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Coupled Fixed Point Theorems with Rational Type Contractive Condition via C-Class Functions and Inverse C_k-Class Functions

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Abstract: The purpose of this paper is to develop coupled fixed point theorems by C-class functions with mixed monotonicity, discuss the existence of the coupled fixed point of two mappings and obtain a coupled coincidence point theorem via inverse C_k-class functions in partially ordered H-metric spaces. This improves the existing results. In addition, some instances and an application are given to illustrate the usability and validity of the results.

Keywords: coupled fixed point; C-class functions; partially ordered H-metric; inverse C_k-class functions

1. Introduction

Since the Banach Contraction Principle, many scholars have developed it by changing the contraction environment and changing the metric type, e.g., [1–11]. For instance, in 2004, Ran and Reurings [1] introduced partial order in metric spaces and proposed some fixed point results in such spaces. In 2006, Bhaskar and Lakshmikantham [2] presented some coupled fixed point theorems for linear contractions in partially ordered metric spaces. In 2009, Lakshmikantham and Ćirić [3] introduced some coupled fixed point results for nonlinear contraction in partially ordered metric spaces. There are a large number of researchers who have considered coupled fixed points well in a variety of contractive conditions [12–16]. In 2006, Mustafa and Sims [17] introduced a class of new metric spaces called G-metric spaces which is equivalent to H-metric spaces in this paper, some fixed point results in such spaces can be found in [18–27]. In 2010, Harjani et al. [28] utilized rational type contractive mapping proposed by Jaggi [29] in partially ordered metric spaces. Shortly after, Chakrabarti [30] generalized these contractive conditions in [28] and extended the results of [28] to partially ordered G-metric spaces.

On the other hand, in 2014, Ansari [31] presented C-class functions and proved some relevant fixed points of complete metric spaces. In 2018, Saleem et al. [32] presented a new notion of inverse C-class functions where the first inequality is opposite to that of C-class functions.

As far as we know, there are also some contractions that have not been studied in partially ordered complete H-metric space. Thus, inspired by [30] and C-class functions, in

Section 3 we intend to study the coupled fixed point theorems and coupled coincidence point theorem in partially ordered H-metric space, which develops some results of [30]. In Section 4, motivated by inverse C-class functions [32], a class of development functions of inverse C-class functions called inverse C_k-class functions is introduced. Furthermore, the existence theorem of a coupled coincidence point by using inverse C_k-class functions is raised.

2. Preliminaries

Denote a partial order relation by $\ll$ on Z. For arbitrary $a, b \in Z$ endowed with the partial order $\ll$, $a \ll b$ and $b \gg a$ are the same. Moreover, $a < b$ means that $a \ll b$ and $a \neq b$.

We first introduce the fundamental definitions and notions which will be used in the following section.

Definition 1 ([17]). *Suppose that $Z \neq \emptyset$ and $H : Z \times Z \times Z \to \mathbf{R}^+$ has the following property:*

$(H1)$ $H(a,b,c) = 0$ *if* $a = b = c$.

$(H2)$ $0 < H(a,a,b)$, *for all* $a, b \in Z$ *with* $a \neq b$.

$(H3)$ $H(a,a,b) \leq H(a,b,c)$, *for all* $a,b,c \in Z$ *with* $c \neq b$.

$(H4)$ $H(a,b,c) = H(p\{a,b,c\})$, *where p is any permutation of a,b,c (a,b,c are symmetrical).*

$(H5)$ $H(a,b,c) \leq H(a,s,s) + H(s,b,c)$, *for all* $a,b,c,s \in Z$.

Then, H is a generalized metric, or in more detail, an H-metric on Z and (Z, H) is an H-metric space.

Proposition 1 ([17]). *Suppose that (Z,H) is an H-metric space; then for any $a,b,c,s \in Z$, the following is obviously appropriate:*

1. Supposing that $H(a,b,c) = 0$, *we have* $a = b = c$.

2. $H(a,b,c) \leq H(a,a,b) + H(a,a,c)$.

3. $H(a,b,b) \leq 2H(b,a,a)$.

4. $H(a,b,c) \leq H(a,s,c) + H(s,b,c)$.

Definition 2 ([17]). *Assume that (Z,H) is an H-metric space. The sequence $\{a_m\} \subseteq Z$ is H-convergent to a if for any arbitrary $\varepsilon > 0$, there exists a positive integer M such that $H(a, a_m, a_m) < \varepsilon$ for $m \geq M$, i.e., $\lim_{m \to +\infty} H(a, a_m, a_m) = 0$.*

Proposition 2 ([17]). *Suppose that (Z,H) is an H-metric space, it follows that for a sequence $\{a_m\} \subseteq Z$ and a point $a \in Z$, the terms in the following are equivalent:*

1. $\{a_m\}$ *is H-convergent to a.*

2. $H(a_m, a_m, a) \to 0$ *when* $m \to +\infty$.

3. $H(a_m, a, a) \to 0$ *when* $m \to +\infty$.

4. $H(a_m, a_l, a) \to 0$ *when* $m, l \to +\infty$.

Proposition 3 ([17]). *Assume that (Z,H) is an H-metric space. It follows that the function $H(a,b,c)$ is jointly continuous with respect to three variables.*

Remark 1. *It is noted that if $\{a_s\}$, $\{b_t\}$, $\{c_w\}$ are sequences in Z such that $\lim_{s \to +\infty} a_s = a$, $\lim_{t \to +\infty} b_t = b$, $\lim_{w \to +\infty} c_w = c$, then $H(a_s, b_t, c_w) \to H(a,b,c)$ when $s,t,w \to +\infty$.*

Definition 3 ([17]). *Suppose that (Z,H) is an H-metric space. We said that the sequence $\{a_s\} \subseteq X$ is H-Cauchy if for an arbitrary $\varepsilon > 0$, there is a positive integer S such that $H(a_s, a_t, a_w) < \varepsilon$ for all $s,t,w \geq S$.*

Proposition 4 ([17]). *Assume that (Z, H) is an H-metric space; the terms as follows are the same:*
1. The sequence $\{a_s\}$ is H-Cauchy.
2. For arbitrary $\varepsilon > 0$, there is a positive integer S satisfying that $H(a_s, a_t, a_w) < \varepsilon$ for all $s, t, w > S$.

Lemma 1. *Let (Z, H) be an H-metric space. Suppose that $\{a_s\}$ is a sequence in Z satisfying that $G(a_s, a_{s+1}, a_{s+1}) \to 0$ when $s \to +\infty$. Suppose that $\{a_s\}$ is not H-Cauchy, then there exist $\varepsilon > 0$ and sequences of the positive integers $s(w)$ and $t(w)$ with $s(w) > t(w) > w$ satisfying that $H(a_{t(w)}, a_{s(w)}, a_{s(w)}) \geq \varepsilon$, $H(a_{t(w)}, a_{s(w)-1}, a_{s(w)-1}) < \varepsilon$, and*

$$\lim_{w\to+\infty} H(a_{t(w)}, a_{s(w)}, a_{s(w)}) = \varepsilon,$$
$$\lim_{w\to\infty} H(a_{s(w)+1}, a_{t(w)+1}, a_{t(w)+1}) = \varepsilon,$$
$$\lim_{w\to\infty} H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+2}) = \varepsilon.$$

Proof. Because $H(a_s, a_{s+1}, a_{s+1}) \to 0$ when $s \to +\infty$, i.e.,

$$\lim_{s\to\infty} H(a_s, a_{s+1}, a_{s+1}) = 0. \tag{1}$$

From the inequality (3) in Proposition 1, it follows that

$$\lim_{s\to\infty} H(a_{s+1}, a_s, a_s) = 0. \tag{2}$$

If $\{a_s\}$ is not H-Cauchy, there are $\varepsilon > 0$ subsequences $\{a_{s(w)}\}$ and $\{a_{t(w)}\}$ of $\{a_s\}$ with $s(w) > t(w) > w$ satisfying that

$$H(a_{t(w)}, a_{s(w)}, a_{s(w)}) \geq \varepsilon, \tag{3}$$

for arbitrary $w \in \mathbf{N}$. Meanwhile, for $t(w)$, we can take $s(w)$ which is the smallest integer by virtue of $s(w) > t(w)$ and (3). Therefore, it follows that

$$H(a_{t(w)}, a_{s(w)-1}, a_{s(w)-1}) < \varepsilon, \tag{4}$$

for all $w \in \mathbf{N}$. With $(H5)$ in Definition 1, it occurs that

$$\begin{aligned}\varepsilon &\leq H(a_{t(w)}, a_{s(w)}, a_{s(w)}) \\ &\leq H(a_{t(w)}, a_{s(w)-1}, a_{s(w)-1}) + H(a_{s(w)-1}, a_{s(w)}, a_{s(w)}) \\ &\leq \varepsilon + H(a_{s(w)-1}, a_{s(w)}, a_{s(w)}).\end{aligned} \tag{5}$$

Taking $w \to \infty$ in (5) and noting that $H(a_s, a_{s+1}, a_{s+1}) \to 0$, it is naturally obtained that

$$\lim_{s\to\infty} H(a_{t(w)}, a_{s(w)}, a_{s(w)}) = \varepsilon. \tag{6}$$

We can obtain

$$\begin{aligned}H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+1}) \leq& H(a_{t(w)+1}, a_{t(w)}, a_{t(w)}) \\ &+ H(a_{t(w)}, a_{s(w)}, a_{s(w)}) + H(a_{s(w)}, a_{s(w)+1}, a_{s(w)+1}).\end{aligned} \tag{7}$$

and

$$\begin{aligned}H(a_{t(w)}, a_{s(w)}, a_{s(w)}) \leq& H(a_{t(w)}, a_{t(w)+1}, a_{t(w)+1}) \\ &+ H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+1}) + H(a_{s(w)+1}, a_{s(w)}, a_{s(w)})\end{aligned} \tag{8}$$

Letting $w \to +\infty$ in expressions (7) and (8) and regrading (1), (2) and (6), we derive

$$\lim_{w\to\infty} H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+1}) = \varepsilon. \tag{9}$$

Similarly, it follows that

$$\begin{aligned} H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+1}) &\leq H(a_{t(w)+1}, a_{s(w)+2}, a_{s(w)+1}) \\ &\leq H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+1}) + H(a_{s(w)+1}, a_{s(w)+2}, a_{s(w)+1}) \end{aligned} \tag{10}$$

from $(H3)$ and $(H5)$. Taking $w \to +\infty$ in expressions (10) and using (1), (2) and (9), we conclude that

$$\lim_{w\to\infty} H(a_{t(w)+1}, a_{s(w)+1}, a_{s(w)+2}) = \epsilon. \tag{11}$$

□

In 2014, in a partially ordered H-metric space, Chakrabarti [30] proposed coupled fixed point theorems for a mapping satisfying mixed monotone property and a nonlinear, rational type contractive condition and presented results for the existence of coupled coincidence points of two mappings via the notion of mixed f-monotonicity [3].

Definition 4 ([3])**.** *Suppose that $(Z, \ll)$ is a set endowed with partial order, $S : Z \times Z \mapsto Z$ and $f : Z \mapsto Z$. S is said to be mixed f-monotone if S is monotone f-non-decreasing with respect to the first variable and is monotone f-non-increasing with respect to the second variable, i.e., for all $s, t \in Z$, $s_1, s_2 \in Z$, $\quad fs_1 \ll fs_2 \rightleftharpoons S(s_1, t) \ll S(s_2, t)$, and $t_1, t_2 \in Z$, $\quad ft_1 \ll ft_2 \rightleftharpoons S(s, t_1) \gg S(s, t_2)$.*

Theorem 1 ([30])**.** *Suppose that Z is a set endowed with a partial order $\ll$ and G is a generalized metric on H satisfying that (Z, H) is completed with respect to the H-metric. Assume that $S : Z \times Z \to Z$ is continuous on Z and mixed monotone. Meanwhile, for all (s, t), (a, b), $(c, d) \in Z \times Z$ with $(s, t) \ll (a, b) \ll (c, d)$,*

$$\begin{aligned} &H(S(s,t), S(a,b), S(c,d)) \\ &\leq [\alpha \frac{H(s, S(s,t), S(s,t)) H(a, S(a,b), S(a,b)) H(c, S(c,d), S(c,d))}{1 + [H(s,a,c)]^2} + \beta H(s,a,c)], \end{aligned} \tag{12}$$

where $8\alpha + \beta < 1$. In case there exists $s_0, t_0 \in Z$ satisfying $s_0 \ll S(s_0, t_0)$ and $t_0 \gg S(t_0, s_0)$, it can be deduced that $(s^*, t^*) \in Z \times Z$ is a coupled fixed point of S, i.e., (s^*, t^*) fulfills $s^* = S(s^*, t^*)$, $t^* = S(t^*, s^*)$.

Theorem 2 ([30])**.** *Assume that the conditions are the same as in Theorem 1, except for the following: if for each $(s, t), (a, b) \in Z \times Z$, there exists a $(c, d) \in Z \times Z$ that can be comparable to (s, t) and (a, b), there exists uniqueness of coupled fixed points of S.*

Theorem 3 ([30])**.** *Suppose that Z is a set with a partial order and H is a metric on Z satisfying that (Z, H) is completed with respect to the H-metric. Assume that $S : Z \times Z \to Z$ and $f : Z \to Z$ is continuous on Z with the fact that S is mixed $f-$monotone. Let $S(Z \times Z) \subseteq f(Z)$; f commutes with S and for each $(s, t), (a, b), (c, d) \in Z \times Z$ with $(s, t) \ll (a, b) \ll (c, d)$ and $fs \ll fa \ll fc$, $ft \gg fb \gg fd$,*

$$\begin{aligned} &H(S(s,t), S(a,b), S(c,d)) \\ &\leq \alpha[\frac{H(fs, S(s,t), S(a,b)) H(fa, S(a,b), S(a,b)) H(fc, S(c,d), S(c,d))}{1 + [H(fs, fa, fc)]^2} + \beta H(fs, fa, fc)], \end{aligned} \tag{13}$$

where $8\alpha + \beta < 1$. If there is $s_0, t_0 \in Z$ such that $fs_0 \ll S(s_0, t_0)$ and $ft_0 \gg S(t_0, s_0)$, so $(s^*, t^*) \in Z \times Z$ is a coupled coincidence point of S and f, i.e., (s^*, t^*) is the solution of $fs^* = S(s^*, t^*)$, $\quad ft^* = S(t^*, s^*)$.

In addition, C-class functions were proposed by Ansari [31]. Improved by this kind of function, a lot of fixed point theorems can be generalized in the existing results.

Definition 5 ([31])**.** *We call a mapping $G : [0, +\infty)^2 \mapsto \mathbf{R}$ a C-class function if it has continuity properties and meets the terms as follows:*

(1) $G(x, y) \leq x$.

(2) $G(x, y) = x$ can deduce that one of the two terms holds: (a) $x = 0$; (b) $y = 0$, for all $x, y \in [0, +\infty)$.

We utilize C to represent the set of all the C-class functions.

Example 1 ([31])**.** *Every $G : [0, +\infty)^2 \mapsto \mathbf{R}$ in the following belongs to C for all $x, y \in [0, +\infty)$:*

(1) $G(x, y) = x - y$, $G(x, y) = x$ leads to $y = 0$.

(2) $G(x, y) = nx$, $n > 1$, $G(x, y) = x$ leads to $x = 0$.

(3) $G(x, y) = \frac{x}{(1+y)^s}$; $s \in (0, +\infty)$, $G(x, y) = x$ leads to $x = 0$ or $y = 0$.

(4) $G(x, y) = \log(y + a^x)/(1 + y)$, $a > 1$, $G(x, y) = x$ leads to $x = 0$ or $y = 0$.

(5) $G(x, y) = \ln(1 + a^x)/2$, $a > e$, $G(x, 1) = x$ leads to $x = 0$.

(6) $G(x, y) = (x + l)^{(1/(1+y)^s)} - l$, $l > 1, s \in (0, +\infty)$, $G(x, y) = x$ leads to $y = 0$.

(7) $G(x, y) = x \log_{y+a} a$, $a > 1$, $G(x, y) = x$ leads to $x = 0$ or $y = 0$.

(8) $G(x, y) = x - (\frac{1+x}{2+x})(\frac{y}{1+y})$, $G(x, y) = x$ leads to $y = 0$.

(9) $G(x, y) = x\beta(x)$, $\beta : [0, +\infty) \mapsto [0, 1)$, and is continuous, $G(x, y) = x$ leads to $x = 0$.

(10) $G(x, y) = x - \frac{y}{t+y}$, $G(x, y) = x$ leads to $y = 0$.

(11) $G(x, y) = x - \phi(x), G(x, y) = x$ leads to $x = 0$ where $\phi : [0, +\infty) \mapsto [0, +\infty)$ is continuous satisfying that $\phi(y) = 0 \rightleftharpoons y = 0$.

(12) $G(x, y) = xk(x, y), G(x, y) = x$ leads to $x = 0$ where $k : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$ is continuous satisfying that $k(y, x) < 1$ for all $x, y > 0$.

(13) $G(x, y) = x - (\frac{2+y}{1+y})y$, $G(x, y) = x$ leads to $y = 0$.

(14) $G(x, y) = \sqrt[n]{\ln(1 + x^n)}$, $G(x, y) = x$ leads to $x = 0$.

(15) $G(x, y) = \psi(x), G(x, y) = x$ leads to $x = 0$ where $\psi : [0, +\infty) \mapsto [0, +\infty)$ is upper semi-continuous satisfying that $\psi(0) = 0$, and $\psi(y) < y$ for $y > 0$.

(16) $G(x, y) = \frac{x}{(1+x)^s}$; $s \in (0, +\infty)$, $G(x, y) = x$ leads to $x = 0$.

(17) $G(x, y) = \theta(s)$; $\theta : \mathbf{R}^+ \times \mathbf{R}^+ \mapsto \mathbf{R}$ is a generalized Mizoguchi–Takahashi-type function; $G(x, y) = x$ leads to $x = 0$.

(18) $G(x, y) = \frac{x}{\beta(1/2)} \int_0^{+\infty} \frac{e^{-s}}{\sqrt{s}+y} ds$, where β is the Euler Gamma function.

Furthermore, Saleem et al. [32] presented a new notion of inverse C-class functions in the following.

Definition 6. *A mapping $G : [0, +\infty)^2 \mapsto \mathbf{R}$ is said to be an inverse C-class function in cases where it is continuous and the terms as follows hold:*

(i) $\quad G(x, y) \geq x$ for all $x, y \in [0, +\infty)$.

(ii) $\quad G(x, y) = x$ yields that one of the two terms holds: (a) $x = 0$; (b) $y = 0$.

C_{inv} denotes the set of all the inverse C-class functions. Some examples are given in [32].

Example 2. *All the mappings $G : [0, +\infty)^2 \mapsto \mathbb{R}$ belong to C_{inv}, for all $x, y \in [0, +\infty)$:*

1. $G(x, y) = x + y, G(x, y) = x$ *leads to* $y = 0$.
2. $G(x, y) = tx$, *for some* $t \in (1, +\infty), G(x, y) = x$ *leads to* $x = 0$.

3. $G(x,y) = x(1+y)^s$, *for some* $s \in (0,+\infty)$, $G(x,y) = x$ *leads to* $s = 0$ *or* $y = 0$.
4. $G(x,y) = \log_a[(y+a^x)(1+y)]$, *for some* $a > 1$, $G(x,y) = x$ *leads to* $y = 0$.
5. $G(x,y) = \varphi(x)$, $G(x,y) = x$ *leads to* $x = 0$, *where* $\varphi : [0,+\infty) \mapsto [0,+\infty)$ *is an upper semi-continuous function satisfying that* $\varphi(0) = 0$ *and* $\varphi(x) > x$, *for any* $x > 0$.

As far as we know, there are also some contractions that have not been studied in partially ordered complete H-metric space. Thus, inspired by [30] and C-class functions, in Section 3, we intend to study the coupled fixed point theorems and coupled coincidence point theorem in partially ordered H-metric space, which develops some results of [30]. In Section 4, the existence theorem of a coupled coincidence point by use of inverse C_k-class functions arises. In Section 5, an application is given to justify the main results.

3. Main Results

In the section, assume a set Z endowed with a partial order $\ll$. The partial order on $Z \times Z$ is induced by demanding that for any $(a,b), (s,t) \in Z \times Z$, $(a,b) \ll (s,t) \rightleftharpoons a \ll s,\ b \gg t$.

Before starting the main results, let us introduce an auxiliary function used in the main result. Ψ_s denotes the set of all functions $\psi : [0,+\infty) \mapsto [0,+\infty)$ which meets terms in the following [31]:

(1) ψ is continuous on $[0,+\infty)$.
(2) $\psi(0) \geq 0$.
(3) $\psi(x) > 0$ for all $x > 0$.

Now, list the main results. Firstly, a rational type contractive condition is presented in a partially ordered complete H-metric space and a coupled fixed point theorem is obtained.

Theorem 4. *Assume that Z is a set endowed with a partial order $\ll$ and we define H as a generalized metric on it. Moreover, (Z,H) is completed with regard to H-metric. Suppose $S : Z \times Z \mapsto Z$ is continuous on Z with the mixed monotone property. For all $(s,t), (a,b), (c,d) \in Z \times Z$ with $(s,t) \ll (a,b) \ll (c,d)$,*

$$
\begin{aligned}
&H(S(s,t), S(a,b), S(c,d)) \qquad (14)\\
&\leq G\Big(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2} + \beta H(s,a,c)],\\
&\psi\Big(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2} + \beta H(s,a,c)]\Big)\Big),
\end{aligned}
$$

where $G \in \mathrm{C}, \psi \in \Psi_s$ and $8\alpha + \beta > 0, \alpha, \beta \geq 0$. If there exists $s_0, t_0 \in Z$ satisfying that $s_0 \ll S(s_0,t_0)$ and $t_0 \gg S(t_0,s_0)$, then S has a coupled fixed point $(s^*,t^*) \in Z \times Z$., i.e., (s^*,t^*) is the solution of $s^* = S(s^*,t^*)$, $t^* = S(t^*,s^*)$.

Proof. Assume that there exist $s_0, t_0 \in Z$ such that $s_0 \ll S(s_0,t_0)$ and $t_0 \gg S(t_0,s_0)$. Define $s_{w+1} = S(s_w,t_w)$, $t_{w+1} = S(t_w,s_w)$, $w \in \mathrm{N} \cup \{0\}$. Because G is mixed monotone, and meanwhile is from the mathematical induction, it yields that

$$s_0 \ll s_1 \ll s_2 \ll \cdots \ll s_w \ll s_{w+1} \ll \cdots .$$

and

$$t_0 \gg t_1 \gg t_2 \gg \cdots \gg t_w \gg t_{w+1} \gg \cdots .$$

Taking $s = s_w, a = b = s_{w-1}, t = t_w, c = d = t_{w-1}$ in inequality (14), we have,

$$\begin{aligned}
&H(s_{w+1}, s_w, s_w) = H(S(s_w, t_w), S(s_{w-1}, t_{w-1}), S(s_{w-1}, t_{w-1})) \\
&\leq G(\frac{1}{8\alpha + \beta}[\alpha \frac{H(s_w, s_{w+1}, s_{w+1}) H(s_{w-1}, s_w, s_w) H(s_{w-1}, s_w, s_w)}{1 + [H(s_w, s_{w-1}, s_{w-1})]^2} + \beta H(s_w, s_{w-1}, s_{w-1})], \\
&\psi(\frac{1}{8\alpha + \beta}[\alpha \frac{H(s_w, s_{w+1}, s_{w+1}) H(s_{w-1}, s_w, s_w) H(s_{w-1}, s_w, s_w)}{1 + [H(s_w, s_{w-1}, s_{w-1})]^2} + \beta H(s_w, s_{w-1}, s_{w-1})])) \\
&\leq \frac{1}{8\alpha + \beta}[\alpha \frac{H(s_w, s_{w+1}, s_{w+1}) H(s_{w-1}, s_w, s_w) H(s_{w-1}, s_w, s_w)}{1 + [H(s_w, s_{w-1}, s_{w-1})]^2} + \beta H(s_w, s_{w-1}, s_{w-1})].
\end{aligned} \tag{15}$$

Now, by inequality (3) in Proposition 1, it follows that

$$\begin{aligned}
&H(s_{w+1}, s_w, s_w) \\
&\leq \frac{1}{8\alpha + \beta}[\alpha \frac{H(s_w, s_{w+1}, s_{w+1}) H(s_w, s_{w-1}, s_{w-1}) H(s_w, s_{w-1}, s_{w-1})}{1 + [H(s_w, s_{w-1}, s_{w-1})]^2} \\
&+ \beta H(s_w, s_{w-1}, s_{w-1})] \\
&\leq \frac{1}{8\alpha + \beta}[4\alpha H(s_w, s_{w+1}, s_{w+1}) + \beta H(s_w, s_{w-1}, s_{w-1})] \\
&\leq \frac{1}{8\alpha + \beta}[8\alpha H(s_{w+1}, s_w, s_w) + \beta H(s_w, s_{w-1}, s_{w-1})].
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&H(s_{w+1}, s_w, s_w) \\
&\leq \frac{1}{8\alpha + \beta}[4\alpha H(s_{w+1}, s_w, s_w) + \beta H(s_w, s_{w-1}, s_{w-1})].
\end{aligned} \tag{17}$$

Reconsidering and rewriting inequality (16), it follows that

$$H(s_{w+1}, s_w, s_w) \leq H(s_w, s_{w-1}, s_{w-1}) \leq \cdots \leq H(s_1, s_0, s_0). \tag{18}$$

In the same manner, we can see that

$$H(t_{w+1}, t_w, t_w) \leq H(t_w, t_{w-1}, t_{w-1}) \leq \cdots \leq H(t_1, t_0, t_0). \tag{19}$$

Hence, the sequence $\{H(s_{w+1}, s_w, s_w)\}$ is positive decreasing and bounded below. Thus, the sequence $\{H(s_{w+1}, s_w, s_w)\}$ converges to $r \geq 0$. We shall prove that $r = 0$.
Letting $w \to +\infty$ in (17), we have

$$r \leq \frac{1}{8\alpha + \beta}(4\alpha r + \beta r) < r.$$

Thus $r = 0$; that is,

$$\lim_{w \to +\infty} H(s_{w+1}, s_w, s_w) = 0. \tag{20}$$

Moreover, by Proposition 1, we derive that

$$\lim_{w \to +\infty} H(s_w, s_{w+1}, s_{w+1}) = 0. \tag{21}$$

From inequality (21) and $(H3)$, there exists $W \in \mathbf{N}$ satisfying that for all $w > W$,

$$
\begin{aligned}
&H(s_{w+2}, s_{w+1}, s_w) \\
&= H(S(s_{w+1}, t_{w+1}), S(s_w, t_w), S(s_{w-1}, t_{w-1})) \\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha \frac{H(s_{w+1}, s_{w+2}, s_{w+2})H(s_w, s_{w+1}, s_{w+1})H(s_{w-1}, s_w, s_w)}{1+[H(s_{w+1}, s_w, s_{w-1})]^2} + \beta H(s_{w+1}, s_w, s_{w-1})], \\
&\psi(\frac{1}{8\alpha+\beta}[\alpha \frac{H(s_{w+1}, s_{w+2}, s_{w+2})H(s_w, s_{w+1}, s_{w+1})H(s_{w-1}, s_{wn}, s_w)}{1+[H(s_{w+1}, s_w, s_{w-1})]^2}] + \beta H(s_{w+1}, s_w, s_{w-1})])) \\
&\leq \frac{1}{8\alpha+\beta}[\alpha \frac{H(s_{w+1}, s_{w+2}, s_{w+2})H(s_w, s_{w+1}, s_{w+1})H(s_{w-1}, s_w, s_w)}{1+[H(s_{w+1}, s_w, s_{w-1})]^2} + \beta H(s_{w+1}, s_w, s_{w-1})] \\
&\leq \frac{1}{8\alpha+\beta}[4\alpha H((s_{w+2}, s_{w+1}, s_w) + \beta H(s_{w+1}, s_w, s_{w-1})] \\
&\leq \frac{1}{8\alpha+\beta}[8\alpha H(s_{w+2}, s_{w+1}, s_w) + \beta H(s_{w+1}, s_w, s_{w-1})].
\end{aligned} \tag{22}
$$

i.e., for $w > W$,

$$H(s_{w+2}, s_{w+1}, s_w) \leq H(s_{w+1}, s_w, s_{w-1}).$$

Hence, the sequence $\{H(s_{w+2}, s_{w+1}, s_w)\}$ is positive decreasing and bounded below. Thus, the sequence $\{H(s_{w+2}, s_{w+1}, s_w)\}$ converges to $p \geq 0$. We shall prove that $p = 0$. Letting $w \to +\infty$ in (22), together with (21) and continuity of ψ, we have

$$p \leq G(\frac{\beta p}{8\alpha+\beta}, \psi(\frac{\beta p}{8\alpha+\beta})) \leq \frac{\beta p}{8\alpha+\beta}.$$

Hence, $p = 0$, which gives us that

$$\lim_{w \to +\infty} H(s_w, s_{w+1}, s_{w+2}) = 0. \tag{23}$$

Next, we shall show that $\{s_w\}$ is an H-Cauchy sequence.
From Lemma 1, (20) and (21), we derive that

$$\lim_{v \to +\infty} H(s_{u(v)}, s_{w(v)}, s_{w(v)}) = \varepsilon, \tag{24}$$
$$\lim_{v \to \infty} H(s_{u(v)+1}, s_{w(v)+1}, s_{w(v)+2}) = \varepsilon. \tag{25}$$

Again, from $(H3)$ and $(H5)$, it follows that

$$
\begin{aligned}
H(s_{u(v)}, s_{w(v)}, s_{w(v)+1}) &\leq H(s_{u(v)}, s_{w(v)}, s_{w(v)}) + H(s_{w(v)}, s_{w(v)}, s_{w(v)+1}) \\
&\leq H(s_{u(v)}, s_{w(v)}, s_{w(v)}) + H(s_{w(v)}, s_{w(v)+1}, s_{w(v)+2}).
\end{aligned} \tag{26}
$$

Hence, taking $v \to +\infty$ and in virtue of the Sandwich Theorem and (23), (24), we have

$$\lim_{v \to +\infty} H(s_{u(v)}, s_{w(v)}, s_{w(v)+1}) = \varepsilon. \tag{27}$$

Since $s_{u(v)} \ll s_{w(v)} \ll s_{w(v)+1}$, putting $s = s_{u(v)}, t = t_{u(v)}, a = s_{w(v)}, b = t_{w(v)}$ and $c = s_{w(v)+1}, d = t_{w(v)+1}$ in (14), for all $v \geq 0$, it follows that

$$\begin{aligned}
&H(s_{u(v)+1}, s_{w(v)+1}, s_{w(v)+2}) \qquad (28)\\
&= H(S(s_{u(v)}, t_{u(v)}), S(s_{w(v)}, t_{w(v)}), S(s_{w(v)+1}, t_{w(v)+1})\\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s_{u(v)}, s_{u(v)+1}, s_{u(v)+1})H(s_{w(v)}, s_{w(v)+1}, s_{w(v)+1})H(s_{w(v)+1}, s_{w(v)+2}, s_{w(v)+2})}{1+[H(s_{u(v)}, s_{w(v)}, s_{w(v)+1})]^2}]\\
&\quad + \beta H(s_{u(v)}, s_{w(v)}, s_{w(v)+1}),\\
&\psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s_{u(v)}, s_{u(v)+1}, s_{u(v)+1})H(s_{w(v)}, s_{w(v)+1}, s_{w(v)+1})H(s_{w(v)+1}, s_{w(v)+2}, s_{w(v)+2})}{1+[H(s_{u(v)}, s_{w(v)}, s_{w(v)+1})]^2}]\\
&\quad + \beta H(s_{u(v)}, s_{w(v)}, s_{w(v)+1})])).
\end{aligned}$$

Letting $u(v) \to +\infty$, $w(v) \to +\infty$ in (28), from (25) and (27), we have

$$\varepsilon \leq G(\frac{\beta\varepsilon}{8\alpha+\beta}, \psi(\frac{\beta\varepsilon}{8\alpha+\beta})) \leq \frac{\beta\varepsilon}{8\alpha+\beta}.$$

Thus, $\varepsilon = 0$, this contracts with the assumption $\varepsilon > 0$. So $\{s_w\}$ is an H-Cauchy sequence in Z.

Sequentially, the same reasoning applies to the sequence $\{t_w\}$; we can also prove that $\{t_w\}$ is an H-Cauchy sequence in Z. Since (Z, H) is completed with regard to H-metric, it leads to that there exist points $s^*, t^* \in Z$ satisfying that $s_w \to s^*$ and $t_w \to t^*$ as $w \to +\infty$.

In the following, it suffices to show that (s^*, t^*) is a coupled fixed point of S. Since S is continuous on Z and H as a metric is continuous in each variable, it follows that

$$\begin{aligned}
H(S(s^*, t^*), s^*, s^*) &= H(\lim_{w\to+\infty} S(s_w, t_w), s^*, s^*)\\
&= H(\lim_{w\to+\infty} s_{w+1}, s^*, s^*)\\
&= \lim_{w\to+\infty} H(s_{w+1}, s^*, s^*)\\
&= 0, \qquad (29)
\end{aligned}$$

which infers that $S(s^*, t^*) = s^*$. Similar to the above proof, note that $H(S(t^*, s^*), t^*, t^*)$ and repeating the arguments used to derive (29), we can show $S(t^*, s^*) = t^*$. This shows us that S has a coupled fixed point (s^*, t^*). □

Remark 2. *Taking $G(s, t) = (8\alpha + \beta)s$ with $0 < 8\alpha + \beta < 1$ in Theorem 4, it easily leads to Theorem 1. In addition, if we take the special function in Example 1 and its other functions about the function G, we can also obtain many results.*

Example 3. *Suppose that $Z = [0, +\infty)$ and define $H : Z \times Z \times Z \to \mathbf{R}^+$*

$$H(s,t,r) = \begin{cases} 0, & \text{if} \quad s = t = r,\\ \max\{s,t,r\}, & \text{otherwise.}\end{cases}$$

We can derive that (Z, H) is completed with regard to H-metric. At the same time, a partial order $\ll$ is defined on Z by the following: for any $s, t \in Z$, $s \ll t$ if $s \geq t$. Simultaneously, define $S : Z \times Z \to Z$

$$S(s,t) = \begin{cases} 1, & \text{if} \quad s \ll t,\\ 0, & \text{otherwise.}\end{cases}$$

If $s, t, a, b, c, d \in Z$ such that $d \ll b \ll t \ll s \ll a \ll c$ with $s \neq 0, a \neq 0, c \neq 0$; then it follows *that $c \leq a \leq s \leq t \leq b \leq d$. The left side of (12) turns out to be $H(0,0,0) = 0$. The right side of (12) becomes*

$$\begin{aligned}
&\alpha \frac{H(s,0,0)H(a,0,0)H(c,0,0)}{1+[H(s,a,c)]^2} + \beta H(s,a,c) \\
&= \alpha \frac{sac}{1+s^2} + \beta s \\
&> 0 \text{ with } \alpha = \frac{1}{16}, \beta = \frac{1}{3}.
\end{aligned} \tag{30}$$

If $s_0 = 0$ and $t_0 = 1$, then $s_0 \ll G(s_0,t_0)$ and $t_0 \gg G(t_0,s_0)$. So all of Theorem 4's conditions are met. It is obvious that S has a coupled fixed point $(0,1)$. In the meantime, S has another coupled fixed point $(1,0)$.

Next, we give the proof of the uniqueness of the coupled fixed point of S in Theorem 4.

Theorem 5. *Assume that the conditions are the same as in Theorem 4, except that, if for each $(s,t),(a,b) \in Z \times Z$, there exists a $(c,d) \in Z \times Z$ that can be comparable to (s,t) and (a,b), there exists uniqueness of coupled fixed points of S.*

Proof. Assume that $(s^*,t^*),(s',t') \in Z \times Z$ are coupled fixed points.

Case I. Suppose that (s^*,t^*) and (s',t') are comparable,

$$\begin{aligned}
H(s^*,s',s') &= H(S(s^*,t^*),S(s',t'),S(s',t')) \\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s^*,S(s^*,t^*),S(s^*,t^*))[H(s',S(s',t'),S(s',t'))]^2}{1+[H(s^*,s',s')]^2} + \beta H(s^*,s',s')], \\
&\quad \psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s^*,S(s^*,t^*),S(s^*,t^*))[H(s',S(s',t'),S(s',t'))]^2}{1+[H(s^*,s',s')]^2} + \beta H(s^*,s',s'))) \\
&= G(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s^*,s^*,s^*)[H(s',s',s')]^2}{1+[H(s^*,s',s')]^2} + \beta H(s^*,s',s')], \\
&\quad \psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s^*,s^*,s^*)[H(s',s',s')]^2}{1+[H(s^*,s',s')]^2} + \beta H(s^*,s',s')])) \\
&= G(\frac{\beta}{8\alpha+\beta}H(s^*,s',s'), \psi(\frac{\beta}{8\alpha+\beta}H(s^*,s',s'))) \\
&\leq \frac{\beta}{8\alpha+\beta}H(s^*,s',s') \leq H(s^*,s',s');
\end{aligned}$$

therefore one of the following two terms holds: (a) $\frac{\beta}{8\alpha+\beta}H(s^*,s',s') = 0$; (b) $\psi(\frac{\beta}{8\alpha+\beta}H(s^*,s',s')) = 0$. Then $H(s^*,s',s') = 0$. So we must have $s^* = s'$. Similar to the above proof, take $H(S(t^*,s^*),S(t',s'),S(t',s'))$ into account; it follows that $t^* = t'$. This shows $(s^*,t^*) = (s',t')$; then the uniqueness of the coupled fixed points is verified.

Case II. Suppose that (s^*,t^*) cannot be comparable to (s',t'); according to the factors of this theorem, there exists $(a,b) \in Z \times Z$ comparable to (s^*,t^*) and (s',t'). In cases where there exists a positive integer u_0 satisfying $S^{u_0}(a,b) = (s^*,t^*)$, then

$$\begin{aligned}
&S^{u_0}(a,b) = (s^*,t^*), \\
&S^{u_0+1}(a,b) = S(s^*,t^*) = s^*, \\
&S^{u_0+2}(a,b) = S^2(s^*,t^*) = S(S(s^*,t^*),S(t^*,s^*)) = S(s^*,t^*) = s^*.
\end{aligned}$$

Continuing this process as above, then $S^u(a,b) = s^*$ for$u \geq u_0 + 1$; therefore $S^u(a,b) \to s^*$ when $u \to +\infty$.

On the other hand, if there does not exist u_0, then for each $u \geq 1$,

$$
\begin{aligned}
&H(S^u(a,b),s^*,s^*) = H(S^u(a,b),S^u(s^*,t^*),S^u(s^*,t^*)) \\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha \frac{H(S^{u-1}(a,b),S^u(a,b),S^u(a,b))[H(S^{u-1}(s^*,t^*),S^u(s^*,t^*),S^u(s^*,t^*))]^2}{1+[H(S^{u-1}(a,b),S^{u-1}(s^*,t^*),S^{u-1}(s^*,t^*))]^2} \\
&\quad +\beta H(S^{u-1}(a,b),S^{u-1}(s^*,t^*),S^{u-1}(s^*,t^*))], \\
&\quad \psi(\frac{1}{8\alpha+\beta}[\alpha \frac{H(S^{u-1}(a,b),S^u(a,b),S^u(a,b))[H(S^{u-1}(s^*,t^*),S^u(s^*,t^*),S^u(s^*,t^*))]^2}{1+[H(S^{u-1}(a,b),S^{u-1}(s^*,t^*),S^{u-1}(s^*,t^*))]^2} \\
&\quad +\beta H(S^{u-1}(a,b),S^{u-1}(s^*,t^*),S^{u-1}(s^*,t^*))])),
\end{aligned} \tag{31}
$$

regarding the fact that $S^u(s,t)) = S(S^{u-1}(s,t),S^{u-1}(t,s))$ for each $(s,t) \in Z \times Z$. Because S has a coupled fixed point (s^*,t^*), it follows that $S^u(s^*,t^*) = s^*$ for any $u \geq 1$ and by virtue of (31), it can be inferred that

$$
\begin{aligned}
&H(S^u(a,b),s^*,s^*) = H(S^u(a,b),S^u(s^*,t^*),S^u(s^*,t^*)) \\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha \frac{H(S^{u-1}(a,b),S^u(a,b),S^u(a,b))[H(s^*,s^*,s^*)]^2}{1+[H(S^{u-1}(a,b),s^*,s^*)]^2} + \beta H(S^{u-1}(a,b),s^*,s^*)], \\
&\quad \psi(\frac{1}{8\alpha+\beta}[\alpha \frac{H(S^{u-1}(a,b),S^u(a,b),S^u(a,b))[H(s^*,s^*,s^*)]^2}{1+[H(S^{u-1}(a,b),s^*,s^*)]^2} + \beta H(S^{u-1}(a,b),s^*,s^*)])) \\
&= G(\frac{\beta}{8\alpha+\beta}H(S^{u-1}(a,b),s^*,s^*),\psi(\frac{\beta}{8\alpha+\beta}H(S^{u-1}(a,b),s^*,s^*))) \\
&\leq \frac{\beta}{8\alpha+\beta}H(S^{u-1}(a,b),s^*,s^*).
\end{aligned} \tag{32}
$$

Reconsidering and rewriting the above expression, it can be concluded that

$$
H(S^u(a,b),s^*,s^*) \leq H(S^{u-1}(a,b),s^*,s^*). \tag{33}
$$

Take $r_u = H(S^u(a,b),s^*,s^*)$, which is positive decreasing and bounded below. Hence, $r_u = H(S^u(a,b),s^*,s^*) \to r \geq 0$ when $u \to +\infty$. Taking $u \to +\infty$ in the equality (32), we have that

$$
r \leq G(\frac{\beta}{8\alpha+\beta}r,\psi(\frac{\beta}{8\alpha+\beta}r)) \leq \frac{\beta}{8\alpha+\beta}r < r. \tag{34}
$$

So $\frac{\beta}{8\alpha+\beta}r = 0$ or $\psi(\frac{\beta}{8\alpha+\beta}r) = 0$. Thus $r = 0$. This proves $S^u(a,b) \to s^*$ when $u \to +\infty$. Analogous to the above, it is easy to show that $S^u(b,a) \to t^*$ when $u \to +\infty$. Replacing s^* by s' and t^* by t' and rewriting the above description, it follows that $S^u(a,b) \to s'$ and $S^u(b,a) \to t'$ when $u \to +\infty$. Therefore, $s^* = s'$ and $t^* = t'$. So the $(s^*,t^*) = (s',t')$ and the uniqueness of the coupled fixed point is verified. □

Remark 3. *Taking $G(s,t) = (8\alpha+\beta)s$ with $0 < 8\alpha+\beta < 1$ in Theorem 5 easily leads to Theorem 2.*

Taking special functions in Theorem 5 easily leads to the following corollaries.

Corollary 1. *Assume that Z is a set endowed with a partial order $\ll$ and we define H as a generalized metric on it. Moreover, (Z,H) is completed with regard to H-metric. Suppose $S: Z \times Z \mapsto Z$ is continuous on Z with the mixed monotone property. For all $(s,t),(a,b),(c,d) \in Z \times Z$ with $(s,t) \ll (a,b) \ll (c,d)$,*

$$\begin{aligned}&H(S(s,t),S(a,b),S(c,d))\\&\leq \frac{1}{8\alpha+\beta}[\alpha\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2}+\beta H(s,a,c)]-\\&\psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2}+\beta H(s,a,c)]),\end{aligned}$$

where $\psi \in \Psi_s$ and $8\alpha+\beta > 0, \alpha, \beta \geq 0$. If there exists $s_0, t_0 \in Z$ satisfying that $s_0 \ll S(s_0,t_0)$ and $t_0 \gg S(t_0,s_0)$, then S has a coupled fixed point $(s^*,t^*) \in Z \times Z$., i.e., (s^*,t^*) is the solution of $s^* = S(s^*,t^*)$, $t^* = S(t^*,s^*)$. Furthermore, if for each $(s,t),(a,b) \in Z \times Z$, there exists a $(c,d) \in Z \times Z$ that can be comparable to (s,t) and (a,b), there exists uniqueness of coupled fixed points of S.

Proof. Take $G(s,t) = s-t$ in Theorem 5; this result obviously holds. □

Corollary 2. *Assume that Z is a set endowed with a partial order $\ll$ and we define H as a generalized metric on it. Moreover, (Z,H) is completed with regard to H-metric. Suppose $S : Z \times Z \mapsto Z$ is continuous on Z with the mixed monotone property. For all $(s,t),(a,b),(c,d) \in Z \times Z$ with $(s,t) \ll (a,b) \ll (c,d)$,*

$$\begin{aligned}&H(S(s,t),S(a,b),S(c,d))\\&\leq \frac{\alpha}{8\alpha+\beta}\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2}+\frac{\beta}{8\alpha+\beta}H(s,a,c),\end{aligned}$$

where $8\alpha+\beta > 0, \alpha, \beta \geq 0$. If there exists $s_0, t_0 \in Z$ satisfying that $s_0 \ll S(s_0,t_0)$ and $t_0 \gg S(t_0,s_0)$, then S has a coupled fixed point $(s^*,t^*) \in Z \times Z$., i.e., (s^*,t^*) is the solution of $s^* = S(s^*,t^*)$, $t^* = S(t^*,s^*)$. Furthermore, if for each $(s,t),(a,b) \in Z \times Z$, there exists a $(c,d) \in Z \times Z$ that can be comparable to (s,t) and (a,b), there exists uniqueness of coupled fixed points of S.

Proof. Take $G(s,t) = s$ in Theorem 5; this result obviously holds. □

Corollary 3. *Assume that Z is a set endowed with a partial order $\ll$ and we define H as a generalized metric on it. Moreover, (Z,H) is completed with regard to H-metric. Suppose $S : Z \times Z \mapsto Z$ is continuous on Z with the mixed monotone property. For all $(s,t),(a,b),(c,d) \in Z \times Z$ with $(s,t) \ll (a,b) \ll (c,d)$,*

$$\begin{aligned}&H(S(s,t),S(a,b),S(c,d))\\&\leq [\frac{\alpha}{8\alpha+\beta}\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2}+\frac{\beta}{8\alpha+\beta}H(s,a,c)]\\&\mu(\frac{\alpha}{8\alpha+\beta}\frac{H(s,S(s,t),S(s,t))H(a,S(a,b),S(a,b))H(c,S(c,d),S(c,d))}{1+[H(s,a,c)]^2}+\frac{\beta}{8\alpha+\beta}H(s,a,c)),\end{aligned}$$

where $\mu : [0,+\infty) \mapsto [0,1)$, and is continuous and $8\alpha+\beta > 0, \alpha, \beta \geq 0$. If there exists $s_0, t_0 \in Z$ satisfying that $s_0 \ll S(s_0,t_0)$ and $t_0 \gg S(t_0,s_0)$, then S has a coupled fixed point $(s^*,t^*) \in Z \times Z$., i.e., (s^*,t^*) is the solution of $s^* = S(s^*,t^*)$, $t^* = S(t^*,s^*)$. Furthermore, if for each $(s,t),(a,b) \in Z \times Z$, there exists a $(c,d) \in Z \times Z$ that can be comparable to (s,t) and (a,b), there exists uniqueness of coupled fixed points of S.

Proof. Take $G(s,t) = s\mu(s)$ in Theorem 5; this result obviously holds. □

In the following, we offer the proof of the existence of a coupled coincidence point for two maps $S : Z \times Z \mapsto Z$ and $f : Z \mapsto Z$.

Theorem 6. *Suppose that Z is a set endowed with a partial order, H is a metric on Z such that (Z, H) is completed with respect to H-metric. Assume that $S : Z \times Z \mapsto Z$ and $f : Z \mapsto Z$ are continuous on Z satisfying that S is mixed f-monotone. If $S(Z \times Z) \subseteq f(Z)$, f commutes with S, also for $(s,t),(a,b),(c,d) \in Z \times Z$ with $(s,t) \ll (a,b) \ll (c,d)$ and $fs \ll fa \ll fc$, $ft \gg fb \gg fd$,*

$$\begin{aligned}
&H(S(s,t),S(a,b),S(c,d))\\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha\frac{H(fs,S(s,t),S(s,t))H(fa,S(a,b),S(a,b))H(fc,S(c,d),S(c,d))}{1+[H(fs,fa,fc)]^2}\\
&\quad+\beta H(fs,fa,fc)],\\
&\quad\psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(fs,S(s,t),S(s,t))H(fa,S(a,b),S(a,b))H(fc,S(c,d),S(c,d))}{1+[H(fs,fa,fc)]^2}\\
&\quad+\beta H(fs,fa,fc)]))
\end{aligned} \tag{35}$$

where $G \in C$, $\psi \in \Psi_u$ and $8\alpha+\beta > 0, \alpha, \beta \geq 0$. Supposing that there is $s_0, t_0 \in Z$ satisfying that $fs_0 \ll S(s_0,t_0)$ and $ft_0 \gg S(t_0,s_0)$, so $(s^*,t^*) \in Z \times Z$ is a coupled coincidence point of S and f, i.e., (s^*,t^*) is the solution of $fs^* = S(s^*,t^*), ft^* = S(t^*,s^*)$.

Proof. In virtue of $S(Z \times Z) \subseteq f(Z)$, taking $s_1, t_1 \in Z$ satisfying that $fs_1 = S(s_0,t_0), ft_1 = S(t_0,s_0)$. Going on with this, $s_2, t_2 \in Z$ can be found with $fs_2 = S(s_1,t_1), ft_2 = S(t_1,s_1)$. Since S is mixed $g-$monotone, it follows that $fs_0 \ll fs_1 \ll fs_2$ and $ft_2 \ll ft_1 \ll ft_0$. In general, it can be shown that for $u \geq 0$,

$$\begin{aligned}
fs_u = S(s_{u-1},t_{u-1}) \ll fs_{u+1} = S(s_u,t_u),\\
ft_{u+1} = S(t_u,s_u) \ll ft_u = S(t_{u-1},s_{u-1}).
\end{aligned}$$

Setting $s = s_u, a = c = s_{u-1}, t = t_u, b = d = t_{u-1}$ in inequality (35), an analysis analogous to that in Theorem 4 shows

$$\begin{aligned}
&H(fs_{u+1},fs_u,fs_u)\\
&\leq G(\frac{1}{8\alpha+\beta}[4\alpha\frac{H(fs_u,fs_{u+1},fs_{u+1})[H(fs_{u-1},fs_u,fs_u)]^2}{1+[H(fs_{u-1},fs_u,fs_u)]^2}+\beta H(fs_u,fs_{u-1},fs_{u-1})],\\
&\quad\psi(\frac{1}{8\alpha+\beta}[4\alpha\frac{H(fs_u,fs_{u+1},fs_{u+1})[H(fs_{u-1},fs_u,fs_u)]^2}{1+[H(fs_{u-1},fs_u,fs_u)]^2}+\beta H(fs_u,fs_{u-1},fs_{u-1})]))\\
&\frac{1}{8\alpha+\beta}[4\alpha H(fs_{u+1},fs_u,fs_u)+\beta H(fs_u,fs_{u-1},fs_{u-1})]\\
&\leq\frac{1}{8\alpha+\beta}[8\alpha\, H(fs_{u+1},fs_u,fs_u)+\beta H(fs_u,fs_{u-1},fs_{u-1})].
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&H(fs_{u+1},fs_u,fs_u)\\
&\leq\frac{1}{8\alpha+\beta}[4\alpha H(fs_{u+1},fs_u,fs_u)+\beta H(fs_u,fs_{u-1},fs_{u-1})].
\end{aligned} \tag{37}$$

Reconsidering and rewriting (36), it follows that

$$H(fs_{u+1},fs_u,fs_u) \leq H(fs_u,fs_{u-1},fs_{u-1}) \leq \cdots \leq H(fs_1,fs_0,fs_0). \tag{38}$$

In the same manner, we also obtain

$$H(ft_{u+1},ft_u,ft_u) \leq H(ft_u,ft_{u-1},ft_{u-1}) \leq \cdots \leq H(ft_1,ft_0,ft_0). \tag{39}$$

Hence, the sequence $\{H(fs_{u+1}, fs_u, fs_u)\}$ is positive decreasing and bounded below. Thus, the sequence $\{H(fs_{u+1}, fs_n, fs_n)\}$ converges to $r \geq 0$. We shall prove that $r = 0$.
Taking $n \to +\infty$ in (37), we have

$$r \leq \frac{1}{8\alpha + \beta}(4\alpha r + \beta r) < r.$$

So $r = 0$, i.e.,

$$\lim_{u \to +\infty} H(fs_{u+1}, fs_u, fs_u) = 0. \tag{40}$$

Further, making use of Proposition 1, we derive that

$$\lim_{u \to +\infty} H(fs_u, fs_{u+1}, fs_{u+1}) = 0. \tag{41}$$

An analysis analogous to that in Theorem 4 infers that $\{fs_u\}$ is Cauchy with regard to H-metric in (Z, H).
Since (Z, H) is completed with regard to H-metric, it leads to a point $s^* \in Z$ satisfying $fs_u \to s^*$ when $u \to +\infty$.
Keeping on as before, we can also show further that $\{ft_u\}$ is Cauchy with regard to H-metric in (Z, H) and (Z, H) is completed with regard to H-metric; there exists a point $t^* \in Z$ such that $ft_u \to t^*$ when $u \to +\infty$.
At last, it suffices to show that S has a coupled coincident point (s^*, t^*). Because S and f commute, it follows that

$$f(fs_{u+1}) = f(S(s_u, t_u)) = S(fs_u, ft_u). \tag{42}$$

$$f(ft_{u+1}) = f(S(t_u, s_u)) = S(ft_u, fs_u). \tag{43}$$

Setting limits when $u \to +\infty$ in (42), (43) and S and f are, respectively, continuous on $Z \times Z$ and Z, we obtain

$$fs^* = \lim_{u \to +\infty} f(fs_{u+1}) = \lim_{u \to +\infty} f(S(s_u, t_u)) = \lim_{u \to +\infty} S(fs_u, ft_u). \tag{44}$$

$$ft^* = \lim_{u \to +\infty} f(ft_{u+1}) = \lim_{u \to +\infty} f(S(t_u, s_u)) = \lim_{u \to +\infty} S(ft_u, fs_u). \tag{45}$$

Because H is continuous with respect to its variables, it follows that

$$\begin{aligned} H(S(s^*, t^*), fs^*, fs^*) &= H(\lim_{u \to +\infty} S(fs_u, ft_u), fs^*, fs^*) \\ &= H(fs^*, fs^*, fs^*) \\ &= 0. \end{aligned} \tag{46}$$

Therefore $fs^* = S(s^*, t^*)$. Moreover, it leads to $ft^* = S(t^*, s^*)$. It shows that S and f have coupled coincidence point (s^*, t^*). □

Remark 4. *Take $G(s, t) = (8\alpha + \beta)s$ with $0 < 8\alpha + \beta < 1$ in Theorem 6; it is easy to obtain Theorem 3. In addition, with the same idea as the above corollaries, by taking special functions G and ψ in Theorem 6, we can also obtain some coupled coincidence point results of the two mappings.*

Example 4. *Let $Z = [0, 1]$ endowed with the natural ordering of real numbers and $H(s, t, r) = \max\{s, t, r\}$; obviously, (Z, H) is completed with respect to H-metric. Definite functions S and g are by*

$$S(s, t) = \begin{cases} \frac{s}{12} + \frac{t}{2}, & \textit{if} \quad s \geq t, \\ 0, & \textit{otherwise}. \end{cases}, \quad f(s) = \frac{2s}{3}.$$

Obviously, there exist $s_0 = t_0 = 0$ such that $fs_0 \leq S(s_0, t_0)$ and $ft_0 \geq S(t_0, s_0)$; furthermore, S and f are continuous functions, $S(Z, Z) \subseteq g(Z)$, S is mixed f-monotone and f commutes with

S. Take $G(s,t) = \frac{1}{8\alpha+\beta}s$, where $\alpha \geq 0$ and $\beta = \frac{7}{8}$. Now, we show S and f satisfy condition (35). Indeed, we take s,t,a,b,c,d with $s \leq a \leq c$ and $t \geq b \geq d$.

Case I: If $s \geq t$, $a \geq b$ and $c \geq d$, we have

$$
\begin{aligned}
&H(S(s,t),S(a,b),S(c,d)) \\
&= H(\frac{s}{12}+\frac{t}{2},\frac{a}{12}+\frac{b}{2},\frac{c}{12}+\frac{d}{2}) \\
&= \max\{\frac{s}{12}+\frac{6t}{12},\frac{a}{12}+\frac{6b}{12},\frac{c}{12}+\frac{6d}{12}\} \\
&\leq \frac{7c}{12} \\
&\leq \frac{7}{8}\max\{\frac{2s}{3},\frac{2a}{3},\frac{2c}{3}\} \\
&= \frac{7}{8}H(fs,fa,fc) \\
&\leq \alpha\frac{H(fs,S(s,t),S(s,t))H(fa,S(a,b),S(a,b))H(fc,S(c,d),S(c,d))}{1+[H(fs,fa,fc)]^2}+\frac{7}{8}H(fs,fa,fc).
\end{aligned}
$$

Case II: If $s \geq t$, $a \geq b$ and $c < d$, we obtain

$$
\begin{aligned}
&H(S(s,t),S(a,b),S(c,d)) \\
&= H(\frac{s}{12}+\frac{t}{2},\frac{a}{12}+\frac{b}{2},0) \\
&= \max\{\frac{s}{12}+\frac{6t}{12},\frac{a}{12}+\frac{6b}{12},0\} \\
&\leq \frac{7c}{12} \\
&\leq \frac{7}{8}\max\{\frac{2s}{3},\frac{2a}{3},\frac{2c}{3}\} \\
&= \frac{7}{8}H(fs,fa,fc) \\
&\leq \alpha\frac{H(fs,S(s,t),S(s,t))H(fa,S(a,b),S(a,b))H(fc,S(c,d),S(c,d))}{1+[H(fs,fa,fc)]^2}+\frac{7}{8}H(fs,fa,fc).
\end{aligned}
$$

Case III: If $s \geq t$, $a < b$ and $c < d$, we obtain

$$
\begin{aligned}
&H(S(s,t),S(a,b),S(c,d)) \\
&= H(\frac{s}{12}+\frac{t}{2},0,0) \\
&= \max\{\frac{s}{12}+\frac{6t}{12},0,0\} \\
&\leq \frac{7c}{12} \\
&\leq \frac{7}{8}\max\{\frac{2s}{3},\frac{2a}{3},\frac{2c}{3}\} \\
&= \frac{7}{8}H(fs,fa,fc) \\
&\leq \alpha\frac{H(fs,S(s,t),S(s,t))H(fa,S(a,b),S(a,b))H(fc,S(c,d),S(c,d))}{1+[H(fs,fa,fc)]^2}+\frac{7}{8}H(fs,fa,fc).
\end{aligned}
$$

Case IV: If $s < t$, $a < b$ and $c < d$, we obtain

$$
\begin{aligned}
&H(S(s,t), S(a,b), S(c,d)) \\
&= H(0,0,0) \\
&= \max\{0,0,0\} \\
&\leq \frac{7}{8}H(fs, fa, fc) \\
&\leq \alpha\frac{H(fs, S(s,t), S(s,t))H(fa, S(a,b), S(a,b))H(fc, S(c,d), S(c,d))}{1+[H(fs, fa, fc)]^2} + \frac{7}{8}H(fs, fa, fc).
\end{aligned}
$$

Similarly, the other cases follow immediately. Therefore, all the conditions of Theorem 6 are satisfied, so S and f have a coupled coincidence point.

4. Coupled Coincidence Point Theorem via Inverse C_k-Class Functions

In the section, if we carry out some changes in the first inequality of Definition 8, we obtain another kind of inverse C-class function, which can be regarded as the generalization of the inverse C-class functions.

Definition 7. *A mapping $G : [0, +\infty)^2 \mapsto \mathbb{R}$ is said to be an inverse C_k-class function in cases where it is continuous and the terms as follows hold:*

(i) $G(x,y) \geq lx$ *for all* $x, y \in [0, +\infty)$ *and some* $l \geq 1$.
(ii) $G(x,y) = lx$ *yields that one of the two terms holds: (a)* $x = 0$; *(b)* $y = 0$.

$\mathrm{C}_{\mathrm{inv}}$ denotes the set of all the inverse C_k-class functions. Every inverse C-class function is equivalent to an inverse C_k-class function when $k = 1$, but the scope of an inverse C_k-class function must be larger than that of an inverse C-class function.

Example 5. *A mapping $G : [0, +\infty)^2 \mapsto \mathbf{R}$ is a function satisfying that $G(x,y) = lx\log_a(\frac{1}{2}la + y)$ for all $x, y \in [0, +\infty)$, $l \geq 1$ and $a > 1$. Then, obviously, G is an inverse C_k-class function when $l \geq 1$, because $G(x,y) \geq lx$ when $l \geq 2$, $a > 1$ and $y \geq 0$ and $G(x,y) = lx \Rightarrow y = 0$ when $l = 2$. Yet it is not an inverse C-class function, because $G(x,y) \geq x$ does not hold if $l = 1$, $y \neq 0$ and $G(x,y) = x$ does not imply $y = 0$ since we can set $l = 1, y = \frac{1}{2}a > 2$.*

Example 6. *We define a mapping $G : [0, +\infty) \mapsto \mathbf{R}$ by $G(x,y) = 2x + y$ for all $x, y \in [0, +\infty)$. So, it is obvious that G is an inverse C_k-class function for $k = 2$, but it is not an inverse C-class function.*

Example 7. *All the mappings $G : [0, +\infty)^2 \mapsto \mathbf{R}$ belong to $\mathrm{C}_{inv\text{-}k}$, for all $x, y \in [0, +\infty)$:*

1. $G(x,y) = kx + ly$, $G(x,y) = kx$ *leads to* $y = 0$ *for some* $k \geq 1$ *and* $l > 0$.
2. $G(x,y) = knx$, $G(x,y) = kx$ *leads to* $x = 0$ *for* $n \in (1, +\infty)$ *and some* $k \geq 1$.
3. $G(x,y) = kx(1+ky)^r$, $G(x,y) = kx$ *leads to* $x = 0$ *or* $y = 0$ *for* $r \in (0, +\infty)$ *and some* $k \geq 1$.
4. $G(x,y) = \log_a[(ky + a^{kx})(1+ky)]$, $G(x,y) = ks$ *leads to* $y = 0$ *for some* $a > 1$ *and some* $k \geq 1$.
5. $G(x,y) = k\varphi(s)$, $G(x,y) = kx$ *leads to* $x = 0$, *where* $\varphi : [0, +\infty) \mapsto [0, +\infty)$ *is an upper semi-continuous function satisfying that* $\varphi(0) = 0$ *and* $\varphi(x) > x$, *for* $x > 0$ *and some* $k \geq 1$.

The purpose of this section is to give the existence theorem of a coupled coincidence point by use of inverse C_k-class functions.

Singh et al. [33], also introduce new notions which are called g-compatible of type (E) and f-compatible of type (E). We all know that f and g are self-mappings on Z. Inspired by Singh's work, we define f-compatible of type (E) and S-compatible of type (E) where $f : Z \mapsto Z$ and $S : Z \times Z \mapsto Z$.

Definition 8. *Two maps $f : Z \mapsto Z$ and $S : Z \times Z \mapsto Z$ in a metric space (Z, d) are called f-compatible of type (E) if $\lim_{u\to+\infty} ffs_u = \lim_{u\to+\infty} fS(s_u, s_u) = S(y, y)$, whenever $\{s_u\}$ is a sequence in Z with the fact that $\lim_{u\to+\infty} fs_u = \lim_{u\to+\infty} S(s_u, s_u) = y$ for some $y \in Z$. Similarly, two self-mappings f and S of a metric space (Z, d) are called S-compatible of type (E) if $\lim_{u\to+\infty} S(S(s_u, s_u), S(s_u, s_u)) = \lim_{u\to+\infty} S(fs_u, fs_u) = fy$, whenever $\{s_u\}$ is a sequence in Z satisfying that $\lim_{u\to+\infty} fs_u = \lim_{u\to+\infty} S(s_u, s_u) = y$ for some $y \in Z$.*

Theorem 7. *Suppose that Z is a set endowed with a partial order; H is a metric on Z satisfying that (Z, H) is completed with respect to H-metric. Assume that $S : Z \times Z \mapsto Z$ and $f : Z \mapsto Z$ are, respectively, continuous on $Z \times Z$ and Z satisfying that (f, S) is S-compatible of type (E) and S is mixed f-monotone. Let $S(Z \times Z) \subseteq f(Z)$; f commutes with S and for $(s, t), (a, b), (c, d) \in Z \times Z$ with $(s, t) \ll (a, b) \ll (c, d)$ and $fs \ll fa \ll fc$, $ft \ll fb \ll fd$,*

$$
\begin{aligned}
& kH(fs, fa, fc) \\
& \geq G\Big(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs, S(s,t), S(s,t))(1 + H(fa, S(a,b), S(a,b)))(1 + H(fc, S(c,d), S(c,d)))}{[1 + H(fs, fa, fc)]^2} \\
& \quad + \beta H(fs, fa, fc)], \\
& \quad \psi(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs, S(s,t), S(s,t))(1 + H(fa, S(a,b), S(a,b)))(1 + H(fc, S(c,d), S(c,d)))}{[1 + H(fs, fa, fc)]^2} \\
& \quad + \beta H(fs, fa, fc)])\Big)
\end{aligned} \tag{47}
$$

where $G \in \mathcal{C}_{inv\text{-}k}, \psi \in \Psi_u$ and $2\alpha + \beta > 0, \alpha, \beta \geq 0$. If there exists $s_0, t_0 \in Z$ satisfying that $fs_0 \ll S(s_0, t_0)$ and $ft_0 \gg S(t_0, s_0)$, then $(x^*, y^*) \in X \times X$ is a a coupled coincidence point of S and f, i.e., (s^*, t^*) is the solution of $fs^* = S(s^*, t^*), ft^* = S(t^*, s^*)$.

Proof. Because $S(Z \times Z) \subseteq f(Z)$, take $s_1, t_1 \in Z$ satisfying that $fs_1 = S(s_0, t_0), ft_1 = S(t_0, s_0)$. Continuing this way, $s_2, t_2 \in Z$ can be found satisfying that $fs_2 = S(s_1, t_1)$, $ft_2 = S(t_1, s_1)$. Since S is mixed $g-$monotone, it can be inferred that $fs_0 \ll fs_1 \ll fs_2$, $ft_2 \ll ft_1 \ll ft_0$. Inductively, it can be shown that for $u \geq 0$,

$$
\begin{aligned}
fs_u &= S(s_{u-1}, t_{u-1}) \ll fs_{u+1} = S(s_u, t_u), \\
ft_{u+1} &= S(t_u, s_u \ll ft_u = S(t_{u-1}, s_{u-1}).
\end{aligned}
$$

Taking $s = s_u, a = c = s_{u-1}, t = t_u, b = d = t_{u-1}$ in inequality (47), we have,

$$
\begin{aligned}
& kH(fs_u, fs_{u-1}, fs_{u-1}) \\
& \geq G(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs_u, fs_{u+1}, fs_{u+1})[1 + H(fs_{u-1}, fs_u, fs_u)]^2}{[1 + H(fs_u, fs_{u-1}, fs_{u-1})]^2} + \beta H(fs_u, fs_{u-1}, fs_{u-1})], \\
& \quad \psi(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs_u, fs_{u+1}, fs_{u+1})[1 + H(fs_{u-1}, fs_u, fs_u)]^2}{[1 + H(fs_u, fs_{u-1}, fs_{u-1})]^2} + \beta H(fs_u, fs_{u-1}, fs_{u-1})])) \\
& \geq k\frac{1}{2\alpha+\beta}[4\alpha\, H(fs_{u+1}, fs_u, fs_u) + \beta H(fs_u, fs_{u-1}, fs_{u-1})], \\
& \geq k\frac{1}{2\alpha+\beta}[2\alpha\, H(fs_{u+1}, fs_u, fs_u) + \beta H(fs_u, fs_{u-1}, fs_{u-1})].
\end{aligned} \tag{48}
$$

and

$$
\begin{aligned}
&kH(fs_u, fs_{u-1}, fs_{u-1}) \\
&\geq G(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs_u, fs_{u+1}, fs_{u+1})[1+H(fs_{u-1}, fs_u, fs_u)]^2}{[1+H(fs_u, fs_{u-1}, fs_{u-1})]^2} + \beta H(fs_u, fs_{u-1}, fs_{u-1})], \\
&\quad \psi(\frac{1}{2\alpha+\beta}[4\alpha\frac{H(fs_u, fs_{u+1}, fs_{u+1})[1+H(fs_{u-1}, fs_u, fs_u)]^2}{[1+H(fs_u, fs_{u-1}, fs_{u-1})]^2} + \beta H(fs_u, fs_{u-1}, fs_{u-1})])) \\
&\geq k\frac{1}{2\alpha+\beta}[4\alpha\, H(fs_{u+1}, fs_u, fs_u) + \beta H(fs_u, fs_{u-1}, fs_{u-1})].
\end{aligned} \tag{49}
$$

Reconsidering and rewriting (48), it follows that

$$H(fs_{u+1}, fs_u, fs_u) \leq H(fs_u, fs_{u-1}, fs_{u-1}) \leq \cdots \leq H(fs_1, fs_0, fs_0). \tag{50}$$

In the same manner, we can see that

$$H(ft_{u+1}, ft_u, ft_u) \leq H(ft_u, ft_{u-1}, ft_{u-1}) \leq \cdots \leq H(ft_1, ft_0, ft_0). \tag{51}$$

Hence, the sequence $\{H(fs_{u+1}, fs_u, fs_u)\}$ is positive decreasing and bounded below. Thus, the sequence $\{H(fs_{u+1}, fs_u, fs_u)\}$ converges to $r \geq 0$. It suffices to prove that $r = 0$.
Taking $u \to +\infty$ in (49), it follows that

$$kr \geq \frac{k}{2\alpha+\beta}(4\alpha r + \beta r) > kr.$$

Therefore, $r = 0$,i.e.,

$$\lim_{u\to+\infty} H(fs_{u+1}, fs_u, fs_u) = 0. \tag{52}$$

Furthermore, making use of Proposition 1, we deduce that

$$\lim_{u\to+\infty} H(fs_u, fs_{u+1}, fs_{u+1}) = 0. \tag{53}$$

An analysis analogous to that in Theorem 4 yields that $\{fs_u\}$ is Cauchy with respect to $H-$metric in (Z, H).
Because (Z, H) is completed with respect to H-metric, it is obvious that there is a point $s^* \in X$ satisfying that $fs_u \to s^*$ when $u \to +\infty$.
Going on as before, we can also show further that $\{ft_u\}$ is Cauchy with respect to H-metric in (Z, H). Since (Z, H) is completed with respect to H-metric, there exists a point $t^* \in Z$ satisfying that $ft_u \to t^*$ when $u \to +\infty$.
At last, it suffices to prove that f and S have a coupled coincident point (s^*, t^*).
Because S and f commute, it follows that

$$f(fs_{u+1}) = f(S(s_u, t_u)) = S(fs_u, ft_u), \tag{54}$$

$$f(ft_{u+1}) = f(S(t_u, s_u)) = S(ft_u, fs_u). \tag{55}$$

Setting limits when $u \to +\infty$ in (54), (55), since S and f are, respectively, continuous on $Z \times Z$ and Z, we obtain

$$fx^* = \lim_{u\to+\infty} f(fs_{u+1}) = \lim_{u\to+\infty} f(S(s_u, t_u)) = \lim_{u\to+\infty} S(fs_u, ft_u) \tag{56}$$

$$ft^* = \lim_{u\to+\infty} f(ft_{u+1}) = \lim_{u\to+\infty} f(S(t_u, s_u)) = \lim_{u\to+\infty} S(ft_u, fs_u) \tag{57}$$

Sequentially, since H is continuous with respect to all its variables, it follows that

$$\begin{aligned} &H(S(s^*,t^*),fs^*,fs^*) \\ &= H(\lim_{u\to+\infty} S(fs_u,ft_u),fs^*,fs^*) \\ &= H(fs^*,fs^*,fs^*) \\ &= 0. \end{aligned} \tag{58}$$

Therefore, $fs^* = S(s^*,t^*)$. Analogously, $ft^* = S(t^*,s^*)$. This shows that (s^*,t^*) is a coupled coincidence point. □

Remark 5. (i) *The contractive condition (47) is different from (35) in Theorem 6. At first, the direction of inequality changes from $\leq$ to $\geq$. In fact, that is the key difference between $G \in$ C and $G \in \mathrm{C}_{inv-k}$. Secondly, the left-hand side of the inequality becomes $kH(fs,fa,fc)$ which is different from $H(S(s,t),S(a,b),S(c,d)$ in Theorem 6. This difference makes the proof of the Cauchy sequence fs_u easier.*

(ii) *There is an open question. Why not use the S-compatibility of type (E) of the pair (f,S). There is also the question whether we can replace the mixed f-monotonicity of (f,S) with S-compatibility of type (E) of (f,S). In other words, it seems that the proof of Theorem 7 could pass without the S-compatibility of type (E) of (f,S).*

5. Application

In this section, Theorem 5 is used to guarantee the existence and uniqueness of solutions to the following integral equations:

$$x(t) = \int_a^b (K_1(t,s)+K_2(t,s))(f_1(s,x(s))+f_2(s,x(s)))ds + h(t), \tag{59}$$

where $f_i : [a,b]\times\mathbf{R} \to \mathbf{R}$, $K_1 : [a,b]\times[a,b] \to [0,+\infty)$, $K_2 : [a,b]\times[a,b] \to (-\infty,0]$ and $h(t) : [a,b] \to \mathbf{R}$ are continuous for all $i = 1,2$. Let $H : C([a,b],\mathbf{R})\times C([a,b],\mathbf{R})\times C([a,b],\mathbf{R}) \to \mathbf{R}$ be a function defined by

$$H(x,y,z) = \max\{(\sup_{t\in[a,b]}|x(t)-y(t)|, \sup_{t\in[a,b]}|x(t)-z(t)|, \sup_{t\in[a,b]}|y(t)-z(t)|)\} \tag{60}$$

where $C([a,b],\mathbf{R})$ by the set of all continuous real-value functions defined on $[a,b]$. It is easy to see that $(C([a,b],\mathbf{R}),H)$ is completed with regard to H-metric. Let $S : C([a,b],\mathbf{R})\times C([a,b],\mathbf{R}) \to C([a,b],\mathbf{R})$ be a mapping defined by

$$S(x(t),y(t)) = \int_a^b (K_1(t,s)+K_2(t,s))(f_1(s,x(s))+f_2(s,y(s)))ds + h(t), \text{ for all } t\in[a,b]. \tag{61}$$

Suppose that the following conditions hold:

(i) There exist two numbers $\lambda > 0$ and $\alpha,\beta \geq 0$ with $8\alpha+\beta > 0$ such that

$$0 \leq f_2(t,u(t)) - f_2(t,v(t)) \leq f_1(t,x(t)) - f_1(t,y(t)) \leq \frac{\lambda\beta}{8\alpha+\beta}(x-y),$$

for all $x \geq y, u \geq v,\ x,y,u,v \in \mathbf{R}$;

(ii) $\sup\limits_{t\in[a,b]} |\int_a^b K_1(t,s)+K_2(t,s)ds| \leq \frac{k}{\lambda}$ where $k \in 0,1$;

(iii) f_i and g are increasing functions for all $i = 1,2$;

(iv) There exists a couple lower-upper solution (x_0,y_0) of Equation (59); that is,

$$x_0(t) \leq \int_a^b (K_1(t,s)+K_2(t,s))(f_1(s,x_0(s))+f_2(s,y_0(s)))ds + h(t)$$

and

$$y_0(t) \geq \int_a^b (K_1(t,s) + K_2(t,s))(f_1(s, y_0(s)) + f_2(s, x_0(s)))ds + h(t).$$

Now we prove the existence of a solution of the mentioned integral equations (59).

Theorem 8. *Under conditions (i)–(iv), (59) has a solution in $C([a,b], \mathbf{R})$.*

Proof. Consider a partial order on $C([a,b], \mathbf{R})$; that is, $x \ll y$ if and only if $x \leq y$. Assume that $x, y, u, v, w, z \in C([a,b], \mathbf{R})$ such that $x(t) \leq u(t) \leq w(t)$ and $y(t) \geq v(t) \geq z(t)$, for all $t \in [a,b]$. By (i) and (ii); we obtain

$$\begin{aligned}
|S(x,y) - S(u,v)| &= |\int_a^b (K_1(t,s) + K_2(t,s))(f_1(s, x(s)) + f_2(s, y(s)))ds + h(t) - \\
&\int_a^b (K_1(t,s) + K_2(t,s))(f_1(s, u(s)) + f_2(s, v(s)))ds + h(t)| \\
&= |\int_a^b (K_1(t,s) + K_2(t,s)(f_1(s, x(s)) - f_1(s, u(s)))ds \\
&+ \int_a^b (K_1(t,s) + K_2(t,s)((f_2(s, y(s)) - f_2(s, v(s)))ds| \\
&= |\int_a^b (K_1(t,s) + K_2(t,s)(f_1(s, u(s)) - f_1(s, x(s)))ds \\
&- \int_a^b (K_1(t,s) + K_2(t,s)((f_2(s, y(s)) - f_2(s, v(s)))ds| \\
&\leq |\int_a^b (K_1(t,s) + K_2(t,s))(f_1(s, u(s)) - f_1(s, x(s)))ds| \\
&\leq \sup_{t \in [a,b]} |\int_a^b K_1(t,s) + K_2(t,s)ds| \sup_{t \in [a,b]} (\frac{\lambda\beta}{8\alpha + \beta}[u(t) - x(t)]) \\
&\leq \frac{k\beta}{8\alpha + \beta} \sup_{t \in [a,b]} [u(t) - x(t)].
\end{aligned}$$

Analogously, we can obtain

$$|S(u,v) - S(w,z)| \leq \frac{k\beta}{8\alpha + \beta} \sup_{t \in [a,b]} [w(t) - u(t)]$$

and

$$|S(x,y) - S(w,z)| \leq \frac{k\beta}{8\alpha + \beta} \sup_{t \in [a,b]} [w(t) - x(t)].$$

Since

$$\begin{aligned}
&H(S(x,y),S(u,v),S(w,z)) \\
&= \max\{(\sup_{t\in[a,b]}|S(x,y)-S(u,v)|, \sup_{t\in[a,b]}|S(x,y)-S(w,z)|, \sup_{t\in[a,b]}|S(u,v)-S(w,z)|)\} \\
&\leq \max\{\frac{k\beta}{8\alpha+\beta}\sup_{t\in[a,b]}[u(t)-x(t)], \frac{k\beta}{8\alpha+\beta}\sup_{t\in[a,b]}[w(t)-x(t)], \frac{k\beta}{8\alpha+\beta}\sup_{t\in[a,b]}[w(t)-u(t)]\} \\
&= \frac{k\beta}{8\alpha+\beta}\max\{\sup_{t\in[a,b]}[u(t)-x(t)], \sup_{t\in[a,b]}[w(t)-x(t)], \sup_{t\in[a,b]}[w(t)-u(t)]\} \\
&= \frac{k\beta}{8\alpha+\beta}H(x,u,w) \\
&\leq \frac{1}{8\alpha+\beta}[\alpha\frac{H(x,S(x,y),S(x,y))H(u,S(u,v),S(u,v))H(w,S(w,z),S(w,z))}{1+[H(x,u,w)]^2}+\beta H(x,u,w)] \\
&- \frac{1-k}{8\alpha+\beta}[\alpha\frac{H(x,S(x,y),S(x,y))H(u,S(u,v),S(u,v))H(w,S(w,z),S(w,z))}{1+[H(x,u,w)]^2}+\beta H(x,u,w)]
\end{aligned}$$

Let $\psi(x) = (1-k)x$ and $G(x,y) = x - y$; we can obtain

$$\begin{aligned}
&H(S(x,y),S(u,v),S(w,z)) \\
&\leq G(\frac{1}{8\alpha+\beta}[\alpha\frac{H(x,S(x,y),S(x,y))H(u,S(u,v),S(u,v))H(w,S(w,z),S(w,z))}{1+[H(x,u,w)]^2}+\beta H(x,u,w)], \\
&\psi(\frac{1}{8\alpha+\beta}[\alpha\frac{H(x,S(x,y),S(x,y))H(u,S(u,v),S(u,v))H(w,S(w,z),S(w,z))}{1+[H(x,u,w)]^2}+\beta H(x,u,w)])),
\end{aligned}$$

Thus, S satisfies condition (14). It is obvious that S is continuous with the mixed monotone property and there exist $x_0(t)$, $y_0(t)$ such that $x_0(t) \ll S(x_0(t), y_0(t))$ and $y_0(t) \gg S(y_0(t), x_0(t))$ by (iii) and (iv). S satisfies all the conditions of Theorem 5; then, S has a unique couple fixed point. Assume that (x, y) is the couple fixed point of S; we can deduce that (y, x) is also a couple fixed point of S. $x = y$ due to the uniqueness of the couple fixed point of S. That is (59) has a unique solution for $C([a, b], \mathbf{R})$. □

6. Conclusions

In conclusion, a rational type contractive condition is proposed in an H-metric space and some coupled fixed point theorems for mappings which are mixed monotone are given. The uniqueness of the coupled fixed point is offered. The existence of coupled coincidence points of two mappings are also obtained. In addition, the coupled coincidence point theorems by virtue of inverse C_k-class functions are proved in the context of mixed f-monotonicity of (f, S). Furthermore, Some examples and an application are given to justify the main results. It is well known that the fixed point theorem plays an important role in ensuring the existence and uniqueness of solutions concerning some equations; we consider possible future research directions.

There are some works in the future:

(i) Discuss the possibility of applying coupled fixed point results to the problems of differential equations, for example, fuzzy and set differential equations and discontinuous differential equations;

(ii) Discuss the properties of the C-class functions for coupled fixed point problems which include data dependence, well-posedness, Ulam–Hyers stability, limit shadowing property.

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