On open (c, ϵ)-balls in topological spaces that capture convergence in non-additive probability measure with probability-one coincidence*

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April 22, 2022

Abstract

We introduce the topological spaces $(L, \tau_{\nu 1})$ which are well-defined for any given subset of random variables L on any given non-additive probability space $(\Omega, \mathcal{F}, \nu)$. A (c, ϵ)-ball at $X \in L$ contains all random variables in L that are sufficiently close to X in the sense that any payoff differences to X smaller than $c > 0$ happen with ν-probability greater than $1 - \epsilon$. We derive two main results concerning (c, ϵ)-balls. Firstly, all (c, ϵ)-balls must be open sets in $(L, \tau_{\nu 1})$ whenever ν is continuous from below and dual-autocontinuous from above. In that case, convergence of sequences of random variables on $(L, \tau_{\nu 1})$ is equivalent to convergence in non-additive probability measure ν with probability-one coincidence. Secondly, an open (c, ϵ)-ball cannot be a convex strict subset of L whenever L has a non-trivial local cone structure and $(\Omega, \mathcal{F}, \nu)$ is dual-nonatomic.

Subject classifications: 60A86; 60B05
Keywords: Convergence in non-additive probability measure; Weak-base topology; Dual-autocontinuity; Dual-nonatomicity

1 Introduction

Fix an arbitrary measurable space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is a sigma-algebra on the state space Ω. A *not necessarily additive* (=non-additive) probability measure on $(\Omega, \mathcal{F})$ is a set function $\nu : \mathcal{F} \to [0, 1]$ that satisfies

- $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$ (normalization);

*We are very grateful to an anonymous referee for helpful comments and suggestions.
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- For all $A, B \in \mathcal{F}$, $A \subseteq B$ implies $\nu(A) \leq \nu(B)$ (monotonicity).

Denote by L^0 the set of all $\mathcal{F}$-measurable real-valued functions (i.e., random variables) defined on the non-additive probability space $(\Omega, \mathcal{F}, \nu)$. Fix an arbitrary subset $L \subseteq L^0$ of random variables. For given $X \in L$ and $c, \epsilon \in \mathbb{Q}_{>0}$ define the following (c, ϵ)-ball at X

$$B_{c,\epsilon}^{\nu 1}(X) = \{Y \in L | \nu(|X - Y| < c) > 1 - \epsilon\} \tag{1}$$

which contains all $Y \in L$ such that any differences between X and Y that happen with ν-probability greater than $1 - \epsilon$ must be smaller than c.

Definition 1. *Given $L \subseteq L^0$ we define the topology*

$$\tau_{\nu 1} = \left\{\mathcal{U} \subseteq L \mid \text{ for all } X \in \mathcal{U}, \mathcal{U} \in \mathcal{W}_X^{\nu 1}\right\} \cup \{\emptyset, L\}$$

such that the weak neighborhood system at $X \in L$ is given as

$$\mathcal{W}_X^{\nu 1} = \left\{\mathcal{V} \subseteq L \mid B_{\frac{1}{m},\frac{1}{m}}^{\nu 1}(X) \subseteq \mathcal{V} \text{ for some } m \in \mathbb{N}_{>0}\right\}.$$

As generalizations of additive probability measures, non-additive probability measures describe subjective beliefs of Choquet expected utility (Schmeidler 1986; 1989; Gilboa 1987) and/or of prospect theory decision makers (Tversky and Kahneman 1992; Wakker 2010). Structural properties of non-additive measures–such as, e.g., *concavity/submodularity* and *convexity/supermodularity*[1] –are used in Choquet expected utility theory to model ambiguity attitudes of decision makers. Such ambiguity attitudes might serve as possible explanations for Ellsberg (1961)-type paradoxa which represent violations of Savage's (1954) axiomatic foundation of subjective expected utility theory. In prospect theory, structural properties of non-additive beliefs might additionally capture inverse-S shaped probability weighting functions which represent the typical transformation of objective (and therefore additive) probability measures elicited in decision theoretic experiments. In our opinion, the topological space $(L, \tau_{\nu 1})$ is relevant in mathematical decision theory because it stands for a plausible model of how decision makers with non-additive subjective belief ν might perceive convergence of random variables on the subset $L \subseteq L^0$ (cf. Section 2).[2]

This short paper presents two theorems for topological spaces $(L, \tau_{\nu 1})$. Theorem 1 shows that every (c, ϵ)-ball is an open set in any topological space $(L, \tau_{\nu 1})$ whenever ν satisfies *continuity from below* combined with *dual-autocontinuity from above*. A sequence $\{X_n\}_{n \in \mathbb{N}_{>0}} \subseteq L$ converges in ν to X with probability-one coincidence iff (=if and only if), for every $c > 0$,

[1]For formal definitions of such structural properties see, e.g., Denneberg (1994, Chapter 2). Additive probability measures are non-additive probability measures that are both concave and convex.

[2]For the special case of subjective additive beliefs given as the Lebesgue measure λ defined on the open unit interval, Assa and Zimper (2018) and Zimper and Assa (2021) discuss utility- and/or risk-measure representations of preferences over random variables that are (semi-)continuous in the topology of convergence in λ.

$\lim_{n\to\infty} \nu(|X_n - X| < c) = 1$. By Corollary 1, convergence on $(L, \tau_{\nu 1})$ is equivalent to convergence in ν with probability-one coincidence whenever ν is continuous from below and dual-autocontinuous from above. Next we investigate properties of convex and open subsets of topological spaces $(L, \tau_{\nu 1})$ where L has a *local cone* structure and $(\Omega, \mathcal{F}, \nu)$ is a *dual-nonatomic* space. By Theorem 2, the convex hull of every open subset containing $X \in L$ must also contain the sum $X + Y$ for all $Y \in L$. For local cones L that are *non-trivial* (in the sense that they contain some random variable whose absolute payoff values are bounded away from zero), Corollary 2 shows that any open (c, ϵ)-ball which is a strict subset of L cannot be a convex set. Corollary 3 applies our analysis to non-trivial local cones given as (i) the set of all random variables, (ii) the set of all non-negative random variables, and (iii) the set of all non-positive random variables.

The remainder of our paper is organized as follows. Section 2 briefly explains the difference of our approach to the existing literature. Theorem 1 is derived and discussed in Section 3. In Section 4 we derive and discuss Theorem 2.

2 Related literature

The existing literature considers convergence in nonadditive measure ν with *probability-zero divergence* in the sense that for every $c > 0$, $\lim_{n\to\infty} \nu(|X_n - X| \geq c) = 0$ (cf. Wu, Ren, and Wu 2011; Ouyang and Zhang 2011; Li 2012; Borzová-Molnárová, Halčinová, and Hutník 2016). While these authors construct weak base topologies with respect to different notions of balls (i.e., different distance functions), all these topologies turn out to be equivalent to the same weak base topology, denoted $\tau_{\nu 0}$, for an arbitrary nonadditive probability measure ν. To be precise, consider, for example, the following definition of a ball

$$B_\epsilon^{\nu,d}(X) = \{Y \in L \mid d(X, Y) < \epsilon\}$$

with respect to the distance function $d\colon L^0 \times L^0 \to [0, 1]$ introduced in Li (2012):

$$d(X, Y) = \inf\{b > 0 \mid \nu(|X - Y| > b) \leq b\}.$$

The topology $\tau_{\nu 0}$ on L is then defined as follows:

$$\tau_{\nu 0} = \left\{\mathcal{U} \subseteq L \mid \text{ for all } X \in \mathcal{U}, \mathcal{U} \in \mathcal{W}_X^{\nu,d}\right\} \cup \{\emptyset, L\}$$

such that the weak neighborhood system at $X \in L$ is given as

$$\mathcal{W}_X^{\nu,d} = \left\{\mathcal{V} \subseteq L \mid B_{\frac{1}{m}}^{\nu,d}(X) \subseteq \mathcal{V} \text{ for some } m \in \mathbb{N}_{>0}\right\}.$$

If ν satisfies additional structural properties (e.g., *Condition (*)* in Wu, Ren, and Wu (2011) or the stronger condition of *uniform autocontinuity from above* in Ouyang and Zhang (2011)), then convergence in the topology $\tau_{\nu 0}$ is equivalent to convergence in ν with probability-zero divergence. Whereas convergence in ν with probability-zero divergence and our notion of convergence in ν with probability-one coincidence are identical concepts for additive probability measures, this is not necessarily the case for arbitrary nonadditive probability measures. Convergence with probability-one coincidence would be the appropriate convergence concept for a decision maker or/and modeler who perceives two random variables X and Y as identical whenever they coincide on some probability one event, i.e., whenever

$$\nu\left(|Y-X|=0\right)=1. \tag{2}$$

In contrast, convergence with probability-zero divergence would correspond to a decision maker or/and modeler who perceives two random variables X and Y as identical if they only diverge on some probability zero event, i.e., whenever

$$\nu\left(|Y-X|>0\right)=0.$$

Example 1. Consider $L=\{X,Y\}$ such that the random variables

	A_1	A_2
X	1	1
Y	1	0

are defined on $(\Omega,\mathcal{F},\nu)$ whereby $\{A_1,A_2\}\subseteq\mathcal{F}$ forms a partition of Ω. Consider, at first, a nonadditive probability measure ν such that

$$\nu\left(A_1\right)=1 \text{ and } \nu\left(A_2\right)>0.$$

Then both random variables coincide on the probability-one event A_1 so that, trivially, $X\rightarrow_{\nu 1}Y$ as well as $Y\rightarrow_{\nu 1}X$. On the other hand, both random variables diverge on the event A_2 which has probability strictly greater zero. Consequently, we neither have $X\rightarrow_{\nu 0}Y$ nor $Y\rightarrow_{\nu 0}X$. As corresponding topologies we obtain

$$\tau_{\nu 1}=\{\emptyset,\{X,Y\}\}\neq\tau_{\nu 0}=\{\emptyset,\{X\},\{Y\},\{X,Y\}\}.$$

Next, consider a nonadditive probability measure ν' such that

$$\nu'\left(A_1\right)<1 \text{ and } \nu'\left(A_2\right)=0.$$

Both random variables only diverge on the probability-zero event A_2 but they do not coincide on any probability-one event (here: Ω). In that case, we have $X\rightarrow_{\nu'0}Y$ as well as $Y\rightarrow_{\nu'0}X$ but neither $X\rightarrow_{\nu'1}Y$ nor $Y\rightarrow_{\nu'1}X$. Note that

$$\tau_{\nu'1}=\{\emptyset,\{X\},\{Y\},\{X,Y\}\}\neq\tau_{\nu'0}=\{\emptyset,\{X,Y\}\}.$$

□

While our topology $\tau_{\nu 1}$ is thus not necessarily equivalent to $\tau_{\nu 0}$ for arbitrary ν, one can easily show the following equivalence relationship between both topological approaches. For arbitrary ν, we have $\tau_{\nu 1}=\tau_{\tilde{\nu}0}$ where $\tilde{\nu}:\mathcal{F}\rightarrow[0,1]$ denotes the *dual* of ν, i.e.,

$$\tilde{\nu}\left(A\right)=1-\nu\left(\Omega-A\right) \text{ for all } A\in\mathcal{F}.$$

3 Convergence on $(L, \tau_{\nu 1})$ versus convergence in ν with probability-one coincidence

3.1 First main result: Theorem 1

The family

$$\left\{\mathcal{B}_X^{\nu 1} \mid X \in L\right\}$$

with

$$\mathcal{B}_X^{\nu 1} = \left\{B_{\frac{1}{m},\frac{1}{m}}^{\nu 1}(X) \mid m \in \mathbb{N}_{>0}\right\} \tag{3}$$

forms a *weak base* for the $\tau_{\nu 1}$-topology whereby

$$B_{\frac{1}{m},\frac{1}{m}}^{\nu 1}(X) \subseteq B_{\frac{1}{n},\frac{1}{n}}^{\nu 1}(X) \text{ for } m \geq n \text{ and } X \in B_{\frac{1}{m},\frac{1}{m}}^{\nu 1}(X) \text{ for all } m \in \mathbb{N}_{>0}.$$

Because (3) is countable for every $X \in L$, $\tau_{\nu 1}$ is *weakly first-countable* (or *g-first countable*) (cf. Arkhangel'skiĭ 1966; Siwiec 1974; Hong 1999).

A set $\mathcal{U}$ is open in $\tau_{\nu 1}$ iff for every $Y \in \mathcal{U}$ there exists some $m_Y \in n \in \mathbb{N}_{>0}$ such that

$$B_{\frac{1}{m_Y},\frac{1}{m_Y}}^{\nu 1}(Y) \subseteq \mathcal{U}.$$

In general, a (c,ϵ)-ball $B_{c,\epsilon}^{\nu 1}(X)$ is not necessarily an open set in the topology $\tau_{\nu 1}$. In a seminal paper, Wang (1984) discusses several continuity and autocontinuity properties of nonadditive measures. Our first main result identifies a new autocontinuity condition on ν which ensures that the (c,ϵ)-balls are open sets in any topological space $(\tau_{\nu 1}, L)$, $L \subseteq L^0$.

Theorem 1. *Fix an arbitrary $(\Omega, \mathcal{F}, \nu)$ and $L \subseteq L^0$. Suppose that ν satisfies the following two conditions:*

(a) *'Continuity from below': for any increasing sequence $\{A_n\}_{n\in\mathbb{N}_{>0}} \subseteq \mathcal{F}$ with $A_1 \subseteq A_2 \subseteq ....$ we have*

$$\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n\in\mathbb{N}_{>0}} A_n\right).$$

(b) *'Dual-autocontinuity from above': For any sequence $\{A_n\}_{n\in\mathbb{N}_{>0}} \subseteq \mathcal{F}$ with $\lim_{n\to\infty}\nu(A_n) = 1$ we have for all $A \in \mathcal{F}$*

$$\lim_{n\to\infty} \nu(A_n \cap A) = \nu(A).$$

Then we have the following:

(i) *All (c,ϵ)-balls (1) are open sets in $\tau_{\nu 1}$; that is, $B_{c,\epsilon}^{\nu 1}(X) \in \tau_{\nu 1}$ for all $c, \epsilon \in \mathbb{Q}_{>0}$ and all $X \in L$.*

(ii) *The family of open sets $\mathcal{B}_X^{\nu 1}$, given by (3), forms a countable neighborhood base at $X \in L$ so that $\tau_{\nu 1}$ is first-countable.*

The following two examples illustrate that the conclusions of Theorem 1 do not necessarily hold for ν that either violate continuity from below or dual-autocontinuity from above.

Example 2. Violation of continuity from below. Let $\Omega = [0, 1)$ with $\mathcal{F}$ as the Borel sigma-algebra and consider the nonadditive measure such that $\nu(A) = 0$ for all $A \in \mathcal{F}$ with $A \neq \Omega$. Observe that ν satisfies 'dual-autocontinuity from above' since the only sequences with $\lim_{n\to\infty} \nu(A'_n) = 1$ must satisfy $A'_n = \Omega$, $n \geq M$ for some M, so that

$$\lim_{n\to\infty} \nu(A \cap A'_n) = \nu(A) = \begin{cases} 0 & \text{if } A \neq \Omega \\ 1 & \text{if } A = \Omega. \end{cases}$$

Construct the countable partition

$$\{A_n\}_{n\in\mathbb{N}_{>0}}$$

of Ω into half-open intervals such that

$$\begin{aligned} A_1 &= \left[0, \frac{1}{2}\right) \text{ and, for all } n \geq 1,\ A_n = \left[a_n, \frac{1+a_n}{2}\right) \\ \text{where } a_n &= \frac{1+a_{n-1}}{2}. \end{aligned} \tag{4}$$

That is, $A_1 = \left[0, \frac{1}{2}\right)$, $A_2 = \left[\frac{1}{2}, \frac{3}{4}\right)$, $A_3 = \left[\frac{3}{4}, \frac{7}{8}\right)$ etc. whereby $\lim_{n\to\infty} a_n = 1$. Let

$$A'_k = \bigcup_{n=1}^{k} A_n = \left[0, \frac{1+a_k}{2}\right)$$

and note that $A'_1 \subseteq A'_2 \subseteq \ldots$ with $\bigcup_{n\in\mathbb{N}_{>0}} A'_n = [0, 1)$ but with $A'_n \neq [0, 1)$ for all $n \in \mathbb{N}_{>0}$. Consequently, ν violates 'continuity from below' because we have

$$\lim_{n\to\infty} \nu(A_n) = 0 < 1 = \nu\left(\bigcup_{n\in\mathbb{N}_{>0}} A_n\right).$$

Next assume that L consists of the following random variables which are all constant on each partition cell A_n, $n \in \mathbb{N}_{>0}$,

$$\begin{aligned} X(\omega) &= \frac{1}{n+1} \text{ for } \omega \in A_n \\ Y(\omega) &= 1 \text{ for } \omega \in \Omega, \\ Z_n(\omega) &= \begin{cases} 1 + \frac{1}{n+1} & \text{if } \omega \in A_n \\ 1 & \text{else.} \end{cases} \end{aligned}$$

Fix $m = 1$. Observe, at first, that $Y \in B_{\frac{1}{m},\frac{1}{m}}(X)$ because of

$$\begin{aligned} |X(\omega) - Y(\omega)| &= \frac{n}{n+1} \text{ for } \omega \in A_n \text{ for all } n \in \mathbb{N}_{>0} \\ &\Longrightarrow \\ \nu(|X - Y| < 1) &= \nu(\Omega) = 1 > 0. \end{aligned}$$

Next note that $Z_n \notin B_{\frac{1}{m},\frac{1}{m}}(X)$ for all $n \in \mathbb{N}_{>0}$ because of

$$\nu(|X - Z_n| < 1) = \nu(\Omega \backslash A_n) = 0 \leq 1 - \frac{1}{n} \text{ for all } n \in \mathbb{N}_{>0}.$$

However, we also have that $Z_n \in B_{\frac{1}{n},\frac{1}{n}}(Y)$ for all $n \in \mathbb{N}_{>0}$ because of

$$\begin{aligned} |Y(\omega) - Z_n(\omega)| &= \begin{cases} \frac{1}{n+1} & \text{if } \omega \in A_n \\ 0 & \text{else} \end{cases} \\ &\Longrightarrow \\ \nu\left(|Y - Z_n| < \frac{1}{n}\right) &= \nu(\Omega) = 1 > 1 - \frac{1}{n} \text{ for all } n \in \mathbb{N}_{>0}. \end{aligned}$$

That is, we cannot find any $n \in \mathbb{N}_{>0}$ such that

$$B_{\frac{1}{n},\frac{1}{n}}(Y) \subseteq B_{\frac{1}{m},\frac{1}{m}}(X)$$

in spite of $Y \in B_{\frac{1}{m},\frac{1}{m}}(X)$. Consequently, the ball $B_{\frac{1}{m},\frac{1}{m}}(X)$ with $m = 1$ cannot be an open set in $\tau_{\nu 1}$.□

Example 3. Violation of dual-autocontinuity from above. Let $\{A_1, A_2\} \subseteq \mathcal{F}$ be some partition of Ω into non-empty sets and assume that ν satisfies

$$\nu(A) = 1 \text{ for all non-empty } A \in \mathcal{F}.$$

On the one hand, ν violates dual-autocontinuity from above because of

$$\lim_{n \to \infty} \nu(A_1 \cap A_2) = 0 \neq 1 = \nu(A_2)$$

in spite of $\lim_{n \to \infty} \nu(A_1) = 1$. On the other hand, continuity from below holds as for any increasing sequence

$$\begin{aligned} \lim_{n \to \infty} \nu(A_n) &= \nu\left(\bigcup_{n \in \mathbb{N}_{>0}} A_n\right) = 0 \text{ if } A_n = \emptyset \text{ for all } n \text{ and} \\ \lim_{n \to \infty} \nu(A_n) &= \nu\left(\bigcup_{n \in \mathbb{N}_{>0}} A_n\right) = 1 \text{ else.} \end{aligned}$$

Next consider $L = \{X, Y, Z\}$ such that

	A_1	A_2
X	1	1
Y	1	0
Z	0	1

Note that we have for any $m \in \mathbb{N}_{>0}$

$$\begin{aligned} B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X) &= \{X, Y, Z\}, \\ B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(Y) &= \{X, Y\}, \\ B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(Z) &= \{X, Z\}, \end{aligned}$$

resulting in weak neighborhood systems

$$\begin{aligned} \mathcal{W}^{\nu 1}_X &= \{\{X, Y, Z\}\}, \\ \mathcal{W}^{\nu 1}_Y &= \{\{X, Y\}, \{X, Y, Z\}\}, \\ \mathcal{W}^{\nu 1}_Z &= \{\{X, Z\}, \{X, Y, Z\}\}. \end{aligned}$$

This weak base generates $\tau_{\nu 1}$ as the trivial topology

$$\tau_{\nu 1} = \{\emptyset, \{X, Y, Z\}\}$$

with neighborhood systems

$$\mathcal{N}^{\nu 1}_X = \mathcal{N}^{\nu 1}_Y = \mathcal{N}^{\nu 1}_Z = \{\{X, Y, Z\}\}.$$

Clearly, the conclusion of Theorem 1 does not hold as not all balls are open sets in $\tau_{\nu 1}$.□

A sequence $\{X_n\}_{n \in \mathbb{N}_{>0}} \subseteq L$ *converges on the topological space* $(L, \tau_{\nu 1})$, denoted $X_n \to_{\tau_{\nu 1}} X$, iff all but finitely many members of this sequence in every neighborhood $\mathcal{V} \in \mathcal{N}^{\nu 1}_X$ whereby the *neighborhood system* at X is defined as

$$\mathcal{N}^{\nu 1}_X = \{\mathcal{V} \subseteq L \mid \mathcal{U} \subseteq \mathcal{V} \text{ for some } \mathcal{U} \in \tau_{\nu 1} \text{ with } X \in \mathcal{U}\}. \tag{5}$$

A sequence $\{X_n\}_{n \in \mathbb{N}_{>0}} \subseteq L$ *converges in* ν *to* X *with probability-one coincidence*, denoted $X_n \to_{\nu 1} X$, iff for every $c > 0$

$$\lim_{n \to \infty} \nu(|X_n - X| < c) = 1. \tag{6}$$

If the $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X)$ are open sets $\mathcal{U} \in \tau_{\nu 1}$ for all $m \in \mathbb{N}_{>0}$, the neighborhood system (5) becomes

$$\mathcal{N}^{\nu 1}_X = \left\{\mathcal{V} \subseteq L \mid B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{V} \text{ for some } m \in \mathbb{N}_{>0}\right\}.$$

In that case, convergence on $(L, \tau_{\nu 1})$ to some $X \in L$ is equivalent to convergence on all the $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls at X, which is the same convergence concept as (6). Theorem 1 therefore implies the following result.

Corollary 1. *If ν is continuous from below and dual-autocontinuous from above, we have*

$$X_n \to_{\tau_{\nu 1}} X \text{ if and only if } X_n \to_{\nu 1} X. \tag{7}$$

3.2 Proof of Theorem 1

Part (ii) of Theorem 1 follows from Part (i) (cf. Aliprantis and Border 2006, p.27). We prove Part (i) of Theorem 1 through a string of propositions. We start out with the following obvious fact.

Proposition 1. *Suppose that $\tau_{\nu 1}$ satisfies the following condition for all $c, \epsilon \in \mathbb{Q}_{>0}$ and all $X \in L$:*

Condition YX: *If $Y \in B^{\nu 1}_{c,\epsilon}(X)$, then there exists some $m \in \mathbb{N}_{>0}$ such that*

$$B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(Y) \subseteq B^{\nu 1}_{c,\epsilon}(X). \tag{8}$$

Then all (c, ϵ)-balls are open sets in $\tau_{\nu 1}$.

Proof. By definition, $B^{\nu 1}_{c,\epsilon}(X)$ is an open set in $\tau_{\nu 1}$ iff for every $Y \in B^{\nu 1}_{c,\epsilon}(X)$

$$B^{\nu 1}_{c,\epsilon}(X) \in \mathcal{W}^{\nu 1}_{Y} = \left\{ \mathcal{V} \subseteq L \mid B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(Y) \subseteq \mathcal{V} \text{ for some } m \in \mathbb{N}_{>0} \right\}. \tag{9}$$

Letting $B^{\nu 1}_{c,\epsilon}(X) = \mathcal{V}$ in (9) shows that $B^{\nu 1}_{c,\epsilon}(X)$ is an open set in $\tau_{\nu 1}$ if (8) is satisfied for all $Y \in B^{\nu 1}_{c,\epsilon}(X)$.□□

Next we identify a continuity condition which ensures that Condition YX is always satisfied.

Proposition 2. *Suppose that ν satisfies the following Continuity Condition for all $X, Y, Z \in L$:*

Condition CC: *Let*

$$\nu(|X - Y| < c) > 0.$$

For any $\delta > 0$, there exists some $m_\delta \in \mathbb{N}$ such that, for all $Z \in L$,

$$\nu\left(|Y - Z| < \frac{1}{m_\delta}\right) > 1 - \frac{1}{m_\delta}$$

implies

$$\nu(|X - Y| < c) - \delta < \nu(|X - Y| + |Y - Z| < c).$$

Then $\tau_{\nu 1}$ satisfies Condition YX for all $c, \epsilon \in \mathbb{Q}_{>0}$ and all $X \in L$.

Proof. Let $Y \in B^{\nu 1}_{c,\epsilon}(X)$ be arbitrarily given so that

$$\nu(|X-Y|<c) > 1-\epsilon.$$

Suppose that Condition CC holds so that for any

$$\delta \in (0, \nu(|X-Y|<c) - (1-\epsilon)), \tag{10}$$

there exists $m_\delta \in \mathbb{N}$ such that, for all $Z \in L$,

$$\nu\left(|Y-Z| < \frac{1}{m_\delta}\right) > 1 - \frac{1}{m_\delta}$$

implies

$$\nu(|X-Y|<c) - \delta < \nu(|X-Y| + |Y-Z| < c).$$

Because of

$$(|X-Y| + |Y-Z| < c) \subseteq (|X-Z| < c),$$

we have, by monotonicity of ν, for all $Z \in B^{\nu 1}_{\frac{1}{m_\delta},\frac{1}{m_\delta}}(Y)$,

$$1-\epsilon < \nu(|X-Y|<c) - \delta \leq \nu(|X-Y| + |Y-Z| < c) \leq \nu(|X-Z|<c).$$

Consequently, $B^{\nu 1}_{\frac{1}{m_\delta},\frac{1}{m_\delta}}(Y) \subseteq B^{\nu 1}_{c,\epsilon}(X)$, i.e., Condition YX holds.□□

Proposition 3. *Suppose that ν is continuous from below as well as dual-autocontinuous from above. Then Condition CC holds for any $L \subseteq L^0$.*

Proof. Step 1. Observe that we have, for all $n \in \mathbb{N}_{>0}$,

$$\left(\left(|X-Y| < c - \frac{1}{n}\right) \cap \left(|Y-Z| < \frac{1}{n}\right)\right) \subseteq (|X-Y| + |Y-Z| < c).$$

Define the increasing sequence $\{A_n\}_{n\in\mathbb{N}_{>0}} \subseteq \mathcal{F}$ such that

$$A_n = \left(|X-Y| < c - \frac{1}{n}\right).$$

By monotonicity of ν, we have

$$\nu\left(A_n \cap \left(|Y-Z| < \frac{1}{n}\right)\right) \leq \nu(|X-Y| + |Y-Z| < c) \tag{11}$$

for all $n \in \mathbb{N}_{>0}$. By *continuity from below*, we have

$$\begin{aligned}\lim_{n\to\infty} \nu(A_n) &= \nu\left(\bigcup_{n\in\mathbb{N}_{>0}} A_n\right) \\ &= \nu(|X-Y|<c)\end{aligned}$$

so that there exists, for every $\varepsilon > 0$, some n_ε such that

$$\nu\left(|X-Y| < c - \frac{1}{n_\varepsilon}\right) > \nu\left(|X-Y| < c\right) - \varepsilon.$$

Fix any $\delta^* > 0$ and let $\varepsilon = \frac{\delta^*}{2}$.

Step 2. By *dual-autocontinuity from above*, there exists for every $n_\varepsilon \in \mathbb{N}_{>0}$ and $\delta > 0$ some n_δ such that

$$\nu\left(|Y-Z| < \frac{1}{n_\varepsilon}\right) > 1 - \frac{1}{n_\delta}$$

implies

$$\begin{aligned} \nu\left(|X-Y| < c\right) - \varepsilon - \delta &< \nu\left(|X-Y| < c - \frac{1}{n_\varepsilon}\right) - \delta \\ &< \nu\left(\left(|X-Y| < c - \frac{1}{n_\varepsilon}\right) \cap \left(|Y-Z| < \frac{1}{n_\varepsilon}\right)\right). \end{aligned}$$

Let $\delta = \frac{\delta^*}{2}$ to obtain, by (11), that

$$\nu\left(|Y-Z| < \frac{1}{n_\varepsilon}\right) > 1 - \frac{1}{n_\delta}$$

implies

$$\nu\left(|X-Y| < c\right) - \delta^* < \nu\left(|X-Y| + |Y-Z| < c\right).$$

Choosing some $m_{\delta^*} \geq \max\left\{n_\varepsilon, n_\delta\right\}$ gives us Condition CC, i.e.,

$$\nu\left(|Y-Z| < \frac{1}{m_{\delta^*}}\right) > 1 - \frac{1}{m_{\delta^*}}$$

implies

$$\nu\left(|X-Y| < c\right) - \delta^* \quad < \quad \nu\left(|X-Y| + |Y-Z| < c\right).$$

□□

Combining the Propositions 1-3 gives us part (i) of Theorem 1. Because all open sets $\mathcal{U} \in \tau_{\nu 1}$ with $X \in \mathcal{U}$ are neighborhoods of X, a combination of Theorem 1 with the following proposition gives us Corollary 1.

Proposition 4.

(i) *For all ν we have that*

$$X_n \to_{\nu 1} X \text{ implies } X_n \to_{\tau_{\nu 1}} X. \tag{12}$$

(ii) *If every $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X)$, $m \in \mathbb{N}_{>0}$, is a neighborhood of X for all $X \in L$ in the topology $\tau_{\nu 1}$, we have that*

$$X_n \to_{\tau_{\nu 1}} X \text{ implies } X_n \to_{\nu 1} X. \tag{13}$$

Proof. Part (i). By the weak base construction of $\tau_{\nu 1}$, there exists for every $\mathcal{U} \in \tau_{\nu 1}$ with $X \in \mathcal{U}$ some m such that $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{U}$. Fix $\mathcal{U} \in \tau_{\nu 1}$ with corresponding $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{U}$ and suppose that $X_n \to_{\nu 1} X$. If $\{X_n\}$ is eventually in every $\left(\frac{1}{m},\frac{1}{m}\right)$-ball, it must eventually be in $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X)$ and therefore in every neighborhood $\mathcal{V}$ of X such that $\mathcal{U} \subseteq \mathcal{V}$. Since this argument applies to every $\mathcal{U} \in \tau_{\nu 1}$ with $X \in \mathcal{U}$, $\{X_n\}$ will be eventually in every neighborhood of X. This gives us the convergence behavior (12).

Part (ii). Suppose now that $X_n \to_{\tau_{\nu 1}} X$. If $\{X_n\}$ is eventually in every neighborhood of X, it will be eventually in every $B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X)$, $m \in \mathbb{N}_{>0}$, whenever all $\left(\frac{1}{m},\frac{1}{m}\right)$-balls at X are neighborhoods of X. This gives us the convergence behavior (13).□□

4 Properties of open subsets of random variables if L has a local cone structure and $(\Omega, \mathcal{F}, \nu)$ is dual-nonatomic

4.1 Second main result: Theorem 2

Our second main result concerns topological spaces $(L, \tau_{\nu 1})$ under the assumptions that (i) L satisfies specific structural properties and (ii) the space $(\Omega, \mathcal{F}, \nu)$ is *dual-nonatomic*.

- $(\Omega, \mathcal{F}, \nu)$ is *nonatomic* iff there exists for every $\epsilon > 0$ some finite partition $\{\Omega_1, ..., \Omega_n\} \subseteq \mathcal{F}$ such that
$$\nu(\Omega_i) < \epsilon \text{ for all } i \in \{1, ..., n\}.$$
- $(\Omega, \mathcal{F}, \nu)$ is *dual-nonatomic* iff there exists for every $\epsilon > 0$ some finite partition $\{\Omega_1, ..., \Omega_n\} \subseteq \mathcal{F}$ such that
$$\nu(\Omega_i^c) > 1 - \epsilon \text{ for all } i \in \{1, ..., n\}.$$

For additive probability measures nonatomicity and dual-nonatomicity are equivalent. The standard example of an additive nonatomic (i.e., dual-nonatomic) probability space is $((0,1), \mathcal{B}, \lambda)$ where $\mathcal{B}$ stands for the Borel-sigma algebra defined on the open unit interval $(0,1)$ and λ is the Lebesgue measure (cf. Problem 2.19(a) in Billingsley 1996).

Next consider the following structural properties of L.

Structural properties of L.

(P1) L is closed under *additivity*, i.e.,
$$X, Y \in L \text{ implies } X + Y \in L.$$

(P2) L is closed under *multiplication with natural numbers*, i.e.,
$$X \in L \text{ implies } nX \in L \text{ for all } n \in \mathbb{N}_{>0}.$$

(P3) L is closed under *locality* in the following sense: for any $A \in \mathcal{F}$,

$$X \in L \text{ implies } X 1_A \in L$$

where 1_A denotes the indicator function.

(P4) L is *non-trivial* if there exists some $Z \in L$ such that, for some $\delta > 0$,

$$|Z(\omega)| \geq \delta \text{ for all } \omega \in \Omega. \tag{14}$$

Theorem 2. *Suppose that $(\Omega, \mathcal{F}, \nu)$ is dual-nonatomic. Consider a topological space $(L, \tau_{\nu 1})$ such that L satisfies the Structural Properties P1, P2, and P3.*

(i) *We have for any open set $\mathcal{U} \in \tau_{\nu 1}$ that*

$$X \in \mathcal{U} \text{ implies } X + Z \in co(\mathcal{U}) \text{ for all } Z \in L.$$

where $co(\mathcal{U})$ denotes the convex hull of $\mathcal{U}$.

(ii) *In particular, we have for any convex and open $\mathcal{U} \in \tau_{\nu 1}$ that*

$$X \in \mathcal{U} \text{ implies } X + Z \in \mathcal{U} \text{ for all } Z \in L. \tag{15}$$

Before we prove Theorem 2, let us demonstrate that Theorem 2 comes with powerful implications.

Corollary 2. *Suppose that $(\Omega, \mathcal{F}, \nu)$ is dual-nonatomic and that L satisfies the Structural Properties P1, P2, and P3. If L additionally satisfies the non-triviality condition P4, then there cannot exist any open ball $B^{\nu 1}_{c,\epsilon}(X)$ in $(L, \tau_{\nu 1})$ that is also a convex strict subset of L.*

Proof. Suppose to the contrary that $B^{\nu 1}_{c,\epsilon}(X) \subsetneq L$ is open and convex. If $Z \in L$, we also have $nZ \in L$ for all $n \in \mathbb{N}_{>0}$. By (15), $X + nZ \in B^{\nu 1}_{c,\epsilon}(X)$ which is equivalent to

$$\begin{aligned} \nu(|X - (X + nZ)| < c) &> 1 - \epsilon \\ &\Leftrightarrow \\ \nu(|nZ| < c) &> 1 - \epsilon \end{aligned} \tag{16}$$

whereby $B^{\nu 1}_{c,\epsilon}(X) \subsetneq L$ requires $\epsilon \leq 1$. Pick some $n \geq \frac{c}{\delta}$ and observe that

$$\begin{aligned} \nu(|nZ| < c) &\leq \nu\left(\frac{c}{\delta}|Z| < c\right) \\ &= \nu(|Z| < \delta) = \nu(\emptyset) \text{ by (14)} \\ &= 0, \end{aligned}$$

which is a contradiction to (16).□□

Recall that *convex cones* are closed under positive linear combinations, i.e.,

$$X, Y \in L \text{ implies } aX + bY$$

for $a, b \in \mathbb{R}_{>0}$. Consequently, all subsets of random variables that are convex cones satisfy the Structural Properties P1 and P2. We call L a *non-trivial local cone* whenever L is a convex cone that additionally satisfies Structural Properties P3 and P4. The set of all random variables L^0 is a non-trivial local cone. Other relevant subdomains of random variables that are non-trivial local cones are the sets of all non-negative and all non-positive random variables, respectively, defined as follows

$$\begin{aligned} L^0_+ &= \left\{ Y \in L^0 \mid 0 \leq Y(\omega) \text{ for all } \omega \in \Omega \right\}, \\ L^0_- &= \left\{ Y \in L^0 \mid Y(\omega) \leq 0 \text{ for all } \omega \in \Omega \right\}. \end{aligned}$$

Recall from Theorem 1 that the (c, ϵ)-balls (1) are open sets in $(L, \tau_{\nu 1})$ whenever ν is continuous from below and uniformly dual-autocontinuous from above. In this case, any (c, ϵ)-ball on $(L, \tau_{\nu 1})$, $L \in \left\{ L^0, L^0_+, L^0_- \right\}$, that is a strict subset of L cannot be a convex set by Corollary 2.

Next, let us write $X \leq Y$ iff, for every $\omega \in \Omega$, either $X(\omega) = Y(\omega)$ or $X(\omega) < Y(\omega)$. Applying Theorem 2 to the non-trivial local cones L^0, L^0_+, and L^0_-, respectively, gives us the following properties of convex and open sets for the corresponding topological spaces.

Corollary 3. *Suppose that $(\Omega, \mathcal{F}, \nu)$ is dual-nonatomic. Denote by $\mathcal{U}$ any convex and open set in the topological space $(L, \tau_{\nu 1})$ such that $X \in \mathcal{U}$.*

(i) *For $L = L^0$ we have that $Y \in \mathcal{U}$ for all $Y \in L^0$.*

(ii) *For $L = L^0_+$ we have that $Y \in \mathcal{U}$ for all $Y \in L^0_+$ such that $X \leq Y$.*

(iii) *For $L = L^0_-$ we have that $Y \in \mathcal{U}$ for all $Y \in L^0_-$ such that $Y \leq X$.*

By Corollary 3(i), the only non-empty, open and convex subset of the topological space $(L^0, \tau_{\nu 1})$ is the set of all random variables L^0 itself. This implication of Theorem 2 extends a well-known result from the analysis of *not locally convex* L^p-spaces, $0 \leq p < 1$–which are defined on some nonatomic additive probability space–to non-additive dual-nonatomic probability spaces.[3]

4.2 Proof of Theorem 2

We start with a lemma.

[3] Compare., e.g., Theorem 13.41(3) in Aliprantis and Border (2006), Paragraph 1.47 in Rudin (1991), Theorem 1 in Day (1940).

Lemma 1. *Consider a topological space $(L, \tau_{\nu 1})$ such that L satisfies the Structural Properties P1 and P3. Then there exists for every open set $\mathcal{U} \in \tau_{\nu 1}$ with $X \in \mathcal{U}$ some $m \in \mathbb{N}_{>0}$ such that, for all $Y \in L$,*

$$\nu\left(A^c\right) > 1 - \frac{1}{m} \text{ implies } X + Y 1_A \in \mathcal{U}. \tag{17}$$

Proof. If L satisfies the Structural Properties P1 and P3, we have $X + Y1_A \in L$ for any $X, Y \in L$ and any $A \in \mathcal{F}$. Let $X \in \mathcal{U}$ for an arbitrary open set $\mathcal{U}$. By the weak base construction of $\tau_{\nu 1}$, there exists some $m \in \mathbb{N}_{>0}$ such that

$$B^{\nu 1}_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{U}.$$

Next observe that, for any $\frac{1}{m} > 0$,

$$\begin{aligned} \nu\left(\left|X + Y1_A - X\right| < \frac{1}{m}\right) &= \nu\left(\left|Y1_A\right| < \frac{1}{m}\right) \\ &\geq \nu\left(A^c\right) \end{aligned} \tag{18}$$

because

$$\begin{aligned} \omega &\in A^c \text{ implies } \left|Y1_A(\omega)\right| = 0 < \frac{1}{m} \\ &\Rightarrow \\ A^c &\subseteq \left(\left|Y1_A\right| < \frac{1}{m}\right). \end{aligned}$$

By (18), $\nu\left(A^c\right) > 1 - \frac{1}{m}$ implies

$$\begin{aligned} \nu\left(\left|X + Y1_A - X\right| < \frac{1}{m}\right) &> 1 - \frac{1}{m} \\ &\Leftrightarrow \\ X + Y1_A &\in B^{\nu 1}_{\frac{1}{m},\frac{1}{m}} \subseteq \mathcal{U} \end{aligned}$$

for any $Y \in L$.□□

Proof of Theorem 2. Fix $X \in L$. If ν is dual-nonatomic, there exists for every $m \in \mathbb{N}_{>0}$ some partition $\{\Omega_1, ..., \Omega_n\} \subseteq \mathcal{F}$ such that, for every $i = 1, ..., n$,

$$\nu\left(\Omega_i^c\right) > 1 - \frac{1}{m}. \tag{19}$$

If L satisfies the Structural Property P2, we have that $nZ \in L$ for any $Z \in L$. Because L also satisfies the Structural Properties P1 and P3, we obtain, by Lemma 1, that there exists for every $\mathcal{U} \in \tau_{\nu 1}$ with $X \in \mathcal{U}$ some m such that (19) implies, for every $i = 1, ..., n$,

$$X + nZ1_{\Omega_i} \in \mathcal{U}.$$

By construction,

$$X + Z = \sum_{i=1}^{n} \frac{1}{n}\left(X + nZ1_{\Omega_i}\right),$$

which gives us the desired result $X + Z \in co\left(\mathcal{U}\right)$.□□

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