Few-Body Systems

Recovering the two-body potential from a given three-body wave function

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Corresponding Author:	Sergei Anatoljevich Rakityansky, PhD University of Pretoria Pretoria, Gauteng SOUTH AFRICA
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	University of Pretoria
Corresponding Author's Secondary Institution:	
First Author:	Vladimir Borisovich Belyaev
First Author Secondary Information:	
Order of Authors:	Vladimir Borisovich Belyaev
	Sergei Anatoljevich Rakityansky, PhD
	Ishmael Mmusi Gopane
Order of Authors Secondary Information:	
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Recovering the two-body potential from a given three-body wave function

V. B. Belyaev*[a], S. A. Rakityansky[a,b], I. M. Gopane[b]

[a] Joint Institute for Nuclear Research, Dubna, Russia

[b] Department of Physics, University of Pretoria, Pretoria, South Africa

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Abstract

A simple way of deducing the two-body potential from a given two- or three-body wave function is suggested. This method makes it possible to numerically obtain an unknown potential acting between the particles A and B when we know the potentials of their interaction with a third particle C and know the characteristics of the three-body bound state (ABC). Using the examples of the systems (nnp) and $(\Lambda\Lambda\alpha)$, we show that even very simple three-body wave functions constructed on the basis of the general reasoning and the knowledge of the binding energies and sizes of these systems, allow us to deduce reasonable and realistic nn and $\Lambda\Lambda$ potentials. Within this approach, any artificially constructed wave function automatically becomes an exact solution of the corresponding Schrödinger equation with the AB-potential that the method produces. This fact suggests yet another possible application of this method when the AB-potential is known. In such a case we can find a bound state solution of the Schrödinger equation by looking for such values of the free parameters in an artificially constructed wave function that minimize the difference between the deduced and the exact AB-potentials.

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1 Introduction

In many publications dealing with the few-body problems, it is a commonplace to say that a three-body system can reveal some additional information about the two-body subsystems involved. It is however difficult to find any practical implementation of such a statement. At the most, the few-body calculations are used to test how good the guess of the unknown two-body forces is. In contrast to this, in the present paper, we suggest a way to directly obtain the two-body potential, using available information on a three-body system where the two bodies in question are included.

What kind of the three-body information are we speaking about? Suppose we want to find out how the particles A and B interact with each other. However for some reason it is not possible to study the AB-scattering. At the same time we know the potentials that describe the interactions of these particles with a third particle, C, and they form a bound state (ABC) whose energy can be determined experimentally. As we will show shortly, in such a situation, in order to obtain the AB-potential, we need to know the three-body wave function. Of course, one cannot calculate the (ABC) wave function if all the two-body forces keeping this system together, are not known. Very often, however, one can make a reasonable assumptions as to the size of the system and its density distribution. Guessing the wave function in this way, one then can obtain a reasonable guess for the two-body potential.

In some cases, when the AB scattering data are not available this might be the only way to "derive" the AB-potential from experimental data. In the present paper, as an example, we look for the singlet $\Lambda\Lambda$ potential.

The advantage of such an approach is that thus constructed potential generates the bound state at exactly the given experimental energy. At this point it should be emphasized that the method we are proposing here is not intended to compete with the sophisticated approaches developed in the inverse scattering theory. Our goal is much more modest. What we are trying to do is to roughly find the shape of the potential when very little is known about it. We therefore cannot hope to deduce the correct angular dependence of the potential if it is non-central, i.e. we assume that the potential is spherically symmetric, $V(\vec{r}) \equiv V(r)$. Due to the same reason it is logical to approximate the wave function by its dominant component.

2 Two-Body Problem

To begin with, let us obtain the potential V that binds a two-body system with the reduced mass μ, assuming that we know the bound-state wave function, $\psi(\vec{r}) = \langle \vec{r}|\psi\rangle$, and the corresponding energy E. The state vector $|\psi\rangle$ obeys the Schrödinger equation,

$$V|\psi\rangle = (E - H_0)|\psi\rangle \ , \tag{1}$$

where H_0 is the free-motion hamiltonian. Writing this equation in the coordinate representation, we immediately find the potential,

$$V(\vec{r}) = \frac{1}{\psi(\vec{r})}\left(E - \frac{\hbar^2}{2\mu}\Delta\right)\psi(\vec{r}) \ . \tag{2}$$

In principle, Eq. (2) gives us a recipe to deduce a non-central potential, if we know E and the corresponding complete wave function depending on the three-dimensional vector $\vec{r}$. However, as was mentioned above, we are only intended to look for the general properties of an unknown potential using an approximate wave function and therefore will ignore various fine details of it. This means that among all possible partial-wave components of ψ we only use the dominant one,

$$\psi(\vec{r}) = \sum_{[\ell']} R_{[\ell']}(r)\mathcal{Y}_{[\ell']}(\hat{\vec{r}}) \approx R_{[\ell]}(r)\mathcal{Y}_{[\ell]}(\hat{\vec{r}}) \ , \tag{3}$$

where $R_{[\ell]}$ is the radial wave function; the multi-index $[\ell] \equiv \{\ell, s, J, M\}$ includes the orbital angular momentum ℓ, the two-body spin s, the total angular momentum J, and its third component M; the symbol $\hat{\vec{r}}$ represents the spherical angles of vector $\vec{r}$; and $\mathcal{Y}_{[\ell]}(\hat{\vec{r}})$ is the spin-angular part of the wave function in a single partial wave,

$$\mathcal{Y}_{[\ell]}(\hat{\vec{r}}) = \sum_{ms_z} C^{JM}_{\ell m s s_z} Y_{\ell m}(\hat{\vec{r}})\chi_{ss_z} \ . \tag{4}$$

Here $C^{JM}_{\ell m s s_z}$ and χ_{ss_z} are the Clebsch-Gordan coefficients and the two-body spin function, respectively. Just to simplify notation, we furtherdown omit where possible the quantum numbers J and M. For the same reason, in some places we omit even the multiindex $[\ell]$. This should not cause confusion since $R(r)$ is supposed to describe the motion in a single partial wave for which we are constructing a spherically symmetric potential, $V(r)$, acting in that state $[\ell]$.

Formally, we can get rid of the angular dependence by projecting Eq. (1) onto a particular spin-angular state,

$$\langle \mathcal{Y}_{[\ell]}|V|\psi\rangle = \langle \mathcal{Y}_{[\ell]}|(E - H_0)|\psi\rangle \ , \tag{5}$$

which replaces Eq. (2) with

$$V(r) = E - \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} + \frac{\hbar^2}{2\mu R(r)}\left[\frac{2}{r}R'(r) + R''(r)\right] \ , \tag{6}$$

where all unnecessary subscripts are omitted and the prime means the derivative with respect to r.

It should be noted that since the bound-state wave function $R(r)$ is a factor in both the numerator and denominator of Eq. (6), the normalization of this function can be arbitrary.

2.1 Exactly solvable example: Coulomb Potential

Let us check if we can recover the Coulomb potential with the help of Eq. (6), starting with exactly known wave function of the ground state of the hydrogen atom,

$$\psi_{100}(\vec{r}) = \frac{2}{a_0^{3/2}} \exp\left(-\frac{r}{a_0}\right) Y_{00}(\hat{\vec{r}}) \ , \tag{7}$$

where

$$a_0 = \frac{\hbar^2}{\mu e^2} \ .$$

The function ψ_{100} describes the state with $\ell = 0$ and the energy

$$E = -\frac{\mu e^4}{2\hbar^2} = -\frac{e^2}{2a_0} \ .$$

When evaluating the derivatives in Eq. (6), we can use non-normalized radial part of the function (7),

$$R(r) = e^{-r/a_0} \ . \tag{8}$$

Performing the differentiations, we obtain:

$$V(r) = -\frac{e^2}{2a_0} - \frac{\hbar^2}{\mu a_0 r} + \frac{\hbar^2}{2\mu a_0^2} = -\frac{e^2}{r} \ , \tag{9}$$

as was expected.

2.2 Numerical example: triplet NN-potential

In Sec. 2.1 it has been demonstrated how Eq. (6) works in the simplest case when all the derivations can be done analytically, which allows us to trace all the steps and to see mutual cancellations of some extra terms. How would this work if analytic derivations were not possible? Can the potential be accurately recovered if the derivatives are evaluated numerically? Do the extra terms still cancel each other? To answer these questions, we consider here another example, namely, the triplet NN-potential that generates the proton-neutron bound state, i.e. the deuteron.

Let us consider the system consisting of a proton and a neutron. When their spins are parallel and the total spin is 1 (the triplet state), the attaraction between these two nucleons is sufficient to bind them in a stable nucleus, the deuteron, with experimentally known binding energy $2.224566\,$MeV [1] and the RMS-radius $2.128\,$fm [2].

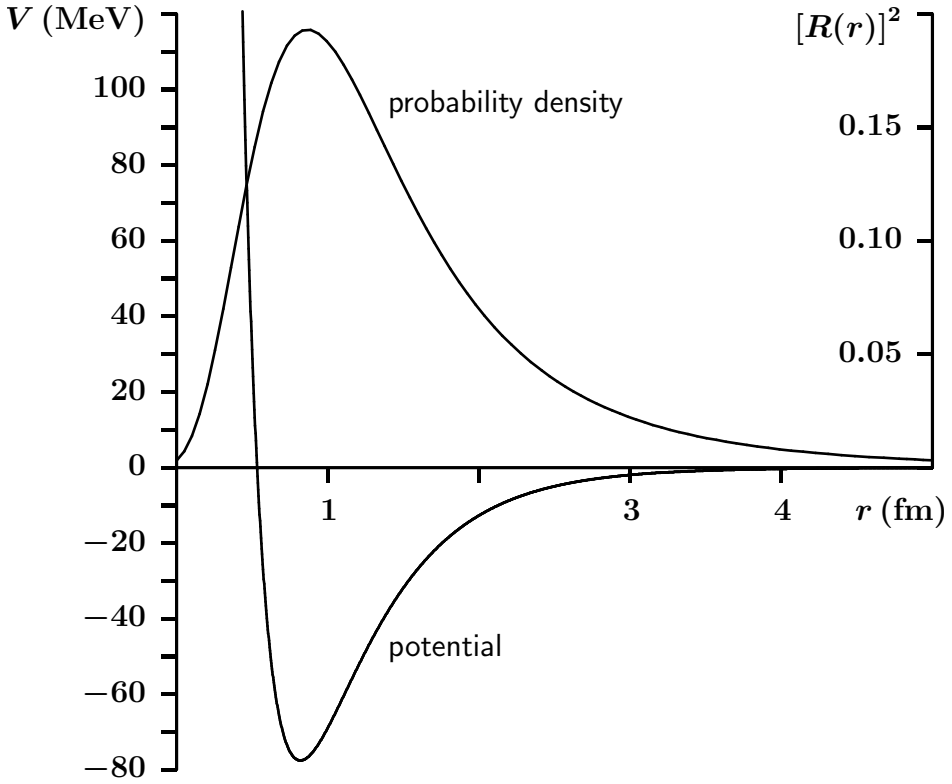

Figure 1: Triplet Malflied-Tjon potential (10) and the radial probability density, $R^2(r)$ [fm^{-3}], for the S-wave bound state generated by this potential.

It is not our intention to accurately describe the deuteron. Its energy and size for us are merely the reference characteristics of a quantum state for which we are going to artificially construct a wave function and then to generate the corresponding two-body potential.

In order to do the test, we choose a simple NN-potential, numerically obtain the deuteron wave function, and using this wave function try to numerically recover the NN-potential from which we started. As such a potential, we use the triplet NN-potential proposed by R. A. Malfliet and J. A. Tjon in Ref. [3] (its slightly modified parameters, which we use, are given in Ref. [4]). It is a combination of two Yukawa terms,

$$V(r) = -\lambda_A \frac{e^{-\mu_A r}}{r} + \lambda_R \frac{e^{-\mu_R r}}{r} \ , \tag{10}$$

where $\lambda_A = 626.885\,\mathrm{MeV}$, $\lambda_R = 1438.72\,\mathrm{MeV}$, $\mu_A = 1.55\,\mathrm{fm}^{-1}$, $\mu_R = 3.11\,\mathrm{fm}^{-1}$, and it is assumed that $\hbar^2/m_N = 41.47\,\mathrm{MeV}\cdot\mathrm{fm}^2$ with m_N being the nucleon mass.

This potential supports an S-wave bound state with $E = -2.2307\,\mathrm{MeV}$ and the RMS-radius $1.985\,\mathrm{fm}$. The Malflied-Tjon potential and the radial dependence of the probability density for the bound state, are shown in Fig. 1. For locating the bound state and calculating its wave function, we used the Jost-function method described in Refs. [5, 6].

Numerically obtained radial wave function $R(r)$ was used then in Eq. (6) to re-construct the potential. Thus recovered potential turned out to be practically indistinguishable from the one shown in Fig. 1.

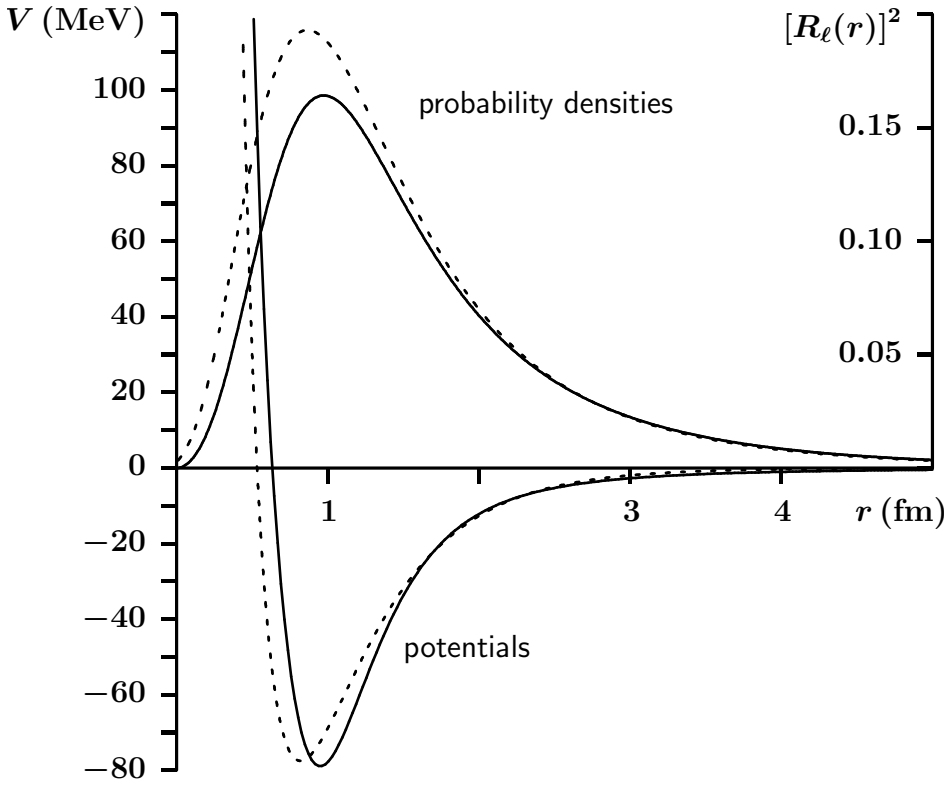

Figure 2: Triplet Malflied-Tjon potential (10) and the corresponding probability density for the deuteron (dashed curves) are compared with a "hand-made" density (solid curve) and the corresponding potential that was obtained from it (solid curve).

In the examples that we used so far the wave function was known or calculated exactly. The main idea of the method we propose here is to uncover general features of a completely unknown potential, when the bound state energy is known and one can guess the space distribution of the probability density. Therefore a question arises: how stable is the method to unavoidable deviations of the guessed wave function from the unknown exact one? To shed

some light on this, let us examine how significant would be the changes in the potential if we slightly distort the wave function of the deuteron.

As a distorted radial wave function, we use the following parametrization (this function is not normalized):

$$R(r) = \frac{r}{r_0} \arctan\left[4(r/r_0)^2\right] e^{-3r/r_0} + \frac{6r_0}{\pi r} \arctan\left[(r/r_0)^2\right] \exp\left(-\sqrt{\frac{2\mu|E|}{\hbar^2}} r\right) \ , \qquad (11)$$

where $r_0 = 1\,\mathrm{fm}$ and $E = -2.2307\,\mathrm{MeV}$ is the same as for the Malfliet-Tjon potential. Since $\varkappa = \sqrt{2\mu|E|/\hbar^2} = 0.2319\,\mathrm{fm}^{-1} \ll 3\,\mathrm{fm}^{-1}$, the first term in Eq. (11) vanishes much faster when $r \to \infty$, and thus this function has the correct asymptotic behaviour determined by the second term.

The radial probability distributions, $[R(r)]^2$, for the Malflied-Tjon deuteron model and for the normalized distorted function, are shown in Fig. 2. The distorted function (11) near the origin vanishes faster, but is slightly above the Malflied-Tjon one at large distances (the latter is not visible in the figure). As a result the distorted probability is shifted to the right, and the RMS-radius is a bit larger, namely, $2.071\,\mathrm{fm}$ (which is a bit closer to the experimental value, $2.128\,\mathrm{fm}$ [2], by the way).

If we denote

$$A(r) = \frac{r}{r_0} \arctan\left[4(r/r_0)^2\right] \qquad (12)$$

and

$$B(r) = \frac{6r_0}{\pi r} \arctan\left[(r/r_0)^2\right] \ , \qquad (13)$$

then

$$R(r) = A(r)e^{-3r/r_0} + B(r)e^{-\varkappa r} \ , \qquad (14)$$

$$R'(r) = \left[A'(r) - \frac{3}{r_0}A(r)\right] e^{-3r/r_0} + \left[B'(r) - \varkappa B(r)\right] e^{-\varkappa r} \ , \qquad (15)$$

and

$$\begin{aligned} R''(r) &= \left[A''(r) - \frac{6}{r_0}A'(r) + \frac{9}{r_0^2}A(r)\right] e^{-3r/r_0} + \qquad (16) \\ &+ \left[B''(r) - 2\varkappa B'(r) + \varkappa^2 B(r)\right] e^{-\varkappa r} \ , \end{aligned}$$

where the derivatives of the auxiliary functions are:

$$A'(r) = \frac{1}{r}A(r) + \frac{8r_0 r^2}{r_0^4 + 16r^4} \ , \qquad (17)$$

$$A''(r) = \frac{1}{r}A'(r) - \frac{1}{r^2}A(r) + 16r_0 r\frac{r_0^4 - 16r^4}{(r_0^4 + 16r^4)^2} \ , \tag{18}$$

$$B'(r) = -\frac{1}{r}B(r) + \frac{12r_0^3}{\pi(r_0^4 + r^4)} \ , \tag{19}$$

$$B''(r) = \frac{1}{r^2}B(r) - \frac{1}{r}B'(r) - \frac{48r_0^3 r^3}{\pi(r_0^4 + r^4)^2} \ . \tag{20}$$

Substituting the above R, R', and R'' in Eq. (6), we obtain the potential, $V(r)$, for which the function (11) is an exact solution of the radial Schrödinger equation with the binding energy $|E|$. For the sake of comparison, we choose $|E| = 2.2307\,\text{MeV}$, i.e. the same as for the potential (10), but in principle we can use any other value for it.

The Malflied-Tjon potential and the potential reconstructed from the distorted function (11) are compared in the same Fig. 2. As we see, the suppression of the probability near the origin results in a larger repulsive core (as one would expect). Despite the fact that the analytic expression for the reconstructed potential is very complicated, there are no drastic changes in the general features of the potential when we change the wave function. In other words, the reconstruction procedure based on Eq. (6) is rather stable against variations (errors) of the wave function.

3 Three-Body Problem

In the case of three interacting particles, the Schrödinger equation formally looks exactly the same as Eq. (1), but now the operator V is the sum of three two-body potentials,

$$V = V_{12} + V_{13} + V_{23} \ , \tag{21}$$

where the subscripts are the particle labels. Let the potentials V_{13} and V_{23} be known while the potential acting between the particles 1 and 2, unknown. It is convenient to re-write the Schrödinger equation in the form,

$$V_{12}|\psi\rangle = (E - V_{13} - V_{23} - H_0)|\psi\rangle \ , \tag{22}$$

where the unknown potential is separated.

Let $|\vec{r}, \vec{\rho}\rangle$ be the state with definite values of the Jacobi coordinates shown in Fig. 3. Multiplying Eq. (22) by $\langle\vec{r}, \vec{\rho}|$ from the left and assuming that all the potentials are local and spherically symmetric, we obtain

$$V_{12}(r)\psi(\vec{r}, \vec{\rho}) = \left[E - V_{13}(r_{13}) - V_{23}(r_{23}) + \frac{\hbar^2}{2\mu_r}\Delta_{\vec{r}} + \frac{\hbar^2}{2\mu_\rho}\Delta_{\vec{\rho}}\right]\psi(\vec{r}, \vec{\rho}) \ , \tag{23}$$

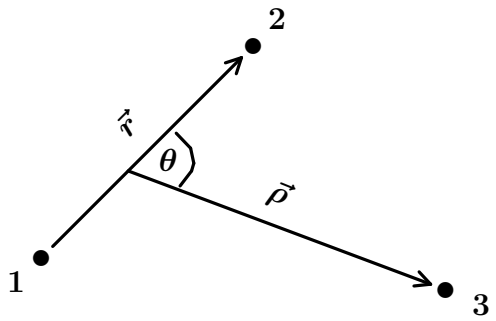

Figure 3: Jacobi coordinates ($\vec{r}$, $\vec{\rho}$) for the three particles 1, 2, and 3.

where $\psi(\vec{r},\vec{\rho}) = \langle \vec{r},\vec{\rho}|\psi\rangle$ is the wave function of the three-body bound state, μ_r and μ_ρ are the reduced masses associated with the motion along the corresponding Jacobi coordinates, and the radial variables r_{13} and r_{23},

$$r_{13} = \left|\frac{m_2}{m_1+m_2}\vec{r}+\vec{\rho}\right| = \sqrt{\left(\frac{m_2}{m_1+m_2}\right)^2 r^2+\rho^2+\frac{m_2}{m_1+m_2}r\rho\cos\theta}\ , \tag{24}$$

$$r_{23} = \left|\frac{m_1}{m_1+m_2}\vec{r}-\vec{\rho}\right| = \sqrt{\left(\frac{m_1}{m_1+m_2}\right)^2 r^2+\rho^2-\frac{m_1}{m_1+m_2}r\rho\cos\theta}\ , \tag{25}$$

are the inter-particle distances in the particle pairs (1,3) and (2,3). Here m_1 and m_2 are the particle masses and θ is the angle between vectors $\vec{r}$ and $\vec{\rho}$.

Perhaps it should be reiterated once more that the method we propose here is intended for just a very rough estimating of the unknown potential that describes the interaction between the particles 1 and 2. When doing this we know exactly the energy, E, of the bound state but need to guess (i.e. to construct "by hand") the corresponding wave function, $\psi(\vec{r},\vec{\rho})$. Of course we can hope to make a reasonable guess only for the ground state and only for the main component of such a function. Similarly to Eq. (3), we therefore take it as

$$\psi(\vec{r},\vec{\rho}) \approx R(r,\rho)Y_{00}(\hat{\vec{r}})Y_{00}(\hat{\vec{\rho}})\chi\ , \tag{26}$$

where it is assumed that all orbital angular momenta are zero and the total angular momentum, J, is built from the particle spins,

$$\chi = |((s_1s_2)s_{12}s_3)JJ^z\rangle\ . \tag{27}$$

In order to leave in Eq. (23) only one "free" variable, r, which the potential V_{12} depends on, we multiply this equation by $\psi^\dagger(\vec{r},\vec{\rho})$ from the left and integrate over $\vec{\rho}$ as well as over the

spherical angles $\hat{\vec{r}}$. When integrating over $\hat{\vec{\rho}}$, we choose the z-component of $\vec{\rho}$ along vector $\vec{r}$. This makes the polar angle of $\vec{\rho}$ coinciding with θ. As a result, we obtain:

$$V_{12}(r) = E - \frac{1}{2D(r)} \int_0^\infty d\rho \int_0^\pi d\theta\, \rho^2 \sin(\theta) R^2(r,\rho) \big[\langle V_{13}(r_{13}) \rangle + \langle V_{23}(r_{23}) \rangle \big] +$$

$$+ \frac{\hbar^2}{2D(r)} \int_0^\infty d\rho\, \rho^2 R(r,\rho) \left[\frac{1}{\mu_r r^2} \partial_r (r^2 \partial_r) + \frac{1}{\mu_\rho \rho^2} \partial_\rho (\rho^2 \partial_\rho) \right] R(r,\rho) \ , \qquad (28)$$

where

$$D(r) = \int_0^\infty R^2(r,\rho) \rho^2 \, d\rho \qquad (29)$$

is the two-body density of the particles 1 and 2; the symbols $\langle V_{13} \rangle$ and $\langle V_{23} \rangle$ denote the averages of the corresponding potentials in the spin space, i.e.

$$\langle V_{13} \rangle = \chi^\dagger V_{13} \chi \qquad \text{and} \qquad \langle V_{23} \rangle = \chi^\dagger V_{23} \chi \ . \qquad (30)$$

For calculating these averages, it is convenient to represent the potentials in terms of the operators $P(s_{ij})$ projecting onto the states with certain values of the two-body spin, s_{ij}, of the particles i and j,

$$V_{13}(r_{13}) = \sum_{s_{13}} \mathscr{V}_{13}^{[s_{13}]}(r_{13}) P(s_{13}) \qquad \text{and} \qquad V_{23}(r_{23}) = \sum_{s_{23}} \mathscr{V}_{23}^{[s_{23}]}(r_{23}) P(s_{23}) \ . \qquad (31)$$

Writing the projection operators in the form

$$P(s_{13}) = |((s_1 s_3) s_{13} s_2) J J^z\rangle \, \langle ((s_1 s_3) s_{13} s_2) J J^z| \qquad (32)$$

and

$$P(s_{23}) = |((s_2 s_3) s_{23} s_1) J J^z\rangle \, \langle ((s_2 s_3) s_{23} s_1) J J^z| \ , \qquad (33)$$

we can express the spin-averaging via the $6j$-symbols (see Ref. [7]) as follows:

$$\langle V_{13}(r_{13}) \rangle = \sum_{s_{13}} \mathscr{V}_{13}^{[s_{13}]}(r_{13}) (2s_{12}+1)(2s_{13}+1) \left\{ \begin{array}{ccc} s_3 & s_1 & s_{13} \\ s_2 & J & s_{12} \end{array} \right\}^2 \ , \qquad (34)$$

$$\langle V_{23}(r_{23}) \rangle = \sum_{s_{23}} \mathscr{V}_{23}^{[s_{23}]}(r_{23}) (2s_{12}+1)(2s_{23}+1) \left\{ \begin{array}{ccc} s_3 & s_2 & s_{23} \\ s_1 & J & s_{12} \end{array} \right\}^2 \ . \qquad (35)$$

As is seen the square of the radial wave function is present in both the numerator and denominator of Eq. (28) and therefore it is not necessary to use it with the proper normalization.

4 Test: recovering nn-potential from the triton-data

In order to test how accurately an unknown potential can be recovered with the help of the procedure described in the preceding section, we consider the three-body bound system nnp (triton), where all the potentials are actually known. However, we pretend that the neutron-neutron potential is unknown and try to recover it using Eq. (28) and the properties of triton.

Referring to Fig. 3, the proton is particle number 3 and the neutrons are particles 1 and 2. Neutron can interact with proton either in the singlet or triplet spin-state via the potentials $\mathscr{V}^s_{NN}$ and $\mathscr{V}^t_{NN}$, respectively. Due to the Pauli principle, the nn sytem with $\ell = 0$ can only be in the singlet state. Therefore, if our method works, then as a result of the recovering we should obtain the singlet NN-potential for the "unknown" nn-interaction.

In the sums (34, 35), we have $s_{12} = 0$ (nn-spin) and $J = 1/2$ (triton spin). The summation involves only two terms and gives:

$$\langle V_{13}(r_{13})\rangle = \frac{1}{4}\mathscr{V}^s_{NN}(r_{13}) + \frac{3}{4}\mathscr{V}^t_{NN}(r_{13}) \ , \tag{36}$$

$$\langle V_{23}(r_{23})\rangle = \frac{1}{4}\mathscr{V}^s_{NN}(r_{23}) + \frac{3}{4}\mathscr{V}^t_{NN}(r_{23}) \ . \tag{37}$$

As the triplet NN-potential, $\mathscr{V}^t_{NN}$, we take the Malflied-Tjon one given by Eq. (10) with the parameters listed just after that equation. The singlet potential, $\mathscr{V}^s_{NN}$, has exactly the same functional form and almost all the same parameters except the first one, namely, $\lambda_A = 513.968\,$MeV for the singlet state [4].

The main question in the proposed procedure of the unknown potential recovering is how to make a reasonable guess of the three-body wave function. We assume that we can only know the binding energy and perhaps (but not for sure and not always) the RMS-radius of the three-body state.

The knowledge of the energy gives us the asymptotic behaviour of the wave function at large distances. Indeed, it is known (see, for example, Ref. [8]) that the main contribution to the asymptotics of a three-body bound-state wave function comes from the out-going hyper-spherical wave, namely,

$$R(r,\rho) \underset{\zeta\to\infty}{\longrightarrow} \sim \frac{e^{-\varkappa\zeta}}{\zeta^{5/2}} \ , \tag{38}$$

where ζ is the hyperradius,

$$\zeta = \sqrt{\frac{\mu_{12}}{\mu}r^2 + \rho^2} \ , \tag{39}$$

and $\varkappa$ is the imaginary part of the momentum corresponding to the negative energy of the

bound state,

$$\varkappa = \sqrt{\frac{2\mu|E|}{\hbar^2}} \ . \tag{40}$$

It is clear that the hyper-spherical wave (38) cannot be used as the function R at all distances. Indeed, the right hand side of Eq. (38) becomes singular when $\zeta \to 0$, which is unphysical. This problem can be circumvented by introducing a positive parameter α in the denominator,

$$R(r,\rho) \ \sim \frac{e^{-\varkappa\zeta}}{\alpha + \zeta^{5/2}} \ , \tag{41}$$

which removes the singularity and keeps the behaviour (38) when $\alpha \ll \zeta^{5/2}$.

The expression (41) is still not a satisfactory choice even as a very rough approximation of the wave function because it is monotonically decreasing while a realistic wave function should have a maximum somewhere at the distances corresponding to the size (RMS-radius) of the system. Such a maximum can appear if we introduce a factor suppressing the probability of finding the particles at short distances. The physical reason for such a factor is the strong repulsion of the particles at distances ~ 1 fm (the hard or soft core in the two-body potentials). In nuclear theory such factors are known as Jastrow factors (see, for example, chapter 3 of the book [9]).

Of course the Jastrow factor can be constructed in many different ways. We found that for our problem of triton a good choice is the following (please take note that this function is not normalized):

$$R(r,\rho) = \frac{1}{r}\arctan\left(\gamma r^{3/2}\right)\frac{e^{-\varkappa\zeta}}{\alpha + \zeta^{5/2}} \ . \tag{42}$$

With the parameters $\gamma = 0.81\,\text{fm}^{-3/2}$ and $\alpha = 29.0\,\text{fm}^{5/2}$ this wave function gives the RMS-radius of triton $1.745\,$fm, which is within the uncertainty interval of its experimental value $(1.7591 \pm 0.0363)\,$fm [10]. As the binding energy of triton, we used the experimental value, $|E| = 8.481798\,$MeV, [11].

Two things should be explained concerning the approximate wave function (42). Firstly, it does not involve explicit Jastrow factors that would suppress approaching of the first and the second neutrons to the proton. In fact, such factors are not needed because the motion along both Jacobi coordinates is in the S-wave states. Thefore the proton can approach any of the two neutrons only at the centre of mass where all three particles meet. Such a configuration is already suppressed by the factor depending on r, and thus no additional Jastrow factors are needed. Secondly, the function (42) is symmetric only with respect to the permutations of the neutrons. This fact might be seen as violation of the Pauli principle. However, protons and neutrons, in fact, are not identical. The isotopic invariance is an approximate symmetry. We therefore are not obliged to construct a completely symmetric wave function.

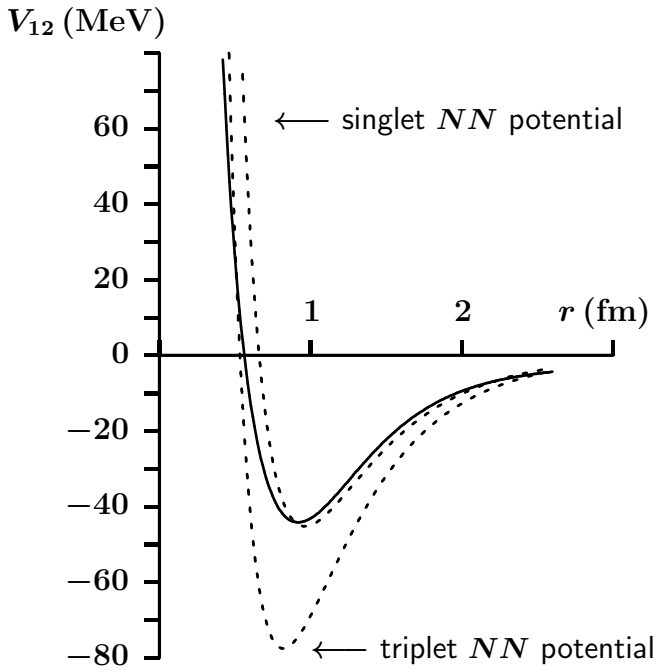

Figure 4: The neutron-neutron potential (solid curve) numerically recovered using Eq. (28) with the approximate (guessed) wave function (42) of triton and with its experimentally known ground-state energy $E = -8.481798\,\mathrm{MeV}$. For the sake of comparison, the singlet and triplet Malfliet-Tjon NN-potentials are shown by the upper and lower dashed curves, respectively.

Eq. (28) involves the first and second derivatives with respect to the radial variables r and ρ. In principle, all necessary differentiations of the function (42) can be done explicitly. However, to avoid cumbersome derivations and to reserve the possibility of changing the functional form of the wave function, we calculated all the derivatives numerically using simple finite difference formulae,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}\ , \qquad f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}\ , \tag{43}$$

which give stable results with the step $h = 10^{-5}$ fm.

The neutron-neutron potential obtained in this way using Eq. (28) is shown in Fig. 4 by the solid curve. For the sake of comparison the singlet and triplet Malfliet-Tjon NN-potentials are shown in that figure by the dashed curves. As is seen, the recovered potential is almost correct, i.e. it almost reproduces the singlet potential. This successful test allows us to hope that the proposed method can indeed give a reliable estimate of a truly unknown potential

in any other physical system even if its wave function is constructed on the basis of general reasoning and intuition.

5 Recovering the $\Lambda\Lambda$-potential

Due to the obvious reasons it is not possible to study the $\Lambda\Lambda$-interaction in the direct collisions of these two particles. The only experimental information from which one can deduce some general features of the forces acting between them, is the information on the double-Λ hypernuclei. Therefore various $\Lambda\Lambda$-potentials can only be constructed on the basis of either the quark or boson-exchange theories. Validity of such potentials is tested in the calculations of the properties of various double hypernuclei whose characteristics are known experimentally.

Examples of the soft-core boson-exchange $\Lambda\Lambda$-potentials can be found in Ref. [12]. They are known as the Nijmegen Soft-Core (NSC97) potentials. Since there are some ambiguities in the constructing of them, several versions of such potentials are available. For the purposes of the few-body calculations these potentials are usually simulated in the coordinate representation by a simple functional form, $V_{\Lambda\Lambda}(r)$, (see, for example, Refs. [13–15]) with the parameters adjusted to make $V_{\Lambda\Lambda}(r)$ either phase-equivalent to the corresponding Nijmegen potentials or to exactly reproduce the measured binding energy of the hypernucleus ${}^{6}_{\Lambda\Lambda}\mathrm{He}$. The observation of this hypernucleus was so important that nowadays it is known as the NAGARA event [16].

The method we are developing in the present work may also make a contribution to the constructing of such a $V_{\Lambda\Lambda}(r)$. Of course we do not expect to propose a very reliable $\Lambda\Lambda$-potential. As it was emphasized from the very beginning, our method can only deduce some general features of a potential, such as, for example, its approximate depth, range, etc. However, in the case of the $\Lambda\Lambda$-interaction even this limited information can be helpful in choosing the most adequate one among many available potentials.

Following the same line of reasoning as in Sec. 4, we construct the following (not normalized) wave function of the hypernucleus ${}^{6}_{\Lambda\Lambda}\mathrm{He}$ in the three-body model $\Lambda\Lambda\alpha$:

$$R(r,\rho) = \arctan\left(\gamma r^{5/2}\right) \frac{e^{-\varkappa\zeta}}{\alpha+\zeta^{5/2}} \ . \tag{44}$$

Here r and ρ are the Jacobi distances shown in Fig. 3, where ${}^{4}\mathrm{He}$ is the particle number 3, and ζ is the hyperradius (39). The momentum parameter (40) corresponds to the binding energy of the three-body system $(\Lambda\Lambda\alpha)$, $|E| = 7.25\,\mathrm{MeV}$ determined in the NAGARA experiment [16]. The other two parameters, $\alpha = 35\,\mathrm{fm}^{5/2}$ and $\gamma = 2.5\,\mathrm{fm}^{-5/2}$, were chosen such that the function (44) gives the following geometric sizes:

$$\sqrt{\langle r^2\rangle} = 3.80\,\mathrm{fm} \ , \qquad \sqrt{\langle \rho^2\rangle} = 2.35\,\mathrm{fm} \ . \tag{45}$$

These RMS-distances are within the corresponding intervals obtained in Refs. [14, 17] for various $\Lambda\Lambda$-potentials, namely,

$$3.09\,\mathrm{fm} \leqslant \sqrt{\langle r^2\rangle} \leqslant 4.09\,\mathrm{fm}\ , \qquad 2.11\,\mathrm{fm} \leqslant \sqrt{\langle \rho^2\rangle} \leqslant 2.35\,\mathrm{fm}\ . \tag{46}$$

We assume that the $\Lambda\alpha$ potential is known. We take it the same as in Ref. [14], namely, as

$$V_{\Lambda\alpha}(r) = \Lambda_1 \exp\left(-r^2/\beta_1^2\right) - \Lambda_2 \exp\left(-r^2/\beta_2^2\right)\ , \tag{47}$$

where $\Lambda_1 = 450.4\,\mathrm{MeV}$, $\beta_1 = 1.25\,\mathrm{fm}$, $\Lambda_2 = 404.9\,\mathrm{MeV}$, $\beta_2 = 1.41\,\mathrm{fm}$. Since the spin of the α-particle is zero, there is only one spin state in the $\Lambda\alpha$-system and therefore the spin-averaging prescribed by Eqs. (34, 35) is trivial, i.e.

$$\langle V_{13}(r_{13})\rangle = V_{\Lambda\alpha}(r_{13})\ , \qquad \langle V_{23}(r_{23})\rangle = V_{\Lambda\alpha}(r_{23})\ .$$

The $\Lambda\Lambda$-subsystem in ${}^{\ 6}_{\Lambda\Lambda}\mathrm{He}$ is assumed to be in the S-wave state and therefore in the singlet spin state. This means that we can only obtain the information on the ${}^1S_0$ potential $V_{\Lambda\Lambda}$.

The $\Lambda\Lambda$-potential recovered from the wave function (44) is shown in Fig. 5 by the solid curve. For the sake of comparison, the dashed curves in the same figure show three different Nijmegen $\Lambda\Lambda$-potentials (which were taken from Ref. [14]) and the corresponding potential (the uppermost dashed curve) from Ref. [15] where it was adjusted in order to reproduce the NAGARA binding energy of ${}^{\ 6}_{\Lambda\Lambda}\mathrm{He}$ and the $\Lambda\Lambda$ bonding energy for this hypernucleus $\Delta B_{\Lambda\Lambda}(A = 6) = 1.01\,\mathrm{MeV}$.

As is seen, our simple estimate supports weaker $\Lambda\Lambda$ attraction of the potential NSC97e as well as of the one used by Hiyama et al. in Ref. [15]. The other two Nijmegen potentials, namely, ND and ESC00, seem incompartible with the parameters of the ${}^{\ 6}_{\Lambda\Lambda}\mathrm{He}$ hypernucleus.

6 Conclusion

In the present paper, we suggest a simple and efficient method for relating a given wave function of a three-body bound state to the corresponding two-body potential acting in a chosen pair of particles, while the potentials in the other two pairs are assumed to be given. The most interesting feature of the presented method is that (thanks to the formal construction) any given wave function becomes an exact solution of the three-body Schrödinger equation for a given binding energy and with the corresponding (numerically obtained) two-body potential in a chosen pair of particles. The method works for a two-body bound state as well. The accuracy of the method is demonstrated using the examples of the bound systems ep, np, nnp, and $\Lambda\Lambda\alpha$.

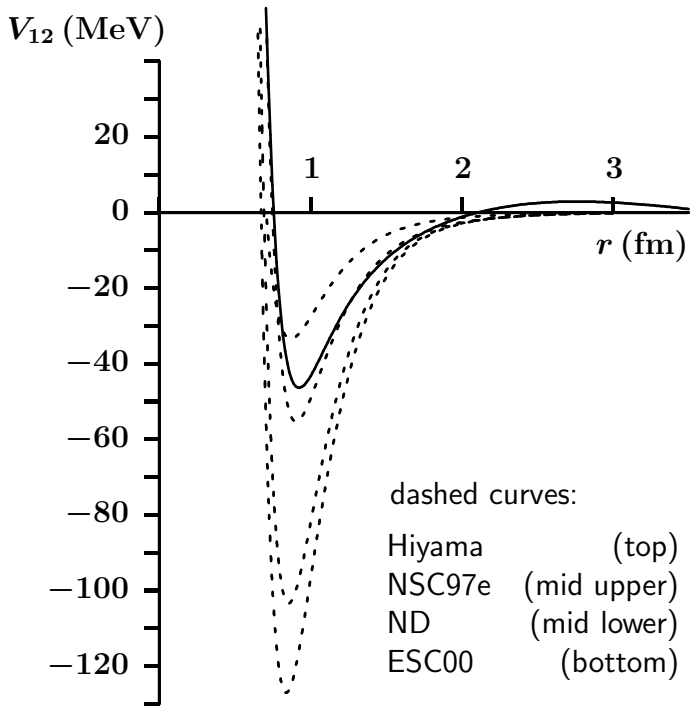

Figure 5: The singlet $\Lambda\Lambda$-potential (solid curve) numerically recovered using Eq. (28) with the approximate (guessed) wave function (44) of the hypernucleus ${}^{6}_{\Lambda\Lambda}\mathrm{He}$ and with its experimentally known ground-state energy $E = -7.25\,\mathrm{MeV}$. For the sake of comparison, three versions of the Nijmegen soft-core one-boson-exchange $\Lambda\Lambda$-potential, namely, NSC97e, ND, and ESC00 as well as the potential used by Hiyama et al. in Ref. [15] are shown by the dashed curves.

Two possible applications of the presented method are envisaged. Firstly, we can deduce an unknown two-body potential from an approximate (guessed) wave function. This may be helpful when very little is known about such a potential and there is no possibility of doing the scattering experiments. Secondly, if the two-body potential is known, we can construct the corresponding three-body wave function. In doing this we can postulate a wave function in a reasonable functional form with some free parameters and then can fix these parameters by minimizing the deviations of the deduced potential from the known (exact) one. The wave function obtained in this way is always a solution of the Schrödinger equation with a given binding energy.

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