ON SOME PRODUCTS OF FINITE GROUPS

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Abstract A classical result of Baer states that a finite group G which is the product of two normal supersoluble subgroups is supersoluble if and only if G' is nilpotent. In this article, we show that if $G = AB$ is the product of supersoluble (respectively, w-supersoluble) subgroups A and B, A is normal in G and B permutes with every maximal subgroup of each Sylow subgroup of A, then G is supersoluble (respectively, w-supersoluble), provided that G' is nilpotent. We also investigate products of subgroups defined above when $A \cap B = 1$ and obtain more general results.

1. Introduction

All groups considered here will be finite.

A significant number of articles investigating the properties of groups expressible as a product of two supersoluble subgroups were published since the 1957 paper by Baer [2] in which he proved that a normal product $G = AB$ of two supersoluble subgroups

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A and B is supersoluble provided that the derived subgroup G' is nilpotent. There has been many generalizations of this theorem. Instead of having normal subgroups, certain permutability conditions were imposed on the factors. The case in which A permutes with every subgroup of B and B permutes with every subgroup of A, that is, when G is a *mutually permutable product* of A and B, is in fact one of the most interesting cases and has been investigated in detail (see [5] for a thorough review of results in this context and also [1] for general results on products).

In this article, we study a weak form of a normal product arising quite frequently in the structural study of mutually permutable products and appears not to have been investigated in detail.

Definition 1.1. *Let $G = AB$ be a product of subgroups A and B. We say that G is a weak normal product of A and B if*

(a) *A is normal in G.*

(b) *B permutes with all the maximal subgroups of Sylow subgroups of A.*

As an important first step in the study of weak normal products $G = AB$ and motivated by the mutually permutable case, we analyse the situation $A \cap B = 1$. In this case, they are semidirect products of A and B.

Definition 1.2. *Let the group $G = AB$ be the weak normal product of A and B with A normal in G. We say that G is a weak direct product of A and B if $A \cap B = 1$. In this case, we write $G = [A]B$.*

We study these products when the factors are supersoluble and widely supersoluble and analyse the behaviour of the residuals associated to these classes of groups. Recall that a widely supersoluble group, or *w-supersoluble* group for short, is defined as a group G such that every Sylow subgroup of G is $\mathbb{P}$-subnormal in G (a subgroup H of a group G is $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G$, such that $|H_i{:}H_{i-1}|$ is a prime for every $i = 1, \ldots, n$).

The class of w-supersoluble groups, denoted $w\mathfrak{U}$, is a subgroup-closed saturated formation containing the subgroup-closed saturated formation $\mathfrak{U}$ of all supersoluble groups. Moreover w-supersoluble groups have a Sylow tower of supersoluble type (see [8, Corollary]).

Our first aim is to show that the saturated formations of all supersoluble groups and w-supersoluble groups are closed under the formation of weak direct products.

Theorem A. *Let $G = [A]B$ be a weak direct product of A and B. If A and B belong to $\mathfrak{U}$, then G is also supersoluble.*

Theorem A will be very useful in the proofs of Theorem B and Theorem C. Also we obtain the following result as a corollary.

Corollary A. *Let $G = [A]B$ be a weak direct product of A and B. If A and B belong to $w\mathfrak{U}$, then G is w-supersoluble.*

Our second aim is to show that the product of the supersoluble (respectively, w-supersoluble) residuals of the factors of weak direct products is just the supersoluble (respectively, w-supersoluble) residual of the group.

Theorem B. *Suppose that $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F} = w\mathfrak{U}$. Let $G = [A]B$ be a weak direct product of A and B. Then*

$$G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}.$$

We now analyse the behaviour of weak normal products with respect to the formations of all supersoluble and w-supersoluble groups. Our next result shows that Baer's theorem can be generalized in this new direction:

Theorem C. *Let $G = AB$ be a weak normal product of A and B. If G' is nilpotent, A is normal in G and $A, B \in \mathfrak{U}$, then $G \in \mathfrak{U}$.*

As a corollary, we obtain the result for $w\mathfrak{U}$-groups.

Corollary B. *Let the group $G = AB$ be a weak normal product of $w\mathfrak{U}$-subgroups A and B. If G' is nilpotent and A is normal in G, then G belongs to $w\mathfrak{U}$.*

Our second objective is to investigate the residuals of weak normal products. Unfortunately, it does not follow that $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$ when G is a weak normal product as the following examples show. Example 1.3(ii) generalizing (i) was communicated to the authors by the referee:

Example 1.3.

(i) Let

$$
\begin{aligned}
A = \langle g_2, g_4, g_5, g_6, g_7 \mid\ & g_2^3 = g_4^3 = g_5^3 = g_6^3 = g_7^3 = 1, \\
& g_4^{g_2} = g_4 g_6, g_5^{g_2} = g_5 g_7, g_6^{g_2} = g_6, g_7^{g_2} = g_7, \\
& g_5^{g_4} = g_5, g_6^{g_4} = g_6, g_7^{g_4} = g_7, \\
& g_6^{g_5} = g_6, g_7^{g_5} = g_7, \\
& g_7^{g_6} = g_7 \rangle.
\end{aligned}
$$

Let $Q = \langle b \rangle \cong C_4$ act on A via

$$g_2^b = g_2, \qquad g_4^b = g_4 g_5, \qquad g_5^b = g_4 g_5^2, \qquad g_6^b = g_6 g_7, \qquad g_7^b = g_6 g_7^2.$$

Let $G = [A]Q$ be the corresponding semidirect product.

Note that $A' = \Phi(A) = \langle g_6, g_7 \rangle$. Let $A_0 = \langle g_4, g_5 \rangle$. Then A_0 is not a normal subgroup of A but is normalized by Q. Let $B = A_0\langle b \rangle$, then $\mathrm{Core}_G(B) = 1$. Furthermore, B permutes with the 13 maximal subgroups of A. The supersoluble residual of G is $\langle g_4, g_5, g_6, g_7 \rangle$, giving a quotient isomorphic to C_{12}. Consequently, $G^{\mathfrak{U}} \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. This group corresponds to `SmallGroup(972, 406)` of GAP.

(ii) Suppose p is a prime number and n is a positive integer, where n is not a multiple of p, and the order of p modulo n is 2. Let $\mathbf{F}_{p^2}$ be the Galois field of order p^2, and note that n is a factor of $p^2 - 1$, so there is an element β of multiplicative order n in $\mathbf{F}_{p^2}$. Let $V = U_0 \oplus U_1$ be a vector space of dimension 2 over $\mathbf{F}_{p^2}$, where U_0 and U_1 are one-dimensional $\mathbf{F}_{p^2}$-subspaces of V. Take elements a and b in the general linear group $\mathrm{GL}_2(p^2)$, with

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}.$$

Then V can be regarded as an $\mathbf{F}_{p^2}\langle a, b\rangle$-module. Let $G = [V]\langle a, b\rangle$ be the corresponding semidirect product. Consider the following subgroups of G:

$$P = \langle a \rangle \simeq \mathbf{C}_p, \quad A = VP, \quad Q = \langle b \rangle \simeq \mathbf{C}_n, \quad B = U_0 Q.$$

As in (i), A is normal in $AB = G$, $A' = \Phi(A) = [V, P] = U_1$, U_0 is an $\mathbf{F}_p Q$-simple module, $B' = [U_0, Q] = U_0$, the core of B in G is 1, the number of maximal subgroups of A is $(p^3 - 1)/(p - 1) = p^2 + p + 1$, A is the unique Sylow p-subgroup of G and all the maximal subgroups of A permute with B, while the supersoluble residual of G is $G^{\mathfrak{U}} = V$, with $G/G^{\mathfrak{U}} \simeq PQ \simeq \mathbf{C}_{pn}$ and $G^{\mathfrak{U}} \neq A^{\mathfrak{U}} B^{\mathfrak{U}}$. This construction can be carried out when $p = 2$ and $n = 3$, giving an Example with $|G| = 2^5 3 = 96$; moreover, there is a maximal subgroup of A which does not permute with the Sylow subgroup Q of B.

We will use Theorem C to prove the following result (note that Example 1.3(ii) suggests the permutability hypothesis in Theorem D and Corollary D):

Theorem D. *Let the group $G = AB$ be a product of the subgroups A and B. Assume that A is a normal subgroup of G and every Sylow subgroup of B permutes with every maximal subgroup of every Sylow subgroup of A. If G' is nilpotent, then $G^{\mathfrak{U}} = A^{\mathfrak{U}} B^{\mathfrak{U}}$.*

An immediate consequence is:

Corollary C. *Let the group $G = AB$ be a product of the subgroups A and B. Assume that A is a normal subgroup of G and every Sylow subgroup of B permutes with every maximal subgroup of every Sylow subgroup of A. If G' is nilpotent, then $G^{w\mathfrak{U}} = A^{w\mathfrak{U}} B^{w\mathfrak{U}}$.*

Denote by $\mathfrak{N}$ the class of all nilpotent groups. A nice result of Monakhov [7, Theorem 1] states that if $G = AB$ is the mutually permutable product of the supersoluble subgroups A and B, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A, B]^{\mathfrak{N}}$. We prove an analogue of this result for weak normal products.

Corollary D. *Let $G = AB$ be a weak normal product of the supersoluble subgroups A and B. If A is normal in G, we have that $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A, B]^{\mathfrak{N}}$.*

2. Preliminary results

It is easy to see that factor groups of weak normal products are also weak normal products. For weak direct products, we have the following:

Lemma 2.1. *Let $G = [A]B$ be a weak direct product of A and B.*

    (a) *If N is a normal subgroup of G such that $N \leqslant A$ or $N \leqslant B$, then $G/N = [AN/N](BN/N)$ is a weak direct product of AN/N and BN/N.*

    (b) *If K is a subgroup of B, then $[A]K$ is a weak direct product of A and K.*

Proof. (a) Let H/N be a Sylow p-subgroup of AN/N. Then $H/N = PN/N$, where P is a Sylow p-subgroup of A. Let K/N be a maximal subgroup of H/N. Then $K = K \cap PN = N(P \cap K)$ and $K/N = N(P \cap K)/N$. Thus,

$$p = |PN/N : (P \cap K)N/N| = \frac{|P||N|}{|P \cap N|} \frac{|P \cap K \cap N|}{|P \cap K||N|} = |P : P \cap K|.$$

Hence, $P \cap K$ is a maximal subgroup of P. Then B permutes with $P \cap K$, and so BN/N permutes with K/N. Therefore, $G/N = [AN/N](BN/N)$ is a weak direct product of AN/N and BN/N.

(b) Let K be any proper subgroup of B and H be any maximal subgroup of a Sylow subgroup of A. By the hypotheses, we have $HB = BH$ and so $H = H(A \cap B) = A \cap HB$. Since A is normal in G, it follows that H is normal in HB and so B normalizes H. Hence, K permutes with H. Therefore, $[A]K$ is a weak direct product of A and K.   □

Our second lemma contains some of the properties of $\mathbb{P}$-subnormal subgroups.

Lemma 2.2. [8, Lemma 1.4] *Let G be a soluble group and H and K two subgroups of G. The following properties hold:*

    (i) *If H is $\mathbb{P}$-subnormal in G and N is normal in G, then HN/N is $\mathbb{P}$-subnormal in G/N.*

    (ii) *If N is normal in G and HN/N is $\mathbb{P}$-subnormal in G/N, then HN is $\mathbb{P}$-subnormal in G.*

    (iii) *If H is $\mathbb{P}$-subnormal in K and K is $\mathbb{P}$-subnormal in G, then H is $\mathbb{P}$-subnormal in G.*

3. Supersoluble and w-supersoluble residuals

We start this section by proving Theorem A.

Proof of Theorem A. Assume that the result is false, and let G be a counterexample of minimal order. Clearly, G is soluble and A and B are proper subgroups of G. Let N be a minimal normal subgroup of G contained in A, then applying Lemma 2.1(a),

$G/N = [A/N](BN/N)$ is a weak direct product of A/N and BN/N. By the minimality of G, $G/N \in \mathfrak{U}$. Since the class of all supersoluble groups is a saturated formation, there exists a unique minimal normal subgroup N of G contained in A, N a p-group for some prime p, $|N| > p$ and $\Phi(A) = 1$. Since A is supersoluble, A has a normal Sylow subgroup, and since N is the unique minimal normal subgroup of G contained in A, it follows that $\mathbf{F}(A)$ is a p-group and $\mathbf{F}(A)$ is an elementary abelian Sylow p-subgroup of A.

Assume that A is not a p-group. Then $\mathbf{F}(A)$ is a completely reducible A-module, and so $\mathbf{F}(A) = N \times Z$, for some A-module Z. Let L be a minimal normal subgroup of A contained in N. Then $N = L \times D$, for some A-module D. Then $\mathbf{F}(A) = L \times DZ$, and DZ is a maximal subgroup of $\mathbf{F}(A)$ because L is of prime order. Therefore, $E = DZ$ permutes with B. Hence, $DZ = A \cap (DZ)B$, and so DZ is normalized by B. Since DZ is also normalized by A, it follows that DZ is a normal subgroup of G. The minimality of N forces $D = 1$ and so N is of prime order, which is a contradiction. Consequently, A is an elementary abelian p-group. Note that A cannot be cyclic since $|N| > p$. Let $1 \neq X$ be a maximal subgroup of A. Arguing as above, we have that X is normal in XB so that X is normalised by B. Hence X is normal in G because A is abelian. Therefore, N is contained in X, and so $N \leq \Phi(A) = 1$, our final contradiction. $\qquad\square$

Proof of Corollary A. Assume, by way of contradiction, that the result fails, and let G be a counterexample of least order. Clearly, G is soluble and A and B are proper subgroups of G. Since the class of all w-supersoluble groups is a saturated formation, we can argue as in Theorem A to conclude that there exists a unique minimal normal subgroup N of G contained in A, and N is a p-group for some prime p. Moreover, $\Phi(A) = 1$, and $A_p = \mathbf{F}(A)$ is the Sylow p-subgroup of A. By the minimality of G, G/N is w-supersoluble. Let P be a Sylow p-subgroup of G. Then P/N is $\mathbb{P}$-subnormal in G/N. By Lemma 2.2(ii), P is $\mathbb{P}$-subnormal in G. Suppose that for every prime $q \neq p$ dividing $|G|$ and every Sylow q-subgroup B_q of B, we have that AB_q is a proper subgroup of G. Let A_q be a Sylow q-subgroup of A such that $G_q = A_q B_q$ is a Sylow q-subgroup of G. Since G/N is w-supersoluble, it follows that $G_q N$ is $\mathbb{P}$-subnormal in G. By Lemma 2.1(b), AB_q satisfies the hypotheses of the theorem. Hence, AB_q is w-supersoluble by the choice of G. Thus, $G_q N \leq AB_q$ is w-supersoluble. Consequently, G_q is $\mathbb{P}$-subnormal in $G_q N$, which is $\mathbb{P}$-subnormal in G. Applying Lemma 2.2(iii), G_q is $\mathbb{P}$-subnormal in G. Therefore, the Sylow subgroups of G are $\mathbb{P}$-subnormal in G, and so G is w-supersoluble, a contradiction. Thus, we may assume there exists $q \neq p$ such that $G = AB_q$. Let $T = A_p G_q = (A_p A_q) B_q$. Since A is normal in G, we have that A_q is normal in G_q and then $A_p A_q$ is normalized by B_q. Moreover, $A_p A_q$ is a w-supersoluble metanilpotent subgroup of G. By [8, Theorem 2.13(1)], $A_p A_q$ is supersoluble. It is clear that T is a weak direct product of the supersoluble subgroups $A_p A_q$ and B_q. Applying Theorem A, it follows that T is supersoluble. Therefore, T is w-supersoluble. But $G_q N \leq T$, which is w-supersoluble. Thus, G_q is $\mathbb{P}$-subnormal in $G_q N$, which is $\mathbb{P}$-subnormal in G. Again the application of Lemma 2.2(iii) yields G_q is $\mathbb{P}$-subnormal in G. If G_r is a Sylow r-subgroup of G for some prime $r \neq p, q$, then G_r is contained in A and so G_r is $\mathbb{P}$-subnormal in A. Since A is also $\mathbb{P}$-subnormal in G, we have that G_r is $\mathbb{P}$-subnormal in G. Consequently, every Sylow subgroup of G is $\mathbb{P}$-subnormal in G, and G is w-supersoluble. This final contradiction completes the proof of the corollary. $\qquad\square$

Proof of Theorem B. Suppose that the result is not true, and let G be a minimal counterexample. Then

(i) $A \in \mathfrak{F}$, $B^{\mathfrak{F}} \neq 1$, $\mathrm{Core}_G(B) = 1$ and $G^{\mathfrak{F}} = B^{\mathfrak{F}} N$ for every minimal normal subgroup N of G such that $N \leqslant A$.

Let N be a minimal normal subgroup of G such that $N \leqslant A$ or $N \leqslant B$. Then $G/N = [AN/N](BN/N)$ is a weak direct product of AN/N and BN/N by Lemma 2.1(a). The minimal choice of G implies that $G^{\mathfrak{F}} N/N = (A^{\mathfrak{F}} N/N)(B^{\mathfrak{F}} N/N)$, that is, $G^{\mathfrak{F}} N = A^{\mathfrak{F}} B^{\mathfrak{F}} N$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$, $AG^{\mathfrak{F}}/G^{\mathfrak{F}}$ and $BG^{\mathfrak{F}}/G^{\mathfrak{F}}$ also belong to $\mathfrak{F}$ and then $A^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$ and $B^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. If $G^{\mathfrak{F}} \cap N = 1$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}(G^{\mathfrak{F}} \cap N) = A^{\mathfrak{F}} B^{\mathfrak{F}}$, a contradiction. Hence, $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}} N$ for every minimal normal subgroup N of G such that $N \leqslant A$ or $N \leqslant B$. If $A^{\mathfrak{F}} \neq 1$, then there exists a minimal normal subgroup N of G contained in $A^{\mathfrak{F}}$ because $A^{\mathfrak{F}}$ is normal in G. This contradiction yields $A \in \mathfrak{F}$ and $G^{\mathfrak{F}} = B^{\mathfrak{F}} N$ for every minimal normal subgroup N of G such that $N \leqslant A$ or $N \leqslant B$. If $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$ by Theorem A and Corollary A, contrary to the assumption. Hence, $B^{\mathfrak{F}} \neq 1$. Suppose that $\mathrm{Core}_G(B) \neq 1$. Let N be a minimal normal subgroup of G contained in B, and let R be a minimal normal subgroup of G contained in A. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}} N \cap B^{\mathfrak{F}} R \leq B \cap B^{\mathfrak{F}} R = B^{\mathfrak{F}}(B \cap R) = B^{\mathfrak{F}}$, a contradiction. Consequently, we have that $\mathrm{Core}_G(B) = 1$.

(ii) $\mathbf{F}(A)$ is a Sylow p-subgroup of A, where p is the largest prime dividing $|A|$.

Since $A \in \mathfrak{F}$, it follows that A is a Sylow tower group of supersoluble type. In particular, $1 \neq \mathbf{O}_p(A)$ is the Sylow p-subgroup of A, where p is the largest prime dividing $|A|$. If $\mathbf{F}(A)$ is not a p-group, then $1 \neq \mathbf{O}_q(A) \leqslant \mathbf{O}_q(G)$. Let N_1 be a minimal normal subgroup of G contained in $\mathbf{O}_p(A)$, and let N_2 be a minimal normal subgroup of G contained in $\mathbf{O}_q(A)$. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}} N_1 = B^{\mathfrak{F}} N_2$, which is a contradiction since $B^{\mathfrak{F}} \cap N_1 = B^{\mathfrak{F}} \cap N_2 = 1$. Therefore, $\mathbf{F}(A) = \mathbf{O}_p(A)$ is the Sylow p-subgroup of A.

(iii) G is soluble, AK belongs to $\mathfrak{F}$ for every proper subgroup K of B; in particular, B is a minimal non-supersoluble group and $B^{\mathfrak{F}}$ is a q-subgroup of B for some prime q.

Suppose that K is a proper subgroup of B. By Lemma 2.1, AK satisfies the hypotheses of the theorem, and so $(AK)^{\mathfrak{F}} = K^{\mathfrak{F}}$ by the minimal choice of G. Since $(AK^x)^{\mathfrak{F}} = (K^x)^{\mathfrak{F}} = (K^{\mathfrak{F}})^x$ for any $x \in B$, it follows that A normalizes $(K^{\mathfrak{F}})^x$. Thus, A normalizes $\langle (K^{\mathfrak{F}})^x \mid x \in B \rangle$. Then $\langle (K^{\mathfrak{F}})^x \mid x \in B \rangle \lhd G$, contrary to $\mathrm{Core}_G(B) = 1$. Hence, $(K^{\mathfrak{F}})^x = 1$. Consequently, AK belongs to $\mathfrak{F}$. This shows that B is $\mathfrak{F}$-critical, and by [8, Theorem 2.9], we have that B is a minimal non-supersoluble group. By [3, Theorem 10], we have that $B^{\mathfrak{F}}$ is a q-group for some prime q. In particular, B and then G are soluble.

(iv) $G^{\mathfrak{F}} = B^{\mathfrak{F}} \times N$ is an elementary abelian p -group.

Applying (iii), it follows that $B^{\mathfrak{F}}$ is a q-group for some prime q. Let N be a minimal normal subgroup of G contained in A. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}} N$ by (i), and N is a p-group by (ii).

Suppose that $B^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$. Then $G^{\mathfrak{F}}/B^{\mathfrak{F}}$ is an elementary abelian p-group. Consequently, the residual X of $G^{\mathfrak{F}}$ associated to the formation of all elementary abelian p-groups is a normal subgroup of G contained in B. Hence, $X \leq \mathrm{Core}_G(B) = 1$, and $G^{\mathfrak{F}}$ is an elementary abelian p-group.

Assume that $p \neq q$. Let N be a minimal normal subgroup of G contained in A. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}} N$, and N is a p-group by (ii). Hence, $B^{\mathfrak{F}}$ is a Sylow q-subgroup of $G^{\mathfrak{F}} = B^{\mathfrak{F}} N$. Applying Frattini's argument, we have that $G = G^{\mathfrak{F}} N_G(B^{\mathfrak{F}}) = N N_G(B^{\mathfrak{F}})$. Since

$\mathrm{Core}_G(B) = 1$, it follows that $N_G(B^{\mathfrak{F}})$ is a proper subgroup of G. Hence, N is not contained in $\Phi(G)$ for each minimal normal subgroup N of G contained in A. If $\Phi(A) \neq 1$, a minimal normal subgroup of G must be contained in $\Phi(A) \leq \Phi(G)$, a contradiction. Therefore, $\Phi(A) = 1$. Let N be a minimal normal subgroup of G contained in A. Then $N = N_1 \times N_2 \times \cdots \times N_r$ is a direct product of minimal normal subgroups of A, and there exists $i \in \{1, 2, \ldots, r\}$ such that N_i is not contained in $\Phi(A)$. Suppose $i = 1$. Let M be a maximal subgroup of A such that $A = N_1 M$ and $N_1 \cap M = 1$. Assume first that A is a p-group. Then BM is a subgroup of G, and $M = BM \cap A$ is a normal subgroup of BM. Hence, M is normalized by B, and so M is a normal subgroup of G. Now $N = N_1(M \cap N)$. But $M \cap N$ is normal in G. The minimality of N yields $N = N_1$ and then $|N| = p$. Thus, $G/C_G(N)$ is abelian. Hence, $G^{\mathfrak{F}}$ centralizes N, and $B^{\mathfrak{F}}$ is a normal subgroup in $G^{\mathfrak{F}}$, and so $G^{\mathfrak{F}}$ is an elementary abelian p-group. This contradiction implies that A is not a p-group. Then $T = \mathbf{F}(A)B$ is a proper subgroup of G which is a weak direct product of $\mathbf{F}(A)$ and B. By the minimality of G, $T^{\mathfrak{F}} = B^{\mathfrak{F}}$. Then $B^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$, and so $G^{\mathfrak{F}}$ is an elementary abelian p-group, a contradiction which shows that $p = q$. Then $B^{\mathfrak{F}}$ is a subnormal subgroup of G. By [6, Lemma A.14.3], N normalizes $B^{\mathfrak{F}}$, and therefore $B^{\mathfrak{F}}$ is a normal subgroup of the elementary abelian p-group $G^{\mathfrak{F}}$.

(v) Final contradiction. By [6, IV, 5.18], since $B^{\mathfrak{F}}$ is abelian, there exists an $\mathfrak{F}$-projector K of B such that $B = B^{\mathfrak{F}}K$ and $K \cap B^{\mathfrak{F}} = 1$. Consider the subgroup $Z = AK$ of G. Applying (iii), Z belongs to $\mathfrak{F}$ and $G = B^{\mathfrak{F}}Z = F(G)Z$. By [6, III, 3.23(b)], there exists a unique $\mathfrak{F}$-projector of G containing Z, E say. Hence, $G = B^{\mathfrak{F}}Z = G^{\mathfrak{F}}E$ and $G^{\mathfrak{F}} \cap E = 1$ by (iii) and [6, IV, 5.18]. In particular, $B^{\mathfrak{F}} \cap Z = 1$. Now $|Z||B^{\mathfrak{F}}| = |E||G^{\mathfrak{F}}| = |E||B^{\mathfrak{F}}||N|$. Hence, $|Z| = |E||N|$. This implies $Z = E$ and then $B^{\mathfrak{F}} = G^{\mathfrak{F}}$, a contradiction. □

Proof of Theorem C. Assume that the result is false and let G be a minimal counterexample. Then every proper epimorphic image of G is supersoluble, and hence G has exactly one minimal normal subgroup N which is not contained in the Frattini subgroup of G. Since G is soluble, it follows that N is abelian, $N = C_G(N) = F(G)$, and there exists a core-free maximal subgroup of G such that $G = NM$ and $N \cap M = 1$. Let p be the prime dividing $|N|$. Then $|N| > p$. Since $1 \neq G'$ is nilpotent, we have that $G' = N$ and M is abelian. However, $\mathbf{O}_p(M) = 1$ by [6, Lemma A.13.6]. Hence, M is a p'-group and N is the Sylow p-subgroup of G. Since $B \neq G$, we have that $N \leq A$. Note that $N = C_A(N) = \mathbf{O}_{p'p}(A)$. Therefore, $A/\mathbf{O}_{p'p}(A) = A/\mathbf{O}_p(A) = A/N$ is abelian of exponent dividing $p - 1$ because A is supersoluble. Assume BN is a proper subgroup of G. Then, by the minimality of G, BN is supersoluble, and so $B_{p'} \cong BN/\mathbf{O}_{p'p}(BN)$ is abelian of exponent dividing $p - 1$. Consequently, M is abelian of exponent dividing $p - 1$. Since N is an irreducible and faithful module for M, we have that N has order p by [6, Theorem B.9.8], a contradiction. Hence, $G = BN$. Now $B \cap N$ is a normal subgroup of G contained in N. Thus, $B \cap N = 1$, and $G = BN$ is the weak direct product of B and N. By Theorem A, G is supersoluble. This contradiction proves the theorem. □

Proof of Corollary B. Note that since G' is nilpotent, A and B are metanilpotent. By [8, Theorem 2.11], A and B are supersoluble. By Theorem C, G is supersoluble, and hence $G \in w\mathfrak{U}$. □

Proof of Theorem D. Suppose the theorem is not true and let (G, A, B) be a counterexample with $|G| + |A| + |B|$ as small as possible. Let N be a minimal normal subgroup of G. It is easy to check that G/N satisfies the hypotheses of the theorem. By the minimality of G, we have that $G^{\mathfrak{U}} N = A^{\mathfrak{U}} B^{\mathfrak{U}} N$. Hence, $G^{\mathfrak{U}} = A^{\mathfrak{U}} B^{\mathfrak{U}} (G^{\mathfrak{U}} \cap N)$. Consequently, $\mathrm{Soc}(G)$ is contained in $G^{\mathfrak{U}}$ and $G^{\mathfrak{U}} = A^{\mathfrak{U}} B^{\mathfrak{U}} N$ for every minimal normal subgroup N of G. Since $G^{\mathfrak{U}}$ is contained in G', we have that $G^{\mathfrak{U}}$ is nilpotent.

Note that $A^{\mathfrak{U}}$ is a normal subgroup of G. If $A^{\mathfrak{U}} \neq 1$, then there exists a minimal normal subgroup N of G such that $N \leq A^{\mathfrak{U}}$ and so $G^{\mathfrak{U}} = A^{\mathfrak{U}} B^{\mathfrak{U}} N = A^{\mathfrak{U}} B^{\mathfrak{U}}$, a contradiction. Hence, we may assume that A is supersoluble and that $G^{\mathfrak{U}} = B^{\mathfrak{U}} N$ for every minimal normal subgroup N of G. If B were supersoluble, then G would be supersoluble by Theorem C, which is a contradiction. Hence, $B^{\mathfrak{U}} \neq 1$. Furthermore, $B^{\mathfrak{U}}$ cannot contain a normal subgroup of G. Hence, $\mathrm{Core}_G(B^{\mathfrak{U}}) = 1$. Let p be the largest prime dividing $|A|$. Since A is a Sylow tower group of supersoluble type, A has a normal Sylow p-subgroup, A_p say, which is also normal in G. Hence, G has a minimal normal subgroup N of G which is a p-group. Since $G^{\mathfrak{U}}$ is nilpotent, we have that $B^{\mathfrak{U}}$ is a subnormal subgroup of G. By [6, Lemma A.14.3], $B^{\mathfrak{U}}$ is normalized by N. Thus, $B^{\mathfrak{U}}$ is a normal subgroup of $G^{\mathfrak{U}}$, and $G^{\mathfrak{U}}/B^{\mathfrak{U}}$ is an elementary abelian p-group. Consequently $B^{\mathfrak{U}}$ contains the residual X of $G^{\mathfrak{U}}$ associated to the formation of all elementary abelian p-groups. Since X is a normal subgroup of G, it follows that $X \leq \mathrm{Core}_G(B^{\mathfrak{U}}) = 1$. Hence, $G^{\mathfrak{U}}$ is an elementary abelian p-group.

Since $\mathrm{Soc}(G)$ is contained in $G^{\mathfrak{U}}$, $\mathbf{O}_{p'}(G) = 1$ and hence $\mathbf{F}(G) = \mathbf{O}_p(G)$. Therefore $G' \leq \mathbf{F}(G)$ is a p-group, and $\mathbf{F}(G)$ is the unique Sylow p-subgroup of G. Moreover, the Hall p'-subgroups of G are abelian (note that G is soluble). Assume $A_p B < G$. Then $A_p B$ satisfies the hypotheses of the theorem. By the choice of G, we have that $(A_p B)^{\mathfrak{U}} = B^{\mathfrak{U}}$. Note that $G' \leq A_p B$. Hence, $A_p B$ is a normal subgroup of G. This implies that $B^{\mathfrak{U}}$ is normal in G, a contradiction. Hence, $G = A_p B$ and A_p and B satisfy the hypotheses of the theorem. If $A \neq A_p$, the choice of (G, A, B) implies that $G^{\mathfrak{U}} = B^{\mathfrak{U}}$, a contradiction. Consequently, we have that $A = A_p$.

Write $T = A B_{p'}$. By Theorem C, T is supersoluble. Moreover, since $G = F(G) B_{p'}$, it follows that every minimal normal subgroup N of G contained in T is a minimal normal subgroup of T. Thus, $|N| = p$. Consequently, N is $\mathfrak{U}$-central in G. By [6, V, 3.2], N is contained in every supersoluble normalizer of G. Let E be one of them. Then $G = G^{\mathfrak{U}} E$ and $G^{\mathfrak{U}} \cap E = 1$. However, $N \leq G^{\mathfrak{U}} \cap E = 1$. This final contradiction proves the theorem. $\square$

Proof of Corollary C. Since $\mathfrak{U} \subseteq w\mathfrak{U}$, we have $G^{w\mathfrak{U}} \leq G^{\mathfrak{U}} \leq G'$. Then $G/G^{w\mathfrak{U}}$ is a metanilpotent w-supersoluble group. Applying [8, Theorem 2.11], we have that $G/G^{w\mathfrak{U}}$ is supersoluble. Hence, $G^{\mathfrak{U}} \leq G^{w\mathfrak{U}}$, and therefore $G^{\mathfrak{U}} = G^{w\mathfrak{U}}$, and the same is true for A and B. Therefore, by Theorem D, $G^{w\mathfrak{U}} = A^{w\mathfrak{U}} B^{w\mathfrak{U}}$, as desired. $\square$

4. An analogue of Monakhov's result

The following two results are the key to prove Corollary D.

Lemma 4.1. [4, **Theorem A**]   *Let the group $G = HK$ be the product of the subgroups H and K. Assume that H permutes with every maximal subgroup of K and K permutes with*

every maximal subgroup of H. If H is supersoluble, K is nilpotent and K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H, then G is supersoluble.

Proposition 4.2. *Let $G = AB$ be a weak normal product of A and B, with A and B supersoluble and A normal in G. Then $B^{'}$ is a subnormal subgroup of G.*

Proof. Assume the result is not true, and let G be a counterexample of minimal order with $|A|$ as small as possible. Let p be the largest prime dividing the order of A. Then A has a normal Sylow p-subgroup A_p, which is also a normal subgroup of G. Let N be a minimal normal subgroup of G such that $N \leq A_p$. It is clear that $A_p B$ satisfies the hypotheses of the theorem. Assume that $A_p B$ is a proper subgroup of G. By the minimality of G, $B^{'}$ is a subnormal subgroup of $A_p B$. Hence, $B' \leq F(A_p B)$. By Lemma 2.1(a), G/N is a weak normal product of A/N and BN/N. By the minimality of G, we have that $B'N$ is a subnormal subgroup of G. Since $N \leq F(A_p B)$, it follows that $B'N \leq F(A_{p'}B)$. Hence, $B'N$ is a subnormal nilpotent subgroup of G. Consequently, $B'N \leq F(G)$. Thus, $B^{'}$ is a subnormal subgroup of G, a contradiction. Hence, we may assume that $G = A_p B$. The minimality of $|A|$ implies that $A = A_p$. Applying now the above Lemma, we conclude that G is supersoluble and therefore G' is nilpotent. Hence, $B^{'}$ is subnormal in G. This final contradiction proves the proposition. $\square$

Proof of Corollary D. Arguing as in [7, Theorem 1], we obtain $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$. Moreover, by [7, Lemma 1(3)], we have that $G' = A'B'[A,B] = (A')^G (B')^G [A,B]$. Since A is a normal subgroup of G, then $A^{'}$ is a subnormal subgroup of G. Also the application of Proposition 4.2 yields $B^{'}$ is subnormal in G and both $A^{'}$ and $B^{'}$ are nilpotent. Hence, $(A')^G (B')^G$ is a normal nilpotent subgroup of G. By [6, II, Lemma II.2.12], $(G')^{\mathfrak{N}} = ((A')^G (B')^G)^{\mathfrak{N}} [A,B]^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}$, as desired. $\square$

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