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# Dissection-like proofs of the law of cosines

Dr. Henry (Maya) Robert Thackeray
Department of Mathematics and Applied Mathematics, University of Pretoria
Pretoria, 0002 South Africa
maya.thackeray@up.ac.za, mayart314@outlook.com

## Introduction

This paper describes some dissection-like proofs of the law of cosines that are analogous to well-known proofs of the Pythagorean theorem (PT).

PT says that $c^2 = a^2 + b^2$ for a right triangle with hypotenuse $c$ and legs $a$ and $b$. To begin, three proofs of PT are briefly recalled.

This paper's initial inspiration was the following proof ([**1**, p. 11], [**3**, proof 9], and [**9**, geometric proofs 90 and 218]). Start with Figure 1(a). Slide the $T$ triangles, each a clone of the given right triangle, to make Figure 1(b). The area in the large square left over after removing the $T$s stays the same, and it is first $c^2$, then $a^2 + b^2$.

Without loss of generality assume $a \geq b$. Figure 1(c) is Figure 1(a) with added lines. The $c^2$ square contains the small square (which has side $a - b \geq 0$) and is contained in the large square (which has side $a + b$). The next two proofs show PT after basic algebra ([**3**, proofs 3 and 4] and [**9**, geometric proofs 219, 220, 225, and 229]). (a) Adding four $T$s (area $ab/2$ each) to the small square (area $(a - b)^2$) gives the middle square (area $c^2$), so $(a - b)^2 + 2ab = c^2$. This part of the figure is Bhaskara's "Behold!". (b) Adding four $T$s (area $ab/2$ each) to the middle square (area $c^2$) gives the large square (area $(a + b)^2$), so $c^2 + 2ab = (a + b)^2$.

The law of cosines says that $c^2 = a^2 + b^2 - 2ab \cos C$ for a triangle with sides $a$, $b$, and $c$ opposite angles $A$, $B$, and $C$ respectively. In similar spirit to the above PT proofs, this paper will give six inter-related proofs of the law of cosines. Proofs 1 to 4 will move parts of a square, like the first PT proof above. Proofs 5 and 6 will use algebra in a situation where two concentric squares have the same side directions, while an additional shape contains one of the squares and is contained in the other, like the second and third PT proofs above.

Some law-of-cosines proofs in the literature divide into cases based on whether $C$ is acute or obtuse, or consider only one of those cases. For example, in Euclid's *Elements* [**5**], Propositions 12 and 13 of Book II are equivalent to the law of cosines for obtuse and acute $C$ respectively. (The word "cosine" and negative numbers were not available to Euclid, who would have been forced to divide the argument into cases based on the sign of what is now called $\cos C$.) Other examples include two dissection proofs due to Rudolf Hunger and Erwin Dintzl for obtuse $C$, associated with a certain tessellation of the plane [**6**, pp. 38-39], as well as related dissection-like arguments in [**15**, section 3.5] proving the acute and obtuse cases separately. Some literature proofs combine the cases by using signed lengths [**4, 12**] or signed areas [**2, 8**].

In contrast to those arguments, this paper will use the angle $\theta = C/2$, which is always acute. This will allow one to avoid division into cases, and also to avoid signed lengths and signed areas. In each of proofs 1 to 6, one of

$$\begin{aligned} c^2 &= (a-b)^2 + 4ab\sin^2\theta \\ c^2 + 2ab\cos^2\theta &= a^2 + b^2 + 2ab\sin^2\theta \\ c^2 + 4ab\cos^2\theta &= (a+b)^2 \end{aligned}$$

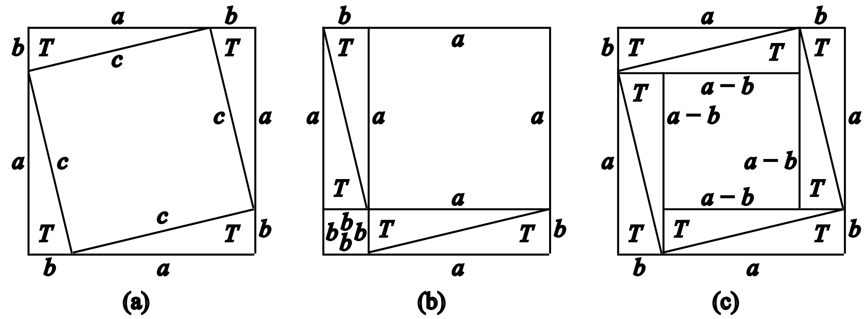



**Figure 1** Diagrams for proofs of PT.

will be shown, and then $1 - 2\sin^2\theta$, $\cos^2\theta - \sin^2\theta$, or $2\cos^2\theta - 1$ respectively will be used as a double-angle formula for $\cos 2\theta$ to deduce the law of cosines.

In keeping with dissection-related literature like [**6**], this paper will tend not to include details of proofs to as rigorous a level as Euclid's *Elements*, except to some extent in the two Appendices. The diagrams will tend to speak for themselves, roughly in the spirit of proofs without words, although, of course, explanations will be given.

## Note: Where will PT be assumed?

Before the proofs, some brief remarks are made about where PT will be assumed.

Proofs 1, 3, and 4 will not assume PT, and proof 2 will assume only the special case of PT dealing with $45°$-$45°$-$90°$ triangles. This means that PT—that is, the case $C = 90°$ of the law of cosines—will automatically be proved in proofs 1, 3, and 4, and proved assuming the $45°$-$45°$-$90°$-triangle case in proof 2. Proofs 5 and 6 will show the acute- and obtuse-angle cases of the law of cosines, but will assume general PT in the form $\cos^2\theta + \sin^2\theta = 1$.

Each of proofs 1 to 6 will use a single double-angle formula for $\cos 2\theta$. That does not involve assuming PT, although any two of the three $\cos 2\theta$ formulas together imply PT immediately as $\cos^2\theta + \sin^2\theta = 1$. It is possible to prove each $\cos 2\theta$ formula by itself, without using one $\cos 2\theta$ formula to prove another, and without assuming PT: see Appendix 1 for such proofs.

The trigonometric ratios $\cos\theta$ and $\sin\theta$ may be defined as the usual ratios of sides of a right triangle. That does not involve assuming PT in the form $\cos^2\theta + \sin^2\theta = 1$. (If two numbers are well defined, that does not imply that the sum of their squares must be 1.) However, these definitions of $\cos\theta$ and $\sin\theta$ do use the fact that if two right triangles have the same sets of angles, then the ratios of corresponding sides are equal.

## The underlying figure

From now on, shapes are congruent if they have the same label.

All six proofs are ultimately derived from Figure 2, which is now briefly described for clarity. The figure has fourfold rotational symmetry, and dimensions toward its left side are marked. The large square has side $(a+b)(\cos\theta + \sin\theta)$. Each $T$ is a clone of the given triangle with sides $a$, $b$, and $c$. Each $T$ is oriented so that the angle bisector of angle $C$ is at right angles to the large square's side touching that $T$. Each $X$ rectangle has sides $a\cos\theta$ and $b\cos\theta$. Each shaded right triangle $I$ or $J$ has hypotenuse $b$ or $a$,

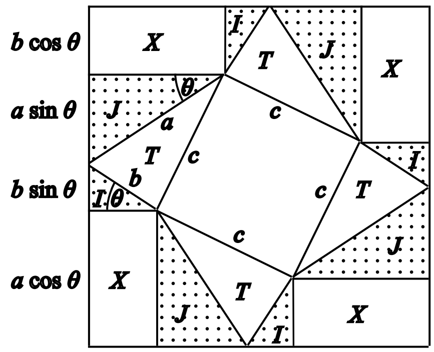



**Figure 2** The diagram from which proofs 1 to 6 are derived.

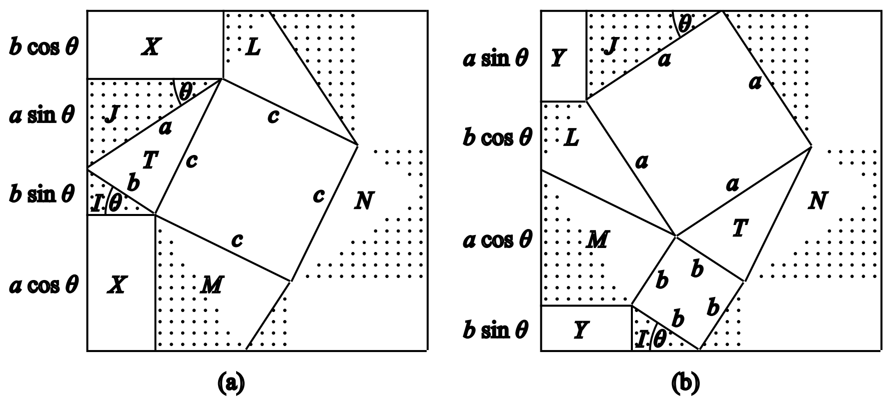



**Figure 3** Diagrams for proof 1.

respectively, and an angle $\theta$ at the vertex touching the figure's central square of side $c$. That square's side next to the topmost $T$ may have negative, zero, or positive slope (for $a > b$, $a = b$, or $a < b$ respectively), except as restricted by the "assumptions without loss of generality" (like $a \geq b$) in proofs 4 to 6.

## Proof 1: Generalizing PT

In Figure 3(a), which is Figure 2 with some lines removed, there are shapes $I$, $J$, $T$, $L$, and $M$, which slide around, and $N$, which is fixed in the large square. The proof simplifies on removing $N$, which is included to show how this argument relates to the others. In Figure 3(b), each $Y$ rectangle has sides $a \sin\theta$ and $b \sin\theta$.

Start with Figure 3(a). Slide the shapes to make Figure 3(b). The area in the large square left over after removing $I$, $J$, $T$, $L$, $M$, and $N$ stays the same, and it is first $c^2 + 2|X|$, then $a^2 + b^2 + 2|Y|$. So $c^2 + 2|X| = a^2 + b^2 + 2|Y|$, which is the law of cosines since $|X| = ab\cos^2\theta$ and $|Y| = ab\sin^2\theta$.*

When $C = 90°$, the shaded $I/J$ rings enclose the squares in Figures 1(a) and 1(b), now tilted by $45°$, and cancelling $2|X| = 2|Y|$ gives PT. In this way, proof 1 generalizes the Introduction's first PT proof. (Note that $\cos^2\theta + \sin^2\theta = 1$ has not been used here.)

*The interested reader may compare this law-of-cosines proof with the two separate cases of [**15**, section 3.5].

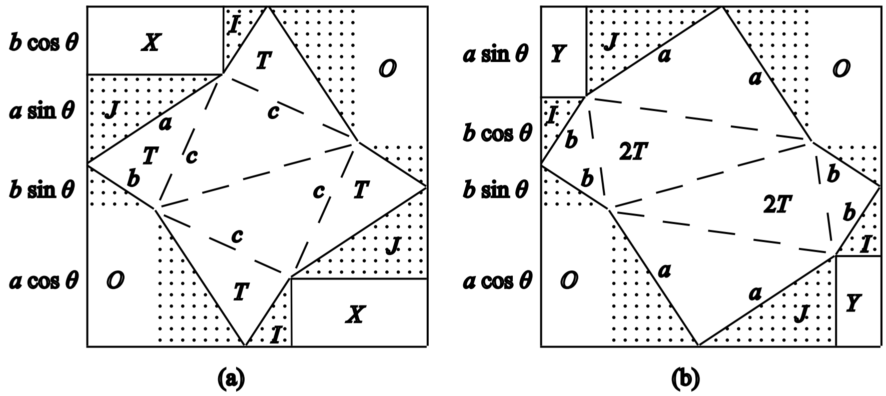



**Figure 4** Diagrams for proof 2.

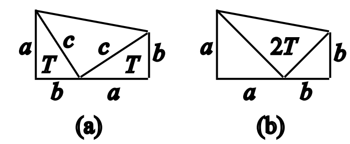



**Figure 5** Diagrams for a proof of PT assuming the $45°$-$45°$-$90°$-triangle case.

## Proof 2: Two butterflies

In this section, assume PT for $45°$-$45°$-$90°$ triangles.

Figures 4(a) and 4(b) have twofold rotational symmetry. The labeled $I$s and $J$s slide around. The $O$s are fixed and superfluous, like $N$. Of each large square's non-$O$ part, only one of the halves cut by the $c^2$ square's diagonal is needed.

The $O$s, $I$s, and $J$s in Figure 4 enclose two six-triangle butterflies. In Figure 4(a), the two central triangles are halves of a $c^2$ square, and there are four outer $T$ triangles. In Figure 4(b), the two central $2T$ triangles are $T$s with lengths scaled by $\sqrt{2}$ (because PT holds for $45°$-$45°$-$90°$ triangles), and the four outer triangles are halves of $a^2$ and $b^2$ squares.

Start with Figure 4(a). Slide the labeled $I$s and $J$s to make Figure 4(b). Then

$$(2(c^2/2) + 4|T|) + 2|X| = (2|2T| + 2(a^2/2) + 2(b^2/2)) + 2|Y|$$

since the area left over after removing the $O$s, labeled $I$s, and labeled $J$s stays the same. Cancel $4|T| = 2|2T|$ to get the law of cosines.

For $C = 90°$, this gives a proof by Floor van Lamoen ([**3**, proof 64] and [**14**]) that PT is implied by its $45°$-$45°$-$90°$-triangle case: Figure 5 gives $c^2/2 + 2|T| = |2T| + (a^2 + b^2)/2$.

## Proof 3: Hinged ring in a large square

Figures 6(a) and 6(b) have fourfold rotational symmetry. (The light gray circles in the background will be explained in the next paragraph. The boundaries of the pieces $P$ and $Q$ are formed not by the circles, but by the darker straight line segments only.) Turning each $P$ by $90°$ and sliding the $Q$s transforms Figure 6(a) into 6(b). The areas left over after removing the $P$s and $Q$s give the law of cosines: $c^2 + 4|X| = (a + b)^2$.

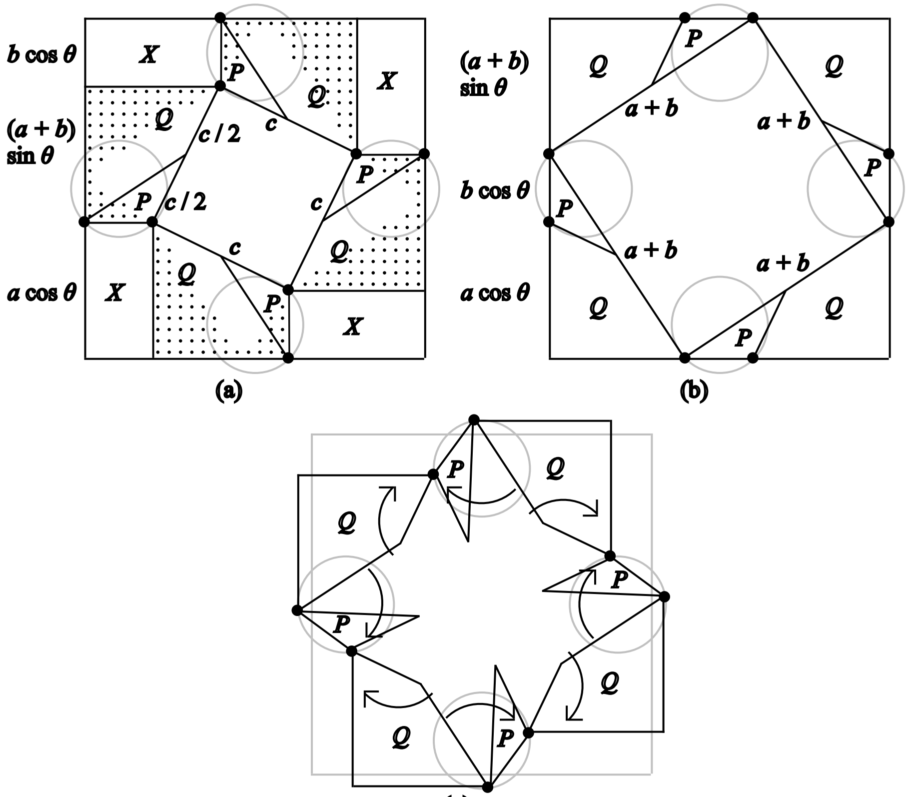



**Figure 6** Diagrams for proof 3.

One way to carry out the transformation between the two diagrams is shown in Figure 6(c). The ring of $P$s and $Q$s is hinged—that is, at each point marked by a large dot, a corner of one piece is permanently attached to the corner of another piece while the turning and sliding are taking place. The points marked by large dots are called hinges, and they move as the attached pieces $P$ and $Q$ move. Figure 6's light gray circles are such that in Figure 6(a), the hinges on each circle are the endpoints of a side of the square with horizontal and vertical sides inscribed in the circle. During the transformation, each circle is fixed in place, the hinges all move clockwise along the circles at the same constant speed (while staying attached to their pieces $P$ and $Q$), and each circle's two hinges are always $90°$ apart along the circle. Each $P$ turns $90°$ about the center of one of the circles, and the $Q$s slide along circular paths without turning.

For a survey of hinged dissections—that is, dissections where the pieces can be attached to one another at hinges while moving from their positions inside one shape to their positions inside another shape—see [**7**].

## Proof 4: Hinged ring in a small square

Without loss of generality, let $a \geq b$. Figure 7(a) is the part of Figure 6(a) that is contained inside a square (side $(a + b)\sin\theta + (a - b)\cos\theta$) with the same edge directions and center as the large square in Figure 6, and with edges passing through the $c^2$ square's corners. Figures 7(a) and 7(b) have fourfold rotational symmetry. Sliding

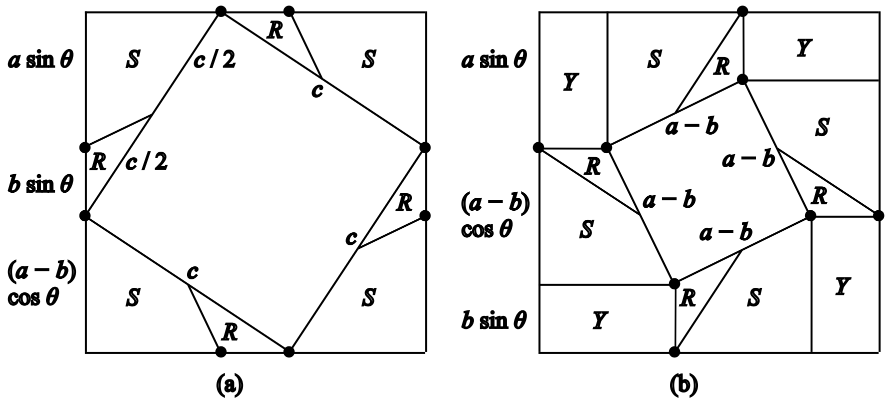



**Figure 7** Diagrams for proof 4.

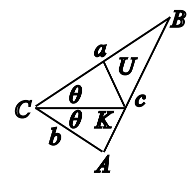



**Figure 8** Triangles $T$ and $U$.

and turning the hinged ring of $R$s and $S$s (in the same way as in proof 3) transforms Figure 7(a) into Figure 7(b). The areas left over after removing the $R$s and $S$s give the law of cosines: $c^2 = (a-b)^2 + 4|Y|$.

Recall the triangle $T$ with sides $a$, $b$, and $c$. In Figure 8, the triangle $U$ is the part of $T$ not in $T$'s reflection about the angle bisector $CK$. The angle bisector theorem says that $|AK|/|KB| = b/a$, so $U$ has sides $ac/(a+b)$, $bc/(a+b)$, and $a-b$.

If proof 3 is applied to triangle $U$ instead of triangle $T$, with $U$'s sides $ac/(a+b)$, $bc/(a+b)$, and $a-b$ playing the role of $T$'s sides $a$, $b$, and $c$ respectively, then after reflecting about a vertical axis, proof 4 is obtained. (Compare Figures 6(a) and 7(b) with each other, and compare Figures 6(b) and 7(a) with each other.)

For a proof of the angle bisector theorem using Figure 8's areas $|\triangle ACK|$ and $|\triangle BCK|$, see [**13**]. Appendix 2 uses a slightly modified version of Figure 8 to prove the triangle-area formula $|T| = ab\cos\theta\sin\theta$. That formula will be used in the next section.

## Proofs 5 and 6: Two algebraic proofs

In this section, assume PT and let $C \neq 90°$.

Without loss of generality, let $a \geq b$ if $C < 90°$, and let $a \leq b$ if $C > 90°$. This is equivalent to assuming that $(a-b)(\cos\theta - \sin\theta) \geq 0$, which is the version of the condition that will be used explicitly.

In Figure 9, which has fourfold rotational symmetry, the solid lines show the arguments' structure. The proofs work whether $\ell$ is to the right of the infinitely long straight line extending the line segment $\ell''$, or $\ell$ is directly above $\ell''$, or $\ell$ is between the straight-line extensions of $\ell'$ and $\ell''$, or $\ell$ is directly above $\ell'$, or $\ell$ is to the left of the straight-line extension of $\ell'$.

(Those five cases occur when $b/a < 1 - \tan\theta$, $b/a = 1 - \tan\theta$, $1 - \tan\theta <$

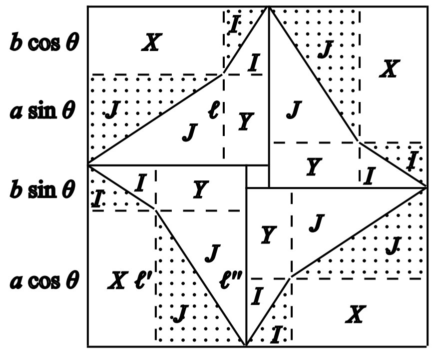



**Figure 9** Diagram for proofs 5 and 6.

$b/a < 1$, $b/a = 1$, and $b/a > 1$ respectively. Proof: The horizontal distance from the large square's left edge to $\ell$ (respectively, to $\ell'$ and to $\ell''$) is $a\cos\theta$ (respectively, $b\cos\theta$ and $b\cos\theta + a\sin\theta$). Now the following double implications hold. (a) $\ell$ is strictly to the right of the extension of $\ell'$ iff $b\cos\theta < a\cos\theta$ iff $b/a < 1$. (b) $\ell$ is strictly to the right of the extension of $\ell''$ iff $b\cos\theta + a\sin\theta < a\cos\theta$ iff $b/a < 1 - \tan\theta$. (c) In (a) and (b), replacing "strictly to the right of" by "on or strictly to the right of" changes each $<$ to $\leq$.)

In Figure 9, the inner boundary of the shaded $I/J$ ring encloses a butterfly (which is convex if and only if $C$ is not acute). The butterfly contains the small square (which has side $(a-b)(\cos\theta - \sin\theta) \geq 0$) and is contained in the large square (which has side $(a+b)(\cos\theta + \sin\theta)$). The next two law-of-cosines proofs use PT in the form $\cos^2\theta + \sin^2\theta = 1$.

Proof 5: Adding $4(|Y| + |I| + |J|)$ to the small square gives the butterfly, so

$$\begin{aligned} c^2 &= (a-b)^2(\cos\theta - \sin\theta)^2 + 4(|Y| + |I| + |J|) - 4|T| \\ &= (a-b)^2(\cos\theta - \sin\theta)^2 + (2a^2 + 2b^2 - 4ab)\cos\theta\sin\theta + 4|Y| \\ &= (a-b)^2((\cos\theta - \sin\theta)^2 + 2\cos\theta\sin\theta) + 4|Y| = (a-b)^2 + 4|Y|. \end{aligned}$$

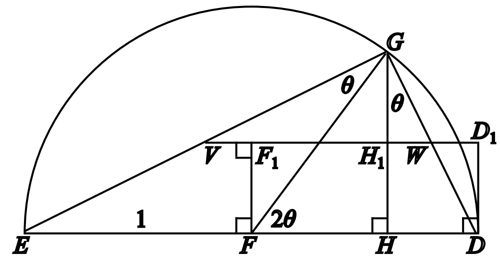
Proof 6: Adding $4(|X| + |I| + |J|)$ to the butterfly gives the large square, so

$$\begin{aligned} c^2 &= (a+b)^2(\cos\theta + \sin\theta)^2 - 4(|X| + |I| + |J|) - 4|T| \\ &= (a+b)^2(\cos\theta + \sin\theta)^2 - (2a^2 + 2b^2 + 4ab)\cos\theta\sin\theta - 4|X| \\ &= (a+b)^2((\cos\theta + \sin\theta)^2 - 2\cos\theta\sin\theta) - 4|X| = (a+b)^2 - 4|X|. \end{aligned}$$

## Appendix 1: Double-angle formulas

This Appendix will prove each double-angle formula for $\cos 2\theta$ separately, in the case of acute $\theta$, without assuming PT. (See also [**10**], from which the $1 - 2\sin^2\theta$ proof is taken and from which the other two proofs are inspired. In [**11**], a $2\cos^2\theta - 1$ proof is given similar to the one below.) Recall that proofs 1 to 6 each use only one double-angle formula, which does not involve assuming PT.

Figure 10 is constructed as follows. Take a semicircle with radius 1, diameter $ED$ and center $F$. Let $G$ be on the semicircle's half-circumference with $\angle DFG = 2\theta$. Let $V$ and $W$ be the midpoints of $EG$ and $GD$ respectively. Take $H$ on $ED$ with $GH \perp ED$. Take the points $F_1$, $H_1$, and $D_1$ on $\overleftrightarrow{VW}$ with $FF_1$, $HH_1$, and $DD_1$ all perpendicular to $\overleftrightarrow{VW}$. Then $\angle EGD = 90^\circ$ and $\angle FEG = \angle FGE = \angle DGH = \theta$.



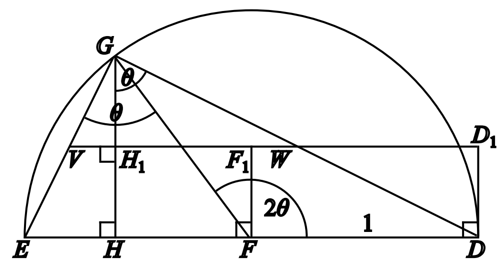




**Figure 10** Diagrams for proofs of double-angle formulas.

The quantity $|EH| - |EF| = |FD| - |HD|$ may be taken as a definition of $\cos 2\theta$ for $0^\circ < 2\theta < 180^\circ$. This definition agrees with the "ratios of sides of a right triangle" definition for $2\theta$ acute.

The equations

$$1 - \cos 2\theta = |HD| = |GD| \sin\theta = 2\sin^2\theta$$

yield $\cos 2\theta = 1 - 2\sin^2\theta$.

The equations

$$1 + \cos 2\theta = |EH| = |EG| \cos\theta = 2\cos^2\theta$$

yield $\cos 2\theta = 2\cos^2\theta - 1$.

Finally, $\triangle VF_1F \cong \triangle WD_1D \cong \triangle WH_1G$ implies $|VF_1| = |H_1W|$, so

$$\begin{aligned} \cos 2\theta &= |VH_1| - |VF_1| = |VH_1| - |H_1W| \\ &= |VG|\cos\theta - |GW|\sin\theta = \cos^2\theta - \sin^2\theta. \end{aligned}$$

## Appendix 2: Area of a triangle

This Appendix uses Figure 11 to prove $|T| = ab\cos\theta\sin\theta$.

Without loss of generality assume $a \geq b$. Drop perpendiculars $AP_A$ and $BP_B$ from $A$ and $B$ respectively to $\overleftrightarrow{CK}$. One has

$$\begin{aligned} a|CK| &= a(|CP_A| + |P_AK|) = ab\cos\theta + a|P_AK| \\ b|CK| &= b(|CP_B| - |KP_B|) = ba\cos\theta - b|KP_B| \end{aligned}$$

and adding these together yields

$$(a+b)|CK| = 2ab\cos\theta + (a|P_AK| - b|KP_B|).$$

One now shows $a|P_AK| = b|KP_B|$. Both sides of that equation are $0$ if $a = b$. If $a > b$, then $\triangle AP_AK$ is similar to $\triangle BP_BK$, so $|P_AK|/|KP_B| = |AK|/|KB| = b/a$ by the angle bisector theorem, so $a|P_AK| = b|KP_B|$.

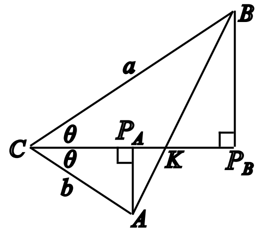



**Figure 11** Diagram for proof of triangle-area formula.

So $(a+b)|CK| = 2ab\cos\theta$. (Another way of phrasing the previous calculation is to say that $|CK|$ is the weighted sum of $|CP_A|$ and $|CP_B|$ with respective weights $a/(a+b)$ and $b/(a+b)$ coming from the angle bisector theorem.)

So

$$|T| = |\triangle BCK| + |\triangle ACK| = (|CK|\cdot|BP_B| + |CK|\cdot|AP_A|)/2$$
$$= |CK|(a\sin\theta + b\sin\theta)/2 = ab\cos\theta\sin\theta.$$

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## REFERENCES

1. Acheson, D. (2002). *1089 and All That: A Journey into Mathematics*. Oxford, UK: Oxford University Press.
2. Bibby, N., French, D. (1988). Pythagoras Extended. A Geometric Approach to the Cosine Rule. *Math. Gaz.* 72(461): 184–188. `http://www.doi.org/10.2307/3618247`
3. Bogomolny, A. (2016). Pythagorean Theorem. `http://www.cut-the-knot.com/pythagoras`
4. Edwards, M. D. (2014). A Possibly New Proof of the Law of Cosines. *Am. Math. Mon*. 121(2): 149. `http://www.doi.org/10.4169/amer.math.monthly.121.02.149`
5. Euclid (translated by Heath, T. L., edited by Densmore, D.) (2017). *Euclid's Elements: All Thirteen Books Complete in One Volume*. Santa Fe, NM: Green Lion Press.
6. Frederickson, G. N. (1997). *Dissections: Plane & Fancy*. Cambridge, UK: Cambridge University Press.
7. Frederickson, G. N. (2002). *Hinged Dissections: Swinging & Twisting*. Cambridge, UK: Cambridge University Press.
8. Howe, R. E. (2013). The Cuoco Configuration. *Am. Math. Mon*. 120(10): 916–923. `http://www.doi.org/10.4169/amer.math.monthly.120.10.916`
9. Loomis, E. S. (1968). *The Pythagorean Proposition: Its Demonstrations Analyzed and Classified and Bibliography of Sources for Data of the Four Kinds of "Proofs"*. Re-issue of 1940 2nd ed. Classics in Mathematics Education. Washington, DC: The National Council of Teachers of Mathematics, Inc. Educational Resources Information Center, `https://files.eric.ed.gov/fulltext/ED037335.pdf`
10. Luzia, N. (posted on site of Bogomolny, A.) (2018). Pythagorean Theorem Via Half-Angle Formulas. `https://www.cut-the-knot.org/pythagoras/Proof109.shtml`
11. Nelsen, R. B. (1989). The Double-Angle Formulas. *Coll. Math. J*. 20(1): 51. `http://www.doi.org/10.2307/2686819`
12. Thompson, S. L. (1951). Note on the Law of Cosines. *Am. Math. Mon*. 58(10): 698–699. `http://www.doi.org/10.2307/2307984`
13. Totten, J. (2000). Megan's proof. *Math. Teach*. 93(6): 525.
14. Van Lamoen, F. (n.d.). A different proof of the Pythagorean theorem. `http://home.planet.nl/~lamoen/wiskunde/pythagorasen.htm`
15. Wikipedia contributors. (2018). Law of cosines. In: *Wikipedia: The Free Encyclopedia*. `https://en.wikipedia.org/w/index.php?title=Law_of_cosines&oldid=862599687`

**Summary.** Inspired by arguments showing the Pythagorean theorem, this paper gives six inter-related dissection-like proofs of the law of cosines. Each proof treats the acute- and obtuse-angle cases together, and uses a trigonometric double-angle formula. Some of the proofs assume the Pythagorean theorem.

**DR. HENRY (MAYA) ROBERT THACKERAY** (ORCID number 0000-0003-4467-9070) completed their B.Sc. and B.Sc. Hons. degrees in mathematics at the University of Pretoria, and their mathematics M.A. and Ph.D. degrees at Princeton University. Thackeray is currently a postdoctoral fellow at the University of Pretoria.