V(0,1)$. The proof is by contradiction. Suppose that $\theta_{in} \in V(0,1)$. Then θ_{in} must satisfy the inequality

$$\|\theta_{in}\| \leq \|\theta'_{in}\|.$$

See Lemma A.2.4 in Appendix A.2. Since $\|\theta_{in}\| = 1$ and $\|\theta'_{in}\| = 0$ the inequality above does not hold as $1 > 0$. Particularly, this contradicts the fact that $V(0,1)$ is the closure of $\mathcal{T}[0,1]$. Therefore, $\theta_{in} \notin V(0,1)$. Since $E_b(0,1) \subset V(0,1)$, it follows that $\theta_{in} \notin E_b(0,1)$. □

From Proposition 2.4.8, there is no weak solution for Problem SC since the necessary condition $\theta_{in} \in E_b(0,1)$ is not satisfied.

The logical conclusion is that the benchmark problem is not well-posed and one cannot expect numerical methods to yield satisfactory results.

Chapter 3

Finite element approximation theory

In 2013, Basson and Van Rensburg generalised the work in [Bak76] by deriving error estimates for the Galerkin finite element approximation for a second order hyperbolic type problem with viscous type damping. Particularly, the authors of [BV13] identified the conditions required in the convergence analysis, using the same existence results as in Chapter 2.

In this chapter, we consider the work done in [BV13]. The initial focus is therefore the Galerkin finite element approximate solution for Problem GH, with viscous type damping.

In convergence analysis, the smoothness of a solution is significant. Specific attention was given to the differentiability of the projection of the exact solution in [BV13]. This must be kept in mind when considering applications. It is important to note that the approach in [BV13] permits the comparison of the required conditions for the existence, uniqueness and convergence of solution.

As previously stated, we restrict our attention to second order hyperbolic type problems with viscous type damping as in [BV13]. In Chapter 2, we examined the existence and uniqueness of a solution for Problem HHE. For this special case, it was established that Assumptions A1, A2, A3 and A4W were required for the existence of a solution. It is important to note that it is necessary to verify the assumptions for each application of the theory, we refer the reader to Section 2.4 for details.

3.1 Galerkin approximation

In this section, we consider the Galerkin finite element approximation of Problem GH, with viscous damping. For this, suppose Assumptions A1, A2, A3 and A4W hold.

For the finite element method (FEM), the solution to Problem GH is approximated on S^h, a finite dimensional subspace of V. We denote the Galerkin approximation by Problem GH^h.

Recall that the interval $\mathcal{J}$ contains zero and can be of the form $(-\infty, \infty)$, $[0, t_*)$ or $(t_*, 0]$, where $t_* \in \mathbb{R}$. For convenience, consider the case $0 < t_* < \infty$.

Problem GH^h.

Given a function $f : [0, t_*] \to X$, find a function $u_h \in C^2[0, t_*]$ such that for each $t \in (0, t_*)$, $u_h(t) \in S^h$ and

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for all} \quad v \in S^h, \tag{3.1.1}$$
$$\text{while} \quad u_h(0) = u_0^h \quad \text{and} \quad u_h'(0) = u_d^h.$$

The initial conditions u_0^h and u_d^h are approximations of u_0 and u_d respectively in S^h. In [BV13], u_0^h and u_d^h are chosen to be the interpolants.

Note that a solution to Problem GH^h exists if f is continuous, particularly, when f is Lipschitz continuous on W.

3.1.1 Projection and fundamental estimate

Our first aim is to obtain an estimate for the semi discrete error $e_h(t) = u(t) - u_h(t)$, the difference between the exact solution and Galerkin approximation. Following [BV13], the semi discrete error is split using a projection.

The projection operator P is defined by

$$b(u - Pu, v) = 0 \quad \text{for each} \quad v \in S^h.$$

P denotes the projection Pu of a function u that is, $(Pu)(t) = Pu(t)$ for each $\mathrm{t} \in (0, \mathrm{t}_*)$. Now, let

$$e(t) = Pu(t) - u_h(t) \text{ and } e_p(t) = u(t) - Pu(t),$$

then

$$e_h(t) = e_p(t) + e(t).$$

To obtain an estimate for $e(t)$ certain properties are required of the projection. From Theorem 2.2.2, $u \in C^1([0, t_*), V) \cap C^2((0, t_*), W)$. For the differentiability of the projection, the authors of [BV13] use properties of the exact solution. They proved that if $u \in C^1([0, t_*), V)$, then $Pu \in C^1[0, t_*)$ and $(Pu)'(t) = Pu'(t)$.

The linear operator P is bounded and maps from V to V. It is not necessarily the case that P maps from W to W, hence an additional assumption for the projection of $u \in C^2((0, t_*), W)$ is required.

Assumption C1. The solution $u \in C(J, V)$ of Problem GH has the property $(Pu) \in C^2(0, t_*)$.

The above condition on the operator P is necessary to obtain the fundamental estimate below, proved in [BV13].

Lemma 3.1.1. *If the solution u of Problem GH satisfies Assumption C1, then for each $t \in [0, t_*]$,*

$$\begin{aligned}\|e(t)\|_W \leq& \sqrt{2}\Big(\|e(0)\|_W + 3t_*\|e_h'(0)\|_W + 3\int_0^{t_*} \|e_p'(t)\|_W \\ &+ 3K_a t_*\|e_h(0)\|_W + 3K_a \int_0^{t_*} \|e_p(t)\|_W\Big),\end{aligned}$$

where $K_a = \left(\dfrac{1}{2\varepsilon}\right)^2$ for some arbitrary ε.

Using approximation theory, one can obtain estimates for the norm of $e_p(t)$. As done in [BV13], we use the following assumption, regarding the error when an element of V is approximated by an element of S^h, to determine estimates for the error of the projection of u. Note that h relates to the dimension of S^h and that $h \to 0$ as the dimension $n \to \infty$.

Assumption C2. There exists a subspace H of V, and a positive integer α_* such that if $u \in H$, then

$$\inf_{v \in S^h} \|u - v\|_V \leq \hat{C} h^{\alpha_*} \|u\|_H,$$

where $\|\cdot\|_H$ is a norm or semi norm associated with H.

The result below for the projection error e_p is a trivial consequence of Assumption C2 combined with Assumption A3.

Proposition 3.1.1. *There exists a subspace H of V and positive constants $\hat{C}$ and α_* such that for $u \in H$*

$$\|u - Pu\|_W \leq C_V \hat{C} h^{\alpha_*} \|u\|_H,$$

where $\|\cdot\|_H$ is a norm or semi norm associated with H.

3.1.2 Error for the semi discrete problem

We use the fact that $e_h(t) = e_p(t) + e(t)$, to obtain a semi discrete error estimate.

Theorem 3.1.1. *If the solution u of Problem GH satisfies Assumption C1, then*

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & \|u(t) - Pu(t)\|_W + \sqrt{2}\Big(\|Pu_0 - u_0\|_W \\ & + 3t_*\|u_d - u_d^h\|_W + 3\int_0^{t_*} \|u'(t) - Pu'(t)\|_W \\ & + (1 + 3K_a t_*)\|u_0 - u_0^h\|_W + 3K_a \int_0^{t_*} \|u(t) - Pu(t)\|_W\Big),\end{aligned}$$

for each $t \in [0, t_*]$.

Finally, we need estimates for terms on the right hand side of Theorem 3.1.1. For instance, consider $\|u_d - u_d^h\|$. If we follow [BV13] and choose $u_d^h = \Pi u_d$ where Π is the interpolation operator, we can use Assumption C2 and obtain

$$\|u_d - \Pi u_d\|_W \leq C_V\|u_d - \Pi u_d\|_V \leq C_V \hat{C} h^{\alpha_*}\|u_d\|_H.$$

The final result follows from combining Theorem 3.1.1 with the projection errors.

Theorem 3.1.2. *Suppose Assumption C2 holds and $u_0^h = \Pi u_0$ and $u_d^h = \Pi\, u_d$. If the solution u of Problem GH satisfies Assumption C1, $u \in \mathcal{L}^2([0, t_*], H)$ and $u' \in \mathcal{L}^2([0, t_*], H)$, then*

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & C_V \hat{C} h^{\alpha_*}\Big(\|u(t)\|_H + 3t_*\sqrt{2}\max\|u'(t)\|_H \\ & + 3K_a t_* \max\|u(t)\|_H + (2 + 3K_a t_*)\|u_0\|_H \\ & + 3t_*\|u_d\|_H\Big),\end{aligned}$$

for each $t \in [0, t_*]$ and some positive constant C_V.

Note that the results above hold in a finite dimensional space. However, the dimension of S^h is not fixed and the norms may not equivalent for a solution u_h of Problem GH^h. Since the semi-discrete error estimate is obtained with respect to the norm of W, the fully discrete error estimate must be determined with respect to the same norm.

3.2 Error for the fully discrete problem

Recall that the solution u of Problem GH is approximated in a finite dimensional subspace of V. The Galerkin approximation is now rewritten to obtain a system of ordinary differential equations (ODEs) which can be solved using a finite difference scheme. In this section, we first consider the difference between the Galerkin approximation u_h and the approximation obtained using the finite difference scheme. Using this estimate together with the error estimate from Section 3.1 we obtain the final estimate.

3.2.1 A system of ordinary differential equations

Suppose S^h is the span of the set $\{\delta_1, \delta_2, \cdots, \delta_n\}$.

Notation
Define the matrices K, L, N and the vector F by

$$\begin{aligned} K_{ij} =& b(\delta_j, \delta_i), L_{ij} = a(\delta_j, \delta_i), \\ N_{ij} =& c(\delta_j, \delta_i) \text{ and } F_i(t) = (f(t), \delta_i)_X \end{aligned}$$

for $i, j = 1, \ldots, n$.

If a function $u_h \in S^h$, that is

$$u^h = \sum_{j=1}^{n} u_j \delta_j, \tag{3.2.1}$$

then the n-tuple $(u_1, u_2, \cdots, u_n)$ denoted by $\bar{u}$, with values in $\mathbb{R}^n$ corresponds to u^h. It is convenient to use d^h and v^h for the initial conditions instead of u_0^h and u_d^h respectively.

Suppose d^h and $v^h \in S^h$ that is, there are n-tuples $\bar{d} = (d_1, d_2, \cdots, d_n)$

and $\bar{v} = (v_1, v_2, \cdots, v_n)$ such that

$$d^h = \sum_{j=1}^{n} d_j \delta_j, \text{ and } v^h = \sum_{j=1}^{n} v_j \delta_j.$$

Using the notation above, we write a system of second order ordinary differential equations for Problem-GH^h.

Problem GH-ODE

Determine $\bar{u} \in C^2[0, t_*]$ such that

$$N\bar{u}'' + L\bar{u}' + K\bar{u} = F(t), \tag{3.2.2}$$
$$\text{while } \bar{u}(0) = \bar{d} \text{ and } \bar{u}'(0) = \bar{v}.$$

Before we consider the error, it is wise to present results that relate the solution u_h of Problem GH^h to $\bar{u}$ a solution of Problem GH-ODE.

Proposition 3.2.1. *If the matrices N, L, K, F and the vectors $\bar{d}$ and $\bar{v}$ are defined as seen above, then the function u_h is a solution of Problem GH^h if and only if the function $\bar{u}$ is a solution of Problem GH-ODE.*

Proposition 3.2.2. *If $F \in C[0, t_*]$ then Problem GH-ODE has a unique solution for each pair of vectors $\bar{d}$ and $\bar{v}$.*

From Propositions 3.2.1 and 3.2.2, if there exists a unique solution $u_h \in C^2[0, t_*]$ of Problem GH^h, then there is a unique solution $\bar{u} \in C^2[0, t_*]$ of Problem GH-ODE. This of course is a result of the conditions imposed on the function $f \in C([0, t_*], X)$.

3.2.2 Fully discrete Galerkin scheme

Assume the interval $[0, t_*]$ is divided into N subintervals of length $\tau = \dfrac{t_*}{N}$ and denote the nodes by t_k. The approximation of u_h at t_k is then denoted by u_k^h.

Notation
For any sequence $\{x_k\} \subset \mathbb{R}_n$:

$$\delta_t x_k = \tau^{-1} \left[x_{k+1} - x_k \right],$$
$$x_{k+\frac{1}{2}} = \frac{1}{2} \left[x_{k+1} + x_k \right].$$

We present the discretised Problem GH^h in variational form, using the central difference average acceleration difference scheme.

Problem GH^h-D

Find a sequence $\{u_k^h\} \subset S^h$, such that for each $k = 0, 1, 2, \cdots, N-1$,

$$\delta_t u_k = v_{k+\frac{1}{2}}, \tag{3.2.3}$$

$$c\left(\delta_t v_k, \psi\right) + a\left(v_{k+\frac{1}{2}}, \psi\right) + b\left(u_{k+\frac{1}{2}}^h, \psi\right) = \frac{1}{2}\left([f(t_k) + f(t_{k+1})], \psi\right)_X, \tag{3.2.4}$$

$$\text{while } u_h(0) = d^h \text{ and } u_h'(0) = v^h,$$

for each $\psi \in S^h$.

Using the finite difference scheme, the system of ODEs is approximated using the problem below.

Problem GH-FD

Find a sequence $\{\bar{u}_k\} \subset \mathbb{R}_n$ such that for each k,

$$\begin{aligned} \bar{u}_{k+1} &= \bar{u}_k + \tau \bar{v}_{k+\frac{1}{2}}, \\ \left(N + \frac{\tau}{2}L + \frac{\tau^2}{4}K\right)\bar{v}_{k+1} &= \left(N - \frac{\tau}{2}L - \frac{\tau^2}{4}K\right)\bar{v}_k - \tau K \bar{u}_k \\ &\quad + \frac{\tau}{2}\left[F(t_k) + F(t_{k+1})\right], \end{aligned}$$

with $\bar{u}_0 = \bar{d}$ and $\bar{v}_0 = \bar{v}$.

The result below was proved in [BV13] to obtain the error estimate for the fully discrete problem.

Theorem 3.2.1. *If $f \in C^2([0, t_*], X)$, then*

$$\begin{aligned} \|u_h(t_k) - u_k^h\|_W \le& 7t_*^2\tau^2 \max \|u_h^{(4)}\|_W + 7t_*\tau^2 \max \|u_h'''\|_W \\ &+ \sqrt{2K_a}\tau^4 \max \|u_h'''\|_W \end{aligned}$$

for each $t \in (0, t_*)$.

Remark. *Note that the continuity of F is a sufficient condition for the existence and $F \in C^m[0, t_*]$ if $f \in C^m([0, t_*], X)$. However, for $\bar{u}$ to have derivatives of order $2 + m$, it is necessary to assume F has derivatives of order m, hence the condition on f in Theorem 3.2.1.*

3.2.3 Convergence and error estimates

To determine the final error estimate the following equation is used:

$$(u(t_k) - u_k^h) = (u(t_k) - u_h(t_k)) + (u_h(t_k) - u_k^h). \tag{3.2.5}$$

The error estimate is therefore obtained using the sum of the semi discrete $(u(t_k) - u_h(t_k))$ and the fully discrete $(u_h(t_k) - u_k^h)$ errors. Since the error estimates were determined using the W norm, we have the following error estimate.

$$\|u(t_k) - u_k^h\|_W \leq \|u(t_k) - u_h(t_k)\|_W + \|u_h(t_k) - u_k^h\|_W.$$

The estimate can be completed using Theorems 3.1.2 and 3.2.1

3.3 Application of convergence theory to the hyperbolic heat conduction problem

An illustration of the application of the convergence theory in Section 3.2 is useful. In this section, we apply the estimates used in the convergence theory to the solution u of Problem HHE (see Section 2.1). Recall the definitions of the bilinear forms and spaces in Section 2.1. Note that Assumptions A1, A2, A3 and A4W hold for the problem as proved in Section 2.4, and therefore the existence of a solution $u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W)$ was established.

3.3.1 Galerkin approximation

For the convergence theory, we construct the finite dimensional subspace S^h of V using the span of piecewise linear basis functions.

The Galerkin finite element approximation of Problem HHE is referred to as Problem HHE^h.

Problem HHE^h

Find a function $u_h(t) \in S^h$ such that for each $t \in [0, t_*]$

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (q(\cdot, t), v) \quad \text{for each } v \in S^h,$$

while $u_h(0) = u_0^h$ and $u_h'(0) = u_d^h$.

Note that Problem HHE^h is a special case of Problem GH^h.

Interpolation

For interpolation error estimates, we use well known results on interpolation theory. This can be found in [OR76, Chapter 6] and [SF73, Chapter 5] for example.

We consider the interpolation operator for piecewise linear basis functions, denoted by Π_l. The following interpolation error estimate replaces Assumption C2.

Corollary 3.3.1. *There exists a constant $\hat{C}_l$ such that if $u \in H^k(0,1)$ for $k \geq 2$, then*

$$\|\Pi_l u - u\|_V \leq \hat{C}_l h |u|_2, \tag{3.3.1}$$

where $|\cdot|_2$ denotes the semi-norm.

Using Assumption A3 and Equation (3.3.1), the error estimate for the projection is

$$\|u - Pu\|_W \leq C_V \|u - Pu\|_V \leq \hat{C}_l h |u|_2.$$

Application of Theorem 3.1.2
Suppose $u_0^h = \Pi_l u_0$ and $u_d^h = \Pi_l u_d$. If the solution u of Problem HHE-W satisfies Assumption C1, $u(t) \in \mathcal{L}^2([0,t_*], H^2(0,1) \cap V)$ and $u'(t) \in \mathcal{L}^2([0,t_*], H^2(0,1) \cap V)$, then

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq & C_V \hat{C}_l h \Big(|u(t)|_2 + 3t_* \sqrt{2} \max |u'(t)|_2 \\ & + 3K_a t_* \max |u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ & + 3t_* |u_d|_2 \Big),\end{aligned}$$

for each $t \in [0, t_*]$.

As previously mentioned in Subsection 3.2.3, the sum of the semi discrete and fully discrete errors yields the final error estimate. For this, we combine the estimates obtained in Theorems 3.1.2 and 3.2.1. Suppose we use the same algorithm as in Problem GH^h-D, we then obtain the following problem.

Problem HHE^h-D

Find a sequence $\{u_k^h\} \subset S^h$, such that for each $k = 0, 1, 2, \cdots, N-1$,

$$\begin{aligned} \delta_t u_k =& v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + a\left(v_{k+\frac{1}{2}}, \psi\right) + b\left(u^h_{k+\frac{1}{2}}, \psi\right) =& \frac{1}{2}\left([f(t_k) + f(t_{k+1})], \psi\right)_X, \\ \text{while } u_0^h = d^h \text{ and } u_d^h = v^h, \end{aligned}$$

for each $\psi \in S^h$.

Application of the combination of Theorems 3.1.2 and 3.2.1
Suppose $u_0^h = \Pi_l u_0$ and $u_d^h = \Pi_l u_d$. If $u(t) \in \mathcal{L}^2([0, t_*], H^2(0, 1) \cap V)$, $u'(t) \in \mathcal{L}^2([0, t_*], H^2(0, 1) \cap V)$, $f \in C^2([0, t_*], \mathcal{L}^2(0, 1))$ and the sequence $\{u_k^h\}$ is a solution of Problem HHE^h-D, then

$$\begin{aligned} \|u(t_k) - u_k^h\|_W \leq& C_V \hat{C}_l h\Big(|u(t)|_2 + 3t_*\sqrt{2}\max|u'(t)|_2 \\ &+ 3K_a t_* \max|u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ &+ 3t_*|u_d|_2\Big) + 7t_*^2\tau^2 \max\|u_h^{(4)}\|_W + 7t_*\tau^2 \max\|u_h'''\|_W \\ &+ \sqrt{2K_a}\tau^4 \max\|u_h'''\|_W, \end{aligned}$$

for each $t_k \in [0, t_*]$.

Chapter 4

Serially connected double beams

In this chapter, we consider a model that consists of two serially connected Timoshenko beams. This model is used in a biological application in Chapter 5.

4.1 Model problem

Consider two beams that are attached at one of their respective endpoints. For convenience, the beams are considered to have a horizontal orientation, so that the beam on the left is referred to as Beam 1, and the other as Beam 2. For convenience, we use subscripts 1 and 2 to distinguish between the beams. The beams are modelled using the Timoshenko theory (see Section 1.3). If the respective lengths of the beams are L_1 and L_2, then the dimensionless length of the each beam is obtained by scaling such that Beam 1 is one dimensionless unit and is denoted by ℓ_1, while the dimensionless length of Beam 2 is

$$\ell_2 = \frac{L_2}{L_1}.$$

Equations of motion

$$\partial_t^2 w_i = \partial_x F_i + q_i,$$
$$\frac{1}{\alpha_i}\,\partial_t^2 \phi_i = F_i + \partial_x M_i.$$

Constitutive equations

$$F_i = \partial_x w_i - \phi_i,$$
$$M_i = \frac{1}{\beta_i}\,\partial_x \phi_i,$$

for $i = 1, 2$.

In the rest of this chapter (unless stated otherwise) Beam 1 is assumed to be embedded in an elastic material and that it does not experience any load. To model this, we let $q_1 = -gw_1$ where g is the parameter for the elastic material. In this case, the reaction of the elastic material is considered as a load, while $q_2(x, t)$ denotes the external load on Beam 2.

Boundary and interface conditions

Boundary conditions

Various boundary conditions are possible for each of the beams. Except where stated otherwise, we restrict our attention to the case where Beam 1 is clamped at its left end, and Beam 2 is free at its right endpoint, that is

Beam 1:

$$w_1(0, t) = \phi_1(0, t) = 0. \tag{4.1.1}$$

Beam 2:

$$F_2(\ell_2, t) = M_2(\ell_2, t) = 0. \tag{4.1.2}$$

In Section 4.3, we consider some of the other possible configurations for the attached Timoshenko beams. Specifically, we are interested in the effect of some boundary conditions on the solvability of a problem.

Interface conditions

The interaction between the beams can be rigid or elastic. For simplicity, we consider a rigid interface between the beams. As such, the continuity conditions at the point of attachment are

$$w_1(\ell_1, t) = w_2(0, t) \text{ and } \phi_1(\ell_1, t) = \phi_2(0, t). \tag{4.1.3}$$

We also have

$$F_1(\ell_1, t) = F_2(0, t) \quad \text{and } M_1(\ell_1, t) = M_2(0, t). \tag{4.1.4}$$

Problem EDB (Embedded Double Beam)

Given functions q_i and positive constats α_i and β_i, find w_i and ϕ_i such that

$$\partial_t^2 w_i = \partial_x F_i + q_i, \tag{4.1.5}$$

$$\frac{1}{\alpha_i}\,\partial_t^2 \phi_i = \partial_x M_i + F_i, \tag{4.1.6}$$

$$F_i = \partial_x w_i - \phi_i, \tag{4.1.7}$$

$$M_i = \frac{1}{\beta_i}\,\partial_x \phi_i, \tag{4.1.8}$$

with boundary conditions (4.1.1) and (4.1.2), interface conditions (4.1.3) and (4.1.4), while

$$w_i(\cdot,0) = w_0^i, \partial_t w_i(\cdot,0) = w_d^i,$$
$$\phi_i(\cdot,0) = \phi_0^i \text{ and } \partial_t \phi_i(\cdot,0) = \phi_d^i,$$

for $i = 1, 2$.

4.2 Existence of a unique solution

In this section we consider the solvability of Problem EDB by applying the existence theory in Chapter 2. As such, we start by writing Problem EDB in variational form.

4.2.1 Variational form

The variational form of Problem EDB is obtained by multiplying Equations (4.1.5) and (4.1.6) respectively by v_i and $\nu_i \in C^1[0,\ell_i]$ and integrating. If $\partial_x F(\cdot,t)$ and $\partial_x M(\cdot,t)$ are integrable, using integration by parts (on the terms on the right side of the equality sign in Equations (4.1.5) and (4.1.6)) yields

$$\int_0^{\ell_i} \partial_t^2 w_i(\cdot,t) v_i = F_i(\ell_i,t) v_i(\ell_i) - F_i(0,t) v_i(0) - \int_0^{\ell_i} F_i(\cdot,t) v_i' + \int_0^{\ell_i} q_i(\cdot,t) v_i, \tag{4.2.1}$$

$$\begin{aligned}\frac{1}{\alpha_i}\int_0^{\ell_i} \partial_t^2\phi_i(\cdot,t)\nu_i =& M_i(\ell_i,t)\nu_i(\ell_i) - M_i(0,t)\nu_i(0) + \int_0^{\ell_i} F_i(\cdot,t)\nu_i \\ &- \int_0^{\ell_i} M_i(\cdot,t)\nu_i',\end{aligned} \tag{4.2.2}$$

for $i = 1, 2$.

Adding Equation (4.2.1) for both beams and substituting the function q_i for each i we obtain

$$\begin{aligned}&\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t)v_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t)v_2 \\ =& F_1(\ell_1,t)v_1(\ell_1) - F_1(0,t)v_1(0) + F_2(\ell_2,t)v_2(\ell_2) - F_2(0,t)v_2(0) \\ &- \int_0^{\ell_1} F_1(\cdot,t)v_1' - \int_0^{\ell_2} F_2(\cdot,t)v_2' - \int_0^{\ell_1} gw_1(\cdot,t)v_1 + \int_0^{\ell_2} q_2(\cdot,t)v_2.\end{aligned} \tag{4.2.3}$$

Similarly, for Equation (4.2.2) we obtain

$$\begin{aligned}&\frac{1}{\alpha_1}\int_0^{\ell_1} \partial_t^2\phi_1(\cdot,t)\nu_1 + \frac{1}{\alpha_2}\int_0^{\ell_2} \partial_t^2\phi_2(\cdot,t)\nu_2 \\ =& M_1(\ell_1,t)\nu_1(\ell_1) - M_1(0,t)\nu_1(0) + M_2(\ell_2,t)\nu_2(\ell_2) - M_2(0,t)\nu_2(0) \\ &+ \int_0^{\ell_1} F_1(\cdot,t)\nu_1 + \int_0^{\ell_2} F_2(\cdot,t)\nu_2 - \int_0^{\ell_1} M_1(\cdot,t)\nu_1' - \int_0^{\ell_2} M_2(\cdot,t)\nu_2'.\end{aligned} \tag{4.2.4}$$

The following test function spaces are defined:

$$\mathcal{T}_B[0,\ell_1] = \{v_1 \in C^1[0,\ell_1] \mid v_1(0) = 0\} \text{ and } \mathcal{T}_A[0,\ell_2] = C^1[0,\ell_2].$$

Remark. *Due to the interface conditions, the following restriction holds for all cases of test functions*

$$v_1(\ell_1) = v_2(0).$$

Next, we define the product space of test functions

$$\mathcal{T}_p = \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_A[0,\ell_2] \times \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_A[0,\ell_2].$$

We present the variational form of Problem EDB.

Problem EDB-V

Given functions q_2 and g and positive constants $\alpha_1, \alpha_2, \beta_1$ and β_2, find w_1, w_2, ϕ_1 and ϕ_2 where $w_1(\cdot,t), \phi_1(\cdot,t) \in \mathcal{T}_B[0,\ell_1]$ and $w_2(\cdot,t)$, $\phi_2(\cdot,t) \in \mathcal{T}_A[0,\ell_2]$ for each $t > 0$ such that

$$\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t)v_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t)v_2 + \frac{1}{\alpha_1}\int_0^{\ell_1} \partial_t^2 \phi_1(\cdot,t)\nu_1 + \frac{1}{\alpha_2}\int_0^{\ell_2} \partial_t^2 \phi_2(\cdot,t)\nu_2$$
$$= -\int_0^{\ell_1} (\partial_x w_1(\cdot,t) - \phi_1(\cdot,t))(v_1' - \nu_1) - \int_0^{\ell_2} (\partial_x w_2(\cdot,t) - \phi_2(\cdot,t))(v_2' - \nu_2)$$
$$- \int_0^{\ell_1} g w_1 v_1 - \int_0^{\ell_1} \frac{1}{\beta_1}\partial_x \phi_1(\cdot,t)\nu_1' - \int_0^{\ell_2} \frac{1}{\beta_2}\partial_x \phi_2(\cdot,t)\nu_2' + \int_0^{\ell_2} q_2(\cdot,t)v_2,$$

for all $\langle v_1, v_2, \nu_1, \nu_2\rangle \in \mathcal{T}_p$, with

$$w_i(\cdot,0) = w_0^i, \partial_t w_i(\cdot,0) = w_d^i,$$
$$\phi_i(\cdot,0) = \phi_0^i \text{ and } \partial_t \phi_i(\cdot,0) = \phi_d^i,$$

for $i = 1,2$.

Recall for Problem EDB $q_1(x,t) = -gw_1(x,t)$ and $q_2(x,t)$ is a specified load. In this case, the reaction of the elastic material is considered as a load, while $q_2(x,t)$ denotes the external load on Beam 2.

Consider a function $u : [0,t_*] \to \mathcal{T}_p$ for each $t_* < \infty$, such that

$$u(t)(x) = \langle w_1(x,t), w_2(x,t), \phi_1(x,t), \phi_2(x,t)\rangle.$$

Similarly, $\tilde{q}(t)(x) = \langle 0, q_2(x,t), 0, 0\rangle$.

We define the following bilinear forms in terms of the $\mathcal{L}^2$ inner products:

$$c(u,v) = (u_1,v_1) + \left(\frac{1}{\alpha_1}u_3, v_3\right) + (u_2,v_2) + \left(\frac{1}{\alpha_2}u_4, v_4\right),$$
$$b(u,v) = (u_1' - u_3, v_1' - v_3) + (u_2' - u_4, v_2' - v_4) + (gu_1, v_1) + \left(\frac{1}{\beta_1}u_3', v_3'\right)$$
$$+ \left(\frac{1}{\beta_2}u_4', v_4'\right).$$

In this application we assume that $\inf |g| > 0$, but $a(u,v) = 0$.

4.2.2 Weak variational form

In Subsection 4.2.1 we derived the variational form of Problem EDB. The aim of this section is to write Problem EDB-V in its weak variational form.

Recall that for Problem EDB-V we have the test function product space

$$\mathcal{T}_p = \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_A[0,\ell_2] \times \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_A[0,\ell_2], \text{ where}$$

$$\mathcal{T}_B[0,\ell_1] = \{v_1 \in C^1[0,\ell_1] \mid v_1(0) = 0\} \text{ and } \mathcal{T}_A[0,\ell_2] = C^1[0,\ell_2],$$

with $v_2 \in \mathcal{T}_A[0,\ell_2]$ and the continuity condition

$$v_1(\ell_1) = v_2(0).$$

The spaces $V_B[0,\ell_1]$ and $V_A[0,\ell_2]$ are the closure of the test functions $\mathcal{T}_B[0,\ell_1]$ and $\mathcal{T}_A[0,\ell_2]$ in the Sobolev spaces $H^1(0,\ell_1)$ and $H^1(0,\ell_2)$ respectively.

It is necessary to define the value of $f(p)$, for a function $f \in H^1(0,\ell_i)$ with $i = 1, 2$. Of use to us is the one-dimensional case of the trace operator. The reader is referred to Appendix A.3 for further details on the trace operator.

To write Problem EDB-V in weak variational form, the product spaces defined below are of importance. The table illustrates the notation used for each space.

$$X = \mathcal{L}^2(0,\ell_1) \times \mathcal{L}^2(0,\ell_2) \times \mathcal{L}^2(0,\ell_1) \times \mathcal{L}^2(0,\ell_2),$$

$$H^1 = H^1(0,\ell_1) \times H^1(0,\ell_2) \times H^1(0,\ell_1) \times H^1(0,\ell_2),$$

$$V_p = V_B(0,\ell_1) \times V_A(0,\ell_2) \times V_B(0,\ell_1) \times V_A(0,\ell_2),$$

$$V = \{u = \langle u_1, u_2, u_3, u_4\rangle \in V_p \mid u_1(\ell_1) = u_2(0) \text{ and } u_3(\ell_1) = u_4(0)\}.$$

Space	Inner product	Norm
$\mathcal{L}^2$	$(\cdot,\cdot)$	$\Vert\cdot\Vert$
X	$(x,y)_X = (x_1,y_1) + (x_2,y_2) + (x_3,y_3) + (x_4,y_4)$	$\Vert\cdot\Vert_X$
H^1	$(\cdot,\cdot)_H = (x_1,y_1)_1 + (x_2,y_2)_1 + (x_3,y_3)_1 + (x_4,y_4)_1$	$\Vert\cdot\Vert_1$

Proposition 4.2.1. *The bilinear form c is an inner product for the space X.*

Proof. Since the bilinear form c is symmetric, it is an inner product if $c(u,u) = 0$ implies $u = 0$. Consider $u \in X$,

$$c(u,u) = (u_1,u_1) + \left(\frac{1}{\alpha_1}u_3, u_3\right) + (u_2,u_2) + \left(\frac{1}{\alpha_2}u_4, u_4\right) \geq B_C\|u\|_X^2, \tag{4.2.5}$$

where $B_C = \min\left\{1, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right\}$. □

Definition 4.2.1 (Inertia space W)**.** *The vector space X equipped with the inner product c is referred to as the Inertia space W. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.*

Proposition 4.2.2. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent.*

Proof. Let $u \in W$,

$$\|u\|_W^2 = (u_1, u_1) + \left(\frac{1}{\alpha_1}u_3, u_3\right) + (u_2, u_2) + \left(\frac{1}{\alpha_2}u_4, u_4\right) \leq B_W \|u\|_X^2,$$

with $B_W = \max\left\{1, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right\}$. Using Inequality (4.2.5) the result follows. □

Corollary 4.2.1. *The space W is complete.*

Proof. Using the remark to Proposition A.1.3 in the Appendix and the fact that $\mathcal{L}^2$ is complete, it follows that X is complete. Since $W = X$ and the norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent the desired result is obtained. □

Recall that $\mathcal{J}$ is an interval that contains zero.

Problem EDB-W

Find $u \in C^1(\mathcal{J}, X)$ such that for each $t \in \mathcal{J}, u(t) \in V, u''(t) \in W$ and

$$c(u''(t), v) + b(u(t), v) = (\tilde{q}(t), v)_X \quad \text{for each } v \in V,$$
$$\text{while } u(0) = u_0 \text{ and } u'(0) = u_d.$$

If the model problem has a classical solution, then

$$u_1(t)(x) = w_1(x,t),\ \ u_2(t)(x) = w_2(x,t),$$
$$u_3(t)(x) = \phi_1(x,t) \text{ and } u_4(t)(x) = \phi_2(x,t).$$

4.2.3 Existence

Next we obtain results in order to define the energy space V and its properties.

The energy space V

To prove Proposition 4.2.4 below, we need the following auxiliary result.

Proposition 4.2.3. *Suppose the function* $f \in C^1[0,\ell]$. *Then,*

$$\|f\| \leq \ell\|f'\| + \sqrt{\ell}|f(0)|.$$

Proof. Let $\sigma(x) = f(0)$ for each $x \in [0,\ell]$ and $h = f - \sigma$. Applying the Triangle inequality on f and substituting σ yields

$$\|f\| = \|h + \sigma\| \leq \|h\| + \|\sigma\| = \|h\| + \sqrt{\ell}|f(0)|. \tag{4.2.6}$$

Using the fact that the function h has a zero in $[0,\ell]$, by Lemma A.2.3 we have

$$\|h\| \leq \ell\|h'\|.$$

It follows from Inequality (4.2.6) and the definition of h

$$\|f\| \leq \|h\| + \sqrt{\ell}|f(0)| \leq \ell\|f'\| + \sqrt{\ell}|f(0)|. \tag{4.2.7}$$

□

Proposition 4.2.4. *There exists a positive constant* B_B *such that*

$$b(u,u) \geq B_B\|u\|_X^2 \text{ for each } u \in V.$$

Proof. For convenience, recall

$$b(u,u) = \|u_1' - u_3\|^2 + \|u_2' - u_4\|^2 + \|\sqrt{g}u_1\|^2 + \frac{1}{\beta_1}\|u_3'\|^2 + \frac{1}{\beta_2}\|u_4'\|^2.$$

For any $u \in V, u_3$ has a zero in $[0,\ell_1]$. Using Lemma A.2.4 we have

$$\|u_3\|^2 \leq \ell_1^2\|u_3'\|^2 \leq B_3 b(u,u), \tag{4.2.8}$$

for some constant $B_3 > 0$.

Let $\eta = \inf\limits_{[0,\ell_1]}|g|$, then $\|\sqrt{g}u_1\|^2 = (gu_1,u_1) \geq \eta\|u_1\|^2$. It follows that

$$\|u_1\|^2 \leq \eta^{-1}\|\sqrt{g}u_1\|^2 \leq B_1 b(u,u), \tag{4.2.9}$$

for some constant $B_1 > 0$.

Recall for any $u \in V$ the following continuity restriction holds

$$u_1(\ell_1) = u_2(0).$$

Applying Lemma A.3.1 on the trace of u_1 yields

$$|u_1(\ell_1)| \leq K_\ell \|u_1\|_1,$$

where $K_\ell = \sqrt{2} \max \left\{ \sqrt{\ell_1}, \left(\sqrt{\ell_1}\right)^{-1} \right\}$.

Using the result in Proposition 4.2.3 and the inequality above on u_2 we have

$$\|u_2\|^2 \leq (\ell_2 \|u_2'\| + K_\ell \sqrt{\ell_2} \|u_1\|_1)^2 \leq B_2 b(u,u), \tag{4.2.10}$$

for some constant $B_2 > 0$. Similarly, for u_4 there exists a constant $B_4 > 0$ such that

$$\|u_4\|^2 \leq (\ell_2 \|u_4'\| + K_\ell \sqrt{\ell_2} \|u_3\|_1)^2 \leq B_4 b(u,u). \tag{4.2.11}$$

Adding Inequalities (4.2.8) to (4.2.11) we obtain

$$\|u\|_X^2 \leq (B_1 + B_2 + B_3 + B_4) b(u,u).$$

□

Collecting the results above we obain the following corollary.

Corollary 4.2.2. *The bilinear form b is an inner product for the space V.*

Remark. *Note that the property $b(u,u) = 0 \iff u = 0$ is satisfied due to the continuity condition.*

Definition 4.2.2 (Energy space V)**.** *The vector space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.*

Proposition 4.2.5. *There exists a positive constant B_V such that*

$$b(u,u) \geq B_V \|u\|_W^2 \text{ for each } u \in V.$$

Proof. Using Propositions 4.2.4 and 4.2.2 for any $u \in V$,

$$b(u,u) \geq B_B (B_W)^{-1} \|u\|_W^2.$$

□

Theorem 4.2.1. *The norms $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent on V.*

Proof. Recall for any $u \in V$,

$$\|u_1\|^2 \leq \eta^{-1}\|\sqrt{g}u_1\|^2, \tag{4.2.12}$$

where $\eta = \inf_{[0,\ell_1]}|g|$. Since u_1 has a zero in $[0, \ell_1]$, from Lemma A.2.3 we have

$$B_1 b(u,u) \geq \|u_1'\|^2 \geq (\eta\ell_1)^{-1}\|u_1\|^2, \tag{4.2.13}$$

for some constant $B_1 > 0$. Also, for some constant $B_3 > 0$,

$$B_3 b(u,u) \geq \|u_3'\|^2 \geq \ell_1^{-1}\|u_3\|^2. \tag{4.2.14}$$

Using

$$\|u_1'\| + \|u_2'\| \leq \|u_1' - u_3\| + \|u_3\| + \|u_2' - u_4\| + \|u_4\|$$

together with (4.2.11) and (4.2.14), we obtain the result that

$$B^* b(u,u) \geq \|u'\|_X^2. \tag{4.2.15}$$

Combining the result in Proposition 4.2.4 and Inequality (4.2.15) yields

$$Bb(u,u) \geq \|u\|_1^2.$$

Next, recall that for any $u \in V$ the energy norm is defined by

$$\|u\|_V^2 = \|u_1' - u_3\|^2 + \|\sqrt{g}u_1\|^2 + \|u_2' - u_4\|^2 + \left\|\frac{1}{\beta_1}u_3'\right\|^2 + \left\|\frac{1}{\beta_2}u_4'\right\|^2. \tag{4.2.16}$$

(Recall that the continuity condition is required to ensure that $\|\cdot\|_V$ is a norm.) Let $\eta_1 = \max_{[0,\ell_1]}|g|$. Using the Triangle inequality on $u \in V$ in Equation (4.2.16) yields

$$\begin{aligned}\|u\|_V^2 &\leq 2\Big(\|u_1'\|^2 + \|u_3\|^2 + \|u_2'\|^2 + \|u_4\|^2\Big) + \eta_1\|u_1\|^2 + \left|\frac{1}{\beta_1}\right|\|u_3'\|^2 + \left|\frac{1}{\beta_2}\right|\|u_4'\|^2 \\ &\leq B_V\|u\|_1^2,\end{aligned}$$

where $B_V = \max\left\{2, \eta_1, \frac{1}{\beta_1}, \frac{1}{\beta_2}\right\}$. □

Corollary 4.2.3. *The space V is complete.*

Proof. Since $V_B(0,\ell_1)$ and $V_A(0,\ell_2)$ are the closures of the test functions in the Hilbert spaces $H^1(0,\ell_1)$ and $H^1(0,\ell_2)$ respectively, the spaces $V_B(0,\ell_1)$ and $V_A(0,\ell_2)$ are complete. Using the remark to Proposition A.1.3 we have that the product space V is complete with respect to the norm $\|\cdot\|_1$. Since the $\|\cdot\|_V$ and $\|\cdot\|_1$ norms are equivalent, it follows from Proposition A.1.2 that the product space V is complete. □

Existence

Proposition 4.2.6. *V is dense in W and W is dense in X.*

Proof. Recall that V is the product space $V_B(0,\ell_1)\times V_A(0,\ell_2)\times V_B(0,\ell_1)\times V_A(0,\ell_2)$, where $V_B(0,\ell_1)$ and $V_A(0,\ell_2)$ are the closures of the test functions in $H^1(0,\ell_i)$.

By Corollary A.1.1 we have that for $i = 1,2$, $C^1[0,\ell_i]$ is dense in $\mathcal{L}^2(0,\ell_i)$. That is, for any $u \in X$ there exists $u^* \in V$ with $u^* = \langle u_1^*, u_2^*, u_3^*, u_4^*\rangle \in C^1[0,\ell_1]\times C^1[0,\ell_2]\times C^1[0,\ell_1]\times C^1[0,\ell_2]$, such that for any $\varepsilon > 0$

$$\|u_i - u_i^*\|^2 < \frac{\varepsilon}{4} \quad \text{for } i = 1,2,3,4.$$

It follows that

$$\|u - u^*\|_X^2 < \varepsilon.$$

Therefore V is dense in X. The result follows from Proposition 4.2.2. □

To apply the theory in Chapter 2, we define Hilbert spaces X, W and V that are specific to Problem EBD-W.

In this application the sets X and W are the same. The definitions are as seen in Subsection 4.2.2.

Theorem 4.2.2. *Suppose $q \in C^1(\mathcal{J}, X)$. If $u_0 \in E_b$ and $u_d \in V$, then there exists a unique weak solution u such that*

$$u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W),$$

for Problem EDB-W.

Proof. As a result of Propositions 4.2.6, 4.2.2 and 4.2.5, Assumptions A1, A2 and A3 are respectively satisfied. Since $a = 0$, from the remark in Section 2.2, Assumption A4W is satisfied. As a consequence of Theorem 2.2.2 a unique weak solution exists for Problem EDB-W. □

4.3 Special cases of the model

Thus far, we have considered the well-posedness of a double beam problem, where Beam 1 is clamped at its left end while embedded in an elastic material. Other configurations are possible for such a problem for instance, varying properties of a beam and boundary conditions. However, such conditions have implications on the existence theory. As motivation for our discussion in Chapter 5, it is useful to consider some configurations and their influence to the theory in Section 4.2.

4.3.1 Cantilever double beam

Consider Problem EDB-W given that $g = 0$. This implies that Beam 1 is not embedded; it is a special case of Problem EDB-W. With the exception of inequality (4.2.9) all the Definitions, Corollaries, Propositions and Theorem 4.2.1 in Section 4.2 all hold. To this end, we consider the result below which is a replacement for Inequality (4.2.9).

Proposition 4.3.1. *Consider Problem EDB-W, where $g = 0$. There exists a positive constant B_N such that*

$$B_N b(u,u) \geq \|u_1\|^2 \text{ for each } u \in V.$$

Proof. Note that

$$\|u_1\|^2 \leq \|u\|_X^2 \text{ for all } u \in V. \tag{4.3.1}$$

From Lemma A.2.4 we have

$$\|u\|_X^2 \leq \ell_1^2\|u_1'\|^2 + \ell_1^2\|u_3'\|^2 \leq 2\ell_1^2\left(\|u_1' - u_3\|^2 + \|u_3'\|^2\right). \tag{4.3.2}$$

Using the definition of b and substituting $g = 0$ we have

$$\begin{aligned} b(u,u) =& \|u_1' - u_3\|^2 + \|u_2' - u_4\|^2 + \frac{1}{\beta_1}\|u_3'\|^2 + \frac{1}{\beta_2}\|u_4'\|^2 \\ &\geq \min\left\{1, \frac{1}{\beta_1}, \frac{1}{\beta_2}\right\}\left(\|u_1' - u_3\|^2 + \|u_2' - u_4\|^2 + \|u_3'\|^2 + \|u_4'\|^2\right). \end{aligned} \tag{4.3.3}$$

Let $B_\alpha = \min\left\{1, \frac{1}{\beta_1}, \frac{1}{\beta_2}\right\}$ and $B_N = \frac{B_\alpha}{2\ell_1^2}$. It follows from Inequalities (4.3.1), (4.3.2) and (4.3.3) that

$$b(u,u) \geq (B_N)^{-1}\|u_1\|^2.$$

□

As previously stated, the estimate in Proposition 4.3.1 replaces Inequality (4.2.9) when $g = 0$. As such, the result in Proposition 4.2.4 still holds.

As before, Problem EDB-W has a unique solution u such that

$$u \in C^1((-\infty, \infty), V) \cap C^2((-\infty, \infty), W),$$

provided $u_0 \in E_b$ and $u_d \in V$. Briefly, the existence theory for Problem EDB-W does not change irrespective of whether Beam 1 is embedded in an elastic material or not.

4.3.2 Partially embedded beam

Unlike in Subsection 4.3.1, we now consider different boundary conditions on Beam 1 when it is embedded in an elastic material. The boundary conditions imposed are:

$$F_1(0,t) = M_1(0,t) = 0. \tag{4.3.4}$$

For clarity, we refer to Problem EDB with boundary conditions (4.3.4) as Problem F-EDB. To obtain the variational form of Problem F-EDB the required test function space is

$$\mathcal{T}_p = \mathcal{T}_A[0,\ell_1] \times \mathcal{T}_A[0,\ell_2] \times \mathcal{T}_A[0,\ell_1] \times \mathcal{T}_A[0,\ell_2].$$

Note that the space $\mathcal{T}_A[0,\ell_i]$ is defined as

$$\mathcal{T}_A[0,\ell_i] = C^1[0,\ell_i] \text{ for } i = 1, 2.$$

The weak variational forms of Problem EDB and Problem F-EDB are similar, but with different test function spaces. We define the product spaces as in Section 4.2 with

$$V_p = V_A[0,\ell_1] \times V_A[0,\ell_2] \times V_A[0,\ell_1] \times V_A[0,\ell_2].$$

With the exception of Proposition 4.2.4 and Theorem 4.2.1, the Definitions, Corollaries and Propositions discussed in Section 4.2 are satisfied. But, to prove Proposition 4.2.4 we require estimates for $u \in V$. By the definition of V, Lemma A.2.3 cannot be applied since the test functions (in $\mathcal{T}_A[0,\ell_i]$) do not necessarily have zeros. The authors of [VZV09] considered a similar problem. Specifically, a non-dimensionless pinned-pinned Timoshenko beam problem. For completeness, we present the boundary conditions for a pinned-pinned beam problem

$$w(0,t) = M(0,t) = 0 \text{ and } w(\ell,t) = M(\ell,t) = 0.$$

The test function space for the problem is

$$\mathcal{T}_p = T_C[0,\ell] \times T_A[0,\ell],$$

where $T_C[0,\ell] = \{v \in C^1[0,\ell] \mid v(0) = v(\ell) = 0\}$ and $T_A[0,\ell] = C^1[0,\ell]$.

A difficulty is encountered when determining an estimate for $\phi \in \mathcal{T}_A[0,\ell]$. This challenge is discussed further by the authors of [VZV09]. Consequently, proving Propositions 4.2.4, 4.2.5 and Theorem 4.2.1 is beyond the scope of this dissertation. Regardless, we assume that Propositions 4.2.4, 4.2.5 and Theorem 4.2.1 hold for the weak variational form of Problem F-EDB.

Our assumption enables us to establish the existence of a unique solution of Problem F-EDB such that if $u_0 \in E_b$ and $u_d \in V$, there is a solution

$$u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W).$$

4.3.3 Conclusion

From our discussion, note that if Beam 1 is clamped, regardless of being embedded in an elastic material or not, the theory in Chapter 2 can be used to prove the existence and uniqueness of a weak solution for Problem EDB.

If the boundary conditions imposed on Beam 1 are as in Equation (4.3.4), there are difficulties encountered in determining some estimates. Nevertheless, we agree with [VZV09] that it is possible to obtain an estimate for u. Therefore, the existence theory in Chapter 2 can be applied and a solution for Problem F-EDB be determined.

Besides the difference in the definition of the spaces V, it is important to note that Problem EDB-W and the weak variational form of Problem F-EDB are the same. For the rest of this discussion, we consider Problem EDB-W as the general problem for a serially double connected beam model, which is rigidly attached at the interface. The imposed boundary conditions on the model are ignored. Consequently, the existence of a unique solution for Problem EDB-W is established.

4.4 Finite element approximation

In this section, the convergence of the finite element method approximation for the solution of Problem EDB-W is considered. Recall that Assumptions A1, A2, A3 and A4W for the existence and uniqueness of a weak solution $u \in C^1(\mathcal{J}, V) \cap C^2(\mathcal{J}, W)$ has been established in Section 4.2.

4.4.1 Galerkin approximation

Suppose $S^h(0, \ell_i)$ is the span of piecewise linear basis functions on $[0, \ell_i]$. Let the product space S^h denote a finite dimensional subspace of V where S^h is defined as:

$$S^h(0, \ell_1) \times S^h(0, \ell_2) \times S^h(0, \ell_1) \times S^h(0, \ell_2).$$

Problem EDB-W^h

Find a function $u_h(t) \in S^h$ such that for each $t \in [0, t_*]$,

$$c(u_h''(t), v) + b(u_h(t), v) = (q(\cdot, t), v) \quad \text{for each } v \in S^h,$$
$$\text{while } u_h(0) = u_0^h \text{ and } u_h'(0) = u_d^h.$$

Using the approach in Chapter 3, we will interpolate to find the initial conditions u_0^h and u_d^h. Note that Problem EDB-W^h is a special case of Problem GH^h.

4.4.2 Convergence and error estimates

Recall Problem EDB-W is undamped. It was discussed in Chapter 2 that the absence of damping is a special case of weak damping. As such, we will proceed, using the results in Chapter 3 to determine error estimates. This subsection is based on the approach in [Bas14].

Suppose that Assumption C1 holds, as in Subsection 3.3.1 we use results on interpolation theory to replace Assumption C2.

We define a generalised interpolation operator Π in the product space H^1 by

$$\Pi u = \langle \Pi_\ell u_1, \Pi_\ell u_2, \Pi_\ell u_3, \Pi_\ell u_4 \rangle \quad \text{for } u \in H^1,$$

where Π_l is the usual interpolation operator for piecewise linear basis functions.

To apply the results in Chapter 3 (and therefore in [BV13]), we require $u \in C^2((0,t_*), V)$ and that $u(t)$ and $u''(t) \in H^2$.

The following interpolation error estimate replaces Assumption C2.

Corollary 4.4.1. *There exists a constant $\hat{C}_\Pi$ such that if $u \in H^k$ for $k \geq 2$, then*

$$\|\Pi_\Pi u - u\|_V \leq \hat{C}_\Pi h |u|_2, \tag{4.4.1}$$

where $|\cdot|_2$ denotes the semi-norm.

Application of Theorem 3.1.2

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If the solution u of Problem EDB-W satisfies Assumption C1, $u(t) \in \mathcal{L}^2([0,t_*], H^2 \cap V)$ and $u'(t) \in \mathcal{L}^2([0,t_*], H^2 \cap V)$, then

$$\begin{aligned} \|u(t) - u_h(t)\|_W \leq & C_V \hat{C}_\Pi h \Big(|u(t)|_2 + 3t_* \sqrt{2} \max |u'(t)|_2 \\ & + 3K_a t_* \max |u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ & + 3t_* |u_d|_2 \Big), \end{aligned}$$

for each $t \in [0,t_*]$.

Suppose we use the same algorithm as in Problem GH^h-D, we then obtain the following problem.

Problem EDB^h-D

Find a sequence $\{u_k^h\} \subset S^h$, such that for each $k = 0, 1, 2, \cdots, N-1$,

$$\begin{aligned} \delta_t u_k =& v_{k+\frac{1}{2}}, \\ c\left(\delta_t v_k, \psi\right) + b\left(u_{k+\frac{1}{2}}^h, \psi\right) =& \frac{1}{2}\left(\left[f(t_k) + f(t_{k+1})\right], \psi\right)_X, \\ \text{while } u_0^h = d^h & \text{ and } u_d^h = v^h, \end{aligned}$$

for each $\psi \in S^h$.

To obtain the final error estimate, we combine the results from Theorems 3.1.2 and 3.2.1 to obtain the result below.

Application of the combination of Theorems 3.1.2 and 3.2.1

Suppose $u_0^h = \Pi u_0$ and $u_d^h = \Pi u_d$. If $u(t) \in \mathcal{L}^2([0,t_*], H^2 \cap V)$, $u'(t) \in \mathcal{L}^2([0,t_*], H^2 \cap V)$, $f \in C^2([0,t_*], \mathcal{L}^2)$ and the sequence $\{u_k^h\}$ is a solution of Problem EDB^h-D, then

$$\begin{aligned}\|u(t_k) - u_k^h\|_W \leq & C_V \hat{C}_\Pi h \Big(|u(t)|_2 + 3t_* \sqrt{2} \max |u'(t)|_2 \\ & + 3K_a t_* \max |u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ & + 3t_* |u_d|_2 \Big) + 7t_*^2 \tau^2 \max \|u_h^{(4)}\|_W + 7t_* \tau^2 \max \|u_h'''\|_W \\ & + \sqrt{2K_a} \tau^4 \max \|u_h'''\|_W,\end{aligned}$$

for each $t_k \in [0, t_*]$.

4.5 Single beam model for serially connected beams

Thus far, we have considered the solvability and convergence of the Galerkin approximation of a serially connected double beam model problem. In this section, we consider it as a single beam where the beam is embedded. This investigation is motivated by the effect of the interface condition imposed on the serially connected double Timoshenko beam model.

4.5.1 Single Timoshenko beam problem

We begin by writing Problem EDB as an embedded single beam model problem. Recall Beam 1 is configured to be one dimensionless unit, while the dimensionless length of Beam 2 is ℓ_2.

Let $I_1 = [0, 1]$ and $I_2 = [1, L^*]$, where $L^* = 1 + \ell_2$.

Consider the function w such that

$$w(x, t) = \begin{cases} w_1(x, t) & \text{on} \quad I_1, \\ w_2(x - 1, t) & \text{on} \quad I_2. \end{cases}$$

The functions ϕ, F, Q and M are defined similarly.

Note that the interval $I_1 \cup I_2 = I$, so that the point $x = 1$ is the interface. That is, the right end of Beam 1 corresponds to the left end of Beam 2.

Remark. *Since the functions w_1 and w_2 are equal at the interface, it is acceptable to define w_1 and w_2 on closed intervals. We have that $w(1, t) = w_1(1, t) = w_2(0, t)$. This is also the case for the functions ϕ, F and M.*

Boundary and interface conditions

Consider the case where the beam is free at the endpoints, that is

$$F(0,t) = M(0,t) = 0 \text{ and } F(L^*,t) = M(L^*,t) = 0. \tag{4.5.1}$$

The interface conditions are:

$$\begin{aligned} w(1,t) =& w_1(1,t) = w_2(0,t), \quad \phi(1,t) = \phi_1(1,t) = \phi_2(0,t), \\ F(1,t) =& F_1(1,t) = F_2(0,t) \text{ and } M(1,t) = M_1(1,t) = M_2(0,t). \end{aligned} \tag{4.5.2}$$

We formulate the model problem.

Problem ESB (Embedded Single Beam)

Given a function Q and positive constants α and β, find w and ϕ such that

$$\partial_t^2 w = \partial_x F + Q, \tag{4.5.3}$$

$$\frac{1}{\alpha}\,\partial_t^2 \phi = \partial_x M + F, \tag{4.5.4}$$

$$F = \partial_x w - \phi, \tag{4.5.5}$$

$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{4.5.6}$$

with boundary conditions (4.5.1), interface conditions (4.5.2) and initial conditions

$$w(\cdot,0) = w_0, \phi(\cdot,0) = \phi_0, \partial_t w(\cdot,0) = w_d \text{ and } \partial_t \phi(\cdot,0) = \phi_d. \tag{4.5.7}$$

The function Q is defined as

$$Q = \begin{cases} -gw(x,t) & \text{on } I_1, \\ q_2(x-1,t) & \text{on } I_2, \end{cases}$$

where g is a function that describes the elasticity of a material.

4.5.2 Variational form

To obtain the variational form of Problem ESB, multiply Equation (4.5.3) by a function $v \in C^1(\bar{I})$, multiply Equation (4.5.4) by a function $\nu \in C^1(\bar{I})$ and integrate. If $\partial_x F$ and $\partial_x M$ are integrable, integrating the terms on the right side of the equality signs in Equation (4.5.3) and (4.5.4) yields

$$\begin{aligned}\int_0^{L^*} \partial_t^2 w(\cdot,t)v =& F(L^*,t)v(L^*) - F(0,t)v(0) - \int_0^{L^*} F(\cdot,t)v' \\ &+ \int_0^{L^*} Q(\cdot,t)v, \qquad (4.5.8)\end{aligned}$$

$$\begin{aligned}\int_0^{L^*} \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)\nu =& M(L^*,t)\nu(L^*) - M(0,t)\nu(0) - \int_0^{L^*} M(\cdot,t)\nu' \\ &+ \int_0^{L^*} F(\cdot,t)\nu. \qquad (4.5.9)\end{aligned}$$

The substitution of boundary conditions (4.5.1) into Equations (4.5.8) and (4.5.9) necessitates the introduction of a test function space

$$\mathcal{T}_A(I) = C^1(\bar{I})$$

Let

$$\mathcal{T}_p = \mathcal{T}_A(I) \times \mathcal{T}_A(I).$$

The $\mathcal{L}^2(I)$ inner product is denoted by $(\cdot,\cdot)$, such that $(\cdot,\cdot) = (\cdot,\cdot)_{I_1} + (\cdot,\cdot)_{I_2}$.

Combining Equations (4.5.8) and (4.5.9) then substituting constitutive equations (4.5.5) and (4.5.6) results in the variational form of Problem ESB.

Problem ESB-V

Given functions g and q_2, find $\langle w,\phi\rangle$ where $\langle w(\cdot,t),\phi(\cdot,t)\rangle \in \mathcal{T}_p$ for each $t > 0$, such that

$$\begin{aligned}\int_I \partial_t^2 w(\cdot,t)v =& -\int_I(\partial_x w(\cdot,t) - \phi(\cdot,t))(\cdot,t)v' - \int_{I_1} gw(\cdot,t)v \\ &+ \int_{I_2} q_2(\cdot,t)v,\end{aligned}$$

$$\int_I \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)\nu = \int_I(\partial_x w(\cdot,t) - \phi(\cdot,t))\nu - \int_I \frac{1}{\beta}\partial_x\phi(\cdot,t)\nu',$$

for all $\langle v,\nu\rangle \in \mathcal{T}_p$ with

$$w(\cdot,0) = w_0,\ \partial_t w(\cdot,0) = w_d, \phi(\cdot,0) = \phi_0 \text{ and } \partial_t\phi(\cdot,0) = \phi_d.$$

Other variational forms of Problem ESB are possible, we discuss this in Chapter 5.

Note that Problem EDB-V and Problem ESB-V are equivalent. Since a solution u of Problem EDB-W is a solution of Problem EDB-V, it follows that u is a solution of Problem ESB-V. For theoretical purposes we treat the problem as a double beam. For computations however, the single beam model is used (see Chapter 5).

Chapter 5

Beam models for tap root systems

In this chapter, we consider an application of the beam models introduced in Chapter 4, namely the embedded single and double beam models. Specifically, the Timoshenko beam theory is used to study the reaction of plants with a tap root to lateral static and dynamic loads. The aim is to compare numerical results obtained for the finite element approximations to the experimental findings in [Enn00].

In his study, Ennos (2000) states that the significance of roots in plants is not limited to the absorption of water and nutrients. Another major function of the roots is to anchor plants. An outcome of the successful work in [Enn00] is that a framework on how to stabilise crops was established.

Recall Problems EDB and ESB considered in Chapter 4. In this chapter, the Galerkin approximations of the problems are examined and compared. The various parameters influencing the dynamics of a plant with a tap root system are also investigated. Finally, in Section 5.4 these results are discussed and compared to the findings in the above mentioned article.

5.1 Single beam model

Recall that the single beam model introduced in Section 4.5 is an outcome of the rigid attachment of two serially connected beams. In this investigation, a plant is modelled as a beam of length L^*, with an embedded part (0 to 1) which represents the root and the rest (1 to L^*) represents the exposed part of the plant.

For convenience, we rewrite the variational form of the embedded single beam model. Recall the interval $I = I_1 \cup I_2$, where $I_1 = [0,1]$ and $I_2 = [1, L^*]$. The space of test functions for Problem ESB-V is denoted by $\mathcal{T}_p = \mathcal{T}_A(I) \times \mathcal{T}_A(I)$, where $\mathcal{T}_A(I) = v \in C^1(I)$.

Problem ESB-V

Given functions $g \in C(I_1), q_2 \in C^1(I_2)$ and positive constants α and β, find $\langle w, \phi\rangle$ where $\langle w(\cdot,t), \phi(\cdot,t)\rangle \in \mathcal{T}_p$ for each $t > 0$, such that for any $\langle v, \nu\rangle \in \mathcal{T}_p$

$$\begin{aligned}\int_I \partial_t^2 w(\cdot,t)v = &- \int_I (\partial_x w(\cdot,t) - \phi(\cdot,t))(\cdot,t)v' - \int_{I_1} g w(\cdot,t)v \\ &+ \int_{I_2} q_2(\cdot,t)v, \qquad (5.1.1)\end{aligned}$$

$$\int_I \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)\nu = \int_I (\partial_x w(\cdot,t) - \phi(\cdot,t))\nu - \int_I \frac{1}{\beta}\partial_x \phi(\cdot,t)\nu', \qquad (5.1.2)$$

with initial conditions

$$w(\cdot,0) = w_0,\ \partial_t w(\cdot,0) = w_d, \phi(\cdot,0) = \phi_0 \text{ and } \partial_t \phi(\cdot,0) = \phi_d.$$

Variational forms

The standard variational form of Problem ESB was introduced in Subsection 4.5.2. When this variational form is used to obtain approximations using the finite element method, we refer to it as the Standard Finite Element Method (SFEM).

Alternatively, only the second constitutive equation (4.5.6) can be substituted into Equation (5.1.2), while the other is multiplied by a smooth function and is used as a separate equation. This method is referred to as the Mixed Finite Element Method (MFEM). A modification of MFEM is obtained when both the constitutive equations are used separately by multiplying them by arbitrary smooth functions. This method is referred to as the Variant Mixed Finite Element Method (VMFEM).

In this section, the VMFEM is used to show how a system of differential equations is obtained. Similar arguments hold when SFEM and MFEM is considered.

For this application, we define the function g to be a constant function c, where c is the measure of elasticity of the soil (see Section 5.3 for more detail).

To start, we present the variant mixed variational form of Problem ESB. For this case, the space of test functions is defined to be

$$\mathcal{T}_p = \mathcal{T}_A(I) \times \mathcal{T}_A(I) \times \mathcal{T}_A(I) \times \mathcal{T}_A(I).$$

Problem ESB-VV

Given a function $q_2 \in C^1(I_2)$ and positive constants α, β and c for each $t > 0$, find $\langle w, \phi, F, M\rangle$ where $\langle w(\cdot,t), \phi(\cdot,t), F(\cdot,t), M(\cdot,t)\rangle \in \mathcal{T}_p$ such that for any $\langle v, \nu, y, z\rangle \in \mathcal{T}_p$

$$\int_I \partial_t^2 w(\cdot,t)v = -\int_I F(\cdot,t)v' - \int_{I_1} cw(\cdot,t)v + \int_{I_2} q_2(\cdot,t)v, \quad (5.1.3)$$

$$\int_I \frac{1}{\alpha}\partial_t^2 \phi(\cdot,t)\nu = \int_I F(\cdot,t)\nu - \int_I M(\cdot,t)\nu', \quad (5.1.4)$$

$$\int_I F(\cdot,t)y = \int_I (\partial_x w(\cdot,t) - \phi(\cdot,t))y, \quad (5.1.5)$$

$$\int_I M(\cdot,t)z = \int_I \frac{1}{\beta}\partial_x \phi(\cdot,t)z, \quad (5.1.6)$$

with initial conditions

$$w(\cdot,0) = w_0,\ \partial_t w(\cdot,0) = w_d, \phi(\cdot,0) = \phi_0 \text{ and } \partial_t\phi(\cdot,0) = \phi_d.$$

5.1.1 Galerkin approximation

In this subsection, we consider the Galerkin approximation of Problem ESB-VV. Recall the intervals $I_1 = [0,1]$, $I_2 = [1, L^*]$ and that $I = I_1 \cup I_2$. To obtain the Galerkin approximation, we consider piecewise linear basis functions on n subintervals of I and denote the basis functions by $\delta_0, \delta_1, \cdots, \delta_n$.

Define the following finite dimensional subspace

$$S^h = span\{\delta_0, \delta_1, \cdots, \delta_n\}.$$

Problem ESB^h

Given a function $q_2^h \in C^1(I_2)$ and positive constants α, β and c, find $w^h(\cdot,t), \phi^h(\cdot,t), F^h(\cdot,t)$ and $M^h(\cdot,t) \in S^h$ such that for $i = 0,\dots,n$

$$\int_I \partial_t^2 w^h(\cdot,t)\delta_i = -\int_I F^h(\cdot,t)\delta_i' - \int_{I_1} cw^h(\cdot,t)\delta_i + \int_{I_2} q_2^h(\cdot,t)\delta_i, \tag{5.1.7}$$

$$\int_I \frac{1}{\alpha}\partial_t^2 \phi^h(\cdot,t)\delta_i = \int_I F^h(\cdot,t)\delta_i - \int_I M^h(\cdot,t)\delta_i', \tag{5.1.8}$$

$$\int_I F^h(\cdot,t)\delta_i = \int_I \partial_x w^h(\cdot,t)\delta_i - \int_I \phi^h(\cdot,t)\delta_i, \tag{5.1.9}$$

$$\int_I M^h(\cdot,t)\delta_i = \int_I \frac{1}{\beta}\partial_x \phi^h(\cdot,t)\delta_i, \tag{5.1.10}$$

with initial conditions w_0^h, w_d^h, ϕ_0^h and $\phi_d^h \in S^h$. As before, we use interpolation to obtain these approximations.

5.1.2 System of ordinary differential equations

We begin by defining the general form of the matrices K, L and N which can be modified according to the restriction on the basis functions. Later, we present a system of ordinary differential equations used to obtain an approximate solution for Problem ESB^h. For convenience, we denote time derivatives by a dot.

Let

$$K_{ji} = \int_0^{L^*} \delta_i'\delta_j',$$
$$L_{ji} = \int_0^{L^*} \delta_i\delta_j' \quad \text{and}$$
$$N_{ji} = \int_0^{L^*} \delta_i\delta_j,$$

for $i, j = 0,\dots,n$.

Since $w^h(t) \in S^h$, we can write it as

$$w^h(x,t) = \sum_{j=0}^{n} w_j(t)\delta_j(x). \tag{5.1.11}$$

The n-tuple $(w_1, w_2, \cdots, w_n)$ is denoted by $\bar{w}$ and corresponds to w^h.

Similarly, we have

$$\phi^h(x,t) = \sum_{j=0}^{n} \phi_j(t)\delta_j(x), \quad F^h(x,t) = \sum_{j=0}^{n} F_j(t)\delta_j(x) \tag{5.1.12}$$

and

$$M^h(x,t) = \sum_{j=0}^{n} M_j(t)\delta_j(x), \tag{5.1.13}$$

which are denoted by $\bar{\phi}, \bar{F}$ and $\bar{M}$ respectively.

Piecewise linear basis functions

As mentioned, we use piecewise linear basis functions. Let $x_0, x_1, \ldots, x_n$ be equally spaced points such that

$$0 = x_0 < x_1 < \cdots < x_m = 1 < x_{m+1} < \cdots < x_n = L^*.$$

The distance between two consecutive points is denoted by h.

For illustration, we only show how Equation (5.1.7) is rewritten. To rewrite Equations (5.1.8) to (5.1.10) similar arguments are used. Substituting Equations (5.1.11) and (5.1.12) into Equation (5.1.7) we have

$$\sum_{j=0}^{n} \ddot{w}_j \int_{x_0}^{x_n} \delta_j\delta_i = -\sum_{j=0}^{n} F_j \int_{x_0}^{x_n} \delta_j\delta_i' - c\sum_{j=0}^{m} w_j \int_{x_0}^{x_m} \delta_j\delta_i + \sum_{j=m}^{n} q_j \int_{x_m}^{x_n} \delta_j\delta_i \tag{5.1.14}$$

for $i = 0, \ldots, n$.

Using the L and N matrices, Equation (5.1.14) takes the form

$$N\ddot{\bar{w}} = -L\bar{F} - cN_g\bar{w} + N_q\bar{q}, \tag{5.1.15}$$

where

$$N_g := \begin{cases} N_{i,j} & \text{for } \ j < m, \\ N_{n,n} & \text{for } \ j = m, \\ 0 & \text{for } \ j > m. \end{cases}$$

$$N_q := \begin{cases} 0 & \text{for } \ j < m, \\ N_{0,0} & \text{for } \ j = m, \\ N_{i,j} & \text{for } \ j > m, \end{cases}$$

for $i, j = 0, \dots, n$.

Equations (5.1.8), (5.1.9) and (5.1.10), respectively become

$$\frac{1}{\alpha} N\ddot{\bar{\phi}} = N\bar{F} - L\bar{M}, \tag{5.1.16}$$

$$N\bar{F} = L^T\bar{w} - N\bar{\phi} \quad \text{and} \tag{5.1.17}$$

$$N\bar{M} = \frac{1}{\beta} L^T \phi. \tag{5.1.18}$$

5.1.3 Static problem

We will consider both the dynamic and static problems. In this subsection, we consider Problem ESB at equilibrium.

Equations (5.1.15) to (5.1.18) take the form

$$N_q\bar{q} = cN_g\bar{w} + L\bar{F}, \tag{5.1.19}$$

$$\bar{0} = N\bar{F} - L\bar{M}, \tag{5.1.20}$$

$$\bar{0} = L^T\bar{w} - N\bar{\phi} - N\bar{F}, \tag{5.1.21}$$

$$\bar{0} = \frac{1}{\beta} L^T\bar{\phi} - N\bar{M}. \tag{5.1.22}$$

This is equivalent to solving, $A_0\bar{u}_0 = \bar{R}_0$ with

$$A_0 = \begin{bmatrix} cN_g & 0 & L & 0 \\ 0 & 0 & N & -L \\ L^T & -N & -N & 0 \\ 0 & \frac{1}{\beta}L^T & 0 & -N \end{bmatrix}, \bar{u}_0 = \begin{bmatrix} \bar{w} \\ \bar{\phi} \\ \bar{F} \\ \bar{M} \end{bmatrix} \text{ and } \bar{R}_0 = \begin{bmatrix} N_q\bar{q} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}.$$

5.2 Static embedded double beam model

In this section we consider the static embedded double beam model. Recall the model discussed in Chapter 4, we have the following equations for the static case:

Equations of motion

$$\begin{aligned} F_i' + q_i &= 0 \text{ and} \\ F_i + M_i' &= 0, \text{ for } i = 1 \text{ and } 2. \end{aligned}$$

Constitutive equations

$$\begin{aligned} F_i &= w_i' - \phi_i, \\ M_i &= \frac{1}{\beta}_i \phi_i', \text{ for } i = 1 \text{ and } 2. \end{aligned}$$

To distinguish between the embedded and clamped beam it is convenient to denote the embedded beam by E, and the cantilever beam by C. This notation will be used throughout this section. Relative to the theory in Chapter 4, let Beam 1 be the embedded beam and Beam 2 be the cantilever beam.

The load on E is $q_E = -cw_E$ where c is a positive constant as in Section 5.1. The wind load on C (q_C) will be specified. For simplicity, homogeneous beams are considered in this model.

5.2.1 Model problem

In this subsection, we present a mathematical model that is simplified by making q_C a constant. The outcome of the simplification is that we can explicitly solve for the shear force and moment of the cantilever beam.

Problem SDB (Static Double Beam)

Find w_i and ϕ_i for each $t > 0$ such that

$$F_i' + q_i = 0, \tag{5.2.1}$$

$$F_i + M_i' = 0, \tag{5.2.2}$$

$$F_i = w_i' - \phi_i \quad \text{and} \tag{5.2.3}$$

$$M_i = \frac{1}{\beta}\phi_i', \tag{5.2.4}$$

for $i = E$ and C.

For the cantilever beam, given a constant q_C, solving Equations (5.2.1) and (5.2.2) yields

$$F_C(x) = -q_C x + C_F \quad \text{and} \quad M_C(x) = \frac{1}{2}q_C x^2 - C_F x + C_M. \tag{5.2.5}$$

In Equation (5.2.5), if C_F and C_M are known, w_C and ϕ_C can be explicitly determined. This is an additional advantage of the simplification of the static double beam model.

Boundary and interface conditions

The boundary conditions on the embedded and cantilever beams respectively are:

$$\begin{aligned} F_E(0) = M_E(0) = 0. \\ F_C(L^*) = M_C(L^*) = 0. \end{aligned} \tag{5.2.6}$$

Problem SDB is a special case of Problem EDB as such, the supposition at the interface holds. Specifically, at the interface we have that w, ϕ, F and M are continuous.

For convenience, we present the interface conditions.

$$\begin{aligned} F_E(1) &= F_C(0) \quad \text{and} \quad M_E(1) = M_C(0) \\ \text{also,}\ w_E(1) &= w_C(0) \quad \text{and} \quad \phi_E(1) = \phi_C(0). \end{aligned}$$

Cantilever beam

Substitution of Equation (5.2.6) into Equation (5.2.5) yields

$$C_F = q_C L^* \quad \text{and} \quad C_M = \frac{1}{2} q_C (L^*)^2.$$

Embedded beam

The following equations follow from Problem SDB.

$$F_E' - c w_E = 0, \tag{5.2.7}$$

$$F_E + M_E' = 0, \tag{5.2.8}$$

and

$$F_E = w_E' - \phi_E, \tag{5.2.9}$$

$$M_E = \frac{1}{\beta} \phi_E'. \tag{5.2.10}$$

Since the interface conditions are

$$F_E(1) = F_C(0) \quad \text{and} \quad M_E(1) = M_C(0),$$

then

$$F_E(1) = C_F \quad \text{and} \quad M_E(1) = C_M.$$

5.2.2 General solution

The equations in (5.2.5) enable us to determine a general solution for the cantilever beam. The existence of a solution for Problem SDB implies that a solution for the embedded beam exists as well.

To determine a general solution for the embedded beam, we substitute Equations (5.2.9) and (5.2.10) into Equations (5.2.7) and (5.2.8) respectively. Using the original notation yields,

$$w'' - \phi' - cw = 0, \tag{5.2.11}$$
$$\frac{1}{\beta}\phi'' - \phi + w' = 0. \tag{5.2.12}$$

Following a similar approach as in [VV06], we let $\bar{u} = [u_1 \; u_2]^T$ and suppose that $e^{mx}\bar{u}$ is a solution of Equations (5.2.11) and (5.2.12); that is

$$\begin{bmatrix} w \\ \phi \end{bmatrix} = e^{mx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$e^{mx} \begin{bmatrix} m^2 - c & -m \\ \beta m & m^2 - \beta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.2.13}$$

For any nontrivial solution $\bar{u}$, the determinant of Equation (5.2.13) is zero. So,

$$m^4 - cm^2 + \beta c = 0. \tag{5.2.14}$$

Let $k = m^2$ and rewrite Equation (5.2.14) as

$$k^2 - ck + \beta c = 0. \tag{5.2.15}$$

Using the quadratic formula to find the roots of Equation (5.2.15) we have

$$\Delta = c^2 - 4\beta c.$$

There are three options:

i. If $\Delta > 0$, then there are two distinct real roots.

ii. If $\Delta = 0$, then there is a repeated root.

iii. If $\Delta < 0$, then there are complex roots.

For simplicity, suppose Equation (5.2.15) has two distinct real roots. Thus, $c > 4\beta$ where c and β are positive. Equation (5.2.15) has real roots r_1 and r_2 such that

$$r_1 = \frac{c + \sqrt{\Delta}}{2} \quad \text{and} \quad r_2 = \frac{c - \sqrt{\Delta}}{2}.$$

Therefore, Equation (5.2.14) has four distinct roots specifically,

$$m_1 = \sqrt{r_1}, \quad m_2 = -\sqrt{r_1}, \quad m_3 = \sqrt{r_2} \quad \text{and} \quad m_4 = -\sqrt{r_2}\ .$$

Let $f(m) = m^4 - cm^2 + \beta c$ and suppose that $c > 4\beta$, then the turning points of f are $p_1 = 0, p_2 = \sqrt{\frac{c}{2}}$ and $p_3 = -\sqrt{\frac{c}{2}}$.

The Newton-Raphson method can be used to approximate the roots of f, provided the starting values are not chosen near the turning points on the interval $[-c - \beta c, c + \beta c]$. The following results were obtained for the choice $\beta = 100, c = 450$ and ten iterative steps were used:

$m_1 = 12.24745,\ m_2 = -12.24745,\ m_3 = 17.32051,$ and $m_4 = -17.32051.$

Let $\bar{u} = \begin{bmatrix} 1 & \left(\frac{m^2 - c}{m}\right) \end{bmatrix}^T$ then,

$$\begin{bmatrix} m^2 - c & -m \\ \beta m & m^2 - \beta \end{bmatrix} \bar{u} = \bar{0}. \tag{5.2.16}$$

Hence, $\bar{u}$ is a solution of Equation (5.2.16) and $e^{mx}\bar{u}$ is a solution of Equations (5.2.11) and (5.2.12).

If m_1, m_2, m_3 and m_4 are the roots of Equation (5.2.14), then the general solution of the system takes the form:

$$\begin{aligned} \begin{bmatrix} w(x) \\ \phi(x) \end{bmatrix} =& a_1 e^{m_1 x} \begin{bmatrix} 1 \\ \dfrac{m_1^2 - c}{m_1} \end{bmatrix} + a_2 e^{m_2 x} \begin{bmatrix} 1 \\ \dfrac{c - m_2^2}{m_2} \end{bmatrix} \\ &+ a_3 e^{m_3 x} \begin{bmatrix} 1 \\ \dfrac{m_3^2 - c}{m_3} \end{bmatrix} + a_4 e^{m_4 x} \begin{bmatrix} 1 \\ \dfrac{c - m_4^2}{m_4} \end{bmatrix}. \end{aligned} \tag{5.2.17}$$

Theoretically, an exact solution for the embedded beam can be calculated. However, the coefficients of Equation (5.2.17) are small (close to zero) as a result of the roots being large.

5.2.3 Variational forms

In Subsection 4.3.2, we established the existence of a unique solution for Problem EDB. A difficulty is encountered in describing a general solution of the static embedded beam as discussed in Subsection 5.2.2, although in Subsection 5.2.1 a general solution of the cantilever beam was formulated. In this subsection, we derive the mixed and standard variational forms of Problem SDB in order to consider the convergence of the Galerkin approximations later. For this discussion, it is convenient to recall the following equations for the embedded beam:

Equations of motion

$$F'_E - cw_E = 0, \quad (5.2.18)$$
$$F_E + M'_E = 0. \quad (5.2.19)$$

Constitutive equations

$$F_E = w'_E - \phi_E, \quad (5.2.20)$$
$$M_E = \frac{1}{\beta}\phi'_E, \quad (5.2.21)$$

with boundary conditions

$$F_E(0) = 0 \ \text{ and } \ M_E(0) = 0 \text{ and}$$

interface conditions

$$F_E(1) = C_F \ \text{ and } \ M_E(1) = C_M.$$

Mixed variational form

To obtain the mixed variational form (using the original notation) multiply Equations (5.2.18), (5.2.19) and (5.2.20) by arbitrary smooth functions v, ν and y respectively. Integrate and use integration by parts.

Equations (5.2.18) and (5.2.19) respectively become

$$0 = F(1)v(1) - F(0)v(0) - \int_0^1 Fv' - \int_0^1 cwv,$$
$$0 = M(1)\nu(1) - M(0)\nu(0) - \frac{1}{\beta}\int_0^1 \phi'\nu' + \int_0^1 F\nu \text{ using } \mathrm{M_E} = \mathrm{M} = \frac{1}{\beta}\phi'.$$

Substituting the boundary and interface conditions we have

$$0 = C_F v(1) - \int_0^1 F v' - \int_0^1 cwv, \tag{5.2.22}$$

$$0 = C_M \nu(1) - \frac{1}{\beta}\int_0^1 \phi' \nu' + \int_0^1 F\nu. \tag{5.2.23}$$

Lastly, Equation (5.2.20) becomes

$$\int_0^1 Fy = \int_0^1 w'y - \int_0^1 \phi y. \tag{5.2.24}$$

Standard variational form

For the standard variational form, substitute Equations (5.2.20) into Equations (5.2.22) and (5.2.23)

$$C_F v(1) - \int_0^1 w'v' - \int_0^1 cwv + \int_0^1 \phi v' = 0, \tag{5.2.25}$$

$$C_M \nu(1) - \frac{1}{\beta}\int_0^1 \phi'\nu' + \int_0^1 w'\nu - \int_0^1 \phi\nu = 0. \tag{5.2.26}$$

Note that for both variational forms, the space of test functions is

$$\mathcal{T}_A[0,1] = C^1[0,1].$$

5.2.4 Galerkin approximations

To obtain the Galerkin approximation for the embedded problem, using the mixed and standard finite element methods we consider the variational forms discussed in Subsection 5.2.3.

Define the subspace S^h by

$$S^h = span\{\delta_0, \delta_1, \cdots, \delta_n\},$$

where δ_i are piecewise linear basis functions on $[0,1]$ as discussed before.

The Galerkin approximations for the static embedded problem are presented below.

Problem SDB^h (MFEM)

Find w^h, ϕ^h and $F^h \in S^h$ such that

$$C_F v(1) = c\int_0^1 w^h v + \int_0^1 F^h v', \tag{5.2.27}$$

$$C_M \nu(1) = \frac{1}{\beta}\int_0^1 (\phi')^h \nu' - \int_0^1 F^h \nu, \tag{5.2.28}$$

$$0 = \int_0^1 (w')^h y - \int_0^1 \phi^h y - \int_0^1 F^h y, \tag{5.2.29}$$

for each v, ν and $y \in S^h$.

Problem SDB^h (SFEM)

Find w^h and $\phi^h \in S^h$ such that

$$C_F v(1) = \int_0^1 (w')^h v' + c\int_0^1 w^h v - \int_0^1 \phi^h v', \tag{5.2.30}$$

$$C_M \nu(1) = -\int_0^1 (w')^h \nu + \frac{1}{\beta}\int_0^1 (\phi')^h \nu' + \int_0^1 \phi^h \nu, \tag{5.2.31}$$

for each v and $\nu \in S^h$.

5.2.5 System of equations

The aim in this subsection is to write the Galerkin approximations for the static embedded beam problem as a system of equations. The notation and method used in this subsection follow from the discussion in Subsection 5.1.2.

MFEM

From the definition of the mass matrix N, Equation (5.2.27) becomes

$$C_F \bar{e}_n = cN\bar{w} + L\bar{F}, \tag{5.2.32}$$

where $\bar{e}_n$ is a row vector with zero entries, except in its n^{th} component where it is 1.

Equation (5.2.28) takes the form

$$C_M \bar{e}_n = \frac{1}{\beta}K\bar{\phi} - N\bar{F}, \tag{5.2.33}$$

and Equation (5.2.29) becomes

$$\bar{0} = L^T\bar{w} - N\bar{\phi} - N\bar{F}. \quad (5.2.34)$$

This is equivalent to solving $A_1\bar{u}_1 = \bar{R}_1$ where

$$A_1 = \left[\begin{array}{c:c:c} cN & 0 & L \\ \hdashline 0 & \frac{1}{\beta}K & -N \\ \hdashline L^T & -N & -N \end{array}\right], \bar{u}_1 = \begin{bmatrix} \bar{w} \\ \bar{\phi} \\ \bar{F} \end{bmatrix} \text{ and } \bar{R}_1 = \begin{bmatrix} C_F\bar{e}_n \\ C_M\bar{e}_n \\ \bar{0} \end{bmatrix}.$$

SFEM

For the SFEM, Equations (5.2.30) and (5.2.31) respectively take the form

$$C_F\bar{e}_n = cN\bar{w} + K\bar{w} - L\bar{\phi} \text{ and} \quad (5.2.35)$$

$$C_M\bar{e}_n = -L^T\bar{w} + \frac{1}{\beta}K\bar{\phi} + N\bar{\phi}. \quad (5.2.36)$$

This is equivalent to solving $A_2\bar{u}_2 = \bar{R}_2$ where

$$A_2 = \left[\begin{array}{c:c} cN + K & -L \\ \hdashline -L^T & \frac{1}{\beta}K + N \end{array}\right], \bar{u}_2 = \begin{bmatrix} \bar{w} \\ \bar{\phi} \end{bmatrix} \text{ and } \bar{R}_2 = \begin{bmatrix} C_F\bar{e}_n \\ C_M\bar{e}_n \end{bmatrix}.$$

5.3 Numerical results

In this section, we first discuss the numerical results obtained for the static embedded double beam model as well as the equilibrium problem for a single beam model. The aim is to compare the results obtained for the static cases of the two beam models when the MFEM and SFEM are implemented. In Subsection 5.3.3, the dynamics of the single beam model is investigated.

5.3.1 Static: Embedded double beam

The numerical results obtained when the MFEM and SFEM are used to approximate the deflection (w) of the static embedded beam are presented in this subsection. The wind load (q_C) on the cantilever beam is assumed to be constant.

The following parameters are used in the numerical results:

$$\begin{aligned} \beta &= 300, \\ L^* &= c = 1, \\ q_C &= 0.01, \\ C_F &= 0.01 \text{ and} \\ C_M &= 0.005. \end{aligned}$$

The number of elements were varied, in order to illustrate convergence. We show $n = 8, n = 16$ and $n = 64$ in the table below.

Table 5.1: Deflection w for MFEM and SFEM

	n=8		n=16		n=64	
x	MFEM	SFEM	MFEM	SFEM	MFEM	SFEM
0	-0.01138	-0.01801	-0.01163	-0.01348	-0.01170	-0.01182
0.25	-0.01852	-0.01974	-0.01875	-0.01911	-0.01882	-0.01884
0.5	-0.01674	-0.01276	-0.01691	-0.01576	-0.01696	-0.01688
0.75	0.02345	0.02641	0.02345	0.02436	0.02346	0.02352
1	0.1465	0.1278	0.1468	0.1411	0.1469	0.1465

From Table 5.1 we conclude that both the mixed and standard finite element methods converge. The rate of convergence for MFEM is faster than the rate of SFEM. It is important to note that there is a sign change in the interval $(0.5, 0.75)$. This point is referred to as the centre of rotation (CoR). This terminology is also used in the literature and is discussed more in Subsection 5.4.

5.3.2 Static: Single beam model

Next, the equilibrium problem for the single beam model is considered and compared to the results obtained in Subsection 5.3.1. To obtain numerical results the wind load q_C is kept constant, and the system of equations as discussed in Subsection 5.1.3 (VMFEM) is implemented.

For this comparison, our investigation is restricted to the results obtained when the VMFEM is implemented for the single beam model and MFEM is used for the static embedded double beam model. The choice of parameters is similar to those discussed in Subsection 5.3.1.

Table 5.2: Approximations of w for the static embedded beam.

	$n = 64$	
x	Single beam	Double beam
0	-0.01170	-0.01170
0.25	-0.01883	-0.01882
0.5	-0.01701	-0.01696
0.75	0.02333	0.02346
1	0.1467	0.1469

It is clear from Table 5.2 that the deflection of the static embedded beam for these two models is not remarkably different. This was expected and confirms the reliability of the results.

For the rest of this subsection, we consider results obtained from the VMFEM on the single beam model.

Thus far, the examination of the behaviour of the embedded beam has been limited to a fixed length of the exposed beam ($L^* = 1$). To this end, we investigate the behaviour of the static embedded beam when the length of the exposed beam is varied.

Table 5.3: Approximate values of w for the embedded beam with varying lengths of the exposed beam.

x	$L^* = 1$	$L^* = 1.5$	$L^* = 2$
0	-0.01170	-0.02079	-0.03205
0.25	-0.01883	-0.03856	-0.06517
0.5	-0.01701	-0.03927	-0.07070
0.75	0.02333	0.03667	0.05111
1	0.1467	0.2850	0.4664

From Table 5.3 it can be seen that the deflection of the embedded beam increases as the length of the exposed part of the beam increases. This agrees with what is expected.

5.3.3 Dynamic single beam model

Finally, in this subsection we examine the motion and the influence of various parameters on the single beam model. The wind load, q_C, is chosen to be a periodic function.

The central difference average acceleration finite difference scheme is applied. To ensure that the number of elements are the only factor influencing the accuracy of the solution, the number of time steps are chosen to be "large enough"so that it does not interfere with the interpretation of the results.

For this experiment, we consider the following initial conditions:

$$\bar{\phi}_0 = \bar{w}_0 = \bar{0}, \quad \text{where} \quad \bar{\phi}_1 = \bar{\phi}_{-1} \text{ and } \bar{w}_1 = \bar{w}_{-1}.$$

The default parameters that are used for the numerical results presented are chosen to be:

$$\begin{aligned} \alpha =& 75, \\ \gamma =& 0.25, \\ c =& 1, \\ L^* =& 1, \\ q_C =& 0.01x \sin\left(\frac{\pi}{4}t\right) \quad \text{and} \\ T_f =& 6. \end{aligned}$$

T_f denotes the final time where the beam is considered in dimensionless units .

As in Subsection 5.3.1, we first determine the number of elements required so that implementation of the MFEM yields reliable results.

Table 5.4: Deflection for w for different number of elements.

x	$n = 16$	$n = 64$	$n = 256$
0	-0.001124	-0.001124	-0.001124
0.5	-0.001300	-0.001244	-0.001244
1	0.009651	0.009818	0.009827
1.5	0.04212	0.04221	0.04221
2	0.08568	0.08568	0.08445

The difference between 64 and 256 elements are not substantial. As such, a default value of $n = 64$ will be used to obtain the numerical results in this subsection.

It is important to note that even though graphs are useful to visualize the motion of the beam, the values on the y-axis are very small. Therefore, the scale on the two axes are not the same so that the figures are an exaggeration of the behaviour of the beam.

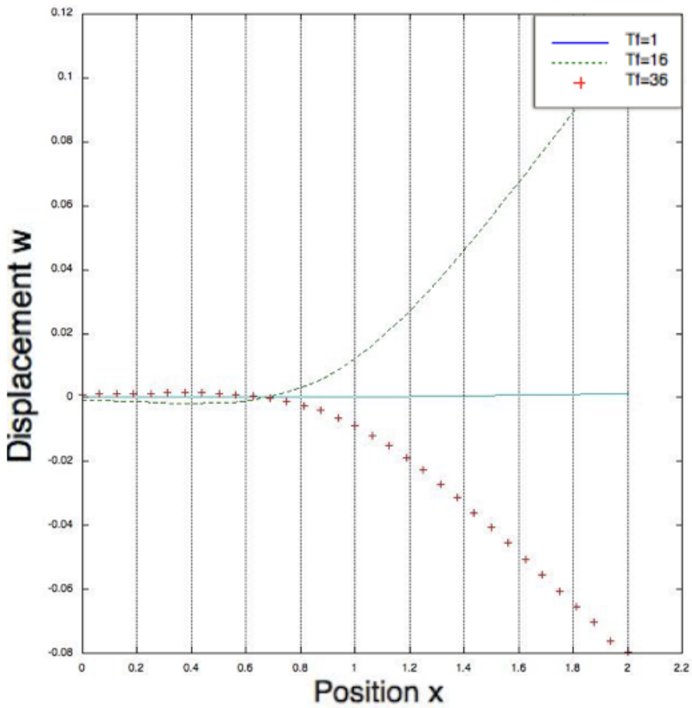

Figure 5.1: Deflection of the beam at different final times.

For a final time of one dimensionless unit, the exposed beam is displaced less than 0.02 dimensionless units. The maximum deflection of the exposed beam is reached after 16 dimensionless time units. This is seen in Figure 5.1 when $T_f = 16$. The exposed beam reaches its maximum deflection in the opposite direction at $T_f = 36$. Note that CoR is located in the same position for the different final times.

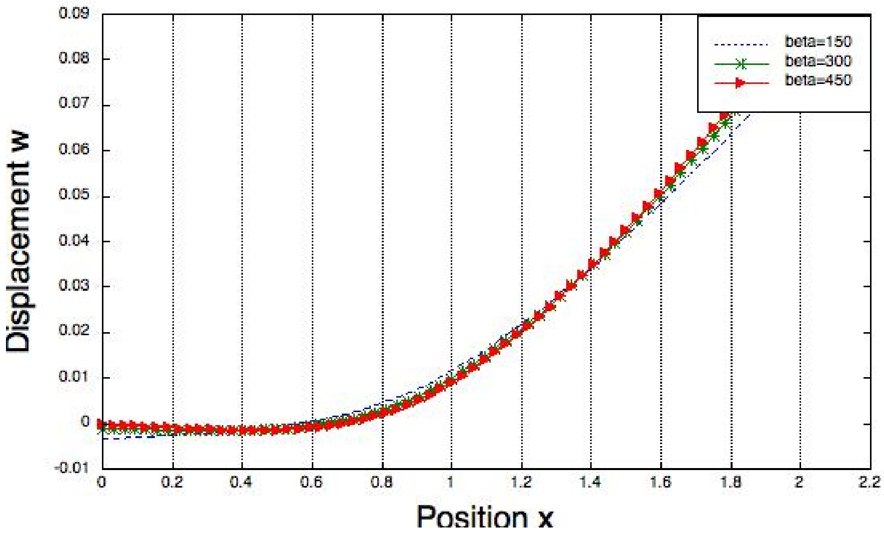

Figure 5.2: Deflection of the beam for varying β values.

Next the shear deformation parameter β is varied. In Figure 5.2. it is observed that for larger values of β the deflection of the beam does not differ significantly. Note that the CoR increases with β however.

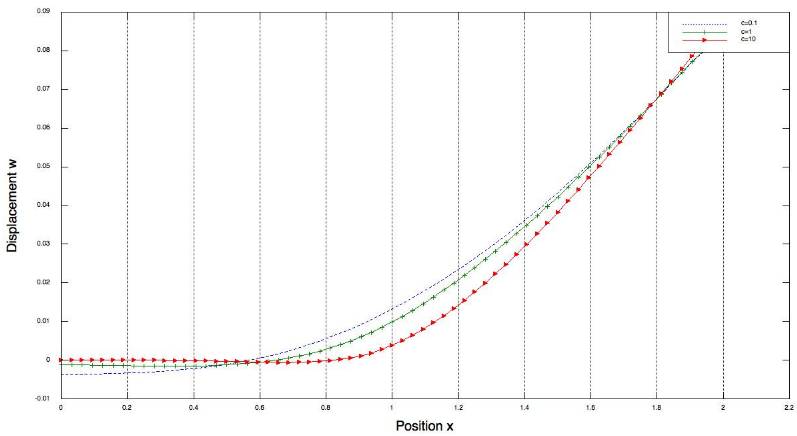

Figure 5.3: Deflection of a single beam for varying c values.

Finally, we investigate the influence of the soil resistance. An increase in the soil resistance results in a shift in the location of the CoR. This may suggest that the resistance of the beam increases when the soil resistance increases.

5.4 Discussion of results

In this section we consider the findings in the articles [SBFBC96], [CE98] and [Enn00] and compare it to what was observed in our numerical experiments.

In their article, the authors of [SBFBC96] state that the critical components in preventing uprooting of an embedded plant are the length and depth of the root system in the soil. According to the experimental findings in [CE98], it is suggested that the size of a plant with a tap root system is the determining factor in a plant's ability to suppress uprooting. We agree with the results in [CE98] and that they support the work by [SBFBC96].

In 2000, Ennos supplemented the findings in [SBFBC96] and [CE98] by describing the significant elements responsible for the resistance to overturning of a simple tap root system. The main identified elements were the soil resistance and the resistance of the tap root itself. In addition, it is stated in [Enn00] that the movement of a plant is relative to a point beneath the stem. This point is referred to as the centre of movement or centre of rotation (CoR).

The results in Table 5.3 suggest that the solution of our model problem support the findings as reported above. It can be observed, from the deflection of the embedded beam, that an increase in the length of the stem results in the resistance of the root increasing to prevent uprooting. In all the simulations the CoR was observed in the embedded beam, which agrees with the point beneath the stem as described by [CE98] and [Enn00].

For the default values, the CoR lies in the dimensionless interval $(0.5, 0.75)$ of the tap root (embedded beam). However, when the soil resistance, c, is increased, the resistance of the tap root results in a shift of the location of the CoR. This is also observed when the dimension of the beam (β) is varied. The graphs in Figures 5.2 and 5.3 illustrate this.

We conclude that the single Timoshenko beam model yields a reliable quantitative description of the experimental findings in [SBFBC96], [CE98] and [Enn00].

Although the results obtained from the model are reliable, there are restrictions to its applications resulting from the assumptions imposed on the model. Future work may include a modification of the single beam model so that the morphology of a plant is taken into account. Also, an examination of the effect of gravity on the dynamics of a plant with branches, leaves and tap root system should be considered.

Chapter 6

Rigid bodies attached to beams

There are a number of articles that investigate the behaviour of a beam with a tip body. For instance, the authors of [AS02] considered a model of a damped beam with a tip body attached at an endpoint. Their proposed model used the Euler-Bernoulli theory. Later, the authors of [BDV14] considered the work by [AS02] and used a variational approach to analyse the motion of a beam with a tip body. The variational approach enabled the authors to prove the existence and uniqueness of a solution for the proposed model.

Other studies include the work by [ZVV04] and Chapter 6 in the PhD thesis [DuT21]. The authors of [ZVV04] investigated the effects of boundary damping on a cantilevered Timoshenko beam with a rigid body attached to it's free end. As such, artificially imposed devices are of significance in dampening oscillations. This process is known as stabilisation. A study that specifically examines the process is by [RA15]. In [DuT21], the focus is on a realistic modelling of the interface between the cantilever Timoshenko beam and the attached body.

According to the authors of [FE17], over the past three decades there were a number of studies that examined the analytical analysis of Euler-Bernoulli beams with intermediate rigid bodies. Also, the analytical and experimental analysis of Timoshenko beams with intermediate rigid bodies had not been covered then.

In their work, [FE17] proposed a mathematical model that consists of two elastic beam segments with an intermediate extended eccentric rigid mass. Timoshenko theory of elasticity was used to obtain the eigenfrequencies and their associated mode shapes. The validity of their model was investigated using analytical and experimental results.

6.1 Dynamics of a beam with a tip body

6.1.1 Dynamics

In this section we discuss the dynamics of a rigid body attached to an endpoint of a beam.

Consider a righthand triad of basis vectors denoted by **(i, j, k)**. The basis vectors are constant for a non-accelerating and non-rotating frame of reference. Assume **i** is in the direction of the axis of the undeformed beam, and that **j** is the direction that the beam executes small transverse vibrations. The deflection of the axis of the beam is denoted by w.

For convenience let the rigid body be attached at the right end of a beam, and the position of the center of mass of the tip body be denoted by $\mathbf{x}_C$ then,

$$\mathbf{x}_C = w(\ell, t)\mathbf{j} + d\cos\theta(t)\mathbf{i} + d\sin\theta(t)\mathbf{j}.$$

The expression for the velocity of the center of mass is

$$\mathbf{v}_C = \partial_t w(\ell, t)\mathbf{j} - d\dot{\theta}(t)\sin\theta(t)\mathbf{i} + d\dot{\theta}(t)\cos\theta(t)\mathbf{j}.$$

It follows that the acceleration is given by

$$\begin{aligned}\mathbf{a}_C = &- d\ddot{\theta}(t)\sin\theta(t)\mathbf{i} - d\dot{\theta}^2(t)\cos\theta(t)\mathbf{i} + \partial_t^2 w(\ell, t)\mathbf{j} \\ &+ d\ddot{\theta}(t)\cos\theta(t)\mathbf{j} - d\dot{\theta}(t)^2\sin\theta(t)\mathbf{j}.\end{aligned}$$

For the linear approximation, it is assumed that θ is small, $\cos\theta \approx 1$ and that $\sin\theta \approx \theta$. Using these assumptions on the acceleration yields

$$\mathbf{a}_C \approx -d\ddot{\theta}(t)\theta(t)\mathbf{i} - d\dot{\theta}^2(t)\mathbf{i} + \partial_t^2 w(\ell, t)\mathbf{j} + d\ddot{\theta}(t)\mathbf{j} - d\dot{\theta}(t)^2\theta(t)\mathbf{j}.$$

Consequently, the transverse component of the velocity is

$$v_2 = \partial_t w(\ell, t) + d\dot{\theta}(t), \tag{6.1.1}$$

while for the acceleration we have

$$a_2 = \partial_t^2 w(\ell, t) + d\ddot{\theta}(t) - d\dot{\theta}(t)^2\theta(t).$$

Using Newton's second law of motion, for the center of mass, results in

$$m\partial_t^2 w(\ell,t) + md\ddot{\theta}(t) - md\dot{\theta}(t)^2\theta(t) = F_B(t), \qquad (6.1.2)$$

where m is the mass of the tip body and $F_B(t)$ is the force acting on the tip body at the interface.

Taking moments about the center of mass yields the equation of motion for rotation

$$J\ddot{\theta}(t) = M_B(t) - dF_B(t), \qquad (6.1.3)$$

where J is the moment of inertia about the center of mass and $M_B(t)$ is the couple on the rigid body.

It is useful to rewrite the equations for the dynamics of a rigid body in dimensionless form.

For θ, we use the scaling $\theta^*(\tau) = \theta(t)$. Importantly, $\dfrac{d\theta^*}{d\tau} = \dfrac{1}{\zeta}\dfrac{d\theta}{dt}$.

From Equations (6.1.2) and (6.1.3) we have

$$\frac{m\ell}{(\zeta^2 AG\kappa^2)}\partial_\tau^2 w^* + \frac{md}{(\zeta^2 AG\kappa^2)}\frac{d^2\theta^*}{d\tau^2} - md\left(\frac{d\theta^*}{d\tau}\right)^2\theta^* = -F^*,$$
$$\frac{J}{(\zeta^2 AG\kappa^2\ell)}\frac{d^2\theta^*}{d\tau^2} + M^* = \frac{d^*\ell AG\kappa^2}{AG\kappa^2\ell}F^*.$$

Recall the dimensionless quantity $\zeta = \sqrt{\dfrac{\rho\ell^2}{G\kappa^2}}$ (defined in Subsection 1.3.2).

Substituting ζ in the result above yields

$$\frac{m}{\rho A\ell}\partial_\tau^2 w^* + \frac{md}{\rho A\ell^2}\frac{d^2\theta^*}{d\tau^2} - \frac{md}{\rho A\ell^2}\left(\frac{d\theta^*}{d\tau}\right)^2\theta^* = -F^*,$$
$$\frac{J}{\rho A\ell^3}\frac{d^2\theta^*}{d\tau^2} + M^* - d^*F^* = 0.$$

Let

$$m_B = \frac{m}{\rho A\ell}, \quad d^* = \frac{d}{\ell}, \quad J_B = \frac{J}{\rho A\ell^3}, \quad F_B^* = \frac{F_B}{AG\kappa^2} \quad \text{and} \quad M_B^* = \frac{M_B}{AG\kappa^2\ell}.$$

Returning to the original notation, the acceleration of the tip body in dimensionless form is

$$a_2 = \partial_t^2 w(1,t) + d\ddot{\theta}(t) - d\dot{\theta}^2\theta.$$

Neglecting the nonlinear term we have

$$a_2 = \partial_t^2 w(1,t) + d\ddot{\theta}(t). \tag{6.1.4}$$

(We assume that θ and $\dot{\theta}$ are small).

The dimensionless form of Equations (6.1.2) and (6.1.3) is

$$m\partial_t^2 w(1,t) + md\ddot{\theta}(t) = F_B(t), \tag{6.1.5}$$

$$J\ddot{\theta}(t) = M_B(t) - dF_B(t). \tag{6.1.6}$$

6.1.2 Models in previous publications

In this section we familiarise ourselves with previous work on Euler-Bernoulli and Timoshenko beam models with an attached tip body at an end of a beam. Recall that a Timoshenko beam model accounts for shear deformation of a cross-section of a beam, while the Euler-Bernoulli beam model does not.

The authors in [AS02] proposed a model that accounted for the centre of mass of the body not being at an end of the beam. As previously mentioned, the authors of [BDV14] considered the model proposed by [AS02]. They used the variational approach. This was done so that they could apply the theory in [VV02] to prove the existence and uniqueness of a solution. In addition, the authors briefly discussed the finite element approximation of a solution.

In their studies [AS02] and [BDV14] considered Kelvin-Voight damping, and proposed the following non-dimensionless interface conditions:

$$m\partial_t^2 w(\ell,t) + md\partial_t^2\partial_x w(\ell,t) + k_0\partial_t w(\ell,t) = F_B(t),$$

$$J\partial_t^2\partial_x w(\ell,t) + dk_1\partial_t\partial_x w(\ell,t) = M_B(t) - dF_B(t),$$

where k_0 and k_1 are damping parameters. Due to function conventions,

$$F_B(t) = -F(\ell,t) \quad \text{and} \quad M_B(t) = -M(\ell,t).$$

In 2014 Basson et al. noted that [AS02] neglected the term $d\partial_t\partial_x w(\ell,t)$ in their expression for the transverse component of the velocity of the tip body. Even more, in [BDV14] a justification is given on why the term should not be neglected.

The general opinion is that the Timoshenko theory is closer to reality. In [ZVV04] the authors considered the effect of boundary damping on the dynamics of a cantilevered Timoshenko beam with a rigid body attached at the free end. Their aim was to investigate the behaviour of the model by establishing the efficiency and accuracy of the finite element method for calculating eigenvalues and eigenmodes.

The authors [ZVV04] chose the centre of mass of the body to be at the end of a beam. Their proposed dimensionless interface conditions are:

$$
\begin{aligned}
m\partial_t^2 w(1,t) &= -F(1,t) - k_0\partial_t w(1,t),\\
J\partial_t^2 \phi(1,t) &= -M(1,t) - k_1\partial_t w(1,t).
\end{aligned}
$$

The uniform exponential stability of a hybrid Timoshenko beam model with a tip body was examined in the study by [RA15]. The aim of their study was to show that a hybrid Timoshenko beam model with a tip body is not uniformly exponentially stable, rather it is polynomially stable.

Our interpretation of the main result in [RA15] is that it supports the empirical results obtained in the study by [ZVV04]. The proofs in [RA15] were achieved through the use of semigroup theory. The proposed interface conditions are similar to those stated by [ZVV04] however, the authors of [RA15] presented them in their non-dimensionless form.

Remark. *There are two possibilities of modelling a beam with a tip body. The first option is when* $\theta(t) \approx \partial_x w(\ell,t)$ *as in [AS02] and [BDV14], the other is when* $\theta(t) = \phi(\ell,t)$ *as in [ZVV04] and [RA15].*

6.2 The intermediate body

The authors of [KT02] claim that two-part beam systems are of practical importance.

According to [FE17], [KT02] considered the transverse vibration of a system consisting of a rigid body carried by two uniform beams of different flexural rigidity, length and pinned at the beams ends. The center of mass of the body was located at the mid-point of the axial width.

The aim in this section is to model an intermediate rigid body between two Timoshenko beams, see Figure 6.1.

6.2.1 The model problem

In this section, we assume the beams are clamped at the extreme ends. We refer to the beam on the left of the body as Beam 1 and the other as Beam 2.

The line L_B connects the endpoints of the deflection curves, and the centre of mass of the body $\mathbf{x}_C$. The angle θ is between the line L_B and the direction of $\mathbf{i}$. In the undeformed state of the beams, the line L_B lies on the neutral axes of the beams and the center of mass of the body also lies on the line.

The distance from the endpoint of Beam 1 to the center of mass of the body is denoted by d_1, similarly the distance from the end of Beam 2 to the center of mass of the body is denoted by d_2.

We assume that the beams can only execute small transverse vibrations. We refer the reader to Section 6.1 for our discussion on a Timoshenko beam with a tip body attached at the right end.

The dimensionless form is the same as in Section 6.1. The lengths are scaled by ℓ, the length of Beam 1. The dimensionless lengths are $\ell_1 = 1$ and ℓ_2.

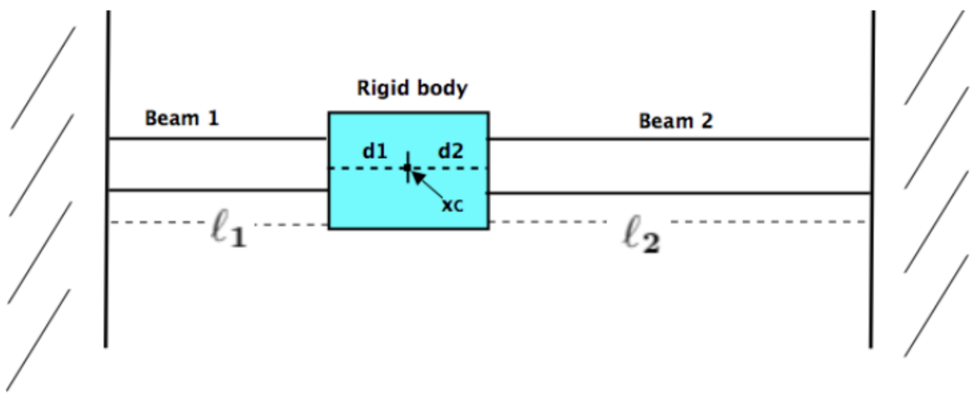

Figure 6.1: Intermediate body between beams (undeformed)

As in Section 6.1, $\theta(t)$ denotes the angle of rotation of the body. Recall that

$$\mathbf{x}_C = w_1(\ell_1, t)\mathbf{j} + d_1 \cos\theta(t)\mathbf{i} + d_1 \sin\theta(t)\mathbf{j},$$

with d replaced by d_1. Similarly,

$$\mathbf{x}_C = w_2(0,t)\mathbf{j} - d_2 \cos\theta(t)\mathbf{i} - d_2 \sin\theta(t)\mathbf{j}.$$

Let $u(t)$ be the transverse displacement of the center of mass of the body in the direction $\bar{e}_2$. Since we consider small vibrations, we assume that

$$u(t) = w_1(\ell_1,t) + d_1\theta(t) = w_2(0,t) - d_2\theta(t).$$

Consequently,

$$u(t) = \frac{d_2 w_1(\ell_1,t) + d_1 w_2(0,t)}{d_1 + d_2}.$$

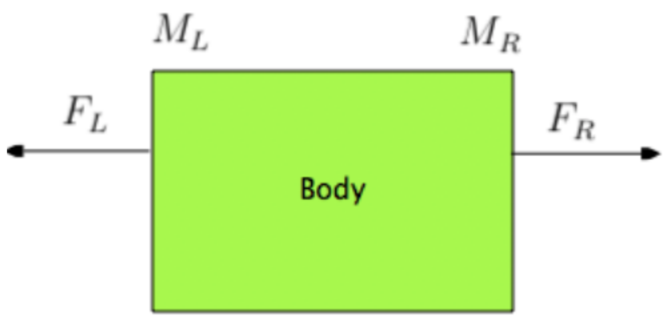

Figure 6.2: Direction of force and moment on the body

Once more, we start with the dynamics of a rigid body. The equations of motion for the body are:

$$m\ddot{u}(t) = F_L(t) + F_R(t), \tag{6.2.1}$$

$$J\ddot{\theta}(t) = M_R(t) + M_L(t) + d_2F_R(t) - d_1F_L(t). \tag{6.2.2}$$

Beam 1 is clamped at its left end and Beam 2 clamped at its right end, it follows that the boundary conditions are

$$w_1(0,t) = \phi_1(0,t) = 0 \text{ and } w_2(\ell_2,t) = \phi_2(\ell_2,t) = 0. \tag{6.2.3}$$

Due to the convention for the functions F_i and M_i,

$$F_L(t) = -F_1(1,t) \text{ and } F_R(t) = F_2(0,t),$$

and

$$M_L(t) = -M_1(1,t) \text{ and } M_R(t) = M_2(0,t).$$

These relations can now be substituted into Equations (6.2.1) and (6.2.2) to formulate interface conditions.

A result of the configuration is that the angle of rotation of the body is equal to the rotation of the cross-section at the ends of each of the beams. In summary, the dimensionless constraints for a model of an intermediate body between two Timoshenko beams are as follows.

Constraints

$$\begin{aligned} \phi_1(1,t) &= \phi_2(0,t) = \theta(t) \\ \ddot{u}(t) &= \partial_t^2 w_1(1,t) + d_1\ddot{\theta}(t) \\ \ddot{u}(t) &= \partial_t^2 w_2(0,t) - d_2\ddot{\theta}(t) \end{aligned} \tag{6.2.4}$$

Problem IB (Intermediate Body)

Given positive constants α_i and β_i for $i = 1, 2$, find w_i and ϕ_i such that

$$\partial_t^2 w_i = \partial_x F_i, \tag{6.2.5}$$

$$\frac{1}{\alpha_i}\partial_t^2 \phi_i = \partial_x M_i + F_i, \tag{6.2.6}$$

$$F_i = \partial_x w_i - \phi_i, \tag{6.2.7}$$

$$M_i = \frac{1}{\beta_i}\partial_x \phi_i, \tag{6.2.8}$$

with interface conditions

$$m\ddot{u}(t) = F_2(0,t) - F_1(\ell_1,t), \tag{6.2.9}$$

$$J\ddot{\theta}(t) = M_2(0,t) - M_1(\ell_1,t) + d_2F_2(0,t) + d_1F_1(\ell_1,t), \tag{6.2.10}$$

and boundary conditions (6.2.3) as well as constraints (6.2.4), while

$$w_i(\cdot,0) = w_i(0), \phi_i(\cdot,0) = \phi_i(0),$$
$$\partial_t w_i(\cdot,0) = w_d^i(0) \text{ and } \partial_t \phi_i(\cdot,0) = \phi_d^i(0).$$

Problem IB is similar to the problem in Chapter 5 of [DuT21].

Remark. *This model ignores the loads on the beams and assumes that the force of gravity experienced by the body is negligible.*

6.2.2 Variational form

To write Problem IB in variational form, multiply Equation (6.2.5) by $v_i \in C^1[0,\ell_i]$ and Equation (6.2.6) by $\psi_i \in C^1[0,\ell_i]$. Using integration by parts on the terms on the right side of Equations (6.2.5) and (6.2.6) yields

$$\int_0^{\ell_i} \partial_t^2 w_i(\cdot,t)v_i = F_i(\ell_i,t)v_i(\ell_i) - F_i(0,t)v_i(0) - \int_0^{\ell_i} F_i(\cdot,t)v_i', \tag{6.2.11}$$

$$\int_0^{\ell_i} \frac{1}{\alpha_i}\partial_t^2 \phi_i(\cdot,t)\psi_i = M_i(\ell_i,t)\psi_i(\ell_i) - M_i(0,t)\psi_i(0) - \int_0^{\ell_i} M_i(\cdot,t)\psi_i' + \int_0^{\ell_i} F_i(\cdot,t)\psi_i, \tag{6.2.12}$$

for $i = 1, 2$.

The space of test functions for Beam 1 is defined by

$$\mathcal{T}_B[0,\ell_1] = \{v \in C^1[0,\ell_1] \mid v(0) = 0\},$$

and the test function space for Beam 2 is defined by

$$\mathcal{T}_C[0,\ell_2] = \{v \in C^1[0,\ell_2] \mid v(\ell_2) = 0\}.$$

To derive the variational form, we assume that v_1 and ψ_1 are in $\mathcal{T}_B[0,\ell_1]$ and v_2 and ψ_2 are in $\mathcal{T}_C[0,\ell_2]$. Adding Equation (6.2.11) for beams 1 and 2 and using the boundary conditions, we have

$$\begin{aligned}&\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t)v_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t)v_2 \\ &= F_1(\ell_1,t)v_1(\ell_1) - F_2(0,t)v_2(0) - \int_0^{\ell_1} F_1(\cdot,t)v_1' - \int_0^{\ell_2} F_2(\cdot,t)v_2'.\end{aligned} \tag{6.2.13}$$

Due to the rotation of the body, a constraint on the test functions is

$$\theta(t) = \psi_1(\ell_1) = \psi_2(0). \tag{6.2.14}$$

For some real number r we also have the following constraints

$$v_1(\ell_1) = r - d_1\psi_1(\ell_1) \text{ and } v_2(0) = r + d_2\psi_2(0). \tag{6.2.15}$$

Now consider the terms containing $F_1(\ell_1,t)$ and $F_2(0,t)$ in Equation (6.2.13). From Equations (6.2.9), (6.2.14) and (6.2.15) we have

$$\begin{aligned}&F(\ell_1,t)v_1(\ell_1) - F_2(0,t)v_2(0) \\ &= F_1(\ell_1,t)(r - d_1\psi_1(\ell_1)) - F_2(0,t)(r + d_2\psi_2(0)) \\ &= (F_1(\ell_1,t) - F_2(0,t))r - F_1(\ell_1,t)d_1\psi_1(\ell_1) - F_2(0,t)d_2\psi_2(0) \\ &= (F_1(\ell_1,t) - F_2(0,t))r - (F_1(\ell_1,t)d_1 + F_2(0,t)d_2)\psi_1(\ell_1) \\ &= -m\ddot{u}(t)r - (F_1(\ell_1,t)d_1 + F_2(0,t)d_2)\psi_1(\ell_1).\end{aligned} \tag{6.2.16}$$

Substituting Equation (6.2.16) into Equation (6.2.13) results in

$$\begin{aligned}\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t)v_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t)v_2 \\ = - m\ddot{u}(t)r - (d_1F_1(\ell_1,t) + d_2F_2(0,t))\psi_1(\ell_1) \\ - \int_0^{\ell_1} F_1(\cdot,t)v_1' - \int_0^{\ell_2} F_2(\cdot,t)v_2'. \end{aligned} \tag{6.2.17}$$

Following the same procedure, adding Equation (6.2.12) for beams 1 and 2 and using the boundary conditions, we have

$$\begin{aligned}&\int_0^{\ell_1} \frac{1}{\alpha_1}\partial_t^2 \phi_1(\cdot,t)\psi_1 + \int_0^{\ell_2} \frac{1}{\alpha_2}\partial_t^2 \phi_2(\cdot,t)\psi_2 \\ &= M_1(\ell_1,t)\psi_1(\ell_1) - M_2(0,t)\psi_2(0) - \int_0^{\ell_1} M_1(\cdot,t)\psi_1' \\ &\quad - \int_0^{\ell_2} M_2(\cdot,t)\psi_2' + \int_0^{\ell_1} F_1(\cdot,t)\psi_1 + \int_0^{\ell_2} F_2(\cdot,t)\psi_2. \end{aligned} \tag{6.2.18}$$

Substituting Equation (6.2.10) into Equation (6.2.18) and using the fact that $\psi_1(\ell_1) = \psi_2(0)$, yields

$$\begin{aligned}&\int_0^{\ell_1} \frac{1}{\alpha_1}\partial_t^2 \phi_1(\cdot,t)\psi_1 + \int_0^{\ell_2} \frac{1}{\alpha_2}\partial_t^2 \phi_2(\cdot,t)\psi_2 \\ &= (d_1F_1(\ell_1,t) + d_2F_2(0,t) - J\ddot{\theta}(t))\psi_1(\ell_1) - \int_0^{\ell_1} M_1(\cdot,t)\psi_1' \\ &\quad - \int_0^{\ell_2} M_2(\cdot,t)\psi_2' + \int_0^{\ell_1} F_1(\cdot,t)\psi_1 + \int_0^{\ell_2} F_2(\cdot,t)\psi_2. \end{aligned} \tag{6.2.19}$$

Finally, adding Equations (6.2.17) and (6.2.19) we have

$$\begin{aligned}&\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t)v_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t)v_2 + \int_0^{\ell_1} \frac{1}{\alpha_1}\partial_t^2 \phi_1(\cdot,t)\psi_1 \\ &+ \int_0^{\ell_2} \frac{1}{\alpha_2}\partial_t^2 \phi_2(\cdot,t)\psi_2 + m\ddot{u}(t)r + J\ddot{\theta}(t)\psi_1(\ell_1) \\ &= - \int_0^{\ell_1} F_1(\cdot,t)v_1' - \int_0^{\ell_2} F_2(\cdot,t)v_2' + \int_0^{\ell_1} F_1(\cdot,t)\psi_1 \\ &+ \int_0^{\ell_2} F_2(\cdot,t)\psi_2 - \int_0^{\ell_1} M_1(\cdot,t)\psi_1' - \int_0^{\ell_2} M_2(\cdot,t)\psi_2'. \end{aligned} \tag{6.2.20}$$

Consider the linear space

$$Z = \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_C[0,\ell_2] \times \mathcal{T}_B[0,\ell_1] \times \mathcal{T}_C[0,\ell_2] \times \mathbb{R} \times \mathbb{R}.$$

The test function space $\mathcal{T}_p$ is a subspace of Z with the condition: $z = \langle z_1, z_2, z_3, z_4, z_5, z_6 \rangle \in \mathcal{T}_p$ if

$$z_5 = \frac{d_2 z_1(\ell_1) + d_1 z_2(0)}{d_1 + d_2} \tag{6.2.21}$$

and

$$z_6 = z_3(\ell_1) = z_4(0). \tag{6.2.22}$$

It is now possible to present the variational form of Problem IB.

Problem IB-V

Given positive constants α_i and β_i for $i = 1, 2$, find $\langle w_1, w_2, \phi_1, \phi_2, u, \theta \rangle$ where

$$\langle w_1(\cdot,t), w_2(\cdot,t), \phi_1(\cdot,t), \phi_2(\cdot,t), u(t), \theta(t) \rangle \in \mathcal{T}_p \text{ for all } t > 0,$$

such that

$$\begin{aligned}
&\int_0^{\ell_1} \partial_t^2 w_1(\cdot,t) z_1 + \int_0^{\ell_2} \partial_t^2 w_2(\cdot,t) z_2 + \int_0^{\ell_1} \frac{1}{\alpha_1} \partial_t^2 \phi_1(\cdot,t) z_3 \\
&\quad + \int_0^{\ell_2} \frac{1}{\alpha_2} \partial_t^2 \phi_2(\cdot,t) z_4 + m\ddot{u}(t) z_5 + J\ddot{\theta}(t) z_6 \\
&\quad = -\int_0^{\ell_1} (\partial_x w_1(\cdot,t) - \phi_1(\cdot,t))(z_1' - z_3) - \int_0^{\ell_1} \frac{1}{\beta_1} \partial_x \phi_1(\cdot,t) z_3' \\
&\quad\quad - \int_0^{\ell_2} (\partial_x w_2(\cdot,t) - \phi_2(\cdot,t))(z_2' - z_4) - \int_0^{\ell_2} \frac{1}{\beta_2} \partial_x \phi_2(\cdot,t) z_4',
\end{aligned} \tag{6.2.23}$$

for each $z \in \mathcal{T}_p$ while

$$\begin{aligned}
w_i(\cdot,0) &= w_i(0), \phi_i(\cdot,0) = \phi_i(0), \\
\partial_t w_i(\cdot,0) &= w_d^i(0), \partial_t \phi_i(\cdot,0) = \phi_d^i(0),
\end{aligned}$$

for $i = 1, 2$.

We define the following bilinear forms in terms of the $\mathcal{L}^2$ inner products:

For f and g in $\mathcal{T}_p$

$$c(f,g) =(f_1,g_1) + (f_2,g_2) + \left(\frac{1}{\alpha_1}f_3,g_3\right) + \left(\frac{1}{\alpha_2}f_4,g_4\right) + mf_5g_5 + Jf_6g_6, \tag{6.2.24}$$

$$b(f,g) =(f_1' - f_3, g_1' - g_3) + (f_2' - f_4, g_2' - g_4) + \left(\frac{1}{\beta_1}f_3', g_3'\right) + \left(\frac{1}{\beta_2}f_4', g_4'\right). \tag{6.2.25}$$

6.3 Weak variational form

To write Problem IB in weak variational form, it is convenient to construct the spaces V and W. To start, introduce the product space

$$X = \mathcal{L}^2(0,\ell_1) \times \mathcal{L}^2(0,\ell_2) \times \mathcal{L}^2(0,\ell_1) \times \mathcal{L}^2(0,\ell_2) \times \mathbb{R} \times \mathbb{R},$$

with inner product

$$(y_1,z_1) + (y_2,z_2) + (y_3,z_3) + (y_4,z_4) + y_5z_5 + y_6z_6.$$

The space X is clearly complete.

6.3.1 Function spaces

In this subsection we investigate the properties of the bilinear forms and functions spaces to verify if Assumptions A1, A2, A3 and A4W are satisfied.

Proposition 6.3.1. *The bilinear form c is an inner product for the space X.*

Proof. Since the bilinear form c is symmetric, it is an inner product if $c(u,u) = 0$ implies $u = 0$. Let $u \in X$, then

$$\begin{aligned} c(u,u) =&(u_1,u_1) + (u_2,u_2) + \left(\frac{1}{\alpha_1}u_3,u_3\right) + \left(\frac{1}{\alpha_2}u_4,u_4\right) \\ &+ mu_5^2 + Ju_6^2 \\ \geq& M_C\left(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \|u_4\|^2 + u_5^2 + u_6^2\right) \\ \geq& M_C\|u\|_X^2, \end{aligned} \tag{6.3.1}$$

where $M_C = \min\left\{1, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, m, J\right\}$. □

Definition 6.3.1 (Inertia space W)**.** *The vector space X equipped with the inner product c is referred to as the Inertia space W. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$.*

Proposition 6.3.2. *The norms $\|\cdot\|_W$ and $\|\cdot\|_X$ are equivalent on W.*

Proof. First for $u \in W$,

$$\begin{aligned}\|u\|_W^2 =&(u_1,u_1) + (u_2,u_2) + \left(\frac{1}{\alpha_1}u_3, u_3\right) + \left(\frac{1}{\alpha_2}u_4, u_4\right) + mu_5^2 \\ &+ Ju_6^2 \\ \le& M_W\left(\|u_1\|^2 + \|u_2\|^2 + \|u_3\|^2 + \|u_4\|^2 + u_5^2 + u_6^2\right) \\ \le& M_W\|u\|_X^2,\end{aligned}$$

where $M_W = \max\left\{1, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, m, J\right\}$. Using Inequality (6.3.1), the result follows. □

Corollary 6.3.1. *The space W is complete.*

Proof. Since X is the product of complete spaces it follows from the remark to Proposition A.1.3 that X is complete. In addition, the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent it follows from Proposition A.1.2 that the space W is complete. □

Problem IB-W

Find u such that for each $t > 0$, $u(t) \in V, u''(t) \in W$ and

$$c(u''(\cdot,t),v) + b(u(\cdot,t),v) = 0 \text{ for each } v \in V,$$
$$\text{while } u(0) = u_0 \text{ and } u'(0) = u_d.$$

6.3.2 The energy space V

First, define the product space

$$H^1 = H^1(0,\ell_1) \times H^1(0,\ell_2) \times H^1(0,\ell_1) \times H^1(0,\ell_2) \times \mathbb{R} \times \mathbb{R},$$

with inner product

$$(x, y)_{H^1} = (x_1, y_1)_1 + (x_2, y_2)_1 + (x_3, y_3)_1 + (x_4, y_4)_1 + x_5 y_5 + x_6 y_6.$$

Define V to be the closure of $\mathcal{T}_p$ in H^1.

Proposition 6.3.3. *There exists a positive constant M_K such that*

$$\|u\|_{H^1} \leq M_K(\|u_1\|_1^2 + \|u_2\|_1^2 + \|u_3\|_1^2 + \|u_4\|_1^2) \text{ for each } u \in V.$$

Proof. Let $u \in V$, then using the definition of the H^1 norm

$$\|u\|_{H^1}^2 = \|u_1\|_1^2 + \|u_2\|_1^2 + \|u_3\|_1^2 + \|u_4\|_1^2 + u_5^2 + u_6^2. \tag{6.3.2}$$

Consider $\|u_1\|_1^2$. Applying Lemma A.2.4 yields $\|u_1\|^2 \leq \ell_1^2 \|u_1'\|^2$ and hence

$$\|u_1\|_1^2 = \|u_1\|^2 + \|u_1'\|^2 \leq (1 + \ell_1^2)\|u_1'\|^2.$$

Similar estimates hold for u_2, u_3 and u_4. Consequently, using Equation (6.3.2) we have

$$\begin{aligned} \|u\|_{H^1}^2 \leq & (1 + \ell_1^2)\|u_1'\|^2 + (1 + \ell_2^2)\|u_2'\|^2 + (1 + \ell_1^2)\|u_3'\|^2 + (1 + \ell_2^2)\|u_4'\|^2 \\ & + u_5^2 + u_6^2 \\ \leq & M_M \left(\|u_1'\|^2 + \|u_2'\|^2 + \|u_3'\|^2 + \|u_4'\|^2 + u_5^2 + u_6^2\right) \end{aligned} \tag{6.3.3}$$

where $M_M = \max\{1, 1 + \ell_1^2, 1 + \ell_2^2\}$.

To estimate u_5^2 and u_6^2 we use Lemma A.3.1. First, since $u_6 = u_3(\ell_1)$

$$|u_6|^2 \leq K_{\ell_1}^2 \|u_3\|_1^2,$$

where $K_{\ell_1} = \sqrt{2} \max\{\sqrt{\ell_1}, (\sqrt{\ell_1})^{-1}\}$.

To estimate u_5^2, we use Equation (6.2.21) and this implies that we must estimate $u_1(\ell_1)$ and $u_2(0)$. Now,

$$|u_1(\ell_1)|^2 \leq K_{\ell_1}^2 \|u_1\|_1^2 \text{ and } |u_2(0)|^2 \leq K_{\ell_2}^2 \|u_2\|_1^2,$$

where $K_{\ell_2} = \sqrt{2} \max\{\sqrt{\ell_2}, (\sqrt{\ell_2})^{-1}\}$.

From Equation (6.2.21) we have

$$|u_5|^2 \leq M_I^2(\|u_1\|_1^2 + \|u_2\|_1^2),$$

where $M_I = (d_1 + d_2)^{-1} \max\{d_2 K_{\ell_1}, d_1 K_{\ell_2}\}$.

It follows from Equation (6.3.3) that

$$\begin{aligned}\|u\|_{H^1} \leq & M_M(\|u_1'\|^2 + \|u_2'\|^2 + \|u_3'\|^2 + K_{\ell_1}^2\|u_3\|^2 + M_I^2(\|u_1\|_1^2 + \|u_2\|_1^2)) \\ \leq & M_K(\|u_1\|_1^2 + \|u_2\|_1^2 + \|u_3\|_1^2 + \|u_4\|_1^2)\end{aligned}$$

where $M_K = M_M \max\{K_{\ell_1}^2, M_I^2\}$. □

Corollary 6.3.2.

$$[u, v]_1 = (u_1, v_1)_1 + (u_2, v_2)_1 + (u_3, v_3)_1 + (u_4, v_4)_1$$

is an inner product for H^1 and the induced norm $||| \cdot |||_1$ is equivalent to $\| \cdot \|_{H^1}$.

Proof. The definition above implies $[u, v]_1$ is an inner product. Clearly, $|||u|||_1 \leq \|u\|_{H^1}$. It follows from Proposition 6.3.3 that
$M_K\|u\|_{H^1} \leq |||u|||_1$. □

Proposition 6.3.4. *There exists a positive constant M_B such that*

$$b(u, u) \geq M_B\|u\|_{H^1}^2 \text{ for each } u \in V.$$

Proof. Applying the Triangle inequality and Lemma A.2.4 on $\|u_1'\|$ we have

$$\begin{aligned}\|u_1'\| &= \|u_1' - u_3 + u_3\| \\ &\leq \|u_1' - u_3\| + \|u_3\| \\ &\leq \|u_1' - u_3\| + \ell_1\|u_3'\|. \end{aligned} \tag{6.3.4}$$

Note that

$$\|u_1'\|^2 \leq \|u_1' - u_3\|^2 + 2(\|u_1' - u_3\|\ell_1\|u_3'\|) + \ell_1^2\|u_3'\|^2.$$

Using Lemma A.2.6 when $\epsilon = 1$ on the term $2(\|u_1' - u_3\|\ell_1\|u_3'\|)$ yields

$$2(\|u_1' - u_3\|\ell_1\|u_3'\|) \leq \|u_1' - u_3\|^2 + \ell_1^2\|u_3'\|^2$$

It follows that

$$\|u_1'\|^2 \leq 2(\|u_1' - u_3\|^2 + \ell_1^2\|u_3'\|^2).$$

Clearly,

$$\begin{aligned}\|u_1'\|^2 + \|u_3'\|^2 \leq & 2(\|u_1' - u_3\|^2 + \ell_1^2\|u_3'\|^2) + \|u_3'\|^2 \\ \leq & M_{\ell_1}\left(\|u_1' - u_3\|^2 + \|u_3'\|^2\right),\end{aligned} \tag{6.3.5}$$

where $M_{\ell_1} = \max\{2, 1 + 2\ell_1^2\}$. Similarly,

$$\|u_2'\|^2 + \|u_4'\|^2 \leq M_{\ell_2}\left(\|u_2' - u_4\|^2 + \|u_4'\|^2\right), \qquad (6.3.6)$$

where $M_{\ell_2} = \max\{2, 1 + 2\ell_2^2\}$.

As in the proof of Proposition 6.3.3 we have

$$|||u|||_1^2 \leq M_K(\|u_1\|_1^2 + \|u_2\|_1^2 + \|u_3\|_1^2 + \|u_4\|_1^2).$$

It follows that

$$\begin{aligned} |||u|||_1^2 \leq & M_K M_{\ell_1}(\|u_1' - u_3\|^2 + \|u_3'\|^2) \\ & + M_K M_{\ell_2}(\|u_2' - u_4\|^2 + \|u_4'\|^2) \\ \leq & M_J(\|u_1' - u_3\|^2 + \|u_2' - u_4\|^2 + \|u_3'\|^2 + \|u_4'\|^2), \end{aligned} \qquad (6.3.7)$$

where $M_J = M_K \max\{M_{\ell_1}, M_{\ell_2}\}$.

From the definition of the bilinear form b we have

$$\begin{aligned} b(u,u) = & \|u_1' - u_3\|^2 + \|u_2' - u_4\|^2 + \frac{1}{\beta_1}\|u_3'\|^2 + \frac{1}{\beta_2}\|u_4'\|^2 \\ \geq & M_A\left(\|u_1' - u_3\|^2 + \|u_3'\|^2 + \|u_2' - u_4\|^2 + \|u_4'\|^2\right), \end{aligned} \qquad (6.3.8)$$

where $M_A = \min\left\{1, \frac{1}{\beta_1}, \frac{1}{\beta_2}\right\}$.

From Equations (6.3.7) and (6.3.8) we have

$$\begin{aligned} \frac{M_J}{M_A} b(u,u) \geq & M_J\left(\|u_1' - u_3\|^2 + \|u_3'\|^2 + \|u_2' - u_4\|^2 + \|u_4'\|^2\right) \\ \geq & M_J |||u|||_1^2. \end{aligned}$$

Finally, using Corollary 6.3.2

$$b(u,u) \geq M_B \|u\|_{H^1}^2,$$

where $M_B = \dfrac{M_A M_K}{M_J}$. □

Corollary 6.3.3. *The bilinear form b is an inner product for the space V.*

Definition 6.3.2 (Energy space V)**.** *The vector space V equipped with the inner product b is referred to as the Energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.*

Proposition 6.3.5. *There exists a positive constant M_E such that*

$$b(u,u) \geq M_E \|u\|_W^2 \text{ for each } u \in V.$$

Proof. From the proof of Proposition 6.3.2 we have $\|u\|_W^2 \leq M_W \|u\|_X^2$ and from the definition of the norms $\|u\|_X^2 \leq \|u\|_{H^1}^2$. Finally, from Proposition 6.3.4 we have $\|u\|_{H^1}^2 \leq M_B^{-1} b(u,u)$. Combining these inequalities we obtain the desired result where $M_E = M_B(M_W)^{-1}$. □

Proposition 6.3.6. *The norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent on V.*

Proof. The equivalence of norms is obtained using the results in Proposition 6.3.3 and 6.3.4.

□

Corollary 6.3.4. *The space V is complete.*

Proof. From Proposition A.1.3 we have that the product space H^1 is complete. Since V is a subset of H^1, V is complete with respect to the norm $\|\cdot\|_1$. It follows from the equivalence of the $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ norms (Proposition A.1.2) that the product space V is complete. □

The following table summarises the spaces and notation:

Space	Inner product
$\mathcal{L}^2(0,\ell_i)$	$(\cdot,\cdot)$
X	$(x,y)_X = (x_1,y_1) + (x_2,y_2) + (x_3,y_3) + (x_4,y_4) + x_5y_5 + x_6y_6$
W	$c(x,y) = (x_1,y_1) + (x_2,y_2) + \left(\frac{1}{\alpha_1}x_3,y_3\right) + \left(\frac{1}{\alpha_2}x_4,y_4\right) + mx_5y_5 + Jx_6y_6$
H^1	$(x,y)_{H^1} = (x_1,y_1)_1 + (x_2,y_2)_1 + (x_3,y_3)_1 + (x_4,y_4)_1 + x_5y_5 + x_6y_6$
V	$b(x,y) = (x_1' - x_3, y_1' - y_3) + (x_2' - x_4, y_2' - y_4) + \left(\frac{1}{\beta_1}x_3', y_3'\right) + \left(\frac{1}{\beta_1}x_4', y_4'\right)$

Existence

Proposition 6.3.7. *V is dense in W and W is dense in X.*

Proof. Recall that V is the closure of test functions $\mathcal{T}_p$ in H^1.

By Corollary A.1.1 we have that $C^1[0, \ell_i]$ is dense in $\mathcal{L}^2(0, \ell_i)$. It follows that for any $u \in X$ we can find $u^* \in V$ with
$u^* = \langle u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^* \rangle \in C^1[0, \ell_1] \times C^1[0, \ell_2] \times C^1[0, \ell_1] \times C^1[0, \ell_2] \times \mathbb{R} \times \mathbb{R}$ such that for any $\varepsilon > 0$

$$\|u_i - u_i^*\|^2 < \frac{\varepsilon}{6} \text{ for } i = 1, 2, 3, 4, 5, 6.$$

It follows that

$$\|u - u^*\|_X^2 < \varepsilon.$$

Therefore, V is dense in X. The result follows from Proposition 6.3.2. □

Theorem 6.3.1. *Suppose $q \in C^1([0, t_*), X)$. If $u_0 \in E_b$ and $u_d \in V$, then there exists a unique weak solution u such that*

$$u \in C^1([0, t_*), V) \cap C^2([0, t_*), W),$$

for Problem IB-W.

Proof. As a result of Propositions 6.3.7, 6.3.2 and 6.3.5, Assumptions A1, A2 and A3 are respectively satisfied. Since $a = 0$, from the remark in Section 2.2, Assumption A4W is satisfied. As a consequence of Theorem 2.2.2 a unique weak solution exists for Problem IB-W. □

6.4 Galerkin approximation

To obtain the Galerkin approximation of Problem IB, we need to define the finite dimensional subspace S^h. Recall that we have the intervals $I_1 = [0, 1]$ and $I_2 = [0, \ell_2]$. These two intervals are divided into n and m subintervals respectively.

Consider piecewise linear basis functions and denote them by $\delta_{i,j}$, where $i = 1$ or 2 denotes the interval and j counts from 0 to the number of subintervals.

Define the following finite dimensional subspaces

$$S_1^h = span\{\delta_{1,1}, \cdots, \delta_{1,n}\}$$
$$S_2^h = span\{\delta_{2,0}, \delta_{2,1}, \cdots, \delta_{2,m-1}\}.$$

The Galerkin approximation of Problem IB is:

Find w_i^h and ϕ_i^h in S_i^h, and real values u and θ such that

$$\int_0^1 \partial_t^2 w_1^h(\cdot,t)z_1 + \int_0^{\ell_1} \partial_t^2 w_2^h(\cdot,t)z_2 + \int_0^1 \frac{1}{\alpha_1}\partial_t^2\phi_1^h(\cdot,t)z_3 + \int_0^{\ell_2} \frac{1}{\alpha_2}\partial_t^2\phi_2^h(\cdot,t)z_4$$
$$= -m\ddot{u}z_5 - J\ddot{\theta}z_6 - \int_0^1 (\partial_x w_1^h(\cdot,t) - \phi_1^h(\cdot,t))(z_1' - z_3) - \int_0^1 \frac{1}{\beta_1}\partial_x\phi_1^h(\cdot,t)z_3'$$
$$- \int_0^{\ell_2} (\partial_x w_2^h(\cdot,t) - \phi_2^h(\cdot,t))(z_2' - z_4) - \int_0^{\ell_2} \frac{1}{\beta_2}\partial_x\phi_2^h(\cdot,t)z_4'$$

for z_1, z_3 in S_1^h; z_2, z_4 in S_2^h and the constraints

$$z_5 = \frac{d_2 z_1(1) + d_1 z_2(0)}{d_1 + d_2} \quad \text{and} \quad z_6 = z_3(1) = z_4(0) \quad \text{hold.}$$

6.4.1 System of ordinary differential equations

In order to obtain the system of ordinary differential equations, we consider the equations for w_1^h, w_2^h, ϕ_1^h and ϕ_2^h separately. First, let $z_1 \neq 0$ and $z_i = 0$ for $i = 2, 3, 4$ in the Galerkin approximation. This results in

$$\int_0^1 \partial_t^2 w_1^h(\cdot,t)z_1 = -m\ddot{u}z_5 - \int_0^1 (\partial_x w_1^h(\cdot,t) - \phi_1^h(\cdot,t))z_1'. \tag{6.4.1}$$

Similarly, we obtain the other three equations

$$\int_0^{\ell_1} \partial_t^2 w_2^h(\cdot,t)z_2 = -\, m\ddot{u}z_5 - \int_0^{\ell_2} (\partial_x w_2^h(\cdot,t) - \phi_2^h(\cdot,t))z_2' \tag{6.4.2}$$

$$\int_0^1 \frac{1}{\alpha_1}\partial_t^2\phi_1^h(\cdot,t)z_3 = -\, J\ddot{\theta}z_6 + \int_0^1 (\partial_x w_1^h(\cdot,t) - \phi_1^h(\cdot,t))(z_3) - \int_0^1 \frac{1}{\beta_1}\partial_x\phi_1^h(\cdot,t)z_3' \tag{6.4.3}$$

$$\int_0^{\ell_2} \frac{1}{\alpha_2}\partial_t^2\phi_2^h(\cdot,t)z_4 = -\, J\ddot{\theta}z_6 + \int_0^{\ell_2} (\partial_x w_2^h(\cdot,t) - \phi_2^h(\cdot,t))z_4) - \int_0^{\ell_2} \frac{1}{\beta_2}\partial_x\phi_2^h(\cdot,t)z_4'. \tag{6.4.4}$$

Since $w_i^h(t)$ and $\phi_i^h(t)$ are in S_i^h, we can write it as

$$w_1^h(x,t) = \sum_{j=1}^{n} w_j^{(1)}(t)\delta_{1,j}(x), \quad \phi_1^h(x,t) = \sum_{j=1}^{n} \phi_j^{(1)}(t)\delta_{1,j}(x),$$
$$w_2^h(x,t) = \sum_{j=0}^{m-1} w_j^{(2)}(t)\delta_{2,j}(x) \quad \text{and} \quad \phi_2^h(x,t) = \sum_{j=0}^{m-1} \phi_j^{(2)}(t)\delta_{2,j}(x).$$

The n-tuple $(w_1^{(1)}, w_2^{(1)}, \cdots, w_n^{(1)})$ is denoted by $\bar{w}_1$ and corresponds to w_1^h. Similar notation is used for the other variables.

As mentioned, we use piecewise linear basis functions. Let $x_j^{[i]}$ be grid points for the finite element mesh, where i denotes which interval (I_1 or I_2) the grid point forms part of. The length of elements on an interval are chosen to be equal and is denoted by $h^{[i]}$.

First, for illustration, we show how Equation (6.4.1) is rewritten, when the interaction with the body is not considered. Substituting w_i^h and ϕ_i^h with the partial sums, we have

$$\sum_{j=1}^{n} \ddot{w}_j^{(1)} \int_{x_0^{[1]}}^{x_n^{[1]}} \delta_{1,j}\delta_{1,i} = -\sum_{j=1}^{n} w_j^{(1)} \int_{x_0^{[1]}}^{x_n^{[1]}} \delta'_{1,j}\delta'_{1,i} - \sum_{j=1}^{n} \phi_j^{(1)} \int_{x_0^{[1]}}^{x_n^{[1]}} \delta_{1,j}\delta'_{1,i} \tag{6.4.5}$$

for $i = 1, \ldots, n$.

Recall the standard finite element matrices K, N and L defined in Subsection 5.1.2. We use superscripts as above to indicate which interval (and therefore which length of elements) is used. Using the matrices, Equation (6.4.5) takes the form

$$N^{[1]}\ddot{\bar{w}}_1 = -K^{[1]}\bar{w}_1 - L^{[1]}\bar{\phi}_1. \tag{6.4.6}$$

Following the same procedure, Equations (6.4.2), (6.4.3) and (6.4.4), respectively become

$$N^{[2]}\ddot{\bar{w}}_2 = -K^{[2]}\bar{w}_2 - L^{[2]}\bar{\phi}_2, \tag{6.4.7}$$
$$\frac{1}{\alpha_1}N^{[1]}\ddot{\bar{\phi}}_1 = (L^{[1]})^T\bar{w}_1 - N^{[1]}\bar{\phi}_1 - \frac{1}{\beta_1}K^{[1]}\ddot{\bar{\phi}}_1 \quad \text{and} \tag{6.4.8}$$
$$\frac{1}{\alpha_2}N^{[2]}\ddot{\bar{\phi}}_2 = (L^{[2]})^T\bar{w}_2 - N^{[2]}\bar{\phi}_2 - \frac{1}{\beta_2}K^{[2]}\ddot{\bar{\phi}}_2. \tag{6.4.9}$$

Finally, the terms with z_5 and z_6 need to be considered. Applying the constraints, it follows from straight forward (but elaborate) calculations that Equation (6.4.6) can be written as

$$\tilde{N}_n^{[1]}\ddot{\bar{w}}_1 + \frac{md_1d_2}{d_1+d_2}\tilde{N}_{nn}^{[1]}\ddot{\bar{\phi}}_1 = -K^{[1]}\bar{w}_1 - L^{[1]}\bar{\phi}_1, \tag{6.4.10}$$

where $\tilde{N}_n^{[1]}$ is the same as the matrix $N^{[1]}$, but the entry $N_{nn}^{[1]}$ is changed by adding $\dfrac{md_2}{d_1+d_2}$ and $\tilde{N}_{nn}^{[1]}$ is the N matrix where everything is zero except for the entry in row n and column n. Applying similar arguments to Equations (6.4.7) and (6.4.8), the rest of the system can be written as

$$\tilde{N}_0^{[2]}\ddot{\bar{w}}_2 + \frac{md_1d_2}{d_1+d_2}\tilde{N}_{00}^{[2]}\ddot{\bar{\phi}}_2 = -K^{[2]}\bar{w}_2 - L^{[2]}\bar{\phi}_2, \tag{6.4.11}$$

$$\frac{1}{\alpha_1}N^{[1]}\ddot{\bar{\phi}}_1 + J\tilde{N}_{nn}^{[1]}\ddot{\bar{\phi}}_1 = (L^{[1]})^T\bar{w}_1 - N^{[1]}\bar{\phi}_1 - \frac{1}{\beta_1}K^{[1]}\ddot{\bar{\phi}}_1 \quad \text{and} \tag{6.4.12}$$

$$\frac{1}{\alpha_2}N^{[2]}\ddot{\bar{\phi}}_2 = (L^{[2]})^T\bar{w}_2 - N^{[2]}\bar{\phi}_2 - \frac{1}{\beta_2}K^{[2]}\ddot{\bar{\phi}}_2. \tag{6.4.13}$$

We substitute into Equation (6.4.12). Equation (6.4.13) does not change where $\tilde{N}_0^{[2]}$ is the same as the matrix $N^{[2]}$, but the entry $N_{00}^{[2]}$ is changed by adding $\dfrac{md_1}{d_1+d_2}$ and $\tilde{N}_{00}^{[2]}$ is the N matrix where everything is zero except for the entry in row 0 and column 0.

Convergence and error estimates

The error estimates for this problem are obtained using similar methods and arguments as in Chapter 4 (application of Theorems in Chapter 3). Therefore we will not repeat everything here, but only explain how the interpolation operator is defined.

Define an interpolation operator Π_I on the product space H^1 by

$$\Pi_I u = \langle \Pi_\ell^{[1]}u_1, \Pi_\ell^{[2]}u_2, \Pi_\ell^{[1]}u_3, \Pi_\ell^{[2]}u_4\rangle \quad \text{for each } u \in H^1,$$

where $\Pi_\ell^{[i]}$ is the usual interpolation operator for piecewise linear basis functions on the interval I_i.

The interpolation and convergence estimates can now be obtained following the same pattern as in Chapter 4. Recall that $u \in C^2((0,t_*),V)$ and $u(t)$ and $u''(t) \in H^2$ are required to apply the general results.

Chapter 7

Hyperbolic heat conduction problem

In this chapter, we consider a multi-dimensional heat transfer model proposed by the authors of [DD19], motivated by the model in [DWJMB08]. (The reader may wish to take another look at Section 1.6.) In [DD19], the authors propose a finite element method for a duplex layered hyperbolic heat equation problem defined on two-dimensional domains. They investigate the well-posedness of the problem, which is presented in its abstract form. They also found [VV02] (the theory in Chapter 2) useful to establish solvability. However, the connection between the article [VV02] and the theory in [DD19] is not clear. (For more detail see Section 7.3.) One of the achievements in this dissertation is to rectify this.

Specifically, we are interested in the verification of the assumptions necessary for the existence of a weak solution, and the estimates used in convergence theory. This investigation will be referred to as the variational approach to existence.

7.1 Model problem

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with a Lipschitz boundary $\partial\Omega$, such that Ω consists of two subdomains i.e., $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where Γ is the boundary of Ω_1. It is important to note that $\partial\Omega_1 \subset \bar{\Omega}_2$.

Suppose the domain Ω consists of open subdomains Ω_1 and Ω_2 with interface Γ.

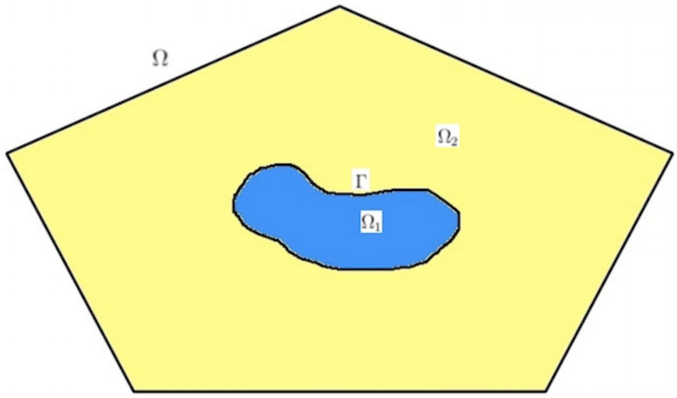

Figure 7.1: Domain

Remark. *It is not immediately clear why the domain should be convex polygonal.*

Consider the hyperbolic heat equation

$$\partial_t^2 T_i = \operatorname{div}(\beta_i \nabla T_i) - \partial_t T_i + Q_i \text{ in } \Omega_i \times (0, t_*] \text{ where } t_* < \infty, \quad (7.1.1)$$

for $i = 1$ and 2, with $T_i(\cdot, t)$ the restriction of $T(\cdot, t)$ to Ω_i.

In Equation (7.1.1), T_i and β_i denote a quantity T and non-negative functions respectively, while Q_i is the source function. The source and non-negative functions are defined in Ω.

For simplicity, we proceed as in [DD19] and define β_i to be a piecewise constant and discontinuous function on Ω. Note that β was previously used in the beam problems, however in this chapter it has different parametric properties.

Remark. *Comparing Equations* (7.1.1) *and* (1.6.2) *(the modified and dimensionless multi-layered SPL model), we see that it is drastically simplified. Consequently, Equation* (7.1.1) *is a special case of a dimensionless multi-layered SPL model. It is important to note that the authors in [DD19] do not give a justification for this choice of parameters. Note that Equation* (7.1.1) *can also be compared to the multi-dimensional model in Section 1.5 of this dissertation.*

Boundary and interface conditions

We consider the boundary condition

$$T(x,t) = 0, \text{ along } \partial\Omega \times [0, t_*) \text{ where } t_* < \infty,$$

As a consequence

$$T_2(x,t) = 0 \text{ along } \partial\Omega. \tag{7.1.2}$$

Note that the boundary of Ω_1 is $\Gamma = \partial\Omega_1 \subset \partial\Omega_2$. The interface conditions along Γ are

$$T_1(x,t) = T_2(x,t), \text{ and} \tag{7.1.3}$$

$$\beta_1 \nabla T_1(x,t) \cdot n_1 = \beta_2 \nabla T_2(x,t) \cdot n_2. \tag{7.1.4}$$

The interface conditions are used to describe the jump of a quantity T across Γ, where $T_1 = T_2$. The unit outward normal to Ω_1 at its boundary is n_1, while n_2 is the unit outward normal to Ω_2. This implies that $n_1 = -n_2$ at the interface Γ.

The model of interest is formulated as

Problem MC (Maxwell-Cattaneo)

Given a non-negative function β, find T for each $t > 0$, such that

$$\partial_t^2 T = \text{div}(\beta \nabla T) - \partial_t T + Q, \text{ in } \Omega \times (0, t_*] \text{ where } t_* < \infty, \tag{7.1.5}$$

$$T(x,t) = 0 \text{ along } \partial\Omega \times [0, t_*), \tag{7.1.6}$$

$$T(x,0) = T_0, \quad \partial_t T(x,0) = T_d, \tag{7.1.7}$$

and interface conditions (7.1.3) and (7.1.4).

7.2 Variational approach to existence

The variational approach is used to determine the solvability of Problem MC. We will examine whether the theory in Chapter 2 can be applied to Problem MC by verifying that Assumptions A1, A2, A3 and A4W are satisfied. Problem MC is similar to Problem HHE in Section 2.1. The spatial domain is now two-dimensional instead of an interval.

7.2.1 Variational form

Unlike in Subsection 1.5.5, the multi-dimensional hyperbolic heat problem is considered on a two-dimensional domain instead of a three-dimensional domain. Even more, in this subsection we consider the variational form of an interfacial multi-dimensional hyperbolic heat problem. Note that $\partial\Omega_2 = \partial\Omega \cup \Gamma$ and $\partial\Omega_1 = \Gamma$.

To obtain the variational form of Problem MC, multiply Equation (7.1.5) by a function $v \in C(\bar{\Omega})$ and integrate. If the restriction of v to Ω_i is contained in $C^1(\Omega_i)$, then Green's formula (see Appendix A.2) may be used on the term on the right side of the equality sign of Equation (7.1.5) to obtain (for $i = 1, 2$)

$$\iint_{\Omega_i} \operatorname{div}(\beta_i \nabla T_i(\cdot,t)) v_i dA = -\iint_{\Omega_i} \beta_i \nabla T_i(\cdot,t) \cdot \nabla v_i dA + \int_{\partial\Omega_i} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds, \qquad (7.2.1)$$

provided that $\nabla T_i(\cdot,t)$ and $\operatorname{div}(\beta_i \nabla T_i(\cdot,t))$ are integrable. Since $T_i(\cdot,t)$ is a solution of Equation (7.1.5), it follows from Equation (7.2.1) that

$$\iint_{\Omega_i} \partial_t^2 T_i(\cdot,t) v_i dA = -\iint_{\Omega_i} \beta_i \nabla T_i(\cdot,t) \cdot \nabla v_i dA - \iint_{\Omega_i} \partial_t T_i(\cdot,t) v_i dA + \int_{\partial\Omega_i} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds + \iint_{\Omega_i} Q_i(\cdot,t) v_i dA, \qquad (7.2.2)$$

for $i = 1$ and 2. To proceed, recall that $\partial\Omega_2 = \partial\Omega \cup \Gamma$ and $\partial\Omega_1 = \Gamma$, hence

$$\int_{\partial\Omega_2} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds = \int_{\partial\Omega} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds + \int_{\Gamma} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds$$

Since $n_1 = -n_2$ on Γ, the boundary integral in Equation (7.2.2) reduces to

$$\int_{\partial\Omega} \beta_i (\nabla T_i(\cdot,t) \cdot n_i) v_i ds.$$

Adding Equation (7.2.2) on each subdomain yields

$$\begin{aligned}\sum_{i=1}^{2}\iint_{\Omega_i}\partial_t^2 T_i(\cdot,t)v_i dA \\ = -\sum_{i=1}^{2}\iint_{\Omega_i}\beta_i\nabla T_i(\cdot,t)\cdot\nabla v_i dA - \sum_{i=1}^{2}\iint_{\Omega_i}\partial_t T_i(\cdot,t)v_i dA \\ + \sum_{i=1}^{2}\iint_{\Omega_i} Q_i(\cdot,t)v_i dA + \int_{\partial\Omega}\beta_2(\nabla T_2(\cdot,t)\cdot n_2)v_2 ds. \qquad (7.2.3)\end{aligned}$$

To define the test functions, introduce $C_p^1(\bar{\Omega})$:

$$v \in C_p^1(\bar{\Omega}) \text{ if } v\,\big|_{\bar{\Omega}_i} \in C^1(\bar{\Omega}_i).$$

The test function space is then

$$\mathcal{T}(\Omega) = \{v \in C_p^1(\bar{\Omega}) \mid v = 0 \text{ along } \partial\Omega\}.$$

Note that $C_p^1(\bar{\Omega}) \subset H^1(\Omega)$.

The $\mathcal{L}^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$ and the bilinear form $b(u,v)$ is defined as

$$b(u,v) = \iint_{\Omega_1}\beta_1\nabla u(\cdot,t)\cdot\nabla v + \iint_{\Omega_2}\beta_2\nabla u(\cdot,t)\cdot\nabla v. \qquad (7.2.4)$$

Problem MC-V

Given a function $Q \in C^1(\bar{\Omega})$ and a non-negative function $\beta \in C(\bar{\Omega})$, find T where $T(\cdot,t) \in \mathcal{T}(\Omega)$ for each $t > 0$, such that for each $v \in \mathcal{T}(\Omega)$

$$\begin{aligned}\left(\partial_t^2 T(\cdot,t),v\right) + (\partial_t T(\cdot,t),v) + b\,(T(\cdot,t),v) = (Q(\cdot,t),v), \qquad (7.2.5)\\ \text{while } T(\cdot,0) = T_0, \quad \partial_t T(\cdot,0) = T_d.\end{aligned}$$

Remark. *The bilinear forms c and a are both equal to the inner product for $\mathcal{L}^2(\Omega)$.*

Remark. *The conditions (7.1.3) and (7.1.4) along $\partial\Omega$ and Γ are defined in the sense of trace (see Appendix A.2)*

Consider the following spaces. First, in this special case

$$W = X = \mathcal{L}^2(\Omega).$$

Next

$$V(\Omega) = \text{Closure of } \mathcal{T}(\Omega) \text{ in } H^1(\Omega).$$

Remark. *Note that in this case the space* $V(\Omega) = H_0^1(\Omega)$, *see Appendix A.2.*

Define $\tilde{q}$ by $\tilde{q}(t) = \langle Q_1(x,t), 0\rangle$.

Problem MC-W

Find $u \in C^2([0,t_*);W)$ such that for each $t > 0, u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{q}(t), v)_X \quad \text{for each } v \in V, \tag{7.2.6}$$

$$\text{while} \quad u(0) = u_0,\ u'(0) = u_d.$$

The formulation of a weak variational problem differs from [DD19]. (Their formulation is not compatible with [VV02].) We emphasise that a solution of Problem MC-W is not necessarily a solution of Problem MC-V.

7.2.2 Properties of function spaces

For the present problem, the vector space X is equipped with the inner product c. The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$ which equals the norm for $\mathcal{L}^2(\Omega)$. Clearly, the space W is complete. To prove Proposition 7.2.2 below, we need the following auxiliary result.

Notation
For any $u \in W$ $\|u\|_{\Omega_i}^2 = (u,u)_{\Omega_i} = \int_{\Omega_i} u_i^2$.

Proposition 7.2.1. *There exists a constant* C_Ω *such that*

$$\|u\|_{\Omega_i} \leq C_\Omega \|\nabla u\|_{\Omega_i} \tag{7.2.7}$$

for each i *and each* $u \in \mathcal{T}(\Omega)$.

Proof. We use Friedrichs' inequality, see Lemma A.2.7 in Appendix A.2. □

The next result is necessary for the existence theory. It is not mentioned in [DD19] although the authors claim to use the theory in [VV02].

Proposition 7.2.2. *There exists a positive constant* D_B *such that*

$$b(u,u) \geq D_B \|u\|_W^2 \quad \textit{for each } u \in V. \tag{7.2.8}$$

Proof. Note that $b(u,u) \geq \beta_{\min}\|\nabla u\|^2$ since

$$b(u,u) = \beta_1\|\nabla u\|^2_{\Omega_1} \; + \; \beta_2\|\nabla u\|^2_{\Omega_2} \geq \beta_{\min}\left(\|u\|^2_{\Omega_1} + \|u\|^2_{\Omega_2}\right).$$

It follows that

$$b(u,u) \geq \beta_{\min} C_\Omega^{-1}\|u\|^2_W$$

using Inequality (7.2.7). This proves Inequality (7.2.8) for each $u \in \mathcal{T}(\Omega)$. The result follows (as before) from the fact that $V(\Omega)$ is the closure of $\mathcal{T}(\Omega)$ in $H^1(\Omega)$. □

Corollary 7.2.1. *The bilinear form b is an inner product for the space $V(\Omega)$.*

Definition 7.2.1 (Energy Space V)**.** *The vector space V equipped with the inner product b is referred to as the Energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.*

Theorem 7.2.1. *The norms $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent on V.*

Proof. Clearly, $|b(u,v)| \leq \beta_{\max}\|u\|_1\|v\|_1$, where $\beta_{\max}$ is the maximum of β_i.

It follows from Propositions 7.2.1 and 7.2.2 that

$$b(u,u) \geq \beta_{\min}\|\nabla u\|^2 + D_B\|u\|^2_W \geq D_M\|u\|^2_1,$$

where $D_M = \min\{\beta_{\min}, D_B\}$. □

Corollary 7.2.2. *The space V is complete.*

7.2.3 Existence

In this section we have a system with weak damping. As mentioned at the beginning of the section, it is necessary to investigate whether the necessary assumptions for existence are satisfied.

Proposition 7.2.3. *$V(\Omega)$ is dense in W.*

Proof. We know that $\mathcal{T}(\Omega) \subset V(\Omega) \subset W$. But $C_0^\infty(\bar{\Omega}) \subset \mathcal{T}(\Omega)$ and $C_0^\infty(\bar{\Omega})$ is dense in W (see Corollary A.1.1). It follows that $\mathcal{T}(\Omega)$ is dense in W and $V(\Omega)$ is dense in W. □

It is necessary to investigate the space E_b. Recall that an element of V is in E_b if there exists a $y \in W$ such that $b(u, v) = c(y, v)$ for each $v \in V$. The first result in this regard is Proposition 7.2.4 below.

Theorem 7.2.2. *Suppose* $\tilde{q} \in C^1([0, t_*), X)$. *If* $u_0 \in E_b$ *and* $u_d \in V$, *then there exists a unique weak solution* u *such that*

$$u \in C^1([0, t_*), V) \cap C^2([0, t_*), W),$$

for Problem MC-W.

Proof. As a result of Proposition 7.2.3, the definition of the spaces X and W as well as Proposition 7.2.2, Assumptions A1, A2 and A3 are respectively satisfied. Since a is symmetric, nonnegative and bounded on W, Assumption A4W is satisfied. Using Theorem 2.2.2 a unique weak solution exists for Problem MC-W provided $u_0 \in E_b, u_d \in V$ and $\tilde{q} \in C^1([0, t_*), X)$. □

Remark. *In [DD19] the necessary conditions to apply Theorem 2.2.2, are not considered. What the authors call a weak solution is not a solution of Problem MC-W. Some authors refer to such a solution as a mild solution, see e.g. [Paz83] or [VS19].*

In this dissertation, the conditions for the application of Theorem 2.2.2 are established except for one. It remains to identify the set E_b. (When is $u_0 \in E_b$?)

Definition 7.2.2. $\mathcal{T}_\Gamma(\Omega)$ *is a subset of* $\mathcal{T}(\Omega)$ *such that* $u|_{\Omega_i} \in C^2(\bar{\Omega}_i)$ *and* u *satisfies the interface conditions on* Γ.

Proposition 7.2.4. *If* $u \in \mathcal{T}_\Gamma(\Omega)$, *then* u *is in* E_b.

Proof. If ∇u and $\text{div}(\nabla u)$ are integrable, then

$$\begin{aligned}\iint_{\Omega_i} \text{div}(\beta_i \nabla u) v dA = &-\iint_{\Omega_i} \beta_i \nabla u \cdot \nabla v dA \\ &+ \int_{\partial\Omega_i} \beta_i (\nabla u \cdot n) v ds,\end{aligned}$$

for each $v \in \mathcal{T}(\Omega)$.

We intend to sum the two identities above over the whole region Ω. First, we consider the boundary term. Recall that $\partial\Omega_2 = \partial\Omega \cup \Gamma$ and

$\partial\Omega_1 = \Gamma$. Since $n_1 = -n_2$ on Γ, the boundary integral reduces to $\int_{\partial\Omega} \beta_2(\nabla u_2 \cdot n_2)v_2 ds$. This term vanishes if $v \in \mathcal{T}(\Omega)$. Consequently

$$\sum_{i=1}^{2} \iint_{\Omega_i} \operatorname{div}(\beta_i \nabla u) v dA = -\sum_{i}^{2} \iint_{\Omega_i} \beta_i \nabla u \cdot \nabla v dA = -b(u, v). \tag{7.2.9}$$

Now, let $F \in \mathcal{L}^2(\Omega)$ such that $F|_{\Omega_i} = \operatorname{div}(\beta_i \nabla u_i)$, then

$$b(u, v) = -\iint_{\Omega} F v dA \text{ for each } v \in V(\Omega).$$

This proves that $u \in E_b$. □

Corollary 7.2.3. *If $u_0 \in \mathcal{T}_\Gamma(\Omega)$ and $u_d \in V(\Omega)$ then there exists a unique weak solution u such that*

$$u \in C^1([0, t_*), V) \cap C^2([0, t_*), W).$$

Remark. *From Subsection 2.3.1 the pair $\langle u(t), u'(t)\rangle \in \mathcal{D}(A)$. From Subsection 2.3.2 $\mathcal{D}(A) = E_b \times V$ for weak damping. It follows that $u(t) \in E_b$ for each t and $u'(t) \in V(\Omega)$. The fact that $u(t) \in E_b$ is important as will become clear in the next section.*

7.3 Approach to existence by Dekka and Dutta

The jump of the temperature flux over Γ creates a problem; for an accurate numerical algorithm one needs a sufficiently regular solution. One may consider following the approach in Chapter 4, but our aim is to study the method in [DD19]. For convenience, recall the model problem and its variational forms.

Problem MC

$$\partial_t^2 u = \nabla \cdot (\beta \nabla u) - \partial_t u + f, \text{ in } \Omega \times (0, t_*] \text{ where } t_* < \infty, \tag{7.3.1}$$
$$u(x, t) = 0 \text{ along } \partial\Omega \times (0, t_*], \tag{7.3.2}$$
$$u(x, 0) = u_0, \quad \partial_t u(x, 0) = u_d. \tag{7.3.3}$$

The domains are defined as in Section 7.1.

Interface conditions

The proposed interface conditions were formulated as

$$[u] = 0, \quad \left[\beta(x)\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \ \text{ along } \Gamma \times [0, t_*], \tag{7.3.4}$$

where $[u]$ is the jump of a quantity u across the interface Γ and $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega_1$.

Recall the weak variational form of the problem as derived in Section 7.2

Problem MC-W

Find $u \in C^2([0, t_*); W)$ such that for each $t > 0, u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{q}(t), v)_X \ \text{ for each } v \in V, \tag{7.3.5}$$

$$\text{while } \ u(0) = u_0,\ u'(0) = u_d.$$

The authors of [DD19] refer to Theorem 2.2.2 (in Chapter 2) for sufficient conditions such that Equation (7.3.5) has a unique solution $u \in \mathcal{L}^2(H_0^1(\Omega) \cap H^2(\mathcal{L}^2(\Omega))$.

7.3.1 Weak and strong solutions

In [DD19], the authors formulate sufficient conditions for a weak solution to be a strong solution. To show the existence of a weak solution, the spaces in Section 7.2 are significant.

Recall that

$$b(w, v) = \int_{\Omega_1} \beta_1(x)\nabla w \cdot \nabla v dx + \int_{\Omega_2} \beta_2(x)\nabla w \cdot \nabla v dx \text{ for all } w, v \in V.$$

To show the existence of a weak solution to Problem MC, the authors in [DD19] introduce similar notation as used in this dissertation (see end of Section 2.1 of this dissertation).

For convenience, some of the notation is repeated here:

1. Let $\mathcal{J}$ be a interval containing zero and Z any Hilbert space.
2. $u^{(k)} \in \mathcal{L}^2(\mathcal{J}, Z)$ if $u^{(k)}(t) \in Z$ for each $t \in \mathcal{J}$ and $\int_{\mathcal{J}} \|u^{(k)}\|_Z^2 < \infty$.

Remark. *The notation in [DD19] is confusing. For instance, the notation for a derivative (for example* $\frac{\partial u}{\partial t}$*) is used for classical derivatives, weak derivatives and functionals.*

Definition 7.3.1 (Weak solution)**.** *Let* $u_0 \in V, u_d \in W$ *and* $f \in \mathcal{L}^2(V')$. *A function* $u \in \mathcal{L}^2(V) \cap H^1(W) \cap H^2(W)$ *is called a weak solution of Equations* (7.3.1) *to* (7.3.3) *if* $u(0) = u_0$ *and* $\partial_t u(0) = u_d$ *with jump conditions (Equation* (7.3.4)*), and it satisfies the following weak formulation*

$$(\partial_t^2 u, v) + (\partial_t u, v) + b(u, v) = \langle f, v\rangle_{V' \times V}, \tag{7.3.6}$$

for all $v \in H_0^1(\Omega)$ *and a.e.* $t \in J$.

The authors of [DD19] refer to Theorem 2 in [VV02] (Theorem 2.2.2 in this dissertation) for sufficient conditions such that Equation (7.3.6) has a unique solution $u \in \mathcal{L}^2(H_0^1(\Omega) \cap H^2(\mathcal{L}^2(\Omega))$. However, for such a u higher regularity is necessary at the interface. The statement regarding the existence of a weak solution is not useful and should be replaced by Corollary 7.2.3. Their weak solution is not a solution of Problem MC-W and is referred to as a mild solution by [Paz83] and [VS19], for example. Furthermore, $\partial_t^2 u$ and $\partial_t u$ do not exist in the sense of a weak derivative and should be interpreted as functionals. A detailed discussion can be found in Section 5 of [VS19].

In [DD19], the authors point out that a weak solution (their definition) is not useful for the application of FEM and introduce the idea of a strong solution. To do this, additional spaces are introduced:

$$\mathcal{X} = \{v \in H_0^1(\Omega) \mid v|_{\Omega_i} \in H^2(\Omega_i)\ i = 1, 2\}$$

and the Banach space

$$\mathcal{Y} = \{v \in \mathcal{L}^2(\Omega) \mid v|_{\Omega_i} \in H^2(\Omega_i)\ i = 1, 2 \text{ and } v = 0 \text{ on } \partial\Omega\},$$

with the corresponding norm

$$\|v\|_{\mathcal{Y}} = \|v\| + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}.$$

This norm is not desirable since we lose some properties of the Hilbert space $\mathcal{Y}$. Instead, we introduce the inner product

$$(u, v)_{\mathcal{Y}} = (u, v) + (u, v)_{H^2(\Omega_1)} + (u, v)_{H^2(\Omega_2)}.$$

Consequently,

$$\|v\|_{\mathcal{Y}}^2 = \|v\|^2 + \|v\|_{H^2(\Omega_1)}^2 + \|v\|_{H^2(\Omega_2)}^2.$$

Note that $\mathcal{X}$ is a subset of $\mathcal{Y}$.

Definition 7.3.2 (Strong solution)**.** *Let* $u_0 \in \mathcal{X}, u_d \in W$ *and* $f \in \mathcal{L}^2(W)$. *A function* $u \in \mathcal{L}^2(\mathcal{X}) \cap H^1(W) \cap H^2(W)$ *is called a strong solution of Equations* (7.3.1) *to* (7.3.3) *if* $u(0) = u_0$ *and* $\partial_t u(0) = u_d$ *with jump conditions (Equation* (7.3.4)*), and the relation*

$$\partial_t^2 u(x,t) + \partial_t u(x,t) - \nabla \cdot (\beta(x)\nabla u(x,t)) = f(x,t), \qquad (7.3.7)$$

holds for a.e. $t \in \mathcal{J}$ *and* $x \in \Omega_i,\ i = 1, 2$.

In [DD19], there are two important results regarding the existence of a strong solution, i.e. Lemma 2.1 and Theorem 2.1. They are displayed below as Lemma 7.3.1 and Theorem 7.3.1.

Lemma 7.3.1. *Let* u *be a weak solution of Equation* (7.3.1)*. Assume* $u_0 \in \mathcal{X}, v_0 \in V, f \in \mathcal{L}^2(W), u \in \mathcal{L}^2(\mathcal{X}) \cap H^1(W) \cap H^2(W)$ *and* Γ *is Lipschitz continuous. Then* u *is a strong solution of Equations* (7.3.1) *to* (7.3.3)*.*

Theorem 7.3.1. *Let* $u_0 \in \mathcal{X}, v_0 \in V$ *and* $f \in \mathcal{L}^2(W)$*, then the interface problem (Equations* (7.3.1) *to* (7.3.3)*) admits a unique strong solution.*

The proofs in [DD19] are incomplete. In this dissertation, we adjust the arguments to prove similar results for the stationary case. These results are Proposition 7.3.1, Proposition 7.3.2 and Lemma 7.3.2.

7.3.2 Improved exposition

In this dissertation, the approach in [VV02] is followed. Rather than a weak solution, one considers a weak formulation of the problem (Problem MC-W) which is similar to the formulation of Problem GH. This can be interpreted as the minimum requirements for a solution, as contained in the formulation of Problem GH. However, the existence Theorems 2.2.1 and 2.2.2 show that a solution of Problem GH automatically satisfies stricter criteria.

Proposition 7.3.1. *If* $u \in \mathcal{X}$ *and satisfies the interface conditions, then* $u \in E_b$*.*

Proof. Since $u \in \mathcal{X}$, we have $u_1 = u|_{\Omega_1} \in H^2(\Omega_1)$. There exists functions $u_1^{(n)} \in C^2(\bar{\Omega}_1)$ such that $\lim_{n\to\infty} u_1^{(n)} = u_1$ with respect to the norm of $H^2(\Omega_1)$. The same is true for u_2.

We may choose $u^{(n)} \in \mathcal{T}_\Gamma$, consequently (7.2.9) is true for $u^{(n)}$. Therefore, for each n, there exists $F^{(n)} \in \mathcal{L}^2(\Omega)$ such that

$$b(u^{(n)}, v^{(n)}) = -\iint_\Omega F^{(n)} v^{(n)} dA \text{ for each } v^{(n)} \in V(\Omega).$$

Finally, the weak partial derivatives less than or equal to two converges in $\mathcal{L}^2(\Omega_i)$ and consequently $u^{(n)}$ converges to u in $\mathcal{L}^2(\Omega)$. It follows from Proposition 7.2.4 that $u \in E_b$. □

Proposition 7.3.2. *If* $u \in \mathcal{X}$ *and* $\nabla \cdot \beta \nabla u = f \in \mathcal{L}^2(\Omega)$, *then*

$$\sum_{i=1}^{2} (f, v)_{\Omega_i} + b(u, v) - \int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] v ds = 0 \text{ for each } v \in V. \qquad (7.3.8)$$

Proof. First, assume that $u_i \in C^2(\bar{\Omega}_i)$. Applying integration by parts we have

$$\begin{aligned} 0 &= \sum_{i=1}^{2} \int_{\Omega_i} f v dx + \sum_{i=1}^{2} \int_{\Omega_i} -\nabla \cdot (\beta_i \nabla u) v dx \\ &= \sum_{i=1}^{2} \int_{\Omega_i} f v dx + \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u \cdot \nabla v dx \\ &\quad - \int_\Gamma \beta_1 \frac{\partial u}{\partial \mathbf{n}} v ds + \int_\Gamma \beta_2 \frac{\partial u}{\partial \mathbf{n}} v ds \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^{2} \int_{\Omega_i} f v dx + \sum_{i=1}^{2} \int_{\Omega_i} \beta_i \nabla u \cdot \nabla v dx - \int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] v ds \\ &= \sum_{i=1}^{2} (f, v)_{\Omega_i} + b(u, v) - \int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] v ds \text{ for each } v \in V. \end{aligned}$$

Thus Equation (7.3.8) holds.

Next, we consider $u_i \in H^2(\Omega_i)$ but $u_i \notin C^2(\bar{\Omega}_i)$ and use the arguments in Proposition 7.3.1. □

For the next result we need the trace operator for $\mathbb{R}^2$, as discussed at the end of Appendix A.3.

Proposition 7.3.3. *If* $u \in \mathcal{X}$, *then the first condition in* (7.3.4) *holds.*

Proof. Since $u_i \in H^2(\Omega_i), u_i \in C(\bar{\Omega}_i)$ and $u_\Gamma(\Omega_i) = u_i|_\Gamma$. In [BF12] it is proved that $u_\Gamma(\Omega_i) = u_\Gamma(\Omega)$ for $i = 1, 2$. Consequently,

$$u_1|_\Gamma - u_2|_\Gamma = \gamma_\Gamma u(\Omega_1) - \gamma_\Gamma u(\Omega_2) = \gamma_\Gamma u(\Omega) - \gamma_\Gamma u(\Omega) = 0.$$

Therefore $[u] = 0$. □

Lemma 7.3.2. *If $u \in E_b$, then $u \in \mathcal{X}$ and it satisfies the interface conditions.*

Proof. For $u \in E_b$ there exists a $y \in W$ such that $b(u, v) = (y, v)$ for each $v \in V$. Since b is positive definite, the weak form $b(u, v) = (y, v)$ has similar properties to elliptic problems. Using [CZ98] it follows that the solution is in $H^2(\Omega_i)$ on each subdomain and therefore $u \in \mathcal{X}$.

Next we show that u also satisfies the jump conditions (7.3.4). From Proposition 7.3.3, the first condition in (7.3.4) holds.

Equation (7.3.8) together with the definition of a weak solution results in

$$\int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] v ds = 0 \text{ for each } v \in V.$$

Since v is arbitrary, u satisfies the second condition in (7.3.4). □

Corollary 7.3.1. *If u is a solution of Problem MC-W (with $u_0 \in E_b$ and $u_d \in V$), then $u(t) \in \mathcal{X}$ for each t and satisfies the interface conditions.*

Due to the results above, the existence of a unique solution of Problem MC-W implies the existence of a unique strong solution for Problem MC. The verification of the assumptions required to indeed show that Problem MC-W has a unique solution is crucial.

7.4 Application of the finite element method

7.4.1 Galerkin approximation

For the finite element approximation, we consider a two-dimensional domain. For simplicity, we consider a special case of Problem MC where Ω_1 and Ω_2 are open rectangles. As before, $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$. The rectangle Ω is subdivided into smaller rectangles (elements) such that no part of Γ is an interior point of an element, i.e. an element is either in Ω_1 or in Ω_2.

Suppose $S^h(\Omega)$ is the span of piecewise bilinear basis functions δ_i on Ω. Let V^h denote a finite dimensional subspace of V, which is also contained in $S^h(\Omega)$.

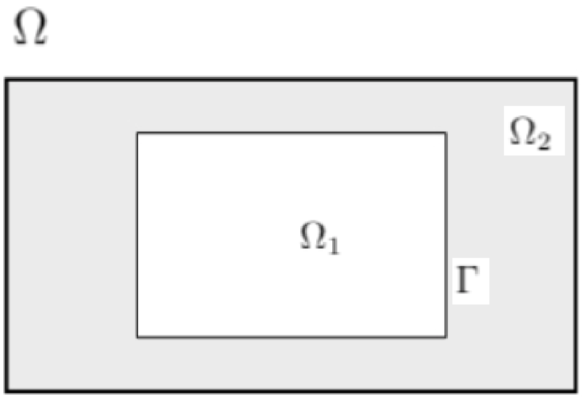

Figure 7.2: The rectangle $\Omega_1 \cup \Omega_2 \cup \Gamma$

Problem MC-W^h

Find a function $u_h(t) \in V^h$ such that for each $t \in [0, t_*]$,

$$c(u_h''(t), v) + a(u_h'(t, v)) + b(u_h(t), v) = (f(\cdot, t), v) \quad \text{for each } v \in V^h,$$
$$\text{while } u_h(0) = u_0^h \text{ and } u_h'(0) = u_d^h.$$

As before, we will interpolate to find the initial conditions u_0^h and u_d^h. Note that Problem MC-W^h is a special case of Problem GH^h.

Since the functions in V^h are continuous on $\bar{\Omega}$, the nodal values of a function in V^h are well defined (including on Γ).

7.4.2 System of ordinary differential equations

To obtain a system of ordinary differential equations, we begin by defining the finite element matrices $N_{ij} = (\delta_j, \delta_i)$ and $K_{ij} = b(\delta_j, \delta_i)$. Recall that for this problem $a(u, v) = c(u, v) = (u, v)$. Due to the definition of $b(u, v)$ (Equation (7.2.4)), one needs to be cautious in assembling the K matrix to ensure that the correct β_i is used, depending on where the relevant elements are located (in Ω_1 or in Ω_2).

Following the same procedure as in Subsection 5.1.2, we obtain the system of ordinary differential equations:

$$N\ddot{\bar{u}} + N\dot{\bar{u}} + K\bar{u} = \bar{Q},$$

where the vector $\bar{Q}$ is due to the external heat source that is experienced in Ω_1 ($Q_i = (f, \delta_i)_{\Omega_1}$ on Ω_1). Finally, a finite difference scheme (for example central difference average acceleration as in Subsection 5.3.3) can be applied to the system.

7.5 Convergence and error estimates

The theory in Chapter 3 can be applied to Problem MC-W. The differences have to do with the interpolation process. Corollary 3.3.1 cannot be used without necessary changes.

To apply the results in Chapter 3, we require $u \in C^2((0, t_*), V)$ and that $u(t)$ and $u''(t) \in H^2$. Define h as the maximum of the diameters of the rectangle elements.

Suppose that Assumption C1 holds, as in Subsection 3.3.1 we use results on interpolation theory to replace Assumption C2.

We use the interpolation operator Π_b in the space $\mathcal{X}$, where Π_b is the usual interpolation operator for piecewise bilinear basis functions on rectangles (see [SF73]).

The following interpolation error estimate on Ω_i replaces Assumption C2.

Corollary 7.5.1. *There exists a constant C_i such that if $u|_{\Omega_i} \in H^k$ for $k \geq 2$, then*

$$\|\Pi_b u - u\|_{1,\Omega_i} \leq C_i h |u|_{2,\Omega_i}, \tag{7.5.1}$$

where $|\cdot|_{2,\Omega_i}$ denotes the semi-norm on Ω_i.

We use the estimate above on each space Ω_i and then combine it to obtain the required result.

Corollary 7.5.2. *There exists a constant $\hat{C}$ such that if $u|_{\Omega_i} \in H^k$ for $k \geq 2$ and $i = 1$ and 2, then*

$$\|\Pi_b u - u\|_V \leq \hat{C} h |u|_2, \tag{7.5.2}$$

where $|\cdot|_2$ denotes the semi-norm.

Proof. Since $\|\cdot\|_V$ is equivalent to $\|\cdot\|_1$ on V we have that

$$\begin{aligned}\|\Pi_b u - u\|_V^2 \leq& \beta_C \|\Pi_b u - u\|_1^2 \\ =& \beta_C(\|\Pi_b u - u\|_{1,\Omega_1}^2 + \|\Pi_b u - u\|_{1,\Omega_2}^2) \\ \leq& \beta_C(C_1^2 h^2 |u|_{2,\Omega_1}^2 + C_2^2 h^2 |u|_{2,\Omega_2}^2) \\ \leq& \hat{C}^2 h^2 |u|_2^2.\end{aligned}$$

The result follows. □

Application of Theorem 3.1.2
Suppose $u_0^h = \Pi_b u_0$ and $u_d^h = \Pi_b u_d$. If the solution u of Problem MC-W satisfies Assumption C1, $u(t) \in \mathcal{L}^2([0, t_*], H^2 \cap V)$ and $u'(t) \in \mathcal{L}^2([0, t_*], H^2 \cap V)$, then

$$\begin{aligned}\|u(t) - u_h(t)\|_W \leq& (D_B)^{-1} \hat{C} h \Big(|u(t)|_2 + 3t_* \sqrt{2} \max |u'(t)|_2 \\ &+ 3K_a t_* \max |u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ &+ 3t_* |u_d|_2 \Big),\end{aligned}$$

for each $t \in [0, t_*]$.

Suppose we use the same algorithm as in Problem GH^h-D, we then obtain the following problem.

Problem MC^h-D

Find a sequence $\{u_k^h\} \subset S^h$, such that for each $k = 0, 1, 2, \cdots, N-1$,

$$\begin{aligned}\delta_t u_k =& v_{k+\frac{1}{2}}, \\ c(\delta_t v_k, \psi) + a\left(v_{k+\frac{1}{2}}, \psi\right) + b\left(u_{k+\frac{1}{2}}^h, \psi\right) =& \frac{1}{2}\left([f(t_k) + f(t_{k+1})], \psi\right)_X, \\ \text{while } u_0^h = d^h \text{ and } u_d^h = v^h,&\end{aligned}$$

for each $\psi \in S^h$.

To obtain the final error estimate, we combine the results from Theorems 3.1.2 and 3.2.1 to obtain the result below.

Application of the combination of Theorems 3.1.2 and 3.2.1
Suppose $u_0^h = \Pi_b u_0$ and $u_d^h = \Pi_b u_d$. If $u(t) \in \mathcal{L}^2([0, t_*], H^2 \cap V)$, $u'(t) \in \mathcal{L}^2([0, t_*], H^2 \cap V)$, $f \in C^2([0, t_*], \mathcal{L}^2)$ and the sequence $\{u_k^h\}$ is

a solution of Problem MC^h-D, then

$$\begin{aligned}\|u(t_k) - u_k^h\|_W \leq &(D_B)^{-1}\hat{C}h\Big(|u(t)|_2 + 3t_*\sqrt{2}\max|u'(t)|_2 \\ &+ 3K_a t_* \max|u(t)|_2 + (2 + 3K_a t_*)|u_0|_2 \\ &+ 3t_*|u_d|_2\Big) + 7t_*^2\tau^2 \max\|u_h^{(4)}\|_W + 7t_*\tau^2 \max\|u_h'''\|_W \\ &+ \sqrt{2K_a}\tau^4 \max\|u_h'''\|_W,\end{aligned}$$

for each $t_k \in [0, t_*]$.

Chapter 8

Conclusion

8.1 Overview

1. Hyperbolic mathematical models

The basic hyperbolic mathematical models used in this dissertation are introduced in Chapter 1. Examples of hyperbolic type models are vibration problems of elastic structures and heat conduction taking into account phase-lag models. Specifically, of interest to us are linear vibration problems which have a variational form that resembles the wave equation. To introduce the concept of variational form, the wave equation was considered in Section 1.2 and the variational form was derived.

A model that has a variational form that resembles that of the wave equation is the Timoshenko beam model. The Timoshenko beam theory was introduced and the standard equations of motion and constitutive equations given. For the purpose of mathematical analysis and numerical approximations, the model was then written in its dimensionless form. In fact, this is preferable for all mathematical models considered in this dissertation. The variational form of the problem is also derived.

Special cases of the Timoshenko beam model were also discussed, particularly the Rayleigh and Euler-Bernoulli beam models. Both models were derived from the equations of motion and constitutive equations of the Timoshenko beam model.

Other models that have a variational form that resembles that of the wave equation include the heat conduction models taking phase-lag into account. First, we discuss the conservation law for heat conduction and derive hyperbolic heat conduction models using the constitutive equation suggested by Cattaneo (1948) and Tzou (1995). The models were presented on one and multi-dimensional domains. The one-dimensional case is exactly the same as a weakly damped wave equation from a mathematical perspective. Tzou suggested a dual phase-lag model which includes the phase-lag in heat flux and gradient of the temperature.

As mentioned above, the heat conduction models were written in their dimensionless form. The variational forms of the hyperbolic heat conduction models were derived. In the variational form the partial differential equations and boundary conditions are absorbed into the bilinear forms.

Specific applications of the hyperbolic heat equation were also discussed for bio-heat transfer in skin. The applications were motivated by the work of Dai et al. in 2008 and Liu et al. in 2012.

2. Second order hyperbolic type problems

In Chapter 2, the notion of a weak variational form is introduced which is necessary in the existence theory used to establish the solvability of a problem. (This is the approach followed in Van Rensburg and Van der Merwe (2002).) The one-dimensional hyperbolic heat equation model was used to illustrate the procedure. The problem was written in its variational form, then related to the weak variational form.

A review of the work by Van Rensburg and Van der Merwe (2002) on the general second order hyperbolic problem was done. The main theorems were stated without proofs but an equivalent first order system was formulated to which semigroup theory was applied. The resulting theorems are stated without proofs. However, additional remarks and discussion were presented to make the theory more readable.

The theory is applied to the one-dimensional hyperbolic heat problem. Function spaces were defined and it was proved that the assumptions are satisfied for the problem of interest. Lastly, the relevant theorem was applied to establish the solvability of the problem.

A problem where a solution does not exist was also analysed. The significance of initial conditions for the existence of a solution was demonstrated.

3. Finite element approximation theory

In Chapter 3, the Galerkin finite element approximation for a weakly damped second order hyperbolic problem was considered. The work in Basson and Van Rensburg (2013) was used. Their approach uses the existence results in Van Rensburg and Van der Merwe (2002) and permits a comparison of the required results for the existence, uniqueness and convergence of a solution.

As the first step to obtain convergence results, a projection operator was defined and used to estimate the semi discrete error, the error between the exact solution and Galerkin approximation. The error for the semi discrete problem was obtained with regards to the Inertia space norm.

The Galerkin approximation was then written as a system of ordinary differential equations which can be solved using a finite difference scheme. The difference between the solution obtained for the finite difference scheme and the Galerkin approximation (fully discrete error) was considered. The sum of the semi discrete and fully discrete errors yields the final error estimate, used to determine convergence.

Finally, the convergence theory was applied to the hyperbolic heat conduction problem introduced in Chapter 2.

4. Serially connected double beams

In Chapter 4, a model that consists of two serially connected Timoshenko beams was considered. The same model may also be used for one beam with different loads on separate parts. The problem is formulated using the standard equations of motion and constitutive equations introduced in Chapter 1. One of the beams was assumed to be embedded in an elastic material, while the other beam is either free or subjected to an external load. In addition, boundary and interface conditions were formulated to obtain a well-posed problem. The model problem is referred to as Problem EDB.

Next, the existence of a solution for the problem was considered. As before, a derivation of the variational form of the problem, then its weak variational form was done. Relevant function spaces were defined and their properties proved to establish the existence and uniqueness of a weak solution. This was achieved using the theory in Chapter 2. The main concern was whether the structure should be modelled as one or two beams.

To apply the finite element approximation theory in Chapter 3, Problem EDB was written in its Galerkin form. Regularity of the solution depends on the choice of the model. A discontinuity of a variable may lead to discontinuities for the weak derivatives of a solution. To apply the convergence theory, the solution must be sufficiently smooth. It was therefore necessary to use the double beam model. The error estimates used in the convergence were also discussed.

For the FEM computation it was shown that the single beam model can be used. As a consequence of the interface conditions imposed on the two beams, Problem EDB may be considered as a single beam. Discontinuities do arise but they do not affect the computations.

5. Beam models for tap root systems

In this chapter, an application of the models in Chapter 4 to a biological problem is considered. Specifically, the embedded double beam models are used to model a plant with a tap root system. The experimental findings in the work by Ennos (2000) is compared to results obtained when applying the Finite element method to the serially connected beam models.

Three forms of FEM were introduced, namely: Standard Finite Element Method (SFEM), Mixed Finite Element Method (MFEM) and Variant Mixed Finite Element (VMFEM). VMFEM was used to illustrate the derivation of a system of ordinary differential equations for the single beam model. This problem was also considered at equilibrium.

Next, the static problem for the double beam model was considered. Assuming that the load in Problem SDB is constant, the force and moment of the beam can be estimated. The approach in Van Rensburg and Van der Merwe (2006) was useful in investigating the possibility of determining a general solution for Problem SDB. Although theoretically a general solution exists, a difficulty is encountered when solving for the embedded beam, due to the coefficients of the linear combination of the general solution being too small.

The variational approach was then used to obtain the Galerkin approximation for Problem SDB. The variational forms for both SFEM and MFEM were obtained. Systems of equations were presented and the numerical results analysed. It was found that MFEM converges faster than SFEM.

A comparison of the numerical results for the Galerkin approximations of the static double beam and static single beam was done. The results did not differ significantly. The double beam model was also used to determine the effects of a change in length of the cantilever beam. It was concluded that as the length of that beam is increased, the deflection of both the embedded and cantilever beams also increased.

Since the single and double beam models for the static case compared well, the single beam model was then used to investigate the dynamics of the beam. The beam was considered to be at rest initially, with a periodic forcing function.

Finally, according to Crook and Ennos (1998) and Ennos (2000), there is a point below the soil on a plant with a tap root system, known as the centre of rotation (CoR). The movement of a plant is relative to this point. The parameter modelling the resistance of the soil in the model was varied and it was observed that the location of the CoR moved up towards the top of the soil as the resistance was increased.

6. Rigid bodies attached to beams

In Chapter 6, a brief summary of various articles regarding beams with an attachment at an endpoint is given. The derivation of the equations used to describe the dynamics of a beam with a tip body is done and various models in previous publications were discussed. Of special interest to us are the interface conditions imposed at the endpoints.

The model problem for an intermediate rigid body between two Timoshenko beams was formulated using the previously mentioned dynamics of a beam with a tip body by defining some constraints. This problem was then written in its variational and weak variational form taking the constraints into account. Properties of the function spaces were proved as well as various other results needed in order to establish the existence and uniqueness of a weak solution using the theory in Chapter 2.

Finally, the Galerkin approximation was considered and a system of ordinary differential equations derived for simulations. The intermediate body and resultant constraints were taken into account.

7. Hyperbolic heat conduction model

In Chapter 7, a hyperbolic heat conduction model is considered. The model is from the work by Dekka and Dutta (2019). The problem is formulated and written in its variational and weak variational form. Properties of the relevant function spaces are derived and the existence of a weak solution is proved using the variational approach.

The approach to existence by Dekka and Dutta (2019) was also investigated since the connection between the article and the results used from the 2002 article by Van Rensburg and Van der Merwe is not clear.

Finally, the application of the finite element method to the hyperbolic heat equation is also considered. First, we obtain the Galerkin approximation and derive a system of ordinary differential equations. Lastly, error estimates are obtained using the results in Chapter 3 in order to establish the convergence of a solution.

8.2 Contributions

In this dissertation, various linear vibration problems which have a variational form that resembles the wave equation are investigated. Heat conduction models taking phase-lag into account is also considered, since it is the same as a weakly damped wave equation from a mathematical perspective.

In order to obtain existence results for the models, a review of the work by Van Rensburg and Van der Merwe (2002) on general second order hyperbolic type problems was done. In addition to presenting the results, additional remarks and a discussion which will assist one to apply the theory is included. An example where a solution does not exist was also analysed in order to demonstrate the significance of initial conditions for the existence of a solution.

General convergence results were also given, using the work from Basson and Van Rensburg (2013). In order to obtain the final error estimates for convergence, the sum of the semi discrete and fully discrete errors were used. A hyperbolic heat conduction problem was used as an illustration of how the theory is applied.

Different models for which the variational forms are second order hyperbolic problems were considered and existence and convergence proved.

First, a model that consists of two serially connected Timoshenko beams was considered. One of the beams was modelled to be embedded in an elastic material, while the other beam is either free or subjected to a given external load. This model may also be adjusted for a single beam with different loads on separate parts. The existence and uniqueness of a solution was obtained by defining relevant function spaces and proving required properties. The main concern was whether the structure should be modelled as one or two beams. To apply the convergence theory it was necessary to use the double beam model since the solution must be sufficiently smooth. It was shown that for the FEM computation however, a single beam model can be used.

These models were then used in a biological application of a plant with a tap root system. Three forms of FEM were introduced and the derivation of a system of ordinary differential equations for the single beam model was illustrated. For the static case of the double beam model, the force and moment of the beam can be explicitly solved and used to determine a general solution. Although it theoretically exists, a difficulty is encountered due to coefficients being too small. Therefore, the variational approach was then used to approximate solutions. A comparison of the results for the static double beam and static single beam showed that the models compare well. The single beam model was then used to investigate the dynamics of the beam. These experiments indicated how the parameter modelling the resistance of the soil influenced the location of the so-called center of rotation which in turn influences the reaction of the plant to external forces.

Models for rigid bodies attached to beams were also investigated. The equations used to describe the dynamics of a beam with a tip body were derived with special attention given to the interface conditions imposed at the endpoints. A model problem for an intermediate rigid body between two Timoshenko beams was considered. This includes constraints at the interface. Again, existence and uniqueness of solutions were established by proving the required properties of the function spaces. A system of ordinary differential equations which can be used for simulations were derived. The intermediate body and resultant constraints were taken into account.

Finally, hyperbolic heat conduction models were considered. Application of the hyperbolic heat equation for bio-heat transfer in skin was discussed. Specifically, a model from the work by Dekka and Dutta (2019) was investigated and their approach to existence of solutions scrutinized. It was found that the link between their article and the results used from the 2002 article by Van Rensburg et al. is incomplete. In this dissertation the exposition of the theory is improved. Convergence and error estimates for the FEM approximation was also established.

Future work

Future work may include modification of the models used to simulate a plant with a tap root system so that the morphology of a plant is taken into account. Also, the effect of gravity on the dynamics of a plant should be examined.

Regarding the heat conduction model, in the article by Deka and Dutta (2019) the example for application of numerical methods did not contain sharp crested waves, which sometimes occur in the hyperbolic heat conduction model (see [SV12]). For a multi-dimensional model it is a serious challenge to adapt the approach used in the one-dimensional model.

Appendix A

Sobolev spaces

A.1 Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $H^m(\Omega)$ is the subspace of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Notation Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$, then $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \alpha_2 \cdots \alpha_n$.

Definition A.1.1 (Weak partial derivative of order m). *If $u \in \mathcal{L}^2(\Omega)$ and there exists a function $v \in \mathcal{L}^2(\Omega)$ such that*

$$(u, D^\alpha \phi) = (-1)^{|\alpha|}(v, \phi) \quad \text{for all} \quad \phi \in C_0^\infty(\Omega),$$

then v is called the weak derivative of order $|\alpha|$ of u and is denoted by $D^m u$.

Definition A.1.2 (Inner product on $H^m(\Omega)$). *For u and $v \in H^m(\Omega)$ the inner product is defined by*

$$(u, v)_m = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v) \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Definition A.1.3 (Norm). *For $u \in H^m(\Omega)$,*

$$\|u\|_m = \sqrt{(u, u)_m} \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Properties of Sobolev spaces

Consider Ω a bounded open interval or a bounded open convex subset of $\mathbb{R}^n$.

Remark. *It is not necessary for Ω to be convex however, it is sufficient for our discussion.*

Theorem A.1.1. *$C_0^\infty(\bar{\Omega})$ is dense in $\mathcal{L}^2(\Omega)$ with respect to the norm of $\mathcal{L}^2(\Omega)$.*

Proof. See [Eva98], [Sho77] or [OR76]. □

Corollary A.1.1. *$C^m(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ with respect to the norm of $\mathcal{L}^2(\Omega)$.*

Theorem A.1.2. *The space $H^m(\Omega)$ is complete.*

Proof. See [Eva98], [Sho77] or [OR76]. □

Proposition A.1.1. *A closed subspace of a Hilbert space is complete.*

Proof. Suppose A is a closed subspace of the Hilbert space X. Let (x_n) be a Cauchy sequence in A. Since $A \subset X$ the sequence (x_n) converges in X. Since A is closed, it follows that (x_n) converges in A. Thus A is complete. □

Proposition A.1.2. *Let X be the space with equivalent norms $\|\cdot\|_a$ and $\|\cdot\|_b$. The space X is complete with respect to the norm $\|\cdot\|_a$ if and only if it is complete with respect to the norm $\|\cdot\|_b$.*

Proof. Suppose X is complete with respect to $\|\cdot\|_a$. Let (x_n) be a Cauchy sequence with respect to $\|\cdot\|_b$. Since the norms are equivalent, (x_n) is a Cauchy sequence with respect to $\|\cdot\|_a$. Due to the completeness, the sequence converges with respect $\|\cdot\|_a$. By the equivalence of norms (x_n) is also convergent with respect to $\|\cdot\|_b$. Therefore, X is complete with respect to the norm $\|\cdot\|_b$. □

Proposition A.1.3. *Let X and Y be Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ respectively. For the product space $X \times Y$ the induced inner product is $(\cdot,\cdot)_{XY} = (\cdot,\cdot)_X + (\cdot,\cdot)_Y$ and the space $X \times Y$ is complete.*

Proof. Denote an element in $X \times Y$ by $w = \langle x, y \rangle$. Consider a Cauchy sequence (w_n) in $X \times Y$. Now (x_n) is a Cauchy sequence in X and (y_n) is a Cauchy sequence in Y. Since both X and Y are closed, it follows that (x_n) converges in X and (y_n) converges in Y with respect to the induced inner product. From the definition of the inner product $\|w\|^2_{XY} = \|x\|^2_X + \|y\|^2_Y$ and we have that (w_n) converges and therefore the product space is complete. □

Remark. *The result in Proposition A.1.3 can clearly be extended to a finite cartesian product. Consider for example the product space $X \times Y \times Z$ where X, Y and Z are Hilbert spaces. Due to Proposition A.1.3 $X \times Y$ is complete with respect to the inner product $(\cdot,\cdot)_{XY} = (\cdot,\cdot)_X + (\cdot,\cdot)_Y$. using Proposition A.1.3 again, it follows that $(X \times Y) \times Z$ is complete with respect to the inner product $(\cdot,\cdot)_{XYZ} = (\cdot,\cdot)_{XY} + (\cdot,\cdot)_Z$.*

A.2 Inequalities

For this dissertation the one dimensional and two dimensional cases are important. For a one-dimensional domain $\Omega = [0, \ell]$.

Lemma A.2.1. *For any $u \in C^1[0,\ell]$ and any two points x and y in $[0,\ell]$ with $x < y$,*

$$|u(y)| \leq \sqrt{\ell}\|u'\| + |u(x)|.$$

Proof. Consider any functions f and $g \in \mathcal{L}^2(\Omega)$, using the Cauchy-Schwartz inequality we have

$$\left(\int_x^y fg\right)^2 \leq \left(\int_x^y f^2\right)\left(\int_x^y g^2\right).$$

Choosing $g = 1$ yields

$$\left(\int_x^y f\right)^2 \leq \left(\int_x^y f^2\right)(y-x) \leq \ell\|f\|^2. \tag{A.2.1}$$

Thus, for each $f \in \mathcal{L}^2(0,\ell)$, $\left|\int_x^y f\right| \leq \sqrt{\ell}\|f\|$.

From the Fundamental Theorem of Calculus we have

$$u(y) - u(x) = \int_x^y u'.$$

It follows from Equation (A.2.1)

$$|u(y)| \leq \left| \int_x^y u' \right| + |u(x)| \leq \sqrt{\ell}\|u'\| + |u(x)|.$$

□

Lemma A.2.2. *For any* $u \in C^1[0,\ell]$ *with a zero in* $[0,\ell]$,

$$\|u\|_{\sup} \leq \sqrt{\ell}\|u'\|.$$

Proof. Suppose $u(x) = 0$, then $|u(y)| \leq \sqrt{\ell}\|u'\|$ by Lemma A.2.1. The result follows since $\sqrt{\ell}\|u'\|$ is an upper bound for $|u|$. □

Lemma A.2.3. *For any* $u \in C^1[0,\ell]$ *with a zero in* $[0,\ell]$,

$$\|u\| \leq \ell\|u'\|.$$

Proof. Using Lemma A.2.2, we have

$$\|u\|^2 = \int_0^\ell \big(u(x)\big)^2 dx \leq \ell\|u\|_{\sup}^2 \leq \ell^2\|u'\|^2.$$

□

Lemma A.2.4. *Let* $p \in [0,\ell]$ *and consider the set*

$$Q = \{u \in C^1[0,\ell] \;\big|\; u(p) = 0\}.$$

Denote the closure of Q *in* $H^1(0,\ell)$ *by* $\bar{Q}$. *For any* $u \in \bar{Q}$,

$$\|u\| \leq \ell\|u'\|.$$

Proof. If $u \in \bar{Q}$, there exists a sequence $\{u_n\}$ in $C^1[0,\ell]$ such that

$$\|u_n - u\|_1 \to 0 \text{ as } n \to \infty \text{ and } \mathrm{u_n(p)} = 0.$$

From the reverse triangle inequality we have

$$\big|\|u_n\|_1 - \|u\|_1\big| \leq \|u_n - u\|_1.$$

Hence, $\|u_n\|_1 \to \|u\|_1$ as $n \to \infty$. It follows

$$\|u_n\| \to \|u\| \text{ and } \|u_n'\| \to \|u'\| \text{ as } n \to \infty.$$

By Lemma A.2.3

$$\|u_n\| \leq \ell \|u'_n\|.$$

Thus taking limits we get directly $\|u\| \leq \ell \|u'\|$.

For the two dimensional case it is required that Ω is a domain that must be open and arcwise connected. We also assume that Green's theorem holds for Ω. □

Definitions

$C^1_+(\Omega)$: The real valued function $u \in C^1_+(\Omega)$ if $u \in C(\bar{\Omega})$ and the partial derivatives $\partial_x u$ and $\partial_y u$ are integrable on Ω.

$C^1_+(\Omega)^2$: The vector valued function $F \in C^1_+(\Omega)^2$ if both components are in $C^1_+(\Omega)$.

$C^2_+(\Omega)$: The real valued function $u \in C^2_+(\Omega)$ if $\nabla u \in C^1_+(\Omega)^2$

Green's formula can be derived from the Green's Theorem.

Lemma A.2.5 (Green's formula)**.** *If $u \in C^2_+(\Omega)$ and $v \in C^1_+(\Omega)$, then*

$$\iint_\Omega -(\nabla^2 u) v dA = \iint_\Omega \partial_x u \partial_x v + \partial_y u \partial_y v dA - \int_{\partial\Omega} v \nabla u \cdot n ds.$$

Lemma A.2.6.

$$ab \leq \frac{1}{2}\Big(\epsilon^2 a^2 + \epsilon^{-2} b^2\Big).$$

Remark. *The notation $\|\nabla u\|$ is used for $\sqrt{\|\partial_x u\|^2 + \|\partial_y u\|^2}$.It is a norm for the vector valued function ∇u but not always a norm for u itself*

Lemma A.2.7 (Friedrichs'inequality)**.** *For any $u \in C^1(\bar{\Omega})$ with u zero on $\partial\Omega$,*

$$\|u\| \leq A \|\nabla u\|$$

where A depends on Ω.

A.3 Trace

A definition for the value $f(p)$ of a function f in $H^1(0,\ell)$ is necessary. To this end, the one-dimensional trace operator is useful. If $f \in C^1[0,\ell]$ and has a zero in $[0,\ell]$. Lemma A.2.2 gives

$$|f(x)| \leq \sqrt{\ell}\|f'\| \qquad \text{for each } x \in [0,\ell]. \tag{A.3.1}$$

If f does not have a zero in $[0,\ell]$ then,

$$|f(x)| \leq 2\ell\|f\|_1 \qquad \text{for each } x \in [0,\ell]. \tag{A.3.2}$$

We require f to be in $H^1(0,\ell)$, but for any $f \in H^1(0,\ell)$ there exists a sequence of functions in $C^1[0,\ell]$ that converges to f. Thus, for any sequence of functions in $C^1[0,\ell]$ that satisfies Equation (A.3.1) or (A.3.2), the limit f is in $H^1(0,\ell)$. More importantly, the result holds for all $f \in C^1(0,\ell)$.

So for any point $p = 0$ or $p = \ell$, a linear operator γ_p is defined on $C[0,\ell]$ by

$$\gamma_p(f) = f(p).$$

If f satisfies Equation (A.3.2), then the linear operator is bounded on $C^1[0,\ell]$ with respect to the $H^1(0,\ell)$ norm, and can be extended to $H^1(0,\ell)$. In this dissertation, we will write $f(p)$ instead of $\gamma_p(f)$.

Lemma A.3.1. *For any $u \in C^1[0,\ell]$*

$$|u(p)| \leq K_\ell\|u\|_1.$$

Proof. Consider $u \in C^1[0,\ell]$. Let $g(x) = 1 - \dfrac{x}{\ell}$ and $v = ug$. Note that $v(0) = u(0)$ and $v(\ell) = 0$. It follows

$$u(0) = \int_0^\ell v' + v(\ell) \tag{A.3.3}$$

Substituting v in Equation (A.3.3)yields

$$\left|\int_0^\ell ug' + u'g\right| \leq \|u\|\|g'\| + \|u'\|\|g\| \leq \|u\|\sqrt{\ell} + \|u'\|\left(\sqrt{\ell}\right)^{-1},$$

Since $(g,g) = \dfrac{\ell}{3} < \ell$ and $(g',g') = \dfrac{1}{\ell}$. Using the fact

$$(\|u\| + \|u'\|)^2 \leq 2\|u\|_1^2$$

we have

$$\|u(0)\| \leq K_\ell \|u\|_1.$$

where $K_\ell = \sqrt{2} \max \left\{ \sqrt{\ell}, \left(\sqrt{\ell}^{-1} \right) \right\}$. □

For more information see [Eva98].

Theorem A.3.1. *Assume Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is in C^1. Then there exists a bounded linear operator T with $\mathcal{D}(T) = H^1(\Omega)$ and $\mathcal{R}(T) \subset \mathcal{L}^2(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Furthermore, there exists a constant C_Ω such that $\|Tu\|_{\partial\Omega} \leq C_\Omega \|u\|_1$ for each $u \in H^1(\Omega)$.*

Proof. The wording of the theorem above is only slightly different from Theorem 1 (Trace Theorem) in [Eva10, Section 5.5]. A complete proof is provided there. □

Corollary A.3.1. *Assume Ω is a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ is in C^1. Then there exists a bounded linear operator T with $\mathcal{D}(T) = H^1(\Omega)$ and $\mathcal{R}(T) \subset \mathcal{L}^2(\partial\Omega)$ such that $Tu = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\bar{\Omega})$. Furthermore, there exists a constant C_Ω such that $\|Tu\|_{\partial\Omega} \leq C_\Omega \|u\|_1$ for each $u \in H^1(\Omega)$.*

Remark. *The operator T is called the trace operator. The notation γ is usually employed to denote the trace operator. A trace operator mapping a subset of Ω onto Γ which may be a part of $\partial\Omega$ is also possible. In fact Γ may even be in the interior of Ω. When there is more than one possibility, $\gamma_\Gamma(A)$ is used to denote the trace operator mapping A onto Γ.*

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