Stress-strength reliability inference for the Pareto distribution with outliers

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Abstract

Estimation of the stress-strength parameter, $\mathcal{R} = \Pr(X < Y)$, is perhaps one of the challenging concepts in the reliability analysis. The estimation of $\mathcal{R}$ often criticized for its lack of stability and robustness against the presence of outliers and extreme values. The issue of estimating $\mathcal{R}$ under the presence of outliers is considered in this contribution for independently distributed random variables X and Y by the Pareto-based models. It is assumed that X has the Pareto distribution in the presence of outliers, whereas the random variable Y follows uncontaminated Pareto distribution. Under various assumptions on the parameters of the model, the maximum likelihood, method of moments, least squares, and modified maximum likelihood estimators are obtained. The shrinkage estimate of the stress-strength reliability parameter is also derived for each case using a prior guess, $\mathcal{R}_0$. We conduct a Monte Carlo simulation study to compare the proposed methods of estimation. Finally, the performance of the postulated methodology is illustrated by analyzing two real-world datasets in the physical and insurance studies.

Keywords: Stress-strength parameter, Outliers, Shrinkage estimation, Pareto distribution, Maximum likelihood estimate, Method of moments estimate, Least squares estimate
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1. Introduction

The stress-strength parameter, originally proposed by Birnbaum (1956), has widely been acknowledged in statistical research to show the system's efficiency. In reliability theory, $\mathcal{R} = \Pr(X < Y)$ is a measure of system failure based on stress X exceeding a strength Y. In fact, a system will be disturbed if stress X exceeds the strength Y. The application of stress-strength parameter can be found within the broad area of sciences, including the reliability of mechanical systems, statistics as well as clinical trials. For instance, by assuming the control group response to a therapeutic approach as Y and the treated group response as X, Hauck et al. (2000) considered $\mathcal{R}$ as a measure of treatment effect in clinical analysis. More details of the stress-strength parameter and its applications can be found in Simonoff et al. (1986) and Kotz and Pensky (2003).

During past decades, the problem of estimating $\mathcal{R}$ has been considered by the researcher in parametric and non-parametric viewpoints with different sampling schemes and distributions for (X, Y). See for instance the works of Ahmad et al. (1997); Awad et al. (1981) and Kundu and Gupta (2005) to name a few. More recently, Hajebi et al. (2012) constructed a confidence interval for $\mathcal{R}$ under generalized exponential distribution. The estimation of stress-strength parameter with the gamma, generalized logistic, and inverted gamma distributions for X and Y were proposed by Huang et al. (2012); Asgharzadeh et al. (2013) and Iranmanesh et al. (2018), respectively. Baklizi (2013, 2014) addressed the interval and Bayes estimations of $\mathcal{R}$ based on the records of the two-parameter exponential distribution. The estimation of $\mathcal{R}$ based on the upper record values in two-parameter bathtub-shaped lifetime distribution was also investigated by Tarvirdizade and Ahmadpour (2016). Moreover, Bai et al. (2021) provided an inference for the stress-strength reliability of multi-state systems by exploiting the generalized survival signature.

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The class of Pareto-based models is one of the well-known classes of distributions in statistical analysis. Specifically, in the stress-strength estimation, many contributions on postulating accurate models were recently published by considering Pareto models. Beg and Singh (1979) computed the minimum variance s-unbiased, Bayesian and ML estimates of $\mathcal{R}$ where X and Y are distributed by the Pareto distribution. Rezaei et al. (2010) considered $\mathcal{R}$ estimation when X and Y were two independent random variables (rvs) followed by the generalized Pareto distributions with different parameters. They obtained the maximum likelihood (ML) estimator of $\mathcal{R}$ and its asymptotic distribution to construct the asymptotic confidence interval. By considering independently distributed X and Y by the two-parameter Pareto distribution, Gunasekera (2014) proposed several generalized variable methods to estimate $\mathcal{R}$. Gunasekera (2014) investigated the generalized size, generalized adjusted and unadjusted powers of the test, and generalized coverage probabilities by conducting a simulation study and comparing p-value as a basis for hypothesis testing. To see more contributions on the stress-strength estimation, the reader is referred to Odat (2010); Ali and Woo (2010) and Wong (2012) to name a few.

Although all aforementioned works on estimating $\mathcal{R}$ have some advantages in practice, they might suffer from the lack of robustness in the presence of outliers. Practical studies in the reliability and stress-strength areas show that the outliers might contaminate variables X and Y since the processor in the life testing may produce some noises. In some applications of $\mathcal{R}$, we should also obtain the treatment effect for a set of response variables that the statistical units are divided by two groups as experiment and control because of removing any other unsuitable effects. In this situation, some observations of the response variable (say k of n) might be followed by another distribution, i.e., data might be contaminated by outliers (Nooghabi and Nooghabi, 2016). A simple way of coping with outliers is to ignore the observations outside of the data range (Nooghabi and Nooghabi, 2016). However, the investigator will lose some information by excluding data points and may obtain misleading results. This paper aims at assuming that the response observations for the experiment group have "good" and outlier points, whereas the observations for the control group do not suffer from contamination. To use all information in the dataset for estimating $\mathcal{R}$, it is supposed that X has the Pareto distribution in the presence of outliers and independently but non-identically Y follows the homogenous case of the Pareto distribution. We derive the ML, method of moments (MM), least squares (LS), and modified maximum likelihood (mixture of MM and ML) estimators of the model's parameters and $\mathcal{R}$.

The paper is therefore organized as follows. Section 2 presents a brief review of the definition of outliers and the Dixit model. In Section 3, we derive a closed-form of the reliability parameter of the Pareto distribution with outliers. The estimation procedure is comprehensively discussed in Section 4 for the various assumptions on the parameters. We conduct a simulation study in Section 5 to compare the obtained estimators. Finally, the superiority of the proposed methodology is illustrated in Section 6 by analyzing two real data examples in the solid-state physics (electron mobility) and motor insurance studies.

2. Outliers: definition and analysis

To present the paper's objective, this section briefly discussed the definition of the outliers in the statistical literature. As an applicable way in dealing with the outliers, the well-known Dixit model is also reviewed.

2.1. Definition

In statistical analysis, outliers usually refer to the observations in a distribution of data that deviate from the other observations. If a dataset contains some outliers, it is also said that the data are contaminated with outliers. It is hard to find a specific and general definition for outliers since the researchers presented various measures to define how far the outliers should be from the usual data points. We refer the reader to Grubbs (1950); Anscombe (1960); Grubbs (1969); Hawkins (1980); Miller (1981) and Barnett and Lewis (1994) to find some definitions. However, a more informative definition can be presented as follows. "*The outlier is an observation that being typical and/or erroneous deviates decidedly from the general behavior of experimental data with respect to the criteria exploited for the analysis.*"

Due to the presence of outliers in the practical situation, several methods and statistical models have recently been introduced for outliers detection and robust statistical inference. These include the works of Kale and Sinha (1971); Veale (1975); Chikkagoudar and Kunchur (1980); Dixit and Jabbari Nooghabi (2011a); Safari et al. (2018) and Safari et al. (2019). In this paper, we will use the well-known Dixit model (Dixit, 1987), described in the next section, as one of the powerful ways of outliers modeling.

2.2. Dixit model

Let $\left\{X_i\right\}_{i=1}^{n}$ be a sequence of non-negative rvs such that for the given combinations $A_1, A_2, \ldots, A_{n-k}$ of the integers $\{1, 2, \ldots, n\}$, we have:

I: The set of independent rvs $\boldsymbol{S}_1 = \left\{X_{A_i}\right\}_{i=1}^{n-k}$ with the probability density function (pdf) or probability mass function (pmf) $f_1(x)$. The set of remaining independent rvs $\boldsymbol{S}_2 = \left\{X_{A_i}\right\}_{i=n-k+1}^{n}$ have also the pdf (pmf) $f_2(x)$. Moreover, it is assumed that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are independent.

II: The combinations $A_1, A_2, \ldots, A_{n-k}$ are chosen at random with equal probability $C^{-1}(n, k) = \frac{k!(n-k)!}{n!}$.

Therefore, the joint pdf of $X_1, X_2, \ldots, X_n$ can be written as

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_1(x_i) \sum_{A_1, A_2, \ldots, A_k} \prod_{j=1}^{k} \frac{f_2(x_{A_j})}{f_1(x_{A_j})} C^{-1}(n, k), \tag{1}$$

where $\sum_{A_1,A_2,\ldots,A_k} = \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} \cdots \sum_{A_k=A_{k-1}+1}^{n}$. For $k = 1$, the Dixit model (1) will reduce to the Kale-Sinha model (Kale and Sinha, 1971). The marginal distribution of X_i can also be obtained as

$$f(x_i) = b f_2(x_i) + \bar{b} f_1(x_i),$$

where $b = kn^{-1}$ and $\bar{b} = 1 - b$. It is clear that the Dixit model does not need any procedure of outliers detection. This advantage of the Dixit model is useful in the practical situation that might help the investigator use it without any concentration on outlier detection.

In the oncoming section, the Pareto distribution with outliers is introduced by exploiting the Dixit model. We also compute the closed form of the stress-strength parameter for the proposed new model.

3. Reliability parameter of the Pareto distribution with outliers

Suppose $X_1, \ldots, X_n$ be a sequence of rvs such that k out of them distributed by the Pareto distribution with pdf

$$f_2(x; \alpha, \beta, \theta) = \frac{\alpha(\beta\theta)^{\alpha}}{x^{\alpha+1}}, \qquad 0 < \beta\theta \leq x, \;\; \alpha > 0, \; \beta > 1, \; \theta > 0,$$

and the remaining $(n-k)$ rvs are distributed by

$$f_1(x; \alpha, \theta) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \qquad 0 < \theta \leq x, \;\; \alpha > 0.$$

Accordingly, the joint pdf of $(X_1, \ldots, X_n)$ with k outliers is given by

$$f(x_1, \ldots, x_n; \alpha, \beta, \theta) = \frac{\alpha^n \theta^{n\alpha} \beta^{k\alpha}}{C(n, k)} \left(\prod_{i=1}^{n} x_i \right)^{-(\alpha+1)} \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} \ldots \sum_{A_k=A_{k-1}+1}^{n} \prod_{j=1}^{k} \mathbb{I}(x_{A_j} - \beta\theta), \tag{2}$$

where the indicator function $\mathbb{I}(\cdot)$ is defined as $\mathbb{I}(x) = 1$ if $x > 0$, and $\mathbb{I}(x) = 0$ otherwise. The marginal pdf of X_i can therefore be obtained as:

$$f(x_i; \alpha, \beta, \theta) = b \frac{\alpha(\beta\theta)^{\alpha}}{x_i^{\alpha+1}} \mathbb{I}(x_i - \beta\theta) + \bar{b} \frac{\alpha\theta^{\alpha}}{x_i^{\alpha+1}} \mathbb{I}(x_i - \theta), \qquad \alpha, \theta > 0, \; \beta > 1,$$

where $(X_1, X_2, ..., X_n)$ are not independent (Dixit and Jabbari Nooghabi, 2011a,b; Nooghabi and Nooghabi, 2016) since the joint pdf (2) is not a multiplication of the marginal densities.

In the stress-strength model, suppose X has the Pareto distribution in the presence of outliers defined in (2) and independently from X, the rv Y be distributed by the homogenous case of the Pareto distribution, i.e.

$$f(y; \nu, \lambda) = \frac{\nu\lambda^{\nu}}{y^{\nu+1}} \mathbb{I}(y - \lambda), \qquad \nu, \lambda > 0.$$

Then, the stress-strength parameter $\mathcal{R}$ based on samples of sizes n and m, respectively taken from X and Y is

$$\mathcal{R} = \Pr(X < Y) = 1 - \frac{\nu}{\alpha + \nu}(b\beta^{\alpha} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\alpha}. \tag{3}$$

Here, α and ν denote the shape parameters, λ and θ are the threshold parameters, and β is the outlier parameter.

4. Estimation of the stress-strength parameter

Practical studies with the Pareto distribution claim that it is reasonable to assume a fixed value for the threshold parameter and estimate the shape parameter based on it. For instance, in the analysis of motor insurance, a claim of at least θ, as compensation, can be made, and claims below it are not entertained. Thus, we can fit the Pareto distribution with parameter α and the known value of θ to claims dataset. Details can be found in Dixit and Jabbari Nooghabi (2011a). We will discuss in the next sections the stress-strength parameter estimation for different scenarios of the model parameters upon the real situations.

4.1. $\mathcal{R}$ estimation when the shape parameters are only unknown

In the first scenario, suppose that the threshold and outlier parameters, λ, θ and β, are known. For a fixed integer value $k \in \{1, 2, \ldots, [(n+1)/2]\}$, we construct a profile log-likelihood function $\ell_{PL}(\alpha, \beta, \theta) = \ln(f(x_1, \ldots, x_n; \alpha, \beta, \theta))$ with respect to k. Here, $[a]$ denotes the greatest integer less than or equal to a, and $\ln(\cdot)$ represents the natural logarithm function. Then, the ML estimate of α is obtained as

$$\hat{\alpha}_{ml1} = \frac{n}{\sum_{i=1}^{n} \ln(X_i) - n\ln(\theta) - k\ln(\beta)} \qquad \text{for} \qquad \sum_{i=1}^{n} \ln(X_i) > \ln(\theta^n \beta^k).$$

Finally, the most plausible value of k corresponds to the maximizer of the likelihood function. Maximizing the log-likelihood function for ν associated with the observation $\boldsymbol{y} = (y_1, \ldots, y_m)$, $\ell_{\boldsymbol{y}}(\nu, \lambda) = \sum_{i=1}^{m} \ln(f(y_i; \nu, \lambda))$, the ML estimate of ν can also be computed as

$$\hat{\nu}_{ml1} = \frac{m}{\sum_{i=1}^{m} \ln(Y_i) - m\ln(\lambda)} \qquad \text{for} \qquad \sum_{i=1}^{m} \ln(Y_i) > \ln(\lambda^m).$$

Consequently, by using the invariant property of the ML estimator and (3), the ML estimate of $\mathcal{R}$ is given

$$\hat{\mathcal{R}}_{ml1} = 1 - \frac{\hat{\nu}_{ml1}}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}}(b\beta^{\hat{\alpha}_{ml1}} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{ml1}}.$$

Now, the first shrinkage estimator of $\mathcal{R}$ can be obtained by minimizing the mean square error (MSE) of the estimator. Let $\tilde{\mathcal{R}}_{11} = \tau_{11}\hat{\mathcal{R}}_{ml1} + (1 - \tau_{11})\mathcal{R}_0$ be the first shrinkage estimator where $\mathcal{R}_0$ is a prior estimate. Therefore, τ_{11} can be obtained by minimizing $\mathrm{MSE}(\tilde{\mathcal{R}}_{11}) = \mathrm{E}[(\tau_{11}\hat{\mathcal{R}}_{ml1} + (1 - \tau_{11})\mathcal{R}_0) - \mathcal{R}]^2$, as

$$\tau_{11} = \frac{(\mathcal{R} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{ml1} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{ml1}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{ml1}) + \mathcal{R}_0^2}, \quad 0 \le \tau_{11} \le 1. \tag{4}$$

Substituting the ML estimate of $\mathcal{R}$ into (4) will lead to

$$\hat{\tau}_{11} = \frac{(\hat{\mathcal{R}}_{ml1} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{ml1} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{ml1}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{ml1}) + \mathcal{R}_0^2}.$$

Hence, the first shrinkage estimator of $\mathcal{R}$ takes the form $\tilde{\mathcal{R}}_{11} = \hat{\tau}_{11}\hat{\mathcal{R}}_{ml1} + (1 - \hat{\tau}_{11})\mathcal{R}_0$. In the following theorem, we present a closed expression of the expectations $\mathrm{E}(\hat{\mathcal{R}}_{ml1})$ and $\mathrm{E}(\hat{\mathcal{R}}_{ml1}^2)$, used for the first shrinkage estimator.

Theorem 1. *The expectations $E(\hat{\mathcal{R}}_{ml1})$ and $E(\hat{\mathcal{R}}^2_{ml1})$ are*

$$
\begin{aligned}
E(\hat{\mathcal{R}}_{ml1}) = 1 - \frac{2}{\Gamma(n)\Gamma(m)} \Bigg\{ & b \sum_{j=0}^{\infty} (-1)^j \sum_{i=0}^{j} C(j,i)(n\alpha)^{\frac{n+i}{2}} [A(\beta,\theta,\lambda)]^{\frac{n-i}{2}} \\
& \times BesselK\left(-n+i, 2\sqrt{n\alpha A(\beta,\theta,\lambda)}\right) \sum_{l=0}^{j-i} C(j-i,l)(-1)^{j-i-l}(m\nu)^{l+1}\Gamma(m-1-l) \\
& + \bar{b} \sum_{j=0}^{\infty} (-1)^j \sum_{i=0}^{j} C(j,i)(n\alpha)^{\frac{n+i}{2}} [A(\theta,\lambda)]^{\frac{n-i}{2}} \\
& \times BesselK\left(-n+i, 2\sqrt{n\alpha A(\theta,\lambda)}\right) \sum_{l=0}^{j-i} C(j-i,l)(-1)^{j-i-l}(m\nu)^{l+1}\Gamma(m-1-l) \Bigg\}, \\
E(\hat{\mathcal{R}}^2_{ml1}) = 1 - 2E(\hat{\mathcal{R}}_{ml1}) + \frac{2}{\Gamma(n)\Gamma(m)} \Bigg\{ & b^2 \sum_{j=0}^{\infty} (-1)^j (j+1) \sum_{i=0}^{j} C(j,i)(n\alpha)^{\frac{n+i}{2}} [2A(\beta,\theta,\lambda)]^{\frac{n-i}{2}} \\
& \times BesselK\left(-n+i, 2\sqrt{2n\alpha A(\beta,\theta,\lambda)}\right) \sum_{l=0}^{j-i} C(j-i,l)(-1)^{j-i-l}(m\nu)^{l+2}\Gamma(m-2-l) \\
& + 2b\bar{b} \sum_{j=0}^{\infty} (-1)^j (j+1) \sum_{i=0}^{j} C(j,i)(n\alpha)^{\frac{n+i}{2}} [A^*(\beta,\theta,\lambda)]^{\frac{n-i}{2}} \\
& \times BesselK\left(-n+i, 2\sqrt{n\alpha A^*(\beta,\theta,\lambda)}\right) \sum_{l=0}^{j-i} C(j-i,l)(-1)^{j-i-l}(m\nu)^{l+2}\Gamma(m-2-l) \\
& + \bar{b}^2 \sum_{j=0}^{\infty} (-1)^j (j+1) \sum_{i=0}^{j} C(j,i)(n\alpha)^{\frac{n+i}{2}} [2A(\theta,\lambda)]^{\frac{n-i}{2}} \\
& \times BesselK\left(-n+i, 2\sqrt{n\alpha 2A(\theta,\lambda)}\right) \sum_{l=0}^{j-i} C(j-i,l)(-1)^{j-i-l}(m\nu)^{l+2}\Gamma(m-2-l) \Bigg\},
\end{aligned}
$$

where $A(\beta,\theta,\lambda) = [\ln(\lambda) - \ln(\beta\theta)]$, $A(\theta,\lambda) = A(1,\theta,\lambda)$, $A^*(\beta,\theta,\lambda) = 2\ln(\lambda) - 2\ln(\theta) - \ln(\beta)$ and $BesselK(\cdot)$ is the Bessel function of the second kind.

Proof. To obtain the expectation of $\hat{\mathcal{R}}_{ml1}$ and $\hat{\mathcal{R}}^2_{ml1}$, the pdfs of $\hat{\alpha}_{ml1}$ and $\hat{\nu}_{ml1}$ are needed. Upon the pdfs of $\sum_{i=1}^{n} \ln(X_i)$ and $\sum_{i=1}^{m} \ln(Y_i)$, one can obtain the pdfs of $\hat{\alpha}_{ml1}$ and $\hat{\nu}_{ml1}$, respectively. Details are available in Appendix A and Dixit and Jabbari Nooghabi (2011a). □

In order to get the second and third ML-based shrinkage estimators of $\mathcal{R}$, we shall use the generalized likelihood ratio test (GLRT) for the hypothesis $H_0 : \mathcal{R} = \mathcal{R}_0$ vs. $H_1 : \mathcal{R} = \mathcal{R}_1$. The p-value of the test and its square root can then be the estimators of the weight. Based on the GLRT for testing H_0 vs. H_1, we reject H_0 when $\Lambda(x,y) < c_1$ or $\Lambda(x,y) > c_2$, where

$$
\Lambda(x,y) = \frac{\sup_{H_0} L(\alpha,\nu)}{\sup_{H} L(\alpha,\nu)}, \quad \text{for} \quad L(\alpha,\nu) \propto \frac{\alpha^n \theta^{n\alpha} \beta^{k\alpha}}{C(n,k)} (\prod_{i=1}^{n} x_i)^{-(\alpha+1)} \nu^m \lambda^{m\nu} (\prod_{i=1}^{m} y_i)^{-(\nu+1)}.
$$

It is clear that $H_0 : \mathcal{R} = \mathcal{R}_0$ is equivalent to $H_0 : \nu^* = \frac{\alpha(1-\mathcal{R}_0)}{(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)}$. Accordingly, the ML estimator of α under H_0 are obtained by maximizing the likelihood function with respect to α when ν^* is replaced in it. This maximization does not have the closed solution for α. However, we can estimate α by numerically solving equation $h(\alpha) = 0$ where

$$h(\alpha) = \frac{n}{\alpha} + k\ln(\beta) + n\ln(\theta) - \sum_{i=1}^{n} \ln(x_i) + \frac{m}{\alpha} - \frac{m\Big[b\beta^{\alpha}\ln(\beta)\left(\frac{\theta}{\lambda}\right)^{\alpha} + \left(\frac{\theta}{\lambda}\right)^{\alpha}\ln\left(\frac{\theta}{\lambda}\right)(b\beta^{\alpha} + \bar{b})\Big] + (1-\mathcal{R}_0)}{(b\beta^{\alpha} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\alpha} - (1-\mathcal{R}_0)}$$
$$- \frac{\Big[b\beta^{\alpha}\ln(\beta)\left(\frac{\theta}{\lambda}\right)^{\alpha} + \left(\frac{\theta}{\lambda}\right)^{\alpha}\ln\left(\frac{\theta}{\lambda}\right)(b\beta^{\alpha} + \bar{b})\Big]\alpha(1-\mathcal{R}_0)\Big[m\ln(\lambda) - \sum_{i=1}^{m}\ln(y_i)\Big]}{\Big[(b\beta^{\alpha} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\alpha} - (1-\mathcal{R}_0)\Big]^2}.$$

The ML estimate of ν under H_0 is obtained by substituting the solution of $h(\alpha) = 0$ in ν^*. As a result, the second and third shrinkage estimations of $\mathcal{R}$, respectively denoted by $\tilde{\mathcal{R}}_{21}$ and $\tilde{\mathcal{R}}_{31}$, take the following formula

$$\tilde{\mathcal{R}}_{21} = \tau_{21}\hat{\mathcal{R}}_{ml1} + (1-\tau_{21})\mathcal{R}_0, \qquad \tilde{\mathcal{R}}_{31} = \tau_{31}\hat{\mathcal{R}}_{ml1} + (1-\tau_{31})\mathcal{R}_0,$$

where $(1-\tau_{21})$ is the p-value of the GLRT and $(1-\tau_{31}) = \sqrt{p\text{-value}}$.

For the known parameters λ, θ and β, the MM estimators of α and ν can be obtained by using the first moment of X and Y, $\mathrm{E}(X) = \frac{\alpha}{\alpha-1}\theta(b\beta + \bar{b})$ and $\mathrm{E}(Y) = \frac{\nu}{\nu-1}\lambda$, respectively. We therefore have

$$\hat{\alpha}_{mm1} = \frac{\bar{X}}{\bar{X} - \theta(b\beta + \bar{b})} \quad \text{and} \quad \hat{\nu}_{mm1} = \frac{\bar{Y}}{\bar{Y} - \lambda},$$

where k was previously obtained using the profile log-likelihood function. Bear in mind that the first moment of the Pareto-based models is negative if the shape parameter is less than one. Therefore, if $\hat{\alpha}_{mm1}, \hat{\nu}_{mm1} < 1$, we will use the first moment of X^{-1} and Y^{-1}, as $\mathrm{E}(X^{-1}) = \frac{\alpha}{\alpha+1}\theta^{-1}(b\beta^{-1} + \bar{b})$ and $\mathrm{E}(Y^{-1}) = \frac{\nu}{\nu+1}\lambda^{-1}$, to find new estimations given by

$$\hat{\alpha}_{mm1} = \frac{\bar{X}_{inv}}{\theta^{-1}(b\beta^{-1} + \bar{b}) - \bar{X}_{inv}} \quad \text{and} \quad \hat{\nu}_{mm1} = \frac{\bar{Y}_{inv}}{\lambda^{-1} - \bar{Y}_{inv}},$$

where $\bar{X}_{inv} = \sum_{i=1}^{n} X^{-1}/n$, and $\bar{Y}_{inv} = \sum_{i=1}^{m} Y^{-1}/m$. So, the moment based estimator of $\mathcal{R}$ is

$$\hat{R}_{mm1} = 1 - \frac{\hat{\nu}_{mm1}}{\hat{\alpha}_{mm1} + \hat{\nu}_{mm1}}(b\beta^{\hat{\alpha}_{mm1}} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{mm1}}.$$

Using the same procedure of calculating the first ML-based shrinkage estimator of $\mathcal{R}$, the MM-based shrinkage estimator is obtained by

$$\tilde{\mathcal{R}}_{41} = \hat{\tau}_{41}\hat{\mathcal{R}}_{mm1} + (1-\hat{\tau}_{41})\mathcal{R}_0, \qquad \text{where} \qquad \hat{\tau}_{41} = \frac{(\hat{\mathcal{R}}_{mm1} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{mm1} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}^2_{mm1}) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{mm1}) + \mathcal{R}_0^2}.$$

There is no closed-form for the expectations in $\hat{\tau}_{41}$ and a Monte Carlo (MC) method should be implemented to approximate them.

Similarly to the MM estimate, we can find the shrinkage estimator of $\mathcal{R}$ related to the least squares estimator. The LS estimates of α and ν can be derived by using the reliability function of X and Y, respectively. It can be shown that the LS estimates of α and ν are

$$\hat{\alpha}_{ls1} = \frac{\sum_{i=1}^{n} z_{x_i}\ln(x_i) - n\bar{z}_x\overline{\ln(x)}}{\sum_{i=1}^{n}\Big[\ln(x_i) - \overline{\ln(x)}\Big]^2}, \qquad \hat{\nu}_{ls1} = \frac{\sum_{j=1}^{m} z_{y_j}\ln(y_j) - m\bar{z}_y\overline{\ln(y)}}{\sum_{j=1}^{m}\Big[\ln(y_j) - \overline{\ln(y)}\Big]^2},$$

where $z_{x_i} = -\ln(1 - F_X(x_i)) = -\ln\left(1 - \frac{i}{n+1}\right)$, $i = 1, 2, \ldots, n$, $\bar{z}_x = \frac{1}{n}\sum_{i=1}^{n} z_{x_i}$, $\overline{\ln(x)} = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i)$, $z_{y_j} = -\ln(1 - F_Y(y_j)) = -\ln\left(1 - \frac{j}{m+1}\right)$, $j = 1, 2, \ldots, m$, $\bar{z}_y = \frac{1}{m}\sum_{j=1}^{m} z_{y_j}$ and $\overline{\ln(y)} = \frac{1}{m}\sum_{j=1}^{m}\ln(y_j)$. So, the LS-based estimator of $\mathcal{R}$ is obtained from (3) as

$$\hat{\mathcal{R}}_{ls1} = 1 - \frac{\hat{\nu}_{ls1}}{\hat{\alpha}_{ls1} + \hat{\nu}_{ls1}}(b\beta^{\hat{\alpha}_{ls1}} + \bar{b})\left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{ls1}},$$

where k was chosen based on the profile log-likelihood function. Finally, the shrinkage estimator of $\mathcal{R}$ for the LS approach is derived as

$$\tilde{\mathcal{R}}_{51} = \hat{\tau}_{51}\hat{\mathcal{R}}_{ls1} + (1-\hat{\tau}_{51})\mathcal{R}_0, \qquad \text{where} \qquad \hat{\tau}_{51} = \frac{(\hat{\mathcal{R}}_{ls1}-\mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{ls1}-\mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}^2_{ls1}) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{ls1}) + \mathcal{R}_0^2}.$$

The expectations in $\tilde{\mathcal{R}}_{51}$ should be approximated by a Monte Carlo approach.

4.2. $\mathcal{R}$ estimation when the outlier parameter is known

In the second scenario, suppose that all parameters except outlier are unknown. The ML, MM, and LS estimator of stress-strength parameter and their corresponding shrinkage estimators can then be obtained as follows.

For a fix integer value $k \in \{1, 2, \ldots, [(n+1)/2]\}$, the ML estimates of α and θ obtained by constructing the profile log-likelihood function $\ell_{PL}(\alpha, \beta, \theta)$, are $\hat{\theta}_{ml2} = X_{(1)}\beta^{-1}$, and

$$\hat{\alpha}_{ml2} = \frac{n}{\sum_{i=1}^n \ln(X_i) - n\ln(X_{(1)}) + (n-k)\ln(\beta)}, \quad \text{for} \quad \sum_{i=1}^n \ln(X_i) > \ln(X^n_{(1)}\beta^{k-n}), \quad \beta > 1, \tag{5}$$

where $X_{(1)}$ denotes the first order statistics of X. We then choose the best value of k corresponds to the maximizer of the likelihood function. Maximizing the log-likelihood function for (ν, λ) associated with the observation $\boldsymbol{y} = (y_1, \ldots, y_m)$, $\ell_{\boldsymbol{y}}(\nu, \lambda)$ leads to the ML estimates $\hat{\lambda}_{ml2} = Y_{(1)}$, and

$$\hat{\nu}_{ml2} = \frac{m}{\sum_{i=1}^m \ln(Y_i) - m\ln(Y_{(1)})}, \quad \text{for} \quad \sum_{i=1}^m \ln(Y_i) > \ln(Y^m_{(1)}), \tag{6}$$

where $Y_{(1)}$ is the first order statistics of Y. From (5), (6) and (3), the ML estimate of $\mathcal{R}$ can be written as

$$\hat{\mathcal{R}}_{ml2} = 1 - \frac{\hat{\nu}_{ml2}}{\hat{\alpha}_{ml2} + \hat{\nu}_{ml2}}(b\beta^{\hat{\alpha}_{ml2}} + \bar{b})\left(\frac{\hat{\theta}_{ml2}}{\hat{\lambda}_{ml2}}\right)^{\hat{\alpha}_{ml2}}.$$

For this scenario, the first shrinkage estimator of $\mathcal{R}$ can be obtained as $\tilde{\mathcal{R}}_{12} = \hat{\tau}_{12}\hat{\mathcal{R}}_{ml2} + (1-\hat{\tau}_{12})\mathcal{R}_0$, same as Section 4.1, where

$$\hat{\tau}_{12} = \frac{(\hat{\mathcal{R}}_{ml2}-\mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{ml2}-\mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}^2_{ml2}) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{ml2}) + \mathcal{R}_0^2}.$$

To calculate $\tilde{\mathcal{R}}_{12}$, we should obtain the expected value $\mathrm{E}(\hat{\mathcal{R}}_{ml2})$ and $\mathrm{E}(\hat{\mathcal{R}}^2_{ml2})$. One can follow the next two lemmas and theorem to compute these two expectations.

Lemma 1. *Let $T = \prod_{i=1}^n X_i$ and $S = \prod_{i=1}^m Y_i$. Then, the joint pdfs of $(X_{(1)}, T)$ and $(Y_{(1)}, S)$ are*

$$\begin{aligned} f_{X_{(1)},T}(x_{(1)}, t) =& \frac{n\alpha^n\beta^{k\alpha}\theta^{n\alpha}t^{-(\alpha+1)}}{(n-2)!x_{(1)}} \\ &\times\Big\{b\left[\ln(t) - n\ln(x_{(1)}) - (k-1)\ln(\beta)\right]^{n-2}\mathbb{I}(x_{(1)} - \beta\theta)\mathbb{I}(t - x^n_{(1)}\beta^{k-1}) \\ &\quad + \bar{b}\left[\ln(t) - n\ln(x_{(1)}) - k\ln(\beta)\right]^{n-2}\mathbb{I}(x_{(1)} - \theta)\mathbb{I}(t - x^n_{(1)}\beta^k)\Big\}, \\ f_{Y_{(1)},S}(y_{(1)}, s) =& \frac{m\nu^m\lambda^{m\nu}s^{-(\nu+1)}}{(m-2)!y_{(1)}}\left[\ln(s) - m\ln(y_{(1)})\right]^{m-2}\mathbb{I}(y_{(1)} - \lambda)\mathbb{I}(s - y^m_{(1)}). \end{aligned}$$

Proof. The joint pdf of $(X_{(1)}, T)$ in the presence of outliers is can be obtained by exploiting Equation (2) and performing the transform $\{x_{(1)} = x_{(1)}, x_{(2)} = x_{(2)}, \ldots, x_{(n-1)} = x_{(n-1)}, x_{(n)} = \frac{t}{x_{(1)}\cdots x_{(n-1)}}\}$. Integrating out with respect to $x_{(2)}, x_{(3)}, \ldots, x_{(n-1)}$, the joint pdf of $(X_{(1)}, T)$ is derived. Similarly, one can calculate the joint pdf of $(Y_{(1)}, S)$ which completes the proof. □

Lemma 2. *The joint pdf of* $(\hat{\theta}_{ml2}, \hat{\alpha}_{ml2}) = (U, W)$ *and the joint pdf of* $(\hat{\lambda}_{ml2}, \hat{\nu}_{ml2}) = (P, Q)$ *are as the following equations, respectively.*

$$
\begin{aligned}
f_{U,W}(u,w) = &\frac{n^2\alpha^n\theta^{n\alpha}u^{-(n\alpha+1)}}{(n-2)!w^2}\exp\left(-\frac{n\alpha}{w}\right) \\
&\times\left\{b\left[\frac{n}{w}-(n-1)\ln(\beta)\right]^{n-2}\mathbb{I}(u-\theta)\mathbb{I}\left(\frac{n}{(n-1)\ln(\beta)}-w\right)+\bar{b}\left[\frac{n}{w}-n\ln(\beta)\right]^{n-2}\mathbb{I}\left(u-\frac{\theta}{\beta}\right)\mathbb{I}\left(\frac{1}{\ln(\beta)}-w\right)\right\}, \\
f_{P,Q}(p,q) = &\frac{\lambda^{m\nu}}{(m-2)!p^{m\nu+1}}\left(\frac{m\nu}{q}\right)^m\exp\left(-\frac{m\nu}{q}\right)\mathbb{I}(q)\mathbb{I}(p-\lambda).
\end{aligned}
$$

Proof. The joint pdfs of (U, W) and (P, Q) are directly obtained from the joint pdfs of $(X_{(1)}, T)$ and $(Y_{(1)}, S)$, respectively, by using some elementary algebra. □

Theorem 2. $E(\hat{\mathcal{R}}_{ml2})$ *and* $E(\hat{\mathcal{R}}^2_{ml2})$ *are as follows.*

$$
\begin{aligned}
E(\hat{\mathcal{R}}_{ml2}) =&1-\frac{\alpha^{n-1}\theta^{n\alpha}\lambda^{m\nu}}{(n-2)!(m-2)!}\sum_{j=0}^{\infty}(-1)^j\sum_{i=0}^{j}C(j,i)\sum_{l=0}^{n-2}C(n-2,l)(-1)^{n-2-l}\sum_{r=0}^{\infty}\frac{n^{r+l}(-\alpha)^r}{r!} \\
&\times\sum_{o=0}^{\infty}\frac{[\ln(\beta)]^{n+r-i-o-1}(n\alpha)^{-o}}{o!(i-l-r+o-1)}\sum_{a=0}^{j-i}C(j-i,a)(-1)^{j-i-a}(m\nu)^{a+1}\Gamma(m-a-1) \\
&\times\left\{b^2\beta^{n\alpha}n^{i-l-r+o}(n-1)^{n+r-i-o-1}A(n,m,o,\alpha,\beta,\theta,\nu,\lambda)+b\bar{b}\beta^{n\alpha}n^{n-2-l}A(n,m,o,\alpha,1,\theta,\nu,\lambda)\right. \\
&\left.+b\bar{b}n^{i-l-r+o}(n-1)^{n+r-i-o-1}A(n,m,o,\alpha,1,\theta,\nu,\lambda)+\bar{b}^2n^{n-2-l}A(n,m,o,\alpha,\beta^{-1},\theta,\nu,\lambda)\right\}, \\
E(\hat{\mathcal{R}}^2_{ml2}) =&1-2E(\hat{R}_{ml2})+\frac{n^2\alpha^n\theta^{n\alpha}\lambda^{m\nu}}{(n-2)!(m-2)!}\sum_{j=0}^{\infty}(-1)^j(j+1)\sum_{i=0}^{j}C(j,i)\sum_{l=0}^{n-2}C(n-2,l)(-1)^{n-2-l} \\
&\times\sum_{r=0}^{\infty}\frac{n^{r+l}(-\alpha)^r}{r!}\sum_{o=0}^{\infty}\frac{2^o[\ln(\beta)]^{n+r-i-o-1}(n\alpha)^{-o-1}}{o!(i-l-r+o-1)}\sum_{a=0}^{j-i}C(j-i,a)(-1)^{j-i-a}(m\nu)^{a+1}\Gamma(m-a-1) \\
&\times\left\{b^3\beta^{n\alpha}n^{i-l-r+o-1}(n-1)^{n+r-i-o-1}A(n,m,o,\alpha,\beta,\theta,\nu,\lambda)+b^2\bar{b}\beta^{n\alpha}n^{n-2-l}A(n,m,o,\alpha,1,\theta,\nu,\lambda)\right. \\
&+2b^2\bar{b}\beta^{0.5n\alpha}n^{i-l-r+o-1}(n-1)^{n+r-i-o-1}A(n,m,o,\alpha,\sqrt{\beta},\theta,\nu,\lambda)+2b\bar{b}^2\beta^{0.5n\alpha}n^{n-2-l}A(n,m,o,\alpha,\sqrt{\beta^{-1}},\theta,\nu,\lambda) \\
&\left.+b\bar{b}^2n^{i-l-r+o-1}(n-1)^{n+r-i-o-1}A(n,m,o,\alpha,1,\theta,\nu,\lambda)+\bar{b}^3n^{n-2-l}A(n,m,o,\alpha,\beta^{-1},\theta,\nu,\lambda)\right\},
\end{aligned}
$$

where

$$
A(n,m,o,\alpha,\beta,\theta,\nu,\lambda)=\frac{\lambda^{-(n\alpha+m\nu)}\Gamma(o+1)}{n\alpha+m\nu}-\sum_{d=0}^{\infty}\frac{(-1)^o(n\alpha)^{d+o}(\beta\theta)^{-(n\alpha+m\nu)}\Gamma\left(d+o+1,-(n\alpha+m\nu)\ln\left(\frac{\beta\theta}{\lambda}\right)\right)}{\Gamma(d+1)(d+o+1)(n\alpha+m\nu)^{d+o+1}},
$$

and $\Gamma(a)$ *and* $\Gamma(a, b)$ *denote the gamma and incomplete gamma functions, respectively.*

Proof. By using the joint pdfs of $(\hat{\theta}_{ml2}, \hat{\alpha}_{ml2})$ and $(\hat{\lambda}_{ml2}, \hat{\nu}_{ml2})$ in Lemmas 2, proof is completed same as the proof of Theorem 1 (see Appendix A). □

When the outlier parameter is only known, the second and third ML-based shrinkage estimators of the stress-strength parameter are obtained though using the GLRT for testing $H_0 : \mathcal{R} = \mathcal{R}_0$ vs. $H_1 : \mathcal{R} = \mathcal{R}_1$. Same as Section 4.1, the p-value of the test and its square root are the estimates of weight in the second and third shrinkage estimators of $\mathcal{R}$. The GLRT test of H_0 vs. H_1 will reject H_0 if $\Lambda'(x, y) < c_3$ or $\Lambda'(x, y) > c_4$, where

$$
\Lambda'(x,y)=\frac{\sup_{H_0}L(\alpha,\theta,\nu,\lambda)}{\sup_{H}L(\alpha,\theta,\nu,\lambda)},\quad\text{where}\quad L(\alpha,\theta,\nu,\lambda)\propto\frac{\alpha^n\theta^{n\alpha}\beta^{k\alpha}}{C(n,k)}\left(\prod_{i=1}^{n}x_i\right)^{-(\alpha+1)}\nu^m\lambda^{m\nu}\left(\prod_{i=1}^{m}y_i\right)^{-(\nu+1)}.
$$

The hypothesis $H_0 : \mathcal{R} = \mathcal{R}_0$ is equivalent to $H_0 : \nu^* = \frac{\alpha(1-\mathcal{R}_0)}{(b\beta^\alpha+\bar{b})(\frac{\theta}{\lambda})^\alpha-(1-\mathcal{R}_0)}$. Consequently, the ML estimators of α, θ, and λ under H_0 should be computed by simultaneously solving equations $h(\alpha) = 0$,

$$h(\theta) = \frac{n\alpha}{\theta} - \frac{m\alpha(b\beta^\alpha + \bar{b})\theta^{\alpha-1}[1 + m\alpha(1-\mathcal{R}_0)\ln(\lambda)]}{\left[(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)\right]\lambda^\alpha} + \frac{m\alpha^2(b\beta^\alpha + \bar{b})(1-\mathcal{R}_0)\sum_{i=1}^m \ln(y_i)\theta^{\alpha-1}}{\left[(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)\right]^2\lambda^\alpha} = 0,$$

$$h(\lambda) = \frac{m\alpha(b\beta^\alpha + \bar{b})\theta^\alpha\lambda^{-\alpha-1} + m\alpha(1-\mathcal{R}_0)\lambda^{-1}}{(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)} + \frac{\alpha^2(b\beta^\alpha + \bar{b})(1-\mathcal{R}_0)\theta^\alpha\lambda^{-\alpha-1}[m\ln(\lambda) - \sum_{i=1}^m \ln(y_i)]}{\left[(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)\right]^2} = 0.$$

We can therefore calculate ν^* and the second and third estimators of $\mathcal{R}$ as

$$\tilde{\mathcal{R}}_{22} = \tau_{22}\hat{\mathcal{R}}_{ml2} + (1-\tau_{22})\mathcal{R}_0, \qquad \tilde{\mathcal{R}}_{32} = \tau_{32}\hat{\mathcal{R}}_{ml2} + (1-\tau_{32})\mathcal{R}_0,$$

where $(1-\tau_{22})$ is the p-value of the GLRT and $(1-\tau_{32}) = \sqrt{p\text{-value}}$.

For the known parameter β, the MM estimators of α, λ, θ, and ν can be obtained by using the first and second moments of X and Y via solving systems of nonlinear equations:

$$\begin{cases} \bar{X} = \mathrm{E}(X) = \dfrac{\alpha}{\alpha-1}\theta(b\beta + \bar{b}); \\ \overline{X^2} = \mathrm{E}(X^2) = \dfrac{\alpha}{\alpha-2}\theta^2(b\beta^2 + \bar{b}), \end{cases} \quad \text{and} \quad \begin{cases} \bar{Y} = \mathrm{E}(Y) = \dfrac{\nu}{\nu-1}\lambda; \\ \overline{Y^2} = \mathrm{E}(Y^2) = \dfrac{\nu}{\nu-2}\lambda^2. \end{cases}$$

The solutions to these nonlinear systems are

$$\begin{cases} \hat{\theta}_{mm2} = \dfrac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}; \\ \hat{\alpha}_{mm2} = \dfrac{\bar{X}}{\bar{X} - \hat{\theta}_{ml2}(b\beta + \bar{b})}, \end{cases} \quad \text{and} \quad \begin{cases} \hat{\lambda}_{mm2} = \dfrac{\overline{Y^2} - \sqrt{(\overline{Y^2})^2 - \bar{Y}^2\overline{Y^2}}}{\bar{Y}}; \\ \hat{\nu}_{mm2} = \dfrac{\bar{Y}}{\bar{Y} - \hat{\lambda}_{mm2}}, \end{cases}$$

where $B_1 = -\bar{X}(b\beta^2 + \bar{b})$, $B_2 = 2\overline{X^2}(b\beta + \bar{b})$, $B_3 = -\bar{X}\overline{X^2}$, $\overline{X^2} = \sum_{i=1}^n X^2/n$, $\overline{Y^2} = \sum_{i=1}^m Y^2/m$ and k was previously obtained based on the profile log-likelihood function. To avoid having negative variance in case $\hat{\alpha}_{mm2}, \hat{\nu}_{mm2} \leq 2$, one should obtain the MM estimators by using the first and second moments of X^{-1} and Y^{-1}, via solving systems of nonlinear equations:

$$\begin{cases} \bar{X}_{inv} = \mathrm{E}(X^{-1}) = \dfrac{\alpha}{\alpha+1}\theta^{-1}(b\beta^{-1} + \bar{b}); \\ \overline{X^2}_{inv} = \mathrm{E}(X^{-2}) = \dfrac{\alpha}{\alpha+2}\theta^{-2}(b\beta^{-2} + \bar{b}), \end{cases} \quad \text{and} \quad \begin{cases} \bar{Y}_{inv} = \mathrm{E}(Y) = \dfrac{\nu}{\nu+1}\lambda^{-1}; \\ \overline{Y^2}_{inv} = \mathrm{E}(Y^{-2}) = \dfrac{\nu}{\nu+2}\lambda^{-2}, \end{cases}$$

where $\overline{X^2}_{inv} = \sum_{i=1}^n X^{-2}/n$, and $\overline{Y^2}_{inv} = \sum_{i=1}^m Y^{-2}/m$. This leads similarly to the solutions to linear systems as

$$\begin{cases} \hat{\theta}_{mm2} = \dfrac{-B_5 - \sqrt{B_5^2 - 4B_4B_6}}{2B_4}; \\ \hat{\alpha}_{mm2} = \dfrac{\bar{X}_{inv}}{\hat{\theta}_{ml2}^{-1}(b\beta^{-1} + \bar{b}) - \bar{X}_{inv}}, \end{cases} \quad \text{and} \quad \begin{cases} \hat{\lambda}_{mm2} = \dfrac{\overline{Y^2}_{inv} - \sqrt{(\overline{Y^2}_{inv})^2 - \bar{Y}_{inv}^2\overline{Y^2}_{inv}}}{\bar{Y}_{inv}\overline{Y^2}_{inv}}; \\ \hat{\nu}_{mm2} = \dfrac{\bar{Y}_{inv}}{\hat{\lambda}_{mm2}^{-1} - \bar{Y}_{inv}}, \end{cases}$$

where $B_4 = \bar{X}_{inv}\overline{X^2}_{inv}$, $B_5 = -2\overline{X^2}_{inv}(b\beta^{-1} + \bar{b})$, $B_6 = \bar{X}_{inv}(b\beta^{-2} + \bar{b})$. By computing the MM estimates of the parameters, we have

$$\hat{\mathcal{R}}_{mm2} = 1 - \frac{\hat{\nu}_{mm2}}{\hat{\alpha}_{mm2} + \hat{\nu}_{mm2}}(b\beta^{\hat{\alpha}_{mm2}} + \bar{b})\left(\frac{\hat{\theta}_{mm2}}{\hat{\lambda}_{mm2}}\right)^{\hat{\alpha}_{mm2}}.$$

Consequently, the MM-based shrinkage estimator of $\mathcal{R}$ can be derived as $\tilde{\mathcal{R}}_{42} = \hat{\tau}_{42}\hat{\mathcal{R}}_{mm2} + (1-\hat{\tau}_{42})\mathcal{R}_0$, where

$$\hat{\tau}_{42} = \frac{(\hat{\mathcal{R}}_{mm2} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{mm2} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{mm2}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{mm2}) + \mathcal{R}_0^2},$$

in which the expectations are approximated by an MC method.

Finally, in order to compute the LS estimate of $\mathcal{R}$ and its corresponding shrinkage estimator, the LS estimate of α, θ, ν and λ obtained by exploiting the reliability function of X and Y are

$$\hat{\alpha}_{ls2} = \frac{\sum_{i=1}^{n} z_{x_i}\ln(x_i) - n\bar{z}_x\overline{\ln(x)}}{\sum_{i=1}^{n}\left[\ln(x_i) - \overline{\ln(x)}\right]^2}, \qquad \hat{\theta}_{ls2} = \exp\left(\frac{\hat{\alpha}_{ls2}\overline{\ln(x)} - \ln\left(b\beta^{\hat{\theta}_{ls2}} + \bar{b}\right) - \bar{z}_x}{\hat{\alpha}_{ls2}}\right),$$
$$\hat{\nu}_{ls2} = \frac{\sum_{j=1}^{m} z_{y_j}\ln(y_j) - m\bar{z}_y\overline{\ln(y)}}{\sum_{j=1}^{m}\left[\ln(y_j) - \overline{\ln(y)}\right]^2}, \qquad \hat{\lambda}_{ls2} = \exp\left(\frac{\hat{\nu}_{ls2}\overline{\ln(y)} - \bar{z}_y}{\hat{\nu}_{ls2}}\right),$$

where $z_{x_i} = -\ln(1 - F_X(x_i)) = -\ln\left(1 - \frac{i}{n+1}\right)$, $i = 1, 2, ..., n$, $\bar{z}_x = \sum_{i=1}^{n} z_{x_i}/n$, $\overline{\ln(x)} = \sum_{i=1}^{n}\ln(x_i)/n$, $z_{y_j} = -\ln(1 - F_Y(y_j)) = -\ln\left(1 - \frac{j}{m+1}\right)$, $j = 1, 2, ..., m$, $\bar{z}_y = \sum_{j=1}^{m} z_{y_j}/m$, $\overline{\ln(y)} = \sum_{j=1}^{m}\ln(y_j)/m$, and k was chosen based on the profile log-likelihood function. The LS estimate of $\mathcal{R}$ is then

$$\hat{\mathcal{R}}_{ls2} = 1 - \frac{\hat{\nu}_{ls2}}{\hat{\alpha}_{ls2} + \hat{\nu}_{ls2}}(b\beta^{\hat{\alpha}_{ls2}} + \bar{b})\left(\frac{\hat{\theta}_{ls2}}{\hat{\lambda}_{ls2}}\right)^{\hat{\alpha}_{ls2}}.$$

The shrinkage estimator of $\mathcal{R}$ with respect to the LS estimators is $\tilde{\mathcal{R}}_{52} = \hat{\tau}_{52}\hat{\mathcal{R}}_{ls2} + (1 - \hat{\tau}_{52})\mathcal{R}_0$, with

$$\hat{\tau}_{52} = \frac{(\hat{\mathcal{R}}_{ls2} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{ls2} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{ls2}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{ls2}) + \mathcal{R}_0^2},$$

in which there is no closed-form for the expectations and an MC approach should be used to approximate them.

4.3. $\mathcal{R}$ estimation when all of the parameters are unknown

The last scenario focuses on estimating $\mathcal{R}$ when it is assumed that none of the model parameters are known. The ML parameter estimates of α, β, and θ can be obtained by maximizing the profile log-likelihood function $\ell_{PL}(\alpha, \beta, \theta)$ for a fix value $k \in \{1, 2, \ldots, [(n+1)/2]\}$. In fact, the ML estimate of $\Theta = (\alpha, \beta, \theta)$ is traditionally obtained by searching the solution of the following function:

$$\Theta = \arg\max_{\Theta} \ln(f(x_1, \ldots, x_n; \alpha, \beta, \theta))$$

However, this optimization is not trivial, especially for β, and numerical method should be exploited. By computing ML estimate of Θ, the value of k with maximum likelihood is chosen as the most plausible value of k. Maximizing the log-likelihood function for (ν, λ) associated with the observation $\boldsymbol{y} = (y_1, \ldots, y_m)$, $\ell_{\boldsymbol{y}}(\nu, \lambda)$ leads to obtain their ML estimates with the same form as (6). As a result, the ML estimate of $\mathcal{R}$ can be computed by (3) under the invariant property of the ML estimator.

To estimate α, β and θ by the MM approach, it is necessary to solve the following systems of nonlinear equations based on the moments of X and X^{-1}:

$$\begin{cases} \bar{X} = \mathrm{E}(X) = \dfrac{\alpha}{\alpha - 1}\theta(b\beta + \bar{b}); \\ \overline{X^2} = \mathrm{E}(X^2) = \dfrac{\alpha}{\alpha - 2}\theta^2(b\beta^2 + \bar{b}); \\ \bar{X}_{inv} = \mathrm{E}(X^{-1}) = \dfrac{\alpha}{\alpha + 1}\theta^{-1}(b\beta^{-1} + \bar{b}); \\ \overline{X^2}_{inv} = \mathrm{E}(X^{-2}) = \dfrac{\alpha}{\alpha + 2}\theta^{-2}(b\beta^{-2} + \bar{b}). \end{cases} \tag{7}$$

By applying some straightforward algebra, the quartic equation with respect to β is calculated as

$$B_7\beta^4 + B_8\beta^3 + B_9\beta^2 + B_{10}\beta + B_{11} = 0,$$

where $B_7 = B_{11} = \bar{X}\bar{X}_{inv} b\bar{b}$, $B_8 = B_{10} = -4\overline{X^2}\ \overline{X^2}_{inv} b\bar{b}$, $B_9 = 3\overline{X^2}\ \overline{X^2}_{inv}\bar{X}\bar{X}_{inv} + (\bar{X}\bar{X}_{inv} - 4\overline{X^2}\ \overline{X^2}_{inv})(b^2 + \bar{b}^2)$. Following Abramowitz et al. (1988) (p. 17) and Pachner (1983) (p. 6.1.), the roots of the quartic equation are

$$\begin{cases} \hat{\beta}_{1mm3} = \dfrac{\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}}{2} - \dfrac{B_8}{4B_7}, & \hat{\beta}_{2mm3} = \dfrac{\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}}{2} - \dfrac{B_8}{4B_7}, \\ \hat{\beta}_{3mm3} = \dfrac{-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}}{2} - \dfrac{B_8}{4B_7}, & \hat{\beta}_{4mm3} = \dfrac{-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}}{2} - \dfrac{B_8}{4B_7}, \end{cases}$$

where z_1, z_2 and z_3 are the roots of cubic equation $z^3 + 2B_{12}z^2 + (B_{12}^2 - 4B_{14})z - B_{13}^2 = 0$, $B_{12} = \frac{B_9}{B_7} - 6\left(\frac{B_8}{4B_7}\right)^2$, $B_{13} = \frac{B_{10}}{B_7} + 2\frac{B_8}{4B_7}\left[4\left(\frac{B_8}{4B_7}\right)^2 - \frac{B_9}{B_7}\right]$ and $B_{14} = \frac{B_{11}}{B_7} + \frac{B_8}{4B_7}\left\{\frac{B_8}{4B_7}\left[\frac{B_9}{B_7} - 3\left(\frac{B_8}{4B_7}\right)^2\right] - \frac{B_{10}}{B_7}\right\}$.

Remark 1. *Note that if all three roots z_1, z_2 and z_3 are real and positive, then all four roots $\hat{\beta}_{1mm3}$, $\hat{\beta}_{2mm3}$, $\hat{\beta}_{3mm3}$, and $\hat{\beta}_{4mm3}$ are real (Pachner, 1983). One may also get more than one feasible solution for β. In this situation, $\hat{\beta}_{mm3}$ can be selected by evaluating the likelihood for each feasible solution and choosing the one that maximizes likelihood.*

By computing $\hat{\beta}_{mm3}$, the MM estimates of α and θ, replacing $\hat{\beta}_{mm3}$ in (7), are obtained as

$$\hat{\alpha}_{mm3} = \sqrt{\frac{\bar{X}\bar{X}_{inv}}{\bar{X}\bar{X}_{inv} - (b\hat{\beta}_{mm3} + \bar{b})(b\hat{\beta}_{mm3}^{-1} + \bar{b})}}, \qquad \hat{\theta}_{mm3} = \frac{\bar{X}(\hat{\alpha}_{mm3} - 1)}{\hat{\alpha}_{mm3}(b\hat{\beta}_{mm3} + \bar{b})}.$$

Details can be found in Dixit and Jabbari Nooghabi (2011b). The MM estimates of ν and λ can be derived by using the same procedure as Section 4.2, as $\hat{\nu}_{mm3} = \hat{\nu}_{mm2}$ and $\hat{\lambda}_{mm3} = \hat{\lambda}_{mm2}$. Therefore, the moment estimator of $\mathcal{R}$ is

$$\hat{\mathcal{R}}_{mm3} = 1 - \frac{\hat{\nu}_{mm3}}{\hat{\alpha}_{mm3} + \hat{\nu}_{mm3}}\left(b\hat{\beta}_{mm3}^{\hat{\alpha}_{mm3}} + \bar{b}\right)\left(\frac{\hat{\theta}_{mm3}}{\hat{\lambda}_{mm3}}\right)^{\hat{\alpha}_{mm3}}.$$

The corresponding MM-based shrinkage estimator of $\mathcal{R}$ is

$$\tilde{\mathcal{R}}_{13} = \hat{\tau}_{13}\hat{\mathcal{R}}_{mm3} + (1 - \hat{\tau}_{13})\mathcal{R}_0, \quad \text{where} \quad \hat{\tau}_{13} = \frac{(\hat{\mathcal{R}}_{mm3} - \mathcal{R}_0)\mathrm{E}(\hat{\mathcal{R}}_{mm3} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{mm3}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{mm3}) + \mathcal{R}_0^2}.$$

The last estimates for the stress-strength parameter $\mathcal{R}$ presented in this paper is based on mixture of ML and MM (MIX) estimations. In this regards, the MIX estimates of the unknown parameters $\alpha, \beta, \theta, \nu$, and λ, denoted by adding the subscript "*mix*" to them, are:

$$\hat{\beta}_{mix} = \hat{\beta}_{mm3}, \quad \hat{\theta}_{mix} = \frac{X_{(1)}}{\hat{\beta}_{mix}}, \quad \hat{\alpha}_{mix} = \frac{n}{\sum_{i=1}^{n}\ln(X_i) - n\ln(X_{(1)}) + (n-k)\ln(\hat{\beta}_{mix})},$$
$$\hat{\lambda}_{mix} = \hat{\lambda}_{mm3}, \quad \hat{\nu}_{mix} = \frac{m}{\sum_{i=1}^{m}\ln(Y_i) - m\ln(\hat{\lambda}_{mix})}.$$

Therefore, the corresponding MIX estimate of $\mathcal{R}$ and its shrinkage estimate are derived as

$$\hat{\mathcal{R}}_{mix} = 1 - \frac{\hat{\nu}_{mix}}{\hat{\alpha}_{mix} + \hat{\nu}_{mix}}\left(b\hat{\beta}_{mix}^{\hat{\alpha}_{mix}} + \bar{b}\right)\left(\frac{\hat{\theta}_{mix}}{\hat{\lambda}_{mix}}\right)^{\hat{\alpha}_{mix}},$$
$$\tilde{\mathcal{R}}_{23} = \hat{\tau}_{23}\hat{\mathcal{R}}_{mix} + (1 - \hat{\tau}_{23})\mathcal{R}_0, \quad \text{where} \quad \hat{\tau}_{23} = \frac{(\hat{\mathcal{R}}_{mix} - R_0)\mathrm{E}(\hat{\mathcal{R}}_{mix} - \mathcal{R}_0)}{\mathrm{E}(\hat{\mathcal{R}}_{mix}^2) - 2\mathcal{R}_0\mathrm{E}(\hat{\mathcal{R}}_{mix}) + \mathcal{R}_0^2}.$$

There is no closed-form for the expectations in $\hat{\tau}_{23}$ and an MC method should be implemented to approximate them.

5. Simulation analysis

We conduct a simulation analysis to check the performance of the proposed estimators discussed in Section 4. For two sample sizes $n = 6$ and 10, X is generated from a Pareto distribution with outliers where the number of outliers is taken to be $k = 1$ and 3. We also generate Y from a Pareto distribution with two sizes $m = 10$ and 30. The true stress-strength reliability parameter $\mathcal{R}$ is set to be 0.5 and 0.8 and three initial points, $\mathcal{R}_0$, are taken for each value of $\mathcal{R}$. It is considered $\mathcal{R}_0 = 0.35, 0.5$, and 0.65 if $\mathcal{R} = 0.5$ and $\mathcal{R}_0 = 0.65, 0.8$, and 0.95 if $\mathcal{R}$=0.8. The presumed parameters set for generating X and Y is $(\alpha, \beta, \theta, \lambda) = (3, 1.5, 1, 1)$ whereas the shape parameter of Y is set to

$$\nu = \frac{\alpha(1-\mathcal{R}_0)}{(b\beta^\alpha + \bar{b})\left(\frac{\theta}{\lambda}\right)^\alpha - (1-\mathcal{R}_0)}.$$

In each replication of 1000 trails, the proposed ML, MM, LS and MIX estimates of $\mathcal{R}$ and their corresponding shrinkage estimations are obtained. To investigate the estimation accuracies, we compute the bias and mean squared error (MSE):

$$\text{bias} = \frac{1}{1000}\sum_{i=1}^{1000}(\mathcal{R}_i^{ES} - \mathcal{R}_{true}) \qquad \text{and} \qquad \text{MSE} = \frac{1}{1000}\sum_{i=1}^{1000}(\mathcal{R}_i^{ES} - \mathcal{R}_{true})^2,$$

where $\mathcal{R}_i^{ES}$ denotes the specific estimate of $\mathcal{R}$ at the ith replication.

The detailed numerical results are reported in Tables 1, 2, and 3 for different three scenarios on parameter discussed in Section 4. Upon inspection of Tables 1 to 3, the following statements can be declared.

- Results depicted in Table 1 suggest that all estimators of $\mathcal{R}$ have small bias and MSE for all sample sizes. Moreover, as n and m increase, the MSE of ML, MM, and LS estimators tend to decrease toward zero. However, the ML estimate of $\mathcal{R}$ has the smallest MSE comparing to the MM and LS estimators. It can also be seen that the shrinkage estimators perform better than the classic estimators, especially for the small sample size. Although the efficiency of the first ML-based shrinkage estimator ($\tilde{\mathcal{R}}_{11}$) is the best, one can order them (from the best to worst) as $\tilde{\mathcal{R}}_{11} \geq \tilde{\mathcal{R}}_{21} \geq \tilde{\mathcal{R}}_{31} \geq \tilde{\mathcal{R}}_{51} \geq \tilde{\mathcal{R}}_{41}$.

- For the known outlier parameter, the results of Table 2 show that the MSE of all estimators tends to decrease as the sample sizes n and m increase. It can be observed that the ML provides a more efficient estimator than the MM and LS methods. It is also clear that the LS estimator has greater MSE and bias than the MM estimator. The numerical outputs in Table 2 also reveal that all types of shrinkage estimators perform much better than the classic estimator, and one can order them (from the best to worst) as $\tilde{\mathcal{R}}_{12} \geq \tilde{\mathcal{R}}_{32} \geq \tilde{\mathcal{R}}_{22} \geq \tilde{\mathcal{R}}_{42} \geq \tilde{\mathcal{R}}_{52}$.

- According to the results of Table 3 which highlights the bias and MSE of the estimators for the third scenarios, we can conclude that all estimators of $\mathcal{R}$ have small bias and MSE for all sample sizes. It is observed that the MSE of all estimators tends to decrease toward zero as n and m are increased. Moreover, the MIX estimator of moment and ML methods is more efficient than the MM estimator. However, this outperformance may significantly be ignorable when the sample sizes increase. The results in Table 3 also show that the shrinkage estimator based on the MIX estimator of moment and ML methods is more efficient than the others, especially when the initial value $\mathcal{R}_0$ is closer to the true value of $\mathcal{R}$. Furthermore, the bias of MIX is less than the MM, and the bias of shrinkage estimators is less than the classic estimators in almost all cases.

6. Actual examples

This section illustrates the usefulness of the proposed methodology by analyzing two real-world datasets in the physical and insurance studies.

Example 1. *The first considered dataset is related to the minority electron mobility for p-type $Ga_{1-x}Al_xAs$ with seven different values of mole fraction, initially reported by Bennett and Filliben (2000). Electron mobility is used for determining the speed of an electron movement through a metal or semiconductor when an electric field pulls it. Depending on the metal or semiconductor density and electric field, some noise could be available, and so the*

Table 1: Simulation results for assessing the accuracy of parameter estimates (bias and MSE (in the parenthesis)) when only the shape parameters are unknown.

n	k	m	$\mathcal{R}$	$\mathcal{R}_0$	$\hat{\mathcal{R}}_{ml1}$	$\tilde{\mathcal{R}}_{11}$	$\tilde{\mathcal{R}}_{21}$	$\tilde{\mathcal{R}}_{31}$	$\hat{\mathcal{R}}_{mm1}$	$\tilde{\mathcal{R}}_{41}$	$\hat{\mathcal{R}}_{ls1}$	$\tilde{\mathcal{R}}_{51}$
6	1	10	0.5	0.35	-0.238841	-0.170210	-0.237053	-0.237948	-0.274981	-0.190513	-0.141050	-0.145051
					(0.128482)	(0.042882)	(0.064461)	(0.068450)	(0.171513)	(0.088256)	(0.143945)	(0.071175)
				0.5	-0.072812	-0.021634	-0.025048	-0.042703	-0.151317	-0.362226	0.011885	-0.004681
					(0.084060)	(0.019401)	(0.028974)	(0.029545)	(0.296953)	(0.176751)	(0.116354)	(0.053162)
				0.65	0.078133	0.105572	0.121190	0.104512	0.002103	0.102458	0.132217	0.129830
					(0.033660)	(0.025179)	(0.027510)	(0.029150)	(0.076813)	(0.066139)	(0.094861)	(0.048752)
10	1	30	0.5	0.35	-0.176820	-0.150491	-0.176432	-0.176631	-0.180304	-0.176863	-0.152105	-0.152531
					(0.038103)	(0.022650)	(0.035771)	(0.037743)	(0.061614)	(0.053475)	(0.080038)	(0.045281)
				0.5	-0.004421	-0.000314	-0.001423	-0.002501	-0.011424	-0.015016	-0.005248	-0.013917
					(0.004932)	(0.000035)	(0.000491)	(0.001562)	(0.005166)	(0.003730)	(0.010981)	(0.002115)
				0.65	0.133140	0.149471	0.142904	0.139063	0.100651	0.072760	0.114331	0.118852
					(0.031812)	(0.022341)	(0.024143)	(0.029052)	(0.073791)	(0.060025)	(0.042123)	(0.032391)
6	3	10	0.5	0.35	-0.420681	-0.337738	-0.219375	-0.286993	-0.599514	-0.237517	-0.055218	-0.109490
					(0.069761)	(0.034179)	(0.052750)	(0.066451)	(1.194876)	(0.185643)	(0.122375)	(0.087860)
				0.5	-0.140562	-0.024380	-0.046860	-0.081131	-0.159015	-0.029861	0.035037	0.022001
					(0.051901)	(0.008391)	(0.012240)	(0.035961)	(0.147501)	(0.054682)	(0.115663)	(0.041981)
				0.65	-0.081512	-0.070170	0.059871	0.005573	0.0162431	-0.054145	0.150060	0.151443
					(0.057660)	(0.025961)	(0.048022)	(0.055356)	(0.440024)	(0.487338)	(0.097442)	(0.072951)
10	3	30	0.5	0.35	-0.197230	-0.150581	-0.158602	-0.170151	-0.226247	-0.182218	-0.126589	-0.134173
					(0.039862)	(0.022681)	(0.025523)	(0.030948)	(0.064112)	(0.040656)	(0.042432)	(0.039351)
				0.5	-0.041032	-0.000561	-0.011490	-0.021701	-0.098943	-0.055217	-0.006209	0.018501
					(0.008602)	(0.000024)	(0.000670)	(0.001401)	(0.019062)	(0.018583)	(0.015586)	(0.013763)
				0.65	0.117001	0.149343	0.136457	0.128856	0.120389	0.044675	0.141123	0.138071
					(0.030297)	(0.022310)	(0.026761)	(0.029352)	(0.062961)	(0.053264)	(0.046101)	(0.031072)
6	1	10	0.8	0.65	-0.213196	-0.155521	-0.175723	-0.190304	-0.292068	-0.201406	-0.175802	-0.170832
					(0.043268)	(0.024413)	(0.033830)	(0.040461)	(0.097456)	(0.136649)	(0.048156)	(0.041873)
				0.8	-0.034190	-0.001530	-0.007972	-0.019560	-0.019501	-0.004740	-0.030480	-0.010573
					(0.006430)	(0.000026)	(0.000609)	(0.002114)	(0.081640)	(0.053601)	(0.097500)	(0.031802)
				0.95	0.139610	0.149765	0.149901	0.149024	0.145946	0.149093	0.141927	0.148558
					(0.040413)	(0.022439)	(0.024176)	(0.028212)	(0.062982)	(0.053253)	(0.050941)	(0.042110)
10	1	30	0.8	0.65	-0.253001	-0.160860	-0.190491	-0.214563	-0.193287	-0.168610	-0.170523	-0.172894
					(0.039041)	(0.017032)	(0.021596)	(0.029657)	(0.040852)	(0.077452)	(0.037297)	(0.032714)
				0.8	-0.007212	-0.000330	-0.002548	-0.005329	-0.068918	-0.020103	0.001326	-0.0617671
					(0.003489)	(0.000001)	(0.000130)	(0.000559)	(0.065554)	(0.040085)	(0.071306)	(0.020262)
				0.95	0.137731	0.149708	0.149920	0.149052	0.148843	0.149251	0.145057	0.148513
					(0.030048)	(0.012415)	(0.022483)	(0.023224)	(0.044375)	(0.032280)	(0.041663)	(0.030134)
6	3	10	0.8	0.65	-0.269396	-0.168578	-0.197835	-0.225553	-0.279942	-0.162781	-0.132490	-0.149271
					(0.063568)	(0.036346)	(0.049347)	(0.056213)	(0.099364)	(0.078275)	(0.074286)	(0.062728)
				0.8	-0.068961	-0.011720	-0.023674	-0.040396	-0.042523	-0.004734	-0.021826	0.002607
					(0.008002)	(0.001369)	(0.005660)	(0.006485)	(0.075406)	(0.054283)	(0.049514)	(0.035215)
				0.95	0.133961	0.149732	0.14986421	0.148754	0.148043	0.148390	0.142215	0.149527
					(0.038823)	(0.020421)	(0.024597)	(0.025138)	(0.054001)	(0.042183)	(0.050964)	(0.032756)
10	3	30	0.8	0.65	-0.178761	-0.150352	-0.161768	-0.168390	-0.175161	-0.161243	-0.157283	-0.163028
					(0.035623)	(0.012614)	(0.026780)	(0.029852)	(0.054253)	(0.046805)	(0.040367)	(0.036825)
				0.8	-0.002971	-0.001433	-0.007124	-0.014605	-0.033492	-0.106683	-0.005136	0.002330
					(0.002323)	(0.000185)	(0.001252)	(0.001840)	(0.047871)	(0.014762)	(0.032417)	(0.006990)
				0.95	0.147790	0.146981	0.152953	0.153240	0.146081	0.151250	0.147042	0.149238
					(0.031942)	(0.012501)	(0.022503)	(0.023674)	(0.048075)	(0.033126)	(0.041824)	(0.028293)

values of minority electron mobility might be contaminated by outliers. Two datasets related to the mole fractions 0.25 ($M_{0.25}$) and 0.30 ($M_{0.30}$) considered in this analysis are; $X = M_{0.25}$: 3.051, 2.779, 2.604, 2.371, 2.214, 2.045, 1.715, 1.525, 1.296, 1.154, 1.016, 0.7948, 0.7007, 0.6292, 0.6175, 0.6449, 0.8881, 1.115, 1.397, 1.506, 1.528, and $Y = M_{0.3}$: 2.658, 2.434, 2.288, 2.092, 1.959, 1.814, 1.530, 1.366, 1.165, 1.041, 0.9198, 0.7241, 0.6403, 0.576, 0.5647, 0.5873, 0.8013, 1.002, 1.250, 1.347, 1.368.

Applying the one-sample Kolmogorov-Smirnov (KS) test, it is observed that the KS statistic for $M_{0.25}$ is 0.20269 with p-value 0.3107 and for $M_{0.30}$ is 0.20141 with p-value 0.3178. One can clearly conclude that these data strongly follow the Pareto distribution since the p-values of the KS test are greater than the 5% significance level. The box-plot

Table 2: Simulation results for assessing the accuracy of parameter estimates (bias and MSE (in the parenthesis)) when only the outlier parameter is known.

n	k	m	$\mathcal{R}$	$\mathcal{R}_0$	$\hat{\mathcal{R}}_{ml2}$	$\tilde{\mathcal{R}}_{12}$	$\tilde{\mathcal{R}}_{22}$	$\tilde{\mathcal{R}}_{32}$	$\hat{\mathcal{R}}_{mm2}$	$\tilde{\mathcal{R}}_{42}$	$\hat{\mathcal{R}}_{ls2}$	$\tilde{\mathcal{R}}_{52}$
6	1	10	0.5	0.35	0.072573	-0.150453	0.076422	0.064521	-0.079867	-0.155651	-0.064719	-0.128011
					(0.011692)	(0.002518)	(0.011035)	(0.010235)	(0.078424)	(0.027817)	(0.091593)	(0.029639)
				0.5	0.140408	-0.000284	0.148627	0.149576	-0.142425	-0.006671	-0.151093	-0.059079
					(0.029293)	(0.000212)	(0.014687)	(0.008611)	(0.066024)	(0.043665)	(1.084480)	(0.226689)
				0.65	0.243901	0.150982	0.163622	0.174851	0.031828	0.137253	-0.107672	-0.006179
					(0.068228)	(0.022501)	(0.053211)	(0.036875)	(0.074158)	(0.069151)	(1.119129)	(1.988091)
10	1	30	0.5	0.35	0.059005	-0.049652	0.063589	0.059125	-0.215158	-0.162838	-0.064891	-0.087829
					(0.007368)	(0.001876)	(0.004682)	(0.003737)	(0.061486)	(0.024707)	(0.078258)	(0.025336)
				0.5	0.135174	-0.008524	0.186516	0.187660	0.084682	-0.010164	0.019253	0.010637
					(0.021661)	(0.000153)	(0.012647)	(0.008477)	(0.065449)	(0.041323)	(0.090729)	(0.071442)
				0.65	0.220152	0.149939	0.219631	0.194772	-0.028627	0.137552	0.012712	0.138210
					(0.051466)	(0.022482)	(0.049824)	(0.033612)	(0.063235)	(0.050429)	(0.093918)	(0.091534)
6	3	10	0.5	0.35	0.116669	-0.150523	0.100902	0.114621	-0.225620	-0.156343	-0.123981	-0.196229
					(0.025615)	(0.022674)	(0.024800)	(0.023402)	(0.088842)	(0.075119)	(0.416763)	(0.293021)
				0.5	0.188868	0.149635	0.189352	0.151689	-0.090881	-0.168002	-0.071923	-0.060324
					(0.045338)	(0.001687)	(0.035338)	(0.015338)	(0.069010)	(0.046727)	(0.274078)	(0.174051)
				0.65	0.290884	0.150421	0.301251	0.290884	0.087361	0.142813	-0.431320	0.134204
					(0.092921)	(0.021871)	(0.053698)	(0.042921)	(0.099718)	(0.071604)	(1.411401)	(0.093341)
10	3	30	0.5	0.35	0.089694	-0.049235	0.077541	0.069694	-0.264795	-0.156273	-0.046886	-0.048575
					(0.011978)	(0.004214)	(0.009624)	(0.007197)	(0.076460)	(0.064791)	(0.112873)	(0.095023)
				0.5	0.163450	-0.058313	0.160451	0.158915	-0.185245	-0.190824	0.025406	0.104663
					(0.030066)	(0.001358)	(0.020067)	(0.015266)	(0.065666)	(0.045982)	(0.070881)	(0.067881)
				0.65	0.247316	0.148357	0.221682	0.191750	-0.085072	0.139422	0.015823	0.087827
					(0.064360)	(0.020381)	(0.051832)	(0.039641)	(0.076260)	(0.070502)	(0.496667)	(0.086291)
6	1	10	0.8	0.65	-0.076165	-0.008541	-0.103821	-0.094438	-0.289678	-0.158238	-0.237829	-0.161074
					(0.011319)	(0.009361)	(0.010926)	(0.010319)	(0.140229)	(0.046088)	(0.238349)	(0.089769)
				0.8	0.048152	0.000437	0.009642	0.007394	-0.147262	-0.015787	-0.128552	-0.0520364
					(0.006598)	(0.002719)	(0.005074)	(0.004139)	(0.081307)	(0.031560)	(0.108306)	(0.095923)
				0.95	0.174661	0.150382	0.169541	0.157224	0.086826	0.149139	-0.057594	-0.164728
					(0.031110)	(0.012163)	(0.030263)	(0.021110)	(0.050352)	(0.048228)	(0.583914)	(0.050715)
10	1	30	0.8	0.65	-0.079857	-0.007622	-0.057821	-0.014322	-0.308482	-0.152226	-0.175696	-0.173413
					(0.008913)	(0.001038)	(0.005621)	(0.003571)	(0.124524)	(0.023240)	(0.159080)	(0.059811)
				0.8	0.029024	-0.000154	0.018921	0.001017	-0.122500	-0.042187	-0.107907	-0.010576
					(0.002533)	(0.000851)	(0.001035)	(0.000982)	(0.056736)	(0.009221)	(0.074966)	(0.014489)
				0.95	0.158960	0.149996	0.150622	0.147911	0.163035	0.149326	-0.034892	0.144636
					(0.025559)	(0.010499)	(0.020173)	(0.015632)	(0.035781)	(0.022307)	(0.125823)	(0.022367)
6	3	10	0.8	0.65	-0.022217	-0.149791	-0.078322	-0.069521	-0.259062	-0.167233	-0.244481	-0.168297
					(0.007465)	(0.004871)	(0.006824)	(0.005843)	(0.141822)	(0.031520)	(0.246917)	(0.045994)
				0.8	0.089226	0.000794	0.009184	0.006533	-0.097718	-0.091953	-0.232917	-0.006712
					(0.012257)	(0.000942)	(0.009631)	(0.001185)	(0.069494)	(0.051760)	(0.241714)	(0.060704)
				0.95	0.185463	0.148661	0.179331	0.170541	0.103975	0.094944	-0.183691	-0.168272
					(0.034625)	(0.011835)	(0.031846)	(0.020371)	(0.138668)	(0.051878)	(0.316852)	(0.061592)
10	3	30	0.8	0.65	-0.043772	-0.011824	-0.038191	-0.021825	-0.334162	-0.161132	-0.148439	-0.153040
					(0.003963)	(0.001225)	(0.003148)	(0.002053)	(0.096793)	(0.028254)	(0.240505)	(0.053502)
				0.8	0.047073	-0.000035	0.009571	0.000715	-0.135552	-0.008444	-0.116259	-0.015086
					(0.003787)	(0.000531)	(0.001162)	(0.000859)	(0.039484)	(0.009509)	(0.094027)	(0.009721)
				0.95	0.0086994	0.009471	0.047812	0.011872	0.075247	0.059411	-0.056637	0.149063
					(0.003803)	(0.001062)	(0.002170)	(0.001982)	(0.078252)	(0.022338)	(0.209551)	(0.023692)

of $M_{0.25}$ in Figure 1 highlights that there are two potential outliers in the data. The Pareto quantile-quantile (Q-Q) plot of $M_{0.25}$ along with the empirical cumulative Pareto probability are also shown in Figure 1, depicting significant evidence that some outliers are available in these data.

Since the number of outliers is not known, we should estimate the unknown parameters α, β, and θ based on the profile-likelihood function. Accordingly, k can be selected as a maximizer of the profile-likelihood. Table 4 shows the likelihood values of the model for different choices of k. It can be observed that the likelihood function is maximized at $k = 2$. By using the MIX estimate of the unknown parameters as $\hat{\alpha}_{mix}$=0.8386112, $\hat{\beta}_{mix}$=1.5959017, $\hat{\theta}_{mix}$=0.3869286, $\hat{\nu}_{mix}$=4.1501706, $\hat{\lambda}_{mix}$=0.9389963, the stress-strength reliability parameter estimate is $\hat{\mathcal{R}}_{mix} = 0.4959972$. *Through an*

Table 3: Simulation results for assessing the accuracy of parameter estimates (bias and MSE (in the parenthesis)) when all of the parameters are unknown.

n	k	m	$\mathcal{R}$	$\mathcal{R}_0$	$\hat{\mathcal{R}}_{mm3}$	$\tilde{\mathcal{R}}_{13}$	$\hat{\mathcal{R}}_{mix}$	$\tilde{\mathcal{R}}_{23}$
6	1	10	0.5	0.35	-0.058729	-0.154536	0.046602	-0.150000
					(0.078286)	(0.024258)	(0.037925)	(0.022500)
				0.5	-0.045705	-0.010123	0.033570	-0.000163
					(0.070752)	(0.001756)	(0.049214)	(0.000003)
				0.65	0.048414	0.136559	0.158010	0.149827
					(0.059336)	(0.040685)	(0.055263)	(0.022451)
10	1	30	0.5	0.35	-0.179518	-0.1533057	-0.027005	-0.150051
					(0.067690)	(0.023679)	(0.027764)	(0.021515)
				0.5	0.080027	-0.003214	0.149293	0.000009
					(0.064111)	(0.000305)	(0.041689)	(0.000001)
				0.65	0.038065	0.139800	0.193459	0.149602
					(0.051150)	(0.031461)	(0.044653)	(0.022389)
6	3	10	0.5	0.35	-0.131981	-0.150012	0.044646	-0.150000
					(0.110758)	(0.032504)	(0.094163)	(0.022500)
				0.5	-0.200143	-0.008174	0.128729	-0.000149
					(0.067783)	(0.000687)	(0.059439)	(0.000002)
				0.65	0.075279	0.149216	0.225678	0.150000
					(0.107065)	(0.032277)	(0.094012)	(0.022500)
10	3	30	0.5	0.35	-0.220490	-0.158953	0.035309	-0.015120
					(0.085705)	(0.003591)	(0.008302)	(0.002480)
				0.5	-0.207898	-0.012052	0.111947	0.000059
					(0.049055)	(0.000194)	(0.019104)	(0.000001)
				0.65	-0.200234	0.111898	0.219468	0.1604821
					(0.088968)	(0.028829)	(0.053095)	(0.019800)
6	1	10	0.8	0.65	-0.264993	-0.165025	-0.150255	-0.150511
					(0.149451)	(0.030391)	(0.066552)	(0.022813)
				0.8	-0.117953	-0.015042	0.010542	-0.000032
					(0.096015)	(0.002775)	(0.019543)	(0.000568)
				0.95	0.058385	0.145539	0.139513	0.141573
					(0.052469)	(0.022722)	(0.037060)	(0.015248)
10	1	30	0.8	0.65	-0.292687	-0.161379	-0.119621	-0.150162
					(0.112389)	(0.027584)	(0.019179)	(0.022550)
				0.8	-0.121166	-0.008142	0.009953	-0.000027
					(0.044829)	(0.001173)	(0.004738)	(0.000052)
				0.95	0.117078	0.146963	0.154106	0.149966
					(0.035904)	(0.021405)	(0.024607)	(0.012489)
6	3	10	0.8	0.65	-0.181028	-0.167975	-0.112637	-0.163547
					(0.095993)	(0.035783)	(0.075774)	(0.025719)
				0.8	-0.101095	-0.039403	0.027296	0.001937
					(0.082270)	(0.002711)	(0.054998)	(0.000074)
				0.95	0.086709	0.149835	0.171011	0.131821
					(0.058173)	(0.022452)	(0.031006)	(0.016914)
10	3	30	0.8	0.65	-0.164233	-0.171668	-0.074320	-0.150482
					(0.045960)	(0.019965)	(0.023855)	(0.013891)
				0.8	-0.273095	-0.009463	0.030637	0.000487
					(0.016744)	(0.001781)	(0.004763)	(0.000035)
				0.95	0.033519	0.145856	0.163014	0.132961
					(0.047423)	(0.021709)	(0.027034)	(0.013591)

MC method for n=21, k=2, m=21 and the obtained MIX parameter estimates, we approximate $\hat{\alpha}_{23}$=0.7666702. Since all values of $M_{0.25}$ are grater than $M_{0.3}$ values, the $\mathcal{R}_0$ is set to 0.999999 and so the MIX-based shrinkage estimator is obtained $\tilde{\mathcal{R}}_{23} = 0.613596$.

Example 2. *By way of the second illustration, we consider insurance claim data. One of the most important services in the insurance industry is motor insurance. In case of an accident, the claiming amount made by the policyholder might be declined by the insurer since it always is far from the claim amount specified by the insurer (indemnity*

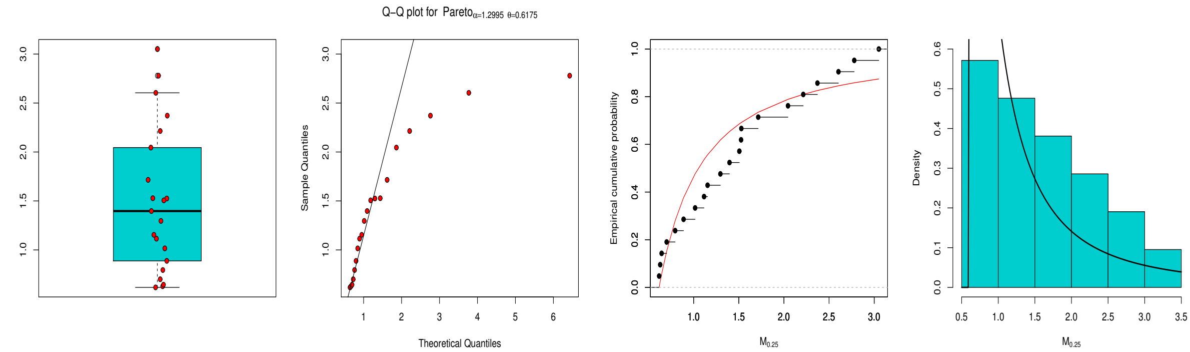

Figure 1: Plots related to the $M_{0.25}$ data. From left to right panels: box-plot, the Pareto Q-Q plot, and empirical cumulative Pareto probability and density plots.

Table 4: The likelihood values of the fitted Pareto distribution with outliers to $M_{0.25}$ for various values of k.

k	1	2	3	4	5
Likelihood	1.131503e-22	2.141748e-16	4.024797e-19	1.465477e-20	1.068233e-21
k	6	7	8	9	10
Likelihood	4.236475e-21	8.028271e-21	1.392202e-20	2.470913e-20	4.715105e-20

Table 5: The likelihood values of the fitted Pareto distribution with outliers to X for various values of k.

k	1	2	3	4	5
Likelihood	4.645267e-147	3.452667e-143	1.342475e-145	8.656568e-146	1.919880e-145
k	6	7	8	9	10
Likelihood	3.301829e-145	5.499681e-145	9.603544e-145	1.831912e-144	3.919700e-144

amount). Hence, it is important for company to estimate $\mathcal{R} = Pr(X < Y)$, where X and Y represent the claim and indemnity amounts, respectively. As previously explained, in the analysis of motor insurance, a claim of at least θ, as compensation, can be made, and claims below it are not entertained. Since the value of vehicles is different, the claim amounts could be vary depending on the damage rate. Suppose that the claims of expensive/severe damaged vehicles are β times higher than the normal ones. Therefore, the claim data can be contaminated by the outliers, whereas the indemnity is always homogenous.

In this experimental example, we consider a sample of the claim amounts and their indemnity amounts of the Iran insurance company. The scaled data by 1000 are; X (claim amounts): 750, 780, 630, 1750, 1450, 3000, 7650, 4210, 890, 950, 1240, 1800, 1630, 9020, 4750, 3250, 1135, 1326, 1280, 760, and Y (indemnity amounts): 830, 750, 650, 1500, 1520, 2700, 7500, 3750, 950, 900, 1300, 1550, 1700, 8200, 4500, 3000, 1200, 1235, 1115, 830. By applying the one-sample KS test, the observed KS statistic for X is 0.14192 with p-value 0.7642 and for Y is 0.13400 with p-value 0.8652. It can be seen that the p-values of the KS test are greater than the 5% significance level, reflecting that these data strongly follow the Pareto distribution. The box-plot of X in Figure 2 highlights that there are two potential outliers in the data. The Pareto quantile-quantile (Q-Q) plot of X along with the empirical cumulative Pareto probability are also shown in Figure 2, depicting significant evidence that some outliers are available in these data.

We estimate the α, β, and θ by constructing the profile-likelihood function for the fix k. Table 5 shows the likelihood values of the model for different choices of k. It can be seen that the likelihood function is maximized at $k = 2$. By using the MIX estimate of the unknown parameters as $\hat{\alpha}_{mix}$=0.4600503, $\hat{\beta}_{mix}$=3.646515, $\hat{\theta}_{mix}$=172767.7, $\hat{\nu}_{mix}$=4.761649, $\hat{\lambda}_{mix}$=1359776, the stress-strength reliability parameter estimate is $\hat{\mathcal{R}}_{mix}$=0.4747673. Through an MC method for n=20, k=2, m=20 and the obtained MIX parameter estimates, we approximate $\hat{\alpha}_{23}$=0.5399776. Since the percent of claim amounts less than indemnity amounts is $\mathcal{R}_0$=0.4, the obtained MIX-based shrinkage estimator is $\tilde{\mathcal{R}}_{23}$ = 0.4403727.

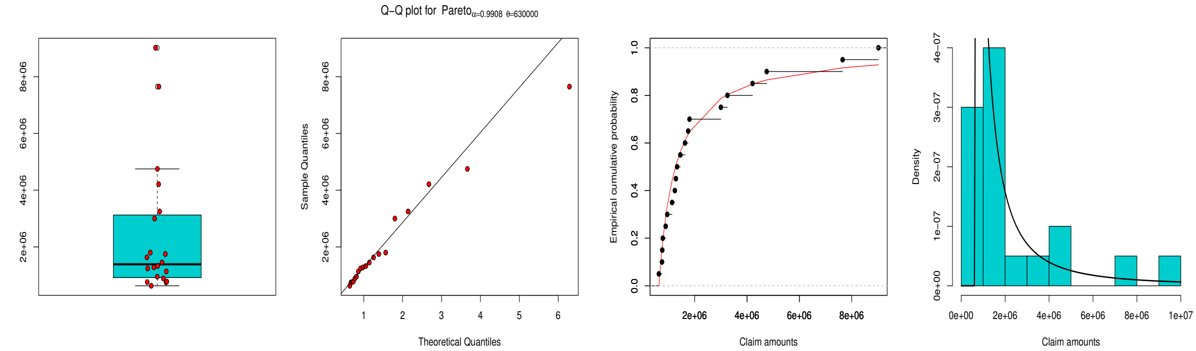

Figure 2: Plots related to the claim amounts data. From left to right panels: box-plot, the Pareto *Q-Q* plot, and empirical cumulative Pareto probability and density plots.

7. Conclusion and future extensions

This paper presented a flexible approach for estimating the stress-strength parameter, $\mathcal{R} = \Pr(X < Y)$, when some outliers contaminated data. It was assumed that X follows the Pareto distribution in the presence of outliers, and independently Y has the homogenous case of the Pareto distribution. The estimation process was derived under three scenarios on the model parameters: 1) Only shape parameters were unknown, 2) Except the outlier parameter, β, all of the parameters were unknown, and 3) The general case, i.e., all the parameters were considered to be unknown. We obtained the ML, MM, LS, MIX, and their corresponding shrinkage estimates.

The accuracy of the proposed method was examined in terms of the bias and MSE by a simulation study. An overall inspection of simulation analysis was that the ML and MIX estimates of $\mathcal{R}$ had the smallest MSE comparing to the MM and LS estimators. Moreover, the shrinkage estimators performed better than the classical estimators, specifically for small sample sizes. We observed that as can be expected, the shrinkage estimators had smaller MSE for the small sample sizes than for the large ones, since as the sample size increases, the precision of the estimators increases, whereas shrinkage estimators are still affected by the prior guess, $\mathcal{R}_0$, which might poorly be made. It was furthermore seen that when all of the parameters were unknown the MIX estimate and its shrinkage estimate were more efficient. Finally, the proposed methodology was illustrated by analyzing two real-world datasets in the physical and insurance studies. All computations were carried out using the statistical software R 4.0.1 in a Win 64 environment with a 2.50 GHz/Intel Core(TM) i5 3120M CPU Processor and 8.0 GB RAM.

The current approach can be extended to the more general case where both rvs X and Y follow the Pareto distribution in the presence of outliers. In addition, it is of interest to develop a new tool for addressing the problem of detecting change points in the stress-strength reliability (Xu et al., 2019) by the Pareto distribution in the presence of outliers as the underlying distributions of X and Y. Due to encountering censored data in many survival fields, the estimation of stress-strength reliability based on the Pareto distribution in the presence of outliers and under various censoring schemes would be an interesting direction for future works (Bai et al., 2018, 2019a,b).

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Appendix A. Proof of Theorems 1

To calculate the first and second moments of $\hat{\mathcal{R}}_{ml1}$, the pdf of $\hat{\alpha}_{ml1}$ and $\hat{\nu}_{ml1}$ should be obtained. Based on the pdf of $\sum_{i=1}^{n} \ln(X_i)$ and $\sum_{i=1}^{m} \ln(Y_i)$ (see Dixit and Jabbari Nooghabi (2011a)), the pdf of $\hat{\alpha}_{ml1}$ and $\hat{\nu}_{ml1}$ can respectively be derived as

$$g(\hat{\alpha}_{ml1}) = \frac{(n\alpha)^n}{\Gamma(n)\hat{\alpha}_{ml1}^{n+1}} \exp\left(-\frac{n\alpha}{\hat{\alpha}_{ml1}}\right), \qquad \hat{\alpha}_{ml1}, \alpha > 0,$$
$$g(\hat{\nu}_{ml1}) = \frac{(m\nu)^m}{\Gamma(m)\hat{\nu}_{ml1}^{m+1}} \exp\left(-\frac{m\nu}{\hat{\nu}_{ml1}}\right), \qquad \hat{\nu}_{ml1}, \nu > 0.$$

Therefore,

$$\begin{aligned} \mathrm{E}(\hat{\mathcal{R}}_{ml1}) &= 1 - \int_0^\infty \int_0^\infty \frac{\hat{\nu}_{ml1}}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} (b\beta^{\hat{\alpha}_{ml1}} + \bar{b}) \left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{ml1}} g(\hat{\alpha}_{ml1}) g(\hat{\nu}_{ml1}) d\hat{\alpha}_{ml1} d\hat{\nu}_{ml1}, \\ &= 1 - \int_0^\infty \underbrace{\left[\int_0^\infty \frac{1}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} (b\beta^{\hat{\alpha}_{ml1}} + \bar{b}) \left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{ml1}} \frac{(n\alpha)^n}{\Gamma(n)\hat{\alpha}_{ml1}^{n+1}} \exp\left(-\frac{n\alpha}{\hat{\alpha}_{ml1}}\right) d\hat{\alpha}_{ml1}\right]}_{Int_{in}} \hat{\nu}_{ml1} g(\hat{\nu}_{ml1})\, d\hat{\nu}_{ml1}. \end{aligned}$$

The inner integral (Int_{in}) is

$$\begin{aligned} Int_{in} =& \frac{(n\alpha)^n}{\Gamma(n)} \int_0^\infty \frac{b\beta^{\hat{\alpha}_{ml1}} + \bar{b}}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} \left(\frac{\theta}{\lambda}\right)^{\hat{\alpha}_{ml1}} \hat{\alpha}_{ml1}^{-n-1} \exp\left(-\frac{n\alpha}{\hat{\alpha}_{ml1}}\right) d\hat{\alpha}_{ml1} \\ =& \frac{(n\alpha)^n}{\Gamma(n)} \left\{ b \int_0^\infty \frac{\hat{\alpha}_{ml1}^{-n-1}}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} \exp\left(\hat{\alpha}_{ml1} \ln\left(\frac{\beta\theta}{\lambda}\right) - \frac{n\alpha}{\hat{\alpha}_{ml1}}\right) d\hat{\alpha}_{ml1} + \bar{b} \int_0^\infty \frac{\hat{\alpha}_{ml1}^{-n-1}}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} \exp\left(\hat{\alpha}_{ml1} \ln\left(\frac{\theta}{\lambda}\right) - \frac{n\alpha}{\hat{\alpha}_{ml1}}\right) d\hat{\alpha}_{ml1} \right\} \\ =& \frac{(n\alpha)^n}{\Gamma(n)} \left\{ b \sum_{j=0}^\infty (-1)^j \sum_{i=0}^j C(j,i)(\hat{\nu}_{ml1} - 1)^{j-i} 2n^{i-n} \hat{\alpha}_{ml1}^{i-n} (n\alpha[\ln(\lambda) - \ln(\beta\theta)])^{\frac{n-i}{2}} BesselK\left(i-n, 2\sqrt{n\alpha[\ln(\lambda) - \ln(\beta\theta)]}\right) \right. \\ &\left. + \bar{b} \sum_{j=0}^\infty (-1)^j \sum_{i=0}^j C(j,i)(\hat{\nu}_{ml1} - 1)^{j-i} 2n^{i-n} \hat{\alpha}_{ml1}^{i-n} (n\alpha[\ln(\lambda) - \ln(\theta)])^{\frac{n-i}{2}} BesselK\left(i-n, 2\sqrt{n\alpha[\ln(\lambda) - \ln(\theta)]}\right) \right\}, \end{aligned}$$

where the last equation is obtained by

$$\frac{1}{\hat{\alpha}_{ml1} + \hat{\nu}_{ml1}} = \sum_{j=0}^\infty (-1)^j (-1 + \hat{\alpha}_{ml1} + \hat{\nu}_{ml1})^j = \sum_{j=0}^\infty (-1)^j \sum_{i=0}^j C(j,i) \hat{\alpha}_{ml1}^i (\hat{\nu}_{ml1} - 1)^{j-i}.$$

The second moment of $\hat{\mathcal{R}}_{ml1}$ can similarly be obtained by

$$\begin{aligned} \mathrm{E}(\hat{\mathcal{R}}_{ml1}^2) =& 1 - 2\mathrm{E}(\hat{\mathcal{R}}_{ml1}) + b^2 \mathrm{E}\left(\frac{\hat{\nu}_{ml1}^2}{(\hat{\alpha}_{ml1} + \hat{\nu}_{ml1})^2} \left(\frac{\beta\theta}{\lambda}\right)^{2\hat{\alpha}_{ml1}}\right) \\ &+ 2b\bar{b}\mathrm{E}\left(\frac{\hat{\nu}_{ml1}^2}{(\hat{\alpha}_{ml1} + \hat{\nu}_{ml1})^2} \beta^{\hat{\alpha}_{ml1}} \left(\frac{\theta}{\lambda}\right)^{2\hat{\alpha}_{ml1}}\right) + \bar{b}^2 \mathrm{E}\left(\frac{\hat{\nu}_{ml1}^2}{(\hat{\alpha}_{ml1} + \hat{\nu}_{ml1})^2} \left(\frac{\theta}{\lambda}\right)^{2\hat{\alpha}_{ml1}}\right). \end{aligned}$$

This completes the proof.

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