Involutions on sheaves of endomorphisms]Involutions on sheaves of endomorphisms of locally finitely presented $\mathscr{O}_X$-modules

Baptiste Calmès, Université d'Artois, Laboratoire de Mathématiques de Lens, France baptiste.calmes@univ-artois.fr

Richard Ng'ambi, University of Pretoria, Department of Mathematics and Applied Mathematics, Hatfield 0002, Pretoria, South Africa u16396392@tuks.co.za

Patrice Ntumba, North-West University, Department of Mathematics and Applied Mathematics, Potchefstroom Campus 2520, South Africa pntumba40@gmail.com

1

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Abstract

We note that, given a coherent $\mathscr{O}_X$-algebra $\mathscr{F}$ such that each affine restriction $\mathscr{F}|_{U_i}$ is associated with some faithful finitely generated projective R_i-algebra A_i, if σ_i is an anti-automorphism of A_i such that $x\sigma_i(x)$ is in R_i for all $x \in A_i$, then $\mathscr{F}$ admits one standard involution $\widetilde{\sigma}$, which commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$. Next, given a locally finitely presented $\mathscr{O}_X$-module $\mathscr{E}$ on an affine scheme X, and an involution of the first kind σ on the sheaf of endomorphisms $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and isomorphisms $\varphi : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ and $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, locally, $\sigma \otimes \mathrm{id} = \Phi \circ m$, where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on any open U in X.

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Patrice Ntumba

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Contents

1 Introduction

The purpose of this note is to explain the basic theory of involutions on sheaves of Azumaya algebras.

Here are some classical references:

- Chapter III, §5 in the book [KO74] of Knus and Ojanguren;
- Chaper III, §5 in the book [Knu91] of Knus;
- Chapters 7 and 11 in the book [For17] of Ford;

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- The Book of Involutions [KMRT98] of Knus, Merkurjev, Rost and Tignol when over a field;
- The article [Gro95] Brauer 1 of Grothendieck;

This paper grew out of an attempt to seek counterparts of classical results pertaining to isomorphisms of some groups associated with central simple algebras in the setting of Azumaya algebras. Of these groups, we may mention the group $\mathrm{Sim}(V, b)$ of similitudes, the group $\mathrm{Iso}(V, b)$ of isometries, the group $\mathrm{PSim}(V, b)$ of projective similitudes, all assigned to a nonsingular symmteric or alternating bilinear vector space, defined over an arbitrary field F.

As is put in [KMRT98, Chap IV], these so-called exceptional isomorphisms are of particular interest as they provide equivalences between categories of algebras with involution. This project requires background work, which underpins results related to central simple algebras endowed with involutions of the first or second kind. The one result states that given a finitely generated left A-module M, where A is a central simple algebra over a field F, and $E \equiv \mathrm{E}nd_A(M)$, then E is a central simple F-algebra and is Brauer equivalent to A and $A \simeq \mathrm{E}nd_E(M)$. Conversely, let G be a central simple F-algebra; if A and G are Brauer equivalent, then there is an A-G-bimodule $M \neq \{0\}$ such that $A = \mathrm{E}nd_G(M)$, $G = \mathrm{E}nd_A(M)$. In the context of Azumaya algebras, we first observe that given an Azumaya right R-algebra A, and M an Azumaya left A-algebra such that $\mathrm{E}nd_R(M)$ is a simple R-module, then $B \equiv \mathrm{E}nd_A(M)$ is an Azumaya R-algebra and $A \otimes_R B$ is isomorphic to $\mathrm{E}nd_R(M)$. Moreover, the R-algebra $\mathrm{E}nd_B(M)$ is Azumaya and isomorphic to A: $A \simeq \mathrm{E}nd_B(M)$. So, the first part of the stated classical result (see [KMRT98, Proposition 1.10]) is also verified in this context. As for the converse, we assume that the ring R is commutative, A is an Azumaya R-algebra, and M is a free left Azumaya A-algebra such that $\mathrm{E}nd_R(M)$ is a simple left R-module. Then, $\mathrm{E}nd_A(M)$ is an Azumaya R-algebra and is Brauer equivalent to the opposite algebra A^{o}. Furthermore, if B is an Azumaya R-algebra Brauer equivalent to the Azumaya R-algebra A, then B is of the form $\mathrm{E}nd_A(M)^{\mathrm{o}}$, where M is both an A-module and R-progenerator.

Next, assigning to each non-singular bilinear form $b : A \times A \to R$, where A is an Azumaya R-algebra, with R a local ring, its adjoint anti-automorphism $\sigma_b : \mathrm{E}nd_R(A) \to \mathrm{E}nd_R(A)$, one induces a bijection between the set of equivalence classes of nonsingular bilinear forms on A modulo multiplication by a unit in R and the set of sdjoint anti-automorphisms of $\mathrm{E}nd_R(A)$. Moreover, R-linear involutions of $\mathrm{E}nd_R(A)$ correspond bijectively to nonsingular bilinear forms that are either symmetric or skew-symmetric. This result generalizes to nonsingular sequilinear forms on A.

Before we consider involutions on sheaves of Azumaya algebras in its generality, we shall first study involutions on sheaves associated with algebras of the form $\mathrm{End}_R(A)$, where R is a local ring and A an Azumaya R-algebra, endowed with a nonsingular bilinear form $b : A \times A \to R$. For such algebra sheaves, we observe that the $\mathscr{O}_X$-linear involutions $\widetilde{\mathrm{End}_R(A)} \to \widetilde{\mathrm{End}_R(A)}$ correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric. Moreover, if the underlying R-module of A is a faithful, finitely generated, and projective module, equipped with an anti-automorphism σ satisfying the property that $x\sigma(x) \in R$, for all $x \in A$, then σ induces an involution $\widetilde{\sigma}$ on the $\mathscr{O}_X$-algebra $\mathscr{F}$ associated with A, which is, in addition, the only standard involution on $\mathscr{F}$, commuting with all automorphisms and anti-automorphisms of $\mathscr{F}$.

Now, for a locally projective quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ of constant rank 2 on a scheme $(X, \mathscr{O}_X)$, that is, the $\mathscr{O}_X$-module $\mathscr{E}$ is associated with a projective R-module of constant rank 2, $\mathscr{E}$ turns out to be a commutative $\mathscr{O}_X$-algebra, endowed with a unique standard involution. Finally, for a locally finitely presented $\mathscr{O}_X$-module on an affine scheme X, and σ an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a sheaf isomorphism φ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}$ onto $\mathscr{E}^*$, and an isomorphism $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, on some appropriate open U in X, $\sigma \otimes \mathrm{id} = \Phi \circ m$, where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on U, and for any open V in U, $\Phi_{VV}(s) = \varphi_V^{-1} s^* \varphi_V$, for any section $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(V)$.

2 Algebraic definition

2.1 Progenerator modules

In this paragraph, we recall some facts on progenerator modules (or simply, progenerators) that are useful for our topic. For an extensive exposition on progenerator modules, we recommend [For17] or [Knu91]. Knus does not use the terminology “progenerator" though, he calls them faithfully projective modules. It is however proved that, given a commutative ring R, an R-module is an R-progenerator if and only if it is finitely generated projective and faithful, (cf. [For17, p. 11, Corollary 1.1.16]).

Definition 2.1. A module M over a ring R (not necessarily commutative) is called a *progenerator*, or *R-progenerator*, when it is finitely generated and projective, and its trace is R, viz

$$\mathfrak{T}_R(M) = \left\{ \sum_{i=1}^{n} f_i(m_i) |\ n \geq 1,\ f_i \in \mathrm{H}om_R(M, R),\ m_i \in M \right\} = R.$$

Equivalently, a finitely generated and projective R-module M is a progenerator if every left R-module is a homomorphic image of a direct sum M^I of copies of M over some index set I, see [For17, Ex. 1.1.11].

The following shows that, for projective modules of finite type over a commutative ring , the condition for being a progenerator entails that the module is nowhere zero locally.

Theorem 2.2. *For any finitely generated projective module M over a* commutative *ring R, the following are equivalent:*

a. *M is a progenerator;*

b. *M is faithful (i.e. has its annihilator reduced to 0);*

c. *For every maximal ideal $\mathfrak{m}$ of R, the module $M/\mathfrak{m}M$ is nonzero;*

d. *For any connected component of R, the module M is nonzero over that connected component. In other words, if S is a factor of R, then $S \otimes_R M$ is nonzero;*

e. $\mathrm{Rank}(M_\mathfrak{p}) \neq 0$*, for every* $\mathfrak{p} \in \mathrm{Spec}(R)$*;*

f. $\mathrm{Rank}(M_\mathfrak{p}) \neq 0$*, for every closed point* $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. See [For17, Cor. 1.1.16] for (a) $\Leftrightarrow$ (b). We now show that (b) $\Rightarrow$ (c). Indeed, since $\mathrm{annih}_R(M) = 0$, then for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}M \neq M$. See [For17, Lemma 1.1.13]. For (c) $\Rightarrow$ (b), by one more use of [For17, Lemma 1.1.13], we note that $\mathfrak{m}M \neq M \Leftrightarrow \mathfrak{m} + \mathrm{annih}_R(M) \neq R$, which, since $\mathrm{annih}_R(M)$ is a two-sided ideal in R, implies that $\mathrm{annih}_R(M) = 0$. Therefore, M is faithful. Next, since the rank of M is a locally constant map on $\mathrm{Spec}(R)$, it is therefore constant on any connected component. Thus, $M \neq 0$ on any connected component iff its rank is not zero on it. Hence, (d) $\Leftrightarrow$ (e). Now, since $\mathrm{Rank}(M_\mathfrak{p}) = \mathrm{Rank}(M_\mathfrak{m})$, for all maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$, (e) $\Leftrightarrow$ (f). Finally, by virtue of the fact that $\mathrm{Rank}(M_\mathfrak{p}) = \dim_{R_\mathfrak{p}} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$, where $R_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, (f) $\Leftrightarrow$ (c). $\square$

Definition 2.3. An R-algebra A is an Azumaya algebra if it satisfies any of the following equivalent properties:

a. A is a central R-algebra and separable over R.

b. A is an R-progenerator and the natural representation

$$\mu : A \otimes_R A^{\mathrm{o}} \to \mathrm{E}nd_R(A),$$

defined by $\mu(a \otimes_R b^{\mathrm{o}})(x) = axb^{\mathrm{o}}$, where a, $b^{\mathrm{o}} \in A$ and $x \in R$, is an isomorphism.

c. For any maximal ideal $\mathfrak{m}$ of R , the quotient $A/\mathfrak{m}A$ is a central simple $R/\mathfrak{m}$-algebra.

Note in particular that by definition, R injects into A.

Example 2.4 ([For17, 7.1.10])**.** When M is a progenerator module over a commutative ring R, the algebra $\mathrm{E}nd_R(M)$ is an Azumaya algebra over R; in particular, when $M = R^n$, we have that $\mathrm{M}_n(R^{\mathrm{o}})$ is an Azumaya algebra over R.

From the geometric point of view, Azumaya algebras over R are exactly étale forms of matrix rings.

Proposition 2.5 ([For17, Theoorem 10.3.9])**.** *The following are equivalent:*

a. A is an Azumaya algebra over R;

b. There is an étale cover $\{S_i \to R, i \in I\}$ such that for any i, there is an isomorphism $A_{S_i} \simeq \mathrm{M}_{n_i}(S_i)$ for some $n_i \geq 1$.

2.2 Modules over Azumaya algebras

When the base ring is a field, an Azumaya algebra is a simple ring, so any module on an Azumaya algebra that is defined over a field decomposes as a direct sum of simple ones; simple modules are indecomposable, and there is only one simple module up to isomorphism (see [Bou58, §7, 2, Prop. 2]). But when the base is more general, what can be said of this category of modules?

Proposition 2.6 ([For17, Ex. 7.6.7] or [DI71, p. 146, Exercise. 1.1])**.** *If A is Azumaya over a* local *ring R, then any two indecomposable A-modules are isomorphic.*

3 Brauer group

In this section, we discuss the Brauer equivalence on Azumaya algebras and questions related to this equivalence.

3.1 Brauer equivalence

Definition 3.1. Two Azumaya algebras A and B over R are said to be *Brauer-equivalent* provided there are progenerator (i.e. nowhere zero) R-modules M and N such that

$$A \otimes_R \mathrm{E}nd_R(M) \simeq B \otimes_R \mathrm{E}nd_R(N).$$

Since $\mathrm{E}nd_R(M \otimes_R N) \simeq \mathrm{E}nd_R(M) \otimes_R \mathrm{E}nd_R(N)$, this is obviously an equivalence relation, and furthermore compatible with the tensor product (over R) of algebras, which thus induces a monoid structure on equivalence classes. It is clear that isomorphic algebras are in the same equivalence class. Actually, this monoid is a group, with neutral element $[R]$. Assuming that the binary operation is given by setting $[A][B] := [A \otimes B]$, the inverse of $[A]$ is $[A^{\mathrm{o}}]$, since

$$[A \otimes_R A^{\mathrm{o}}] = [\mathrm{E}nd_R(A)] = [R]$$

and A is an R-progenerator. See [For17, Theorem 7.1.4].

It is easy to show:

Proposition 3.2. *a. An Azumaya algebra is neutral if and only if there is a progenerator M such that $A \simeq \mathrm{E}nd_R(M)$.*

b. Azumaya algebras A and B are Brauer-equivalent if and only if there is an R-progenerator M such that $A \otimes B^{\mathrm{o}} \simeq \mathrm{E}nd_R(M)$.

Proof. This is [For17, Prop. 7.3.4]. □

We recall that Brauer equivalence relates to Morita equivalence. Succinctly, two algebras A and B are *Morita equivalent* if their corresponding module categories are equivalent, that is, there are additive functors S and T

$$\mathfrak{Mod}_A \underset{S}{\overset{T}{\rightleftarrows}} \mathfrak{Mod}_B$$

such that $ST \simeq Id_{\mathfrak{Mod}_A}$, and $TS \simeq Id_{\mathfrak{Mod}_B}$. It follows that $T \simeq \ \otimes_A P$, $S \simeq \ \otimes_B Q$, where $P = T(A)$ and $Q = S(B)$. See, for instance, [Bas68, p.

60]. Given a semilocal and connected ring R, H. Bass in [Bas64, Corollary 17.2] shows that *two Azumaya R-algebras A and B are Brauer equivalent if and only if they are Morita equivalent as R-algebras.*

Let A be an Azumaya R-algebra, let M be a left A-module and let $B = \mathrm{E}nd_A(M)$. The action of B on M by evaluation of functions endows M with a right B-module structure (endomorphisms of M are written on the right-hand side of the arguments) commuting with that of A; in other words, M becomes an $A \otimes_R B^{\circ}$-module. More precisely, for all $a \in A$, $f \in B$, and $m \in M$, $(a \otimes f)m = amf$.

For the purpose of the next lemma, recall that a progenerator over a progenerator is a progenerator [For17, Proposition 1.1.8], that is, given a ring homomorphism $R \to S$ such that S is an R-progenerator, then any S-progenerator is also an R-progenerator.

Lemma 3.3. *Let A be an Azumaya right R-algebra, and let M be an Azumaya left A-algebra. Then $B := \mathrm{E}nd_A(M)$ is an Azumaya R-algebra. Furthermore, the natural morphism*

$$\begin{array}{lcl} A \otimes_R B & \to & \mathrm{E}nd_R(M) \\ a \otimes f & \mapsto & (m \mapsto f(am) = af(m)) \end{array}$$

is an isomorphism; therefore, B is Brauer equivalent to the opposite Azumaya R-algebra A°.

Proof. Since A is an R-progenerator, and $\mathrm{E}nd_A(M)$ is an A-progenerator, $\mathrm{E}nd_A(M)$ is an R-progenerator. Moreover, $\mathrm{E}nd_A(M)$, as an R-algebra, is R-central; therefore, $\mathrm{E}nd_A(M)$ is an Azumaya R-algebra (see [For17, Theorem 7.1.4]), and consequently, the tensor product $A \otimes_R B$ is an Azumaya R-algebra. Finally, that the map in the statement of Lemma 3.3 is an isomorphism of Azumaya R-algebras is easy to see. $\square$

By symmetry, if A is an Azumaya left R-algebra and M is a right A-module , then $B := \mathrm{E}nd_A(M)$ is an Azumaya right R-algebra and $B \otimes_R A \simeq \mathrm{E}nd_R(M)$.

Corollary 3.4. *Under the hypotheses of Lemma 3.3, the R-algebra $E := \mathrm{E}nd_B(M)$, where $B = \mathrm{E}nd_A(M)$, is Azumaya and is isomorphic to $A : A \simeq \mathrm{E}nd_B(M)$.*

Proof. By virtue of Lemma 3.3, B is an Azumaya left R-algebra; since M is an Azumaya right B-algebra, it follows that $E := \mathrm{E}nd_B(M)$ is an Azumaya right R-algebra. Furthermore, since $E \otimes_R B \simeq \mathrm{E}nd_R(M) \simeq A \otimes_R B$, it follows that $A \simeq \mathrm{E}nd_B(M)$, and the proof is complete. □

We then have (compare with [KMRT98, Proposition (1.10)][1]):

Conversely to Lemma 3.3, one has

Lemma 3.5. *Let R be a commutative ring, and A, B Azumaya R-algebras. Then, if B is Brauer equivalent to A, B is of the form $\mathrm{End}_A(M)^{\mathrm{o}}$, where M is both an A-module and R-progenerator.*

Proof. By [For17, Proposition 7.3.4], there is an isomorphism $\phi : A \otimes_R B^{\mathrm{o}} \simeq \mathrm{E}nd_R(M)$, where M is an R-progenerator. Plainly, by setting $(a \otimes 1)(x) = ax$, for all $a \in A$ and $x \in M$, it turns out that M is an A-module. From [For17, Theorem. 7.2.3], it follows that the commutant of A (identified to its image via ϕ) in $\mathrm{E}nd_R(M)$ is exactly B. In other words, since

$$\mathrm{E}nd_R(M)^{\phi(A)} \equiv \mathrm{E}nd_R(M)^A = \{f \in \mathrm{E}nd_R(M) |\, fa = af,\ a \in A\} = \mathrm{E}nd_A(M),$$

it follows that

$$B^{\mathrm{o}} \simeq \phi(B^{\mathrm{o}}) = \mathrm{E}nd_A(M).$$

□

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3.2 Involutions on Azumaya algebras

Let R be a *local* ring and A an Azumaya R-algebra of finite rank (so A as an R-module is of finite rank). As is well known, a bilinear form $b : A \times A \to R$ is called *nonsingular* if the induced map

$$\hat{b} : A \to A^* := \mathrm{H}om_R(A, R),$$

defined by

$$\hat{b}(x)(y) = b(x, y),$$

[1] Beware that in [KMRT98] the convention on morphisms of page xvii explains why the opposite on B does not appear.

for all x, $y \in A$, is a linear isomorphism. For any $f \in \mathrm{E}nd_R(A)$, we define $\sigma_b(f) \in \mathrm{E}nd_R(A)$ by setting

$$\sigma_b(f) = \hat{b}^{-1} \circ f^t \circ \hat{b}, \tag{3.1}$$

where $f^t \in \mathrm{E}nd_R(A^*)$ is the transpose of f. Also it is standard to define $\sigma_b(f)$ by requiring it to verify the property:

$$b\big(x, f(y)\big) = b\big(\sigma_b(f)(x), y\big),$$

for all x, $y \in A$. It is clear that σ_b turns out to be an anti-automorphism of $\mathrm{E}nd_R(A)$; it is called the *adjoint anti-automorphism* with respect to the nonsingular bilinear form b.

Theorem 3.6. *Let R be a local ring and A an Azumaya R-algebra of finite rank. Moreover, let Λ be the map that sends each nonsingular bilinear form $b : A \times A \to R$ onto its adjoint anti-automorphism σ_b. Then, Λ induces a bijection $\widetilde{\Lambda}$ between the set of equivalence classes of nonsingular bilinear forms on A modulo multiplication by a unit of R and the set of adjoint anti-automorphisms of* $\mathrm{End}_R(A)$*. Under the map $\widetilde{\Lambda}$, the R-linear involutions of* $\mathrm{E}nd_R(A)$ *correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.*

Proof. The proof is standard; see, for instance, [KMRT98, pp 1-2, Theorem]. Indeed, from relation (3.1), it is clear that for any unit α in R, $\sigma_{\alpha b} = \sigma_b$; therefore, the map Λ induces a well-defined map $\widetilde{\Lambda} : [b] \mapsto \sigma_b$, where $[b]$ denotes the equivalence class containing b.

Now, let's show that $\widetilde{\Lambda}$ is one-to-one. To this end, note that if b, b' are nonsingular bilinear forms on A, the isomorphism $v \equiv \hat{b}^{-1} \circ \hat{b}'$ is such that

$$b'(x, y) = b\big(v(x), y\big),$$

for all x, $y \in A$; whence, one has, for all $f \in \mathrm{E}nd_R(A)$,

$$\sigma_b(f) = v \circ \sigma_{b'}(f) \circ v^{-1},$$

which can be rewritten as

$$\sigma_b = \mathrm{I}nt(v) \circ \sigma_{b'},$$

where

$$\mathrm{I}nt(v)(f) = v \circ f \circ v^{-1},$$

for all $f \in \mathrm{E}nd_R(A)$. Therefore, if $\sigma_b = \sigma_{b'}$, then v is a unit in R, and $[b] = [b']$.

Next, let's fix a nonsingular bilinear form b on A. It follows that for any linear anti-automorphism ν of $\mathrm{E}nd_R(A)$, $\sigma_b \circ \nu^{-1}$ is an R-linear automorphism of $\mathrm{E}nd_R(A)$. Since $\mathrm{E}nd_R(A)$ is an Azumaya R-algebra (see [For17, Theorem 7.1.4] along with [For17, Proposition 7.1.10]) and R is local, by the Skolem-Noether theorem ([For17, Corollary 7.8.15]), $\sigma_b \circ \nu^{-1}$ is an inner automorphism, that is, $\sigma_b \circ \nu^{-1} = \mathrm{I}nt(u)$, for some R-linear isomorphism $u \in \mathrm{E}nd_R(A)$. Then, ν is an adjoint anti-automorphism for the bilinear form b' defined by

$$b'(x, y) = b(u(x), y),$$

which ends the proof of the first part of the theorem.

Finally, if b is a nonsingular bilinear form on A with adjoint anti-automorphism σ_b, then the nonsingular bilinear form b' :

$$b'(x, y) = b(y, x) \quad \text{for all } x,\ y \in A$$

satisfies the equation

$$\sigma_{b'} = \sigma_b^{-1}.$$

Therefore, b and b' are scalar multiples of each other if and only if $\sigma_b^2 = 1$; it follows that if $b' = \varepsilon b$, for some unit ε, then $\varepsilon^2 = 1$. Hence, b is symmetric or skew-symmetric. □

Theorem 3.6 generalizes to nonsingular sesquilinear forms on A. In fact, let R be a local ring, endowed with a *conjugation* ϑ, let A be an Azumaya R-algebra of finite rank (any *involution on A clearly induces a conjugation on R*), and let $h : A \times A \to R$ be a sesquilinear form with respect to ϑ, that is, h is $\mathbb{Z}$-bilinear and is such that, for x, $y \in A$, and α, $\beta \in R$, $h(\alpha x, y\beta) = \vartheta(\alpha)h(x, y)\beta$. It is clear that the argument for the proof of Theorem 3.6 applies mutatis mutandis in this context as well. Furthermore, the *R-linear involutions of $End_R(A)$ correspond to nonsingular sesquilinear forms which are either hermitian or skew-hermitian.* (A *hermitian form* on A (with respect to the conjugation ϑ) is a sesquilinear map $h : A \times A \to R$ such that $h(y, x) = \vartheta(h(x, y))$, for all x, $y \in A$. The map h is called *skew-hermitian* if $h(y, x) = -\vartheta(h(x, y))$, for all x, $y \in A$.)

More generally, suppose that R is a local ring and a PID, A is an R-Azumaya algebra, M is a finitely free right A-module, and $\vartheta : A \to A$ is an involution (of any kind). Then, *for every nonsingular hermitian or* *skew-hermitian form* $h : M \times M \to A$*, there exists a unique involution* $\sigma_h : \mathrm{End}_A(M) \to \mathrm{End}_A(M)$ *such that*

$$h(x, f(y)) = h(\sigma_h(f)(x), y),$$

for all $x,\ y \in M$. The *involution* σ_h is called the adjoint involution with respect to h. See [KMRT98, Proposition 4.1].

On considering the framework of Azumaya algebras over schemes, Theorem 3.6 can be stated as follows:

Theorem 3.7. *Let* R *be a local ring and* A *an Azumaya* R*-algebra of finite rank. The map that sends each nonsingular bilinear form* $b : A \times A \to R$ *onto its adjoint anti-automorphism*

$$\widetilde{\sigma_b} : \widetilde{\mathrm{E}nd_R(A)} \to \widetilde{\mathrm{E}nd_R(A)}$$

is a bijection. Moreover, the $\mathscr{O}_X$*-linear involutions of* $\widetilde{\mathrm{E}nd_R(A)}$ *correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.*

Proof. That $\widetilde{\sigma_b}$ exists and is a sheaf anti-automorphism stems from the fact that the map

$$\mathrm{H}om_R(\mathrm{E}nd_R(A), \mathrm{E}nd_R(A)) \to \mathrm{H}om_{\mathscr{O}_X}(\widetilde{\mathrm{E}nd_R(A)}, \widetilde{\mathrm{E}nd_R(A)}), \quad \sigma \to \widetilde{\sigma}$$

is bijective. See [Bos13, p. 258, Proposition 2]. □

Now, let A be an R-algebra (*not necessarily Azumaya*), and $X = \mathrm{Spec}(R)$; let us introduce *involutions* on $\mathscr{O}_X$-algebras $\mathscr{F}$ associated with A. An involution of $\mathscr{F}$ is an $\mathscr{O}_X$-anti-automorphism of order 2, that is an $\mathscr{O}_X$-endomorphism of $\mathscr{F}$ such that, for any given sections s, t of $\mathscr{F}$ over the same open subset U of X, $\sigma(st) = \sigma(t)\sigma(s)$ and $\sigma^2 = \mathrm{id}$.

Definition 3.8. Let R be a ring, A an R-algebra such that the canonical morphism $R \to A$ is injective, $X = \mathrm{Spec}(R)$, and $\mathscr{F}$ the $\mathscr{O}_X$-algebra associated with A. An involution σ of $\mathscr{F}$ is called a *standard* $\mathscr{O}_X$*-involution* (or simply *standard involution*) provided that, for every open U in X, σ_U is a standard involution, that is, the morphism $\mathscr{O}_X(U) \longrightarrow \mathscr{F}(U)$ is injective, and $a\sigma_U(a) \equiv a\sigma(a) \in \mathscr{O}_X(U)$ for all $a \in \mathscr{F}(U)$; the scalar $a\sigma(a)$ is called the *norm* of a, and is often denoted by $\mathscr{N}_U(a) \equiv \mathscr{N}(a)$. By *trace* of $a \in \mathscr{F}(U)$, we mean the element $a + \sigma(a) = \mathscr{N}(a+1) - \mathscr{N}(a) - 1 \in \mathscr{O}_X(U)$, which is usually denoted by $\mathrm{tr}_U(a) \equiv \mathrm{tr}(a)$.

Whenever the circumstances at hand are clear, we shall write σ for any component σ_U of a sheaf morphism σ, so that the condition $a\sigma_U(a) \in \mathscr{O}_X(U)$ of Definition 3.8 simply becomes $a\sigma(a) \in \mathscr{O}_X$.

The terminology of Definition 3.8 is classical; cf. [HM08, p. 40, Definition 1.13.7]. On the other hand, let us notice that, cf. [Bos13, p. 258, Proposition 2], the mapping $\mathrm{H}om_R(A, A) \to \mathrm{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$, sending any endomorphism φ of A onto its corresponding endomorphism $\widetilde{\varphi}$ of $\mathscr{F}$, where for any $f \in R$, $\widetilde{\varphi}_{D(f)} = \varphi \otimes 1_{R_f}$, is bijective; on the strength of this bijection, we note that *for any commutative ring R, an endomorphism $\sigma : A \to A$ is an involution if and only if its image $\widetilde{\sigma}$ is an involution of $\mathscr{F}$.* Indeed, for any $\mathfrak{p} \in X$, let $\frac{a}{s}, \frac{b}{t} \in A_{\mathfrak{p}}$; clearly, one has that $\widetilde{\sigma}_{\mathfrak{p}}\big(\frac{a}{s}\frac{b}{t}\big) = \frac{\sigma(ab)}{st} = \widetilde{\sigma}_{\mathfrak{p}}\big(\frac{b}{t}\big)\widetilde{\sigma}_{\mathfrak{p}}\big(\frac{a}{s}\big)$. Furthermore, since the mapping $A \mapsto \widetilde{A}$ yields an exact fully faithful functor from the category of R-modules to the category of $\mathscr{O}_X$-modules (cf. [Har77, p. 110, Proposition 5.2]), it follows that $\widetilde{\sigma}^2 = 1$. The converse is straightforward. For any open set U in X, the U-th component of $\widetilde{\sigma}$ is obtained by setting

$$\widetilde{\sigma}_U = \varprojlim_{D(f)\subseteq U} \widetilde{\sigma}_{D(f)} = \varprojlim_{f\in R \text{ with } D(f)\subseteq U} \sigma_f,$$

which is an involution on $\mathscr{F}(U)$. Because of the bijective correspondence above, we shall often identify σ with $\widetilde{\sigma}$ whenever there is no confusion.

Now, let us note that the natural morphism $\iota : R \to A$ gives rise to the sheaf morphism $\widetilde{\iota} : \mathscr{O}_X \to \mathscr{F}$, where, if $\iota_f : R_f \to A_f$ denotes the localization of ι at $f \in R$, then, for any open set U in X,

$$\widetilde{\iota}_U = \varprojlim_{D(f)\subseteq U} \widetilde{\iota}_{D(f)} = \varprojlim_{f\in R \text{ with } D(f)\subseteq U} \iota_f.$$

It is clear from the universal property of projective limits that $\widetilde{\iota}$ is a functor on the category of open subsets of X. Now, since injectiveness (of morphisms of modules) is a local property, and since, for all $x \in X$, $\mathscr{O}_{X,x} \simeq R_x$ (cf. [Bos13, p. 248, Proposition 9.]) and

$$\mathscr{F}_x = \varinjlim_{D(f)\ni x} \mathscr{F}(D(f)) = A \otimes_R \varinjlim_{D(f)\ni x} R_f = A \otimes_R R_x = A_x,$$

it follows that the natural morphism $\iota : R \to A$ is injective if and only if the induced morphism $\widetilde{\iota} : \mathscr{O}_X \to \mathscr{F}$ is injective; thus, we have:

Lemma 3.9. *Let R be a ring, $X = \operatorname{Spec}(R)$, and A an R-algebra. Then, an endomorphism σ of A is a standard involution if and only if $\widetilde{\sigma}$ is a standard involution of $\mathscr{F} = \widetilde{A}$.*

Proof. Indeed, let U be open in X; as a projective limit of the projective system $\{\mathscr{F}(D(f)) : f \in R \text{ and } D(f) \subseteq U\}$, $\mathscr{F}(U)$ is contained in $\prod_{D(f)\subseteq U} \mathscr{F}(D(f))$. Moreover, let $x \in \mathscr{F}(U)$, then $\rho_f^U(x) = \frac{a}{f^n} \in A_f$, for some $a \in A$ and $n \in \mathbb{N}$, and where $\rho_f^U : \mathscr{F}(U) \to \mathscr{F}(D(f))$ is a restriction map for the $\mathscr{O}_X$-module $\mathscr{F}$; in fact, $\rho_f^U = \operatorname{pr}_f$, where pr_f is the natural projection of $\prod_{D(f)\subseteq U} \mathscr{F}(D(f))$ onto $\mathscr{F}(D(f))$. On the other hand, $\rho_f^U(\widetilde{\sigma}(x)) = \sigma_f(\frac{a}{f^n})$; whence $\rho_f^U(x\widetilde{\sigma}(x)) = \frac{a}{f^n}\sigma_f(\frac{a}{f^n}) \in R_f = \mathscr{O}_X(D(f))$. It follows that $x\widetilde{\sigma}(x) \in \mathscr{O}_X(U)$ for all $x \in \mathscr{F}(U)$; in other words, $\widetilde{\sigma}$ is a standard involution of $\mathscr{F}$ whenever σ is a standard involution of A. The converse is immediate. $\square$

Corollary 3.10. *Let R be a ring, $X = \operatorname{Spec}(R)$, A an R-algebra whose underlying R-module is faithful, finitely generated, and projective; and let σ be an anti-automorphism of A such that $x\sigma(x) \in R$ for all $x \in A$. Then, σ induces an involution $\widetilde{\sigma}$ on the $\mathscr{O}_X$-algebra $\mathscr{F}$ associated with A; it is, in addition, the only standard involution of $\mathscr{F}$. Moreover, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$.*

Proof. By [HM08, p. 40, Lemma 1.13.8], σ turns out to be the only standard involution of A, and commutes with all automorphisms and anti-automorphisms of A. Moreover, by Lemma 3.9, $\widetilde{\sigma}$ is the only standard involution of $\mathscr{F}$; since the mapping $\sim$ is a functor, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$. $\square$

Corollary 3.11. *Let $(X, \mathscr{O}_X)$ be a scheme and $\mathscr{F}$ a coherent $\mathscr{O}_X$-algebra such that if $\mathscr{U} := (U_i)_{i\in I}$ is a covering of X by open affine subsets $U_i = \operatorname{Spec}(R_i)$, then, for each i, the restriction $\mathscr{F}|_{U_i}$ is associated with some faithful finitely generated projective R_i-algebra A_i. Moreover, let σ_i, $i \in I$, be an anti-automorphism of A_i such that $x\sigma_i(x) \in R_i$, for all $x \in A_i$. Then, $\mathscr{F}$ admits exactly one standard involution $\widetilde{\sigma}$; in addition, $\widetilde{\sigma}$ commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$.*

Proof. According to Corollary 3.10, let $\widetilde{\sigma_i}$ be the only standard involution of the $\mathscr{O}|_{U_i}$-algebra $\mathscr{F}|_{U_i}$, where, by hypothesis, $\mathscr{F}|_{U_i}$ is the $\mathscr{O}|_{U_i}$-algebra associated with the faithful finitely generated projective R_i-algebra A_i. The morphism $\widetilde{\sigma} : \mathscr{F} \to \mathscr{F}$ such that $\widetilde{\sigma}|_{U_i} = \widetilde{\sigma}_i$ is well defined. Indeed, for all i, j such that $U_i \cap U_j \neq \emptyset$, $\widetilde{\sigma}_i|_{U_i \cap U_j} = \widetilde{\sigma}_j|_{U_i \cap U_j}$. Clearly, $\widetilde{\sigma}$ is the only standard involution on $\mathscr{F}$, and it commutes with all automorphisms and anti-automorphisms of $\mathscr{F}$. □

Corollary 3.12. *Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{I}$ be an $\mathscr{O}_X$-ideal generated by nowhere-zero global sections $(f_1, \ldots, f_n)$. The direct product $\mathscr{L} = \prod_{i=1}^n \mathscr{O}_{X,f_i}$ of the sheaves of rings of fractions $\mathscr{O}_{X,f_i}$ is faithfully flat if and only if $\mathscr{I} = \mathscr{O}_X$. Whenever $\mathscr{I} = \mathscr{O}_X$, the sheaf of rings $\mathscr{L}$ is called a* Zariski extension *of $\mathscr{O}_X$.*

Proof. For all $i = 1, \ldots, n$, the ring sheaf extension $\mathscr{O}_X \to \mathscr{O}_{X,f_i}$ is flat; therefore $\mathscr{L}$ is flat. Now, suppose that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{E} = 0$ for some $\mathscr{O}_X$-module $\mathscr{E}$; since $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{E} = 0$ if and only if $(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{E})_x = \mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{E}_x = 0_x = 0$, for all $x \in X$, it is sufficient to show that $\mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{E}_x = 0$ implies $\mathscr{E}_x = 0$. But then $\mathscr{I}_x$ is the ideal of $\mathscr{O}_{X,x}$ generated by germs $(f_{1,x}, \ldots, f_{n,x})$, $\mathscr{L}_x = \left(\prod_{i=1}^n \mathscr{O}_{X,f_i}\right)_x = \prod_{i=1}^n \left(\mathscr{O}_{X,x}\right)_{f_{i,x}}$ is faithfully flat if and only if $\mathscr{I}_x = \mathscr{O}_{X,x}$ (see [HM08, p. 24, Corollary 1.10.6]). □

Note that the notation $f_{i,x}$ in the above proof means the germ defined by the section f_i at the point $x \in X$. On the other hand, $\mathscr{O}_{X,f_i}$ is the sheaf obtained by sheafifying the presheaf, given by the assignment

$$U \mapsto \mathscr{O}_{X,f_i}(U),$$

where, for any open subset U of X,

$$\mathscr{O}_{X,f_i}(U) \equiv \mathscr{O}_X(U)_{f_i} = \left\{ \frac{s}{\rho_U^X(f_i)^n} = \frac{s}{(f_i|_U)^n} = \frac{s}{f_i^n|_U};\ s \in \mathscr{O}_X(U),\ n \geq 0 \right\}.$$

For the sake of the sequel, we recall that a *quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ on a scheme* $(X, \mathscr{O}_X)$ is called *locally projective if, for all $x \in X$, there exists an open affine neighborhood U of x such that $\mathscr{E}|_U$ is isomorphic to a direct summand of $\left(\mathscr{O}_X|_U\right)^{(I)}$, for some indexing set I,* (see [RG71, 3.1, 2nd part]). It is also proved in the same paper that a quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ is locally projective if and only if, for all open affine subschemes $U = \operatorname{Spec}(R) \subseteq X$, the restriction $\mathscr{E}|_U$ is isomorphic to some associated sheaf $\widetilde{P}$, where P is a projective R-module. We say that the *locally projective quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ is of constant rank n if, for any open affine subscheme U of X, the associated R-module P of $\mathscr{E}|_U$ is of constant rank n.*

Another useful result at the center of the proof of Theorem 3.13 is concerned with glueing of sheaves. Indeed, given any topological space X, suppose that $(U_i)_{i\in I}$ is an open covering of X, and, for each $i \in I$, $\mathscr{F}_i$ is a sheaf on U_i such that, for each $i,\ j \in I$, there is given an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i\cap U_j}$ satisfying properties: (1) $\varphi_{ii} = \mathrm{id}$, for all i, and (2) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$, for all $i,\ j,\ k \in I$. Then, there is a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that, for each $i,\ j$, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. This result is stated in [Har77, p.69].

Theorem 3.13. *Let X be a scheme and $\mathscr{E}$ a locally projective quasi-coherent $\mathscr{O}_X$-module of constant rank 2. Then, $\mathscr{E}$ is a commutative $\mathscr{O}_X$-algebra, endowed with a unique standard involution.*

Proof. Let $(U_i)_{i\in I} \equiv (U_i, \mathscr{O}_X|_{U_i})$ be an affine open covering of X. For $i \in I$, let P_i be a projective R-module with the property that $\mathscr{E}|_{U_i} \simeq \widetilde{P_i}$. Since P_i is a projective module of constant rank 2, it is a known fact that P_i is a commutative algebra, endowed with a unique standard involution σ_i, (cf. [HM08, p. 42, Theorem 1.13.10]). By Lemma 3.9, $\widetilde{\sigma_i}$ is a standard involution of $\widetilde{P_i}$, $i \in I$. But then $\widetilde{\sigma_i} \in \mathscr{H}om_{\mathscr{O}_X}(\widetilde{P_i}, \widetilde{P_i})(U_i)$, it follows that, for any pair (i, j) in $I \times I$ with $i \neq j$, $\widetilde{\sigma_i}|_{U_i\cap U_j} = \widetilde{\sigma_j}|_{U_i\cap U_j}$ is the unique involution on $\widetilde{P_i}|_{U_i\cap U_j} \simeq \widetilde{P_j}|_{U_i\cap U_j}$. The collection $(\widetilde{P_i}, \varphi_{ij})$, where φ_{ij} is the isomorphism $\widetilde{P_i}|_{U_i\cap U_j} \simeq \widetilde{P_j}|_{U_i\cap U_j}$, is a glueing data for sheaves of sets with respect to the covering $X = \cup_{i\in I} U_i$. Thus, there is a sheaf of sets $\mathscr{F}$ on X together with isomorphisms

$$\varphi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \widetilde{P_i},$$

that is,

$$\mathscr{F} \simeq \mathscr{E}.$$

Since $\widetilde{\sigma_i}|_{U_i\cap U_j} = \widetilde{\sigma_j}|_{U_i\cap U_j}$, there is a unique standard involution $\widetilde{\sigma}$ on $\mathscr{E}$ such that $\widetilde{\sigma}|_{U_i} = \widetilde{\sigma_i}$, $i \in I$. □

4 Main results

In this section, we purpose to discuss involutions of the first kind on $\mathscr{O}_X$-algebras $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$, where $\widetilde{M}$ is the sheaf of modules associated with an R-module M on an affine scheme $X = \mathrm{Spec}(R)$. Let N be another R-module and assume that $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{N})$ a sheaf isomorphism. For any open U of X, set

$$\alpha_U : \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})(U) \times \widetilde{N}(U) \to \widetilde{N}(U)$$

by

$$\alpha_U(f, s) = \varphi_{UU}(f_U)(s) \equiv \varphi(f)(s),$$

for any $f \equiv (f_V)_{U \supseteq V,\ open} \in \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})(U)$ and $s \in \widetilde{N}(U)$. The sheaf morphism $\alpha \equiv (\alpha_U)_{X \supseteq U,\ open}$ defines a left $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$-structure on $\widetilde{N}$; we denote $\widetilde{N}$ endowed with the left structure $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$-structure by ${}_{\varphi}\widetilde{N}$. In the similar way, we define ${}_{\varphi^{-1}}\widetilde{M}$. See [Knu91, p. 171, (8.2)].

For the purpose of the sequel, we recall the following (see [NY14]): Let X be a topological space, $\mathscr{A} \equiv (\mathscr{A}, \pi, X)$ a sheaf of unital and commutative algebras and $\mathscr{S} \equiv (\mathscr{S}, \pi|_{\mathscr{S}}, X)$ a sheaf of submonoids in $\mathscr{A}$. A *sheaf of algebras of fractions of $\mathscr{A}$ by $\mathscr{S}$* is a sheaf of algebras, denoted $\mathscr{S}^{-1}\mathscr{A}$, such that, for every $x \in X$, the corresponding stalk $(\mathscr{S}^{-1}\mathscr{A})_x$ is an algebra of fractions of $\mathscr{A}_x$ by $\mathscr{S}_x$.

In this context, we also recall the following:

Theorem 4.1. [NY14] *For all $\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$ on a topological space X, the $(\mathscr{S}^{-1}\mathscr{A})$-morphism*

$$\vartheta : \mathscr{S}^{-1}\mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}om_{\mathscr{S}^{-1}\mathscr{A}}(\mathscr{S}^{-1}\mathscr{E}, \mathscr{S}^{-1}\mathscr{F}),$$

given by

$$\vartheta_x(f/s)(e/t) = f(e)/st,$$

where $x \in X$, $s, t \in \mathscr{S}_x$, $e \in (\mathscr{S}^{-1}\mathscr{E})_x$, $f \in \mathscr{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{F})_x$, is an $(\mathscr{S}^{-1}\mathscr{A})$-isomorphism , whenever $\mathscr{E}$ is a locally finitely presented $\mathscr{A}$-module.

In the same vein, we recall the following isomorphism, whose proof may be found in [CF15, p.33, Lemme 2.4.1.6].

Lemma 4.2. *The natural map*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})(U) \to \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{E}|_U(U), \mathscr{F}|_U(U)) = \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{E}(U), \mathscr{F}(U)),$$

where U is open in X, is an isomorphism of modules if and only if the $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{F}$ are free or locally free of finite type and $X = \mathrm{Spec}(R)$.

Now, let us recall that an R-module M is *locally finitely presented* if $R = (f_i)$, $i \in I$, that is, R, as an ideal, is generated by some elements $f_i \in R$, and, for every $i \in I$, there is a presentation

$$R_{f_i}^{\mu_i} \longrightarrow R_{f_i}^{\eta_i} \longrightarrow M_{f_i} \longrightarrow 0\ ,$$

where η_i, $\mu_i \in \mathbb{N}$. It is known that, *given an R-module M, if M is locally finitely presented, then it is finitely presented.*

Lemma 4.3. *Let M be a locally of finite presentation R-module. Then,*

$$\widetilde{\mathrm{E}nd_R(M)} \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M}),$$

where $X = \mathrm{Spec}(R)$ and $\widetilde{M}$ the sheaf of modules associated with M.

Proof. For any $f \in R$, by Theorem 4.1,

$$\widetilde{\mathrm{E}nd_R(M)}(D(f)) = \mathrm{E}nd_{R_f}(M_f),$$

and

$$\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})(D(f)) = \mathrm{E}nd_{\mathscr{O}_X|_{D(f)}}(\widetilde{M}|_{D(f)}),$$

whence we have $\widetilde{\mathrm{E}nd_R(M)}(D(f)) \xrightarrow{\sim} \mathrm{E}nd_{\mathscr{O}_X|_{D(f)}}(\widetilde{M}|_{D(f)})$. (See Lemma 4.2 for the aforementioned isomorphism.) Moreover, since the $D(f)$ form a basis for the Zariski topology on X, the sought isomorphism follows. $\square$

Now, suppose that M and N are locally finitely presented progenerator R-modules such that $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M}) \to \mathscr{E}nd_{\mathscr{O}_X}(\widetilde{N})$ is an isomorphism; so the component $\varphi_X : \mathrm{E}nd_R(M) \to \mathrm{E}nd_R(N)$ is an R-module isomorphism. By [Knu91, p. 181, Lemma 8.2.1], there exist an invertible R-module L and an isomorphism $\rho : M \otimes L \to {}_{\varphi_X}N$ of $\mathrm{E}nd_R(M)$-modules such that $\varphi_X(f) = \rho(f \otimes 1)\rho^{-1}$, for every $f \in \mathrm{E}nd_R(M)$. (The $\mathrm{E}nd_R(M)$-structure on $M \otimes L$ is given by the assignment $(f, m \otimes l) \mapsto f(m) \otimes l$.) By the isomorphism ([Bos13, p. 258, Proposition 2])

$$\mathrm{H}om_R(\mathrm{E}nd_R(M), \mathrm{E}nd_R(N)) \xrightarrow{\sim} \mathrm{H}om_{\mathscr{O}_X}(\widetilde{\mathrm{E}nd_R(M)}, \widetilde{\mathrm{E}nd_R(N)})$$

given by $\alpha \mapsto \widetilde{\alpha}$, one has $\widetilde{\varphi_X} : \widetilde{\mathrm{E}nd_R(M)} \to \widetilde{\mathrm{E}nd_R(N)}$. But then, by virtue of Lemmas 4.2 and 4.3, $\widetilde{\varphi_X} = \varphi$; thereafter, by [Har77, p. 110, Proposition 5.2], the isomorphism $\widetilde{\rho} : \widetilde{M} \otimes \widetilde{L} \xrightarrow{\sim} {}_{\varphi}\widetilde{N}$ is such that

$$\widetilde{\varphi_X(f)} = \widetilde{\rho}(\widetilde{f \otimes 1})\widetilde{\rho}^{-1},$$

for every $f \in \mathrm{E}nd_R(M)$.

Thus, we have proved the following

Lemma 4.4. *Let M and N be locally finitely presented progenerator R-modules such that sheaves $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{M})$ and $\mathscr{E}nd_{\mathscr{O}_X}(\widetilde{N})$ are isomorphic (via an isomorphism φ), where $\widetilde{M}$ ($\widetilde{N}$, resp.) is the associated sheaf of R-modules for M (N, resp.) on the affine scheme $X = \mathrm{Spec}(R)$. Then , there exist an invertible R-module L and an isomorphism $\widetilde{\rho} : \widetilde{M} \otimes \widetilde{L} \xrightarrow{\sim} {}_\varphi\widetilde{N}$ such that $\widetilde{\varphi_X(f)} = \widetilde{\rho}(\widetilde{f \otimes 1})\widetilde{\rho}^{-1}$, for every $f \in \mathrm{E}nd_R(M)$.*

This result can be generalized to the following context: Let $\mathscr{E}$, $\mathscr{F}$ be locally finitely presented progenerator $\mathscr{O}_X$-modules on an affine scheme $X = \mathrm{Spec}(R)$, and $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$. For any open subset $U \subseteq X$, $\mathscr{F}(U)$ carries a left $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$-module structure; in fact, by Lemma 4.2, $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ is isomorphic to $\mathrm{E}nd_{\mathscr{O}_X(U)}(\mathscr{E}(U))$, and the action of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ on $\mathscr{F}(U)$ into $\mathscr{F}(U)$, is given by $(f, s) \mapsto \varphi_U(f)(s)$, for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ and $s \in \mathscr{F}(U)$. Hence, $\mathscr{F}$ turns out to be a left $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-module on X, and we shall denote it by ${}_\varphi\mathscr{F}$. In a similar way, ${}_{\varphi^{-1}}\mathscr{E}$ denotes $\mathscr{E}$ endowed with the right $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$-structure obtained through φ^{-1}.

The sought generalization can now be formulated as follows:

Lemma 4.5. *Let $\mathscr{E}$ and $\mathscr{F}$ be locally finitely presented progenerator $\mathscr{O}_X$-modules, where $X = \mathrm{Spec}(R)$, and let $\varphi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$. Then, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and an isomorphism $\rho : \mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} {}_\varphi\mathscr{F}$ of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-modules such that $\varphi_x(f) = \rho_x(f \otimes 1)\rho_x^{-1}$, for all $x \in X$, and $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})_x = \mathrm{E}nd_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$.*

Proof. By virtue of a variant of the well-known Morita equivalence for $\mathscr{O}_X$-stacks (see [KS06, p. 475, Theorem 19.5.4]), functors $(\ ) \otimes_{\mathscr{O}_X} \mathscr{E} : \mathfrak{M}_{\mathscr{O}_X} \to {}_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}\mathfrak{M}$ and $\mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E},\ ) : {}_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}\mathfrak{M} \to \mathfrak{M}_{\mathscr{O}_X}$ are inverse equivalences; consequently, the $\mathscr{O}_X$-module $\mathscr{L} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_\varphi\mathscr{F})$ is invertible with inverse $\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\varphi\mathscr{F}, \mathscr{E})$. We thus obtain an $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-isomorphism $\rho : \mathscr{L} \otimes \mathscr{E} \xrightarrow{\sim} {}_\varphi\mathscr{F}$ as in the classical case (see [Knu91, p. 171, Lemma 8.2.1]). For all $x \in X$, the localization $\rho_x : \mathscr{L}_x \otimes \mathscr{E}_x \to (\varphi\mathscr{F})_x$ is bijective, i.e., $\mathrm{H}om_{\mathrm{E}nd_{\mathscr{O}_x}(\mathscr{E}_x)}(\mathscr{E}_x, {}_{\varphi_x}\mathscr{F}_x) \otimes \mathscr{E}_x \to {}_{\varphi_x}\mathscr{F}_x$ is bijective. The equality $\varphi_x(f) = \rho_x(f \otimes 1)\rho_x^{-1}$, for all $x \in X$, and $f \in \mathrm{E}nd_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$, is guaranteed by [Knu91, p. 171, Lemma 8.2.1]. $\square$

Lemma 4.5 does not hold at the level of sections in general: for, the sheaf $\mathscr{E}\otimes\mathscr{L}$ is generated by the presheaf $(U\mapsto\mathscr{E}(U)\otimes\mathscr{L}(U))_{X\supseteq U,\ open}$, and $\mathscr{E}(U)\otimes\mathscr{L}(U)$ is not in general bijective to ${}_{\varphi_U}\mathscr{F}(U)$. However, section-wise, one may relax the conditions on progenerator $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{F}$ to obtain the following lemma.

Lemma 4.6. *Let $\mathscr{E}$ and $\mathscr{F}$ be locally finitely free progenerator $\mathscr{O}_X$-modules on $X=\operatorname{Spec}(R)$, and let $\varphi:\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})\xrightarrow{\sim}\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$. Then, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and an isomorphism $\rho:\mathscr{E}\otimes\mathscr{L}\xrightarrow{\sim}{}_{\varphi}\mathscr{F}$ of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$-modules such that, for every open set U in X, $\varphi_{UU}(s)=\rho_U(\widetilde{s\otimes 1})\rho_U^{-1}$, for all $s\in\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, and where $\widetilde{s\otimes 1}$ stands for the section of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}\otimes\mathscr{L})$ over U, corresponding to $s\otimes 1$ through sheafification.*

Before we succinctly go through different types of involutions on sheaves of Azumaya algebras, let us first recall the concept of Azumaya $\mathscr{O}_X$-algebra with involution on an affine scheme $X=\operatorname{Spec}(R)$.

Definition 4.7. An *Azumaya $\mathscr{O}_X$-algebra $(\mathscr{A},\sigma)$ with involution of the first kind* is a sheaf of Azumaya R-algebras on a scheme $X=\operatorname{Spec}(R)$ with an $\mathscr{O}_X$-linear involution σ.

Remark 4.8. If (M,σ) is an R-module with involution of the first kind σ, it is easy to see that $\widetilde{\sigma}$ is an $\mathscr{O}_X$-linear involution on the corresponding sheaf of R-modules $\widetilde{M}$. In fact, for any $f\in R$, $m\in M$, and $p\in\mathbb{N}$, $\widetilde{\sigma}_{D(f)}\left(\dfrac{m}{f^p}\right)=\dfrac{\sigma(m)}{f^p}$.

Let A *be an Azumaya R-algebra of constant rank* n^2 and with involution σ of the first kind. By [For17, p. 395, Corollary 10.3.10], *there exists a commutative faithfully flat étale R-algebra S such that $A\otimes_R S$ is isomorphic to* $\mathrm{M}_n(S)$. Let φ be an isomorphism $A\otimes_R S\xrightarrow{\sim}\mathrm{M}_n(S)$ that makes S into a faithfully flat splitting R-algebra of A, it induces an involution $\kappa=\varphi\circ(\sigma\otimes 1)\circ\varphi^{-1}$ on $\mathrm{M}_n(S)$. On considering the sheaves associated with $A\otimes_R S$ and $\mathrm{M}_n(S)$, respectively, on $X=\operatorname{Spec}(R)$, we have $\widetilde{A}\otimes_{\mathscr{O}_X}\widetilde{S}\equiv\widetilde{A}\otimes_{\widetilde{R}}\widetilde{S}\xrightarrow{\sim}\widetilde{\mathrm{M}_n(S)}$. By virtue of [Har77, p. 110, Proposition 5.2], $\widetilde{\kappa}=\widetilde{\varphi}\circ(\widetilde{\sigma}\otimes 1)\circ\widetilde{\varphi^{-1}}=\widetilde{\varphi}\circ(\widetilde{\sigma}\otimes 1)\circ(\widetilde{\varphi})^{-1}$ is the induced involution on $\widetilde{\mathrm{M}_n(S)}$. The map $\Gamma:\widetilde{\mathrm{M}_n(S)}\to\widetilde{\mathrm{M}_n(S)}$, given by $\Gamma_U(s)=\widetilde{\kappa}_U(s^t)$, where $s\in\widetilde{\mathrm{M}_n(S)}(U)$ and s^t means the transpose of s, is clearly an automorphism of $\widetilde{\mathrm{M}_n(S)}$ and corresponds to the automorphism $x\mapsto\kappa(x^t)$ of $\mathrm{M}_n(S)$. By choosing S such that $\kappa(x)=vx^tv^{-1}$, for any $x\in$

$\widetilde{\mathrm{M}_n(S)}$ and for some $v \in \widetilde{\mathrm{GL}_n(S)}$, for any open U in X, $\kappa_U(s) = us^tu^{-1}$, where $s \in \widetilde{\mathrm{M}_n(S)}(U)$ and $u \in \widetilde{\mathrm{GL}_n(S)}(U)$. In [Knu91, p. 170], there is $\varepsilon \in \mu_2(S)$ ($\mu_2(S) = \{x \in S|\ x^2 = 1\}$) such that $v^t = \varepsilon v$. Next, let us show that the corresponding

$$U \mapsto \mu_2(\widetilde{S}(U)) \tag{4.1}$$

yields a complete presheaf (of groups). That the correspondence given by (4.1) is a presheaf is clear. In order to show the *completeness* of this presheaf, let U be an open subset of X, and $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ an open covering of U; moreover, let s, $t \in \mu_2(\widetilde{S}(U))$ such that

$$\rho^U_{U_\alpha}(s) \equiv s|_{U_\alpha} \equiv s_\alpha = t_\alpha \equiv t|_{U_\alpha} \equiv \rho^U_{U_\alpha}(t), \quad \alpha \in I,$$

where the $(\rho^U_{U_\alpha})_{\alpha \in I}$ are the restriction maps of the sheaf $\widetilde{S}$. Since $\mu_2(\widetilde{S}(U)) \subseteq \widetilde{S}(U)$ and $\widetilde{S}$ is the $\mathscr{O}_X$-module attached to the R-algebra S, $s = t$.

On the other hand, consider any sequence

$$(s_\alpha) \in \prod_{\alpha \in I} \mu_2(\widetilde{S}(U_\alpha)) \subseteq \prod_{\alpha \in I} \widetilde{S}(U_\alpha)$$

such that

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\alpha|_{U_\alpha \cap U_\beta},$$

for any α, $\beta \in I$, with $U_\alpha \cap U_\beta \neq \emptyset$. There is an element $s \in \widetilde{S}(U)$ such that

$$s|_{U_\alpha} = s_\alpha, \quad \alpha \in I.$$

Thus,

$$(s^2)|_{U_\alpha} = \rho^U_{U\alpha}(s^2) = \rho^U_{U_\alpha}(s)\rho^U_{U_\alpha}(s) = 1|_{U_\alpha}, \quad \alpha \in I.$$

One infers that $s^2 = 1 \in \widetilde{S}(U)$, so that $s \in \mu_2(\widetilde{S}(U))$. Hence, the presheaf is complete, and the proof is finished.

Going back to the involution $\widetilde{\kappa}$, it follows that, given any open U in X, the equation $\kappa_U(s) = us^tu^{-1}$, where $u \in \widetilde{\mathrm{GL}_n(S)}(U)$, entails, for some $\varepsilon \in \mu_2(\widetilde{S}(U))$, $u^t = \varepsilon u$. As in the classical case, an involution $s \mapsto us^tu^{-1}$ of the $\mathscr{O}_X$-module $\widetilde{M_n(S)}$, where $u^t = \varepsilon u$, $s \in \widetilde{M_n(S)}(U)$, $u \in \widetilde{\mathrm{GL}_n(S)}(U)$, and $\varepsilon \in \mu_2(\widetilde{S}(U))$ is said to be *of type* ε *on the open subset* U, and κ_U is denoted κ_u.

Lemma 4.9. *Let $\mathscr{E}$ be a sheaf of modules over a scheme X, and σ an $\mathscr{O}_X$-endomorphism of $\mathscr{E}$. Then, σ is an involution if and only if , for every $x \in X$, $\sigma_x : \mathscr{E}_x \to \mathscr{E}_x$ is an involution.*

Proof. It is known that σ is bijective if and only if σ_x is bijective for all $x \in X$. (See, for instance, [Bos13, p. 233, Proposition 3].) Therefore, we need only show that σ is an anti-isomorphism if and only if σ_x is an anti-isomorphism, for all $x \in X$. The *only-if* part is easily seen from the characterization of stalks. To settle the *if* part, observe that, if U is an open neighborhood of x, and $\mathscr{U}(x)$ denotes the set of all open sets containing x, and f, $g \in \mathscr{E}(U)$,

$$\varinjlim_{V\in\mathscr{U}(x)} \sigma_V(f\cdot g) = \sigma_x(f_x\cdot g_x) = \sigma_x(g_x)\sigma_x(f_x) = \varinjlim_{V\in\mathscr{U}(x)} \sigma_V(g)\cdot \varinjlim_{V\in\mathscr{U}(x)} \sigma_V(f)$$

$$= \varinjlim_{V\in\mathscr{U}(x)} \sigma_V(g)\sigma_V(f),$$

which entails that, for some open neigbourhood V^x of x in U, $\sigma_{V^x}(f|_{V^x}\cdot g|_{V^x}) = \sigma_{V^x}(g|_{V^x})\sigma_{V^x}(f|_{V^x})$. By Sheaf Axiom (S1), $\sigma_U(f\cdot g) = \sigma_U(g)\sigma_U(f)$. For the last displayed equality, see [Bou68, p. 211, (35)]. □

Definition 4.10. Let X be a scheme. An *Azumaya $\mathscr{O}_X$-algebra $\mathscr{E}$ with involution of the first kind* is a sheaf of Azumaya algebras with involution of the first kind on X, i.e., an involution that leaves the center elementwise invariant.

It is clear that, given an open set U in X and sections f, g of an $\mathscr{O}_X$-algebra $\mathscr{E}$ over U, if $f_x\cdot g_x = g_x\cdot f_x$, for all $x\in U$, $f\cdot g = g\cdot f$. It follows that an involution σ of $\mathscr{E}$ fixes the center of $\mathscr{E}$ elementwise if and only if , for every $x \in X$, σ_x keeps fixed the center of $\mathscr{E}_x$ elementwise. Hence, σ is an involution of the first kind on $\mathscr{E}$ if and only if, for every $x\in X$, σ_x is an involution of $\mathscr{E}_x$ of the first kind.

Lemma 4.11. *Let $\mathscr{E}$ be a locally finitely presented $\mathscr{O}_X$-module on an affine scheme $X = \operatorname{Spec}(R)$, and let σ be an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. Then, there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a sheaf isomorphism φ of $\mathscr{E}\otimes_{\mathscr{O}_X}\mathscr{L}$ onto $\mathscr{E}^*$, and an $\mathscr{O}_X$-isomorphism $\Phi : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$ such that, on every open U in X where $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$,*

$$\sigma\otimes \mathrm{Id} = \Phi\circ m, \tag{4.2}$$

where m is the natural isomorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \otimes_{\mathscr{O}_X} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L}) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ on U, and , for any open V in U , and any section s of $\mathscr{E}nd_{\mathscr{O}_X}\mathscr{E}$ over V, $\Phi(s) \equiv \Phi_{VV}(s) = \varphi_V^{-1} s^* \varphi_V \equiv \varphi^{-1} s^* \varphi$, and s^* is the image of s through the natural morphism $\mathscr{E}nd_{\mathscr{O}_X}\mathscr{E} \to \mathscr{E}nd_{\mathscr{O}_X}\mathscr{E}^*$.

Proof. As in the proof of Lemma 4.5, by letting $\Phi = \tau\sigma$, where τ is the anti-automorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$, and by virtue of Morita theory, the $\mathscr{O}_X$-algebra $\mathscr{L} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}}(\mathscr{E}, {}_{\Phi}\mathscr{E}^*)$ is invertible. It is clear that Φ locally satisfies Equation (4.2). $\square$

In order to proceed further, let us recall the corresponding (sheaf-theoretic) notion of a center of a group. Precisely, let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathscr{E}$ a vector sheaf on X of constant rank n. The $\mathscr{O}_X$-module $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ is also clearly locally free and of constant rank n^2 (see [Mal98, p. 138, Equation (6.26)]). On considering the correspondence

$$U \mapsto Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)) \simeq Z(\mathscr{O}_X{}^{n^2}|_U) \simeq Z((\mathscr{O}_X|_U)^{n^2})$$

(where $Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U))$ consists of all $\vartheta \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ such that $\vartheta \circ \varphi = \varphi \circ \vartheta$, for all $\varphi \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$) together with the obvious restriction maps yields a complete presheaf, called the *(pre)sheaf of centers of groups.* On any local gauge U of the vector sheaf $\mathscr{E}$, one has

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})|_U \simeq \mathscr{O}_X{}^{n^2}|_U \simeq \mathrm{M}_n(\mathscr{O}_X)|_U;$$

therefore,

$$Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)) \simeq Z(\mathrm{M}_n(\mathscr{O}_X(U))).$$

Lemma 4.12. *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{E}$ a locally finitely presented $\mathscr{O}_X$-module with involution of the first kind σ on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. Then, for any local gauge U,*

$$(\varphi^*\eta \otimes 1)\varphi^{-1} \in Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)(U)),$$

where η is the canonical $\mathscr{O}_X$-isomorphism $\mathscr{E} \to \mathscr{E}^{**}$, and φ is the $\mathscr{O}_X$-isomorphism $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ of Lemma 4.11. Furthermore, for some $\varepsilon \in \mu_2(\mathscr{O}_X(U))$,

$$\varepsilon\varphi^*\eta \otimes 1 = \varphi. \tag{4.3}$$

(N.B. For any open open V in X, $\eta(s)(u) \equiv s^{**}(u) := u(s)$, where $s \in \mathscr{E}(V)$, and $u \in \mathscr{E}^*(V) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})(V) = \mathrm{Hom}_{\mathscr{O}_X|_V}(\mathscr{E}|_V, \mathscr{E}|_V)$.)

Proof. From Equation (4.2),

$$\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi,$$

for all $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, where U is a local gauge of $\mathscr{L}$. Since $\sigma^2 = 1$, it follows that

$$s \otimes 1 = \varphi^{-1} \sigma(s)^* \varphi. \tag{4.4}$$

On transposing (4.4), we obtain

$$s^* \otimes 1 = \varphi^* \sigma(s)^{**} (\varphi^{-1})^*.$$

But then

$$\sigma^{**}(s^{**}) = \sigma(s)^{**} = \eta \sigma(s) \eta^{-1},$$

so

$$(\varphi^* \eta)^{-1} \circ (s^* \otimes 1) \circ (\varphi^* \eta) = \sigma(s). \tag{4.5}$$

Tensoring (4.5) with 1 yields, under the assumption $\mathscr{L}^* \otimes \mathscr{L} \simeq \mathscr{O}_X$, which allows one to identify $s^* \otimes 1 \otimes 1$ with s^*,

$$(\varphi^* \eta \otimes 1)^{-1} \circ s^* \circ (\varphi^* \eta \otimes 1) = \sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi.$$

Hence,

$$(\varphi^* \eta \otimes 1) \varphi^{-1} \in Z(\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)).$$

By virtue of (4.2), and since $Z(\mathrm{M}_n(\mathscr{O}_X(U))) \simeq \mathscr{O}_X(U)$,

$$(\varphi^* \eta \otimes 1) \varphi^{-1} = \varepsilon$$

for some $\varepsilon \in \mathscr{O}_X(U)$. It is clear that ε must be invertible, that is, $\varepsilon \in \mathscr{O}_X(U)^\bullet = \mathscr{O}_X^\bullet(U)$, with $\mathscr{O}_X(U)^\bullet$ the group of units of the unital ring $\mathscr{O}_X(U)$. ($\mathscr{O}_X^\bullet$ is the sheaf on X generated by the presheaf defined by the correspondence

$$U \mapsto \mathscr{O}_X^\bullet \simeq \mathscr{O}_X(U)^\bullet,$$

where U varies over the Zariski topology of X. (See [Mal98, p. 282, Lemma 1.1]).) $\square$

Corollary 4.13. *The section $\varepsilon \in \mathscr{O}_X(U)$ satisfies the condition: $\varepsilon^2 = 1$.*

Proof. First, note that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{E}(U)\otimes\mathscr{L}(U) \arrow[r, "\varphi_U"] \arrow[d, "\eta_U\otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}^*(U)\otimes\mathscr{O}_X(U)\simeq\mathscr{E}^*(U)\ , \\
\mathscr{E}^{**}(U)\otimes\mathscr{L}(U) \arrow[r, "\varphi_U^*\otimes 1_{\mathscr{L}(U)}"'] & \mathscr{E}^*(U)\otimes\mathscr{L}^*(U)\otimes\mathscr{L}(U) \arrow[u, "1_{\mathscr{E}^*(U)}\otimes\mu_U"']
\end{tikzcd}$$

where μ is the canonical isomorphism $\mathscr{L}^*\otimes\mathscr{L}\xrightarrow{\sim}\mathscr{O}_X$, and ε, in the center of the diagram, means that the diagram commutes up to a factor ε.

Note that, on U, $\mathscr{E}^*(U)=\mathscr{E}(U)^*$, $\mathscr{E}^{**}(U)=\mathscr{E}(U)^{**}$, and $\mathscr{L}^*(U)=\mathscr{L}(U)^*$. On transposing the diagram above, one obtains:

$$\begin{tikzcd}
\mathscr{E}(U)^{**} \arrow[r, "\varphi_U^*"] \arrow[d, "1_{\mathscr{E}(U)^{**}}\otimes\mu_U^*"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}(U)^*\otimes\mathscr{L}(U)^* \\
\mathscr{E}(U)^{**}\otimes\mathscr{L}(U)^{**}\otimes\mathscr{L}(U)^* \arrow[r, "\varphi_U^{**}\otimes 1_{\mathscr{L}(U)^*}"'] & \mathscr{E}(U)^{***}\otimes\mathscr{L}(U)^*. \arrow[u, "\eta_U^*\otimes 1_{\mathscr{L}(U)^*}"']
\end{tikzcd}$$

Tensoring with $\mathscr{L}(U)$ and taking into account the isomorphism $\mathscr{L}(U)^*\otimes\mathscr{L}(U)\simeq\mathscr{O}_X(U)$ yields:

$$\begin{tikzcd}
\mathscr{E}(U)^{**}\otimes\mathscr{L}(U) \arrow[r, "\varphi_U^*\otimes 1_{\mathscr{L}(U)}"] \arrow[d, "1_{\mathscr{E}(U)^{**}}\otimes\mu_U^*\otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}(U)^* \\
\mathscr{E}(U)^{**}\otimes\mathscr{L}(U)^{**}\otimes\mathscr{L}(U)^*\otimes\mathscr{L}(U) \arrow[r, "\varphi_U^{**}"'] & \mathscr{E}(U)^{***}. \arrow[u, "\eta_U^*"']
\end{tikzcd}$$

Superposing the first diagram with the last one, one obtains

$$\begin{tikzcd}
\mathscr{E}(U)\otimes\mathscr{L}(U) \arrow[r, "\varphi_U"] \arrow[d, "\eta_U\otimes 1_{\mathscr{L}(U)}"'] \arrow[dr, phantom, "\varepsilon"] & \mathscr{E}^*(U)\otimes\mathscr{O}_X(U)\simeq\mathscr{E}^*(U)\ . \\
\mathscr{E}^{**}(U)\otimes\mathscr{L}(U) \arrow[r, "\varphi_U^*\otimes 1_{\mathscr{L}(U)}"'] \arrow[d, "1_{\mathscr{E}(U)^{**}}\otimes\mu_U^*\otimes 1_{\mathscr{L}(U)}"'] & \mathscr{E}^*(U)\otimes\mathscr{L}^*(U)\otimes\mathscr{L}(U) \arrow[u, "1_{\mathscr{E}(U)^*}"'] \\
\mathscr{E}(U)^{**}\otimes\mathscr{L}(U)^{**}\otimes\mathscr{L}(U)^*\otimes\mathscr{L}(U) \arrow[r, "\varphi_U^{**}"'] & \mathscr{E}(U)^{***} \arrow[u, "\eta_U^*"']
\end{tikzcd}$$

From the outer contour, one has: $\eta_U^*\varphi_U^{**}\eta_U=\varepsilon^2\varphi_U$ or, equivalently, $\varphi_U^{**}\eta_U=\varepsilon^2\eta_U\varphi_U$. But then, $\varphi_U^{**}\eta_U=\eta_U\varphi_U$, hence, $\varepsilon^2=1$. $\square$

Theorem 4.14. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, $\mathscr{E}$ a locally finitely presented $\mathscr{O}_X$-module, and $\sigma : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ an involution of the first kind. Moreover, let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and φ an isomorphism $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ such that $\sigma \otimes Id = \Phi \circ m$, where $\Phi = \tau\sigma$ with τ the anti-$\mathscr{O}_X$-automorphism $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}^*)$, and $m : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E} \otimes \mathscr{L})|_U \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})|_U$, with U a local gauge of $\mathscr{L}$. Then, for any $x \in X$, there is $u \in \mathscr{L}_x$ such that*

$$\sigma_x(f) = u^{-1} \circ f^* \circ u,$$

for any $f \in \mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$, i.e., $\sigma_x = \sigma_{x,u^{-1}}$. Furthermore, for any local gauge V of $\mathscr{L}$ at x, there is a unit $\varepsilon \in \mathscr{O}_X(V)$ such that

$$\varepsilon_x u(q)(p) = u(p)(q),$$

for all p, $q \in \mathscr{E}_x$.

Proof. For all $x \in X$, observe that $\mathscr{E}_x$ is a finitely presented $\mathscr{O}_{X,x}$-module (see [Mal98, p. 101, (1.54) and (1.55)]); since $\mathscr{L}_x$ is invertible over a local ring $\mathscr{O}_{X,x}$, it is necessarily free. Therefore, $\mathscr{L}_x \simeq u\mathscr{O}_{X,x} \simeq \mathscr{O}_{X,x}$ for some $u \in \mathscr{L}_x$. By Lemma 4.11, $\mathscr{L} = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_\Phi\mathscr{E}^*)$, where $\Phi = \tau\sigma$. Since $\mathscr{E}$ is locally finitely presented, $\mathscr{L}_x = \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E}, {}_\Phi\mathscr{E}^*)_x \simeq \mathrm{H}om_{\mathrm{E}nd_{\mathscr{O}_{X,x}(\mathscr{E}_x)}}(\mathscr{E}_x, {}_{\Phi_x}(\mathscr{E}_x)^*)$. Since $\sigma(s) \otimes 1 = \varphi^{-1}s^*\varphi$, for any section s of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, or equivalently, $\varphi \circ (\sigma(s) \otimes 1) = s^* \circ \varphi$. Stalk-wise, we have that, for any $x \in X$, $\varphi_x \circ (\sigma_x(s_x) \otimes 1) = s_x^* \circ \varphi_x \in \mathrm{H}om_{\mathrm{E}nd_{\mathscr{O}_{X,x}}(\mathscr{E}_x)}(\mathscr{E}_x \otimes \mathscr{L}_x, \mathscr{E}_x^*)$, where $\varphi_x(p \otimes u) = u(p)$, for all $p \in \mathscr{E}_x$; hence, $(s_x^* \circ \varphi_x)(p \otimes u) = (s_x^* \circ u)(p) \in \mathscr{E}_x^*$. On the other hand, $(\varphi_x \circ (\sigma_x(s_x) \otimes 1))(p \otimes u) = u(\sigma_x(s_x)(p))$. Thus, $s_x^* \circ u = u \circ \sigma_x(s_x)$ and $\sigma_x(s_x) = u^{-1} \circ s_x^* \circ u$. If $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$ is represented by the matrix sheaf $\mathrm{M}_n(\mathscr{O}_X) \simeq \mathscr{O}_X^{n^2}$ (rank$\mathscr{E} = n$), then $\mathscr{E}nd_{\mathscr{O}_{X,x}}(\mathscr{E}_x) \simeq \mathscr{O}_{X,x}^{n^2}$, we have $\sigma_x = \sigma_{x,u^{-1}}$.

Now, $\varphi_x^*\eta_x \otimes 1 : \mathscr{E}_x \otimes \mathscr{L}_x \to (\mathscr{E}_x \otimes \mathscr{L}_x)^* \otimes \mathscr{L}_x \simeq \mathscr{E}_x^*$ maps $p \otimes u$ onto $\varphi_x^*(\eta_x(p)) \otimes u$. Since $\mathscr{L}_x$ is free of rank 1, we may use u to identify $\mathscr{E}_x \otimes \mathscr{L}_x$ with $\mathscr{E}_x$; then $\varphi_x^*\eta_x : \mathscr{E}_x \to \mathscr{E}_x^*$ maps p onto $\varphi_x^*(\eta_x(p)) \in \mathscr{E}_x^*$, which is the mapping $q \mapsto \eta_x(p)(u(q)) = u(q)(p)$. On the other hand, since $\mathscr{E}_x \otimes \mathscr{L}_x \xrightarrow{\sim} \mathscr{E}_x$, we may assume φ_x to be an isomorphism $\mathscr{E}_x \to \mathscr{E}_x^*$; therefore, $\varphi_x(p)(q) = u(p)(q)$. It follows that

$$\varepsilon_x u(q)(p) = u(p)(q),$$

for all p, $q \in \mathscr{E}_x$. □

Note that one arrives at a similar result section-wise when one considers any vector sheaf $\mathscr{E}$ of finite rank on a locally ringed space $(X, \mathscr{O}_X)$.

Theorem 4.15. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, $\mathscr{E}$ a vector sheaf of finite rank n on X, and σ an involution of the first kind on $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$. Moreover, let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $\varphi \equiv (\varphi_V)_{X \supseteq V,\ open}$ an isomorphism $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ such that $\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi$, for any $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$ and any $\varphi_U : (\mathscr{E} \otimes \mathscr{L})(U) \xrightarrow{\sim} \mathscr{E}^*(U)$ or $\varphi_U : \mathscr{E}(U) \xrightarrow{\sim} \mathscr{E}^*(U)$, where the open subset U of X is chosen such that both $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq \mathscr{O}_X^n|_U$ are satisfied. Then, there is $u \in \mathscr{L}(U)$ such that*

$$\sigma(f) = u^{-1} \circ f^* \circ u,$$

for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U) = \mathrm{End}_{\mathscr{O}_X|_U}(\mathscr{E}|_U)$. Furthermore, there is a unit $\varepsilon \in \mathscr{O}_X(U)$ such that

$$\varepsilon u(t)(r) = u(r)(t), \tag{4.6}$$

for all r, $t \in \mathscr{E}(U)$.

Proof. Let $x \in X$, V an open neighborhood of x such that $\mathscr{E}|_V \simeq \mathscr{O}_X^n|_V$, and W a local gauge of $\mathscr{L}$ at x, i.e., $\mathscr{L}|_W \simeq \mathscr{O}_X|_W$. Define: $U = V \cap W$. Since $\mathscr{O}_X(U)$ is a local ring and $\mathscr{L}(U)$ is invertible over $\mathscr{O}_X(U)$, $\mathscr{L}(U)$ is free. Therefore, $\mathscr{L}(U) \simeq u\mathscr{O}_X(U) \simeq \mathscr{O}_X(U)$, for some $u \in \mathscr{L}(U)$. For any section $s \in \mathscr{E}(U)$, $\varphi \circ (\sigma(s) \otimes 1) = s^* \circ \varphi \in \mathscr{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})}(\mathscr{E} \otimes \mathscr{L}, \mathscr{E}^*)(U) = \mathrm{H}om_{\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})|_U}((\mathscr{E} \otimes \mathscr{L})|_U, \mathscr{E}^*|_U) = \mathrm{H}om_{\mathscr{E}nd_{\mathscr{O}_X|_U}(\mathscr{E}|_U)}(\mathscr{E}|_U \otimes \mathscr{L}|_U, \mathscr{E}^*|_U) = \mathrm{H}om_{\mathscr{E}nd_{\mathscr{O}_X|_U}(\mathscr{E}|_U)}(\mathscr{E}|_U, \mathscr{E}^*|_U)$. For any $r \in \mathscr{E}(U)$, $\varphi(r \otimes u) = u(r)$; therefore, $(s^* \circ \varphi)(r \otimes u) = (s^* \circ u)(r) \in \mathscr{E}^*(U)$. On another side, $(\varphi \circ (\sigma(s) \otimes 1))(r \otimes u) = \varphi(\sigma(s)(r) \otimes u) = u(\sigma(s)(r))$. Thus, $s^* \circ u = u \circ \sigma(s)$ or $\sigma(s) = u^{-1} \circ s^* \circ u$.

Since $\mathscr{L}^* \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X$, one has

$$\varphi^* \eta \otimes 1 : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \xrightarrow{\sim} \mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{L}^* \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{E}^*;$$

therefore, for any open U in X with $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq {\mathscr{O}_X}^n|_U$, any section $r \otimes u$ of the $\mathscr{O}_X$-module $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}$ maps onto $\varphi^*(\eta(r)) \otimes u$. Since $\mathscr{L}|_U$ is free of rank 1, we may use a suitably chosen u, namely any nowhere-zero section, as an isomorphism $(\mathscr{E} \otimes \mathscr{L})|_U \xrightarrow{\sim} \mathscr{E}|_U$; then $\varphi^*\eta|_U : \mathscr{E}|_U \to \mathscr{E}^*|_U \simeq (\mathscr{E}|_U)^*$ maps r onto $\varphi^*(\eta(r)) \in \mathscr{E}^*(U) = \mathscr{E}(U)^*$, which in turn maps a section t in $\mathscr{E}^*(U)$ onto $\eta(r)(u(t)) = u(t)(r)$. On the other hand, since $\mathscr{E}|_U \otimes_{\mathscr{O}_X|_U} \mathscr{L}|_U \xrightarrow{\sim} \mathscr{E}|_U$, we may assume φ to be an isomorphism $\mathscr{E}|_U \to \mathscr{E}^*|_U$;

therefore, $\varphi(r)(t) = u(r)(t)$. From Equation (4.3), it follows that, for some $\varepsilon \in \mathscr{O}_X(U)^\bullet$,

$$\varepsilon u(t)(r) = u(r)(t),$$

for all r, $t \in \mathscr{E}(U)$. $\square$

Corollary 4.16. *Let R be a commutative ring such that the induced ringed space $(X, \mathscr{O}_X)$ is a locally ringed space; let $\mathscr{E}$ be a vector sheaf of finite rank n on X, σ an involution of the first kind on the vector sheaf $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$, and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module such that $\mathscr{E} \otimes \mathscr{L} \xrightarrow{\sim} \mathscr{E}^*$ is an isomorphism φ with $\sigma(s) \otimes 1 = \varphi^{-1} s^* \varphi$, for any $s \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, where U is any open subset of X such that $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$ and $\mathscr{E}|_U \simeq {\mathscr{O}_X}^n|_U$. Then, on identifying $\mathscr{E}|_U$ with $(\mathscr{E}|_U)^* = \mathscr{E}^*|_U$ with the help of some section u of $\mathscr{L}$, where $\sigma(f) = u^{-1} \circ f^* \circ u$, for any $f \in \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U)$, and identifying $\mathscr{E} \otimes \mathscr{E}^*$ with $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})$,*

$$\sigma(r \otimes s) = \varepsilon u^{-1}(s) \otimes u(r),$$

for $\varepsilon \in \mathscr{O}_X^\bullet$, $r \in \mathscr{E}(U)$ and $s \in \mathscr{E}^*(U)$.

Proof. For the sake of containedness, we recall that, given any $\mathscr{O}_X$-modules $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ with $\mathscr{E}$ or $\mathscr{G}$ being locally finitely free, the functorial homomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \tag{4.7}$$

is an isomorphism, (see [GW10, p. 177, Proposition 7.7]). In particular, for any vector sheaf $\mathscr{E}$ of finite rank on X,

$$\mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{E} \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) = \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E}).$$

It follows that since, for some section u of $\mathscr{L}$, one has: $u : \mathscr{E}|_U \xrightarrow{\sim} \mathscr{E}^*|_U$, and $\sigma(r \otimes s) = u^{-1} \circ (r \otimes s)^* \circ u$, where $r \otimes s \in \mathscr{E}(U) \otimes \mathscr{E}^*(U) = (\mathscr{E} \otimes \mathscr{E}^*)(U) \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{E})(U) = \mathrm{End}_{\mathscr{O}_X|_U}(\mathscr{E}|_U)$. The transpose $(r \otimes s)^* : \mathscr{E}^*|_U \to \mathscr{E}^*|_U$ is such that $(r \otimes s)^*(u(t)) = u(t) \circ (r \otimes s)$, for any section t of $\mathscr{E}$ on U. It is clear that, for any $z \in \mathscr{E}(U)$,

$$(u(t) \circ (r \otimes s))(z) = u(t)(s(z)r) = u(t)(r)s(z),$$

viz.

$$u(t) \circ (r \otimes s) = u(t)(r)s.$$

Consequently, on using (4.6), one has

$$\sigma(r \otimes s) = u(t)(r)u^{-1}(s) = \varepsilon u(r)(t)u^{-1}(s).$$

Thus,

$$\sigma(r \otimes s) = \varepsilon u^{-1}(s) \otimes u(r),$$

for $r \in \mathscr{E}(U)$ and $s \in \mathscr{E}^*(U)$. □

References

[Bas64] H. Bass, *R-theory and stable algebra,* Publications mathématiques de l' I.H.È.S. **22** (1964), 5–60. ↑8

[Bas68] ———, *Algebraic R-theory.*, W. A. Benjamin, Inc., New York, 1968. ↑7

[Bos13] S. Bosch, *Algebraic Geometry and Commutative Algebra*, Springer, London, 2013. ↑12, 13, 18, 22

[Bou68] N. Bourbaki, *Elements of Mathematics, Theory of Sets*, Hermann, Paris, 1968. ↑22

[Bou58] ———, *Éléments de mathématique. Algèbre. Chapitre 8*, Hermann, Paris, 1958. ↑6

[CF15] B. Calmès and J. Fasel, *Groupes classiques*, Autours des schémas en groupes. Vol. II, 2015, pp. 1–133. MR3525594 ↑17

[For17] T. J. Ford, *Separable algebras*, Graduate Studies in Mathematics, vol. 183, American Mathematical Society, Providence, RI, 2017. MR3618889 ↑2, 4, 5, 6, 7, 8, 9, 11, 20

[GW10] U. Görtz and T. Wedhorn, *Algebraic Geometry I. Schemes with Examples and Exercises*, Springer Fachmedien, Germany, 2010. MR2675155 ↑28

[Gro95] A. Grothendieck, *Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses* {MR0244269 (39 #5586a), Séminaire Bourbaki, Vol. 9, 1995, pp. Exp. No. 290, 199–219. MR1608798 ↑3

[KS06] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006. ↑19

[Knu91] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 294, Springer-Verlag, Berlin, 1991. ↑2, 4, 17, 18, 19, 21

[KMRT98] M. A. Knus, A.S. Merkurjev, M. Rost, and J. P. Tignol, *The Book of Involutions*, Amer. Math. Soc. Colloquium Publications, Amer. Math. Soc., 1998. ↑3, 9, 10, 12

[KO74] M.-A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Mathematics, Vol. 389, Springer-Verlag, Berlin, 1974. MR0417149 (54 #5209) ↑2

[Har77] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977. MR0463157 (57 #3116) ↑13, 16, 18, 20

[HM08] J. Helmstetter and A. Micali, *Quadratic Mappings and Clifford Algebras*, Birkhauser, 2008. MR2408410 ↑13, 14, 15, 16

[Mal98] A Mallios, *Geometry of Vector Sheaves, An Axiomatic Approach to Differential Geometry. Volume I: Vector Sheaves. General Theory*, Kluwer Academic Publishers, 1998. MR1628461 ↑23, 24, 26

[NY14] P.P. Ntumba and B. Yizengaw, *On the commutativity of the Clifford and "extension of scalars" functors*, Topology and its Applications **168** (2014), 159–179. MR3196848 ↑17

[RG71] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de platification d' un module*, Inventiones Math. **13** (1971), 1–89. MR0308104 ↑15

[DI71] Frank DeMeyer and Edward Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin-New York, 1971. MR0280479 ↑6