The reciprocal relationship between conceptual and procedural knowledge – a case study of two calculus problems

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Abstract
The literature describes different stances concerning the focus on how mathematics should be taught, with some preferring a conceptual knowledge approach and others a procedural knowledge approach. The current study investigated the relationship between students' conceptual and procedural knowledge in a calculus context. To better understand the relationship between students' conceptual and procedural knowledge, we conducted a content analysis of possible student responses to two mathematical problems and then analysed student work. Students (*n* =192) were enrolled for a first-year mathematics module which forms part of an extended engineering degree in South Africa. The solutions to the two problems were analysed based on the number and nature (conceptual or procedural) of experts and students' steps to solve each problem. Each step in the solution was categorised based on the approaches used to solve the problem. The study found that solutions are not unique and could follow more than one approach. More importantly, the study found that the relationship between conceptual and procedural knowledge is complex and integrated as solutions require both procedural and conceptual knowledge. The findings reveal that calculus problems cannot be uniquely described as mainly conceptual or procedural. Both procedural and conceptual thinking is required to solve calculus problems and is often iterative. Student techniques to solve the calculus problems included algebraic, graphical and unexpected approaches. The analyses of student solutions suggest that lecturers and teachers should compare and discuss multiple solution strategies with their students to enhance mathematical proficiency and understanding.

Keywords:
Calculus teaching and learning;
Case study;
Conceptual and procedural knowledge;
Multiple solution strategies
Reciprocal relationship;

Introduction

The literature describes different stances concerning how mathematics should be taught and raises questions about whether the focus should be on concepts and applications or on developing skills in carrying out procedures first. The debates about differences between concept-driven versus skills-orientated approaches have led to tension and the so-called '*math wars*' between mathematics education researchers globally (Brown et al., 2002; Sowder, 2007; Star, 2005; Wu, 1999), as well as in South Africa (Engelbrecht et al., 2009; Engelbrecht et al., 2005). These tensions have been described by Star (2005):

> Whether developing skills with symbols leads to conceptual understanding or whether basic understanding should precede symbolic representation and skill practice is one of the fundamental disagreements between the opposing sides of the so-called math wars (Star, 2005, p.404).

The mathematics education researchers advocating concept-driven reform emphasise understanding mathematics and using mathematics to solve real-world problems (Sowder, 2007). This perspective focuses on concepts, relations between concepts and contextual applications and favours reasoning, critical thinking and problem-solving skills, including using different methods to solve a problem. Brown et al. (2002) argue that teaching for conceptual understanding leads to developing procedures and skills, but the knowledge of procedures does not necessarily promote conceptual understanding.

In contrast, skills-oriented mathematics education researchers prefer direct instruction that promotes procedures and skills, focusing on numerical and symbolic calculations and manipulations. The assumption is that as students develop skills in carrying out procedures, their conceptual understanding will implicitly be enhanced (Wu, 1999). The group advocating a procedural focus argues that the concept-driven mathematics approach is "fuzzy". It lacks rigour, and mathematical skills are best learned through the drill and practice of procedures (Sowder, 2007).

The two groups hold different beliefs about mathematics and how it should be learned and taught (Sowder, 2007). In procedural knowledge, teaching focuses on definitions, symbols and isolated skills, whereas teaching for conceptual understanding begins with posing problems that require students to reason flexibly (Brown et al., 2002). The two paradigms are in opposition to each other, and teaching approaches are influenced by teachers' prior experiences and their paradigms.

This paper examines the mathematical knowledge and different problem-solving approaches used during the problem-solving process in the context of two problems from a calculus course. The following research question is explored: Can we categorise solutions to the two mathematical problems using a mainly conceptual or procedural approach?

The study is located within a first-year mathematics module presented to engineering students enrolled in an extended curriculum programme in South Africa. The authors investigate two problems and their solutions presented by three mathematics education specialists and examples of student solutions.

Conceptual and procedural knowledge and problem-solving approaches

Secondary and tertiary mathematics courses in many North American universities are often taught with a heavy procedural focus, resulting in a lack of development of conceptual knowledge and limiting a student's ability to apply procedures to unfamiliar problems and problem-solving (Maciejewski & Star, 2016). Researchers developed an interest in moving away from rote learning and procedures toward initiatives to improve first-year students' calculus learning, focusing on concepts. Understanding the concept(s) that underpin a procedure will make the procedure more robust and applicable in other contexts.

Conceptual knowledge is described as relations between concepts and operations within a specific domain (Kilpatrick et al., 2001). Conceptual knowledge is described as abstract, general, explicit, or implicit understanding of principles (Rittle-Johnson, 2017). Procedural knowledge is the ability to perform step-by-step procedures to solve problems (Star et al., 2016). Both types of knowledge develop procedural flexibility, described as knowledge of multiple methods and choosing the most appropriate method based on specific problem properties (Kilpatrick et al., 2001; Rittle-Johnson, 2017; Star, 2005). Mathematics competence or proficiency requires both types of mathematical knowledge and procedural flexibility (Rittle-Johnson, 2017, Kilpatrick et al., 2001).

While procedural knowledge is described as routine procedures focusing on the accuracy of procedures and answers, rules and techniques (Rittle-Johnson & Schneider, 2016), conceptual knowledge requires the integration of related concepts. Engelbrecht et al. (2009) noted that some students 'proceduralised' mathematical solutions were assumed to be conceptual since the students used procedural methods. Therefore, the description of being 'conceptual' or 'procedural' is not necessarily a property of the problem but rather depends on the specific solution (Engelbrecht et al., 2009).

A collaboration project between South Africa and Sweden investigated whether the teaching emphasis in undergraduate mathematics courses for engineering students would benefit from being conceptual rather than the traditional, more procedural approach (Engelbrecht et al., 2009; Engelbrecht et al., 2012). According to the study, the high school mathematics focus in these two countries is primarily routine and procedural. As a result, these first-year mathematics students have often experienced mathematics as a subject focusing on procedures and manipulation, and few students have been exposed to deeper conceptual thinking. Therefore university lecturers may claim that these students have little comprehension of basic pre-calculus concepts and that stronger students are only better at procedural thinking (Engelbrecht et al., 2005; Engelbrecht et al., 2012). The collaborative study between South Africa and Sweden assumed that the problem-solving approach used to solve a mathematical problem could be mainly conceptual or procedural (Bergsten et al., 2015; Bergsten et al., 2017; Engelbrecht et al., 2012). The terms conceptual and procedural approaches are distinguished below:

Conceptual approach: This includes translations between verbal, visual (graphical), numerical, and formal/algebraic mathematical expressions (representations), linking relationships, and interpretations and applications of concepts to mathematical situations.

Procedural approach: This includes symbolic and numerical calculations, employing (given) rules, algorithms, formulae, and symbols.

Problem-solving that takes on procedural approaches focuses on the accuracy of calculations, routine procedures and notation. In contrast, that based on conceptual approaches focuses on translations between different representations, linking relationships, interpretations and applications. Some studies (Chappell & Killpatrick, 2003) have looked at whether the type of approach influences achievement. The authors investigated student achievement in calculus for two groups of students where instruction followed either a procedural or a conceptual teaching approach. The study favoured the concept-based environment since students subjected to the conceptual approach scored significantly higher than students who followed the procedural approach on both procedural and conceptual assessments.

Although some believe that the two approaches are separate and that students take either a procedural or a conceptual approach, there are links between the two approaches. Procedures could connect concepts through reasoning and representations, e.g. graphs (Davis, 2005). Students should be exposed to discussions about different strategies. Evaluating, comparing, and explaining numerous strategies for solving the same problem promotes student learning (Star et al., 2016). Comparing different solution methods for solving the same problem supports procedural flexibility across students and develops conceptual and procedural knowledge among students with prior knowledge of one of the methods (Durkin et al., 2017).

Mathematical solution approaches could be described as bidirectional causal relations since solution methods show that procedural and conceptual steps alternate (Rittle-Johnson et al., 2015) but in no specific order (Rittle-Johnson et al., 2016). The analysis of solution methods in a study showed that some steps repeat, indicating iterative relations between concepts and procedures (Rittle-Johnson, 2017). In the current study, we analyse the approaches used by experts and students to solve two problems to understand how the procedural and conceptual approaches are used and how they relate to each other. The first problem on functions is based on solving an inequality involving the absolute value function and quadratic expressions. The second task is a calculus problem where students must calculate the local extrema.

The literature confirms that students have misconceptions about absolute functions and inequalities. A recent Wewe (2020) study confirms that students have difficulties mastering the concept of absolute value inequalities. Student errors include omitting the absolute value sign and changing the absolute value bars to parentheses, in other words changing the absolute value function into a different function (Almog & Ilany, 2012; Aziz et al., 2019), as well as incorrect use of the logical connectors *and* ($\cap$, intersection) and *or* ($\cup$, union) and not reaching the final solution (Almog & Ilany, 2012).

White and Mesa (2014) categorise finding extreme values as routine optimisation tasks, in particular, 'recognise and apply procedure tasks' since students could approach this task procedurally by finding the derivative or setting it equal to zero. A study by Mkhatshwa (2019) found that the zeros of the derivative function were found by: factorising the derivative and equating each factor to zero, the quadratic formula, graphing the derivative and calculating the *x*-intercepts. On the other hand, if the student understands the derivative's graphical meaning and the critical points' conceptual importance, problem-solving can be done with reasoning instead of by memorising a procedure (White & Mesa, 2014).

Methodology

The analysis is part of a more extensive, mixed-method study investigating the relationship between procedural and conceptual knowledge in calculus (Hechter, 2020). The study examined the types of knowledge required to solve 33 calculus problems. The qualitative approach involves a content analysis of the solutions to the tasks, followed by a quantitative Rasch analysis. A test with 33 items was developed and administered to 192 first-year engineering students enrolled in a mathematics module, where 76% were male. Subject matter experts were asked to comment on the items, confirming that the instrument and items were well-developed. Quantitative evidence for the reliability and validity of the whole instrument was reported in a separate study, where the instrument can also be obtained (Hechter, 2020).

This paper presents a case study of two selected problems from the more extensive study and written student responses for the tasks. These items were selected since student work presented alternative, interesting methods and rich data related to conceptual and procedural problem-solving approaches. The two problems are:

1. Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation.
2. If a stone is thrown vertically upwards, the position function of the stone is given by $s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds. Calculate:
   a) the time t when the stone will reach its maximum height
   b) the maximum height of the stone (before it falls to the ground).

Data Analysis

The problems were given to three subject specialists who provided their solution approaches to the problems. These solutions were subjected to a content analysis using categories that expand the conceptual and procedural approaches described by Engelbrecht et al. (2012) and Bergsten et al. (2015, 2017) for the topics of functions and differentiation. Table 1

shows the codes for the associated categories we used to characterise the solutions to the problems chosen for this paper. We have distinguished conceptual and procedural steps using the letters C and P, respectively.

Table 1

Conceptual and procedural problem-solving categories

Code	Conceptual and procedural problem-solving categories
C_1	translations between verbal, visual, numerical, and formal/algebraic mathematical expressions
C_{2F}	linking relationships wrt functions: functions $\Leftrightarrow$ inverse functions, equation of a function
C_{2D}	linking relationships wrt differentiation: $f \Leftrightarrow f' \Leftrightarrow f'', D_{f'} \subseteq D_f$, $f'(x) = 0 \Rightarrow$ f local extrema, $f'(x) > 0 \Rightarrow$ f increasing, $f'(x) < 0 \Rightarrow$ f decreasing, $f''(x) = 0 \Rightarrow$ a possible point of inflexion, $f''(x) > 0 \Rightarrow$ f concave up, $f''(x) < 0 \Rightarrow$ f concave down, link position function (displacement) $\Rightarrow$ velocity (speed) $\Rightarrow$ acceleration
C_{3F}	interpretation of concepts wrt functions: definitions, functions and relations, inverses, domain and range, restrictions, inequalities (quadratic and higher-order), incl. intersection and union, the turning point of a parabola (extreme), an axis of symmetry, *x*-intercepts
C_{3D}	interpretation of concepts wrt differentiation: gradient, continuity, differentiability, point of inflexion, concavity
C_4	applications of concepts to mathematical situations
P_1	symbolic and numerical calculations, substitution
P_{2F}	rules wrt functions, expressions (e.g. division by zero), exponential laws (e.g. $a^0 = 1$), log laws, equations (e.g. $a.b = 0 \Rightarrow a = 0$ or $b = 0$), inequalities (e.g. division by -1), graph of the parabola, factorisation, $\sqrt{a^2} = \lvert a \rvert$
P_{2D}	differentiation rules
P_3	algorithms (set of rules), e.g. long division or completing the square
P_4	formulae, e.g. quadratic formula and turning point formula
P_5	symbols (including notation)

Source: Extracted from Hechter (2020)

We analysed the solutions according to the number of conceptual and procedural steps used to solve the problem. The numbers of conceptual and procedural problem-solving categories were coded and counted, resulting in a label for each step of the solution approach. Coding and counting categories need further explanation. A problem-solving category is only counted *once* when the *exact* procedure/concept is repeated for a particular approach in a problem solution. The repeated step is shaded in grey, as shown below:

$$f(x) = \begin{cases} x^2 \\ 5x \end{cases} \Rightarrow f'(x) = \begin{cases} 2x \\ 5 \end{cases} \qquad \begin{cases} P_{2D}\text{: differentiation rules} \\ P_{2D}\text{: differentiation rules} \end{cases}$$

Result: C = 0 P = 1 (conceptual steps 0, procedural steps 1)

A problem-solving category should be counted more than once when the *same* category requires *different thinking skills* for a procedure/concept in a particular step in the solution, e.g.

$(x-3)(x+4) < 0$ $C_{3F}[1]$: interpretation *quadratic* inequality

$|x+1| \leq 0$ $C_{3F}[2]$: interpretation *absolute value* inequality

Result: C = 2 P = 0 (conceptual steps 2, procedural steps 0)

The *quadratic* and *absolute value* functions are different functions that require similar but also alternative reasoning skills to solve the respective inequalities. The square brackets indicate the different conceptual skills. If a problem solution follows more conceptual than procedural steps, it could be classified as a conceptual problem; similarly, a problem that uses more procedural than conceptual steps is procedural. Analyses of 192 written student solutions followed the expert analyses. The number of students who completed Problem 1 was 178, and 182 did Problem 2.

Ethical Considerations

We obtained clearance from the engineering faculty's ethics committee. Students signed a consent form and voluntarily agreed to participate in the study and that their results could be used for research purposes. The test formed part of the students' continuous assessment marks and was a natural part of their learning and assessment.

Results

The expected solution responses to the problems on functions and application of differentiation analyses are shown first. We then share students' written responses to illustrate alternative strategies used by the students. Note that solution approaches cannot be uniquely described since the responses indicate that students used various solution methods.

Results of the analysis of responses of subject experts and students to Problem 1

Problem 1 appears in Figure 1.

Figure 1

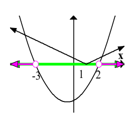
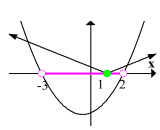
Instructions for Problem 1

Solve $\frac{|x-1|}{x^2+x-6} \geq 0$. Write your final answer in interval notation.

In Table 2, we show mathematical experts' content analysis of Problem 1. We include graphical demonstrations to explain the solution steps. The analysis revealed that the algebraic approach had nine non-repeated steps (C=5 P=4). Repeated steps, shaded in grey, were not counted twice. Analyses of student work indicated that only three of the 178 students who attempted the problem scored the full two marks (1.7%). Students struggled with this item; the mean was 10.7% (weighted average). Some 17% of the students could gain one mark out of two, but most failed to solve the problem, and 7% did not even try to answer even though the assessment counted as class marks.

Table 2

Analysis of steps used in Approach 1 to Problem 1

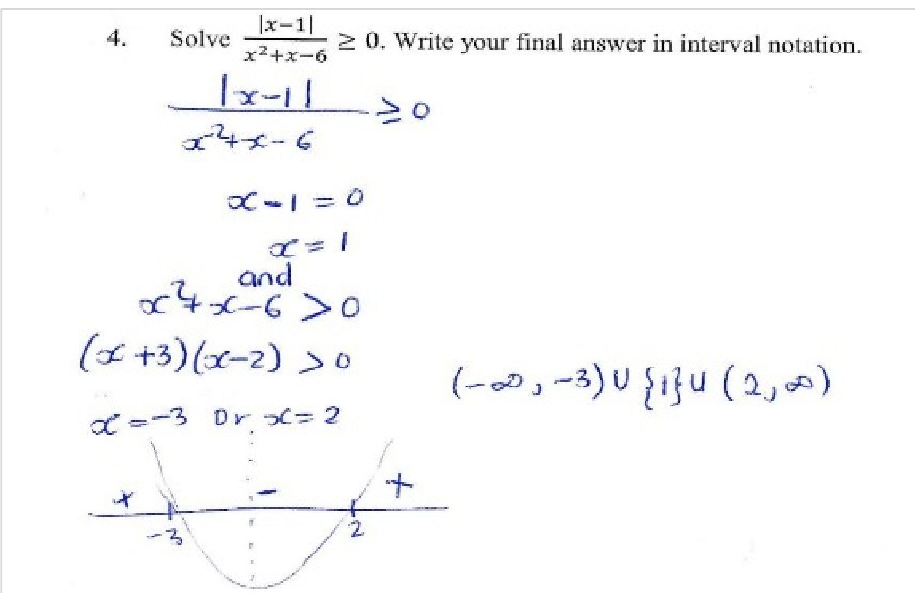
Approach 1: Algebraic method		Result: C=5 P=4
$\frac{|x-1|}{x^2+x-6} \geq 0$ 	**P_{2F}[1]**	P_{2F}[1]: inequality rules $(\frac{+}{+}$ or $\frac{-}{-})$
	P_{2F}[2]	P_{2F}[2]: factorisation
$(x-2)(x+3) > 0$ **and** $|x-1| \geq 0$	**C_1**	C_1: translations (graph)
Graph of parabola	**C_{3F} [1]**	C_{3F} [1]: interpretation quadratic inequality
$x \in (-\infty,-3) \cup (2,\infty)$ and $x \in R$	**C_{3F} [2]**	C_{3F} [2]: interpretation absolute value ineq
$\Rightarrow x \in (-\infty,-3) \cup (2,\infty)$ **[and]**	**C_{3F} [3]**	C_{3F} [3]: interpretation inequality (and)
OR ($\cup$)	**P_{2F}[1]**	P_{2F}[1]: inequality rules $(\frac{+}{+}$ or $\frac{-}{-})$
$\frac{|x-1|}{x^2+x-6} \geq 0$ 	**C_1**	C_1: translations (graph)

$(x-2)(x+3)<0$ **and** $\lvert x-1\rvert \le 0$	**C_{3F} [1]**	C_{3F} [1]: interpretation quadratic inequality
Graph of parabola	**C_{3F} [2]**	C_{3F} [2]: interpretation absolute value inequality
$x\in(-3,2)$ and $x-1=0\Rightarrow x=1$	**P_1**	P_1: numerical calculations
$\Rightarrow x=1$ **[and]**	**C_{3F} [3]**	C_{3F} [3]: interpretation inequality (and)
	C_{3F} [4]	C_{3F} [4]: interpretation inequality (or)
Final answer: $x\in(-\infty,-3)\cup\{1\}\cup(2,\infty)$ **[OR]**	**P_5**	P_5: notation

The three successful students did not use the first approach but instead a graphical method, although the algebraic method was taught as a preferable method. Interestingly, the three students who achieved full marks on this item performed more poorly on the test overall than those with higher personal proficiency (Hechter, 2020). Figure 2 presents an example of a student response who used the graphical approach.

Figure 2

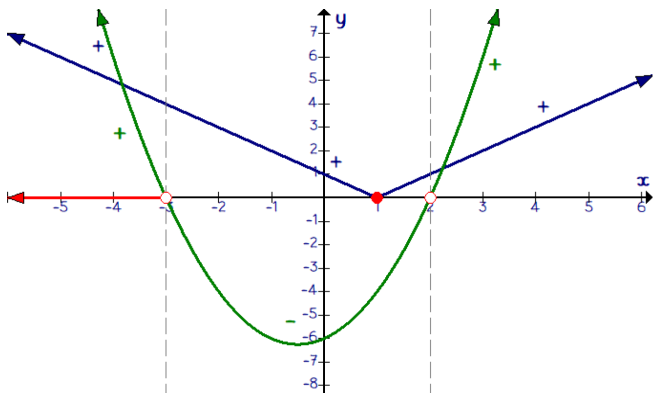
Student response to Problem 1 showing Approach 2 (graphical method)



As can be seen from the student who used the graphical approach, fewer steps were required for the approach, and the student relied on both procedural and conceptual knowledge.

Table 3

Analysis of steps used in Approach 2 (graphical method) to Problem 1 (graphical method)

Approach 2: Graphical method	Result: C=5 P=1
	C_1[1]: translations (graph of absolute value) C_1[2]: translations (graph of parabola) C_{3F}[1]: interpretation x-intercepts parabola C_{3F}[2]: interpretation x-intercepts absolute value C_{3F}[3]: interpretation inequality (graphical) therefore $\geq \;\Rightarrow\; ++$ or $--$ P_5: notation

The analysis of the graphical method presents an alternative, shorter solution method with fewer procedural steps and fewer technical obstacles. The steps analysed in Table 3 are not based on subject matter experts but student solutions. Table 4 summarises the 178 students' approaches to solving Problem 1. Most students used algebraic approaches (91.6%). Only 15 (8.4%) of the students used graphs to attempt the problem.

Table 4

Content analysis of student strategies to solve Problem 1 and common errors

Method	*n* (178)	%	Common errors per approach	Example
Algebra	163	91.6%	Conceptual error (34% of students) Absolute value changed to straight-line function	$\lvert x-1\rvert \geq 0$ $\Rightarrow x-1 \geq 0$ $\Rightarrow x \geq 1$
Graphs	12	6.7%	Procedural error (6% of students) Quadratic inequality solved incorrectly	$x^2+x-6>0$ $\Rightarrow (x-2)(x+3)>0$ $\text{Error}: x>-3 \text{ and } x>3$
	3	1.7%		

For more than a third of students, the most common error was a conceptual error, where the absolute value function was changed to become a straight-line function (Almog & Ilany, 2012; Aziz, 2019). The procedural error was less common than the conceptual error. The content analysis for both approaches confirms that Problem 1 could be classified as a conceptual problem focusing primarily on interpreting function concepts, including *x*-intercepts and inequalities.

Results of the analysis of responses of subject experts and students to Problem 2

Problem 2 appears in Figure 3.

Figure 3

Instructions for Problem 2

If a stone is thrown vertically upwards, the position function of the stone is given by $s(t) = 30t - 5t^2 + 20$, where s is in metres and t is in seconds.

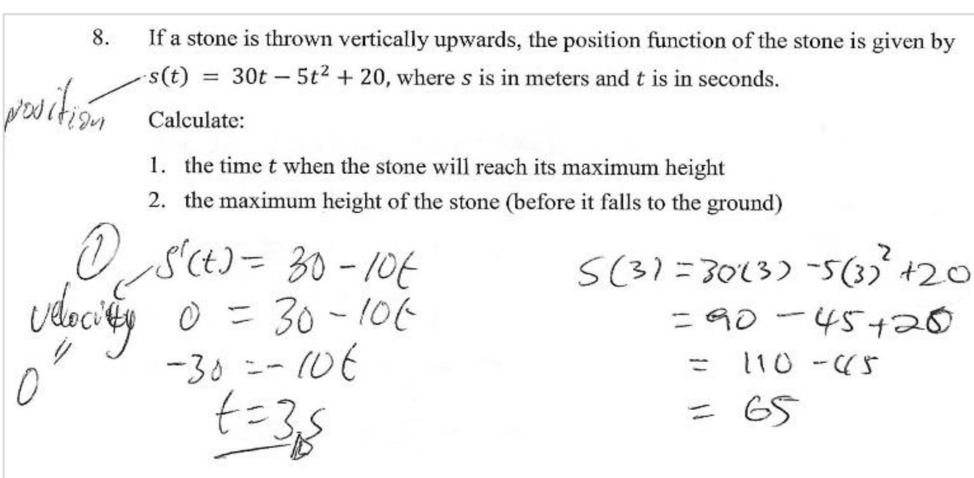
Calculate:

1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground)

The expectation was that students would use their knowledge of differentiation, mainly position function and velocity concepts, to solve the problem. This approach is referred to as Approach 1 and is seen in the student work presented in Figure 4, where calculations refer to the position function and velocity.

Figure 4

Student response to Problem 2 showing Approach 1



The analysis of the student's work is demonstrated with Approach 1 in Table 5.

Table 5

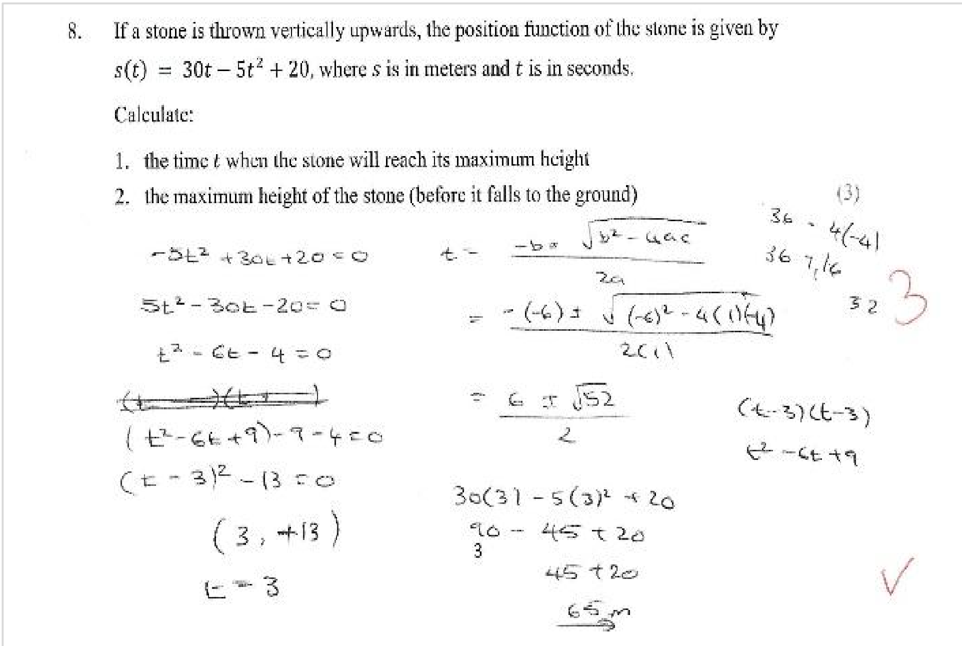
Analysis of steps used in Approach 1 to Problem 2

Approach 1		Result: C=5 P=3
$s(t) = 30t - 5t^2 + 20$ (position function)	$C_4[1]$	$C_4[1]$: context - position function stone
$s'(t) = 30 - 10t$ (velocity function)	P_{2D}	P_{2D}: differentiation rules
$s'(t) = 0$ (velocity function = 0)	$C_4[2]$	$C_4[2]$: context - velocity function
		$\text{time}_{\text{velocity zero}} \Rightarrow \text{time}_{\text{max-height}}$
$\Rightarrow 30 - 10t = 0$	$C_{2D}[1]$	$C_{2D}[1]$: link $f'(x) = 0 \Rightarrow f$ local extrema
$\Rightarrow -10t = -30$	$P_1[1]$	$P_1[1]$: numerical calculations
$\Rightarrow t = 3s$		
$s(3) = 30t - 5t^2 + 20$	$C_4[3]$	$C_4[3]$: context - position function
(position function at t = 3)	$C_{2D}[2]$	max height $\Rightarrow \text{time}_{\text{velocity zero}}$
		$C_{2D}[2]$: link position function (max height) and velocity (zero) at t = 3
$\Rightarrow s(3) = 90 - 45 + 20$	$P_1[2]$	$P_1[2]$: substitution into position function
$\Rightarrow s(3) = 65\,m$ (max height)	$P_1[1]$	$P_1[1]$: numerical calculations
	$C_4[1]$	$C_4[1]$: context - position function stone

Students also used other methods such as the turning point formula and the parabola's axis of symmetry to calculate the time when the maximum height would be reached (Figure 5 and Figure 6).

Figure 5

Student response to Problem 2 showing the use of the turning point formula (Approach 2)



8. If a stone is thrown vertically upwards, the position function of the stone is given by

$s(t) = 30t - 5t^2 + 20$, where s is in meters and t is in seconds.

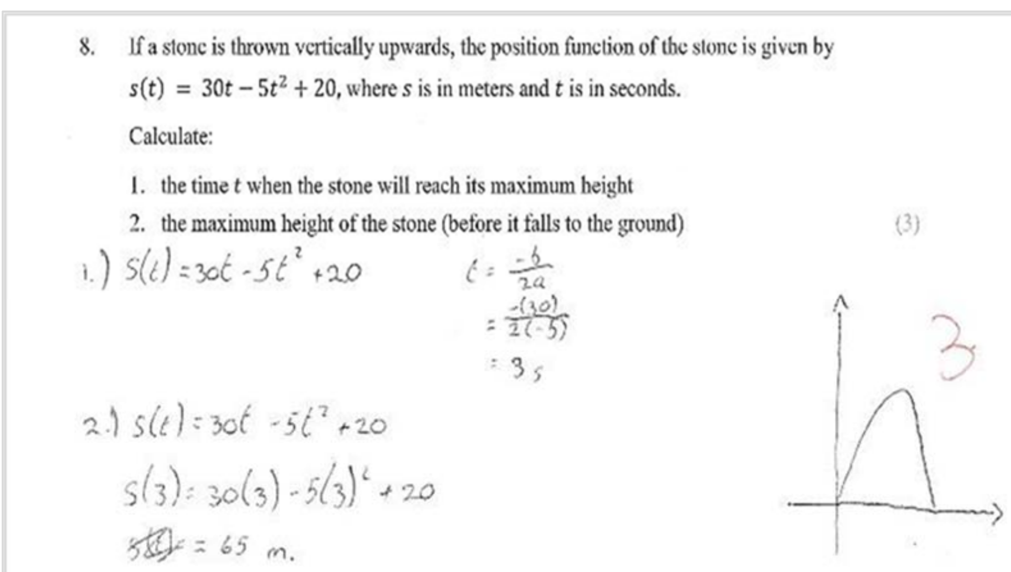
Calculate:

1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground) (3)

Here the student relied on completing the square and the turning point method to reach the correct answer. Another alternative is shown in Figure 6, where a student relied on the parabola's axis of symmetry to solve the problem.

Figure 6

Student response to Problem 2 showing the use of axis of symmetry of the parabola (Approach 3)



8. If a stone is thrown vertically upwards, the position function of the stone is given by

$s(t) = 30t - 5t^2 + 20$, where s is in meters and t is in seconds.

Calculate:

1. the time t when the stone will reach its maximum height
2. the maximum height of the stone (before it falls to the ground) (3)

The 'completing of the square and the turning point formula approach' used to calculate (time; maximum height) is analysed and presented as Approach 2 in Table 6.

Table 6

Analysis of steps used in Approach 2 to Problem 2

Approach		Result: C=2 P=5
$s(t) = 30t - 5t^2 + 20$	$P_1[1]$	$P_1[1]$: numerical calculations (division by -5)
$\Rightarrow t^2 - 6t - 4$	P_3	P_3: completing the square
$\Rightarrow (t - 6t + 9) - 4 - 9$	P_2	P_{2F}: factorisation
$\Rightarrow (t-3)^2 - 13$	P_4	P_4: Turning point formula
TP: (3,-13)		C_{3F}: interpretation of turning point ($p;q$) of a parabola: $p \Rightarrow$ time $_{\text{max-height}}$
$\Rightarrow$ maximum height at $t = 3s$	C_{3F} C_4	C_4: contextual applications: time, max-height
$\Rightarrow s(3) = 65m$ (max height)	$P_1[2]$	$P_1[2]$: substitution into position function
	$P_1[1]$	$P_1[1]$: numerical calculations

Table 7 displays Approach 3, which uses the axis of symmetry of a parabola to calculate (time; maximum height).

Table 7

Analysis of steps used in Approach 3 to Problem 2

Approach		Result: C=2 P=3
$s(t) = 30t - 5t^2 + 20$	P_4	P_4: formula for the axis of symmetry
$\Rightarrow x = \frac{-b}{2a}$	$P_1[1]$	$P_1[1]$: substitution into formula
$\Rightarrow x = \frac{-30}{2(-5)}$	$P_1[2]$	$P_1[2]$: numerical calculations
$\Rightarrow x = 3$	C_{3F}	C_{3F}: interpretation of axis of symmetry of a parabola (extreme value)
$\Rightarrow$ maximum height at $t = 3s$	C_4	C_4: context - time, max height
$\Rightarrow s(3) = 65m$ (max height)	$P_1[1]$	$P_1[1]$: substitution into position function
	$P_1[2]$	$P_1[2]$: numerical calculations

Table 8 summarises the 182 students' approaches to solving Problem 2.

Table 8

Content analysis of student strategies to solve Problem 2 and common errors

Method	*n* (182)	%	Common error per approach
Calculus	172	94.5%	Conceptual error (6%) $s(t) = 30t - 5t^2 + 20 = 0$ then factorise with quadratic formula, etc
Turning point formula	3	1.7%	
Axis of symmetry	7	3.8%	

The students performed well in this item, and the item mean was 81.3% (weighted average). The reason for the good results may be because many students (procedurally) know that the expression $s'(t)$ is associated with the local extreme value(s), in this case, the time when the stone would reach maximum height. Most students used calculus procedures knowledge to solve Problem 2 (94.5%), but some used secondary school knowledge to solve the problem. These students used the turning point formula (1.7%) and a parabola's axis of symmetry (3.8%) to calculate the time it took for the stone to reach its maximum height. The most common error in the student responses was a conceptual error where students assumed the maximum height was reached when the position function was zero instead of when the velocity function was zero (Table 8). Procedural errors were found but were less common than conceptual ones. The analysis of the steps in Approach 1 suggested more conceptual than procedural steps; however, the analysis of the student solutions identified more procedural than conceptual steps (Approach 2 and 3). The problem can therefore not be described as mainly procedural or mainly conceptual.

Discussion

On the one hand, the analyses confirm that Problem 1 can be classified as a conceptual problem since, for both approaches, more conceptual than procedural steps were used to obtain the answer. Most students attempted to do the problem following an algebraic approach. However, the three students who successfully completed the problem used graphs to obtain the answer. The most common error (conceptual) was when students changed the absolute value function into a straight-line function, changing the question's nature (Almog & Ilany, 2012; Aziz, 2019). The procedural error of wrongly solving the quadratic inequality was found in a few answers (6%). Students' responses to this problem showed more conceptual than procedural errors.

On the other hand, Problem 2 cannot be described as mainly procedural or mainly conceptual because the number of conceptual and procedural steps differed across the different approaches. For Approach 1, there were more conceptual than procedural steps (C=5 P=3, C > P), and Approach

2 and 3 proposed more procedural than conceptual steps (C=2 P=5, C=2 P=3, C < P). Most students used the position function's derivative to solve the problem (94.5%), showing a preference for a conceptual approach (Chappell & Killpatrick, 2003). However, it is essential to note that 5.5% of students used the parabola's turning point and axis of symmetry to reach the correct answer. Approaches 2 and 3 could be used since the position function is quadratic – if a polynomial of a different degree were chosen, the alternative methods would not have been possible. To solve the problem, a few students made the conceptual error of equating the position function to zero. These students procedurally knew that they had to equate a function to zero but made a conceptual error when choosing the position function instead of the velocity function.

One of the findings from the analysis is that the two problems taken from a calculus course cannot be uniquely described as mainly conceptual or procedural. The analysis of solution approaches for the two selected items confirms that the relations between the types of knowledge within problem solutions are integrated (Kilpatrick et al., 2001), complex and reciprocal. Solution approaches draw on both procedural and conceptual steps. Furthermore, evidence confirms that solution methods are not unique and could follow more than one solution approach. The evidence supports Star and Stylianides's (2013) assertion that the nature of mathematical problems and solutions is not absolute. Procedural and conceptual steps are present and appear to be bidirectional since the procedural and conceptual knowledge categories alternate (Rittle-Johnson et al., 2015) but in no specific order (Rittle-Johnson et al., 2016), and some solution steps repeat (Rittle-Johnson, 2017).

The analysis of the responses to Problem 2 provides disconfirming evidence for the statement that the approach used to solve a mathematical problem is classified as either mainly conceptual or mainly procedural (Bergsten et al., 2015; Bergsten et al., 2017; Engelbrecht et al., 2012). Furthermore, an approach that would be conceptualised for one student could be seen as procedural for another since the solution selected could depend on whether the person saw the problem beforehand. Some students' familiar and procedural problems could be unfamiliar and conceptual to other students. Problem 2 is an example of a contextual problem that could be seen as such a problem since many students (procedurally) know that the expression $s'(t)$ is required in order to find the local extreme value(s) (White & Mesa, 2014). Nonetheless, the most common error in student responses was when the position function (instead of the velocity function) was equated to zero. These students procedurally know how to solve the problem but struggle to apply the procedure. As an analogue to this argument, a final year school examination question may be procedural for students whose teachers have taught and tested similar problems but conceptual for students whose teachers have avoided such questions.

A second finding that emerged from the study is that student work shows alternative solutions that should alter teaching practice to promote learning. The responses to Problem 1 illustrated alternative student methods, and these could be used to inform comparison and explanation of different strategies. The algebraic method was the preferred method to teach solving inequalities in class; however, the three students who successfully solved the inequality used graphs to reason and justify their answers. They used a shorter method with fewer procedural steps and fewer technical obstacles than Approach 1. If teachers draw upon the teaching strategy of comparing the two approaches and connecting the inequality rules to the graphs of the functions, this may assist student learning.

Students' responses for Problem 2 suggested additional methods for solving the contextual problem using the parabola's turning point or axis of symmetry. The turning point and axis of symmetry methods (x, extreme y – value) could be connected to the extreme y-value of the position function ($s(t)$) at t). The local extreme will be for a t - value where $s'(t) = 0$ (velocity function = 0), that is, where the gradient is zero. This is the t-value where the gradient of the graph function s changes from positive to negative. Explaining and comparing the different methods will connect the local extreme of the position function (where velocity function = 0) to the extreme value of the parabola, which will enhance the conceptual understanding of calculus, particularly the relationship between the position and velocity function.

The two problems exemplify how different student approaches could influence future teaching instruction. The solutions to both the problems show how different problem-solving approaches could be connected through reasoning and explanation using procedures, concepts and different representations (Davis, 2005). Comparing and explaining multiple strategies for solving the same problem will deepen and promote student learning (Star et al., 2016). Furthermore, these practices could enhance students' procedural flexibility and develop conceptual and procedural knowledge among students with prior knowledge of one of the methods (Durkin et al., 2017).

Conclusion

The concept-driven and skills-oriented perspectives should not stand in opposition; teaching and learning strategies should focus on both concepts and procedures. The evidence suggests that calculus problem solutions include multiple solution strategies. The recommendation is that calculus teaching strategies evaluate, explain and compare multiple problem-solving methods. Lecturers should facilitate learning by asking probing questions that link procedures and concepts through reasoning, comparison and justification (Davis, 2005). Furthermore, we recommend examining and analysing more calculus problem solutions using the defined mathematical problem-solving categories to investigate multiple solution methods further.

A limitation of the study may be that the content analysis was a subjective interpretation by three mathematical education experts. Hence, the results may not represent the views of a wider group. Furthermore, the current study only examined two calculus questions answered by a small, homogeneous sample of engineering students. The extent of the response analysis could be improved by expanding the number of respondents taken from a more heterogeneous population. Future research could include further empirical evidence, analysis of student solutions when solving calculus problems, and further investigation of the teaching strategy of comparing and explaining multiple problem solutions when learning calculus.

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