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A maximum likelihood estimation approach for spliced distributions obtained through quantile splicing

by

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Abstract

This mini-dissertation proposes constructing a family of spliced distributions at a point different from the median, hence $k = \frac{1}{4}$ instead of $k = \frac{1}{2}$, using the method of quantile splicing proposed by Mac'Oduol et al. (2020). General results of these families of distributions are developed and the maximum likelihood approach is explored and investigated for estimation purposes. Moreover, a numerical application is presented in order to illustrate the implementation and application of the proposed method.

Declaration of originality

I, *Jeanne-Louise van der Sande,* declare that this mini-dissertation, submitted in partial fulfillment of the degree *MSc Advanced Data Analytics,* at the University of Pretoria, is my own work and has not been previously submitted at this or any other tertiary institution.

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Contents

1 Introduction:

1.1 Overview

The quantile function, also referred to as the inverse CDF, can be used to define quantile-based distributions, since closed-form expressions for the probability density functions (PDFs) and/or cumulative distribution functions (CDFs) are difficult to obtain. Such distributions include the logistic, cosine, normal, uniform and Student's $t(2)$ distribution.

Hastings Jr et al. (1947) and Tukey (1960) applied quantile-based methods to the lambda distribution by joining their quantile functions together. The generalized lambda distributions (GLDs) arose from there by generalizing the results obtained by Ramberg and Schmeiser (1972) and Ramberg and Schmeiser (1974), for the lambda distributions.

Various techniques have since arisen with the objective of generating asymmetric distributions or skewing the existing distributions. These methods rely on whether the distributions have existing CDF, PDF, or quantile functions, and intend on increasing the flexibility of these distributions to provide a better model to fit the data.

A skew logistic model was developed by Balakrishnan et al. (2017) where the PDF of the half logistic distribution to the left of its location parameter was chosen, which includes a single positive shape parameter. This was then joined to the PDF of the half logistic distribution to the right of the location parameter. This proposed skew logistic's single and product moments were derived using the order statistics and moments of the half logistic distribution. Thereafter, the properties of the skew logistic distribution's order statistics were calculated for any sample size n, based on the aforementioned results.

Subsequently, the development of asymmetric families of distributions was suggested by Mac'Oduol et al. (2020), based on utilizing quantile functions of symmetric univariate distributions as kernels. As a result, the general formula for the L-moments of a two-piece distribution was derived. The method of L-moments can be utilised to obtain the parameter estimates of the new family of distributions. This reduces the computational difficulty encountered when single and product moments are used for estimation.

The method of maximum likelihood estimation has not yet been explored for quantile splicing. The principle of maximum likelihood estimation is defined as selecting values from our sample that are most likely to be observed as the parameter estimates, before any observations have been made. The log of the likelihood is often easier to maximize, and equates to the same estimate, since the log function is strictly increasing, as stated in Pan and Fang (2002).

Maximum likelihood (ML) is a very reliable method that generates satisfactory estimates when there are large samples, however these estimators can be bias. As the sample size increases, the bias decreases, so this can be resolved. The disadvantage of using ML estimation is that the error terms need to emanate from a particular, stated distribution.

1.2 Aims and objectives:

The general formula in Mac'Oduol et al. (2020) will be used in this mini-dissertation to create a family of spliced distributions, at the point $k = \frac{1}{4}$. The general form of the PDF, CDF and quantile functions will then be derived, before investigating the properties of the spliced distributions. Thereafter, the main goal is to acquire the maximum likelihood approach for spliced families of distributions, since the method of L-moments estimation has already been applied by Mac'Oduol et al. (2020). Lastly, the developed maximum likelihood estimate results will be applied and tested with real data to measure the efficiency of this estimation method.

1.3 Outline of the dissertation:

In Chapter 2, an extensive overview is given for the introduction of families of distributions. The skewing methods for general distributions is discussed with examples of such distributions and their properties. Thereafter, the beta-generated distributions and it's special cases are considered. The properties of this family of distributions are also discussed.

Chapter 3 introduces two-piece families of distributions and introduces the method quantile splicing. The general form of the CDF, PDF and quantile function of the two-piece families of distributions is given. The general forms will be related to existing two-piece families of distributions, stating their CDF, PDF and quantile functions and the properties of these distributions. The general form of the CDF, PDF and quantile functions of spliced distributions will be given. Thereafter two-piece logistic, cosine and Student's $t(2)$ distributions, which have been spliced at $k = \frac{1}{2}$, are given. The general form of the r^{th} order L-moments for the two-piece family of distributions is given, as well as those of the two-piece logistic, cosine and Student's $t(2)$ distributions, when $k = \frac{1}{2}$.

Chapter 4 makes use of the results obtained in Chapter 3 for the PDF, CDF and quantile functions for the two-piece family of distributions, by replacing $k = \frac{1}{4}$. The quantile measures and the distributional forms are proposed, and examples of two-piece families of distributions, when $k = \frac{1}{4}$ will be considered. The method of L-moments estimation will also be discussed.

In Chapter 5, the method of maximum likelihood estimation for quantile-based functions will be discussed. The theoretical results are then applied to the two-piece logistic, cosine and Student's $t(2)$ distributions, to obtain MLE estimates.

In Chapter 6, the method of L-moments estimation and the method of maximum likelihood estimation for quantile-based distributions are applied to a dataset to obtain parameter estimates and compare the fit of the models to the data.

Chapter 7 summarises the different techniques discussed as well as the results obtained in this mini-dissertation.

1.4 Contributions of the dissertation:

The contributions per chapter, are outlined below.

Chapter 4

- The general formulae for the CDF, PDF and quantile functions of two-piece distributions is used to derive the results for the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{4}$, in Chapter 4.3.1, 4.3.2 and 4.3.3.

- The quantile-based measures of distributional form for the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{4}$, are derived in Chapter 4.3.1, 4.3.2 and 4.3.3.

- The r^{th} order L-moments are derived using the general formula, for the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{4}$, in Chapter 4.3.1, 4.3.2 and 4.3.3. These results are derived in full in the Appendix of this mini-dissertation.

- The L-skewness and L-kurtosis ratios for the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{4}$, are obtained and plotted in Chapter 4.3.1, 4.3.2 and 4.3.3.

Chapter 5

- The log-likelihood functions and the partial and second derivatives of the log-likelihood function for the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{2}$ and when $k = \frac{1}{4}$, are derived in Chapter 5.3.1, 5.3.2 and 5.3.3.

2 Literature review:

Statistical distributions and their developments have been monitored for years prior to Vicari and Kotz (2005) examining the initial build-out of statistical distributions. The normal distribution was discovered by de Moivre (1733), whereas the normal probability integral $\Phi(x)$ was suggested, in tabular form by Laplace (1774), after he discovered that the normal distribution was an approximation of the hypergeometric distribution.

Thereafter, estimation methods based on the normal distribution were formulated by Gauss (1809) and by Gauss (1816). Pearson used a method of higher moments and attained a generalization of the normal distribution, which originated from his work in Pearson (1895). He also derived the Pearson curves, which are still being used to date, to visualize probability transformations for a variety of shapes. The method of differential equations was also used by Pearson (1895) to generate statistical distributions for nonsymmetric data, as referred to in Lee et al. (2013).

The Pearson system of continuous distributions applies to every PDF, $f(x)$, that satisfies the following differential equation

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{a+x}{b_0+b_1x+b_2x^2}, \tag{1}$$

where the location, scale and shape parameters of the PDF depends on the constants a, b_0, b_1 and b_2 and $x \in \mathbb{R}$.

Hereafter, numerous methods were introduced to create families of distributions, such as the translation method by Johnson (1949). Johnson suggested generating distributions using a system that utilized the method of normalization transformation. This resulted in a random variable which has the general form

$$Z = \gamma + \delta f\Big(\frac{X-\xi}{\lambda}\Big), \quad \delta > 0,\ \lambda > 0, \tag{2}$$

where $f(\cdot)$ is defined as the transformation function, Z represents a standardized normal random variable and $\gamma, \xi \in \mathbb{R}$.

The following families were proposed as transformation functions:

- The lognormal family (S_L):

$$Z = \gamma + \delta \ln(X-\xi), \tag{3}$$

  where $X \geq \xi$.

- The family of bounded distributions (S_B):

$$Z = \gamma + \delta \ln\Big(\frac{X-\xi}{\xi+\lambda-X}\Big), \tag{4}$$

where $\xi \leq X \leq \xi + \lambda$.

- The family of unbounded distributions (S_U):

$$Z = \gamma + \delta \times \ln\Bigg(\Big(\frac{X-\xi}{\lambda}\Big) + \Bigg[\Big(\frac{X-\xi}{\lambda}\Big)^2 + 1\Bigg]^{\frac{1}{2}}\Bigg)$$
$$= \gamma + \delta \sinh^{-1}\Big(\frac{X-\xi}{\lambda}\Big), \tag{5}$$

where $-\infty < X < \infty$.

Thereafter, univariate distributions were derived from the above mentioned families of distributions, namely the normal, gamma, lognormal, exponential and beta distributions.

The rest of this chapter takes an in-depth look at the different methods used to introduce flexibility to distributions with respect to their distributional forms.

2.1 Skewing methods for general distributions

2.1.1 Azzalini's skewing method

Azzalini (1985) suggested constructing a skew distribution by combining two symmetric distributions, namely the skew normal family of distributions.

Lemma 1. *Suppose X is a random variable with a PDF that is symmetric about* 0 *and Y be a random variable with CDF $G(.)$, such that $G(.)$ is absolutely continuous with a symmetric first derivative. Then it follows from Lee et al. (2013) that*

$$\frac{1}{2} = P(Y - \lambda X < 0)$$
$$= E_X[P(Y < \lambda X | X = x)]$$
$$= \int_{-\infty}^{\infty} f_0 \; G(\lambda x) dy, \tag{6}$$

where $\lambda \in \mathbb{R}$. ■

From Eq.(6) it follows that

$$2 f_0 G(\lambda x) = 1, \tag{7}$$

where $-\infty < x < \infty$.

Corollary 1. *Suppose that $X \sim N(0,1)$, then it follows that the random variable X_λ follows a skew normal distribution, denoted by $X_\lambda \sim SN(\lambda)$, where λ represents the asymmetry parameter. The PDF is given as*

$$f_{X_\lambda}(x;\lambda) = 2\psi(x)\Phi(\lambda x), \tag{8}$$

where $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\psi(x)$ and $\Phi(x)$ represent the $N(0,1)$ PDF and CDF, respectively. ■

Properties of the skew-normal (SN(λ)) distribution:

Let X_λ represent a $SN(\lambda)$ random variable. Let $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and assume that the random variables A and B are independent standard normal random variables,

- $SN(0) \sim N(0,1)$.
- $|X_\lambda| \cong |Z|$.
- $X_\lambda^2 \sim \chi_1^2$.
- $-X_\lambda \sim SN(-\lambda)$.
- $M_\lambda(t) = 2e(\frac{t^2}{2})\Phi(\delta t)$ is the moment generating function of X_λ.
- The conditional distribution of A given $B < \lambda A$ follows as SN(λ).
- $$\begin{aligned} X_\lambda &\cong \delta\ |A| + \sqrt{(1-\delta^2)}B \\ &\cong \delta\ A(0) + \sqrt{(1-\delta^2)}B, \end{aligned}$$

  where $A(0)$ denotes that A is truncated below 0.
- $X_\lambda \xrightarrow{d} |Z|$ as $\lambda \to \infty$ and $X_\lambda \xrightarrow{d} -|Z|$ as $\lambda \to -\infty$.
- Let X be a random variable with PDF, $g(x)$, then $X^2 \sim \chi^2$ if and only if a skewed function exists, $\pi(x)$ such that $f(x;\lambda) = 2\phi(x)\Phi(\lambda x)$ is true.
- Suppose X and Y are independent and identically distributed $SN(\lambda)$ random variables. $F(\cdot)$ denotes the CDF of two independent and identically distributed (*iid*) random variables, X_1 and X_2, which both have finite moments. It then follows that $F(\cdot)$ is skew-normally distributed if and only if $X_1^2 \sim \chi^2$ and $X_2^2 \sim \chi^2$ and if $(X_1+X_2)^2 \doteq (X+Y)^2$.

Another method used to create skewed distributions by using symmetric random variables is observed in Lee et al. (2013), by setting $f_0(\cdot)$ as the PDF of a symmetric random variable X and $\pi \in [0,1]$ representing the skewing function, such that $\pi(x) + \pi(-x) = 1$. Therefore,

$$f(x) = f_0(x)\pi(x) \tag{9}$$

is considered to be a valid PDF.

2.1.2 Weighted skewing approach

Chang and Genton (2007) suggested a weighted skewing approach which generates a skew symmetric family of distributions. The PDF is manipulated by a non-negative and multiplicative weighting function, after which the observed data follows as a random sample originating from a weighted distribution.

Definition 1. *Suppose X is a symmetric random variable with PDF $f(x;\beta)$, where $w(x;\beta,\alpha)$ is a weight function, with α and β defined as an unknown parameter. Then the general form of the PDF from the weighted distribution follows from Lee et al. (2013) as*

$$g(x;\beta,\alpha) = f(x;\beta)\frac{w(x;\beta,\alpha)}{E(w(x;\beta,\alpha))}. \tag{10}$$

■

Remark. *When $E(w(x;\beta,\alpha)) = \frac{1}{2}$, Eq.(10) is the PDF of a skewed symmetric distribution represented by the form given in Eq.(9).*

The epsilon-$SN(\lambda,\epsilon)$ distribution from Mudholkar and Hutson (2000) introduces an additional shape parameter, $\epsilon \in [0,1)$, to control the skewness. This family of distributions has a PDF given by

$$g(x) = \begin{cases} \psi\Big(\dfrac{x}{1+\epsilon}\Big), & x < 0, \\ \psi\Big(\dfrac{x}{1+\epsilon}\Big), & x \geq 0, \end{cases} \tag{11}$$

where $x \in \mathbb{R}$ and $\psi(\cdot)$ is the $N(0,1)$ CDF.

The above Epsilon-SN and SN family from Eq.(8) were combined to form the extended family of skew distributions in Salinas et al. (2007) with the following PDF,

$$f(x|\lambda,\beta) = \begin{cases} 2f_0\Big(\dfrac{x}{1+\beta}\Big)\Big[\dfrac{\beta}{1+\beta} + \Big(\dfrac{1-\beta}{1+\beta}\Big)G\Big(\dfrac{\lambda x}{1+\beta}\Big)\Big] & x < 0, \\ 2f_0\Big(\dfrac{x}{1-\beta}\Big)G\Big(\dfrac{\lambda x}{1+\beta}\Big) & x \geq 0, \end{cases} \tag{12}$$

where $f_0(\cdot)$ and $G(\cdot)$ are defined as in Eq.(7), whilst $\lambda \in \mathbb{R}$ and $\beta \in [0,1)$ are skewness parameters.

Salinas et al. (2007) introduced the extended skew-exponential power distribution by setting $f_0(\cdot)$ as a symmetric exponential power density with $G(\cdot)$ defined as the CDF of a normal density such that $G'(\cdot)$ is symmetric, including scale and location parameters.

2.1.3 The inverse scaling method

Fernández and Steel (1998) proposed a skewing method that can be implemented on any unimodal, symmetric and continuous distribution. This method proposed the inverse scaling of the PDF, of any continuous distribution, on both sides of the mode. The unimodality is unaffected and allows for increased flexibility with respect to the shape of the distribution with only one scalar parameter.

Definition 2. *Let $f(\cdot)$ represent the PDF of a unimodal distribution symmetric about 0. The PDF is defined by Lee et al. (2013) as*

$$g(x) = \begin{cases} cf(\alpha x), & x \geq 0, \\ cf(\dfrac{x}{\alpha}), & x < 0, \end{cases} \tag{13}$$

where $\alpha, c > 0$. ■

Whenever $\alpha \neq 0$ and $\frac{g(x \geq 0|\alpha)}{g(x<0|\alpha)}$, then this distribution becomes skewed. It can be noted that α controls the mass of the probability on either side of the mode, whilst 'c' serves as a normalizing factor that ensures $g(x)$ is a valid PDF and is defined as $c = \frac{2\alpha}{(1+\alpha^2)}$.

Fernández and Steel (1998) suggested the introduction of two parameters to better control the flexibility of the distribution by using α_1 and α_2 instead of α and $\frac{1}{\alpha}$. The following class of skewed distributions follows, with $\gamma \in (0, \infty)$, such that

$$p(\epsilon|\gamma) = \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ f\left(\frac{\epsilon}{\gamma}\right) \, I_{[0,\infty)}(\epsilon) + f(\gamma\epsilon) \, I_{(-\infty,0)}(\epsilon) \right\}. \tag{14}$$

2.1.4 The transformation of the inverse probability integral

Skewness is introduced to symmetric, univariate distributions via a suggested general framework by Ferreira and Steel (2006), which utilises the transformation of the inverse probability integral.

Lemma 2. *Suppose $f(\cdot)$ and $F(\cdot)$ represent the PDF and CDF of a univariate, symmetric distribution, respectively. Similarly, let $p(\cdot)$ and $P(\cdot)$ represent the PDF and CDF, respectively, of a bounded distribution on $(0, 1)$. The resulting skew family of distributions defined by Ferreira and Steel (2006) has a PDF that takes on the general form given by*

$$g(x|f, p) = f(x)p[F(x)], \; x \in \mathbb{R}, \tag{15}$$

where $g(x|f, p)$ is a weighted version of the PDF $f(x)$, with the weight function $p[F(x)]$. ■

If $p(\cdot)$ originates from the uniform distribution, then $g(\cdot) = f(\cdot)$. The family of distributions, from Eq.(7), can be linked to Eq.(15) if

$$p(y|\lambda) = 2G(\lambda F^{-1}(y))^{-1}. \tag{16}$$

Special cases:

- The inverse scale family of distributions by Fernández and Steel (1998) can be obtained from Eq.(15) when
$$p(y|\alpha) = \frac{2}{\alpha + \frac{1}{\alpha}} \; \frac{f[\alpha^{sign(0.5-y)} \; F^{-1}(y)]}{f[F^{-1}(y)]}, \tag{17}$$
where $sign(\cdot)$ returns the sign of a real number, either positive or negative.

- The beta-generated family of distributions suggested by Eugene et al. (2002) is a special form of Eq.(15) when
$$p(y|\alpha, \beta) = [\beta(\alpha, \beta)]^{-1} \; y^{\alpha-1} \; (1-y)^{\beta-1}. \tag{18}$$

Properties:

- $P(\cdot)$ is independent from $F(\cdot)$.
- If $P(\cdot)$ is uniform, then $P(\cdot)$ leads to a symmetric $G(\cdot)$.
- The mode of $F(\cdot)$ is equal to the unique mode of $G(\cdot)$.
- Skewness exists close to the mode of the distribution such that the tail behaviour on both tails of $G(\cdot)$ are identical.
- The skewness measure is independent of $F(\cdot)$, and is given by $AG = 1 - 2P(X < mode)$.
- An odd function of $P(\cdot)$ exists due to the measure of skewness.

2.2 Beta-generated distributions

The introduction of the use of the beta distribution, as the source of developing beta-generated distributions originated from Eugene et al. (2002). The beta distribution is used by the beta-generated family, to generate distributions that have more parameters, to fit a larger variety of shapes. In this case, the skewness is not controlled by a certain parameter, but by the combined shape parameters.

Definition 3. *A beta-class random variable, X, has CDF that takes on the form*
$$G(x) = \int_0^{F(x)} b(t)dt, \tag{19}$$
where $F(x)$ represents the CDF of any continuous random variable, and $b(t)$ represents the PDF of the beta random variable, given as
$$b(t) = \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha, \beta)}, \tag{20}$$
with $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha,\beta)}$, as defined in Lee et al. (2013). ■

Definition 4. *The PDF of a random variable originating from the beta-class distribution has the following general form*

$$g(x) = \frac{1}{\beta(\alpha,\beta)}\ f(x)\ F^{\alpha-1}(x)\ (1-F(x))^{\beta-1}, \tag{21}$$

where $x \in (0,1)$, $\alpha > 0$ and $\beta > 0$ represent the shape parameters, as defined by Lee et al. (2013). ■

Remark. *The distributions for which order statistics exist for a random variable X, with CDF $F(x)$, is generalized by this family of distributions. Eq.(21) represents the α^{th} order statistic of a random sample with size $(\alpha+\beta-1)$, only if α and β are integers.*

Special cases

The following distributions are examples of the beta-generated distributions, referring to Eq.(19) and Eq.(21):

1. The skew-t distribution by Jones (2001) is an example of a beta-generated distribution with PDF given by

$$f(x;\alpha,\beta) = \Big(\frac{1}{B(\alpha,\beta)\ \sqrt{\alpha+\beta}\ 2^{\alpha+\beta-1}}\Big)\ \Big(1+\frac{x}{\sqrt{\alpha+\beta+x^2}}\Big)^{\alpha+\frac{1}{2}}\Big(1-\frac{x}{\sqrt{\alpha+\beta+x^2}}\Big)^{\beta+\frac{1}{2}}, \tag{22}$$

where $x \in R$, $\beta > 0$ and $\alpha > 0$, with $F(x) = \frac{1+\frac{x}{\sqrt{\alpha+\beta+x^2}}}{2}$.

2. The log-F distribution, or better known as the beta-logistic distribution, has the following PDF

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\ \frac{e^{\alpha x}}{(1+e^x)^{\alpha+\beta}}, \quad -\infty < x < \infty, \tag{23}$$

with $F(x) = \frac{e^x}{(1+e^x)}$, as stated in Jones (2004).

3. The generalized beta-Type I distribution from McDonald (1984) has PDF

$$g(x) = \frac{\alpha\ [(\frac{x}{\beta})^\alpha]^{\alpha-1}[1-(\frac{x}{\beta})^\alpha]^{\beta-1}}{B(\alpha,\beta)}, \quad 0 \le x \le \beta, \tag{24}$$

where $F(x) = (\frac{x}{\beta})^\alpha$.

4. The generalized beta-Type II distribution from Sepanski and Kong (2008) has PDF

$$g(x) = \frac{\alpha\ [(\frac{x}{\beta})^\alpha]^{\alpha-1}}{B(\alpha,\beta)\ [1+(\frac{x}{\beta})^\alpha]^{\alpha+\beta}}, \quad x > 0, \tag{25}$$

where $F(x) = 1 - \frac{1}{[1+(\frac{x}{\beta})^\alpha]}$.

The properties of beta-generated distributions:

Suppose X is a random variable with CDF, $F(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of parameters.

1. Property of transformation:

    If $U \sim Uni(0,1)$, it follows that $X = F^{-1}(U)$. Suppose that $Y \sim \text{Beta}(\alpha, \beta)$ is defined on $(0,1)$. A beta-generated random variable is then defined as $X = F^{-1}(Y)$, with the following parameters $X \sim G_F$ and (θ, α, β).

2. Use the beta-generated distribution to produce a random sample:

    Using the transformation property, one can simply generate a random sample from a beta-generated distribution by first generating a random sample from a $\text{Beta}(\alpha, \beta)$ distribution, and then using the inverse CDF (quantile function) to obtain the beta-generated distribution values.

3. Skewness:

    (a) If $f(x)$ is symmetric, it follows that $g(x)$ is also symmetric, only when $\alpha = \beta$.

    (b) When $\alpha > \beta$, it follows that $g(x)$ is skewed to the right, whereas $g(x)$ is skewed to the left when $\alpha < \beta$.

4. Tail weight:

    The tail weight is a concept that measures the behaviour of the shape of a distribution, with respect to it's extreme values, as stated by Dato (2017).

    (a) When $\alpha, \beta < 1$, then $f(x)$ has heavy symmetric tails. Bimodality occurs when α or β decreases.

    (b) When $\alpha, \beta > 1$, then $f(x)$ has long symmetric tails with a greater peak, which occurs at larger values of α or β.

    (c) If it is true that $f(\cdot)$ has power tails, such that $f \sim x^{-(\gamma+1)}$, with $\gamma > 0$ and x containing large values, then it follows from Jones (2004) that the tails of $g(x)$ behave as $g \sim x^{-\beta\gamma-1}$. An example of this occurring is the skew-t distribution, where $\gamma = 2$. As $\beta \to 0$, the tail weight increases rapidly towards the limit of x^{-1}, no matter the value of γ.

    (d) If $f(\cdot)$ consists of exponential tails, such that $f \sim e^{-\lambda x}$, where $\lambda > 0$, it follows that $g(x) \sim e^{-\beta\lambda x}$. The log-$F$ distribution falls within this category.

    (e) If the tails of $f(\cdot)$ are normal, such that $f \sim e^{-\frac{x^2}{2}}$, with $(1-F) \sim \frac{f(\cdot)}{x}$ then it follows that $g(x) \sim \frac{e^{-\frac{\beta x^2}{2}}}{x^{\beta-1}}$.

5. Modes:

If $f(x)$ is unimodal, then it follows that $g(x)$ will also be unimodal, if $\alpha = \beta \geq 1$. However $g(x)$ could also be bimodal.

6. Hazard function shapes:

   Let $h_g(x) = \frac{g(x)}{[1-G(x)]}$ be the hazard function of a beta-generated distribution. By incorporating the property of the incomplete beta function, I_x, given by

$$I_x(\alpha, \beta) = \beta(\alpha, \beta) - I_{1-x}(\beta, \alpha), \tag{26}$$

   then the hazard function of the beta-generated family of distributions follows as

$$h_g(x) = \frac{F^{\alpha-1}(x)\ (1 - F(x))^{\beta-1}}{I_{1-F(x)}(\beta, \alpha)}\ f(x). \tag{27}$$

   The shape of $h_g(x)$ varies, but can be flexible. Lee et al. (2013) revealed that the beta-Weibull distribution has a hazard function which can be monotonically decreasing, increasing, in a bathtub shape or concave.

7. Entropy:

   The Shannon entropy is defined as $\eta_x = -E_X[log(g(X))]$. If a random variable follows the beta-generated distribution, then the Shannon entropy is obtained as

$$\eta_x = log\beta(\alpha, \beta) + (\alpha - 1)\kappa(\alpha, \beta) + (\beta - 1)\kappa(\beta, \alpha) - E_Y[log(F^{-1}(Y))], \tag{28}$$

   where $Y \sim Beta(\alpha, \beta)$, $\kappa(\alpha, \beta) = \psi(\alpha + \beta) - \psi(\alpha)$ and $\psi(.)$ is the digamma function.

3 Two-piece families of distributions:

3.1 Introduction

The general form of the CDF, PDF and quantile functions of the two piece distribution is given and related to the different formulations given in Fernández and Steel (1998), Arellano-Valle et al. (2005) and Nassiri and Loris (2013). The properties of the different two piece distributions are stated, as well as the method of moments estimations and maximum likelihood estimation. Different variations of the asymmetry parameter leads to the distribution considered in Mac'Oduol et al. (2020), as well as the other proposed distributions mentioned above.

Definition 5. *The general form of the PDF, CDF and quantile function of a two piece distribution is defined by Fernández and Steel (1998) and is given as*

$$f_T(x;\alpha,\beta,\lambda) = 2\left(\lambda\ \frac{1}{\alpha} f_X\left(\frac{x}{\alpha}\right) I_{(x\leq 0)} + (1-\lambda)\frac{1}{\beta} f_X\left(\frac{x}{\beta}\right) I_{(x>0)}\right), \tag{29}$$

$$F_T(x;\alpha,\beta,\lambda) = \begin{cases} 2\lambda F_X\left(\dfrac{x}{\alpha}\right) & \text{for } x \leq 0, \\ 2\lambda - 1 + 2(1-\lambda) F_X\left(\dfrac{x}{\beta}\right) & \text{for } x > 0, \end{cases} \tag{30}$$

and

$$Q_T(x;\alpha,\beta,\lambda) = \begin{cases} \alpha Q_X\left(\dfrac{x}{2\lambda}\right) & \text{for } x \leq \lambda, \\ \beta Q_X\left(\dfrac{x-2\lambda+1}{2(1-\lambda)}\right) & \text{for } x > \lambda, \end{cases} \tag{31}$$

respectively, where $\lambda \in [0,1]$, $\alpha > 0$ and $\beta > 0$. ■

In Chapter 3.2, the CDF, PDF and quantile function of the proposed two piece distribution in Fernández and Steel (1998), that relate to the general form given in Eq.(29), are discussed. The properties and method of moments estimation are also stated. In Chapter 3.3, the CDF, PDF and quantile function of the proposed two-piece distribution in Arellano-Valle et al. (2005), that relate to the general form given in Eq.(29), is discussed. The properties and method of moments estimation are also given.

In Chapter 3.4, the PDF, CDF and quantile function of the proposed two-piece distribution in Nassiri and Loris (2013), that relate to the general form given in Eq.(29), are discussed. The properties and method of moments estimation is discussed. In Chapter 3.5 the method of quantile splicing is introduced and discussed.

3.2 Generalisation of the skewed exponential power distribution

Fernández and Steel (1998) proposed an approach to construct an asymmetric family of distributions, which transforms any symmetric distribution into the desired skew distribution. If a unimodal distribution that is symmetric around 0, with density f and scalar index $\gamma \in (0, \infty)$ exists, then a probability density in this class of skew distributions follows as

$$f_\gamma(y) = \frac{2}{\gamma + \frac{1}{\gamma}} \begin{cases} f(\gamma y) & \text{if } y \leq 0, \\ f(\frac{y}{\gamma}) & \text{if } y > 0. \end{cases} \tag{32}$$

This relates to the general form of a two piece distribution given in Eq.(29), if $\alpha = \frac{1}{\gamma}$ and $\beta = \gamma$.

The CDF and quantile function are obtained, using the general form given in Eq.(30) and Eq.(31) with $\alpha = \frac{1}{\gamma}$ and $\beta = \gamma$, respectively, as

$$F_\gamma\Big(y; \frac{1}{\gamma}, \gamma, \lambda\Big) = \begin{cases} 2\lambda \; F(\gamma y) & \text{if } y \leq 0, \\ 2\lambda - 1 + 2(1-\lambda) \; F(\frac{y}{\gamma}) & \text{if } y > 0. \end{cases} \tag{33}$$

and

$$Q_\gamma\Big(y; \frac{1}{\gamma}, \gamma, \lambda\Big) = \begin{cases} \frac{1}{\gamma} \; Q\Big(\frac{y}{2\lambda}\Big) & \text{if } y \leq \lambda, \\ \gamma \; Q\Big(\frac{y - 2\lambda + 1}{2(1\lambda)}\Big) & \text{if } y > \lambda. \end{cases} \tag{34}$$

Properties:

The general distributional properties, as found in Fernández and Steel (1998), are as follows:

- **Location:**

  The median of the location-scale skew distribution is

$$\begin{aligned} me &= Q(\frac{1}{2}|\gamma, \lambda) \\ &= \gamma \; Q\Big(\frac{\gamma - 2.\frac{1}{2} + 1}{2(1-\lambda)}\Big) \\ &= \gamma \; Q\Big(\frac{\gamma}{2(1-\lambda)}\Big). \end{aligned}$$

- **Spread:**

  The spread function, $S_t(s)$, is obtained for $\frac{1}{2} \leq s \leq 1$ as

$$\begin{aligned} S_t(s) &= Q(s) - Q(1-s) \\ &= \gamma \; Q\Big(\frac{s - 2\lambda + 1}{2(1-\lambda)}\Big) - \frac{1}{\gamma} \; Q\Big(\frac{1-s}{2\lambda}\Big). \end{aligned}$$

- **Shape:**

The γ-functional is derived as

$$\gamma(s) = \frac{Q(s) + Q(1-s) - 2me}{Q(s) - Q(1-s)}$$
$$= \frac{\gamma\, Q\Big(\frac{s-2\lambda+1}{2(1-\lambda)}\Big) + \frac{1}{\gamma}\, Q\Big(\frac{1-s}{2\lambda}\Big) - 2\gamma\, Q\Big(\frac{\gamma}{2(1-\lambda)}\Big)}{\gamma\, Q\Big(\frac{s-2\lambda+1}{2(1-\lambda)}\Big) - \frac{1}{\gamma}\, Q\Big(\frac{1-s}{2\lambda}\Big)},$$

for $\frac{1}{2} \leq s \leq 1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread function, for $\frac{1}{2} \leq u < v \leq 1$, is obtained as

$$R(u, v) = \frac{S_t(u)}{S_t(v)}$$
$$= \frac{\gamma\, Q\Big(\frac{u-2\lambda+1}{2(1-\lambda)}\Big) - \frac{1}{\gamma}\, Q\Big(\frac{1-u}{2\lambda}\Big)}{\gamma\, Q\Big(\frac{v-2\lambda+1}{2(1-\lambda)}\Big) - \frac{1}{\gamma}\, Q\Big(\frac{1-v}{2\lambda}\Big)}.$$

3.3 A general family of skew distributions

The general family of skew distributions, including Eq.(32) as a special case, was introduced by Arellano-Valle et al. (2005). The PDF of this family of skew distributions, for a given density f that is symmetric around 0, that contains a parameter $\alpha \in \mathbb{R}$ as well as positive asymmetric functions given by $a(.)$ and $b(.)$, is defined as

$$f_\alpha(y) = \frac{2}{a(\alpha) + b(\alpha)} \begin{cases} f\Big(\dfrac{y}{b(\alpha)}\Big) & \text{if } y \leq 0, \\ f\Big(\dfrac{y}{a(\alpha)}\Big) & \text{if } y > 0. \end{cases} \tag{35}$$

This relates to the general form of a two-piece distribution given in Eq.(29), if $\alpha = b(\alpha)$, $\beta = a(\alpha)$ and $\lambda = \frac{b(\alpha)}{a(\alpha)+b(\alpha)}$. If the asymmetric functions are chosen such that $a(\alpha) = b(\alpha)$, then Eq.(35) reduces to a scale family of the density f, which does not allow for modification of the underlying symmetry.

The CDF of the skew family of distributions is defined as

$$F(y|\alpha) = \begin{cases} \dfrac{2b(\alpha)}{a(\alpha) + b(\alpha)} F\Big(\dfrac{y}{b(\alpha)}\Big) & \text{if } y \leq 0, \\ \dfrac{b(\alpha) - a(\alpha)}{a(\alpha) + b(\alpha)} + \dfrac{2a(\alpha)}{a(\alpha) + b(\alpha)} F\Big(\dfrac{y}{a(\alpha)}\Big) & \text{if } y > 0, \end{cases} \tag{36}$$

which can be obtained using the general form in Eq.(30) with $\alpha = b(\alpha)$, $\beta = a(\alpha)$ and $\lambda = \frac{b(\alpha)}{a(\alpha)+b(\alpha)}$.

The quantile function is derived using the general formula given in Eq(31), with $\alpha = b(\alpha)$, $\beta = a(\alpha)$

and $\lambda = \frac{b(\alpha)}{a(\alpha)+b(\alpha)}$, such that

$$Q(y|\alpha) = \begin{cases} b(\alpha)\; Q\Big(\dfrac{y(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big), & \text{if } y \leq \dfrac{b(\alpha)}{a(\alpha)+b(\alpha)}, \\ a(\alpha)\; Q\Big(\dfrac{y(a(\alpha)+b(\alpha))+a(\alpha)-b(\alpha)}{2a(\alpha)}\Big), & \text{if } y > \dfrac{b(\alpha)}{a(\alpha)+b(\alpha)}, \end{cases} \tag{37}$$

Properties of the skew family of distributions:

The following properties hold for the suggested skew family of distributions proposed by Fernández and Steel (1998):

- The skew family has a median of

$$F^{-1}\Big(\frac{1}{2}|\alpha\Big) = \begin{cases} b(\alpha)F^{-1}\left(\dfrac{a(\alpha)+b(\alpha)}{4b(\alpha)}\right) & \text{if } a(\alpha) < b(\alpha), \\ a(\alpha)F^{-1}\left(\dfrac{3a(\alpha)-b(\alpha)}{4a(\alpha)}\right) & \text{if } a(\alpha) \geq b(\alpha). \end{cases} \tag{38}$$

- The mode is obtained as

$$H(0|\alpha) = \frac{b(\alpha)}{a(\alpha)+b(\alpha)}. \tag{39}$$

A generalisation of the family of skew distributions occurs once a scale parameter $\sigma > 0$ and a location parameter $\mu \in \mathbb{R}$ are introduced.

Theorem 1. *Suppose X is a skew random variable, $X \sim S(f, \alpha)$. Then it follows from Arellano-Valle et al. (2005) that the family of location-scale skew distributions can be represented such that $Z = \mu + \sigma X$, with $\mu \in \mathbb{R}$ and $\sigma > 0$. The probability density follows as*

$$f_{\boldsymbol{\theta}}(z) = \frac{2}{\sigma(a(\alpha)+b(\alpha))} \begin{cases} f\Big(\dfrac{z-\mu}{\sigma b(\alpha)}\Big) & \text{if } z \leq \mu, \\ f\Big(\dfrac{z-\mu}{\sigma a(\alpha)}\Big) & \text{if } z > \mu, \end{cases} \tag{40}$$

with $\boldsymbol{\theta} = (\mu, \sigma, \alpha)$. The probability density function is denoted as $Z \sim S(f, \mu, \sigma, \alpha)$ or $Z \sim Sf(\mu, \sigma, \alpha)$.

Properties of the location-scale skew family of distributions:

The general distributional properties are stated by Arellano-Valle et al. (2005) and follow as:

- **Location:**

The median is

$$F^{-1}\Big(\frac{1}{2}|\alpha\Big) = \begin{cases} \mu + \sigma b(\alpha)F^{-1}\left(\dfrac{a(\alpha)+b(\alpha)}{4b(\alpha)}\right) & \text{if } a(\alpha) < b(\alpha), \\ \mu + \sigma a(\alpha)F^{-1}\left(\dfrac{3a(\alpha)-b(\alpha)}{4a(\alpha)}\right) & \text{if } a(\alpha) \geq b(\alpha). \end{cases}$$

- **Spread:**

  The spread function, $S_t(s)$, is obtained for $\frac{1}{2} \leq s \leq 1$, as

$$\begin{aligned} S_t(s) &= Q(s) - Q(1-s) \\ &= a(\alpha)\ Q\Big(\frac{s(a(\alpha)+b(\alpha))+a(\alpha)-b(\alpha)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-s)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big) \\ &= a(\alpha)\ Q\Big(\frac{a(\alpha)(1+s)+b(\alpha)(1-s)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-s)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big). \end{aligned}$$

- **Shape:**

  The γ-functional is obtained as:

$$\begin{aligned} \gamma(s) &= \frac{Q(s)+Q(1-s)-2me}{Q(s)-Q(1-s)} \\ &= \frac{a(\alpha)\ Q\Big(\frac{s(a(\alpha)+b(\alpha))+a(\alpha)-b(\alpha)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-s)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big) - 2a(\alpha)\ Q\Big(\frac{3a(\alpha)-b(\alpha)}{4a(\alpha)}\Big)}{a(\alpha)\ Q\Big(\frac{a(\alpha)(1+s)+b(\alpha)(1-s)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-s)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big)} \end{aligned}$$

  for $\frac{1}{2} \leq s \leq 1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread function, for for $\frac{1}{2} \leq u < v \leq 1$, is obtained as

$$\begin{aligned} R(u,v) &= \frac{S_t(u)}{S_t(v)} \\ &= \frac{a(\alpha)\ Q\Big(\frac{a(\alpha)(1+u)+b(\alpha)(1-u)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-u)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big)}{a(\alpha)\ Q\Big(\frac{a(\alpha)(1+v)+b(\alpha)(1-v)}{2a(\alpha)}\Big) - b(\alpha)\ Q\Big(\frac{(1-v)(a(\alpha)+b(\alpha))}{2b(\alpha)}\Big)} \end{aligned}$$

3.4 Location-scale family of asymmetric densities

Nassiri and Loris (2013) constructed an asymmetric density by using a given density f that is symmetric around 0, with positive real parameters λ_1 and λ_2 and defined it as

$$f_{\lambda_1,\lambda_2}(y) = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} \begin{cases} f(\lambda_1 y) & \text{if } y \leq 0, \\ f(\lambda_2 y) & \text{if } y > 0. \end{cases} \tag{41}$$

This relates to the general form of a two piece distribution given in Eq.(29), if $\alpha = \frac{1}{\lambda_1}$, $\beta = \frac{1}{\lambda_2}$ and $\lambda = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. The probability density function given in Eq.(41), is symmetric when $\lambda_1 = \lambda_2$, and if $\lambda_1 = \lambda_2 = 1$ holds, then it results in the special case $f_{\lambda_1,\lambda_2}(y) = f$. When $\lambda_1 > \lambda_2$, a right-skew density is obtained. The opposite follows such that when $\lambda_1 < \lambda_2$ then a left-skew density is obtained. The family given by Arellano-Valle et al. (2005) in Eq.(33), is another special case of the Nassiri and Loris (2013) family in Eq.(41), with $\lambda_1 = \frac{1}{b(\alpha)}$ and $\lambda_2 = \frac{1}{a(\alpha)}$.

Definition 6. *The CDF of Y is obtained as*

$$F_{\lambda_1,\lambda_2}(y) = \begin{cases} \dfrac{2\lambda_2}{\lambda_1+\lambda_2}F(\lambda_1 y) & \text{if } y < 0, \\ \dfrac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2} + \dfrac{2\lambda_1}{\lambda_1+\lambda_2}F(\lambda_2 y) & \text{if } y \geq 0. \end{cases} \tag{42}$$

■

This is a special case of the general form of the CDF given in Eq.(30), obtained when selecting $\alpha = \frac{1}{\lambda_1}$, $\beta = \frac{1}{\lambda_2}$ and substituting this into the special case of Eq.(30) given below:

$$F_T\Big(y; \alpha, \beta, \lambda = \frac{\alpha}{\alpha+\beta}\Big) = \begin{cases} \dfrac{2\alpha}{\alpha+\beta}F(\dfrac{y}{\alpha}) & \text{for } y < 0, \\ \dfrac{\alpha-\beta}{\alpha+\beta} + \dfrac{2\beta}{\alpha+\beta}F(\dfrac{y}{\beta}) & \text{for } y \geq 0. \end{cases} \tag{43}$$

The quantile function of the CDF, given by Eq.(42), follows as

$$Q_{\lambda_1,\lambda_2}(p) = \begin{cases} \dfrac{1}{\lambda_1}F^{-1}\Big(\dfrac{\lambda_1+\lambda_2}{2\lambda_2}p\Big) & \text{if } p < \dfrac{\lambda_2}{\lambda_1+\lambda_2}, \\ \dfrac{1}{\lambda_2}\Big(\dfrac{\lambda_1+\lambda_2}{2\lambda_1}p - \dfrac{\lambda_2-\lambda_1}{2\lambda_1}\Big) & \text{if } P \geq \dfrac{\lambda_2}{\lambda_1+\lambda_2}. \end{cases} \tag{44}$$

This relates to the general form of the quantile function given in Eq.(31), obtained when selecting $\alpha = \frac{1}{\lambda_1}$, $\beta = \frac{1}{\lambda_2}$ and substituting this into the special case of Eq.(31) given below:

$$Q_T\Big(y; \alpha, \beta, \lambda = \frac{\alpha}{\alpha+\beta}\Big) = \begin{cases} \alpha Q_Y\Big(\dfrac{y(\alpha+\beta)}{2\alpha}\Big) & \text{for } y < \dfrac{\alpha}{\alpha+\beta}, \\ \beta Q_Y\Big(\dfrac{y(\alpha+\beta)-(\beta-\alpha)}{2\beta}\Big) & \text{for } y \geq \dfrac{\alpha}{\alpha+\beta}. \end{cases} \tag{45}$$

The reference symmetric density function f in Eq.(41) is a standardized form of the density given by the location-scale family of densities, including the standard Laplace density as well as the standard normal density. Introducing $\phi > 0$, a scale parameter, and $\mu \in \mathbb{R}$, a location parameter, it follows that

$$f_{\lambda_1,\lambda_2}(y; \mu, \phi) = \frac{2\lambda_1\lambda_2}{\phi(\lambda_1+\lambda_2)} \begin{cases} f\Big(\lambda_1(\dfrac{\mu-y}{\phi})\Big) & \text{if } y \leq \mu, \\ f\Big(\lambda_2(\dfrac{y-\mu}{\phi})\Big) & \text{if } y > \mu, \end{cases} \tag{46}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^+$. Let $\mu = 0$ and $\phi = 1$, then it follows that $f_{\lambda_1,\lambda_2}(y;\mu,\phi) = f_{\lambda_1,\lambda_2}(y)$.

Corollary 2. *Assuming that f is contained in the location-scale family of symmetric densities, it follows that if $Y \sim f_{\lambda_1,\lambda_2}(.;\mu,\phi)$, then for any $\beta_1, \beta_2 \in \mathbb{R}$ it follows that $\beta_1 + \beta_2 Y \sim f_{\lambda_1,\lambda_2}(.;\beta_1 + \beta_2\mu, |\beta_1|\phi)$.* ■

Remark. *Suppose Y is a random variable with the density function $f_{\lambda_1,\lambda_2}(.;\mu,\phi)$, as in Eq.(46). Let F and F^{-1} represent the CDF and the quantile function of the standard symmetric density function, f, that is symmetric around 0. Since f possesses the symmetric property, it follows that $F(0) = 0.5$ and $F^{-1}(0.5) = 0$.*

Definition 7. *Suppose Y is a random variable with an asymmetric probability density function, $f_{\lambda_1,\lambda_2}(.;\mu,\phi)$ as in Eq.(46), then the CDF of Y follows as*

$$F_{\lambda_1,\lambda_2}(y;\mu,\phi) = \begin{cases} \dfrac{2\lambda_2}{\lambda_1+\lambda_2}\ F\Big(\lambda_1(\dfrac{y-\mu}{\phi})\Big) & \text{if } y < \mu, \\ \dfrac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2} + \dfrac{2\lambda_1}{\lambda_1+\lambda_2}\ F\Big(\lambda_2(\dfrac{y-\mu}{\phi})\Big) & \text{if } y \geq \mu. \end{cases} \tag{47}$$

For any value of $\beta \in (0,1)$, the quantile function of Y follows as

$$F^{-1}_{\lambda_1,\lambda_2}(\beta) = \begin{cases} \mu + \dfrac{\phi}{\lambda_1}\ F^{-1}\Big(\dfrac{\beta(\lambda_1+\lambda_2)}{2\lambda_2}\Big) & \text{if } \beta < \dfrac{\lambda_2}{\lambda_1+\lambda_2}, \\ \mu + \dfrac{\phi}{\lambda_2}\ F^{-1}\Big(\dfrac{\beta(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big) & \text{if } \beta \geq \dfrac{\lambda_2}{\lambda_1+\lambda_2}, \end{cases} \tag{48}$$

with

$$F^{-1}_{\lambda_1,\lambda_2}\Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big) = \mu, \tag{49}$$

as stated by Nassiri and Loris (2013). ■

The maximum value of Y is found at $F^{-1}_{\lambda_1,\lambda_2}(1)$, whilst the minimum value of Y is at $F^{-1}_{\lambda_1,\lambda_2}(0)$.

Theorem 2. *Suppose Y is a random variable with an asymmetric density $f_{\lambda_1,\lambda_2}(.;\mu,\phi)$. The r^{th} central moment of Y, given that $r \in \mathbb{R}$, follows as*

$$E(Y-\mu)^r = \frac{\phi^r}{(\lambda_1+\lambda_2)}\Big[\frac{\lambda_1^{r+1} + (-1)^r\ \lambda_2^{r+1}}{\lambda_1^r\lambda_2^r}\Big]\mu_r, \tag{50}$$

where

$$\mu_r = 2\int_0^\infty s^r f(s)ds. \tag{51}$$

The mean and variance follow as

$$E(Y) = \mu + \frac{\phi(\lambda_1-\lambda_2)}{\lambda_1\lambda_2}\mu_1 \tag{52}$$

and

$$Var(Y) = \frac{\phi^2}{\lambda_1^2\lambda_2^2}[(\lambda_1-\lambda_2)^2(\mu_2-\mu_1^2) + \lambda_1\lambda_2\mu_2], \tag{53}$$

respectively, whilst the skewness and kurtosis moment-ratios are obtained from Nassiri and Loris (2013) as

$$\gamma_{sk} = \frac{(\lambda_1 - \lambda_2)[(\lambda_1 - \lambda_2)^2(\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3) + \lambda_1\lambda_2(2\mu_3 - 3\mu_1\mu_2)]}{[(\lambda_1 - \lambda_2)^2(\mu_2 - \mu_1^2) + \lambda_1\lambda_2\mu_2]^{\frac{3}{2}}} \tag{54}$$

and

$$\gamma_{ku} = \frac{(\lambda_1^5 + \lambda_2^5)\mu_4 - (\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2[4(\lambda_1^2 + \lambda_2^2)\mu_1\mu_3 - 6(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2)\mu_1^2\mu_2 + 3(\lambda_1 - \lambda_2)^2\mu_1^4]}{(\lambda_1 + \lambda_2)[(\lambda_1 - \lambda_2)^2(\mu_2 - \mu_1^2) + \lambda_1\lambda_2\mu_2]^2}, \tag{55}$$

respectively. ■

Remark. *It is very clear that the kurtosis and the skewness in Eq.*(55) *and Eq.*(54) *respectively, does not depend on* μ *and* ϕ. *They depend only on* λ_1, λ_2 *and* μ_1, μ_2, μ_3 *and* μ_4.

Properties of the location-scale family of asymmetric densities:

The general distributional properties, as found in Nassiri and Loris (2013), are

- **Location:**

  The location-scale skew distribution has the following median

$$\begin{aligned} me =& F^{-1}(0.5) \\ =& 0. \end{aligned}$$

- **Spread:**

  The spread function, $S_t(s)$, is obtained for $\frac{1}{2} \leq s \leq 1$, as

$$\begin{aligned} S_t(s) =& Q(s) - Q(1-s) \\ =& \mu + \frac{\phi}{\lambda_2}\, Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big) - \Big(\mu + \frac{\phi}{\lambda_1}\, Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)\Big) \\ =& \phi\Big(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\Big) Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big) Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big) \end{aligned}$$

- **Shape:**

The γ-functional value is derived as:

$$\begin{aligned}
\gamma(s) &= \frac{Q(s)+Q(1-s)-2me}{Q(s)-Q(1-s)} \\
&= \frac{\mu+\frac{\phi}{\lambda_2}\,Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)+\mu+\frac{\phi}{\lambda_1}\,Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)-2(0)}{\phi\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)} \\
&= \frac{2\mu}{\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)}+\frac{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}} \\
&= \frac{2\mu}{\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)Q\Big(\frac{s(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-s)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)}+\frac{\lambda_2+\lambda_1}{\lambda_1-\lambda_2},
\end{aligned}$$

for $\frac{1}{2} \leq s \leq 1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread function, for for $\frac{1}{2} \leq u < v \leq 1$, is obtained as

$$\begin{aligned}
R(u,v) &= \frac{S_t(u)}{S_t(v)} \\
&= \frac{\phi\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)Q\Big(\frac{u(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-u)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)}{\phi\Big(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\Big)Q\Big(\frac{v(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-v)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)} \\
&= \frac{Q\Big(\frac{u(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-u)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)}{Q\Big(\frac{v(\lambda_1+\lambda_2)+(\lambda_1-\lambda_2)}{2\lambda_1}\Big)Q\Big(\frac{(1-v)(\lambda_1+\lambda_2)}{2\lambda_2}\Big)}.
\end{aligned}$$

3.5 The method of quantile splicing

Quantile splicing is the method that refers to joining two quantile functions at a location point of choice. Moreover, taking the median of a univariate symmetric distribution and joining the quantile functions of the two half distributions at this point, is the method used to obtain the quantile function of a two-piece distribution.

In Mac‘Oduol et al. (2020), quantile splicing was utilized to obtain the general results of the two-piece families of distribution's r^{th} order L-moments. There exists an association between the L-moments of the parent distribution as well as the the half distribution, which is disclosed by the general form, and is attained by the order statistics.

Lemma 3. *Suppose Y is a folded random variable, for which $Y = |X|$, $Z = -Y$ and $0 < y < \infty$. Then the quantile functions of Y and Z follows from Mac‘Oduol et al. (2020) as*

$$Q_Y(p) = Q_X\Big(\frac{1+p}{2}\Big), \qquad 0 < p < 1 \tag{56}$$

and

$$Q_Z(p) = Q_X\left(\frac{p}{2}\right), \qquad 0 < p < 1, \tag{57}$$

respectively. ■

The half distributions used in the quantile functions are referred to as kernels in this method, hence if there is an unknown CDF, this method can be applied to any univariate symmetric distribution. In order to make use of the method of quantile splicing, the domains of the location parameter needs to be attained on the left and the right side, $-\infty < \mu < \infty$.

Lemma 4. *Let X be a continuous random variable, originating from a symmetric distribution, defined on $(-\infty, \infty)$ and let $0 < k < 1$. It follows from Eq.(56) and Eq.(57) that the quantile function for the piecewise distribution and for any given value of k, is defined as*

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma(Q_X(p) - Q_X(k)) & \text{for } p \leq k, \\ \mu + \sigma(Q_X(p) - Q_X(k)) & \text{for } p > k, \end{cases} \tag{58}$$

where $-\infty < \mu < \infty$ is the location parameter, $\alpha > 0$ is defined as the shape parameter and $\sigma > 0$ follows as the scale parameter.

Definition 8. *The CDF of X is obtained as*

$$F_T(x) = \begin{cases} F_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(k)\right) & \text{for } x \leq \mu, \\ F_X\left(\frac{x-\mu}{\sigma} + Q_X(k)\right) & \text{for } x > \mu. \end{cases} \tag{59}$$

The PDF can be obtained from the CDF as

$$f_T(x) = \begin{cases} \frac{1}{\alpha\sigma}\ f_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(k)\right) & \text{for } x \leq \mu, \\ \frac{1}{\sigma}\ f_X\left(\frac{x-\mu}{\sigma} + Q_X(k)\right) & \text{for } x > \mu. \end{cases} \tag{60}$$

■

Mac'Oduol et al. (2020) considered the case $k = \frac{1}{2}$, where the quantile splicing occurred at the median. Suppose we let the scaling factor be $k = \frac{1}{2}$, as in Mac'Oduol et al. (2020). Then we get the following results.

Lemma 5. *Suppose X is a continuous random variable, as in Lemma 4, with the scaling factor $k = \frac{1}{2}$.*

The quantile function follows as

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma(Q_X(p) - Q_X(\frac{1}{2})) & \text{for } p \leq \frac{1}{2}, \\ \mu + \sigma(Q_X(p) - Q_X(\frac{1}{2})) & \text{for } p > \frac{1}{2}, \end{cases} \tag{61}$$

as in Eq.(58)*. The CDF is obtained as*

$$F_T(x) = \begin{cases} F_X\left(\frac{x-\mu}{\alpha\sigma}\right) & \text{for } x \leq \mu, \\ F_X\left(\frac{x-\mu}{\sigma}\right) & \text{for } x > \mu. \end{cases} \tag{62}$$

The PDF follows from the CDF as

$$f_T(x) = \begin{cases} \frac{1}{\alpha\sigma}\ f_X\left(\frac{x-\mu}{\alpha\sigma}\right) & \text{for } x \leq \mu, \\ \frac{1}{\sigma}\ f_X\left(\frac{x-\mu}{\sigma}\right) & \text{for } x > \mu. \end{cases} \tag{63}$$

■

The results were applied to the logistic, cosine and Student's $t(2)$ distribution and recorded in Table 1. Here are a few examples of two-piece distributions and their CDF, PDF and quantile functions.

Distribution	$F_T(x)$	$f_T(x)$	$Q_T(s)$
Logistic	$=\begin{cases} \frac{e^{(\frac{x-\mu}{\alpha\sigma})}}{1+e^{(\frac{x-\mu}{\alpha\sigma})}},\ s\leq\frac{1}{2} \\ \frac{e^{(\frac{x-\mu}{\sigma})}}{1+e^{(\frac{x-\mu}{\sigma})}},\ s>\frac{1}{2} \end{cases}$	$=\begin{cases} \frac{e^{(\frac{x-\mu}{\alpha\sigma})}}{\alpha\sigma\left(1+e^{(\frac{x-\mu}{\alpha\sigma})}\right)^2},\ x\leq\mu \\ \frac{e^{(\frac{x-\mu}{\sigma})}}{\sigma\left(1+e^{(\frac{x-\mu}{\sigma})}\right)^2},\ x>\mu \end{cases}$	$=\begin{cases} \mu+\alpha\sigma\log\left(\frac{s}{1-s}\right),\ s\leq\frac{1}{2} \\ \mu+\sigma\log\left(\frac{s}{1-s}\right),\ s>\frac{1}{2} \end{cases}$
Cosine	$=\begin{cases} 0,\ x\leq\mu-\alpha\sigma, \\ \sin^2\left(\frac{\pi}{2}\left(\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)\right),\ \mu-\alpha\sigma<x<\mu, \\ \sin^2\left(\frac{\pi}{2}\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right),\ \mu<x<\mu+\sigma, \\ 1,\ x\geq\mu+\sigma \end{cases}$	$=\begin{cases} \frac{\pi}{4\alpha\sigma}\sin\left(\pi\left(\frac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\right)\right),\ \mu-\alpha\sigma<x<\mu \\ \frac{\pi}{4\sigma}\sin\left(\pi\left(\frac{x-(\mu-\sigma)}{2\sigma}\right)\right),\ \mu<x<\mu+\sigma \\ 0,\ \text{elsewhere} \end{cases}$	$=\begin{cases} \mu+\alpha\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s})-1\right)\ s\leq\frac{1}{2} \\ \mu+\sigma\left(\frac{4}{\pi}\arcsin(\sqrt{s})-1\right)\ s>\frac{1}{2} \end{cases}$
Student's $t(2)$	$=\begin{cases} \frac{1}{2}\left(1+\frac{\frac{x-\mu}{\alpha\sigma}}{\sqrt{2+\left(\frac{x-\mu}{\alpha\sigma}\right)^2}}\right),\ x\leq\mu, \\ \frac{1}{2}\left(1+\frac{\frac{x-\mu}{\sigma}}{\sqrt{2+\left(\frac{x-\mu}{\sigma}\right)^2}}\right),\ x>\mu. \end{cases}$	$=\begin{cases} \frac{1}{\alpha\sigma}\left(2+\left(\frac{x-\mu}{\alpha\sigma}\right)^2\right)^{-\frac{3}{2}},\ x\leq\mu, \\ \frac{1}{\sigma}\left(2+\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{3}{2}},\ x>\mu \end{cases}$	$=\begin{cases} \mu+\alpha\sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right),\ s\leq\frac{1}{2}, \\ \mu+\sigma\left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}}\right),\ s>\frac{1}{2}. \end{cases}$

Table 1: The CDF, PDF and quantile functions of the logistic, cosine and Student's $t(2)$ distribution

3.5.1 Properties of the two-piece families of distributions

The general distributional properties for the two-piece families of distributions for the logistic, cosine and Student's $t(2)$ distribution found in Mac'Oduol et al. (2020) are given as follows:

1. **Logistic distribution:**

   - **Location:**

     The median is

$$
\begin{aligned}
me =& Q_T(\frac{1}{2}) \\
=& \mu + \sigma\Big(\frac{\frac{1}{2}}{1-\frac{1}{2}}\Big) \\
=& \mu + \sigma\log(1) \\
=& \mu.
\end{aligned}
$$

   - **Spread:**

     The spread function is obtained as

$$
\begin{aligned}
S_t(s) =& Q_T(s) - Q_T(1-s) \\
=& \Big\{\mu + \sigma\log\Big(\frac{s}{1-s}\Big)\Big\} - \Big\{\mu + \sigma\alpha\log\Big(\frac{1-s}{1-(1-s)}\Big)\Big\} \\
=& \sigma(1+\alpha)\log\Big(\frac{1-s}{1-(1-s)}\Big)
\end{aligned}
$$

     for $\frac{1}{2} < s < 1$.

   - **Shape:**

     The γ-functional is derived by

$$
\begin{aligned}
\gamma_T(s) =& \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
=& \frac{\mu + \sigma\log\left(\frac{s}{1-s}\right) + \mu + \alpha\sigma\log\left(\frac{1-s}{1-(1-s)}\right) - 2\mu}{\sigma(1+\alpha)\log\left(\frac{1-s}{1-(1-s)}\right)} \\
=& \frac{\sigma\log\left(\frac{s}{1-s}\right) - \alpha\sigma\log\left(\frac{1-s}{1-(1-s)}\right)}{\sigma(1+\alpha)\log\left(\frac{1-s}{1-(1-s)}\right)} \\
=& \frac{1-\alpha}{1+\alpha}
\end{aligned}
$$

     for $\frac{1}{2} < s < 1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread functions, for $\frac{1}{2} < a < b < 1$ is

$$R_T(a,b) = \frac{S_T(a)}{S_T(b)} = \frac{\sigma(1+\alpha)\log\left(\frac{a}{1-a}\right)}{\sigma(1+\alpha)\log\left(\frac{b}{1-b}\right)} = \frac{\log\left(\frac{a}{1-a}\right)}{\log\left(\frac{b}{1-b}\right)}.$$

2. **Cosine distribution:**

- **Location:**

  The median is

$$\begin{aligned} me =& Q_T(\frac{1}{2}) \\ =& \mu + \sigma\left(\frac{4}{\pi}\text{arcsin}\left(\sqrt{\frac{1}{2}}\right) - 1\right) \\ =& \mu + \sigma\left(\frac{4}{\pi}.\frac{\pi}{4} - 1\right) \\ =& \mu. \end{aligned}$$

- **Spread:**

  The spread function is obtained as

$$\begin{aligned} S_t(s) =& Q_T(s) - Q_T(1-s) \\ =& \left\{\mu + \sigma\left(\frac{4}{\pi}\text{arcsin}\left(\sqrt{\frac{1}{2}}\right) - 1\right)\right\} - \left\{\mu + \sigma\alpha\left(\frac{4}{\pi}\text{arcsin}\left(\sqrt{\frac{1}{2}}\right) - 1\right)\right\} \\ =& \frac{4}{\pi}\sigma\left(\text{arcsin}(\sqrt{s}) - \alpha\left(\frac{\pi}{2} - \text{arcsin}(\sqrt{s})\right)\right) - \sigma(1-\alpha) \\ =& \frac{4}{\pi}\sigma(1+\alpha)\text{arcsin}(\sqrt{s}) - \sigma(1+\alpha) \\ =& \sigma\left(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - 1\right)(1+\alpha), \end{aligned}$$

  for $\frac{1}{2} < s < 1$.

- **Shape:**

The γ-functional is derived by

$$\begin{aligned}\gamma_T(s) =&\frac{Q_T(s)+Q_T(1-s)-2me}{S_T(s)}\\ =&\frac{\big(\mu+\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)\big)+\big(\mu+\alpha\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s})-1\big)\big)-2\mu}{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)(1+\alpha)}\\ =&\frac{\frac{4}{\pi}\sigma\big(\text{arcsin}(\sqrt{s})+\alpha\big(\frac{\pi}{2}-\text{arcsin}(\sqrt{s})\big)\big)-\sigma(1-\alpha)}{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)(1+\alpha)}\\ =&\frac{\frac{4}{\pi}\big(\text{arcsin}(\sqrt{s})+\alpha\text{arcsin}(\sqrt{s})\big)-(1+\alpha)}{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)(1+\alpha)}\\ =&\frac{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)(1-\alpha)}{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{s})-1\big)(1+\alpha)}\\ =&\frac{1-\alpha}{1+\alpha},\end{aligned}$$

for $\frac{1}{2}<s<1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread functions, for $\frac{1}{2}<a<b<1$ is

$$R_T(a,b)=\frac{S_T(a)}{S_T(b)}=\frac{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{a})-1\big)(1+\alpha)}{\sigma\big(\frac{4}{\pi}\text{arcsin}(\sqrt{b})-1\big)(1+\alpha)}=\frac{\big(\frac{4}{\pi}\text{arcsin}(\sqrt{a})-1\big)}{\big(\frac{4}{\pi}\text{arcsin}(\sqrt{b})-1\big)}.$$

3. **Student's $t(2)$ distribution:**

- **Location:**

  The median is

$$\begin{aligned}me =&Q_T(\frac{1}{2})\\ =&\mu+\sigma\left(\frac{2.\frac{1}{2}-1}{(2.\frac{1}{2}(1-\frac{1}{2}))^{\frac{1}{2}}}\right)\\ =&\mu.\end{aligned}$$

- **Spread:**

The spread function is derived as

$$
\begin{aligned}
S_t(s) =& Q_T(s) - Q_T(1-s) \\
=& \left\{ \mu + \sigma \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right) \right\} - \left\{ \mu + \sigma\alpha \left(\frac{2(1-s)-1}{(2(1-s)(1-(1-s)))^{\frac{1}{2}}}^{\frac{1}{2}} \right) \right\} \\
=& \sigma \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right) + \sigma\alpha \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right) \\
=& \sigma(1+\alpha) \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right),
\end{aligned}
$$

for $\frac{1}{2} < s < 1$.

- **Shape:**

  The γ-functional is obtained as

$$
\begin{aligned}
\gamma_T(s) =& \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
=& \frac{\mu + \sigma \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right) + \mu + \alpha\sigma \left(\frac{2(1-s)-1}{(2(1-s)(1-(1-s)))^{\frac{1}{2}}} \right) - 2\mu}{\sigma(1+\alpha) \left(\frac{2s-1}{(2s(1-s))^{\frac{1}{2}}} \right)} \\
=& \frac{1-\alpha}{1+\alpha},
\end{aligned}
$$

  for $\frac{1}{2} < s < 1$.

- **Ratio-of-spread functions:**

  The ratio-of-spread functions, for $\frac{1}{2} < a < b < 1$ is

$$
R_T(a,b) = \frac{S_T(a)}{S_T(b)} = \frac{\left(\frac{2a-1}{(2a(1-a))^{\frac{1}{2}}} \right)}{\left(\frac{2b-1}{(2b(1-b))^{\frac{1}{2}}} \right)} = \frac{(2a-1).(2b(1-b))^{\frac{1}{2}}}{(2b-1).(2a(1-a))^{\frac{1}{2}}}
$$

3.5.2 r^{th} order L-moments

The r^{th} order L-moments for the two-piece distributions has a general form stated by the following theorem.

Theorem 3. *Suppose there exists a random variable, T, that follows a two-piece (TP) distribution, which is denoted by $T \sim TP(\mu, \sigma, \alpha)$, defined by its quantile function $Q_T(s)$. The r^{th} order L-moment, for $r \geq 1$, has the following general expression*

$$
L_{T:r} = \mu^* + \sigma \left(L_{X:r} - 0.5(1-\alpha) \times \sum_{j=1}^{r} c_{j-1}^{(r-1)} \frac{\mu_{j:j}}{j} \right), \tag{64}
$$

where μ^* is the location parameter for which $-\infty < \mu < \infty$ is true, if $r = 1$, and if $r > 1$ then $\mu^* = 0$. $\mu_{j:j}$ represents the expected value of the j^{th} order statistic from a half distribution that has a sample size n and $c_{j-1}^{(r-1)}$ represents the $(j-1)^{th}$ coefficient of the r^{th} order shifted Legendre polynomial.

Proof. See Mac'Oduol et al. (2020) for detailed results. □

Table 2 represents the polynomial coefficients for $j = 1, 2, ..., r-1$.

	0	1	2	3	$\cdots$	j
0	$c_0^{(0)}$					
1	$c_0^{(1)}$	$c_1^{(1)}$				
2	$c_0^{(2)}$	$c_1^{(2)}$	$c_2^{(2)}$			
3	$c_0^{(3)}$	$c_1^{(3)}$	$c_2^{(3)}$	$c_3^{(3)}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$r-1$	$c_0^{(r-1)}$	$c_1^{(r-1)}$	$c_2^{(r-1)}$	$c_3^{(r-1)}$		$c_j^{(r-1)}$

Table 2: The coefficients of a polynomial with degree $(r-1)$.

Theorem 4. *The first 4 L-moments of T are obtained by substituting $r = 1, 2, 3, 4$, respectively, into Eq.(64), to obtain*

$$
\begin{aligned}
L_{T:1} &= \mu + \sigma\Big(L_{X:1} - 0.5(1-\alpha)c_0^{(0)}L_{Z:1}\Big) \\
L_{T:2} &= \sigma\bigg(L_{X:2} - 0.5(1-\alpha)\times\Big(c_0^{(1)}L_{Z:1} + \frac{c_1^{(1)}}{2}(L_{Z:1} + L_{Z:2})\Big)\bigg) \\
L_{T:3} &= \sigma\bigg(L_{X:3} - 0.5(1-\alpha)\times\Big(L_{Z:1}\Big(c_0^{(2)} + \frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{3}\Big) + L_{Z:2}\Big(\frac{c_1^{(2)}}{2} + \frac{c_2^{(2)}}{2}\Big) + L_{Z:3}\Big(\frac{c_2^{(2)}}{6}\Big)\Big)\bigg) \\
L_{T:4} &= \sigma\bigg(L_{X:3} - 0.5(1-\alpha)\times\Big(L_{Z:1}\Big(c_0^{(3)} + \frac{c_1^{(3)}}{2} + \frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{4}\Big) + \frac{L_{Z:2}}{2}(c_1^{(3)} + c_2^{(3)} + \frac{9}{10}c_3^{(3)}) \\
&\quad + \frac{L_{Z:3}}{2}\Big(\frac{c_2^{(3)}}{3} + \frac{c_3^{(3)}}{2}\Big) + \frac{L_{Z:4}}{20}c_3^{(3)}\Big)\bigg)
\end{aligned}
$$

Proof. See Mac'Oduol et al. (2020) for detailed results. □

The coefficients for the two-piece L-moments are given in Table 3for $r = 1, 2, 3, 4$.

$r-1$ \ $j-1$	0	1	2	3
0	1			
1	-1	2		
2	1	-6	6	
3	-1	12	-30	20

Table 3: The coefficients of a polynomial with degree $(r-1)$ for values $r = 1, 2, 3, 4$.

The L-moments of the logistic, cosine and Student's $t(2)$ distribution are as follows:

Distribution	l_1	l_2	l_3	l_4
Logistic	$L_{T:1} = \log(2)(1-\alpha)$	$L_{T:2} = \frac{1}{2}(1+\alpha)$	$L_{T:3} = \frac{1}{4}(1-\alpha)$	$L_{T:4} = \frac{1}{12}(1+\alpha)$
Cosine	$L_{T:1} = \left(2 - \frac{4}{\pi}\right)(1-\alpha)$	$L_{T:2} = \frac{1}{2}(1+\alpha)$	$L_{T:3} = \frac{2}{3\pi}(1-\alpha)$	$L_{T:4} = \frac{1}{32}(1+\alpha)$
Student's $t(2)$	$L_{T:1} = \frac{2}{\pi}(1-\alpha)$	$L_{T:2} = \frac{1}{2}(1+\alpha)$	$L_{T:3} = \frac{1}{\pi}(1-\alpha)$	$L_{T:4} = \frac{3}{16}(1+\alpha)$

Table 4: The L-moments of the logistic, cosine and Student's $t(2)$ distribution

3.6 Method of estimation

Based on the existence of a closed-form expression for the L-moments of the logistic, cosine and Student's $t(2)$ distributions, the method of L-moments can be applied, as stated by Gilchrist (2000).

4 Extended results using quantile splicing

4.1 Introduction

The general results from Mac'Oduol et al. (2020) are used to obtain the PDF, CDF and quantile functions for the family of two-piece distributions, by replacing $k = \frac{1}{4}$. This means the family of distributions consists of members whose quantile functions are spliced at the 1^{st} quantile, or the 25^{th} percentile. This will be proposed in Chapter 4.2.1, in addition to the quantile measures and their distributional form. In Chapter 4.2.2, the L-moments results are used to acquire a general formula of the r^{th} L-moments when $k = \frac{1}{4}$, as well as obtaining a general formula for the first 4 L-moments. Chapter 4.2.3 will consist of examples of two-piece families of distributions where $k = \frac{1}{4}$, defined by their PDF, CDF and quantile functions. Lastly, in Chapter 4.2.4 the L-moments estimation method is discussed.

4.2 General results

4.2.1 Definition

Lemma 6. *Let X be a random variable from a symmetric, uniform distribution with quantile function, $Q_T(p)$. Suppose T is a random variable from a two-piece family of distributions, spliced at the lower quantile ($k = \frac{1}{4}$) and denoted as $T \sim TP(\mu, \sigma, \alpha)$, where $\sigma > 0$, $-\infty < \mu < \infty$ and $\alpha > 0$ are the scale, location and shape parameters, respectively. The quantile function of T is defined as*

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma(Q_X(p) - Q_X(\frac{1}{4})) & \text{for } p \leq \frac{1}{4}, \\ \mu + \sigma(Q_X(p) - Q_X(\frac{1}{4})) & \text{for } p > \frac{1}{4}, \end{cases} \tag{65}$$

as in Eq.(31)*. whereas the CDF is obtained as*

$$F_T(x) = \begin{cases} F_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(\frac{1}{4})\right) & \text{for } x \leq \mu, \\ F_X\left(\frac{x-\mu}{\sigma} + Q_X(\frac{1}{4})\right) & \text{for } x > \mu, \end{cases} \tag{66}$$

and the PDF is defined as

$$f_T(x) = \begin{cases} \frac{1}{\alpha\sigma}\ f_X\left(\frac{x-\mu}{\alpha\sigma} + Q_X(\frac{1}{4})\right) & \text{for } x \leq \mu, \\ \frac{1}{\sigma}\ f_X\left(\frac{x-\mu}{\sigma} + Q_X(\frac{1}{4})\right) & \text{for } x > \mu. \end{cases} \tag{67}$$

■

4.2.2 Quantile measures of the distributional form

The quantile measures of the distributional form can be constructed, due to quantile splicing, to investigate the properties of the two-piece families of distributions with regards to their location, scale and shape.

- **Location:** The median is the chosen measure of location defined by

$$me = Q_T\left(\frac{1}{2}\right). \tag{68}$$

- **Spread:** The spread function summarises the span that is generated by the two-piece family of distributions. This function is strictly increasing and location-invariant. The spread function is defined as

$$S_T(s) = Q_T(s) - Q_T(1-s), \tag{69}$$

  for $\frac{1}{2} < s < 1$.

  For $\frac{1}{2} < s < 1$, it can be noted that $Q_T(s) > Q_T(1-s)$, hence $S(s) > 0$. If these conditions hold, the requirements are met for $S(s)$ to be a valid spread function. Special cases of the spread function exist, namely the inter-decile range (IDR) and the inter-quartile range (IQR), for which $s = \frac{9}{10}$ and $s = \frac{3}{4}$, respectively.

- **γ-functional:**

  The γ-functional is defined as:

$$\gamma_T(s) = \frac{Q_T(s) + Q_T(1-s) - 2Q_T(\frac{1}{2})}{Q_T(s) - Q_T(1-s)} = \frac{Q_T(s) + Q_T(1-s) - 2me}{Q_T(s) - Q_T(1-s)}, \tag{70}$$

  for $\frac{1}{2} < s < 1$.

  The functional value increases as the numerator increases and the opposite also holds. If $s = \frac{3}{4}$ in Eq.(70), then the resulting equation is equal to the Bowley's quartile-based measure of skewness proposed by Bowley (1902).

- **Ratio-of-spread function:**

  This is an additional measure of kurtosis that describes the location of the probability mass that exists in the distribution's tails. It gets measured for any set of values a and b, such as

$$R_T(a,b) = \frac{S_T(a)}{S_T(b)}, \tag{71}$$

  for $\frac{1}{2} < a < b < 1$. Since $S_T(a) > S_T(b)$ holds for $\frac{1}{2} < a < b < 1$, it follows that $R_T(a,b) > 1$.

4.2.3 r^{th} order L-moments

Theorem 5. *Let T represent a random variable that follows a two-piece distribution, denoted by $T \sim TP(\mu, \sigma, \alpha)$, for which the location parameter exists for $-\infty < \mu < \infty$, the spread parameter is represented by $\sigma > 0$ and $\alpha > 0$ is the asymmetry parameter. Let $0 < k < 1$, then the general form of the r^{th} order L-moments follows as*

$$L_{T:r} = \mu^* + \sigma(L_{X:r} - Q_X(k)) - k\sigma(1-\alpha)\left(\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)} \mu_{j:j}}{j} - Q_X(k)\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)}}{j}\right), \tag{72}$$

where $Q_X(k)$ represents the quantile function of X, $\mu_{j:j}$ is the expected value of the greatest observation in a sample of size r, originating from the k^{th} piece distribution, which is generated from X, the parent distribution and the coefficients of the shifted scaled Legendre polynomials are represented by $c_{j-1}^{(r-1)}$, for $j = 1, 2, ..., r$ and $r \geq 1$.

Proof. See Mac'Oduol et al. (2020) for the detailed proofs. □

Remark. *Note that when $r = 1$, it follows that the location parameter $\mu^* = \int_0^1 \mu P_{r-1}^*(p)dp = \mu$, and when $r > 1$ it equals 0.*

Using Eq.(72) when $k = \frac{1}{4}$, the following results are obtained

Lemma 7. *Let T be a random variable as in Theorem 5 and let the scaling factor $k = \frac{1}{4}$. Then the general form of the r^{th} order L-moments follows from Eq.(72) as*

$$L_{T:r} = \mu^* + \sigma\Big(L_{X:r} - Q_X\Big(\frac{1}{4}\Big)\Big) - \Big(\frac{1}{4}\Big)\sigma(1-\alpha)\left(\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)} \mu_{j:j}}{j} - Q_X\Big(\frac{1}{4}\Big)\frac{\sum_{j=1}^{r} c_{j-1}^{(r-1)}}{j}\right) \tag{73}$$

and has the following L-skewness ratio:

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}}, \tag{74}$$

and the following L-kurtosis ratio:

$$tau_{T:4} = \frac{L_{T:4}}{L_{T:2}}. \tag{75}$$

■

4.3 Examples of two-piece distributions spliced at the lower quartile

Examples of the two-piece families of distributions are proposed, stating their PDF, CDF and quantile functions, as well as their L-moments, when $k = \frac{1}{4}$. The logistic, cosine and Student's $t(2)$ distribution are considered.

4.3.1 Logistic distribution:

- **Definition and distributional properties.**

  The two-piece logistic distribution has the following quantile function, CDF and PDF respectively, using Eq.(58), Eq.(59) and Eq.(60), when $k = \frac{1}{4}$

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma\left(\log\left(\dfrac{p}{1-p}\right) - \log\left(\dfrac{1}{3}\right)\right), & p \leq \dfrac{1}{4}, \\ \mu + \sigma\left(\log\left(\dfrac{p}{1-p}\right) - \log\left(\dfrac{1}{3}\right)\right), & p > \dfrac{1}{4}, \end{cases} \tag{76}$$

$$F_T(x) = \begin{cases} \dfrac{e^{\left(\frac{x-\mu}{\alpha\sigma} + \log\left(\frac{1}{3}\right)\right)}}{1 + e^{\left(\frac{x-\mu}{\alpha\sigma} + \log\left(\frac{1}{3}\right)\right)}}, & x \leq \mu, \\ \dfrac{e^{\left(\frac{x-\mu}{\sigma} + \log\left(\frac{1}{3}\right)\right)}}{1 + e^{\left(\frac{x-\mu}{\sigma} + \log\left(\frac{1}{3}\right)\right)}}, & x > \mu, \end{cases} \tag{77}$$

and

$$f_T(x) = \begin{cases} \dfrac{e^{\left(\frac{x-\mu}{\alpha\sigma} + \log\left(\frac{1}{3}\right)\right)}}{\alpha\sigma\left(1 + e^{\left(\frac{x-\mu}{\alpha\sigma} + \log\left(\frac{1}{3}\right)\right)}\right)^2}, & x \leq \mu, \\ \dfrac{e^{\left(\frac{x-\mu}{\sigma} + \log\left(\frac{1}{3}\right)\right)}}{\sigma\left(1 + e^{\left(\frac{x-\mu}{\sigma} + \log\left(\frac{1}{3}\right)\right)}\right)^2}, & x > \mu, \end{cases} \tag{78}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$.

Figure 1 depicts the PDF of the two-piece logistic distribution, for varying values of α.

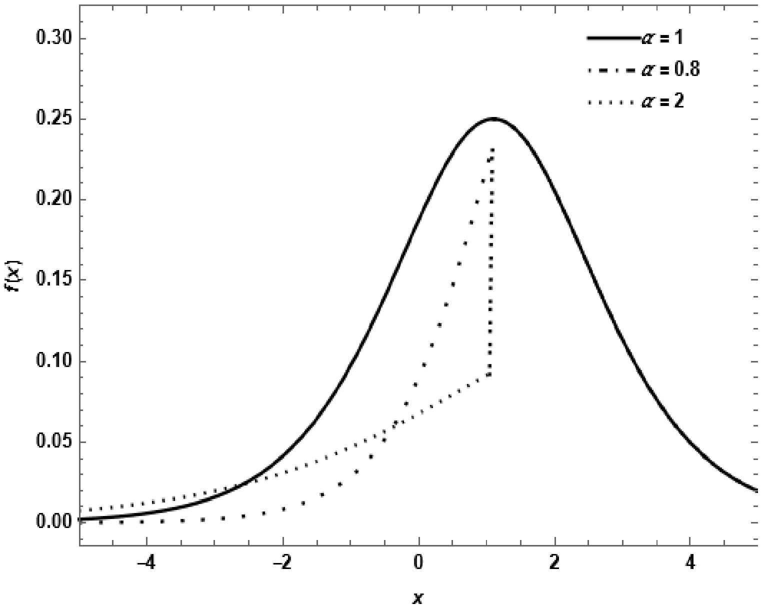


Figure 1: The two-piece logistic distribution probability density curves, with $L_1 = 0$ and $L_2 = 1$, for varying values of $\alpha > 0$.

- **Quantile-based measures of distributional form:**

  The quantile-based measures of the two-piece logistic distribution are given as

  - **Location:** The median is

$$\begin{aligned} me =& Q_T\Big(\frac{1}{2}\Big) \\ =& \mu + \sigma\Big(\log\Big(\frac{\frac{1}{2}}{1-\frac{1}{2}}\Big) - \log\Big(\frac{1}{3}\Big)\Big) \\ =& \mu - \sigma \log\Big(\frac{1}{3}\Big). \end{aligned}$$

  - **Spread:** The spread function is obtained as

$$\begin{aligned} S_t(s) =& Q_T(s) - Q_T(1-s) \\ =& \Big\{\mu + \sigma\Big(\log\Big(\frac{s}{1-s}\Big) - \log\Big(\frac{1}{3}\Big)\Big)\Big\} - \Big\{\mu + \alpha\sigma\Big(\log\Big(\frac{1-s}{1-(1-s)}\Big) - \log\Big(\frac{1}{3}\Big)\Big)\Big\} \\ =& \sigma\Big(\log\Big(\frac{s}{1-s}\Big) - \log\Big(\frac{1}{3}\Big)\Big) - \alpha\sigma\Big(\log\Big(\frac{1-s}{s}\Big) - \log\Big(\frac{1}{3}\Big)\Big), \end{aligned}$$

    for $\frac{1}{2} < s < 1$.

  - **γ-functional:**

The γ-functional is derived by

$$\begin{aligned}\gamma_T(s) =&\frac{Q_T(s)+Q_T(1-s)-2me}{S_T(s)}\\ =&\frac{\left\{\mu+\sigma\left(\log\left(\frac{s}{1-s}\right)-\log\left(\frac{1}{3}\right)\right)\right\}+\left\{\mu+\alpha\sigma\left(\log\left(\frac{1-s}{1-(1-s)}\right)-\log\left(\frac{1}{3}\right)\right)\right\}-2\left(\mu-\sigma\log\left(\frac{1}{3}\right)\right)}{\sigma\left(\log\left(\frac{s}{1-s}\right)-\log\left(\frac{1}{3}\right)\right)-\alpha\sigma\left(\log\left(\frac{1-s}{s}\right)-\log\left(\frac{1}{3}\right)\right)}\\ =&\frac{\sigma\left(\log\left(\frac{s}{1-s}\right)-\log\left(\frac{1}{3}\right)\right)+\alpha\sigma\left(\log\left(\frac{1-s}{s}\right)-\log\left(\frac{1}{3}\right)\right)+2\sigma\log\left(\frac{1}{3}\right)}{\sigma\left(\log\left(\frac{s}{1-s}\right)-\log\left(\frac{1}{3}\right)\right)-\alpha\sigma\left(\log\left(\frac{1-s}{s}\right)-\log\left(\frac{1}{3}\right)\right)}\\ =&\frac{\sigma\left(\log\left(\frac{s}{1-s}\right)+\log\left(\frac{1}{3}\right)\right)+\alpha\sigma\left(\log\left(\frac{1-s}{s}\right)-\log\left(\frac{1}{3}\right)\right)}{\sigma\left(\log\left(\frac{s}{1-s}\right)-\log\left(\frac{1}{3}\right)\right)-\alpha\sigma\left(\log\left(\frac{1-s}{s}\right)-\log\left(\frac{1}{3}\right)\right)},\end{aligned}$$

for $\frac{1}{2} < s < 1$.

- **Ratio-of-spread function:**

  The ratio-of-spread functions, for $\frac{1}{2} < a < b < 1$ is

$$R_T(a,b)=\frac{S_T(a)}{S_T(b)}=\frac{\sigma\left(\log\left(\frac{a}{1-a}\right)-\log\left(\frac{1}{3}\right)\right)-\alpha\sigma\left(\log\left(\frac{1-a}{a}\right)-\log\left(\frac{1}{3}\right)\right)}{\sigma\left(\log\left(\frac{b}{1-b}\right)-\log\left(\frac{1}{3}\right)\right)-\alpha\sigma\left(\log\left(\frac{1-b}{b}\right)-\log\left(\frac{1}{3}\right)\right)}.$$

- r^{th} **order L-moments:**

Theorem 6. *The first four L-moments of a two-piece logistic distribution, when $k=\frac{1}{4}$ follows as*

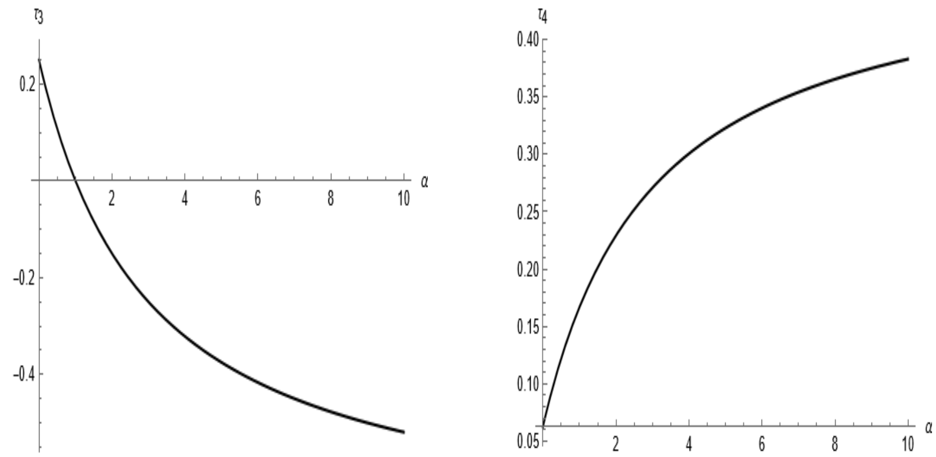
$$\begin{aligned}L_{T:1} =&1.38629-0.28768\alpha,\\ L_{T:2} =&0.75+0.25\alpha,\\ L_{T:3} =&0.1875-0.1875\alpha,\\ L_{T:4} =&0.04685+0.11979\alpha.\end{aligned}$$

The L-skewness and L-kurtosis ratios for the two-piece logistic distribution, when $k=\frac{1}{4}$, are given below

$$\tau_{T:3}=\frac{L_{T:3}}{L_{T:2}}=\frac{0.1875-0.1875\alpha}{0.75+0.25\alpha},\qquad \tau_{T:4}=\frac{L_{T:4}}{L_{T:2}}=\frac{0.04685+0.11979\alpha}{0.75+0.25\alpha}.$$

Proof. See Appendix for detailed proofs. □

Figure 2 presents the L-skewness and L-kurtosis ratios, for different values of α.



(a) L-skewness ratio plot (b) L-kurtosis ratio plot

Figure 2: The two-piece logistic distribution L-skewness and L-kurtosis ratio plots

It can be observed that $\tau_{T:3}$ is a decreasing function as the value of α increases. This implies that the distribution becomes heavily skewed as $\alpha \to \infty$. Since $k = \frac{1}{4}$, the L-kurtosis ratio is no longer a constant value. As the value of $\alpha \to \infty$, the L-kurtosis ratio tends to a constant value of 0.479166.

4.3.2 Cosine distribution:

- **Definition and distributional properties:**

The two-piece cosine distribution has the following quantile function, CDF and PDF respectively, using Eq.(58), Eq.(59) and Eq.(60), when $k = \frac{1}{4}$:

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma\left(\dfrac{4}{\pi}\arcsin(\sqrt{p}) - \dfrac{2}{3}\right), & p \leq \dfrac{1}{4}, \\ \mu + \sigma\left(\dfrac{4}{\pi}\arcsin(\sqrt{p}) - \dfrac{2}{3}\right), & p > \dfrac{1}{4}. \end{cases} \tag{79}$$

$$F_T(x) = \begin{cases} 0, & 0 < \mu - \alpha\sigma, \\ \sin^2\Big(\dfrac{\pi}{2}\Big(\dfrac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big) - \dfrac{1}{3}\Big), & \mu - \alpha\sigma, < x < \mu, \\ \sin^2\Big(\dfrac{\pi}{2}\Big(\dfrac{x-(\mu-\sigma)}{2\sigma}\Big) - \dfrac{1}{3}\Big), & \mu < x < \mu + \sigma, \\ 1, & x \geq \mu + \sigma. \end{cases} \tag{80}$$

$$f_T(x) = \begin{cases} \dfrac{\pi}{4\alpha\sigma}\sin\Big(\pi\Big(\dfrac{x-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big)-\dfrac{1}{3}\Big), & \mu-\alpha\sigma < x < \mu, \\ \dfrac{\pi}{4\sigma}\sin\Big(\pi\Big(\dfrac{x-(\mu-\sigma)}{2\sigma}\Big)-\dfrac{1}{3}\Big), & \mu < x < \mu+\sigma, \\ 0 & \text{elsewhere} \end{cases} \tag{81}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$.

Figure 3 represents the PDF of the two-piece logistic distribution, for different values of $\alpha > 0$.

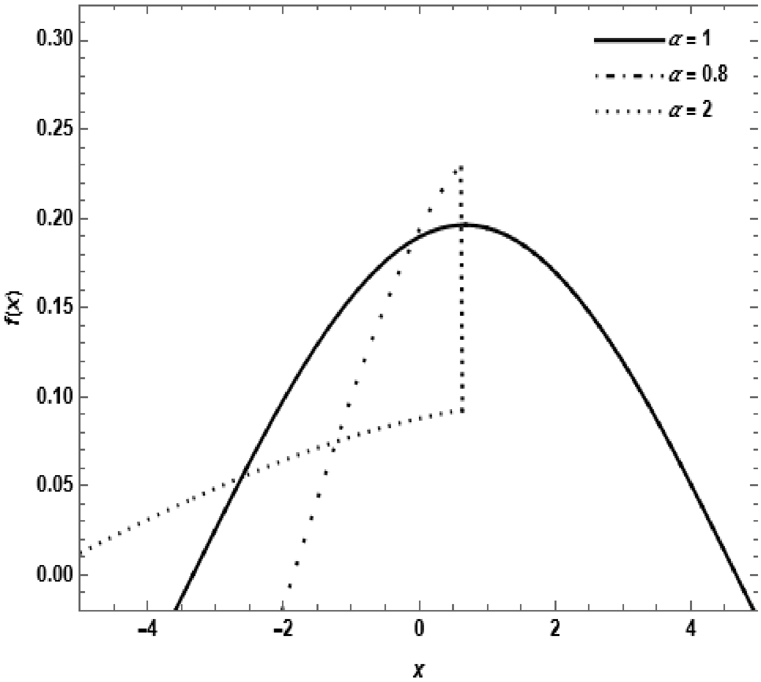


Figure 3: The two-piece cosine distribution probability density curves, with $L_1 = 0$ and $L_2 = 1$, for different values of $\alpha > 0$.

- **Quantile-based measures of distributional form:**
  - **Location:** The median is

$$\begin{aligned} me =& Q_T\Big(\frac{1}{2}\Big) \\ =& \mu + \sigma\left(\frac{4}{\pi}\arcsin\Big(\sqrt{\frac{1}{2}}\Big) - \frac{2}{3}\right) \\ =& \mu + \frac{\sigma}{3}. \end{aligned}$$

  - **Spread:**

The spread function is obtained as

$$\begin{aligned}S_t(s) =&Q_T(s) - Q_T(1-s)\\=&\Bigg\{\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{2}{3}\Big)\Bigg\} - \Bigg\{\mu + \sigma\alpha\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big)\Bigg\}\\=&\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{2}{3}\Big) - \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big),\end{aligned}$$

for $\frac{1}{2} < s < 1$.

– **Shape:**

The γ-functional is derived by

$$\begin{aligned}\gamma_T(s) =&\frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)}\\=&\frac{\Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{2}{3}\Big)\Big) + \Big(\mu + \sigma\alpha\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big)\Big) - 2\Big(\mu + \frac{\sigma}{3}\Big)}{\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{2}{3}\Big) - \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big)}\\=&\frac{\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{4}{3}\Big) + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big)}{\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{s}) - \frac{2}{3}\Big) - \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-s}) - \frac{2}{3}\Big)},\end{aligned}$$

for $\frac{1}{2} < s < 1$.

– **Ratio-of-spread functions:**

The ratio-of-spread functions, for $\frac{1}{2} < a < b < 1$ is

$$R_T(a,b) = \frac{S_T(a)}{S_T(b)} = \frac{\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{a}) - \frac{2}{3}\Big) - \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-a}) - \frac{2}{3}\Big)}{\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{b}) - \frac{2}{3}\Big) - \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}(\sqrt{1-b}) - \frac{2}{3}\Big)}.$$

- r^{th} **order L-moments:**

Theorem 7. *The first four L-moments of a two-piece cosine distribution, when $k = \frac{1}{4}$ follows as*

$$\begin{aligned}L_{T:1} =&1.56401 - 0.230676\alpha,\\L_{T:2} =&0.804499 + 0.195501\alpha,\\L_{T:3} =&0.137832 - 0.137832\alpha,\\L_{T:4} =&-0.0143277 + 0.076825\alpha.\end{aligned}$$

The L-skewness and L-kurtosis ratios are defined and given below for the two-piece cosine distribu-

tion

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{0.137832 - 0.137832\alpha}{0.804499 + 0.195501\alpha}, \qquad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{0.137832 - 0.137832\alpha}{0.804499 + 0.195501\alpha}.$$

Proof. See Appendix for detailed proofs. □

The plots of these L-skewness and L-kurtosis ratios, for different values of α, are illustrated in Figure 4.

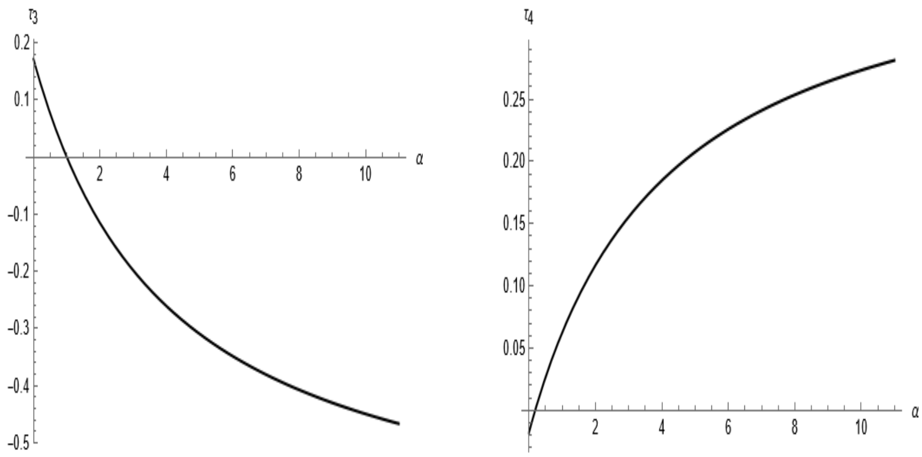


Figure 4: The L-skewness and L-kurtosis ratio plots for the two-piece cosine distribution

It can be observed that $\tau_{T:3}$ is a decreasing function when $\alpha > 0$. The level of kurtosis, $\tau_{T:4}$ is increasing for all values of $\alpha > 0$. Due to the scaling factor, $k = \frac{1}{4}$ that was introduced, the level of kurtosis is not constant for the cosine two-piece distribution, but as $\alpha \to \infty$, the value of the L-kurtosis tends to a constant value of 0.392978.

4.3.3 Student's $t(2)$ distribution:

- **Definition and distributional properties:**

  The two-piece Student's $t(2)$ distribution has the following quantile function, CDF and PDF respectively, using Eq.(58), Eq.(59) and Eq.(60), when $k = \frac{1}{4}$:

$$Q_T(p) = \begin{cases} \mu + \alpha\sigma\left(\dfrac{2p-1}{(2p(1-p))^{\frac{1}{2}}} + \dfrac{\sqrt{6}}{3}\right), & p \leq \dfrac{1}{4}, \\ \mu + \sigma\left(\dfrac{2p-1}{(2p(1-p))^{\frac{1}{2}}} + \dfrac{\sqrt{6}}{3}\right), & p > \dfrac{1}{4}, \end{cases} \tag{82}$$

$$F_T(x) = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{\frac{x-\mu}{\alpha\sigma} - \frac{\sqrt{6}}{3}}{\sqrt{2 + \left(\frac{x-\mu}{\alpha\sigma}\right)^2 - \frac{\sqrt{6}}{3}}}\right), & x \le \mu, \\ \dfrac{1}{2}\left(1 + \dfrac{\frac{x-\mu}{\sigma} - \frac{\sqrt{6}}{3}}{\sqrt{2 + \left(\frac{x-\mu}{\sigma}\right)^2 - \frac{\sqrt{6}}{3}}}\right), & x > \mu. \end{cases} \tag{83}$$

$$f_T(x) = \begin{cases} \dfrac{1}{\alpha\sigma}\left(2 + \left(\dfrac{x-\mu}{\alpha\sigma}\right)^2 - \dfrac{\sqrt{6}}{3}\right)^{-\frac{3}{2}}, & x \le \mu, \\ \dfrac{1}{\sigma}\left(2 + \left(\dfrac{x-\mu}{\sigma}\right)^2 - \dfrac{\sqrt{6}}{3}\right)^{-\frac{3}{2}}, & x > \mu. \end{cases} \tag{84}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $\alpha > 0$.

Figure 5 depicts the PDF of the two-piece Student's $t(2)$ distribution, for varying values of $\alpha > 0$.

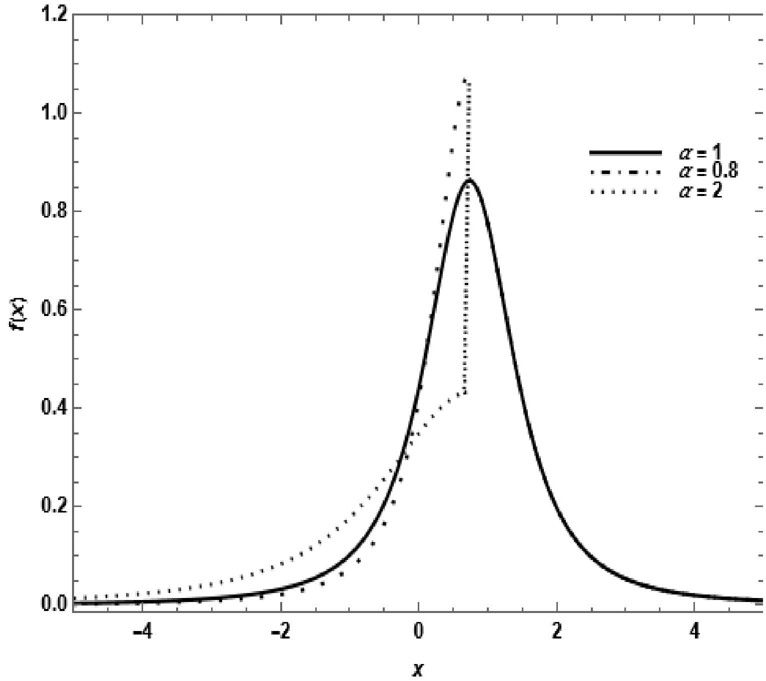


Figure 5: The two-piece Student's $t(2)$ distribution probability density curves, with $L_1 = 0$ and $L_2 = 1$, for different values of $\alpha > 0$.

- **Quantile-based measures of distributional form:**
  - **Location:**

The median is

$$\begin{aligned}
me =& Q_T(\frac{1}{2}) \\
=& \mu + \sigma\left(\frac{2.\frac{1}{2}-1}{\sqrt{(2.\frac{1}{2}(1-\frac{1}{2}))}} - \frac{\sqrt{6}}{3}\right) \\
=& \mu - \sigma\Big(\frac{\sqrt{6}}{3}\Big).
\end{aligned}$$

– **Spread:**

The spread function is derived as

$$\begin{aligned}
S_t(s) =& Q_T(s) - Q_T(1-s) \\
=& \left\{\mu + \sigma\left(\frac{2s-1}{\sqrt{2s(1-s)}} - \frac{\sqrt{6}}{3}\right)\right\} - \left\{\mu + \alpha\sigma\left(\frac{2(1-s)-1}{\sqrt{(2(1-s)(1-(1-s))}} - \frac{\sqrt{6}}{3}\right)\right\} \\
=& \sigma\left(\frac{2s-1}{\sqrt{2s(1-s)}} - \frac{\sqrt{6}}{3}\right) - \sigma\alpha\left(\frac{2-2s-1}{\sqrt{s(2-2s)}} - \frac{\sqrt{6}}{3}\right) \\
=& \sigma\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right) + \sigma\alpha\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right) \\
=& \sigma(1+\alpha)\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right),
\end{aligned}$$

for $\frac{1}{2} < s < 1$.

– **Shape:**

The γ-functional is obtained as

$$\begin{aligned}
\gamma_T(s) =& \frac{Q_T(s) + Q_T(1-s) - 2me}{S_T(s)} \\
=& \frac{\left\{\mu + \sigma\left(\frac{2s-1}{\sqrt{2s(1-s)}} - \frac{\sqrt{6}}{3}\right)\right\} - \left\{\mu + \alpha\sigma\left(\frac{2(1-s)-1}{\sqrt{2(1-s)(1-(1-s))}} - \frac{\sqrt{6}}{3}\right)\right\} - 2\Big(\mu - \sigma\Big(\frac{\sqrt{6}}{6}\Big)\Big)}{\sigma(1+\alpha)\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right)} \\
=& \frac{\sigma\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right) + \sigma\alpha\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right) + 2\sigma\left(\frac{\sqrt{6}}{3}\right)}{\sigma(1+\alpha)\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right)} \\
=& \frac{\sigma\left(\frac{2s-1}{\sqrt{2s-s^2}} + \frac{\sqrt{6}}{3}\right) + \sigma\alpha\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right)}{\sigma(1+\alpha)\left(\frac{2s-1}{\sqrt{2s-s^2}} - \frac{\sqrt{6}}{3}\right)},
\end{aligned}$$

for $\frac{1}{2} < s < 1$.

– **Ratio-of-spread functions:**

The ratio-of-spread functions, for $\frac{1}{2} < a < b < 1$ is

$$R_T(a,b) = \frac{S_T(a)}{S_T(b)} = \frac{\sigma(1+\alpha)\left(\frac{2a-1}{\sqrt{2a-a^2}} - \frac{\sqrt{6}}{3}\right)}{\sigma(1+\alpha)\left(\frac{2b-1}{\sqrt{2b-b^2}} - \frac{\sqrt{6}}{3}\right)} = \frac{\left(\frac{2a-1}{\sqrt{2a-a^2}} - \frac{\sqrt{6}}{3}\right)}{\left(\frac{2b-1}{\sqrt{2b-b^2}} - \frac{\sqrt{6}}{3}\right)}.$$

- r^{th} **order L-moments:**

Theorem 8. *The first four L-moments of a two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$ follows as*

$$\begin{aligned} L_{T:1} &= 1.10266 - 0.36755\alpha, \\ L_{T:2} &= 0.66667 + 0.33333\alpha, \\ L_{T:3} &= 0.27566 - 0.27566\alpha, \\ L_{T:4} &= 0.16855 + 0.211145\alpha. \end{aligned}$$

The L-skewness and L-kurtosis ratios are defined, for the two-piece Student's $t(2)$ distribution, as

$$\tau_{T:3} = \frac{L_{T:3}}{L_{T:2}} = \frac{0.27566 - 0.27566\alpha}{0.66667 + 0.33333\alpha}, \qquad \tau_{T:4} = \frac{L_{T:4}}{L_{T:2}} = \frac{0.16855 + 0.211145\alpha}{0.66667 + 0.33333\alpha}.$$

Proof. See Appendix for detailed proofs. ∎

Figure 6 depicts the L-skewness and L-kurtosis ratios, for different values of α.

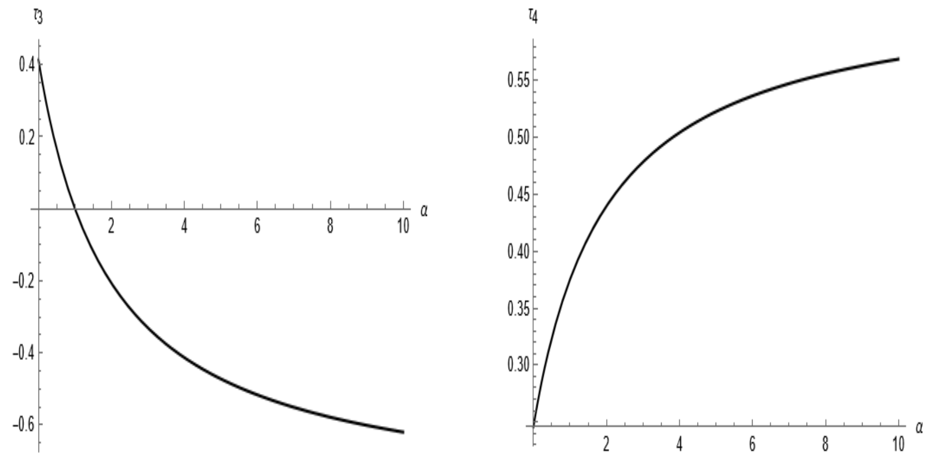


Figure 6: The L-skewness and L-kurtosis ratio plots for the two-piece Student's $t(2)$ distribution

It can be observed that $\tau_{T:3}$ is a decreasing function as α increases. $\tau_{T:4}$ is an increasing function as α increases. Since $k = \frac{1}{4}$, the L-kurtosis is no longer a constant ratio for this two-piece distribution, however, as $\alpha \to \infty$, the L-kurtosis ratio tends to a constant value of 0.633434.

4.4 Method of L-moments estimation

Assuming that there does not exist any closed-form expressions for the PDF or CDF, it makes the process of matching a data set to this model more difficult. Hence, another estimation method, other than maximum likelihood estimation and the method of moments estimation is considered. Let $S(\underline{\theta})$ represents a set of functions that represents the properties of the population. $S(\underline{\theta})$ is dependent on the quantile function, $Q_X(r;\theta)$, and its parameters $\underline{\theta}$. Therefore, the number of sample quantities matches the number of functions in $S(\underline{\theta})$.

If the model is a quantile-based distribution, then making use of the quantile function is very beneficial. The occurrence of the method of percentiles for specific percentiles arose due to this advantage, where $S(\underline{\theta}) = (me)$, IQR or the quantile function $Q(r;\theta)$.

The method of L-moments matches $S(\underline{\theta})$ for the sample quantities, which are the sample L-moments for this case, as suggested by Hosking (1990). Due to the L-moments that have a closed-form expression in Eq.(64), this method is applied in order to find the parameter estimates for the two-piece families of distributions. During this process, four sample quantities are considered and matched to the population functions that need to be estimated, denoted by ℓ_i, for $i = 1,2,3,4$. ℓ_1 is matched to the location parameter, ℓ_2 is matched to the scale parameter, ℓ_3 is matched to the L-skewness ratio (τ_3) and lastly ℓ_4 is matched to the L-kurtosis ratio (τ_4).

These values are defined by the sample order statistics of size$=n$. Thereafter, the U-statistics are used to estimate these values, by defining the average of the sub-samples as a function, with a size of r.

U-statistics are a direct result obtained from the observed data, of size n.

Lemma 8. *Let $X_1, X_2, X_3, ..., X_n$ represent a sample of size n. The ordered sample follows as $X_{(1)} < X_{(2)} < X_{(3)} < ... < X_{(n)}$. The r^{th} sample ℓ-moment, defined by Hosking (1990), is given as*

$$\ell_r = \binom{n}{r}^{-1} \underset{1<i_1<i_2<i_3<...<i_r<n}{\sum\sum\sum\cdots\sum\sum} r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k:n}}, \tag{85}$$

for $r = 1, 2, 3, ..., n$.

The first four ℓ-moments obtained from Eq.(85) *are*

$$\begin{aligned} l_1 =& \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}, \\ l_2 =& \frac{1}{2} \binom{n}{2}^{-1} \sum_{i>j}^{n} \sum (x_i - x_j), \\ l_3 =& \frac{1}{3} \binom{n}{3}^{-1} \sum_{i>j>k}^{n} \sum\sum (x_i - 2x_j + x_k), \\ l_4 =& \frac{1}{4} \binom{n}{4}^{-1} \sum_{i>j>k>l}^{n} \sum\sum\sum (x_i - 3x_j + 3x_k - x_l). \end{aligned} \tag{86}$$

The sample has the following L-skewness ratio:

$$t_3 = \frac{l_3}{l_4}, \tag{87}$$

and the following L-kurtosis ratio:

$$t_3 = \frac{l_4}{l_2}. \tag{88}$$

■

Parameter estimates are obtained by matching the estimates in Lemma 8 to the population values of any univariate two-piece distribution,by following the subsequent steps.

Step 1:

Applying Eq.(77) to the observed data set, the first four sample L-moments can be obtained. Using Eq.(87) and Eq.(88), the sample L-kurtosis and L-skewness ratios are calculated, respectively. Thereafter the effectiveness of this proposed two-piece distribution is verified by fitting the t_3 and t_4 values to the $(\tau_{T:3}, \tau_{T:4})$-space of the two-piece distribution. If they lie within this space, we conclude that the proposed two-piece distribution may be fit to the data set.

If there exists an additional shape parameter in the two-piece distribution, then the $(\tau_{T:3}a, \tau_{T:4})$-space is a region enclosed by all combinations of the two shape parameters. If t_3 and t_4 exist within this region, then the estimation process can continue.

Step 2:

Solve $\hat{\alpha}$, the asymmetry parameter estimate.. Set $t_3 = \tau_{T:3}$.

Step 3:

Solve $\hat{\sigma}$, the scale parameter estimate, by substituting the parameter estimates found in Step 2 into the formula for $L_{T:2}$. Equate this expression to l_2 and solve appropriately.

Step 4:

Solve $\hat{\mu}$, the location parameter estimate, by substituting the parameter estimates found in Step 2 and Step 3 into the formula for $L_{T:1}$ from the two-piece distribution and equate this to ℓ_1 and solve the unknown parameter estimate accordingly.

Step 5:

The standard errors of the parameter estimates are calculated, using the bootstrap method, for $N = 10000$ samples.

5 Maximum likelihood estimation

5.1 Introduction:

According to Gilchrist (2000), the general idea behind the likelihood function is that an observation x_i, in a small interval dx_i, has the probability of occurrence $f(x_i : \theta)\ dx_i$, where $f(x_i : \theta)$ represents the PDF with parameter θ. If a set of n independent observations exists, then the probability changes to the product of the marginal PDFs of the independent observations, $\prod(f(x_i : \theta)\ dx_i)$.

Therefore, the likelihood function is defined as

$$\begin{aligned} l(\theta) &= \prod(f(x_i : \theta)\ dx_i) \\ &= f(x_1 : \theta)f(x_2 : \theta)...f(x_n : \theta). \end{aligned} \tag{89}$$

The likelihood is large if the value selected for θ is near the true value, whilst the likelihood is small if the value selected for θ is a far off, relative to the true value.

5.2 MLE for quantile-based distributions

The likelihood function for quantile functions is defined as

$$\mathrm{L}(\theta) = f_p(p_{(1)} : \theta)f_p(p_{(2)} : \theta)...f_p(p_{(n)} : \theta), \tag{90}$$

with p_r defined as $x_r = Q(p_{(r)} : \theta)$. This corresponds to the p-value which generates the observed x for a given value of θ. The log-likelihood is defined as

$$\begin{aligned} l(\theta) &= \ln[\mathrm{L}(\theta)] \\ &= \sum \ln\ f_p(p_{(r)} : \theta). \end{aligned} \tag{91}$$

The log-likelihood is a maximized goodness-of-fit criterion. The method of maximum likelihood selects the parameter to maximise the likelihood, or rather, the log-likelihood. The advantage of utilizing the method of maximum likelihood is that it produces estimators with useful properties. These properties include:

- The maximum likelihood estimators $\hat{\theta}$, obtained for big samples, are unbiased, follow an approximately normal distribution and have a minimum variance value.

- If a minimum variance and the unbiased property holds theoretically for an estimator from a small sample, then the estimator is granted the maximum likelihood.

- The properties of the variance of the obtained estimators are derivable.
- The more observations obtained in the sample implies that the estimators become closer to the true value, hence they are consistent estimators.

Proposition:

Considering the family of distributions obtained by the transformation $X = \nu + \delta^{-1}Z$, where $\delta > 0$, is used as shown by Kim (2005). A random variable, X, has the following density

$$h(x;\theta,\nu,\delta) = c_\theta \delta\phi(\delta(x-\nu))\Phi(\theta\delta|x-\nu|). \tag{92}$$

Suppose there exists a random sample emanating from the density in Eq.(92) that exists, such as $X_1, X_2, ..., X_n$. The log-likelihood of the parameters θ, ν and δ follows as

$$L_n(\theta,\delta,\nu) = n\ln c_\theta + n\ln\delta + \sum_{i=1}^{n}\ln\phi(\delta(x_i-\nu)) + \sum_1 \ln\Phi(\delta\theta(x_i-\nu)) + \sum_2 \ln\Phi(-\delta\theta(x_i-\nu)), \tag{93}$$

where the summation over all observations is represented by $\sum_1$ for $x_i - \nu \geq 0$ and $\sum_2$ for $x_i - \nu < 0$.

After reparametrisation, such that $\alpha = \delta\nu$, and keeping θ fixed, the log-likelihood L_n changes to

$$L_n = \text{constant} + n\ln\delta + \sum_{i=1}^{n}\ln\phi(\omega_i) + \sum_1 \ln\Phi(u_i) + \sum_2 \ln\Phi(v_i), \tag{94}$$

where $\omega_i = \delta x_i - \alpha$, $u_i = \theta\delta x_i - \theta\alpha$ and $v_i = -\theta\delta x_i - \theta\alpha$.

Due to the log-concavity of Eq.(94), keeping θ fixed, it follows that δ and α have unique solutions. Applying the same methodology as Azzalini (1985), we estimate by solving for δ and α, keeping θ fixed, the following equations

$$\frac{\partial L_n}{\partial\delta} = n\delta^{-1} - \sum_{i=1}^{n}\omega_i x_i + \theta\Big\{\sum_1 x_i\eta(u_i) - \sum_2 x_i\eta(v_i)\Big\} = 0 \tag{95}$$

$$\frac{\partial L_n}{\partial\alpha} = \sum_{i=1}^{n}\omega_i - \theta\Big\{\sum_1 \eta(u_i) - \sum_2 \eta(v_i)\Big\} = 0, \tag{96}$$

with $\eta(u_i) = \frac{\phi(u_i)}{\Phi(u_i)}$ and $\eta(v_i) = \frac{\phi(v_i)}{\Phi(v_i)}$. This can be solved via the Newton-Raphson method. A range of values for θ is selected, and this step is repeated with this range of values for θ to generate a profile likelihood, from which the value of θ can be estimated.

The sample information matrix obtained via the Newton-Raphson procedure is

$$\frac{-\partial^2 L_n}{\partial \delta^2} = n\delta^{-2} + \sum_{i=1}^{n} x_i^2 + \theta^2 \Big\{ \sum_1 [x_i^2 u_i \eta(u_i) + x_i^2 \eta(u_i)^2] + \sum_2 [x_i^2 v_i \eta(v_i) + x_i^2 \eta(v_i)^2] \Big\}, \tag{97}$$

$$\frac{-\partial^2 L_n}{\partial \alpha^2} = -n + \theta^2 \Big\{ \sum_1 [u_i \eta(u_i) + \eta(u_i)^2] + \sum_2 [v_i \eta(v_i) + \eta(v_i)^2] \Big\}, \tag{98}$$

$$\frac{-\partial^2 L_n}{\partial \alpha \partial \delta} = -\sum_{i=1}^{n} x_i - \theta^2 \Big\{ \sum_1 [x_i u_i \eta(u_i) + x_i \eta(u_i)^2] + \sum_2 [x_i v_i \eta(v_i) + x_i \eta(v_i)^2] \Big\}. \tag{99}$$

■

The proposition holds for the more general model, $X_i = \tau_i'\beta + \delta^{-1} Z_i$, for $i = 1, 2, ..., n$ and where β is a p-dimensional parameter, τ_i represents the p-vector of covariates and $Z_1, Z_2, ..., Z_n$ are *iid* random variables that follow a two-piece skew-normal distribution with the shape parameter θ. The maximum likelihood estimators may also be derived using the profile log-likelihood in Eq.(94).

The method of maximum likelihood estimation requires maximising the log-likelihood function, in terms of an additional parameter r, where $0 \leq r \leq n$, which is included in the log-likelihood defined in the following theorem.

Theorem 9. *The log-likelihood $l(\theta, \sigma^2, \epsilon)$, includes an integer $r = r(x_{(1)}, x_{(2)}, ..., x_{(n)})$ for $0 \leq r \leq n$, and may be represented by*

$$l(\theta, \sigma^2, \epsilon) = \begin{cases} -\dfrac{n}{2}\log 2\pi\sigma^2 - \dfrac{1}{8\sigma^2}[\displaystyle\sum_{i=1}^{n}(x_{(i)} - \theta)^2] & \text{if } r = 0, n, \\ -\dfrac{n}{2}\log 2\pi\sigma^2 - \dfrac{1}{2\sigma^2}[\displaystyle\sum_{i=1}^{r} \frac{(x_{(i)} - \theta)^2}{(1+\epsilon)^2} + \sum_{i=r+1}^{n} \frac{(x_{(i)} - \theta)^2}{(1+\epsilon)^2}] & \text{if } 1 \leq r \leq n, \end{cases} \tag{100}$$

where the order statistics of an epsilon-skew normal population, ESN$(\theta, \sigma, \epsilon)$, sample is represented by $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ and the log-likelihood's form originates from the PDF of the ESN$(\theta, \sigma, \epsilon)$ distribution. ■

This theorem originates from Mudholkar and Hutson (2000). The form obtained when $r = 0$ is equivalent to the case when $\epsilon = -1$ in the PDF. Likewise, when $r = n$ is equivalent to when $\epsilon = 1$ in the PDF of the ESN$(\theta, \sigma, \epsilon)$. The log-likelihood corresponds to the half-normal distributions, when $r = 1$ and $r = n$.

Lemma 9. *The maximum likelihood estimate of $(\hat{\theta}, \hat{\sigma}^2, \hat{\epsilon})$, when $r = 0$ and $r = n$ follows as*

$$(\hat{\theta}, \hat{\sigma}^2, \hat{\epsilon}) = \begin{cases} (x_{(1)}, s_0^2, -1) & \text{if } r = 0, \\ (x_{(n)}, s_n^2, 1) & \text{if } r = n, \end{cases} \tag{101}$$

where $s_0^2 = \sum_{i=2}^{n} \frac{(x_{(i)} - x_{(1)})^2}{4n}$ and $s_n^2 = \sum_{i=1}^{n-1} \frac{(x_{(i)} - x_{(n)})^2}{4n}$.

If $0 \leq r \leq 1$, then the local minima of the following

$$h_j(\boldsymbol{x}, \theta) = \frac{1}{4}\left\{\Big[\sum_{i=1}^{j}(x_{(i)} - \theta)^2\Big]^{\frac{1}{3}} + \Big[\sum_{i=j+1}^{n}(x_{(i)} - \theta)^2\Big]^{\frac{1}{3}}\right\}^3, \tag{102}$$

for $j = 1, 2, ..., n-1$, can be used to establish the local maxima of the log-likelihood of the ESN(θ, σ, ϵ) family, as stated by Mudholkar and Hutson (2000).

Proof. If $0 \leq r \leq n$ holds for a fixed value of σ^2, then the log-likelihood $l(\theta, \sigma^2, \epsilon)$ in Eq.(100) is maximised where the local minima is as follows

$$h_j(\boldsymbol{x}, \theta, \epsilon) = \sum_{i=1}^{j}\frac{(x_{(i)} - \theta)^2}{(1+\epsilon)^2} + \sum_{i=j+1}^{n}\frac{(x_{(i)} - \theta)^2}{(1-\epsilon)^2}, \tag{103}$$

for $-1 < \epsilon < 1$ and $j = 1, 2, ..., n-1$. Equating the derivative of Eq.(103) to 0, with respect to ϵ, follows as

$$\sum_{i=1}^{j}\frac{(x_{(i)} - \theta)^2}{(1+\epsilon)^3} + \sum_{i=j+1}^{n}\frac{(x_{(i)} - \theta)^2}{(1-\epsilon)^3} = 0. \tag{104}$$

If ϵ is removed between Eq.(103) and Eq.(104), the minimum of $h_j(\boldsymbol{x}, \theta, \epsilon)$, in terms of ϵ is

$$h_j(\boldsymbol{x}, \theta) = \frac{1}{4}\left\{\Big[\sum_{i=1}^{j}(x_{(i)} - \theta)^2\Big]^{\frac{1}{3}} + \Big[\sum_{i=j+1}^{n}(x_{(i)} - \theta)^2\Big]^{\frac{1}{3}}\right\}^3. \tag{105}$$

Therefore, $h_j(\boldsymbol{x}, \theta)$ needs to be minimised in terms of θ, for $x_{(j)} \leq \theta \leq x_{(j+1)}$. The following derivative of $h_j(\boldsymbol{x}, \theta)$ exists, in terms of θ

$$h_j'(\boldsymbol{x}, \theta) = -\frac{2}{3}\left\{\sum_{i=1}^{j}(x_{(i)} - \theta)^2\Big[\sum_{i=1}^{j}(x_{(i)} - \theta)^2\Big]^{-\frac{2}{3}} + \sum_{i=j+1}^{n}(x_{(i)} - \theta)^2\Big[\sum_{i=j+1}^{n}(x_{(i)} - \theta)^2\Big]^{-\frac{2}{3}}\right\}. \tag{106}$$

The solution of $h_j'(\boldsymbol{x}, \theta) = 0$, for $j = 1, 2, ..., n-1$ is clearly the local stationary points of the log-likelihood. $\square$

5.3 Theoretical results to obtain MLE estimates for two-piece families of distributions

5.3.1 Two-piece logistic distribution

Using the PDF given in Mac'Oduol et al. (2020) for the two-piece logistic distribution, the log-likelihood function of the two-piece logistic distribution, when $k = \frac{1}{2}$, follows as:

$$\begin{aligned}
l(\boldsymbol{X}) &= -\sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha\sigma} - \ln(\alpha\sigma) - \ln\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)^2\Big]_{I[X_i\leq\mu]} + \Big[\frac{X_i-\mu}{\sigma} - \ln(\sigma) - \ln\Big(1+e^{\frac{X_i-\mu}{\sigma}}\Big)^2\Big]_{I[X_i>\mu]} \\
&= -\sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha\sigma} - \ln(\alpha\sigma) - 2\ln\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)\Big]_{I[X_i\leq\mu]} + \Big[\frac{X_i-\mu}{\sigma} - \ln(\sigma) - 2\ln\Big(1+e^{\frac{X_i-\mu}{\sigma}}\Big)\Big]_{I[X_i>\mu]} \\
&= n\ln(\alpha\sigma) - \sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha\sigma} - 2\ln\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)\Big]_{I[X_i\leq\mu]} + n\ln(\sigma) - \sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\sigma} - 2\ln\Big(1+e^{\frac{X_i-\mu}{\sigma}}\Big)\Big]_{I[X_i>\mu]}
\end{aligned} \tag{107}$$

The partial derivatives are derived with respect to α, σ and μ, from Eq.(107), such that:

$$\begin{aligned}
\frac{\partial l(\boldsymbol{X})}{\partial\alpha} &= \frac{n}{\alpha\sigma}\sigma - \sum_{i=1}^{n}\Big[-\frac{X_i-\mu}{\alpha^2\sigma} + 2\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}\left(\frac{X_i-\mu}{\alpha^2\sigma}\right)}{1+e^{\frac{X_i-\mu}{\alpha\sigma}}}\Big)\Big]_{I[X_i\leq\mu]} \\
&= \frac{n}{\alpha} - \sum_{i=1}^{n}\Big[\Big(\frac{X_i-\mu}{\alpha^2\sigma}\Big)\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-1}{1+e^{\frac{X_i-\mu}{\alpha\sigma}}}\Big)\Big]_{I[X_i\leq\mu]}
\end{aligned} \tag{108}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial\sigma} = \frac{2n}{\sigma} - \sum_{i=1}^{n}\Big[\Big(\frac{X_i-\mu}{\alpha\sigma^2}\Big)\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-1}{1+e^{\frac{X_i-\mu}{\alpha\sigma}}}\Big)\Big]_{I[X_i\leq\mu]} + \Big[\Big(\frac{X_i-\mu}{\sigma^2}\Big)\Big(\frac{e^{\frac{X_i-\mu}{\sigma}}-1}{1+e^{\frac{X_i-\mu}{\sigma}}}\Big)\Big]_{I[X_i>\mu]} \tag{109}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial\mu} = -\sum_{i=1}^{n}\Big[\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-1}{1+e^{\frac{X_i-\mu}{\alpha\sigma}}}\Big)\Big(\frac{1}{\alpha\sigma}\Big)\Big]_{I[X_i\leq\mu]} + \Big[\Big(\frac{e^{\frac{X_i-\mu}{\sigma}}-1}{1+e^{\frac{X_i-\mu}{\sigma}}}\Big)\Big(\frac{1}{\sigma}\Big)\Big]_{I[X_i>\mu]} \tag{110}$$

The second derivatives are derived with respect to α, σ and μ, from Eq.(108), Eq.(109)and Eq.(110) to obtain:

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n}\Big[\Big(\frac{2(X_i-\mu)}{\alpha^4\sigma^2}\Big)\left(\frac{\alpha\sigma e^{\frac{2(X_i-\mu)}{\alpha\sigma}} + (X_i-\mu)e^{\frac{X_i-\mu}{\alpha\sigma}} - \alpha\sigma}{\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)^2}\right)\Big]_{I[X_i\leq\mu]} \tag{111}$$

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial \sigma^2} = -\frac{2n}{\sigma^2} - \sum_{i=1}^{n}\Big[\Big(\frac{-2(X_i-\mu)}{\alpha^2\sigma^4}\Big)\left(\frac{\alpha\sigma e^{\frac{2(X_i-\mu)}{\alpha\sigma}} + (X_i-\mu)e^{\frac{X_i-\mu}{\alpha\sigma}} - \alpha\sigma}{\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)^2}\right)\Big]_{I[X_i\leq\mu]} - \Big[\Big(\frac{2(X_i-\mu)}{\sigma^4}\Big)$$
$$\left(\frac{\sigma e^{\frac{2(X_i-\mu)}{\sigma}} + (X_i-\mu)e^{\frac{X_i-\mu}{\sigma}} - \sigma}{\Big(1+e^{\frac{X_i-\mu}{\sigma}}\Big)^2}\right)\Big]_{I[X_i>\mu]} \tag{112}$$

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial \mu^2} = -\sum_{i=1}^{n}\Big[\left(\frac{-2e^{\frac{X_i-\mu}{\alpha\sigma}}}{\alpha\sigma\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}}\Big)^2}\right)\Big]_{I[X_i\leq\mu]} - \Big[\left(\frac{2e^{\frac{X_i-\mu}{\sigma}}}{\sigma\Big(1+e^{\frac{X_i-\mu}{\sigma}}\Big)^2}\right)\Big]_{I[X_i>\mu]} \tag{113}$$

Using the PDF given in Eq.(78), the log-likelihood function of the two-piece logistic distribution, when $k=\frac{1}{4}$, follows as:

$$l(\boldsymbol{X}) = -\sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha\sigma} + \log\Big(\frac{1}{3}\Big) - \ln(\alpha\sigma) - \ln\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}+\log(\frac{1}{3})}\Big)^2\Big]_{I[X_i\leq\mu]} + \Big[\frac{X_i-\mu}{\sigma} + \log\Big(\frac{1}{3}\Big) - \ln(\sigma)$$
$$- \ln\Big(1+e^{\frac{X_i-\mu}{\sigma}+\log(\frac{1}{3})}\Big)^2\Big]_{I[X_i>\mu]}$$
$$= -2n\log\Big(\frac{1}{3}\Big) + n\ln(\alpha\sigma) - \sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha\sigma} - 2\ln\Big(1+e^{\frac{X_i-\mu}{\alpha\sigma}+\log(\frac{1}{3})}\Big)\Big]_{I[X_i\leq\mu]} + n\ln(\sigma) - \sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\sigma}$$
$$- 2\ln\Big(1+e^{\frac{X_i-\mu}{\sigma}+\log(\frac{1}{3})}\Big)\Big]_{I[X_i>\mu]} \tag{114}$$

The partial derivatives are derived with respect to α, σ and μ, from Eq.(114), such that:

$$\frac{\partial l(\boldsymbol{X})}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\Big[\frac{X_i-\mu}{\alpha^2\sigma}\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-3}{e^{\frac{X_i-\mu}{\alpha\sigma}}+3}\Big)\Big]_{I[X_i\leq\mu]} \tag{115}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial \sigma} = \frac{2n}{\sigma} - \sum_{i=1}^{n}\Big[\Big(\frac{X_i-\mu}{\alpha\sigma^2}\Big)\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-3}{e^{\frac{X_i-\mu}{\alpha\sigma}}+3}\Big)\Big]_{I[X_i\leq\mu]} - \sum_{i=1}^{n}\Big[\Big(\frac{X_i-\mu}{\sigma^2}\Big)\Big(\frac{e^{\frac{X_i-\mu}{\sigma}}-3}{e^{\frac{X_i-\mu}{\sigma}}+3}\Big)\Big]_{I[X_i>\mu]} \tag{116}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial \mu} = -\sum_{i=1}^{n}\Big[\Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-3}{e^{\frac{X_i-\mu}{\alpha\sigma}}+3}\Big)_{I[X_i\leq\mu]} + \Big(\frac{e^{\frac{X_i-\mu}{\alpha\sigma}}-3}{e^{\frac{X_i-\mu}{\alpha\sigma}}+3}\Big)_{I[X_i>\mu]}\Big] \tag{117}$$

The second derivatives are derived with respect to α, σ and μ, from Eq.(115), Eq.(116) and Eq.(117) to obtain:

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n}\Big[\Big(\frac{2(X_i-\mu)}{\alpha^4\sigma^2}\Big)\left(\frac{\alpha\sigma e^{\frac{2(X_i-\mu)}{\alpha\sigma}} + (3X_i-3\mu)e^{\frac{X_i-\mu}{\alpha\sigma}} - 9\alpha\sigma}{\Big(e^{\frac{X_i-\mu}{\alpha\sigma}}+3\Big)^2}\right)\Big]_{I[X_i\leq\mu]} \tag{118}$$

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial\sigma^2} = -\frac{2n}{\sigma^2} - \sum_{i=1}^{n}\Big[\Big(\frac{-2(X_i-\mu)}{\alpha^2\sigma^4}\Big)\left(\frac{\alpha\sigma e^{\frac{2(X_i-\mu)}{\alpha\sigma}} + (3X_i-3\mu)e^{\frac{X_i-\mu}{\alpha\sigma}} - 9\alpha\sigma}{\left(e^{\frac{X_i-\mu}{\alpha\sigma}}+3\right)^2}\right)\Big]_{I[X_i\leq\mu]} - \Big[\Big(\frac{2(X_i-\mu)}{\sigma^4}\Big)$$
$$\left(\frac{\sigma e^{\frac{2(X_i-\mu)}{\sigma}} + (3X_i-3\mu)e^{\frac{X_i-\mu}{\sigma}} - 9\sigma}{\left(e^{\frac{X_i-\mu}{\sigma}}+3\right)^2}\right)\Big]_{I[X_i>\mu]} \tag{119}$$

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial\mu^2} = -\sum_{i=1}^{n}\Big[\left(\frac{-6e^{\frac{X_i-\mu}{\alpha\sigma}}}{\alpha\sigma\left(e^{\frac{X_i-\mu}{\alpha\sigma}}+3\right)^2}\right)\Big]_{I[X_i\leq\mu]} - \Big[\left(\frac{6e^{\frac{X_i-\mu}{\sigma}}}{\sigma\left(e^{\frac{X_i-\mu}{\sigma}}+3\right)^2}\right)\Big]_{I[X_i>\mu]} \tag{120}$$

5.3.2 Two-piece cosine distribution

Using the PDF given in Mac‘Oduol et al. (2020) for the two-piece cosine distribution, the log-likelihood function of the two-piece cosine distribution, when $k=\frac{1}{2}$, follows as:

$$\begin{aligned}
l(\boldsymbol{X}) =& -\sum_{i=1}^{n}\Big[\ln(\pi) - \ln(4\alpha\sigma) + \ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big)\Big)\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]} + \Big[\ln(\pi) - \ln(4\sigma) \\
&+ \ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\sigma)}{2\sigma}\Big)\Big)\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]} \\
=& -2n\ln(\pi) + n\ln(4\alpha\sigma) + n\ln(4\sigma) - \sum_{i=1}^{n}\Big[\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big)\Big)\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]} \\
&+ \Big[\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\sigma)}{2\sigma}\Big)\Big)\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned} \tag{121}$$

The partial derivatives are derived with respect to α, σ and μ, from Eq.(121), such that:

$$\frac{\partial l(\boldsymbol{X})}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\Big[\Big(\frac{\pi(X_i-\mu)}{2\alpha^2\sigma}\Big)\Big(\frac{\cos\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}{\sin\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}\Big)\Big]_{II[\mu-\alpha\sigma<X_i\leq\mu]} \tag{122}$$

$$\begin{aligned}
\frac{\partial l(\boldsymbol{X})}{\partial\sigma} =& \frac{2n}{\sigma} + \sum_{i=1}^{n}\Big[\Big(\frac{\pi(X_i-\mu)}{2\alpha\sigma^2}\Big)\Big(\frac{\cos\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}{\sin\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]} \\
&+ \sum_{i=1}^{n}\Big[\Big(\frac{\pi(X_i-\mu)}{2\sigma^2}\Big)\Big(\frac{\cos\left(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\right)}{\sin\left(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\right)}\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned} \tag{123}$$

$$\begin{aligned}
\frac{\partial l(\boldsymbol{X})}{\partial\mu} =& \sum_{i=1}^{n}\Big[\Big(\frac{\pi}{2\alpha\sigma}\Big)\Big(\frac{\cos\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}{\sin\left(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\right)}\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]} \\
&+ \sum_{i=1}^{n}\Big[\Big(\frac{\pi}{2\sigma}\Big)\Big(\frac{\cos\left(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\right)}{\sin\left(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\right)}\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned} \tag{124}$$

Using the PDF given in Eq.(81), the log-likelihood function of the two-piece cosine distribution, when $k=\frac{1}{4}$, follows as:

$$\begin{aligned}
l(\boldsymbol{X}) =& -\sum_{i=1}^{n}\Big[\ln(\pi)-\ln(4\alpha\sigma)+\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big)-\frac{1}{3}\Big)\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]}+\Big[\ln(\pi)-\ln(4\sigma)\\
&+\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\sigma)}{2\sigma}\Big)-\frac{1}{3}\Big)\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}\\
=& n\ln(\pi)+n\ln(4\alpha\sigma)-\sum_{i=1}^{n}\Big[\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\alpha\sigma)}{2\alpha\sigma}\Big)-\frac{1}{3}\Big)\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]}+\Big[\ln(\pi)-\ln(4\sigma)\\
&+\ln\Big(\sin\Big(\pi\Big(\frac{X_i-(\mu-\sigma)}{2\sigma}\Big)-\frac{1}{3}\Big)\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned}\tag{125}$$

The partial derivatives are derived with respect to α,σ and μ, from Eq.(125), such that:

$$\frac{\partial l(\boldsymbol{X})}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\Big[\Big(\frac{3\pi(X_i-\mu)}{2\alpha^2\sigma}\Big)\Big(\frac{\cos\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)}{\Big(3\sin\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)-1\Big)}\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]}\tag{126}$$

$$\begin{aligned}
\frac{\partial l(\boldsymbol{X})}{\partial\sigma}=&\frac{2n}{\sigma}+\sum_{i=1}^{n}\Big[\Big(\frac{3\pi(X_i-\mu)}{2\alpha\sigma^2}\Big)\Big(\frac{\cos\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)}{\Big(3\sin\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)-1\Big)}\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]}\\
&+\sum_{i=1}^{n}\Big[\Big(\frac{3\pi(X_i-\mu)}{2\sigma^2}\Big)\Big(\frac{\cos\Big(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\Big)}{\Big(3\sin\Big(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\Big)-1\Big)}\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned}\tag{127}$$

$$\begin{aligned}
\frac{\partial l(\boldsymbol{X})}{\partial\mu}=&\sum_{i=1}^{n}\Big[\Big(\frac{\pi}{2\alpha\sigma}\Big)\Big(\frac{\cos\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)}{\Big(\sin\Big(\frac{\pi}{2\alpha\sigma}(X_i-(\mu-\alpha\sigma))\Big)-\frac{1}{3}\Big)}\Big)\Big]_{I[\mu-\alpha\sigma<X_i\leq\mu]}\\
&+\sum_{i=1}^{n}\Big[\Big(\frac{\pi}{2\sigma}\Big)\Big(\frac{\cos\Big(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\Big)}{\Big(\sin\Big(\frac{\pi}{2\sigma}(X_i-(\mu-\sigma))\Big)-\frac{1}{3}\Big)}\Big)\Big]_{I[\mu<X_i\leq\mu+\sigma]}
\end{aligned}\tag{128}$$

5.3.3 Two-piece Student's $t(2)$ distribution

Using the PDF given in Mac'Oduol et al. (2020) for the two-piece Student's $t(2)$ distribution, the log-likelihood function of the two-piece Student's $t(2)$ distribution, when $k=\frac{1}{2}$, follows as:

$$\begin{aligned}
l(\boldsymbol{X}) =& -\sum_{i=1}^{n}\Big[\ln(1)-\ln(\alpha\sigma)+\ln\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2\Big)^{-\frac{3}{2}}\Big]_{I[X_i\leq\mu]}+\Big[\ln(1)-\ln(\sigma)+\ln\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2\Big)^{-\frac{3}{2}}\Big]_{I[X_i>\mu]}\\
=& n\ln(\alpha\sigma)+\frac{3}{2}\sum_{i=1}^{n}\Big[\ln\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2\Big)\Big]_{I[X_i\leq\mu]}+n\ln(\sigma)+\frac{3}{2}\sum_{i=1}^{n}\Big[\ln\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2\Big)\Big]_{I[X_i>\mu]}
\end{aligned}\tag{129}$$

The partial derivatives are derived with respect to α, σ and μ, from Eq.(129), such that:

$$\begin{aligned}\frac{\partial l(\boldsymbol{X})}{\partial \alpha} =& \frac{n}{\alpha\sigma}\sigma + \frac{3}{2}\sum_{i=1}^{n}\Big[\Big(\frac{1}{2+\left(\frac{X_i-\mu}{\alpha\sigma}\right)^2}\Big)\Big(\frac{(X_i-\mu)^2}{\sigma^2}\times\frac{-2}{\alpha^3}\Big)\Big]_{I[X_i\leq\mu]}\\ =& \frac{n}{\alpha} - 3\sum_{i=1}^{n}\Big[\Big(\frac{(X_i-\mu)^2}{\alpha^3\sigma^2\Big(2+\left(\frac{X_i-\mu}{\alpha\sigma}\right)^2\Big)}\Big)\Big]_{I[X_i\leq\mu]}\end{aligned} \tag{130}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial \sigma} = \frac{2n}{\sigma} - 3\sum_{i=1}^{n}\Big[\Big(\frac{(X_i-\mu)^2}{\alpha^2\sigma^3\Big(2+\left(\frac{X_i-\mu}{\alpha\sigma}\right)^2\Big)}\Big)\Big]_{I[X_i\leq\mu]} + \Big[\Big(\frac{(X_i-\mu)^2}{\sigma^3\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)}\Big)\Big]_{I[X_i>\mu]} \tag{131}$$

$$\frac{\partial l(\boldsymbol{X})}{\partial \mu} = -3\sum_{i=1}^{n}\Big[\Big(\frac{(X_i-\mu)}{\alpha^2\sigma^2\Big(2+\left(\frac{X_i-\mu}{\alpha\sigma}\right)^2\Big)}\Big)\Big]_{I[X_i\leq\mu]} + \Big[\Big(\frac{(X_i-\mu)}{\sigma^2\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)}\Big)\Big]_{I[X_i>\mu]} \tag{132}$$

The second derivatives are derived with respect to α, σ and μ, from Eq.(130), Eq.(131) and Eq.(132) to obtain:

$$\frac{\partial^2 l(\boldsymbol{X})}{\partial \alpha^2} = -\frac{n}{\alpha^2} + 3\sum_{i=1}^{n}\Big[\frac{(X_i-\mu)^2(X_i^2-2X_i\mu+6\alpha^2\sigma^2+\mu^2)}{\alpha^2(X_i^2-2X_i\mu+2\alpha^2\sigma^2+\mu^2)^2}\Big]_{I[X_i\leq\mu]} \tag{133}$$

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{X})}{\partial \sigma^2} =& -\frac{2n}{\sigma^2} + \sum_{i=1}^{n}\Big[\Big(\frac{3(X_i-\mu)^2(X_i^2-2X_i\mu+6\alpha^2\sigma^2+mu^2)}{\sigma^2(X_i^2-2X_i\mu+2\alpha^2\sigma^2+mu^2)^2}\Big)\Big]_{I[X_i\leq\mu]}\\ &+\Big[\Big(\frac{9(X_i-\mu)^2}{\sigma^4\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)} - \frac{6(X_i-\mu)^2}{\sigma^6\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)^2}\Big)\Big]_{I[X_i>\mu]}\end{aligned} \tag{134}$$

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{X})}{\partial \mu^2} =& \sum_{i=1}^{n}\Big[\frac{-3(X_i^2-2X_i\mu-2\alpha^2\sigma^2+\mu^2)}{(X_i^2-2X_i\mu+2\alpha^2\sigma^2+\mu^2)^2}\Big]_{I[X_i\leq\mu]} + \Big[\Big(\frac{3}{\sigma^2\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)}\Big)\\ &-\Big(\frac{6(X_i-\mu)^2}{\sigma^4\Big(2+\left(\frac{X_i-\mu}{\sigma}\right)^2\Big)^2}\Big)\Big]_{I[X_i>\mu]}\end{aligned} \tag{135}$$

Using the PDF given in Eq.(84), the log-likelihood function of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$ follows as:

$$
\begin{aligned}
l(\boldsymbol{X}) =& -\sum_{i=1}^{n}\Big[\ln(1)-\ln(\alpha\sigma)+\ln\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)^{-\frac{3}{2}}\Big]_{I[X_i\leq\mu]}+\Big[\ln(1)-\ln(\sigma)+\ln\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2 \\
&-\frac{\sqrt{6}}{3}\Big)^{-\frac{3}{2}}\Big]_{I[X_i>\mu]} \\
=& n\ln(\alpha\sigma)+\frac{3}{2}\sum_{i=1}^{n}\Big[\ln\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)\Big]_{I[X_i\leq\mu]}+n\ln(\sigma)+\frac{3}{2}\sum_{i=1}^{n}\Big[\ln\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)\Big]_{I[X_i>\mu]}
\end{aligned}
\tag{136}
$$

The partial derivatives are derived with respect to α, σ and μ, from Eq.(136), such that:

$$
\frac{\partial l(\boldsymbol{X})}{\partial\alpha} = \frac{n}{\alpha}-3\sum_{i=1}^{n}\Big[\frac{(X_i-\mu)^2}{\alpha^3\sigma^2\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big]_{I[X_i\leq\mu]}
\tag{137}
$$

$$
\frac{\partial l(\boldsymbol{X})}{\partial\sigma} = \frac{2n}{\sigma}-3\sum_{i=1}^{n}\Bigg(\Big[\frac{(X_i-\mu)^2}{\alpha^2\sigma^3\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big]_{I[X_i\leq\mu]}+\Big[\frac{(X_i-\mu)^2}{\sigma^3\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big]_{I[X_i>\mu]}\Bigg)
\tag{138}
$$

$$
\frac{\partial l(\boldsymbol{X})}{\partial\mu} = -3\sum_{i=1}^{n}\Bigg(\Big[\frac{(X_i-\mu)^2}{\alpha^2\sigma^2\Big(2+\Big(\frac{X_i-\mu}{\alpha\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big]_{I[X_i\leq\mu]}+\Big[\frac{(X_i-\mu)^2}{\sigma^2\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big]_{I[X_i>\mu]}\Bigg)
\tag{139}
$$

The second derivatives are derived with respect to α, σ and μ, from Eq.(137), Eq.(138) and Eq.(139) to obtain:

$$
\frac{\partial^2 l(\boldsymbol{X})}{\partial\alpha^2} = -\frac{n}{\alpha^2}-27\sum_{i=1}^{n}\Big[\frac{(X_i-\mu)^2((\sqrt{6}-6)\alpha^2\sigma^2-X_i^2+2X_i\mu-\mu^2)}{\alpha^2((\sqrt{6}-6)\alpha^2\sigma^2-3X_i^2+6X_i\mu-3\mu^2)^2}\Big]_{I[X_i\leq\mu]}
\tag{140}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{X})}{\partial\sigma^2} =& -\frac{2n}{\sigma^2}+\sum_{i=1}^{n}\Big[\Big(\frac{-27(X_i-\mu)^2((\sqrt{6}-6)\alpha^2\sigma^2-X_i^2+2X_i\mu-\mu^2)}{\sigma^2((\sqrt{6}-6)\alpha^2\sigma^2-3X_i^2+6X_i\mu-3\mu^2)^2}\Big)\Big]_{I[X_i\leq\mu]} \\
&+\Big[\Big(\frac{9(X_i-\mu)^2}{\sigma^4\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)}\Big)-\Big(\frac{6(X_i-\mu)^4}{\sigma^6\Big(2+\Big(\frac{X_i-\mu}{\sigma}\Big)^2-\frac{\sqrt{6}}{3}\Big)^2}\Big)\Big]_{I[X_i>\mu]}
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{X})}{\partial\mu^2} =& \sum_{i=1}^{n}\Big[\frac{-18(\sqrt{6}-6)\alpha^2\sigma^2(X_i-\mu)}{\Big((6-\sqrt{6})\alpha^2\sigma^2+3\mu^2-6X_i\mu+3X_i^2\Big)^2}\Big]_{I[X_i\leq\mu]} \\
&+\Big[\frac{-18(\sqrt{6}-6)\sigma^2(X_i-\mu)}{\Big((6-\sqrt{6})\sigma^2+3\mu^2-6X_i\mu+3X_i^2\Big)^2}\Big]_{I[X_i>\mu]}
\end{aligned}
\tag{142}
$$

6 Application

6.1 Introduction

The fit of the skew logistic distribution proposed by Balakrishnan et al. (2017) can be determined using the L-moments method or the method of maximizing the likelihood function numerically, since there exists an explicit CDF, PDF and quantile function. The dataset used in this application chapter is from Hand et al. (1993), which reports the pulse rates, in beats per second, of 50 patients in a hospital.

In Chapter 6.2, the method of L-moments estimation was applied to the dataset for the suggested two-piece logistic when $k = \frac{1}{2}$ and the two-piece logistic when $k = \frac{1}{4}$, to compare the best estimated parameters for the dataset. In Chapter 6.3, the method of maximum likelihood estimation was applied to the dataset for the suggested two-piece Student's $t(2)$ when $k = \frac{1}{2}$ and the two-piece Student's $t(2)$ when $k = \frac{1}{4}$, to find the best estimated parameters for the dataset.

6.2 Descriptive results

Figure 7 depicts the histogram of the pulse rates dataset.

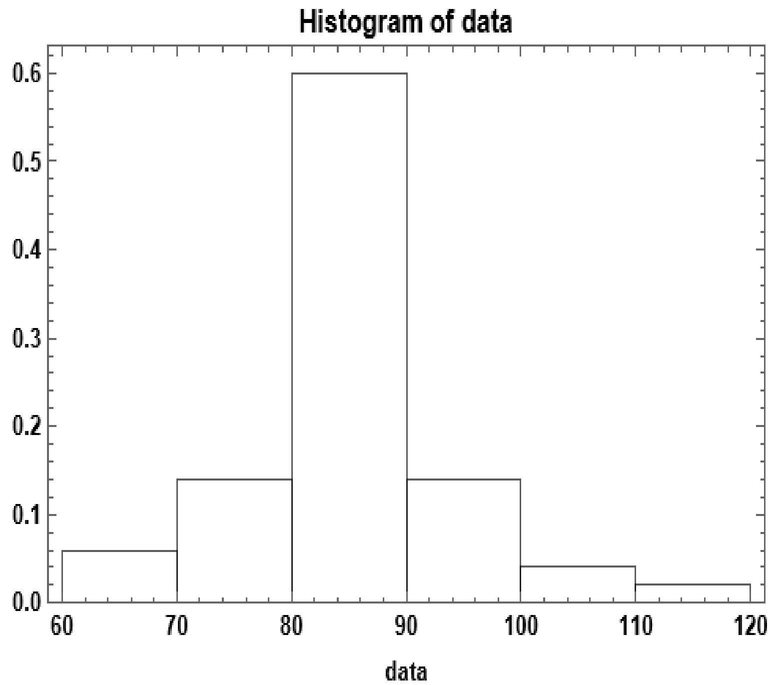


Figure 7: The pulse rates, in beats per minute, of 50 patients in a hospital.

The descriptive statistics for the pulse rates dataset is given in Table 5.

Descriptive statistics measure	Value
Mean	82.3
Variance	79.8469
Skewness moment ratio (α_3)	1.23788
Kurtosis moment ratio (α_4)	6.25636

Table 5: The descriptive statistics measures of the pulse-rates of 50 patients, in beats per minute, in a hospital

The data are clearly skewed to the right, since $\alpha_3 > 1$, and since the kurtosis ratio, $\alpha_4 > 3$, it can be concluded that the pulse rates dataset is leptokurtic, when compared to a normal distribution.

The average scaled absolute error (ASAE), introduced by Castillo and Hadi (1997), is defined as a measure that compares the fit of different models to a dataset. The ASAE is defined as

$$ASAE = \frac{1}{n}\sum_{i=1}^{n}\frac{|x_{i:n} - \hat{Q}(S_{i:n})|}{(x_{n:n} - x_{1:n})}, \qquad i = 1, 2, ..., n, \tag{143}$$

where $\hat{Q}(S_{i:n})$ is the empirical quantile function of the fitted distribution.

The ASAE measure may be used to compare models which are defined by their CDF or PDF or models that are quantile-based. The smaller ASAE value is preferred, as this represents the distribution that has the better fit to the data.

The Akaike's information criterion (AIC) and Bayesian information criterion (BIC) are the most popular criteria used to determine which model is the best fit to the dataset. The AIC is defined as

$$AIC = -2l(\boldsymbol{X}) + 2\kappa, \tag{144}$$

where κ is the number of parameters estimated and $l(\boldsymbol{X})$ is defined as the log-likelihood function.

BIC is defined as

$$AIC = \log(n\kappa) - 2l(\boldsymbol{X}), \tag{145}$$

where n is the sample size. The model with the minimum AIC value is preferred, and the model with the smaller BIC value is preferred, as stated by Vrieze (2012).

6.3 Method of L-moments estimation

6.3.1 Two-piece logistic distribution

The parameter estimates and ASAE value obtained with the method of L-moments is given in Table 6.

Distribution	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	ASAE
Two-Piece Logistic, $k = \frac{1}{2}$	79.9594	6.18347	0.45391	0.04831
Two-Piece Logistic, $k = \frac{1}{2}$	77.6189	7.87184	0.33054	0.12077

Table 6: Parameter estimates of the pulse rates dataset using the method of L-moments estimation

The probability density plots, using the parameter estimates, is given in Figure 8 for the two-piece logistic when $k = \frac{1}{2}$ and Figure 9 for the two-piece logistic when $k = \frac{1}{4}$, respectively.

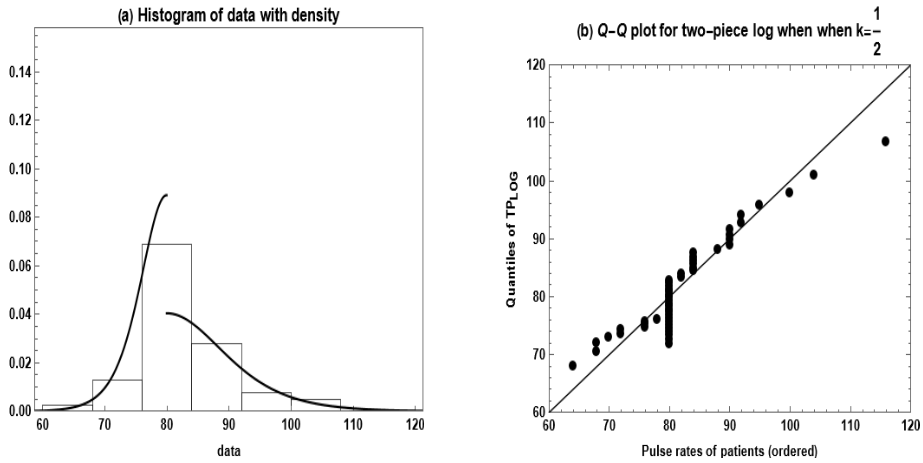


Figure 8: The probability density plot and the QQ-plot given for the two-piece logistic, when $k = \frac{1}{2}$

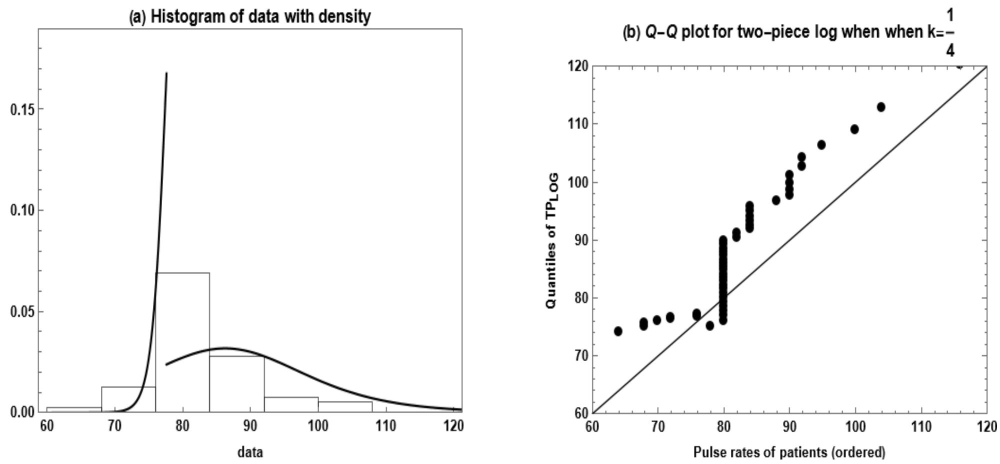


Figure 9: The probability density plot and the QQ-plot given for the two-piece logistic, when $k = \frac{1}{4}$

The density plotted for the two-piece logistic distribution, when $k = \frac{1}{2}$ provides the best fit to the histogram of the data, since it does not overfit the data as the density of the two-piece logistic distribution, when $k = \frac{1}{4}$ does. The QQ-plot in Figure 8 is closer to a straight line than that of the QQ-plot in Figure 9. When comparing the ASAE values, the value for $k = \frac{1}{2}$ is 0.04831, which is much lower than that of the value for $k = \frac{1}{4}$, which is 0.12077. We can conclude that the better fit to the dataset is by the two-piece logistic distribution, when $k = \frac{1}{2}$.

6.3.2 Two-piece Student's $t(2)$

The parameter estimates and ASAE value obtained with the method of L-moments is given in Table 6.

Distribution	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	ASAE
Two-Piece Student's $t(2)$, $k = \frac{1}{2}$	80.612	5.821	0.544	0.0242
Two-Piece Student's $t(2)$, $k = \frac{1}{4}$	77.119	5.516	0.4448	0.078

Table 7: Parameter estimates of the pulse rates dataset using the method of L-moments estimation

The probability density plots, using the parameter estimates obtained via the method of L-moments estimation, is given in Figure 10 for the two-piece Student's $t(2)$ when $k = \frac{1}{2}$ and Figure 11 for the two-piece Student's $t(2)$ when $k = \frac{1}{4}$, respectively.

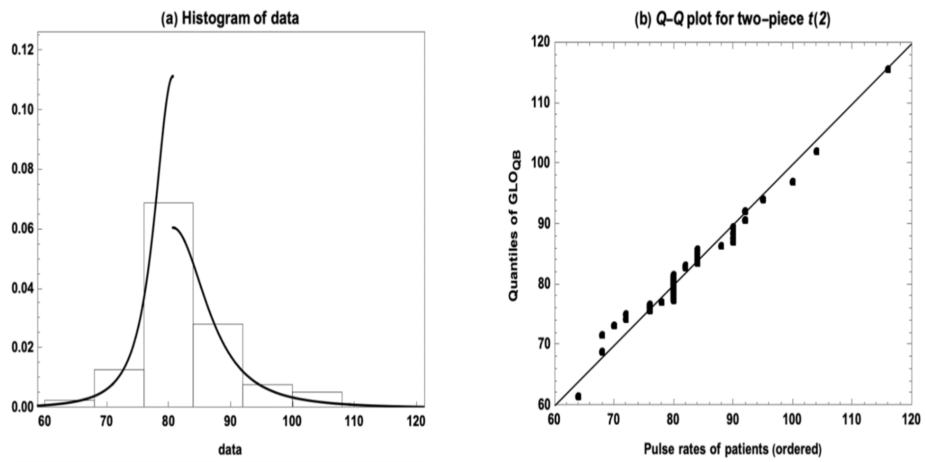


Figure 10: The probability density plot and the QQ-plot given for the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$

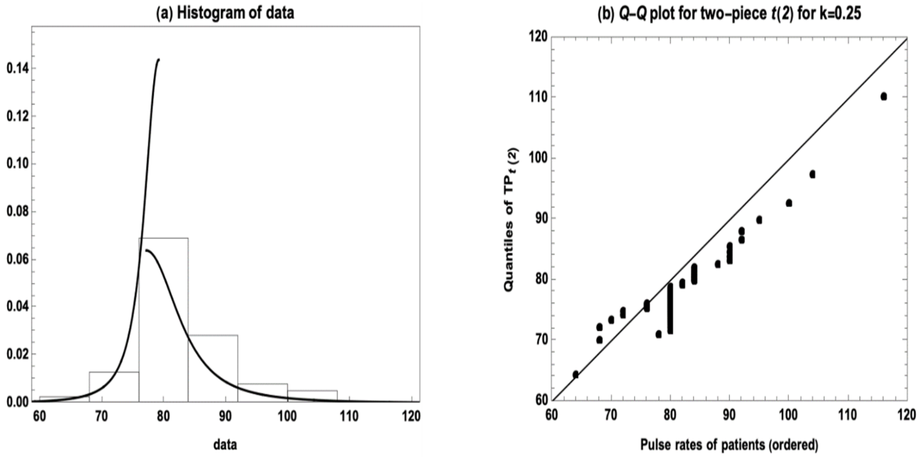


Figure 11: The probability density plot and the QQ-plot given for the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$

The QQ-plot in Figure 10 is closer to a straight line, than that of the QQ-plot in Figure 11, and there is less underfitting in the density plot over the histogram of the data in Figure 10, therefore the better fit is obtained by using the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$. When comparing the ASAE values, the value for $k = \frac{1}{4}$ is 0.078 while $k = \frac{1}{2}$ is 0.0242. Since this value is smaller, it shows the two-piece Student's $t(2)$ distribution when $k = \frac{1}{2}$ provides the better fit.

6.4 Maximum likelihood estimation

6.4.1 Two-piece logistic

The parameter estimates, the AIC, BIC and loglikelihood values obtained with the method of maximum likelihood estimation are given in Table 8.

Distribution	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	**AIC**	**BIC**	**Loglikelihood**
Two-Piece Logistic, $k = \frac{1}{2}$	79.5893	6.28687	0.33054	-713.80937	-706.16128	360.90469
Two-Piece Logistic, $k = \frac{1}{2}$	79.9999	10.81089	0.34094	-772.44516	-764.79707	390.22258

Table 8: Parameter estimates of the pulse rates dataset, for the two-piece logistic distribution when $k = \frac{1}{2}$ and $k = \frac{1}{4}$.

The probability density plots are given in Figure 12 for the two-piece logistic when $k = \frac{1}{2}$ and Figure 13 for the two-piece logistic when $k = \frac{1}{4}$, respectively.

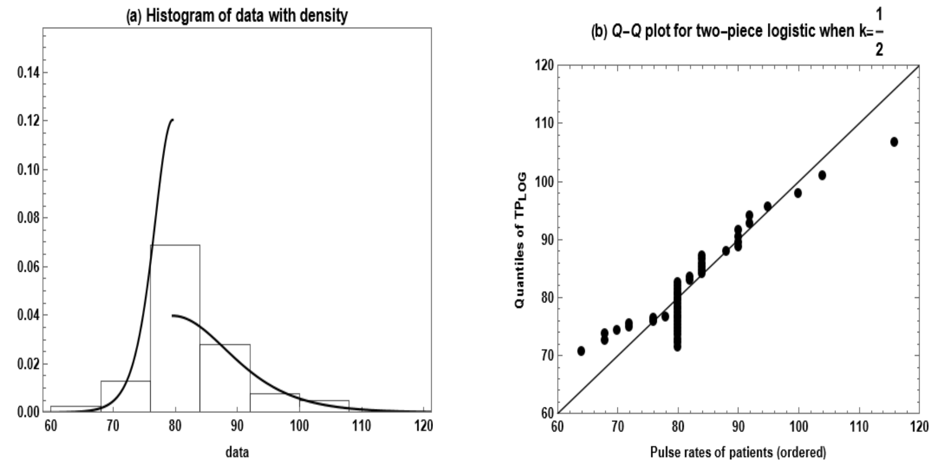


Figure 12: The probability density plot and the QQ-plot given for the two-piece logistic, when $k = \frac{1}{2}$

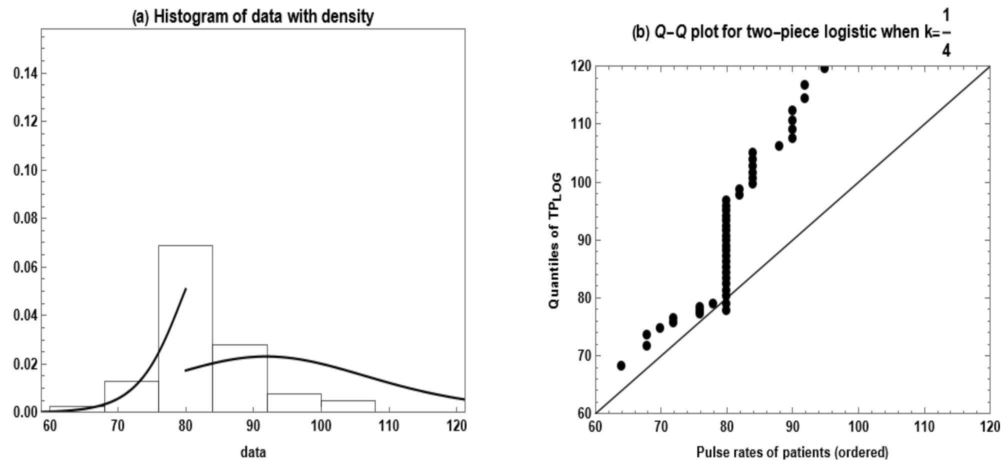


Figure 13: The probability density plot and the QQ-plot given for the two-piece logistic, when $k = \frac{1}{4}$

The density plotted for the two-piece logistic distribution, when $k = \frac{1}{2}$ fits the histogram of the dataset better, as it overfits the data. The QQ-plot of the two-piece logistic distribution, when $k = \frac{1}{2}$, given in Figure 12, lies more closely to the straight line that that of the QQ-plot of the two-piece logistic distribution, when $k = \frac{1}{4}$, given in Figure 13. The AIC value for the two-piece logistic distribution, when $k = \frac{1}{2}$, is -713.80937, compared to the AIC value of -772.44516 obtained for the two-piece logistic distribution, when $k = \frac{1}{4}$. The smallest AIC value is preferred, therefore the two-piece logistic distribution, when $k = \frac{1}{4}$, is preferred for the pulse rates data with the obtained parameter estimates.

6.4.2 Two-piece Student's $t(2)$

The parameter estimates, the AIC, BIC and loglikelihood values obtained with the method of maximum likelihood estimation are given in Table 6.

Distribution	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	**AIC**	**BIC**	**Loglikelihood**
Two-Piece Student's $t(2)$, $k = \frac{1}{2}$	80.0004	6.99999	0.29999	-125.80595	-118.15785	66.90297
Two-Piece Student's $t(2)$, $k = \frac{1}{4}$	80.9687	10.82799	0.29390	-695.16621	-687.51812	351.58310

Table 9: Parameter estimates of the pulse rates dataset, for the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$ and $k = \frac{1}{4}$.

The probability density plots are given in Figure 14 for the two-piece Student's $t(2)$ when $k = \frac{1}{2}$ and Figure 15 for the two-piece Student's $t(2)$ when $k = \frac{1}{4}$, respectively.

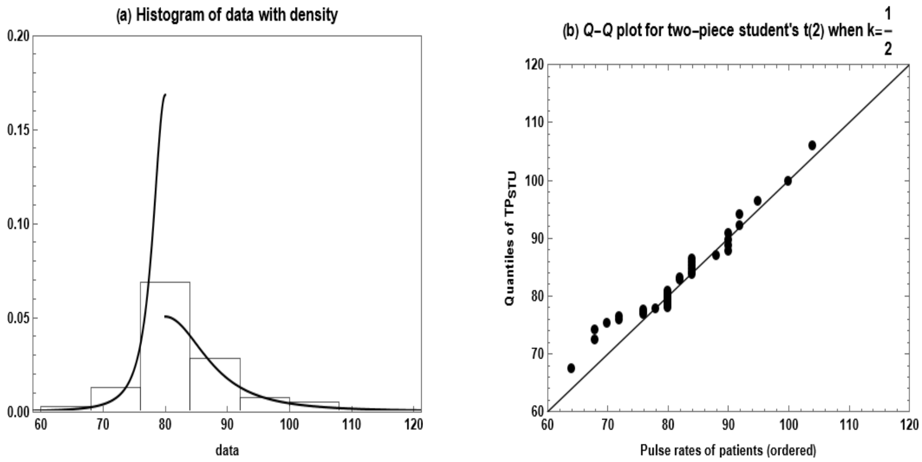


Figure 14: The probability density plot and the QQ-plot given for the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$

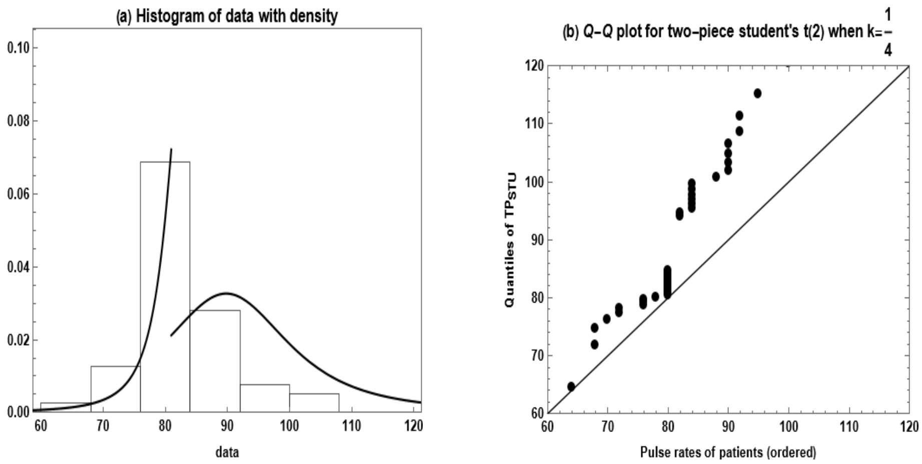


Figure 15: The probability density plot and the QQ-plot given for the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$

The density plot of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$, overfits the histogram of the dataset on the left of mode, but is a better fit to the dataset on the right of the mode. The density plot of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$, underfits the histogram of the dataset to the left of the mode and overfits the histogram of the dataset to the right of the mode.

The AIC value of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$ is -125.80595, compared to the AIC value of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{4}$, which is -695.16621, is bigger. The QQ-plot of the two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$ gives a better fit to the dataset, with the parameter estimates obtained using the method of maximum likelihood estimation.

7 Conclusion

In this mini-dissertation, the method of maximum likelihood estimation was applied to two-piece distributions obtained through quantile splicing. These distributions include the two-piece logistic, cosine and Student's $t(2)$ distribution, when $k = \frac{1}{4}$. The results obtained using the method of L-moments for the above-mentioned distributions is compared to the results obtained using the method of maximum likelihood estimation.

The method of maximum likelihood estimation for quantile-based distributions is explored, by deriving the loglikelihood functions together with the first and second derivatives of the loglikelihood functions for the two-piece logistic, cosine and Student's $t(2)$ distributions, when $k = \frac{1}{2}$ and when $k = \frac{1}{4}$. These results are applied to the pulse rates dataset, obtained from Hand et al. (1993). The results obtained after applying the method of L-moments estimation and the method of maximum likelihood estimation to the pulse rates dataset, for the two-piece logistic and two-piece Student's $t(2)$ distribution, when $k = \frac{1}{2}$ and $k = \frac{1}{4}$, are compared.

Although the method of maximum likelihood estimation did not provide significant results compared to those obtained using the method of L-moments, it has now been investigated. The method of maximum likelihood estimation is a laborious method to apply to two-piece distributions, since the derivations in the different quartiles are tedious to determine.

There exists other estimation methods that can be applied to two-piece distributions such as the Bayes estimator explored in Fernández and Steel (1998), method-of-moments estimation used by Mudholkar and Hutson (2000) as well as probability-weighted moments, proposed by Gilchrist (2000) as a straightforward method to apply.

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A Appendix

This chapter contains the derivations of the r^{th} order L-moments results given for the two-piece univariate distributions in Chapter 4.3.1-4.3.3. The scaling factor is equated to $k = \frac{1}{4}$ to obtain the L-moments of the two-piece distribution that is spliced at the lower quartile.

A.0.1 Two-piece logistic distribution

Theorem 10. *Let T be a real-valued standardized random variable that originates from the two-piece logistic distribution, defined as $T \sim L_{TP}(0, 1, \alpha)$, where the location parameter is $\mu = 0$, the scale parameter is $\sigma = 1$, and $\alpha > 0$ is the asymmetric parameter. Suppose $k = \frac{1}{4}$, then the first 4 L-moments of the two-piece logistic distribution follows as*

$$\begin{aligned}
L_{T:1} =& 1.38629 - 0.28768\alpha, \\
L_{T:2} =& 0.75 + 0.25\alpha, \\
L_{T:3} =& 0.1875 - 0.1875\alpha, \\
L_{T:4} =& 0.04685 + 0.11979\alpha.
\end{aligned}$$

Proof. For $r = 1$, the first moment of T is

$$\begin{aligned}
L_{T:1} =& \int_0^{\frac{1}{4}} \mu + \alpha\sigma \log\Big(\frac{u}{1-u}\Big)\ du + \int_{\frac{1}{4}}^{1} \mu + \sigma \log\Big(\frac{u}{1-u}\Big)\ du - \int_0^{\frac{1}{4}} \mu + \alpha\sigma \log\Big(\frac{k}{1-k}\Big)\ du \\
&- \int_{\frac{1}{4}}^{1} \mu + \sigma \log\Big(\frac{k}{1-k}\Big)\ du \\
=& \int_0^{\frac{1}{4}} \alpha \log\Big(\frac{u}{1-u}\Big)\ du + \int_{\frac{1}{4}}^{1} \log\Big(\frac{u}{1-u}\Big)\ du - \int_0^{\frac{1}{4}} \alpha \log\Big(\frac{1}{3}\Big)\ du - \int_{\frac{1}{4}}^{1} \log\Big(\frac{1}{3}\Big)\ du \\
=& \alpha\Big[\log(1-u) + u \log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_0^{\frac{1}{4}} + \Big[\log(1-u) + u \log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_{\frac{1}{4}}^{1} - \alpha \log\Big(\frac{1}{3}\Big)\ [\frac{1}{4} - 0] \\
&- \log\Big(\frac{1}{3}\Big)\ [1 - \frac{1}{4}] \\
=& \alpha(-0.562335) + (0.562335) + 0.274653\alpha + 0.823959 \\
=& 1.38629 - 0.28768\alpha
\end{aligned}$$

The second L-moment of T, when $P_1(p) = 2p - 1$ and $r = 2$, is derived as

$$
\begin{aligned}
L_{T:2} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \log\Big(\frac{u}{1-u}\Big)\Big)(2u-1)\ du + \int_{\frac{1}{4}}^{1} \Big(\mu + \sigma \log\Big(\frac{u}{1-u}\Big)\Big)(2u-1)\ du \\
& - \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \log\Big(\frac{k}{1-k}\Big)\Big)(2u-1)\ du - \int_{\frac{1}{4}}^{1} \Big(\mu + \sigma \log\Big(\frac{k}{1-k}\Big)\Big)(2u-1)\ du \\
=& \int_0^{\frac{1}{4}} \Big(\alpha \log\Big(\frac{u}{1-u}\Big)\Big)(2u-1)\ du + \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(2u-1)\ du - \int_0^{\frac{1}{4}} \Big(\alpha \log\Big(\frac{1}{3}\Big)\Big)(2u-1)\ du \\
& - \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{1}{3}\Big)\Big)(2u-1)\ du \\
=& 2\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u)\ du - \alpha \int_0^{\frac{1}{4}} \log\Big(\frac{u}{1-u}\Big)\ du + 2\int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u)\ du - \int_{\frac{1}{4}}^{1} \log\Big(\frac{u}{1-u}\Big)\ du \\
& - \alpha \log\Big(\frac{1}{3}\Big) \int_0^{\frac{1}{4}} (2u-1)\ du - \log\Big(\frac{1}{3}\Big) \int_{\frac{1}{4}}^{1} (2u-1)\ du \\
=& 2\alpha\Big[\frac{1}{2}\Big(u + \log(1-u) + u^2\log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_0^{\frac{1}{4}} - \alpha\Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big] \Big|_0^{\frac{1}{4}} \\
& + 2\Big[\frac{1}{2}\Big(u + \log(1-u) + u^2\log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_{\frac{1}{4}}^{1} - \Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big] \Big|_{\frac{1}{4}}^{1} - \alpha \log\Big(\frac{1}{3}\Big)\Big[\frac{2u^2}{2} - u\Big] \Big|_0^{\frac{1}{4}} \\
& - \log\Big(\frac{1}{3}\Big)\Big[\frac{2u^2}{2} - u\Big] \Big|_{\frac{1}{4}}^{1} \\
=& \alpha(-0.106345) - \alpha(-0.562335) + (1.106235) - (0.562335) - 0.20599\alpha + 0.20599 \\
=& 0.75 + 0.25\alpha
\end{aligned}
$$

When $r = 3$ and $P_2(p) = 6p^2 - 6p + 1$, the third L-moment follows as

$$
\begin{aligned}
L_{T:3} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \, \log\Big(\frac{u}{1-u}\Big)\Big)(6u^2 - 6u + 1) \; du + \int_{\frac{1}{4}}^{1} \Big(\mu + \sigma \, \log\Big(\frac{u}{1-u}\Big)\Big)(6u^2 - 6u + 1) \; du \\
& - \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \, \log\Big(\frac{k}{1-k}\Big)\Big)(6u^2 - 6u + 1) \; du - \int_{\frac{1}{4}}^{1} \Big(\mu + \sigma \, \log\Big(\frac{k}{1-k}\Big)\Big)(6u^2 - 6u + 1) \; du \\
=& \int_0^{\frac{1}{4}} \Big(\alpha \, \log\Big(\frac{u}{1-u}\Big)\Big)(6u^2 - 6u + 1) \; du + \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(6u^2 - 6u + 1) \; du \\
& - \int_0^{\frac{1}{4}} \Big(\alpha \, \log\Big(\frac{1}{3}\Big)\Big)(6u^2 - 6u + 1) \; du - \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{1}{3}\Big)\Big)(6u^2 - 6u + 1) \; du \\
=& 6\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^2) \; du - 6\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u) \; du + \alpha \int_0^{\frac{1}{4}} \log\Big(\frac{u}{1-u}\Big) \; du \\
& + 6 \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^2) \; du - 6 \int_{\frac{1}{4}}^{1} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u) \; du + \int_{\frac{1}{4}}^{1} \log\Big(\frac{u}{1-u}\Big) \; du \\
& - \alpha \, \log\Big(\frac{1}{3}\Big) - \int_0^{\frac{1}{4}} (6u^2 - 6u + 1) \; du - \log\Big(\frac{1}{3}\Big) \int_{\frac{1}{4}}^{1} (6u^2 - 6u + 1) \; du \\
=& 6\alpha \Big[\frac{1}{6}\Big(2\log(1-u) + 2u + u^2 + 2u^3 \log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_0^{\frac{1}{4}} - 6\alpha \Big[\frac{1}{2}\Big(u + \log(1-u) + u^2 \log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_0^{\frac{1}{4}} \\
& + \alpha \Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_0^{\frac{1}{4}} + 6\Big[\frac{1}{6}\Big(2\log(1-u) + 2u + u^2 + 2u^3 \log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_{\frac{1}{4}}^{1} \\
& - 6\Big[\frac{1}{2}\Big(u + \log(1-u) + u^2 \log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_{\frac{1}{4}}^{1} + \Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big)\Big] \Big|_{\frac{1}{4}}^{1} \\
& - \alpha \, \log\Big(\frac{1}{3}\Big)\Big[\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big] \Big|_0^{\frac{1}{4}} - \log\Big(\frac{1}{3}\Big)\Big[\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big] \Big|_{\frac{1}{4}}^{1} \\
=& \alpha(-0.047196) - \alpha(-0.319036) + \alpha(-0.562335) + (3.047196) - (3.319036) + (0.562335) \\
& + 0.102995\alpha - 0.102995 \\
=& 0.1875 - 0.1875\alpha
\end{aligned}
$$

The fourth L-moment of T, where $r = 4$ and $P_3(p) = 20p^3 - 30p^2 + 12p - 1$, is

$$
\begin{aligned}
L_{T:4} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \log\Big(\frac{u}{1-u}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \int_{\frac{1}{4}}^1 \Big(\mu + \sigma \log\Big(\frac{u}{1-u}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma \log\Big(\frac{k}{1-k}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_{\frac{1}{4}}^1 \Big(\mu + \sigma \log\Big(\frac{k}{1-k}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
=& \int_0^{\frac{1}{4}} \Big(\alpha \log\Big(\frac{u}{1-u}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du + \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\alpha \log\Big(\frac{1}{3}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du - \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{1}{3}\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
=& 20\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^3)\ du - 30\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^2)\ du + 12\alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u)\ du \\
&- \alpha \int_0^{\frac{1}{4}} \Big(\log\Big(\frac{u}{1-u}\Big)\Big)\ du + 20 \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^3)\ du - 30 \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u^2)\ du \\
&+ 12 \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{u}{1-u}\Big)\Big)(u)\ du - \int_{\frac{1}{4}}^1 \Big(\log\Big(\frac{u}{1-u}\Big)\Big)\ du - \alpha\log\Big(\frac{1}{3}\Big) \int_0^{\frac{1}{4}} (20u^3 - 30u^2 + 12u - 1)\ du \\
&- \log\Big(\frac{1}{3}\Big) \int_{\frac{1}{4}}^1 (20u^3 - 30u^2 + 12u - 1)\ du \\
=& 20\alpha\Big[\frac{1}{24}\Big(6\log(1-u) + 6u + 3u^2 + 2u^3 + 6u^4\log\Big(\frac{u}{1-u}\Big)\Big)\Big]\ \Big|_0^{\frac{1}{4}} \\
&- 30\alpha\Big[\frac{1}{6}\Big(2\log(1-u)\Big) + 2u + u^2 + 2u^3\log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_0^{\frac{1}{4}} + 12\alpha\Big[\frac{1}{2}\Big(u + \log(1-u) + u^2\log\Big(\frac{u}{1-u}\Big)\Big)\Big]\ \Big|_0^{\frac{1}{4}} \\
&- \alpha\Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_0^{\frac{1}{4}} + 20\Big[\frac{1}{24}\Big(6\log(1-u) + 6u + 3u^2 + 2u^3 + 6u^4\log\Big(\frac{u}{1-u}\Big)\Big)\Big]\ \Big|_{\frac{1}{4}}^1 \\
&- 30\Big[\frac{1}{6}\Big(2\log(1-u)\Big) + 2u + u^2 + 2u^3\log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_{\frac{1}{4}}^1 + 12\Big[\frac{1}{2}\Big(u + \log(1-u) + u^2\log\Big(\frac{u}{1-u}\Big)\Big)\Big]\ \Big|_{\frac{1}{4}}^1 \\
&- \Big[\log(1-u) + u\log\Big(\frac{u}{1-u}\Big)\Big]\ \Big|_{\frac{1}{4}}^1 - \alpha\log\Big(\frac{1}{3}\Big)\Big[\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big]\ \Big|_0^{\frac{1}{4}} \\
&- \log\Big(\frac{1}{3}\Big)\Big[\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big]\ \Big|_0^{\frac{1}{4}} \\
=& \alpha(-0.027576) - \alpha(-0.235979) + \alpha(-0.638072) - \alpha(-0.562335) + (9.19424) - (15.236) + (6.63807) \\
&- (0.562335) - 0.012874\alpha + 0.012874 \\
=& 0.04685 + 0.11979\alpha
\end{aligned}
$$

These results were obtained using Gradshteyn et al. (1996) (2.723.1, 2.729.1-2.729.4) □

A.0.2 Two-piece cosine distribution

Theorem 11. *Let T be a real-valued standardized random variable following a two-piece cosine distribution, represented by $T \sim COS_{TP}(0, 4, \alpha)$, where the location parameter $\mu = 0$, 4 represents the* *scale parameter σ and $\alpha > 0$ is the asymmetry parameter. Applying a transformation of variables, let $z = \arcsin(\sqrt{u})$. It then follows that $u = \sin^2(z)$ and $du = \sin(2z)$. z is evaluated on the interval $(0, \frac{\pi}{6})$ and $(\frac{\pi}{6}, \frac{\pi}{2})$. The first 4 L-moments of the cosine distribution, with a scaling factor of $k = \frac{1}{4}$ follows as*

$$\begin{aligned}
L_{T:1} =& 1.56401 - 0.230676\alpha, \\
L_{T:2} =& 0.804499 + 0.195501\alpha, \\
L_{T:3} =& 0.137832 - 0.137832\alpha, \\
L_{T:4} =& -0.0143277 + 0.076825\alpha.
\end{aligned}$$

Proof. For $r = 1$, the first moment of T is

$$\begin{aligned}
L_{T:1} =& \int_0^{\frac{1}{4}} \mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)du + \int_{\frac{1}{4}}^1 \mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)du - \int_0^{\frac{1}{4}} \mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)du \\
&- \int_{\frac{1}{4}}^1 \mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)du \\
=& \int_0^{\frac{1}{4}} 4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)du + \int_{\frac{1}{4}}^1 4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)du - \int_0^{\frac{1}{4}} 4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)du \\
&- \int_{\frac{1}{4}}^1 4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)\ du \\
=& 4\alpha\int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)du - 4\alpha\int_0^{\frac{1}{4}} 1\ du + 4\int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)du - 4\int_0^{\frac{1}{4}} 1\ du + \frac{4\alpha}{3}\int_0^{\frac{1}{4}} 1\ du + \frac{4}{3}\int_{\frac{1}{4}}^1 1\ du \\
=& 4\alpha\int_0^{\frac{\pi}{6}} \frac{4}{\pi}\ z\ \sin(2z)\ dz - 4\alpha\Big(\frac{1}{4} - 0\Big) + 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4}{\pi}\ z\ \sin(2z)\ dz - 4\Big(1 - \frac{1}{4}\Big) + \frac{4\alpha}{3}\Big(\frac{1}{4} - 0\Big) + \frac{4}{3}\Big(1 - \frac{1}{4}\Big) \\
=& 4\alpha\Big[\frac{4}{\pi}\Big(- \frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} - \alpha + 4\Big[\frac{4}{\pi}\Big(- \frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - 3 + \frac{\alpha}{3} + 1 \\
=& 4\alpha\Big[\frac{\sqrt{3}}{2\pi} - \frac{1}{6}\Big] - \alpha + 4\Big[\frac{7}{6} - \frac{\sqrt{3}}{2\pi}\Big] - 2 + \frac{\alpha}{3} \\
=& 1.56401 - 0.230676\alpha
\end{aligned}$$

The second L-moment of T, when $P_1(p) = 2p-1$ and $r=2$, is obtained as

$$\begin{aligned}
L_{T:2} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}-1\Big)\Big)(2u-1)\ du + \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}-1\Big)\Big)(2u-1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k}-1\Big)\Big)(2u-1)\ du - \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k}-1\Big)\Big)(2u-1)\ du \\
=& \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}-1\Big)\Big)(2u-1)\ du + \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}-1\Big)\Big)(2u-1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}}-1\Big)\Big)(2u-1)\ du - \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}}-1\Big)\Big)(2u-1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(2u)\ du - \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du - \int_0^{\frac{1}{4}} (2u)\ du + \int_0^{\frac{1}{4}} 1\ du\Big] \\
&+ 4\Big[\int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(2u)\ du - \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du - \int_0^{\frac{1}{4}} (2u)\ du + \int_0^{\frac{1}{4}} 1\ du\Big] \\
&- \int_0^{\frac{1}{4}} 4\alpha\Big(\frac{-1}{3}\Big)(2u-1)\ du - \int_{\frac{1}{4}}^1 4\Big(\frac{-1}{3}\Big)(2u-1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{\pi}{6}} \frac{8}{\pi}\ z\ \sin^2(z)\ \sin(2z)\ dz - \int_0^{\frac{\pi}{6}} \frac{4}{\pi}\ z\ \sin(2z)\ dz - \Big(\frac{2u^2}{2}\Big)\ \Big|_0^{\frac{1}{4}} + \Big(\frac{1}{4}-0\Big)\Big] \\
&+ 4\Big[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{8}{\pi}\ z\ \sin^2(z)\ \sin(2z)\ dz - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4}{\pi}\ z\ \sin(2z)\ dz - \Big(\frac{2u^2}{2}\Big)\ \Big|_{\frac{1}{4}}^1 + \Big(1-\frac{1}{4}\Big)\Big] \\
&+ \frac{4\alpha}{3}\Big[\frac{2u^2}{2}-u\Big]\ \Big|_0^{\frac{1}{4}} + \frac{4}{3}\Big[\frac{2u^2}{2}-u\Big]\ \Big|_0^{\frac{1}{4}} \\
=& 4\alpha\Bigg(\Big[\frac{1}{8\pi}\Big(-16z\ \cos(2z) + 4z\cos(4z) + 8\sin(2z) - \sin(4z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} - \Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} \\
&- \Big(\frac{1}{16}\Big) + \Big(\frac{1}{4}\Big)\Bigg) + 4\Bigg(\Big[\frac{1}{8\pi}\Big(-16z\ \cos(2z) + 4z\cos(4z) + 8\sin(2z) - \sin(4z)\Big)\Big]\ \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&- \Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \Big(\frac{15}{16}\Big) + \Big(\frac{3}{4}\Big)\Bigg) + \frac{4\alpha}{3}\Big(\frac{-3}{16}\Big) + \frac{4}{3}\Big(\frac{3}{16}\Big) \\
=& 4\alpha\Bigg(\Big(\frac{7\sqrt{3}}{16\pi} - \frac{5}{24}\Big) - \Big(\frac{\sqrt{3}}{2\pi} - \frac{1}{6}\Big) - \Big(\frac{1}{16}\Big) + \Big(\frac{1}{4}\Big)\Bigg) + 4\Bigg(\Big(\frac{7}{48}\Big(10 - \frac{3\sqrt{3}}{\pi}\Big)\Big) - \Big(\frac{7}{6} - \frac{\sqrt{3}}{2\pi}\Big) - \Big(\frac{15}{16}\Big) + \Big(\frac{3}{4}\Big)\Bigg) \\
&- \frac{\alpha}{4} + \frac{1}{4} \\
=& 0.804499 + 0.195501\alpha
\end{aligned}$$

When $r = 3$ and $P_2(p) = 6p^2 - 6p + 1$, the third L-moment follows as

$$
\begin{aligned}
L_{T:3} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(6u^2 - 6u + 1)\ du + \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(6u^2 - 6u + 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)\Big)(6u^2 - 6u + 1)\ du - \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)\Big)(6u^2 - 6u + 1)\ du \\
=& \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(6u^2 - 6u + 1)\ du + \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(6u^2 - 6u + 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)\Big)(6u^2 - 6u + 1)\ du - \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)\Big)(6u^2 - 6u + 1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(6u^2)\ du - \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(6u)\ du + \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du \\
&- \int_0^{\frac{1}{4}} (6u^2 - 6u + 1)\ du\Big] + 4\Big[\int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(6u^2)\ du - \int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(6u)\ du \\
&+ \int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du - \int_{\frac{1}{4}}^1 (6u^2 - 6u + 1)\ du\Big] + \frac{4\alpha}{3}\int_0^{\frac{1}{4}} (6u^2 - 6u + 1)\ du + \frac{4}{3}\int_{\frac{1}{4}}^1 (6u^2 - 6u + 1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{\pi}{6}} \frac{24}{\pi}\ z\ \sin^4(z)\ \sin(2z)\ dz - \int_0^{\frac{\pi}{6}} \frac{24}{\pi}\ z\ \sin^2(z)\ \sin(2z)\ dz + \int_0^{\frac{\pi}{6}} \frac{4}{\pi}\ z\ \sin(2z)\ dz \\
&- \Big(\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big)\ \Big|_0^{\frac{1}{4}}\Big] + 4\Big[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{24}{\pi}\ z\ \sin^4(z)\ \sin(2z)\ dz - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{24}{\pi}\ z\ \sin^2(z)\ \sin(2z)\ dz \\
&+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4}{\pi}\ z\ \sin(2z)\ dz - \Big(\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big)\ \Big|_0^{\frac{1}{4}}\Big] + \frac{4\alpha}{3}\Big(\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big)\ \Big|_0^{\frac{1}{4}} + \frac{4}{3}\Big(\frac{6u^3}{3} - \frac{6u^2}{2} + u\Big)\ \Big|_{\frac{1}{4}}^1 \\
=& 4\alpha\Bigg(\Big[\frac{1}{24\pi}\Big(-90z\ \cos(2z) + 36z\ \cos(4z) - 6z\ \cos(6z) + 45\sin(2z) - 9\sin(4z) + \sin(6z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} \\
&- \Big[\frac{3}{8\pi}\Big(-16z\ \cos(2z) + 4z\ \cos(4z) + 8\sin(2z) - \sin(4z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} + \Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} \\
&- \frac{1}{32} + \frac{3}{16} - \frac{1}{4}\Bigg) + 4\Bigg(\Big[\frac{1}{24\pi}\Big(-90z\ \cos(2z) + 36z\ \cos(4z) - 6z\ \cos(6z) + 45\sin(2z) - 9\sin(4z) \\
&+ \sin(6z)\Big)\Big]\ \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \Big[\frac{3}{8\pi}\Big(-16z\ \cos(2z) + 4z\ \cos(4z) + 8\sin(2z) - \sin(4z)\Big)\Big]\ {}_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&+ \Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\ \cos(2z) + \frac{1}{4}\sin(2z)\Big)\Big]\ \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{27}{32} + \frac{27}{16} - \frac{3}{4}\Bigg) + \frac{4\alpha}{3}\Big(\frac{3}{32}\Big) + \frac{4}{3}\Big(-\frac{3}{32}\Big) \\
=& 4\alpha\Big[\Big(\frac{3\sqrt{3}}{4\pi} - \frac{19}{48}\Big) - \Big(\frac{21\sqrt{3}}{16\pi} - \frac{5}{8}\Big) + \Big(\frac{\sqrt{3}}{2\pi} - \frac{1}{6}\Big) - \frac{1}{32} + \frac{3}{16} - \frac{1}{4}\Big] + 4\Big[\Big(\frac{151}{48} - \frac{3\sqrt{3}}{4\pi}\Big) - \frac{7}{16}\Big(10 - \frac{3\sqrt{3}}{\pi}\Big) \\
&+ \Big(\frac{7}{6} - \frac{\sqrt{3}}{2\pi}\Big) - \frac{27}{32} + \frac{27}{16} - \frac{3}{4}\Big] + \frac{\alpha}{8} - \frac{1}{8} \\
=& 0.137832 - 0.137832\alpha
\end{aligned}
$$

The fourth L-moment of T, where $r = 4$ and $P_3(p) = 20p^3 - 30p^2 + 12p - 1$, follows as

$$
\begin{aligned}
L_{T:4} =& \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\mu + \alpha\sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_{\frac{1}{4}}^1 \Big(\mu + \sigma\Big(\frac{4}{\pi}\text{arcsin}\sqrt{k} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
=& \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{u} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big(4\alpha\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_{\frac{1}{4}}^1 \Big(4\Big(\frac{4}{\pi}\text{arcsin}\sqrt{\frac{1}{4}} - 1\Big)\Big)(20u^3 - 30u^2 + 12u - 1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(20u^3)\ du - \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(30u^2)\ du + \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(12u)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du - \int_0^{\frac{1}{4}} (20u^3 - 30u^2 + 12u - 1)\ du\Big] + 4\Big[\int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(20u^3) \\
&- \int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(30u^2)\ du + \int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)(12u)\ du - \int_{\frac{1}{4}}^1 \Big(\frac{4}{\pi}\text{arcsin}\sqrt{u}\Big)\ du \\
&- \int_{\frac{1}{4}}^1 (20u^3 - 30u^2 + 12u - 1)\ du\Big] + \frac{4\alpha}{3}\int_0^{\frac{1}{4}} (20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \frac{4}{3}\int_{\frac{1}{4}}^1 (20u^3 - 30u^2 + 12u - 1)\ du \\
=& 4\alpha\Big[\int_0^{\frac{\pi}{6}} \frac{80}{\pi}\ z\sin^6(z)\ \sin(2z)\ dz - \int_0^{\frac{\pi}{6}} \frac{120}{\pi}\ z\sin^4(z)\ \sin(2z)\ dz + \int_0^{\frac{\pi}{6}} \frac{48}{\pi}\ z\sin^2(z)\ \sin(2z)\ dz \\
&- \int_0^{\frac{\pi}{6}} \frac{4}{\pi}\ z\sin(2z)\ dz - \Big(\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big)\ \Big|_0^{\frac{1}{4}}\Big] + 4\Big[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{80}{\pi}\ z\sin^6(z)\ \sin(2z)\ dz \\
&- \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{120}{\pi}\ z\sin^4(z)\ \sin(2z)\ dz + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{48}{\pi}\ z\sin^2(z)\ \sin(2z)\ dz - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4}{\pi}\ z\sin(2z)\ dz \\
&- \Big(\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big)\ \Big|_{\frac{1}{4}}^1\Big] + \frac{4\alpha}{3}\Big(\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big)\ \Big|_0^{\frac{1}{4}} \\
&+ \frac{4}{3}\Big(\frac{20u^4}{4} - \frac{30u^3}{3} + \frac{12u^2}{2} - u\Big)\ \Big|_{\frac{1}{4}}^1 \\
=& 4\alpha\Bigg(\Big[\frac{5}{768\pi}\Big(-1344z\ \cos(2z) + 672z\ \cos(4z) - 192z\ \cos(6z) + 24z\cos(8z) + 672\sin(2z) - 168\sin(4z) \\
&+ 32\sin(6z) - 3\sin(8z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} - \Big[\frac{5}{24\pi}\Big(-90z\ \cos(2z) + 36z\ \cos(4z) - 6z\ \cos(6z) + 45\sin(2z) - 9\sin(4z) \\
&+ \sin(6z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} + \Big[\frac{3}{4\pi}\Big(-16z\ \cos(2z) + 4z\ \cos(4z) + 8\sin(2z) - \sin(4z)\Big)\Big]\ \Big|_0^{\frac{\pi}{6}} - \Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\ \cos(2z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\sin(2z)\Big)\Big]\Big|_0^{\frac{\pi}{6}}-\frac{-3}{256}\Bigg)+4\Bigg(\Big[\frac{5}{768\pi}\Big(-1344z\cos(2z)+672z\cos(4z)-192z\cos(6z)+24z\cos(8z)\\
&+672\sin(2z)-168\sin(4z)+32\sin(6z)-3\sin(8z)\Big)\Big]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}-\Big[\frac{5}{24\pi}\Big(-90z\cos(2z)+36z\cos(4z)-6z\cos(6z)\\
&+45\sin(2z)-9\sin(4z)+\sin(6z)\Big)\Big]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}+\Big[\frac{3}{4\pi}\Big(-16z\cos(2z)+4z\cos(4z)+8\sin(2z)-\sin(4z)\Big)\Big]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\\
&-\Big[\frac{4}{\pi}\Big(-\frac{1}{2}z\cos(2z)+\frac{1}{4}\sin(2z)\Big)\Big]\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}-\frac{3}{256}\Bigg)+\frac{4\alpha}{3}\Big(\frac{-3}{256}\Big)+\frac{4}{3}\Big(\frac{3}{256}\Big)\\
=&4\alpha\Big[\frac{5}{512\pi}\Big(169\sqrt{3}-92\pi\Big)-\Big(\frac{15\sqrt{3}}{4\pi}-\frac{95}{48}\Big)+\Big(\frac{21\sqrt{3}}{8\pi}-\frac{5}{4}\Big)-\Big(\frac{\sqrt{3}}{2\pi}-\frac{1}{6}\Big)+\frac{3}{256}\Big]\\
&+4\Big[\frac{5}{512\pi}\Big(-169\sqrt{3}+836\pi\Big)-\Big(\frac{755}{48}-\frac{15\sqrt{3}}{4\pi}\Big)+\frac{7}{8}\Big(10-\frac{3\sqrt{3}}{\pi}\Big)-\Big(\frac{7}{6}-\frac{\sqrt{3}}{2\pi}\Big)-\frac{3}{256}\Big]-\frac{\alpha}{64}+\frac{1}{64}\\
=&-0.0143277+0.076825\alpha
\end{aligned}
$$

These results were obtained using Gradshteyn et al. (1996) (1.321.1, 1.321.2, 1.323.2, 1.335.1, 2.633.1). □

A.0.3 Two-piece Student's $t(2)$ distribution

Theorem 12. *Let T be a random variable that is real-valued and standardized and that follows a two-piece Student's $t(2)$ distribution, denoted by $T \sim t(2)_{TP}(0, \frac{2\sqrt{2}}{\pi}, \alpha)$, where te location parameter is $\mu = 0$, the scale parameter $\sigma = \frac{2\sqrt{2}}{\pi}$ and the asymmetry parameter is given by $\alpha > 0$. The first 4 L-moments of the Student's $t(2)$ distribution, for $k = \frac{1}{4}$ follows as*

$$
\begin{aligned}
L_{T:1} =&1.10266-0.36755\alpha,\\
L_{T:2} =&0.66667+0.33333\alpha,\\
L_{T:3} =&0.27566-0.27566\alpha,\\
L_{T:4} =&0.16855+0.211145\alpha.
\end{aligned}
$$

Proof. The first moment of T, when $r = 1$, is

$$
\begin{aligned}
L_{T:1} =& \int_0^{\frac{1}{4}} \mu + \alpha\sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du + \int_{\frac{1}{4}}^{1} \mu + \sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du - \int_0^{\frac{1}{4}} \mu + \alpha\sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\ du \\
&- \int_{\frac{1}{4}}^{1} \mu + \sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\ du \\
=& \int_0^{\frac{1}{4}} \frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du + \int_{\frac{1}{4}}^{1} \frac{2\sqrt{2}}{\pi}\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du - \int_0^{\frac{1}{4}} \frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{-\sqrt{6}}{3}\Big)\ du \\
&- \int_{\frac{1}{4}}^{1} \frac{2\sqrt{2}}{\pi}\Big(\frac{-\sqrt{6}}{3}\Big)\ du \\
=&\alpha\Big[-\frac{4}{\pi}\Big(\sqrt{-u(u-1)}\Big)\Big]\ \Big|_0^{\frac{1}{4}} + \Big[-\frac{4}{\pi}\Big(\sqrt{-u(u-1)}\Big)\Big]\ \Big|_{\frac{1}{4}}^{1} + \frac{4\sqrt{3}}{3\pi}\alpha\Big(\frac{1}{4}-0\Big) + \frac{4\sqrt{3}}{3\pi}\Big(1-\frac{1}{4}\Big) \\
=&1.10266 - 0.36755\alpha
\end{aligned}
$$

The second L-moment of T, when $P_1(p) = 2p - 1$ and $r = 2$, is

$$
\begin{aligned}
L_{T:2} =& \int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](2u-1)\ du + \int_{\frac{1}{4}}^{1} \Big[\mu + \sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](2u-1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](2u-1)\ du - \int_{\frac{1}{4}}^{1} \Big[\mu + \sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](2u-1)\ du \\
=& \int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](2u-1)\ du + \int_{\frac{1}{4}}^{1} \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](2u-1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{-\sqrt{6}}{3}\Big)\Big](2u-1)\ du - \int_{\frac{1}{4}}^{1} \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{-\sqrt{6}}{3}\Big)\Big](2u-1)\ du \\
=&\frac{2\sqrt{2}}{\pi}\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(2u)\ du - \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] + \frac{2\sqrt{2}}{\pi}\Big[\int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(2u)\ du \\
&- \int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] + \frac{4\sqrt{3}}{3\pi}\alpha\int_0^{\frac{1}{4}}(2u-1)\ du + \frac{4\sqrt{3}}{3\pi}\int_{\frac{1}{4}}^{1}(2u-1)\ du \\
=&\frac{2\sqrt{2}}{\pi}\alpha\left(\Big(\frac{-9\sqrt{3}+4\pi}{24\sqrt{2}}\Big) + \frac{\sqrt{\frac{3}{2}}}{2}\right) + \frac{2\sqrt{2}}{\pi}\left(\frac{1}{48}\Big(9\sqrt{6}+8\sqrt{2}\pi\Big) - \Big(\frac{\sqrt{\frac{3}{2}}}{2}\Big)\right) + \frac{4\sqrt{3}}{3\pi}\alpha\Big(-\frac{3}{16}\Big) \\
&+ \frac{4\sqrt{3}}{3\pi}\Big(\frac{3}{16}\Big) \\
=&0.66667 + 0.33333\alpha
\end{aligned}
$$

When $r = 3$ and $P_2(p) = 6p^2 - 6p + 1$, the third L-moment follows as

$$
\begin{aligned}
L_{T:3} = &\int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](6u^2 - 6u + 1)\ du + \int_{\frac{1}{4}}^1 \Big[\mu + \sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](6u^2 - 6u + 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](6u^2 - 6u + 1)\ du - \int_{\frac{1}{4}}^1 \Big[\mu + \sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](6u^2 - 6u + 1)\ du \\
= &\int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](6u^2 - 6u + 1)\ du + \int_{\frac{1}{4}}^1 \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](6u^2 - 6u + 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{-\sqrt{6}}{3}\Big)\Big](6u^2 - 6u + 1)\ du - \int_{\frac{1}{4}}^1 \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{-\sqrt{6}}{3}\Big)\Big](6u^2 - 6u + 1)\ du \\
= &\frac{2\sqrt{2}}{\pi}\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(6u^2)\ du - \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(6u)\ du + \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] \\
&+ \frac{2\sqrt{2}}{\pi}\Big[\int_{\frac{1}{4}}^1 \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(6u^2)\ du - \int_{\frac{1}{4}}^1 \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(6u)\ du + \int_{\frac{1}{4}}^1 \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] \\
&+ \frac{4\sqrt{3}}{3\pi}\alpha\int_0^{\frac{1}{4}} (6u^2 - 6u + 1)\ du + \frac{4\sqrt{3}}{3\pi}\int_{\frac{1}{4}}^1 (6u^2 - 6u + 1)\ du \\
= &\frac{2\sqrt{2}}{\pi}\alpha\Bigg(\Big(\frac{-15\sqrt{3} + 8\pi}{16\sqrt{2}}\Big) - \Big(\frac{-9\sqrt{3} + 4\pi}{8\sqrt{2}}\Big) + \Big(-\frac{\sqrt{\frac{3}{2}}}{2}\Big)\Bigg) + \frac{2\sqrt{2}}{\pi}\Bigg(\Big(\frac{15}{16}\sqrt{\frac{3}{2}} + \frac{\pi}{\sqrt{2}}\Big) - \Big(\frac{9}{8}\sqrt{\frac{3}{2}} + \frac{\pi}{\sqrt{2}}\Big) \\
&+ \Big(\frac{\sqrt{\frac{3}{2}}}{2}\Big)\Bigg) + \frac{4\sqrt{3}}{3\pi}\alpha\Big(\frac{3}{32}\Big) + \frac{4\sqrt{3}}{3\pi}\Big(-\frac{3}{32}\Big) \\
= &0.27566 - 0.27566\alpha
\end{aligned}
$$

The fourth L-moment of T, where $r = 4$ and $P_3(p) = 20p^3 - 30p^2 + 12p - 1$, follows as

$$
\begin{aligned}
L_{T:4} =& \int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \int_{\frac{1}{4}}^{1} \Big[\mu + \sigma\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\mu + \alpha\sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_{\frac{1}{4}}^{1} \Big[\mu + \sigma\Big(\frac{2k-1}{\sqrt{2k(1-k)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
=& \int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \int_{\frac{1}{4}}^{1} \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_0^{\frac{1}{4}} \Big[\frac{2\sqrt{2}}{\pi}\alpha\Big(\frac{-\sqrt{6}}{3}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
&- \int_{\frac{1}{4}}^{1} \Big[\frac{2\sqrt{2}}{\pi}\Big(\frac{-\sqrt{6}}{3}\Big)\Big](20u^3 - 30u^2 + 12u - 1)\ du \\
=& \frac{2\sqrt{2}}{\pi}\alpha\Big[\int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(20u^3)\ du - \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(30u^2)\ du + \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(12u)\ du \\
&- \int_0^{\frac{1}{4}} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] + \frac{2\sqrt{2}}{\pi}\Big[\int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(20u^3)\ du - \int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(30u^2)\ du \\
&+ \int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)(12u)\ du - \int_{\frac{1}{4}}^{1} \Big(\frac{2u-1}{\sqrt{2u(1-u)}}\Big)\ du\Big] + \frac{4\sqrt{3}}{3\pi}\alpha\int_0^{\frac{1}{4}} (20u^3 - 30u^2 + 12u - 1)\ du \\
&+ \frac{4\sqrt{3}}{3\pi}\int_{\frac{1}{4}}^{1} (20u^3 - 30u^2 + 12u - 1)\ du \\
=& \frac{2\sqrt{2}}{\pi}\alpha\Bigg(-\frac{5}{128\sqrt{2}}\Big(73\sqrt{3} - 40\pi\Big) + \frac{5}{16\sqrt{2}}\Big(15\sqrt{3} - 8\pi\Big) + \frac{1}{8}\Big(-9\sqrt{6} + 4\sqrt{2}\pi\Big) - \Big(-\frac{\sqrt{\frac{3}{2}}}{2}\Big)\Bigg) \\
&+ \frac{2\sqrt{2}}{\pi}\Bigg(\frac{5}{128\sqrt{2}}\Big(73\sqrt{3} + 80\pi\Big) - \frac{5}{32}\Big(15\sqrt{6} + 16\sqrt{2}\pi\Big) + \Big(\frac{9}{4}\sqrt{\frac{3}{2}} + 4\sqrt{2}\pi\Big) - \Big(\frac{\sqrt{\frac{3}{2}}}{2}\Big)\Bigg) \\
&+ \frac{4\sqrt{3}}{3\pi}\alpha\Big(-\frac{3}{256}\Big) + \frac{4\sqrt{3}}{3\pi}\Big(\frac{3}{256}\Big) \\
=& 0.16855 + 0.211145\alpha
\end{aligned}
$$

These results were obtained using Wolfram Research, Inc. (2022). □